A FRACTIONAL LAPLACE EQUATION: REGULARITY OF SOLUTIONS AND FINITE ELEMENT APPROXIMATIONS

GABRIEL ACOSTA AND JUAN PABLO BORTHAGARAY

ABSTRACT. In this work we deal with the Dirichlet homogeneous problem for the *integral* fractional Laplacian on a bounded domain $\Omega \subset \mathbb{R}^n$. Namely, we deal with basic analytical aspects required to convey a complete Finite Element analysis of the problem

$$\begin{cases} (-\Delta)^s u = f \text{ in } \Omega, \\ u = 0 \text{ in } \Omega^c, \end{cases} \tag{1}$$

where the fractional Laplacian of order s is defined by

$$(-\Delta)^s u(x) = C(n,s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\, dy$$

and $C(n,s)$ is a normalization constant.

Independently of the Sobolev regularity of the source f, solutions of (1) are not expected to be in a better space than $H^{s+\min\{s,1/2-\epsilon\}}(\Omega)$ (see [2, 4]). However, by building on Hölder estimates developed in [3], we were able to obtain further regularity results in a novel framework of weighted fractional Sobolev spaces, leading to a priori estimates in terms of the Hölder regularity of the data [1].

After developing a suitable polynomial interpolation theory in these weighted fractional spaces, optimal order of convergence in the energy norm for the standard linear finite element method is proved for graded meshes. Numerical experiments are in agreement with our theoretical predictions, and illustrate the optimality of the aforementioned estimates.

Keywords: Fractional Laplacian, Finite Elements, Weighted Fractional Norms, Graded Meshes

REFERENCES

[1] G. Acosta and J. P. Borthagaray. A fractional Laplace equation: regularity of solutions and Finite Element approximations. Preprint available at http://arxiv.org/abs/1507.08970, 2015.

[2] G. Grubb. Fractional Laplacians on domains, a development of Hörmander's theory of μ-transmission pseudodifferential operators. *Advances in Mathematics*, 268:478-528, 2015.

[3] X. Ros-Oton and J. Serra, The Dirichlet problem for the fractional Laplacian: Regularity up to the boundary. *Journal de Mathématiques Pures et Appliquées*, 101(3):275-302, 2014.

[4] M. I. Vishik and G. I. Eskin, Convolution equations in a bounded region. *Uspehi Mat. Nauk*, 20:89-152, 1965. English translation in *Russian Math. Surveys*, 20:86-151, 1965.

IMAS - CONICET AND DEPARTAMENTO DE MATEMÁTICA, FCEYN - UNIVERSIDAD DE BUENOS AIRES, ARGENTINA.

E-mail address: gacosta@dm.uba.ar

IMAS - CONICET AND DEPARTAMENTO DE MATEMÁTICA, FCEYN - UNIVERSIDAD DE BUENOS AIRES, ARGENTINA.

E-mail address: jpbortha@dm.uba.ar