HIGH-ORDER SOBOLEV ORTHOGONAL POLYNOMIALS, APPROXIMATION THEORY AND SPECTRAL METHODS IN THE BALL

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Abstract. We introduce orthogonal polynomials with respect to high-order weighted Sobolev inner products in the ball equivalent to those of the form

$$(f,g) \mapsto \sum_{|\gamma|\leq m} \int_{B^d} D^\gamma f(x)\,\overline{D^\gamma g(x)}\,(1-\|x\|^2)^\alpha\,\mathrm{d}x, \tag{1}$$

where $B^d \subset \mathbb{R}^d$ is the Euclidean unit ball, $\alpha > -1$ and $D^\gamma = \partial^{|\gamma|}/(\partial x_1^{\gamma_1} \cdots \partial x_d^{\gamma_d})$. The Sobolev spaces defined through such inner products appear naturally in the analysis of partial differential equations with singular coefficients. However, we switch to a family of *ad hoc* inner products whose orthogonal polynomials are eigenfunctions of explicit Sturm–Liouville-type problems, yet induce norms equivalent to those induced by the family (1). Thus, the approximation bounds for the modified inner product orthogonal projections can be derived through simple, classical arguments and are still relevant for the original use cases.

We focus on *spaces* as opposed to *bases* of orthogonal polynomials, thus circumventing the long computations often necessary in the treatment of multivariate orthogonal polynomials.

We present orthogonal decompositions and differential operators connecting different spaces of Sobolev orthogonal polynomials across polynomial degrees, differentiability levels (m) and boundary singularity parameters (α).

We discuss consequences on the numerical analysis of spectral methods for a class of elliptic problems with singular coefficients posed on the ball and its Cartesian powers.

Keywords: Orthogonal projection; Weighted Sobolev space; Unit ball; Orthogonal polynomials.

Mathematics Subject Classifications (2010): 41A10, 41A63, 42C10, 65N15, 65N35.

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