ON THE ANALYTICAL SOLUTION OF THE RIEMANN PROBLEM FOR A SCALAR BALANCE LAW WITH A NON-CONVEX FLUX AND DISCONTINUOUS SOURCE TERM

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ABSTRACT. This paper is concerned with the explicit construction the Riemann problem

$$\begin{aligned} u_t + (f(u))_x &= g(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+, \\ u(x,0) &= u_0(x), \quad x \in \mathbb{R}, \end{aligned}$$

where the flux is of the form $f(u) = u(u-1)(u+1)$ and the source term and the initial condition are given by

$$g(x) = g_R H(x) + g_L H(-x), \quad u_0(x) = u_R H(x) + u_L H(-x), \quad (g_L, g_R, u_L, u_R) \in \mathbb{R}^4,$$

with H is the Heaviside function defined by $H(x) = 0$ for $x \in \mathbb{R}^-$ and $H(x) = 1$ for $x \in \mathbb{R}_0^+$. The main result of this work is the following theorem: "*There is a partition of $\mathbb{R}^4$ characterizing the different types of possible entropic solution in terms of (u_L, u_R, g_L, g_R).*" The proof is constructive and the analysis is developed in several Lemmas. First, we apply the characteristics method (the characteristic curves for the non-convex flux are cubic polynomials) and introduce a classification of the different types of waves. A systematic discussion of the all possible type of waves at $t = 0$, implies the existence of sixteen cases. Then, we give an advance of the solution of several cases and the analytic construction of these solution types. Basically, and in a broad sense, shock and rarefaction wave are formed at $t = 0$. The evolution of shock curve is completely characterized by analyzing the initial value problem obtained by the application of Rankine-Hugoniot condition. The rarefaction wave solution is explicitly obtained by the characteristics method. Finally, by a unification of the sixteen cases, we obtain the required partition of $\mathbb{R}^4$.

Keywords: Riemann problem, discontinuous source term, non-convex flow.

Mathematics Subject Classifications (2010): 35L50, 35L67, 35Q53

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