NEARBY PRECONDITIONING FOR MULTIPLE REALISATIONS OF THE HELMHOLTZ EQUATION

IVAN G. GRAHAM, OWEN R. PEMBERY, AND EUAN A. SPENCE

Abstract. Let $A^{(j)}$, $j = 1, 2$, be the Galerkin matrices corresponding to the $h$-FEM discretisation of the exterior Dirichlet problem for the heterogeneous Helmholtz equations

$$\nabla \cdot (A^{(j)} \nabla u^{(j)}) + k^2 n^{(j)} u^{(j)} = -f.$$  

In this work we answer the following question: How small must $\|A^{(1)} - A^{(2)}\|_{L^\infty}$ and $\|n^{(1)} - n^{(2)}\|_{L^\infty}$ be (in terms of $k$-dependence) for GMRES applied to either $(A^{(1)})^{-1} A^{(2)}$ or $A^{(2)} (A^{(1)})^{-1}$ to converge in a $k$-independent number of iterations for arbitrarily large $k$? (In other words, for $A^{(1)}$ to be a good left- or right-preconditioner for $A^{(2)}$.)

We prove that, if

$$k\|A^{(1)} - A^{(2)}\|_{L^\infty} \text{ and } k\|n^{(1)} - n^{(2)}\|_{L^\infty} \text{ are both sufficiently small}$$

then $A^{(1)}$ is a good preconditioner for $A^{(2)}$ when using weighted GMRES, and numerical experiments show the conditions (1) are sharp. Moreover, numerical experiments show that the conditions (1) are sharp for standard GMRES, but to prove $A^{(1)}$ is a good preconditioner for $A^{(2)}$ for standard GMRES we require a slightly stronger condition on $A^{(1)}$ and $A^{(2)}$ than that in (1).

Our motivation for tackling this question comes from calculations in uncertainty quantification (UQ) for the Helmholtz equation with random coefficients $A$ and $n$. Such a calculation requires the solution of many deterministic Helmholtz problems, each with different $A$ and $n$. The answer to the question above dictates to what extent a previously-calculated preconditioner of one of the Galerkin matrices can be used as a preconditioner for other Galerkin matrices. The extent to which one can reuse preconditioners reduces the cost of the UQ calculation.

Department of Mathematical Sciences, University of Bath
E-mail address: i.g.graham@bath.ac.uk

Department of Mathematical Sciences, University of Bath
E-mail address: o.r.pembery@bath.ac.uk

Department of Mathematical Sciences, University of Bath
E-mail address: e.a.spence@bath.ac.uk