# A POSTERIORI ERROR ESTIMATION WITH POINT SOURCES IN SOBOLEV NORMS

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Abstract. There are some particular physical situations where the mathematical model involves the solution of an elliptic or parabolic problem with one or more Dirac $\delta$-functions as a source term.

We consider as a model problem a divergence form second order equation on a two-dimensional polygonal domain $\Omega$

$$-\nabla \cdot A\nabla u = \sum_{j=1}^{N} d_j \, \delta_{x_j} \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega,$$

where $\delta_{x_j}$ denotes the Dirac's $\delta$-function at a certain point $x_j \in \Omega$, and $A$ is uniformly elliptic and Lipschitz. We are interested in the finite element approximation of the exact solution $u$ to this problem and the development of a posteriori error estimates. A posteriori error estimates are computable quantities, obtained in terms of the computed solution and problem data that can be used to make judicious mesh modifications in order to equidistribute error and numerical effort.

Since $\delta_{x_j}$ does not belong to $H^{-1}$, the solution will not be in $H^1$ and the general theory for a posteriori error estimation in energy norm is not applicable. Using duality techniques we were able to prove sharp (global) upper and (local) lower a posteriori bounds for the error in $H^{1-\alpha}$, $0 < \alpha < \frac{1}{2}$, and $W^1_p$, $1 < p < 2$.

Amazingly, the asymptotic form for the error estimates only depends on the computed solution, but not on data. The location of $x_j$ in $\Omega$ and the values $d_j$ do not show up in the formula, but they only affect the value of the estimator through the computed solution.

We will report on the mathematical analysis and achievement of these bounds, and illustrate with some numerical experiments.

**Keywords**: adaptive finite elements, a posteriori error estimates, Dirac delta source term.

**Mathematics Subject Classifications (2000)**: 65N12, 65N15, 65N30, 65N50, 65Y20.

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