THE SPACE STRUCTURE OF ORDER REDUCTION IN HIGH-ORDER RUNGE–KUTTA METHODS FOR INITIAL BOUNDARY VALUE PROBLEMS

RODOLFO R. ROSALES, BENJAMIN SEIBOLD, DAVID SHIROKOFF, AND DONG ZHOU

Abstract. The phenomenon of order reduction for time step integrators has been of interest for many years — e.g., see [1] through [13], (intended as a sample, and not inclusive). Here we will concentrate on describing the space structure associated with the order reduction phenomenon for pde initial-boundary-value problems that occurs with many Runge-Kutta time-stepping schemes. First, we will illustrate the phenomena with some numerical examples. Second, we will introduce a geometric explanation of the mechanics of the phenomenon: the approximation error develops boundary layers, induced by a mismatch between the approximation error in the interior and at the boundaries. Third, we will describe how an analysis of the modes of the numerical scheme explains under which circumstances the boundary layers persist over many time steps, leading to order reduction. Fourth, we will provide a new condition on the Butcher tableau, called *weak stage order*, which can recover the full order and is compatible with diagonally implicit Runge-Kutta schemes. An example scheme satisfying the condition will be shown. Note that a second, somewhat less appealing, approach to avoid order reduction is the use of modified boundary conditions. Finally, we will describe some fundamental differences between order reduction for pde's and for stiff ode's. In particular, why the reduction orders differ.

Keywords: Order reduction, Time stepping schemes, Initial boundary value problems,

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Department of Mathematics, Massachusetts Institute of Technology (MIT)
E-mail address: rrr@math.mit.edu

Department of Mathematics, Temple U.
E-mail address: seibold@temple.edu

Department of Mathematics, New Jersey Institute of Technology (NJIT)
E-mail address: david.g.shirokoff@njit.edu

Department of Mathematics, California State University, Los Angeles
E-mail address: dzhou11@calstatela.edu