A SYSTEMATIC CONSTRUCTION OF THE SOLUTION FOR THE RIEMANN PROBLEM FOR THE BURGERS EQUATION WITH DISCONTINUOUS SOURCE TERM

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Abstract. This paper is concerned with the explicit construction of the Riemann problem arising in the theory of Radiation Hydrodynamics:

$$u_t + \left(\frac{u^2}{2}\right)_x = g(x), \quad u(x,0) = u_0(x), \quad (x,t) \in \mathbb{R} \times \mathbb{R}^+,$$

where the source term and the initial condition are defined as follows

$$g(x) = g_R H(x) + g_L H(-x), \quad u_0(x) = u_R H(x) + u_L H(-x), \quad (g_L, g_R, u_L, u_R) \in \mathbb{R}^4,$$

with H is the Heviside function, i.e. $H(x) = 0$ for $x \in \mathbb{R}^-$ and $H(x) = 1$ for $x \in \mathbb{R}_0^+$. The main result of this work is the following theorem: "*There is a partition of* $\mathbb{R}^4$ *characterizing the different types of possible entropic solution in terms of* (u_L, u_R, g_L, g_R)*.*" The proof is constructive and developed in sixty Lemmas. First, we apply the characteristics method and introduce a classification of the different type of waves. A systematic discussion of the all possible type of waves at $t = 0$, implies the existence of sixty type of solutions. Then, we analyze in detail the analytic construction of these solution types. Basically, and in a broad sense, a shock a rarefaction wave or a contact discontinuity are formed at $t = 0$. The evolution of shock curve is completely characterized by analyzing the initial value problem obtained by the application of Rankine-Hugoniot condition. The rarefaction wave solution is explicitly obtained by the characteristics method. Here, a subcase of rarefaction wave, called "vacuum wave", requires a regularization of the source term and the initial condition before of the application of the characteristics method. The contact discontinuity wave solution is obtained by application of the shock and rarefaction construction techniques. Finally, by a unification of the sixty Lemmas we obtain the required partition of $\mathbb{R}^4$.

Keywords: Riemann problem, discontinuous source term, Burgers equation

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References

[1] F. Beixiang, T. Pingfan, and W. Ya-Guang Wang. The Riemann problem of the Burgers equation with a discontinuous source term. *J. Math. Anal. Appl.*, 395(1):307–335, 2012.

[2] S. N. Kružkov. First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255, 1970.

[3] Ch. Rohde and F. Xie. Global existence and blowup phenomenon for a 1D radiation hydrodynamics model problem. *Math. Methods Appl. Sci.*, 35(5):564–573, 2012.

[4] A. Tello. Analytic solution of a Riemann Problem arising in Radiation hydrodynamics (in spanish). *Master's thesis*, Universidad del Bío-Bío, 2014.

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