BILINEAR OPTIMAL CONTROL FOR THE FRACTIONAL LAPLACIAN: ERROR ESTIMATES ON LIPSCHITZ DOMAINS

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Abstract. Given a desired state $u_\Omega \in L^2(\Omega)$ and a regularization parameter $\lambda > 0$, we introduce the cost functional

$$J(u,q) = \frac{1}{2}\|u - u_\Omega\|^2_{L^2(\Omega)} + \frac{\lambda}{2}\|q\|^2_{L^2(\Omega)}. \tag{1}$$

Let $f \in L^2(\Omega)$ be a fixed function. We shall be concerned with the following optimal control problem: Find $\min J(u,q)$ subject to the fractional elliptic PDE

$$(-\Delta)^s u + qu = f \text{ in } \Omega, \quad u = 0 \text{ in } \Omega^c, \tag{2}$$

where $\Omega^c = \mathbb{R}^d \setminus \Omega$, and the control constraints

$$q \in \mathbb{Q}_{ad}, \quad \mathbb{Q}_{ad} := \{v \in L^\infty(\Omega) : a \leq v(x) \leq b \text{ for a.e. } x \in \Omega\}. \tag{3}$$

Here, the control bounds a and b are such that $0 < a < b$ and $(-\Delta)^s$ corresponds to the integral definition of the fractional Laplace operator, namely:

$$(-\Delta)^s w(x) := C(d,s)\text{p.v.}\int_{\mathbb{R}^d} \frac{w(x) - w(y)}{|x-y|^{d+2s}}\mathrm{d}y, \quad C(d,s) := \frac{2^{2s} s\Gamma(s + \frac{d}{2})}{\pi^{\frac{d}{2}}\Gamma(1-s)},$$

where p.v. stands for the Cauchy principal value and $C(d,s)$ is a normalization constant.

In this work we establish the existence of optimal solutions and analyze first and, necessary and sufficient, second order optimality conditions. Regularity estimates for optimal variables are also analyzed. We devise two strategies of finite element discretization: a semidiscrete scheme where the control variable is not discretized and a fully discrete scheme where the control variable is discretized with piecewise constant functions. For both solution techniques, we analyze convergence properties of discretizations and derive error estimates.

Keywords: optimal control, fractional diffusion, integral fractional Laplacian, first and second order optimality conditions, regularity estimates, finite elements, convergence, error estimates.

Mathematics Subject Classifications (2010): 35R11, 49J20, 49K20, 49M25, 65K10, 65N15, 65N30.

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