IMPLICIT-EXPLICIT SCHEMES FOR DEGENERATE CONVECTION-DIFFUSION PDE

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Abstract. When using the method of lines for the approximate solution of nonlinear, possibly strongly degenerate, convection-diffusion partial differential equations, Implicit-Explicit (IMEX) [11] Runge-Kutta (RK) methods, that combine an explicit RK scheme for the time integration of the convective part with a diagonally implicit one for the diffusive part, are suitable for, at least, the following reasons:

(1) The stability restrictions, coming from the explicitly treated convective part, are much less severe than those that would be deduced from an explicit treatment of the diffusive term.
(2) Since the convective terms may be dominant in some spatio-temporal regions, care must be taken for its appropriate high-order upwind approximation. This entails a fairly sophisticated discretization, whose implicit treatment would be highly intricate and could lead to badly behaved nonlinear systems.

In [8] a scheme of this type is proposed, where the nonlinear and nonsmooth systems of algebraic equations arising in the implicit treatment of the degenerate diffusive part are solved by smoothing of the diffusion coefficients combined with a damped Newton-Raphson method with a line search strategy for globalizing convergence.

This nonlinearly implicit method is robust but associated with considerable effort of implementation and possibly CPU time. To overcome these burdens while keeping the advantageous stability properties of IMEX-RK methods, a second variant of these methods is proposed in [2], in which the diffusion terms are discretized in a way that more carefully distinguishes between stiff and nonstiff dependence, such that in each time step only a linear system needs to be solved, still maintaining high order accuracy in time, which makes these methods much simpler to implement.

These Linearly Implicit-Explicit Runge-Kutta (LIMEX-RK) schemes, based on partitioned Runge-Kutta methods, may be advantageous in some cases, but are not advisable in those cases where special structure of the diffusive terms would be lost. This is the case of some nonlinear convection-diffusion equations with nonlocal flux and possibly degenerate diffusion that arise in various contexts including interacting gases, granular flows, flow in porous media and collective behavior in biology [13, 14].

The work in [4] is concerned with numerical methods for a nonlinear nonlocal equation with a gradient flow structure, whose numerical solution by an explicit finite difference method is costly due to the necessity of discretizing a local spatial convolution for each evaluation of the convective numerical flux, and due to the disadvantageous Courant-Friedrichs-Lewy (CFL) condition incurred by the diffusion term. Based on explicit schemes for such models devised in [9] a second-order implicit-explicit Runge-Kutta (IMEX-RK) method can be formulated.

The first-order version of this method is proven to yield uniquely solvable nonlinear systems of equations that maintain the positivity of the solution (which typically are probability distribution functions or densities). This result would not be obtained for LIMEX-RK methods, since the special structure of the diffusion term (a Laplacian of a nonlinear function) plays a key role in the proof for the existence and uniqueness of a solution.

In this talk a survey of these techniques will be given, some recent successful applications of them [3, 6, 7] will be reported and some future applications, as multispecies nonlinear nonlocal equations [10] with cross-diffusion or Navier-Stokes-Cahn-Hilliard equations [12, 1], will be presented.

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