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# A DEEPER INVESTIGATION ON VIRTUAL ELEMENT ACCURACY: THE ROLE OF BULK AND BOUNDARY APPROXIMATIONS

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ABSTRACT. The Virtual Element Method (VEM) was introduced in [?] as a generalization of the finite element method (FEM) that is able to cope with general polytopal meshes; since its introduction, the VEM enjoyed a large success in the numerical analysis and engineering communities.

The present talk, based on [?], will not deal with advanced applications of the method but rather with its foundations, and may be of interest in general for schemes making use of polygonal meshes. Standard h-interpolation (and convergence) estimates for shape regular meshes in FEM and VEM involve the diameter $h_E$ of elements as the main grid parameter; in the presence of triangular (or quadrilateral) shape regular meshes this gives a complete picture. Instead, in the presence of more general meshes one may wonder if polygons with many small edges (and an associated “richer” discrete space such as those used in VEM or Polygonal FEM) can yield, in some sense, better interpolation properties and if this will reflect also on the final error among the discrete and exact solutions. Basically, the answer is no, but the investigation allows to shed more light on the matter and develop an interesting variant.

Looking into the interpolation capabilities of the VEM space, by a refined analysis we show that the $H^1$ interpolation error on each element (polygon) $E$ can be split into a boundary contribution and a bulk contribution. Although for basic VEM spaces the bulk contribution will dominate the error, this investigation leads to the following idea: if one increases the degree of the VEM only inside the element then the bulk approximation order improves. For such “enriched” VEM, elements with small edges indeed lead to more accurate interpolation in a sense that we will make precise.

In the VEM setting, in order for such refined interpolation property to reflect also on an improved convergence property, one needs also to ameliorate the stability estimates of the scheme. Indeed, standard VEM stabilization estimates assume a bounded number of edges, an hyphotesis that we are able to eliminate leading to final convergence estimates that show an improvement in the presence of many small edges (with respect to standard estimates looking only at the element diameter). Furthermore, we will show some numerical test both for quadrilateral/Voronoi meshes with edge subdivision and on meshes generated by an agglomeration procedure.

**Keywords**: Virtual element method, bubble enrichment, interpolation and stability estimates

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