FINITE ELEMENT COMPLEXES FOR THE STOKES EQUATION

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Abstract. The Stokes equation is a basic model in fluid mechanics and can be obtained from the Navier-Stokes equation by taking solutions that are constant in time and neglecting the convective, non-linear term. The discretization of the Stokes equation by finite element methods has attracted sustained attention at least since the seventies [1]. From the point of view of mixed finite elements, the challenge is to define two finite element spaces $X_h \subseteq \mathrm{H}^1(S)^n$ and $Y_h \subseteq \mathrm{L}^2(S)$ (where S is the domain) such that the divergence operator gives a surjection $\mathrm{div} : X_h \to Y_h$, with a right inverse uniformly bounded in h. Since this has proved difficult, the goal has been relaxed in several ways, for instance by non-conforming methods and methods that do not guarantee the incompressibility of the flow [3].

We approach this problem through the lens of constructing subcomplexes of the de Rham complex with sufficient regularity, such as:

$$\mathrm{H}^2(S) \xrightarrow{\mathrm{grad}} \mathrm{H}^1(\mathrm{curl}, S) \xrightarrow{\mathrm{curl}} \mathrm{H}^1(S) \xrightarrow{\mathrm{div}} \mathrm{L}^2(S). \tag{1}$$

We achieve the goal as initially formulated by defining composite elements that are piecewise polynomial with respect to different simplicial refinements at each index in the complex. For instance the complexes typically start with a variant of Clough-Tocher elements for scalar functions of class $\mathrm{C}^1(S)$. The gluing conditions between cells, that insure the required regularities of the fields, is taken care of by a framework of finite element systems, which is a natural discrete analogue of sheaf theory. It guarantees the existence of degrees of freedom that provide commuting interpolators. This is joint work with Kaibo Hu [2].

Keywords: Stokes equation, mixed finite elements, de Rham complex

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