SOLVING THE CURL-DIV SYSTEM USING DIVERGENCE-FREE OR CURL-FREE FINITE ELEMENTS

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ABSTRACT. Aim of this talk is the analysis of the following three problems and of their mutual relations:

(a) (see [1]) finding finite element potentials, namely, solving by means of finite elements the problems $\operatorname{grad}\varphi = \mathbf{f}$, $\operatorname{curl}\mathbf{A} = \mathbf{B}$, $\operatorname{div}\mathbf{v} = q$;

(b) (see [2]) finding suitable basis functions for the spaces of curl-free or divergence-free finite elements;

(c) (see [3]) devising simple finite element schemes for the solution of the curl–div system, which reads

$$\begin{cases} \operatorname{curl}\mathbf{u} = \mathbf{J} & \text{in } \Omega \\ \operatorname{div}\mathbf{u} = g & \text{in } \Omega \\ \mathbf{u}\times\mathbf{n} = \mathbf{a} \quad (\text{or } \mathbf{u}\cdot\mathbf{n} = b) & \text{on } \partial\Omega\,. \end{cases} \tag{1}$$

The curl–div system can be rewritten as follows (let us focus on the boundary condition $\mathbf{u}\times\mathbf{n} = \mathbf{a}$ on $\partial\Omega$; moreover, for the ease of presentation, the domain Ω is assumed to have a simple topological shape, say, it is homeomorphic to a cube). The first step is to find a vector field $\mathbf{u}^\star$ satisfying $\operatorname{div}\mathbf{u}^\star = g$ in Ω. Then by simple integration by parts it follows that for each $\mathbf{v}\in H(\operatorname{curl};\Omega)$ the vector field $\mathbf{w} = \mathbf{u} - \mathbf{u}^\star$ satisfies

$$\int_\Omega \mathbf{J}\cdot\mathbf{v} = \int_\Omega \operatorname{curl}\mathbf{u}\cdot\mathbf{v} = \int_\Omega \mathbf{w}\cdot\operatorname{curl}\mathbf{v} + \int_\Omega \mathbf{u}^\star\cdot\operatorname{curl}\mathbf{v} - \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v}\,.$$

Introducing the space $\mathcal{W}_0 = \{\mathbf{v}\in H(\operatorname{div};\Omega)\,|\,\operatorname{div}\mathbf{v} = 0 \text{ in } \Omega\} = \operatorname{curl}[H(\operatorname{curl};\Omega)]$, the vector field $\mathbf{w}$ is thus a solution to

$$\mathbf{w}\in\mathcal{W}_0 \;:\; \textstyle\int_\Omega \mathbf{w}\cdot\operatorname{curl}\mathbf{v} = \int_\Omega \mathbf{J}\cdot\mathbf{v} - \int_\Omega \mathbf{u}^\star\cdot\operatorname{curl}\mathbf{v} + \int_{\partial\Omega}\mathbf{a}\cdot\mathbf{v} \quad \forall\,\mathbf{v}\in H(\operatorname{curl};\Omega)\,, \tag{2}$$

and this solution is easily proved to be unique.

The numerical approximation of the curl–div system(problem (c)) is therefore based on the finite element solution of $\operatorname{div}\mathbf{u}^\star = g$ (problem (a)) and on devising a suitable finite element basis for $\mathcal{W}_0\cap N_h$, where $N_h\subset H(\operatorname{curl};\Omega)$ are the Nédélec edge elements (problem (b)).

Similar results hold for the curl–div system with $\mathbf{u}\cdot\mathbf{n} = b$ imposed on the boundary; in that case, one must find a finite element solution of $\operatorname{curl}\mathbf{u}_\star = \mathbf{J}$ and suitable basis functions for curl-free finite elements.

Keywords: finite element potentials, constrained finite elements, curl–div system

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