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AN AUGMENTED MIXED FINITE ELEMENT METHOD FOR INCOMPRESSIBLE ELASTICITY AND GMRES ITERATION OF ITS COMPRESSIBLE COUNTERPART

UN MÉTODO DE ELEMENTOS FINITOS MIXTOS PARA ELASTICIDAD INCOMPRESIBLE E ITERACIÓN POR GMRES DE SU CONTRAPARTE COMPRESIBLE

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Abstract

In this thesis we build on recent results on the a priori and a posteriori error analysis of a so called augmented mixed finite element method for the linear elasticity problem in three distinct directions.

The first of those directions is the derivation of such an augmented mixed finite element method for the incompressible linear elasticity problem and the corresponding a priori error analysis. The incompressible linear elasticity problem can be seen as a limiting case of the standard linear elasticity problem, yet crucially, the incompressibility condition to be fulfilled demands a treatment of its own. Similarly as before, the present approach is based on the introduction of the Galerkin least-squares type terms arising from the constitutive and equilibrium equations, and from the relations defining the pressure in terms of the stress tensor and the rotation in terms of the displacement, all them multiplied by stabilization parameters. We show that these parameters can be suitably chosen so that the resulting augmented variational formulation is defined by a strongly coercive bilinear form, whence the associated Galerkin scheme becomes well posed for any choice of finite element subspaces.

The second direction, closely related to the first, is the derivation of a reliable and efficient residual-based a posteriori error estimator for the augmented mixed finite element scheme for the incompressible linear elasticity problem. An adaptive algorithm is then proposed and shown to be capable of localizing singularities and large stress regions of the solution.

Finally, for the third direction, we focus on the augmented mixed finite element method for the standard linear elasticity problem and propose the use of the preconditioned GMRES method to solve efficiently the large and sparse linear systems that arise. The spectral properties of the stiffness matrix are used to show how standard preconditioners can directly be used.

Numerical examples are provided for each followed direction.

Resumen

En esta memoria se desarrollan tres líneas de investigación sobre la base de resultados recientes en torno al análisis de error a priori y a posteriori del así llamado método de elementos finitos mixtos aumentados para el problema de elasticidad lineal tradicional.

La primera de estas líneas de investigación es la derivación de un método de elementos finitos mixtos aumentados semejante al arriba mencionado para el problema de elasticidad lineal incompresible y su corresponiente análisis de error a priori. El problema de elasticidad lineal incompresible puede verse como un caso límite del problema de elasticidad lineal tradicional. Sin embargo, la condición de incompresibilidad que se debe observar requiere de un tratamiento ad hoc. Análogamente al caso del problema de elasticidad tradicional, el enfoque aquí presentado se basa en la introducción de términos de tipo cuadrados mínimos de Galerkin que surgen de las ecuaciones de equilibrio y constitutiva y de las relaciones que definen la presión en términos del tensor de esfuerzos y la rotación en términos del desplazamiento. Todos estos términos se introducen multiplicados por parámetros de estabilización. Se demuestra que estos parámetros pueden ser elegidos de manera que la formulación variacional aumentada resulta ser representada por una forma bilineal fuertemente elíptica. Por lo tanto el esquema de Galerkin asociado queda bien planteado cualquiera sea la elección de los subespacios de elementos finitos.

La segunda línea de investigación, muy relacionada con la primera, es la derivación de un estimador de error a posteriori para el esquema de elementos finitos mixtos aumentados para el problema de elasticidad lineal. El estimador cumple con las desigualdades de confiabilidad y eficiencia. Se propone un algoritmo adaptativo que exhibe la capacidad de localizar singularidades y regiones de esfuerzos altos de la solución.

Finalmente, la tercera línea de investigación concierne al método de elementos finitos mixtos aumentados para el problema de elasticidad lineal tradicional. Para resolver de forma eficiente los sistemas lineales grandes y ralos que surgen se propone el uso del método GMRES precondicionado. Se usan las propiedades espectrales de la matriz de rigidez para mostrar cómo pueden utilizarse precondicionadores ya existentes en la literatura.

Se proveen ejemplos numéricos para cada una de las líneas de investigación seguidas.

Contents

CHAPTER 1

Introduction

1. Purpose and context of this work

The stabilization of dual-mixed variational formulations through the application of diverse procedures has been widely investigated during the last two decades. In particular, the augmented variational formulations, also known as Galerkin least-squares methods, and which go back to [**21**] and [**22**], have already been extended in different directions. Some applications to elasticity problems can be found in [**24**] and [**14**], and a non-symmetric variant was considered in [**19**] for the Stokes problem. In addition, stabilized mixed finite element methods for related problems, including Darcy and incompressible flows, can be seen in [**3**], [**11**], [**23**], [**31**], [**32**], and [**34**]. For an abstract framework concerning the stabilization of general mixed finite element methods, we refer to [**13**].

On the other hand, a new stabilized mixed finite element method for plane linear elasticity with homogeneous Dirichlet boundary conditions was presented and analyzed in [**27**]. The approach there is based on the introduction of suitable Galerkin least-squares terms arising from the constitutive and equilibrium equations, and from the relation defining the rotation in terms of the displacement. It is shown that the resulting continuous and discrete augmented formulations are well posed, and that the latter becomes locking-free. Moreover, since the augmented variational formulation is strongly coercive, arbitrary finite element subspaces can be utilized in the discrete scheme, which constitutes one of its main advantages. In particular, Raviart-Thomas spaces of lowest order for the stress tensor, piecewise linear elements for the displacement, and piecewise constants for the rotation can be used. The corresponding extension to the case of non-homogeneous Dirichlet boundary conditions was provided recently in [**28**]. In addition, a residual based a posteriori error analysis yielding a reliable and efficient estimator for the augmented method from [**27**], is provided in the recent work [**8**]. A posteriori error analyses of the traditional mixed finite element methods for the elasticity problem can be seen in [**15**] and the references therein.

Yet another source of concern when solving any kind of Galerkin scheme for a linear variational problem is the solution and preconditioning of the linear systems that arise. Studies for primal and dual (saddle-point) structures go back to [**6**] and [**35**], respectively. Problems with saddle point structure are particularly keen to become ill-conditioned and thus have been thoroughly studied ([**41**], [**42**], [**25**], [**26**] and references therein). See [**9**] for a survey of preconditioning methods for saddle point problems. Among the Krylov subspace methods for non-symmetric problems GMRES holds a number of advantages. Accordingly it has been the focus of research like [**30**] and [**38**].

The purpose of this piece of work is threefold: The first two purposes are the extensions of the results from [**27**] and [**8**] to the case of incompressible elasticity. The third purpose is to provide an efficient iterative method for the solution of linear systems such as those appearing on [**27**].

The rest of this work is organized as follows. In Section 3 we give a brief introduction to linear elasticity. Then, in Section 4 we describe the first boundary value problem of interest, establish its dual-mixed variational formulation, and prove that it is well-posed. Then, in

Sections 5 and 6 we introduce and analyze the continuous and discrete augmented formulations, respectively. Next, in Section 7 we develop the residual-based a posteriori error analysis of our augmented mixed finite element method. Finally, several numerical results confirming the theoretical properties predicted in Section 6, and the reliability and efficiency of the estimator are provided in Section 10. The capability of the corresponding adaptive algorithm to localize the singularities and the large stress regions of the solution is also illustrated here. In Section 11 we describe the augmented mixed finite element method for standard elasticity introduced in [**27**] in order to develop an efficient way of solving the associated linear systems in Section 12. Section 13 shows numerical results concerning the solution of the linear systems appearing in Section 11.

2. Notation

We present some notations to be used below. Given any Hilbert space U, U^2 and $U^{2\times 2}$ denote, respectively, the space of vectors and square matrices of order 2 with entries in U. In particular, $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write as usual $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{2} \tau_{ii}$, $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\,\mathbf{I}$, and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. h is the mesh size for the finite element spaces. We denote by $\lambda_{\min}(\mathcal{X})$ and $\lambda_{\max}(\mathcal{X})$ the eigenvalues of smallest and largest absolute value of a square matrix $\mathcal{X}$, respectively. Bold capitals ($\mathbf{A}$, $\mathbf{D}$, ...) denote the Gram matrices of the bilinear forms / induced operators represented by their plain capitals counterparts (A, D, ...). Similarly, we denote members of Hilbert spaces ($\mathbf{s}$, $\mathbf{u}_h$, p, ...) and their coefficient vectors with respect to a given finite element subspace basis ($\vec{\mathbf{s}}$, $\vec{\mathbf{u}}_h$, $\vec{p}$, ...). The adjoint of an operator A is denoted by A^* and the symmetric part of a matrix A is denoted by $A^{\mathtt{sym}}$. We put duality pairings within square brackets, inner products within angle brackets, and the dual of a Hilbert space H is denoted by H'.

Also, in what follows we utilize the standard terminology for Sobolev spaces and norms, employ $\mathbf{0}$ to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

3. Path from the mechanics of continuous media to the model problems

We draw heavily in this section from the excellent treatise on the linear theories of elasticiy by M. Gurtin [**29**], where proofs of all the statements made on this section can be found, as well as a comprehensive bibliography on the subject.

The theory of elasticity concerns itself with the mechanics of solid bodies, bound to the following assumptions:

i. The body is a continuous media. That is, its constituent particles are considered to be infinitesimally small, and biunivocally identified with the points of the geometrical region enclosed by the body on each instant of time.
ii. The body has a reference particle configuration to which it will restore itself if no external body of surface forces are applied.

Let Ω denote the region in $\mathbb{R}^2$ (since we are dealing with plane elasticity throughout this thesis) enclosed by the body in the reference particle configuration. A deformation of Ω is a smooth homeomorphism $\boldsymbol{\chi}$ of Ω into the region $\boldsymbol{\chi}(\Omega)$. The point $\boldsymbol{\chi}(\mathbf{x})$ is the place occupied by the material point $\mathbf{x}$ in the deformation $\boldsymbol{\chi}$, while

$$\mathbf{u}(\mathbf{x}) = \boldsymbol{\chi}(\mathbf{x}) - \mathbf{x} \tag{3.1}$$

is the displacement of $\mathbf{x}$.

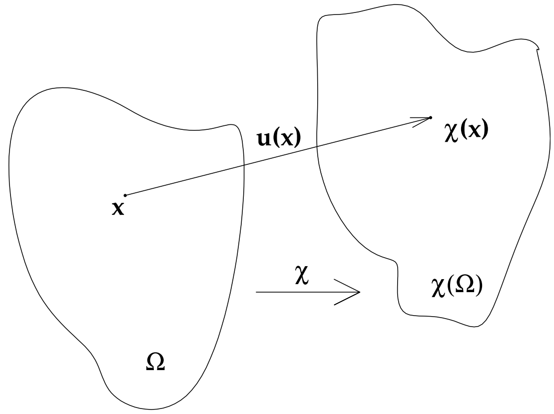

FIGURE 1.1

In order to arrive at a number of simplifications and, more crucially, at a meaningful linear theory of elasticity, we must make the assumption that the deformation is small with respect to the spatial scales involved in such a way that

$$\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^{\mathrm{t}}\right) \approx \frac{1}{2}\left(\nabla\boldsymbol{\chi}\nabla\boldsymbol{\chi}^{\mathrm{t}} - \mathbf{I}\right) \tag{3.2}$$

or, equivalently,

$$\nabla\mathbf{u}\nabla\mathbf{u}^{\mathrm{t}} \ll \nabla\mathbf{u}. \tag{3.3}$$

Then, the rigid displacements (namely, those that leave the relative position among the particles unchanged) are adequately characterized by

$$\boldsymbol{\varepsilon}(\mathbf{u}) = 0. \tag{3.4}$$

The body is subject to a system of forces that, for the stationary case, must be in equilibrium. Let $\mathbf{f}(\mathbf{x})$ denote the sum of the body forces per unit volume exerted by external bodies on each point $\mathbf{x}$ of Ω. And, given an oriented surface $\mathcal{S}$ in Ω with unit normal ν we denote by $\mathbf{s}_\nu(\mathbf{x})$ the force per unit area at $\mathbf{x}$ exerted by the portion of Ω on the side of $\mathcal{S}$ where $\mathbf{x}$ points on the portion of Ω on the other side.

It can be shown that $\mathbf{s}_\nu$ has the form $\boldsymbol{\sigma}\nu$, where $\boldsymbol{\sigma}$, called the stress tensor, is a second order tensor field. The divergence theorem, the equilibrium of forces and regularity assumptions lead to

$$\mathbf{div}(\boldsymbol{\sigma}) \;=\; -\mathbf{f} \tag{3.5}$$

on each point of Ω. The small displacement assumption leads, through the need to comply with the equilibrium of angular momentum, to

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\mathrm{t}}. \tag{3.6}$$

It is worth noting that the work expended by the forces deforming Ω is

$$W = \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\varepsilon}(\mathbf{u}) \tag{3.7}$$

Therefore upon a rigid displacement no work associated to the deformation is expended.

At last we show the constitutive relations that play a part on this thesis. The first of these is the constitutive relation for a linear isotropic elastic material

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) + \lambda \mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u})), \tag{3.8}$$

where $\mu, \lambda > 0$ are called the Lamé constants.

The small displacement assumption above makes a measure of local volume change out of $\mathrm{div}(\mathbf{u})$. Thus, in the limit $\lambda \longrightarrow \infty$, if the work expended is to remain bounded, $\mathrm{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) = \mathrm{div}(\mathbf{u})$ must be zero, making the material under consideration incompressible.

In that case the stress tensor is determined up to an arbitrary pressure p,

$$\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) - p\mathbf{I}. \tag{3.9}$$

CHAPTER 2

A priori analysis for augmented incompressible elasticity

4. The problem and its dual mixed-formulation

Let Ω be a bounded and simply connected polygonal domain in $\mathbb{R}^2$ with boundary Γ. Our goal is to determine the displacement $\mathbf{u}$, the stress tensor $\boldsymbol{\sigma}$, and the pressure-like unknown p of a linear incompressible material occupying the region Ω, under the action of an external force. In other words, given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$, we seek a symmetric tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$ and a scalar field p such that

$$\begin{gathered}\boldsymbol{\sigma} \,=\, 2\,\mu\,\boldsymbol{\varepsilon}(\mathbf{u}) - p\,\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}(\boldsymbol{\sigma}) \,=\, -\,\mathbf{f} \quad \text{in} \quad \Omega\,,\\ \operatorname{div}(\mathbf{u}) \,=\, 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma\,,\end{gathered} \tag{4.1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\,(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}})$ is the linearized strain tensor, μ is the shear modulus, and $\mathbf{div}$ stands for the usual divergence operator div acting along each row of the tensor.

Since $\operatorname{tr}(\boldsymbol{\varepsilon}(\mathbf{u})) = \operatorname{div}(\mathbf{u})$ in Ω, we find from the first equation in (4.1) that the incompressibility condition $\operatorname{div}(\mathbf{u}) = 0$ in Ω can be stated in terms of the stress tensor and the pressure as follows

$$p + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}) \,=\, 0 \quad \text{in} \quad \Omega\,. \tag{4.2}$$

Next, we choose to impose weakly the symmetry of $\boldsymbol{\sigma}$ through the introduction of the infinitesimal rotation tensor $\boldsymbol{\gamma} := \frac{1}{2}\,(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}})$ as a further unknown (see [**2**] and [**39**]), which yields

$$\frac{1}{2\mu}\,(\boldsymbol{\sigma} + p\mathbf{I}) \,=\, \boldsymbol{\varepsilon}(\mathbf{u}) \,=\, \nabla\mathbf{u} - \boldsymbol{\gamma} \quad \text{in} \quad \Omega\,. \tag{4.3}$$

Note that (4.2) and (4.3) imply the modified constitutive equation

$$\frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} = \boldsymbol{\varepsilon}(\mathbf{u}) \quad \text{in} \quad \Omega\,. \tag{4.4}$$

Then, testing equations (4.3) and (4.2) and weakly taking care of the equilibrium equation of (4.1) and the symmetry of $\boldsymbol{\sigma}$ gives rise to the problem: Find $(\boldsymbol{\sigma}, p, (\mathbf{u}, \boldsymbol{\gamma}))$ in $H(\mathbf{div};\Omega) \times L^2(\Omega) \times Q$ such that

$$\begin{aligned}\frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma} : \boldsymbol{\tau} + \frac{1}{2\mu}\int_\Omega p\operatorname{tr}(\boldsymbol{\tau}) + \frac{1}{2\mu}\int_\Omega q\operatorname{tr}(\boldsymbol{\sigma}) + \frac{1}{\mu}\int_\Omega p\,q + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} = 0,\\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} = -\int_\Omega \mathbf{f}\cdot\mathbf{v},\end{aligned}$$

for all $(\boldsymbol{\tau}, q, (\mathbf{v}, \boldsymbol{\eta})) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times Q$, where

$$H(\mathbf{div};\Omega) \,:=\, \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\} \quad \text{and} \quad Q := [L^2(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}\,,$$

with

$$[L^2(\Omega)]^{2\times 2}_{\mathtt{asym}} := \left\{\boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\right\}\,.$$

Now, noting that

$$\boldsymbol{\sigma} : \boldsymbol{\tau} \,+\, p\,\mathrm{tr}(\boldsymbol{\sigma}) \,+\, q\,\mathrm{tr}(\boldsymbol{\tau}) \,+\, 2\,pq \,=\, \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \,+\, 2\,\left(p + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})\right)\left(q + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\right)\,,$$

the last system can be written in the more compact form: Find $(\boldsymbol{\sigma}, p, (\mathbf{u}, \boldsymbol{\gamma}))$ in $H(\mathbf{div};\Omega) \times L^2(\Omega) \times Q$ such that

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \frac{1}{\mu}\int_\Omega \Big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\Big)\Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\Big) + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} = 0,& \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} = -\int_\Omega \mathbf{f}\cdot\mathbf{v},& \end{aligned} \tag{4.5}$$

for all $(\boldsymbol{\tau}, q, (\mathbf{v}, \boldsymbol{\eta})) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times Q$. At this point we observe that for any $c \in \mathbb{R}$, $(c\mathbf{I}, -c, (\mathbf{0}, \mathbf{0}))$ is a solution of the homogeneous version of system (4.5). Hence, in order to avoid this non-uniqueness we consider the decomposition

$$H(\mathbf{div};\Omega) \;=\; H_0 \,\oplus\, \mathbb{R}\,\mathbf{I}\,, \tag{4.6}$$

where $H_0 := \Big\{ \boldsymbol{\tau} \in H(\mathbf{div};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0 \Big\}$, and require from now on that $\boldsymbol{\sigma} \in H_0$. The following lemma guarantees that the test space can also be restricted to H_0.

Lemma 4.1. *Any solution of* (4.5) *with* $\boldsymbol{\sigma} \in H_0$ *is also solution of: Find* $(\boldsymbol{\sigma}, p, (\mathbf{u}, \boldsymbol{\gamma})) \in H_0 \times L^2(\Omega) \times Q$ *such that*

$$\begin{aligned} \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \frac{1}{\mu}\int_\Omega \Big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\Big)\Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\Big) + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} = 0,& \\ \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} = -\int_\Omega \mathbf{f}\cdot\mathbf{v},& \end{aligned} \tag{4.7}$$

for all $(\boldsymbol{\tau}, q, (\mathbf{v}, \boldsymbol{\eta})) \in H_0 \times L^2(\Omega) \times Q$. *Conversely, any solution of* (4.7) *is also a solution of* (4.5).

Proof. It is immediate that any solution of (4.5) with $\boldsymbol{\sigma} \in H_0$ is also a solution of (4.7). Conversely, let $(\boldsymbol{\sigma}, p, (\mathbf{u}, \boldsymbol{\gamma}))$ be a solution of (4.7). Because of (4.6) it suffices to prove that $(\boldsymbol{\sigma}, p, (\mathbf{u}, \boldsymbol{\gamma}))$ also satisfies (4.5) if tested with $(\mathbf{I}, 0, (\mathbf{0}, \mathbf{0}))$. This requires that $\int_\Omega \big(p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\big)$ vanishes which can be seen to be true by selecting $(\boldsymbol{\tau}, q, (\mathbf{v}, \boldsymbol{\eta})) = (\mathbf{0}, 1, (\mathbf{0}, \mathbf{0})) \in H_0 \times L^2(\Omega) \times Q$ in (4.7). $\square$

Furthermore, we now let $H := H_0 \times L^2(\Omega)$, consider a constant $\kappa_0 > 0$, and introduce a generalized version of (4.7): Find $((\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}))$ in $H \times Q$ such that

$$\begin{aligned} a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) + b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) = 0 \qquad & \forall\, (\boldsymbol{\tau}, q) \,\in\, H\,, \\ b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) = -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad & \forall\, (\mathbf{v}, \boldsymbol{\eta}) \,\in\, Q\,, \end{aligned} \tag{4.8}$$

where $a : H \times H \longrightarrow \mathbb{R}$ and $b : H_0 \times Q \longrightarrow \mathbb{R}$ are the bounded bilinear forms defined by

$$a((\boldsymbol{\zeta}, r), (\boldsymbol{\tau}, q)) := \frac{1}{2\mu}\int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} + \frac{\kappa_0}{\mu}\int_\Omega \Big(r + \frac{1}{2}\mathrm{tr}(\boldsymbol{\zeta})\Big)\Big(q + \frac{1}{2}\mathrm{tr}(\boldsymbol{\tau})\Big) \tag{4.9}$$

and

$$b(\boldsymbol{\zeta}, (\mathbf{v}, \boldsymbol{\eta})) := \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\zeta}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\zeta} \tag{4.10}$$

for $(\boldsymbol{\zeta}, r)$, $(\boldsymbol{\tau}, q)$ in H and $(\mathbf{v}, \boldsymbol{\eta})$ in Q. Note that (4.7) corresponds to (4.8) with $\kappa_0 = 1$.

In order to show that the formulations (4.8) are independent of $\kappa_0 > 0$, we prove next that they are all equivalent to the simplified version arising after replacing the incompressibility condition (4.2) into (4.8) (equivalently, taking $\kappa_0 = 0$ in (4.8)), that is: Find $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H_0 \times Q$ such that

$$\begin{aligned} a_0(\boldsymbol{\sigma}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) &= 0 \qquad && \forall\, \boldsymbol{\tau} \in H_0\,, \\ b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) &= -\int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad && \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in Q\,, \end{aligned} \tag{4.11}$$

where $a_0 : H_0 \times H_0 \longrightarrow \mathbb{R}$ is the bounded bilinear form defined by

$$a_0(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \frac{1}{2\mu} \int_\Omega \boldsymbol{\zeta}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{\tau}) \in H_0 \times H_0\,.$$

Lemma 4.2. *Problems (4.8) and (4.11) are equivalent. Indeed,* $((\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma})) \in H \times Q$ *is a solution of (4.8) if and only if* $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H_0 \times Q$ *is a solution of (4.11) and* $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$.

Proof. It suffices to take $\boldsymbol{\tau} = \mathbf{0}$ in (4.8) and then use that the traces of the tensor-valued functions in $H(\mathbf{div}; \Omega)$ live in $L^2(\Omega)$ as the pressure test functions do. □

The following lemmata will be useful in order to prove well-posedness of (4.8) and (4.11).

Lemma 4.3. *There exists a positive constant* β*, depending only on* Ω *such that*

$$\sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} \geq \beta\, \|(\mathbf{v}, \boldsymbol{\eta})\|_Q \tag{4.12}$$

for all $(\mathbf{v}, \boldsymbol{\eta})$ *in* Q.

Proof. See Lemma 4.3 in [**7**] for a detailed proof. □

Lemma 4.4. *There exists* $c_1 > 0$*, depending only on* Ω*, such that*

$$c_1 \|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} \leq \left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \quad \forall\, \boldsymbol{\tau} \in H_0, \tag{4.13}$$

Proof. See Lemma 3.1 in [**4**] or Proposition 3.1 of Chapter IV in [**12**]. □

We are now in a position to state the following theorem.

Theorem 4.5. *Problem (4.11) has a unique solution* $(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma})) \in H_0 \times Q$*. Moreover, there exists a positive constant* C*, depending only on* Ω*, such that*

$$\|(\boldsymbol{\sigma}, (\mathbf{u}, \boldsymbol{\gamma}))\|_{H(\mathbf{div};\Omega)\times Q} \leq C\, \|\mathbf{f}\|_{[L^2(\Omega)]^2}\,.$$

Proof. It suffices to prove that the bilinear forms a_0 and b satisfy the hypotheses of the Babuška-Brezzi theory. Indeed, given $(\mathbf{v}, \boldsymbol{\eta})$ in Q it is easy to see that

$$\sup_{\substack{\boldsymbol{\tau} \in H_0 \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}} = \sup_{\substack{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \\ \boldsymbol{\tau} \neq 0}} \frac{\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\eta} : \boldsymbol{\tau}}{\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}}\,, \tag{4.14}$$

which, together with Lemma 4.3, proves the continuous inf-sup condition for b. Now, let V be the kernel of the operator induced by b, that is

$$\begin{aligned} V := &\ \{\boldsymbol{\tau} \in H_0 : \quad b(\boldsymbol{\tau}, (\mathbf{v}, \boldsymbol{\eta})) = 0 \quad \forall\, (\mathbf{v}, \boldsymbol{\eta}) \in Q\} \\ = &\ \left\{\boldsymbol{\tau} \in H_0 : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \quad \text{and} \quad \boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{t}} \quad \text{in} \quad \Omega\right\}. \end{aligned}$$

It follows, applying Lemma 4.4, that for each $\boldsymbol{\tau} \in V$ there holds

$$a_0(\boldsymbol{\tau}, \boldsymbol{\tau}) = \frac{1}{2\,\mu} \left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|^2_{[L^2(\Omega)]^{2\times 2}} \geq \frac{c_1}{2\,\mu} \|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} = \frac{c_1}{2\,\mu} \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)}\,,$$

which shows that the bilinear form a_0 is strongly coercive in V. Finally, a straightforward application of the classical result given by Theorem 1.1 in Chapter II of [**12**] completes the proof. □

THEOREM 4.6. *Problem* (4.8) *has a unique solution* $((\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma})) \in H \times Q$, *independent of* κ_0, *and there holds* $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$. *Moreover, there exists a constant* $C > 0$, *depending only on* Ω, *such that*

$$\|((\boldsymbol{\sigma}, p), (\mathbf{u}, \boldsymbol{\gamma}))\|_{H\times Q} \leq C\, \|\mathbf{f}\|_{[L^2(\Omega)]^2} \,.$$

PROOF. It is a direct consequence of Lemma 4.2, which gives the equivalence between (4.8) and (4.11), and Theorem 4.5, which yields the well-posedness of (4.11). □

5. The augmented dual-mixed variational formulations

In the following we enrich the formulations (4.8) and (4.11) with residuals arising from the modified constitutive equation (4.4), the equilibrium equation, and the relation defining the rotation as a function of the displacement. More precisely, as in [**27**] we substract the second from the first equation in both (4.8) and (4.11) and then add the Galerkin least-squares terms given by

$$\kappa_1 \int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}) - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}}\right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}) + \frac{1}{2\mu}\boldsymbol{\tau}^{\mathtt{d}}\right) = 0, \tag{5.1}$$

$$\kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{div}(\boldsymbol{\tau}) = -\kappa_2 \int_\Omega \mathbf{f} \cdot \mathbf{div}(\boldsymbol{\tau}), \tag{5.2}$$

and

$$\kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)\right) : \left(\boldsymbol{\eta} + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathtt{t}}\right)\right) = 0, \tag{5.3}$$

for all $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in H_0 \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$, where $(\kappa_1, \kappa_2, \kappa_3)$ is a vector of positive parameters to be specified later. We notice that (5.1) and (5.3) implicitly require now the displacement $\mathbf{u}$ to live in the smaller space $[H_0^1(\Omega)]^2$.

In this way, instead of (4.8) we propose the following augmented variational formulation: Find $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H} := H_0 \times L^2(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ such that

$$A((\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) = F(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) \qquad \forall\, (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}\,, \tag{5.4}$$

where the bilinear form $A : \mathbf{H} \times \mathbf{H} \longrightarrow \mathbb{R}$ and the functional $F : \mathbf{H} \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} A((\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \;&:=\; a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) \;+\; b(\boldsymbol{\tau}, (\mathbf{u}, \boldsymbol{\gamma})) \;-\; b(\boldsymbol{\sigma}, (\mathbf{v}, \boldsymbol{\eta})) \\ +\;\; \kappa_1 &\int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}) - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}}\right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}) + \frac{1}{2\mu}\boldsymbol{\tau}^{\mathtt{d}}\right) \;+\; \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{div}(\boldsymbol{\tau}) \\ +\;\; \kappa_3 &\int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)\right) : \left(\boldsymbol{\eta} + \frac{1}{2}\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^{\mathtt{t}}\right)\right) \end{aligned} \tag{5.5}$$

and

$$F(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) := \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \kappa_2\, \mathbf{div}(\boldsymbol{\tau}))\,. \tag{5.6}$$

Similarly, instead of (4.11) we propose: Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0 := H_0 \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ such that

$$A_0((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) = F_0(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0\,, \tag{5.7}$$

where the bilinear form $A_0 : \mathbf{H}_0 \times \mathbf{H}_0 \longrightarrow \mathbb{R}$ and the functional $F_0 : \mathbf{H}_0 \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} A_0((\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) &:= a_0(\boldsymbol{\sigma},\boldsymbol{\tau}) + b(\boldsymbol{\tau},(\mathbf{u},\boldsymbol{\gamma})) - b(\boldsymbol{\sigma},(\mathbf{v},\boldsymbol{\eta})) \\ &+ \kappa_1 \int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}) - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathrm{d}}\right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}) + \frac{1}{2\mu}\boldsymbol{\tau}^{\mathrm{d}}\right) + \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \mathbf{div}(\boldsymbol{\tau}) \\ &+ \kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)\right) : \left(\boldsymbol{\eta} + \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\right)\right) \end{aligned} \tag{5.8}$$

and

$$F_0(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) := \int_\Omega \mathbf{f} \cdot (\mathbf{v} - \kappa_2\, \mathbf{div}(\boldsymbol{\tau}))\,. \tag{5.9}$$

The analogue of Lemma 4.2 is given now.

Lemma 5.1. *Problems (5.4) and (5.7) are equivalent. Indeed,* $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}$ *is a solution of (5.4) if and only if* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$ *is a solution of (5.7) and* $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$.

Proof. It suffices to take $(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}) = (\mathbf{0},\mathbf{0},\mathbf{0})$ in (5.4) and then use again that the traces of the tensor-valued functions in $H(\mathbf{div};\Omega)$ live in $L^2(\Omega)$ as the pressure test functions do. □

In what follows we aim to show the well-posedness of (5.7). The main idea is to choose the vector of parameters $(\kappa_1, \kappa_2, \kappa_3)$ such that A_0 be strongly coercive on $\mathbf{H}_0$ with respect to the norm $\|\cdot\|_{\mathbf{H}_0}$ defined by

$$\|(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{H}_0} := \left\{ \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} + |\mathbf{v}|^2_{[H^1(\Omega)]^2} + \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \right\}^{1/2}.$$

We first notice, after simple computations, that

$$\int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{v}) - \frac{1}{2\mu}\boldsymbol{\tau}^{\mathrm{d}}\right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}) + \frac{1}{2\mu}\boldsymbol{\tau}^{\mathrm{d}}\right) = \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} - \frac{1}{4\mu^2}\left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|^2_{[L^2(\Omega)]^{2\times 2}},$$

and that

$$\begin{aligned} &\int_\Omega \left(\boldsymbol{\eta} - \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\right)\right) : \left(\boldsymbol{\eta} + \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\right)\right) \\ &\qquad = \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} + \|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} - |\mathbf{v}|^2_{[H^1(\Omega)]^2}, \end{aligned}$$

which gives

$$\begin{aligned} A_0((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) &= \frac{1}{2\mu}\left(1 - \frac{\kappa_1}{2\mu}\right)\left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|^2_{[L^2(\Omega)]^{2\times 2}} + \kappa_2\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \\ &+ (\kappa_1 + \kappa_3)\,\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} - \kappa_3\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} + \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}}\,. \end{aligned} \tag{5.10}$$

Now, Korn's first inequality (see, e.g., Theorem 10.1 in [**33**]) establishes that

$$\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^2} \geq \frac{1}{2}\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} \qquad \forall\mathbf{v} \in [H^1_0(\Omega)]^2, \tag{5.11}$$

and hence (5.10) yields

$$\begin{aligned} A_0((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) &\geq \frac{1}{2\mu}\left(1 - \frac{\kappa_1}{2\mu}\right)\left\|\boldsymbol{\tau}^{\mathrm{d}}\right\|^2_{[L^2(\Omega)]^{2\times 2}} + \kappa_2\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} \\ &+ \frac{(\kappa_1 - \kappa_3)}{2}\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} + \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}}\,. \end{aligned}$$

Then, choosing κ_1 and κ_2 such that

$$0 < \kappa_1 < 2\mu \quad \text{and} \quad 0 < \kappa_2\,,$$

and applying Lemma 4.4, we deduce that

$$A_0((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) \,\geq\, \alpha_2\,\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} \,+\, \frac{(\kappa_1-\kappa_3)}{2}\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} \,+\, \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}}\,,$$

where

$$\alpha_2 \,:=\, \min\left\{c_1\alpha_1,\frac{\kappa_2}{2}\right\},\qquad \alpha_1 := \min\left\{\frac{1}{2\mu}\left(1-\frac{\kappa_1}{2\mu}\right),\frac{\kappa_2}{2}\right\},$$

and c_1 is the constant that appears in Lemma 4.4. In addition, choosing the parameter κ_3 such that $0\,<\,\kappa_3\,<\,\kappa_1$, we find that

$$A_0((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})) \,\geq\, \alpha\,\|(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})\|^2_{\mathbf{H}_0} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}))\,\in\,\mathbf{H}_0\,, \tag{5.12}$$

where

$$\alpha \,:=\, \min\left\{\alpha_2,\frac{(\kappa_1-\kappa_3)}{2},\kappa_3\right\}.$$

As a consequence of the above analysis, we obtain the following main results.

Theorem 5.2. *Assume that there hold*

$$0\,<\,\kappa_1\,<\,2\,\mu\,,\quad 0\,<\,\kappa_2\,,\quad \textit{and}\quad 0\,<\,\kappa_3\,<\,\kappa_1\,.$$

Then, the augmented variational formulation (5.7) *has a unique solution* $(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})\,\in\,\mathbf{H}_0$*. Moreover, there exists a positive constant* C*, depending only on* μ *and* $(\kappa_1,\kappa_2,\kappa_3)$*, such that* $\|(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})\|_{\mathbf{H}_0}\,\leq\,C\,\|F_0\|_{\mathbf{H}_0'}\,\leq\,C\,\|\mathbf{f}\|_{[L^2(\Omega)]^2}$.

Proof. It is clear from (5.8) and (5.12) that A_0 is bounded and strongly coercive on $\mathbf{H}_0$ with constants depending on μ and $(\kappa_1,\kappa_2,\kappa_3)$. Also, the linear functional F_0 (cf. (5.9)) is clearly continuous with norm bounded by $(1\,+\,\kappa_2)\,\|\mathbf{f}\|_{[L^2(\Omega)]^2}$. Therefore, the assertion is a simple consequence of the Lax-Milgram Lemma. □

Theorem 5.3. *Assume that there hold*

$$0\,<\,\kappa_1\,<\,2\,\mu\,,\quad 0\,<\,\kappa_2\,,\quad \textit{and}\quad 0\,<\,\kappa_3\,<\,\kappa_1\,.$$

Then the augmented variational formulation (5.4) *has a unique solution* $(\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma})\,\in\,\mathbf{H}$*, independent of* κ_0*, and there holds* $p=-\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$*. Moreover, there exists a positive constant* C*, depending only on* μ *and* $(\kappa_1,\kappa_2,\kappa_3)$*, such that* $\|(\boldsymbol{\sigma},p,\mathbf{u},\boldsymbol{\gamma})\|_{\mathbf{H}}\,\leq\,C\,\|\mathbf{f}\|_{[L^2(\Omega)]^2}$.

Proof. It is a direct consequence of Lemma 5.1 and Theorem 5.2. □

We end this section by emphasizing that the introduction of the augmented formulations (5.4) and (5.7) is motivated by the possibility of using arbitrary finite element subspaces in the definition of the associated Galerkin schemes. This is certainly guaranteed by the strong coerciveness of the resulting bilinear form, as already proved for A_0 (cf. (5.12)) and as will be proved for A in the next section. We also remark here that at first glance it could seem, due to Lemmata 4.2 and 5.1, that there is actually no need of considering the continuous variational formulations (4.8) and (5.4) since the equivalent ones, given respectively by (4.11) and (5.7), are clearly simpler. Nevertheless, as we show below in Section 6, the main interest in (4.8) and particularly in the corresponding augmented formulation (5.4) lies in the associated Galerkin scheme, which provides more flexibility for choosing the pressure finite element subspace.

6. The augmented mixed finite element methods

We now let $H^{\boldsymbol{\sigma}}_{0,h}$, H^p_h, $H^{\mathbf{u}}_{0,h}$ and $H^{\boldsymbol{\gamma}}_h$ be arbitrary finite element subspaces of H_0, $L^2(\Omega)$, $[H^1_0(\Omega)]^2$ and $[L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$, respectively, and define

$$\mathbf{H}_h := H^{\boldsymbol{\sigma}}_{0,h} \times H^p_h \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_h \quad \text{and} \quad \mathbf{H}_{0,h} := H^{\boldsymbol{\sigma}}_{0,h} \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_h .$$

In addition, let κ_0, κ_1, κ_2, and κ_3 be given positive parameters. Then, the Galerkin schemes associated with (5.4) and (5.7) read: Find $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_h$ such that

$$A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) = F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \quad \forall\, (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h , \tag{6.1}$$

and: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h}$ such that

$$A_0((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) = F_0(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \quad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h} . \tag{6.2}$$

The following theorem provides the unique solvability, stability, and convergence of (6.2).

THEOREM 6.1. *Assume that the parameters κ_1, κ_2, and κ_3 satisfy the assumptions of Theorem* 5.2 *and let $\mathbf{H}_{0,h}$ be any finite element subspace of $\mathbf{H}_0$. Then, the Galerkin scheme* (6.2) *has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h}$, and there exist positive constants C, $\tilde{C}$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \le C \sup_{\substack{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h} \\ (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \neq 0}} \frac{|F_0(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)|}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0}} \le C \, \|\mathbf{f}\|_{[L^2(\Omega)]^2} ,$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \le \tilde{C} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h}} \|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0} . \tag{6.3}$$

PROOF. Since A_0 is bounded and strongly coercive on $\mathbf{H}_0$ (cf. (5.8) and (5.12)) with constants depending on μ and $(\kappa_1, \kappa_2, \kappa_3)$, the proof follows from a straightforward application of the Lax-Milgram Lemma and Cea's estimate. $\square$

In order to define an explicit finite element subspace of $\mathbf{H}_0$, we now let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T such that $\bar{\Omega} = \cup \{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $\ell \ge 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials of total degree at most ℓ defined on S. Also, for each $T \in \mathcal{T}_h$ we define the local Raviart-Thomas space of order zero

$$\mathbb{RT}_0(T) := \operatorname{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \subseteq [\mathbb{P}_1(T)]^2,$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a generic vector of $\mathbb{R}^2$, and let $\tilde{H}^{\boldsymbol{\sigma}}_h$ be the corresponding global space, that is

$$\tilde{H}^{\boldsymbol{\sigma}}_h := \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_T \in [\mathbb{RT}_0(T)^{\mathtt{t}}]^2 \quad \forall\, T \in \mathcal{T}_h \right\} . \tag{6.4}$$

Then we let $\tilde{\mathbf{H}}_{0,h} := \tilde{H}^{\boldsymbol{\sigma}}_{0,h} \times \tilde{H}^{\mathbf{u}}_{0,h} \times \tilde{H}^{\boldsymbol{\gamma}}_h$, where

$$\tilde{H}^{\boldsymbol{\sigma}}_{0,h} := \left\{ \boldsymbol{\tau}_h \in \tilde{H}^{\boldsymbol{\sigma}}_h : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) = 0 \right\} , \tag{6.5}$$

$$\tilde{H}^{\mathbf{u}}_{0,h} := \left\{ \mathbf{v}_h \in [C(\bar{\Omega})]^2 : \quad \mathbf{v}_h|_T \in [\mathbb{P}_1(T)]^2 \quad \forall\, T \in \mathcal{T}_h , \quad \mathbf{v}_h = 0 \text{ on } \partial\Omega \right\} , \tag{6.6}$$

and

$$\tilde{H}^{\boldsymbol{\gamma}}_h := \left\{ \boldsymbol{\eta}_h \in [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}} : \quad \boldsymbol{\eta}_h|_T \in [\mathbb{P}_0(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h \right\} . \tag{6.7}$$

The approximation properties of these subspaces are given as follows (see [**12**], [**16**], [**27**]):

($\mathrm{AP}^{\boldsymbol{\sigma}}_{0,h}$) *For each* $r \in (0,1]$ *and for each* $\boldsymbol{\tau} \in [H^r(\Omega)]^{2\times 2} \cap H_0$ *with* $\mathbf{div}(\boldsymbol{\tau}) \in [H^r(\Omega)]^2$ *there exists* $\boldsymbol{\tau}_h \in \tilde{H}^{\boldsymbol{\sigma}}_{0,h}$ *such that*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{H(\mathbf{div};\Omega)} \,\le\, C\, h^r \,\Big\{ \|\boldsymbol{\tau}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\tau})\|_{[H^r(\Omega)]^2} \Big\} \,.$$

($\mathrm{AP}^{\mathbf{u}}_{0,h}$) *For each* $r \in [1,2]$ *and for each* $\mathbf{v} \in [H^{1+r}(\Omega)]^2 \cap [H^1_0(\Omega)]^2$ *there exists* $\mathbf{v}_h \in \tilde{H}^{\mathbf{u}}_{0,h}$ *such that*

$$\|\mathbf{v} - \mathbf{v}_h\|_{[H^1(\Omega)]^2} \,\le\, C\, h^r \,\|\mathbf{v}\|_{[H^{1+r}(\Omega)]^2} \,.$$

($\mathrm{AP}^{\boldsymbol{\gamma}}_{h}$) *For each* $r \in [0,1]$ *and for each* $\boldsymbol{\eta} \in [H^r(\Omega)]^{2\times 2} \cap [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ *there exists* $\boldsymbol{\eta}_h \in \tilde{H}^{\boldsymbol{\gamma}}_{h}$ *such that*

$$\|\boldsymbol{\eta} - \boldsymbol{\eta}_h\|_{[L^2(\Omega)]^{2\times 2}} \,\le\, C\, h^r \,\|\boldsymbol{\eta}\|_{[H^r(\Omega)]^{2\times 2}} \,.$$

Then, we have the following result providing the rate of convergence of (6.2) with $\mathbf{H}_{0,h} = \tilde{\mathbf{H}}_{0,h}$.

THEOREM 6.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \tilde{\mathbf{H}}_{0,h}$ *be the unique solutions of the continuous and discrete augmented formulations* (5.7) *and* (6.2)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, $\mathbf{u} \in [H^{1+r}(\Omega)]^2$, *and* $\boldsymbol{\gamma} \in [H^r(\Omega)]^{2\times 2}$, *for some* $r \in (0,1]$. *Then there exists* $C > 0$, *independent of* h, *such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \,-\, (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \,\le$$

$$C\, h^r \,\Big\{ \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^r(\Omega)]^2} + \|\mathbf{u}\|_{[H^{1+r}(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^r(\Omega)]^{2\times 2}} \Big\} \,.$$

PROOF. It follows from the Cea estimate (6.3) and the approximation properties ($\mathrm{AP}^{\boldsymbol{\sigma}}_{0,h}$), ($\mathrm{AP}^{\mathbf{u}}_{0,h}$), and ($\mathrm{AP}^{\boldsymbol{\gamma}}_{h}$). □

We now go back to the general situation and state the discrete analogue of Lemma 5.1, which gives a sufficient condition for the equivalence between (6.1) and (6.2).

LEMMA 6.3. *Assume that the pressure finite element subspace* H^p_h *contains the traces of the members of the stress tensor finite element subspace* $H^{\boldsymbol{\sigma}}_{0,h}$*, that is,*

$$\mathrm{tr}(H^{\boldsymbol{\sigma}}_{0,h}) \,\subseteq\, H^p_h, \tag{6.8}$$

Then, problems (6.1) *and* (6.2) *are equivalent:* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_{\mathbf{h}}) \in \mathbf{H}_h$ *is a solution of* (6.1) *if and only if* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h}$ *is a solution of* (6.2) *and* $p_h \,=\, -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)$.

PROOF. Let $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_{\mathbf{h}}) \in \mathbf{H}_h$ be a solution of (6.1). It is clear from (6.8) that $p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)$ belongs to H^p_h. Then, taking $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \,=\, (\mathbf{0}, p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h), \mathbf{0}, \mathbf{0}) \in \mathbf{H}_h$, we find from (6.1) that

$$\frac{\kappa_0}{\mu} \int_\Omega \Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h) \Big)^2 \,=\, 0\,,$$

which yields $p_h \,=\, -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)$. Conversely, given $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_{\mathbf{h}}) \in \mathbf{H}_{0,h}$ a solution of (6.2), we let $p_h \,:=\, -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)$ and see that $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_{\mathbf{h}}) \in \mathbf{H}_h$ becomes a solution of (6.1). □

A particular example of finite element subspaces satisfying (6.8) is given by (cf. (6.5))

$$H^{\boldsymbol{\sigma}}_{0,h} \,:=\, \tilde{H}^{\boldsymbol{\sigma}}_{0,h} \quad \text{and} \quad H^p_h \,:=\, \big\{ q_h \,\in\, L^2(\Omega): \quad q_h|_T \,\in\, \mathbb{P}_1(T) \quad \forall\, T \,\in\, \mathcal{T}_h \big\} \,.$$

Anyway, it becomes clear from Lemma 6.3 that the augmented scheme (6.1) makes sense only for pressure finite element subspaces not satisfying the condition (6.8). According to the above, we now aim to show that (6.1) is well-posed when an arbitrary finite element subspace

$\mathbf{H}_h$ of $\mathbf{H}$ is considered. The idea, similarly as for A_0, is to choose κ_0, κ_1, κ_2, and κ_3 such that A be strongly coercive on $\mathbf{H}$ with respect to the norm $\|\cdot\|_{\mathbf{H}}$ defined by

$$\|(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}} := \left\{ \|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} + \|q\|^2_{L^2(\Omega)} + |\mathbf{v}|^2_{[H^1(\Omega)]^2} + \|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}} \right\}^{1/2}.$$

In fact, we first notice that

$$A((\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) = \frac{1}{2\mu}\left(1 - \frac{\kappa_1}{2\,\mu}\right) \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} + \frac{\kappa_0}{\mu}\left\|q + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\right\|^2_{L^2(\Omega)}$$
$$+ \quad \kappa_2\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} + (\kappa_1 + \kappa_3)\,\|\boldsymbol{\varepsilon}(\mathbf{v})\|^2_{[L^2(\Omega)]^{2\times 2}} - \kappa_3\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} + \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}},$$

which, using again Korn's first inequality, employing the estimate

$$\left\|q + \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\right\|^2_{L^2(\Omega)} \geq \frac{1}{2}\,\|q\|^2_{L^2(\Omega)} - \left\|\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\right\|^2_{L^2(\Omega)} \geq \frac{1}{2}\,\|q\|^2_{L^2(\Omega)} - \frac{1}{2}\,\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}},$$

and taking $\kappa_0 > 0$, yields

$$A((\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \geq \frac{1}{2\mu}\left(1 - \frac{\kappa_1}{2\,\mu}\right) \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{[L^2(\Omega)]^{2\times 2}} - \frac{\kappa_0}{2\,\mu}\,\|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}}$$
$$+ \kappa_2\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2} + \frac{\kappa_0}{2\,\mu}\,\|q\|^2_{L^2(\Omega)} + \frac{(\kappa_1 - \kappa_3)}{2}\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} + \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}}.$$

Then, choosing κ_1 and κ_2 such that

$$0 < \kappa_1 < 2\,\mu \quad \text{and} \quad 0 < \kappa_2\,,$$

and applying Lemma 4.4, we deduce that

$$A((\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \geq \left(c_1\,\alpha_1 - \frac{\kappa_0}{2\,\mu}\right) \|\boldsymbol{\tau}\|^2_{[L^2(\Omega)]^{2\times 2}} + \frac{\kappa_2}{2}\,\|\mathbf{div}(\boldsymbol{\tau})\|^2_{[L^2(\Omega)]^2}$$
$$+ \quad \frac{\kappa_0}{2\,\mu}\,\|q\|^2_{L^2(\Omega)} + \frac{(\kappa_1 - \kappa_3)}{2}\,|\mathbf{v}|^2_{[H^1(\Omega)]^2} + \kappa_3\,\|\boldsymbol{\eta}\|^2_{[L^2(\Omega)]^{2\times 2}},$$

where c_1 is the constant from Lemma 4.4 and

$$\alpha_1 := \min\left\{\frac{1}{2\mu}\left(1 - \frac{\kappa_1}{2\mu}\right), \frac{\kappa_2}{2}\right\}.$$

Hence, choosing the parameters κ_0 and κ_3 such that

$$0 < \kappa_0 < 2\,\mu\,c_1\,\alpha_1 \quad \text{and} \quad 0 < \kappa_3 < \kappa_1\,,$$

we find that

$$A((\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \geq \alpha\,\|(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})\|^2_{\mathbf{H}} \qquad \forall\,(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \in \mathbf{H}\,, \tag{6.9}$$

where

$$\alpha := \min\left\{\alpha_2, \frac{\kappa_0}{2\,\mu}, \frac{(\kappa_1 - \kappa_3)}{2}, \kappa_3\right\} \quad \text{and} \quad \alpha_2 := \min\left\{c_1\,\alpha_1 - \frac{\kappa_0}{2\,\mu}, \frac{\kappa_2}{2}\right\}.$$

We are now in a position to establish the following result.

Theorem 6.4. *Assume that there hold*

$$0 < \kappa_0 < 2\,\mu\,c_1\,\alpha_1\,, \quad 0 < \kappa_1 < 2\,\mu\,, \quad 0 < \kappa_2\,, \quad \textit{and} \quad 0 < \kappa_3 < \kappa_1\,.$$

In addition, let $\mathbf{H}_h$ *be any finite element subspace of* $\mathbf{H}$*. Then, the Galerkin scheme* (6.1) *has a unique solution* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_h$*, and there exist positive constants* C, $\tilde{C}$*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \,\leq\, C \sup_{\substack{(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h \\ (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \neq 0}} \frac{|F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)|}{\|(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}}} \,\leq\, C\, \|\mathbf{f}\|_{[L^2(\Omega)]^2}\,,$$

and

$$\|(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \,\leq\, \tilde{C} \inf_{(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_h} \|(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}}\,.$$

PROOF. Since A is bounded and strongly coercive on $\mathbf{H}$ (cf. (5.5) and (6.9)) with constants depending on μ and $(\kappa_0, \kappa_1, \kappa_2, \kappa_3)$, the proof follows from a straightforward application of the Lax-Milgram Lemma, and Cea's estimate. $\square$

In order to consider an explicit Galerkin scheme (6.1), we now let

$$\tilde{H}_h^p \,:=\, \left\{ q_h \in L^2(\Omega) : \quad q_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h \right\}\,,$$

and define

$$\tilde{\mathbf{H}}_h \,:=\, \tilde{H}_{0,h}^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}}\,, \tag{6.10}$$

where $\tilde{H}_{0,h}^{\boldsymbol{\sigma}}$, $\tilde{H}_{0,h}^{\mathbf{u}}$, and $\tilde{H}_h^{\boldsymbol{\gamma}}$ are given, respectively, by (6.5), (6.6), and (6.7).

The approximation property of $\tilde{H}_h^p$ is given as follows (see [**12**], [**16**]):

(AP_h^p) *For each* $r \in [0,1]$ *and for each* $q \in H^r(\Omega)$ *there exists* $q_h \in \tilde{H}_h^p$ *such that*

$$\|q - q_h\|_{L^2(\Omega)} \,\leq\, C\, h^r\, \|q\|_{H^r(\Omega)}\,.$$

Then, we have the following theorem providing the rate of convergence of (6.1) with $\mathbf{H}_h = \tilde{\mathbf{H}}_h$.

THEOREM 6.5. *Let* $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}$ *and* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \tilde{\mathbf{H}}_h$ *be the unique solutions of the continuous and discrete augmented formulations* (5.4) *and* (6.1)*, respectively. Assume that* $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, $\mathbf{u} \in [H^{r+1}(\Omega)]^2$*, and* $\boldsymbol{\gamma} \in [H^r(\Omega)]^{2\times 2}$*, for some* $r \in (0,1]$*. Then there exists* $C > 0$*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \leq \\ C h^r \left\{ \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^r(\Omega)]^2} + \|\mathbf{u}\|_{[H^{r+1}(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^r(\Omega)]^{2\times 2}} \right\}.$$

PROOF. We first notice, according to Theorem 5.3 and the hypothesis on $\boldsymbol{\sigma}$, that $p = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma})$ belongs to $H^r(\Omega)$ and that $\|p\|_{H^r(\Omega)} \,\leq\, C\, \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}}$. Then, the proof follows from the Cea estimate from Theorem 6.4 and the approximation properties ($\mathrm{AP}_{0,h}^{\boldsymbol{\sigma}}$), ($\mathrm{AP}_h^p$), ($\mathrm{AP}_{0,h}^{\mathbf{u}}$), and ($\mathrm{AP}_h^{\boldsymbol{\gamma}}$). $\square$

At this point we would like to emphasize the main features of our augmented Galerkin schemes (6.1) and (6.2), as compared to each other, besides the fact that both of them can be implemented with any finite element subspace of $\mathbf{H}$ and $\mathbf{H}_0$, respectively. In fact, it is important to notice on one hand that (6.2) allows an explicit and simple definition of the whole vector of parameters $(\kappa_1, \kappa_2, \kappa_3)$ (cf. Theorem 5.2), whereas the choice of κ_0 in (6.1) depends on the unknown constant c_1 from Lemma 4.4. On the other hand, it is clear that (6.1) provides more flexibility for approximating the pressure since the corresponding finite element subspace H_h^p can be chosen arbitrarily, whereas (6.2) needs a postprocess to compute

p_h in terms of $\boldsymbol{\sigma}_h$, either simply as $p_h := -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ or projecting $-\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ onto some finite element subspace.

We end this section by mentioning that a useful discussion on the actual implementation of augmented Galerkin schemes of the present kind can be seen in [**27**].

CHAPTER 3

A posteriori error analysis for augmented incompressible elasticity

7. The residual error estimator

In this section we derive a residual based a posteriori error estimator for (6.1), much in the spirit of [**8**]. The analysis for (6.2) is contained in what follows, and hence we omit details.

First we introduce several notations. Given $T \in \mathcal{T}_h$, we let $\mathcal{E}(T)$ be the set of its edges, and let $\mathcal{E}_h$ be the set of all edges of the triangulation $\mathcal{T}_h$. Then we write $\mathcal{E}_h = \mathcal{E}_{h,\Omega} \cup \mathcal{E}_{h,\Gamma}$, where $\mathcal{E}_{h,\Omega} := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$ and $\mathcal{E}_{h,\Gamma} := \{e \in \mathcal{E}_h : e \subseteq \Gamma\}$. In what follows, h_e stands for the length of the edge e. Further, given $\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2}$ such that $\boldsymbol{\tau}|_T \in C(T)$ on each $T \in \mathcal{T}_h$, an edge $e \in \mathcal{E}(T) \cap \mathcal{E}_{h,\Omega}$, and the unit tangential vector $\mathbf{t}_T$ along e, we let $J[\boldsymbol{\tau}\mathbf{t}_T]$ be the corresponding jump across e, that is, $J[\boldsymbol{\tau}\mathbf{t}_T] := (\boldsymbol{\tau}|_T - \boldsymbol{\tau}|_{T'})|_e \mathbf{t}_T$, where T' is the other triangle of $\mathcal{T}_h$ having e as an edge. Abusing notation, when $e \in \mathcal{E}_{h,\Gamma}$, we also write $J[\boldsymbol{\tau}\mathbf{t}_T] := \boldsymbol{\tau}|_e \mathbf{t}_T$. We recall here that $\mathbf{t}_T := (-\nu_2, \nu_1)^{\mathtt{t}}$, where $\boldsymbol{\nu}_T := (\nu_1, \nu_2)^{\mathtt{t}}$ is the unit outward vector normal to ∂T. Analogously, we define the normal jumps $J[\boldsymbol{\tau}\boldsymbol{\nu}_T]$. In addition, given scalar, vector and tensor valued fields v, $\boldsymbol{\varphi} := (\varphi_1, \varphi_2)$ and $\boldsymbol{\tau} := (\tau_{ij})$, respectively, we let

$$\mathbf{curl}(v) := \begin{pmatrix} -\frac{\partial v}{\partial x_2} \\ \frac{\partial v}{\partial x_1} \end{pmatrix}, \quad \underline{\mathbf{curl}}(\boldsymbol{\varphi}) := \begin{pmatrix} \mathbf{curl}(\varphi_1)^{\mathtt{t}} \\ \mathbf{curl}(\varphi_2)^{\mathtt{t}} \end{pmatrix}, \text{ and } \operatorname{curl}(\boldsymbol{\tau}) := \begin{pmatrix} \frac{\partial \tau_{12}}{\partial x_1} - \frac{\partial \tau_{11}}{\partial x_2} \\ \frac{\partial \tau_{22}}{\partial x_1} - \frac{\partial \tau_{21}}{\partial x_2} \end{pmatrix}.$$

Then, letting $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}$ and $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_h$ be the unique solutions of the continuous and discrete augmented formulations (5.4) and (6.1), respectively, we define for each $T \in \mathcal{T}_h$ a local error indicator θ_T as follows:

$$\begin{aligned}
\theta_T^2 := {} & \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + \left\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\right\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\|^2_{[L^2(T)]^{2\times 2}} \\
& + h_T^2 \|\operatorname{curl}\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} + \boldsymbol{\gamma}_h\Big)\|^2_{[L^2(T)]^2} + h_T^2\|\mathbf{curl}\Big(p_h + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)\Big)\|^2_{[L^2(T)]^2} \\
& + h_T^2\|\operatorname{curl}\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\|^2_{[L^2(T)]^2} \\
& + \sum_{e \in \mathcal{E}(T)} h_e \|J\Big[\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \boldsymbol{\gamma}_h\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2} \\
& + \sum_{e \in \mathcal{E}(T)} h_e \|J\Big[\Big(p_h + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2} + \sum_{e \in \mathcal{E}(T)} h_e\|J\Big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2}
\end{aligned}$$

$$
\begin{aligned}
&+ h_T^2 \|\mathbf{div}\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\|^2_{[L^2(T)]^2} \\
&+ h_T^2 \|\mathbf{div}\Big(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}\right)\Big)\|^2_{[L^2(T)]^2} \\
&+ \sum_{e \in \mathcal{E}(T)\cap \mathcal{E}_{h,\Omega}} h_e \|J\Big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\boldsymbol{\nu}_T\Big]\|^2_{[L^2(e)]^2} \\
&+ \sum_{e \in \mathcal{E}(T)\cap \mathcal{E}_{h,\Omega}} h_e \|J\Big[\Big(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}\right)\Big)\boldsymbol{\nu}_T\Big]\|^2_{[L^2(e)]^2} .
\end{aligned} \tag{7.1}
$$

The residual character of each term on the right hand side of (7.1) is quite clear. As usual the expression $\boldsymbol{\theta} := \left\{ \sum_{T\in\mathcal{T}_h} \theta_T^2 \right\}^{1/2}$ is employed as the global residual error estimator.

The following theorem is the main result of this section.

THEOREM 7.1. *Let $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}$ and $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_h$ be the unique solutions of* (5.4) *and* (6.1)*, respectively. Then there exist positive constants $C_{\mathtt{eff}}$ and $C_{\mathtt{rel}}$, independent of h, such that*

$$
C_{\mathtt{eff}}\, \boldsymbol{\theta} \;\le\; \|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \;\le\; C_{\mathtt{rel}}\, \boldsymbol{\theta} . \tag{7.2}
$$

The efficiency of the global error estimator (lower bound in (7.2)) is proved below in Subsection 9 and the reliability of the global error estimator (upper bound in (7.2)) is derived now.

8. Reliability of the a posteriori error estimator

We begin with the following preliminary estimate.

LEMMA 8.1. *There exists $C > 0$, independent of h, such that*

$$
\begin{aligned}
&C\,\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|_{\mathbf{H}} \le \\
&\sup_{\substack{(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}\backslash\{\mathbf{0}\} \\ \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}}} \frac{A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}))}{\|(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}}} + \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(\Omega)]^2}
\end{aligned} \tag{8.1}
$$

PROOF. Let us define $\boldsymbol{\sigma}^* = \boldsymbol{\varepsilon}(\mathbf{z})$, where $\mathbf{z} \in [H_0^1(\Omega)]^2$ is the unique solution of the boundary value problem: $-\mathbf{div}(\boldsymbol{\varepsilon}(\mathbf{z})) = \mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)$ in Ω, $\mathbf{z} = 0$ on Γ. It follows that $\boldsymbol{\sigma}^* \in H_0$ and the corresponding continuous dependence result establishes the existence of $c > 0$ such that

$$
\|\boldsymbol{\sigma}^*\|_{H(\mathbf{div};\Omega)} \le c\,\|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(\Omega)]^2} . \tag{8.2}
$$

In addition, $\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*) = -\mathbf{f} - \mathbf{div}(\boldsymbol{\sigma}_h) + (\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)) = \mathbf{0}$ in Ω. Let α and M be the coercivity and boundedness constants of A. Then, using the coercivity of A we find that

$$
\begin{aligned}
&\alpha\,\|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)\|^2_{\mathbf{H}} \\
&\quad \le A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)) \\
&\quad \le A((\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)) \\
&\qquad - A((\boldsymbol{\sigma}^*, 0, \mathbf{0}, \mathbf{0}), (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h - \boldsymbol{\sigma}^*, p - p_h, \mathbf{u} - \mathbf{u}_h, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h)),
\end{aligned}
$$

which, employing the boundedness of A, yields

$$\begin{aligned}
&\alpha \,\|(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h-\boldsymbol{\sigma}^*, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h)\|_{\mathbf{H}} \\
&\quad \le \sup_{\substack{(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})\in\mathbf{H}\backslash\{\mathbf{0}\}\\ \mathbf{div}(\boldsymbol{\tau})=\mathbf{0}}} \frac{A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta}))}{\|(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{H}}} + M\,\|\boldsymbol{\sigma}^*\|_{H(\mathbf{div};\Omega)}\,. \qquad (8.3)
\end{aligned}$$

Hence, (8.1) follows straightforwardly from the triangle inequality, (8.2) and (8.3). □

It remains to bound the first term on the right hand side of (8.1). To this end, we will make use of the well known Clément interpolation operator, $I_h : H^1(\Omega) \longrightarrow X_h$ (cf. [**17**]), with X_h given by

$$X_h := \left\{ v_h \in C(\bar{\Omega}) : v_h|_T \in \mathbb{P}_1(T) \quad \forall\, T \in \mathcal{T}_h \right\},$$

which satisfies the standard local approximation properties stated below in Lemma 8.2. It is important to remark that I_h is defined in [**17**] so that $I_h(v) \in X_h \cap H_0^1(\Omega)$ for all $v \in H_0^1(\Omega)$.

LEMMA 8.2. *There exist constants $C_1, C_2 > 0$, independent of h, such that for all $v \in H^1(\Omega)$ there holds*

$$\|v - I_h(v)\|_{L^2(T)} \le C_1 h_T \|v\|_{H^1(\tilde{\omega}_T)} \quad \forall\, T \in \mathcal{T}_h,$$

and

$$\|v - I_h(v)\|_{L^2(e)} \le C_2 h_e^{1/2} \|v\|_{H^1(\tilde{\omega}_e)} \quad \forall\, e \in \mathcal{E}_h,$$

where $\tilde{\omega}_T$ and $\tilde{\omega}_e$ are the union of all elements sharing at least one point with T and e, respectively.

PROOF. See [**17**]. □

We now let $(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta}) \in \mathbf{H}$, $(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta}) \neq \mathbf{0}$, such that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$ in Ω. Since Ω is connected, there exists a stream function $\boldsymbol{\varphi} := (\varphi_1, \varphi_2) \in [H^1(\Omega)]^2$ such that $\int_\Omega \varphi_1 = \int_\Omega \varphi_2 = 0$ and $\boldsymbol{\tau} = \underline{\mathbf{curl}}(\boldsymbol{\varphi})$. Then, denoting $\boldsymbol{\varphi}_h := (I_h(\varphi_1), I_h(\varphi_2))$, we define $\boldsymbol{\tau}_h := \underline{\mathbf{curl}}(\boldsymbol{\varphi}_h)$.

It can be seen that, since $\boldsymbol{\tau}_h$ has $[H^1(T)]^{2\times 2}$-regularity on each triangle (in fact, it is piecewise constant), and its rows have continuous normal components across each interior edge, $\boldsymbol{\tau}_h$ has a $L^2(\Omega)$ divergence, which is zero. Thus, $\boldsymbol{\tau}_h$ belongs to $\tilde{H}_h^{\boldsymbol{\sigma}}$ (cf. (6.4)). The decomposition $\boldsymbol{\tau}_h = \boldsymbol{\tau}_{h,0} + d_h \mathbf{I}$, holds, where $\boldsymbol{\tau}_{h,0} \in \tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ (cf. (6.5)) and $d_h = \frac{\int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h)}{2|\Omega|} \in \mathbb{R}$.

We also define $\mathbf{v}_h := (I_h(v_1), I_h(v_2)) \in H_0^{\mathbf{u}}$, the vector Clément interpolant of $\mathbf{v} := (v_1, v_2) \in [H_0^1(\Omega)]^2$. From the Galerkin orthogonality, it follows that

$$\begin{aligned}
&A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})) = \\
&\qquad\qquad A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau}-\boldsymbol{\tau}_{h,0},q,\mathbf{v}-\mathbf{v}_h,\boldsymbol{\eta})). \qquad (8.4)
\end{aligned}$$

Also, from (5.5), the orthogonality between symmetric and asymmetric tensors, and as a consequence, again, of the Galerkin orthogonality, it follows that

$$\begin{aligned}
&A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(d_h\mathbf{I},0,\mathbf{0},\mathbf{0})) \\
&\quad = \frac{\kappa_0}{\mu}\int_\Omega \left(p-p_h+\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\right)\frac{1}{2}\mathrm{tr}(d_h\mathbf{I}) \\
&\quad = A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\mathbf{0},d_h,\mathbf{0},\mathbf{0})) \\
&\quad = 0\,.
\end{aligned} \qquad (8.5)$$

Hence, (8.4), (8.5) and (6.1) give

$$\begin{aligned}
&A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})) \\
&\qquad = A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau}-\boldsymbol{\tau}_h,q,\mathbf{v}-\mathbf{v}_h,\boldsymbol{\eta})) \qquad (8.6)\\
&\qquad = F(\boldsymbol{\tau}-\boldsymbol{\tau}_h,q,\mathbf{v}-\mathbf{v}_h,\boldsymbol{\eta}) - A((\boldsymbol{\sigma}_h,p_h,\mathbf{u}_h,\boldsymbol{\gamma}_h),(\boldsymbol{\tau}-\boldsymbol{\tau}_h,q,\mathbf{v}-\mathbf{v}_h,\boldsymbol{\eta})),
\end{aligned}$$

which, after some algebraic manipulations, yields that

$$
\begin{aligned}
& A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \gamma-\gamma_h), (\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta})) \\
&\quad = \int_\Omega (\mathbf{f}+\mathbf{div}(\boldsymbol{\sigma}_h))\cdot(\mathbf{v}-\mathbf{v}_h) + \int_\Omega \left(\frac{1}{2}(\boldsymbol{\sigma}_h-\boldsymbol{\sigma}_h^{\mathtt{t}}) - \kappa_3(\gamma_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right))\right) : \boldsymbol{\eta} \\
&\quad - \int_\Omega \left\{\kappa_1\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h^{\mathtt{d}} + (\boldsymbol{\sigma}_h^{\mathtt{d}})^{\mathtt{t}})\right) + \kappa_3(\gamma_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right))\right\} : \nabla(\mathbf{v}-\mathbf{v}_h) \\
&\quad - \int_\Omega \left\{\left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \gamma_h\right) + \frac{\kappa_0}{2\mu}\left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{I} + \frac{\kappa_1}{2\mu}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\right\} : (\boldsymbol{\tau}-\boldsymbol{\tau}_h) \\
&\quad - \frac{\kappa_0}{\mu}\int_\Omega \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right) p.
\end{aligned}
\tag{8.7}
$$

The rest of reliability consists in deriving suitable upper bounds for each one of the terms appearing on the right hand side of (8.7). We begin by noticing that direct applications of the Cauchy-Schwarz inequality give

$$
\left|\int_\Omega \frac{1}{2}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}) \colon \boldsymbol{\eta}\right| \le \left\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\right\|_{[L^2(\Omega)]^{2\times 2}} \|\boldsymbol{\eta}\|_{[L^2(\Omega)]^{2\times 2}}\,, \tag{8.8}
$$

$$
\left|\int_\Omega (\gamma_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)) \colon \boldsymbol{\eta}\right| \le \left\|\gamma_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\right\|_{[L^2(\Omega)]^{2\times 2}} \|\boldsymbol{\eta}\|_{[L^2(\Omega)]^{2\times 2}}\,, \tag{8.9}
$$

and

$$
\left|\int_\Omega \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right) p\right| \le \left\|p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right\|_{L^2(\Omega)} \|p\|_{L^2(\Omega)}\,. \tag{8.10}
$$

The decomposition $\bar{\Omega} = \bigcup_{T\in\mathcal{T}_h} T$ and the use of integration by parts formulae on each element are employed next to handle the terms from the third and the fourth rows of (8.7). We first replace $\boldsymbol{\tau}-\boldsymbol{\tau}_h$ by $\underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)$ and use that $\mathrm{curl}(\nabla\mathbf{u}_h) = \mathbf{0}$ on each triangle $T\in\mathcal{T}_h$, to obtain

$$
\begin{aligned}
\int_\Omega \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \gamma_h\right) : (\boldsymbol{\tau}-\boldsymbol{\tau}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \gamma_h\right) : \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) \\
&= \sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}\left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} + \gamma_h\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) \\
&\quad - \sum_{e\in\mathcal{E}_h}\left\langle J\left[\left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \gamma_h\right)\mathbf{t}_T\right], \boldsymbol{\varphi}-\boldsymbol{\varphi}_h\right\rangle_{[L^2(e)]^2}\,,
\end{aligned}
\tag{8.11}
$$

$$
\begin{aligned}
\int_\Omega \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{I} \colon (\boldsymbol{\tau}-\boldsymbol{\tau}_h) &= \sum_{T\in\mathcal{T}_h}\int_T \left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{I} \colon \underline{\mathbf{curl}}(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) \\
&= \sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h) \\
&\quad - \sum_{e\in\mathcal{E}_h}\left\langle J\left[\left(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{t}_T\right], \boldsymbol{\varphi}-\boldsymbol{\varphi}_h\right\rangle_{[L^2(e)]^2}\,,
\end{aligned}
\tag{8.12}
$$

and

$$\int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} \right) : (\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \sum_{T\in\mathcal{T}_h} \int_T \left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} \right) : \underline{\mathbf{curl}}(\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)$$
$$= \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl} \left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} \right) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h)$$
$$- \sum_{e\in\mathcal{E}_h} \left\langle J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} \right) \mathbf{t}_T \right], \boldsymbol{\varphi} - \boldsymbol{\varphi}_h \right\rangle_{[L^2(e)]^2} . \quad (8.13)$$

On the other hand, using that $\mathbf{v} - \mathbf{v}_h = \mathbf{0}$ on Γ, we get

$$\int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}} \right) : \nabla(\mathbf{v} - \mathbf{v}_h)$$
$$= -\sum_{T\in\mathcal{T}_h} \int_T \mathbf{div} \left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}} \right) \cdot (\mathbf{v} - \mathbf{v}_h)$$
$$+ \sum_{e\in\mathcal{E}_{h,\Omega}} \left\langle J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}} \right) \boldsymbol{\nu}_T \right], \mathbf{v} - \mathbf{v}_h \right\rangle_{[L^2(e)]^2} , \quad (8.14)$$

and

$$\int_\Omega \left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right) \right) : \nabla(\mathbf{v} - \mathbf{v}_h)$$
$$= -\sum_{T\in\mathcal{T}_h} \int_T \mathbf{div} \left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right) \right) \cdot (\mathbf{v} - \mathbf{v}_h)$$
$$+ \sum_{e\in\mathcal{E}_{h,\Omega}} \left\langle J\left[\left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right) \right) \boldsymbol{\nu}_T \right], \mathbf{v} - \mathbf{v}_h \right\rangle_{[L^2(e)]^2} . \quad (8.15)$$

In what follows we apply again the Cauchy-Schwarz inequality, Lemma 8.2 and the fact that the number of triangles is bounded independently of h in both $\tilde{\omega}_T$ and $\tilde{\omega}_e$ to derive the estimates for the expression $\int_\Omega (\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)) \cdot (\mathbf{v} - \mathbf{v}_h)$ in (8.7) and the right hand sides of (8.11), (8.12), (8.13), (8.14), and (8.15), with constants C independent of h. Indeed, we easily have

$$\left| \int_\Omega (\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)) \cdot (\mathbf{v} - \mathbf{v}_h) \right| \le C \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \, \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|_{[L^2(T)]^2}^2 \right\}^{1/2} \|\mathbf{v}\|_{[H^1(\Omega)]^2} . \quad (8.16)$$

In addition, for the terms containing the stream funcion $\boldsymbol{\varphi}$ (cf. (8.11), (8.12), (8.13)), we get

$$\left| \sum_{T\in\mathcal{T}_h} \int_T \operatorname{curl} \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} + \boldsymbol{\gamma}_h \right) \cdot (\boldsymbol{\varphi} - \boldsymbol{\varphi}_h) \right|$$
$$\le C \left\{ \sum_{T\in\mathcal{T}_h} h_T^2 \left\| \operatorname{curl} \left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} + \boldsymbol{\gamma}_h \right) \right\|_{[L^2(T)]^2}^2 \right\}^{1/2} \|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2} , \quad (8.17)$$

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}_h}\int_T \mathbf{curl}\left(p_h+\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\right| \\ \leq C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\left\|\mathbf{curl}\left(p_h+\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\right\|^2_{[L^2(T)]^2}\right\}^{1/2}\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\,,\end{aligned}\tag{8.18}$$

$$\begin{aligned}\left|\sum_{T\in\mathcal{T}_h}\int_T \mathrm{curl}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}}-\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\cdot(\boldsymbol{\varphi}-\boldsymbol{\varphi}_h)\right| \\ \leq C\left\{\sum_{T\in\mathcal{T}_h} h_T^2\left\|\mathrm{curl}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}}-\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\right\|^2_{[L^2(T)]^2}\right\}^{1/2}\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\,,\end{aligned}\tag{8.19}$$

$$\begin{aligned}\left|\sum_{e\in\mathcal{E}_h}\left\langle J\left[\left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}-\nabla\mathbf{u}_h+\boldsymbol{\gamma}_h\right)\mathbf{t}_T\right],\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\right\rangle_{[L^2(e)]^2}\right| \\ \leq C\left\{\sum_{e\in\mathcal{E}_h} h_e\left\|J\left[\left(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}-\nabla\mathbf{u}_h+\boldsymbol{\gamma}_h\right)\mathbf{t}_T\right]\right\|^2_{[L^2(e)]^2}\right\}^{1/2}\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\,,\end{aligned}\tag{8.20}$$

$$\begin{aligned}\left|\sum_{e\in\mathcal{E}_h}\left\langle J\left[\left(p_h+\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{t}_T\right],\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\right\rangle_{[L^2(e)]^2}\right| \\ \leq C\left\{\sum_{e\in\mathcal{E}_h} h_e\left\|J\left[\left(p_h+\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\right)\mathbf{t}_T\right]\right\|^2_{[L^2(e)]^2}\right\}^{1/2}\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\,,\end{aligned}\tag{8.21}$$

and

$$\begin{aligned}\left|\sum_{e\in\mathcal{E}_h}\left\langle J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}}-\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\mathbf{t}_T\right],\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\right\rangle_{[L^2(e)]^2}\right| \\ \leq C\left\{\sum_{e\in\mathcal{E}_h} h_e\left\|J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}}-\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\right)\mathbf{t}_T\right]\right\|^2_{[L^2(e)]^2}\right\}^{1/2}\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\,.\end{aligned}\tag{8.22}$$

We observe here, due to the equivalence between $\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}$ and $\|\nabla\boldsymbol{\varphi}\|_{[L^2(\Omega)]^{2\times 2}}$, that

$$\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}\leq C\,\|\nabla\boldsymbol{\varphi}\|_{[L^2(\Omega)]^{2\times 2}}=C\,\|\underline{\mathbf{curl}}(\boldsymbol{\varphi})\|_{[L^2(\Omega)]^{2\times 2}}=C\,\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}\,,$$

which allows to replace $\|\boldsymbol{\varphi}\|_{[H^1(\Omega)]^2}$ by $\|\boldsymbol{\tau}\|_{H(\mathbf{div};\Omega)}$ in the above estimates (8.17) - (8.22).

Similarly, for the terms on the right hand side of (8.14) and (8.15), we find that

$$\left| \sum_{T\in\mathcal{T}_h} \int_T \mathbf{div}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\right)\cdot(\mathbf{v}-\mathbf{v}_h)\right|$$
$$\leq C\left\{\sum_{T\in\mathcal{T}_h} h_T^2 \left\|\mathbf{div}\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\right)\right\|^2_{[L^2(T)]^2}\right\}^{1/2} \|\mathbf{v}\|_{[H^1(\Omega)]^2}\,, \quad (8.23)$$

$$\left| \sum_{T\in\mathcal{T}_h} \int_T \mathbf{div}\left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\right)\cdot(\mathbf{v}-\mathbf{v}_h)\right|$$
$$\leq C\left\{\sum_{T\in\mathcal{T}_h} h_T^2 \left\|\mathbf{div}\left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\right)\right\|^2_{[L^2(T)]^2}\right\}^{1/2} \|\mathbf{v}\|_{[H^1(\Omega)]^2}\,, \quad (8.24)$$

$$\left| \sum_{e\in\mathcal{E}_{h,\Omega}} \left\langle J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\right)\boldsymbol{\nu}_T\right], \mathbf{v}-\mathbf{v}_h\right\rangle_{[L^2(e)]^2}\right|$$
$$\leq C\left\{\sum_{e\in\mathcal{E}_{h,\Omega}} h_e \left\|J\left[\left(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\right)\boldsymbol{\nu}_T\right]\right\|^2_{[L^2(e)]^2}\right\}^{1/2} \|\mathbf{v}\|_{[H^1(\Omega)]^2}\,, \quad (8.25)$$

and

$$\left| \sum_{e\in\mathcal{E}_{h,\Omega}} \left\langle J\left[\left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\right)\boldsymbol{\nu}_T\right], \mathbf{v}-\mathbf{v}_h\right\rangle_{[L^2(e)]^2}\right|$$
$$\leq C\left\{\sum_{e\in\mathcal{E}_{h,\Omega}} h_e \left\|J\left[\left(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\right)\boldsymbol{\nu}_T\right]\right\|^2_{[L^2(e)]^2}\right\}^{1/2} \|\mathbf{v}\|_{[H^1(\Omega)]^2}\,. \quad (8.26)$$

Therefore, placing (8.17) - (8.22) (resp. (8.23) - (8.26)) back into (8.11) - (8.13) (resp. (8.14) and (8.15)), employing the estimates (8.8), (8.9), (8.10) and (8.16), and using the identities

$$\sum_{e\in\mathcal{E}_{h,\Omega}} \int_e = \frac{1}{2}\sum_{T\in\mathcal{T}_h}\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} \int_e$$

and

$$\sum_{e\in\mathcal{E}_h} \int_e = \sum_{e\in\mathcal{E}_{h,\Omega}} \int_e + \sum_{T\in\mathcal{T}_h}\sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Gamma}} \int_e,$$

we conclude from (8.7) that

$$\sup_{\substack{(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})\in\mathbf{H}\setminus\{\mathbf{0}\}\\ \mathbf{div}(\boldsymbol{\tau})=\mathbf{0}}} \frac{A((\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, p-p_h, \mathbf{u}-\mathbf{u}_h, \boldsymbol{\gamma}-\boldsymbol{\gamma}_h),(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta}))}{\|(\boldsymbol{\tau},q,\mathbf{v},\boldsymbol{\eta})\|_{\mathbf{H}}} \leq C\boldsymbol{\theta}. \quad (8.27)$$

This inequality and Lemma 8.1 complete the proof of reliability of $\boldsymbol{\theta}$.

We remark that when the finite element subspace $\mathbf{H}_h$ is given by (6.10), that is, when $\boldsymbol{\sigma}_h|_T \in [\mathbb{RT}_0(T)]^2$, $p_h|_T \in \mathbb{P}_0(T)$, $\mathbf{u}_h|_T \in [\mathbb{P}_1(T)]^2$, and $\boldsymbol{\gamma}_h|_T \in [\mathbb{P}_0(T)]^{2\times 2}$, then the expression (7.1) for θ_T^2 simplifies to

$$\begin{aligned}
\theta_T^2 := {} & \|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} + \left\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\right\|^2_{[L^2(T)]^{2\times 2}} + \|\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}\right)\|^2_{[L^2(T)]^{2\times 2}} \\
& + h_T^2 \|\mathrm{curl}\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\|^2_{[L^2(T)]^2} + h_T^2 \|\mathbf{curl}\Big(\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\|^2_{[L^2(T)]^2} \\
& + \sum_{e\in\mathcal{E}(T)} h_e \|J\Big[\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla \mathbf{u}_h + \boldsymbol{\gamma}_h\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2} \\
& + \sum_{e\in\mathcal{E}(T)} h_e \|J\Big[\Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2} + \sum_{e\in\mathcal{E}(T)} h_e \|J\Big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\mathbf{t}_T\Big]\|^2_{[L^2(e)]^2} \\
& + h_T^2 \|\mathbf{div}\Big(\frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\|^2_{[L^2(T)]^2} \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_e \|J\Big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\boldsymbol{\nu}_T\Big]\|^2_{[L^2(e)]^2} \\
& + \sum_{e\in\mathcal{E}(T)\cap\mathcal{E}_{h,\Omega}} h_e \|J\Big[\Big(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}\right)\Big)\boldsymbol{\nu}_T\Big]\|^2_{[L^2(e)]^2}.
\end{aligned} \tag{8.28}$$

9. Efficiency of the a posteriori error estimator

In this section we proceed as in [**8**] and apply results ultimately based on inverse inequalities (see [**16**]) and the localization technique introduced in [**40**], which is based on triangle-bubble and edge-bubble functions, to prove the efficiency of our a posteriori estimator $\boldsymbol{\theta}$ (lower bound of the estimate (7.2)).

Our goal is to estimate the thirteen terms defining the error indicator θ_T^2 (cf. (7.1)). Using $\mathbf{f} = -\mathbf{div}(\boldsymbol{\sigma})$, the symmetry of $\boldsymbol{\sigma}$, and $\boldsymbol{\gamma} = \frac{1}{2}\left(\nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathtt{t}}\right)$, we first observe that there hold

$$\|\mathbf{f} + \mathbf{div}(\boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2} = \|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|^2_{[L^2(T)]^2}\,, \tag{9.1}$$

$$\left\|\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^{\mathtt{t}}\right\|^2_{[L^2(T)]^{2\times 2}} \le 4\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|^2_{[L^2(T)]^{2\times 2}}\,, \tag{9.2}$$

and

$$\|\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla \mathbf{u}_h - (\nabla \mathbf{u}_h)^{\mathtt{t}}\right)\|^2_{[L^2(T)]^{2\times 2}} \le 2\left\{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|^2_{[L^2(T)]^{2\times 2}} + |\mathbf{u} - \mathbf{u}_h|^2_{[H^1(T)]^2}\right\}. \tag{9.3}$$

The upper bounds of the remaining ten terms, which depend on the mesh parameters h_T and h_e, will be derived next. To this end we will make use of Lemmata 9.1 - 9.4 below. Lemma 9.1 is required for the terms involving the curl and **curl** operators, Lemma 9.2 handles the terms involving tangential jumps across the edges of $\mathcal{T}_h$, Lemma 9.3 is required for the terms containing the **div** operator, and Lemma 9.4 is used to take care of the terms encompassing normal jumps across the edges of $\mathcal{T}_h$. For their proofs we refer to [**8**] and references therein. In what follows, we let

$$w_e := \cup\{T' \in \mathcal{T}_h : \quad e \in \mathcal{E}(T')\}\,.$$

LEMMA 9.1. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \ge 0$ on each $T \in \mathcal{T}_h$. In addition, let $\boldsymbol{\rho} \in [L^2(\Omega)]^{2\times 2}$ be such that* $\mathrm{curl}(\boldsymbol{\rho}) = \mathbf{0}$ *on each $T \in \mathcal{T}_h$. Then, there exists $c > 0$, independent of h, such that for any $T \in \mathcal{T}_h$*

$$\|\mathrm{curl}(\boldsymbol{\rho}_h)\|_{[L^2(T)]^2} \le c h_T^{-1}\,\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{[L^2(T)]^{2\times 2}}\,. \tag{9.4}$$

LEMMA 9.2. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$. Then, there exists $c > 0$, independent of h, such that for any $e \in \mathcal{E}_h$*

$$\|J[\boldsymbol{\rho}_h \mathbf{t}_T]\|_{[L^2(e)]^2} \leq c h_e^{-1/2} \, \|\boldsymbol{\rho}_h\|_{[L^2(\omega_e)]^{2\times 2}} \,. \tag{9.5}$$

LEMMA 9.3. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$. Then, there exists $c > 0$, independent of h, such that for any $T \in \mathcal{T}_h$*

$$\|\mathbf{div}(\boldsymbol{\rho}_h)\|_{[L^2(T)]^2} \leq c h_T^{-1} \, \|\boldsymbol{\rho}_h\|_{[L^2(T)]^{2\times 2}} \,. \tag{9.6}$$

LEMMA 9.4. *Let $\boldsymbol{\rho}_h \in [L^2(\Omega)]^{2\times 2}$ be a piecewise polynomial of degree $k \geq 0$ on each $T \in \mathcal{T}_h$. Then, there exists $c > 0$, independent of h, such that for any $e \in \mathcal{E}_h$*

$$\|J[\boldsymbol{\rho}_h \boldsymbol{\nu}_T]\|_{[L^2(e)]^2} \leq c h_e^{-1/2} \, \|\boldsymbol{\rho}_h\|_{[L^2(\omega_e)]^{2\times 2}} \,. \tag{9.7}$$

We now complete the proof of efficiency of $\boldsymbol{\theta}$ by conveniently applying Lemmata 9.1 - 9.4 to the corresponding terms defining θ_T^2.

LEMMA 9.5. *There exist $C_1, C_2, C_3 > 0$, independent of h, such that for any $T \in \mathcal{T}_h$*

$$h_T^2 \|\mathrm{curl}\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} + \boldsymbol{\gamma}_h\Big)\|_{[L^2(T)]^2}^2 \leq C_1 \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(T)]^{2\times 2}}^2 \right\}, \tag{9.8}$$

$$h_T^2 \|\mathbf{curl}\Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\|_{[L^2(T)]^2}^2 \leq C_2 \left\{ \|p - p_h\|_{L^2(T)}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}}^2 \right\}, \tag{9.9}$$

and

$$h_T^2 \|\mathrm{curl}\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\|_{[L^2(T)]^2}^2 \leq C_3 \left\{ |\mathbf{u} - \mathbf{u}_h|_{[H^1(T)]^2}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}}^2 \right\}. \tag{9.10}$$

PROOF. Applying Lemma 9.1 with $\boldsymbol{\rho}_h := \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} + \boldsymbol{\gamma}_h$ and $\boldsymbol{\rho} := \nabla\mathbf{u} = \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} + \boldsymbol{\gamma}$, and then using the triangle inequality and the continuity of $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^{\mathtt{d}}$ we obtain (9.8). Similarly, (9.9) and (9.10) follow from Lemma 9.1 with $\boldsymbol{\rho}_h := p_h\mathbf{I} + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\mathbf{I}$ and $\boldsymbol{\rho} := p\mathbf{I} + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})\mathbf{I} = \mathbf{0}$ (cf. (4.2)), and $\boldsymbol{\rho}_h := \boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}$ and $\boldsymbol{\rho} := \boldsymbol{\varepsilon}(\mathbf{u})^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} = \mathbf{0}$ (cf. (4.4)), respectively. □

LEMMA 9.6. *There exist $C_4, C_5, C_6 > 0$, independent of h, such that for any $e \in \mathcal{E}_h$*

$$\begin{aligned} h_e \|J\Big[\Big(\frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \boldsymbol{\gamma}_h\Big)\mathbf{t}_T\Big]\|_{[L^2(e)]^2}^2 \\ \leq C_4 \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\omega_e)]^{2\times 2}}^2 + |\mathbf{u} - \mathbf{u}_h|_{[H^1(\omega_e)]^2}^2 + \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(\omega_e)]^{2\times 2}}^2 \right\}, \end{aligned} \tag{9.11}$$

$$h_e \|J\Big[\Big(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\Big)\mathbf{t}_T\Big]\|_{[L^2(e)]^2}^2 \leq C_5 \left\{ \|p - p_h\|_{L^2(\omega_e)}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\omega_e)]^{2\times 2}}^2 \right\}, \tag{9.12}$$

and

$$h_e \|J\Big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}\Big)\mathbf{t}_T\Big]\|_{[L^2(e)]^2}^2 \leq C_6 \left\{ |\mathbf{u} - \mathbf{u}_h|_{[H^1(\omega_e)]^2}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(\omega_e)]^{2\times 2}}^2 \right\}. \tag{9.13}$$

PROOF. The estimate (9.11) follows from Lemma 9.2 with $\boldsymbol{\rho}_h := \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}} - \nabla\mathbf{u}_h + \boldsymbol{\gamma}_h$, introducing $\mathbf{0} = \frac{1}{2\mu}\boldsymbol{\sigma}^{\mathtt{d}} - \nabla\mathbf{u} + \boldsymbol{\gamma}$ (cf. (4.2) - (4.4)) in the resulting estimate and applying the triangle inequality and the continuity of $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^{\mathtt{d}}$. Analogously, estimate (9.12) (resp. (9.13)) is obtained from Lemma 9.2 defining $\boldsymbol{\rho}_h$ as $(p_h + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h))\mathbf{I}$ (resp. $\boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}$) and then introducing $\mathbf{0} = (p + \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}))\mathbf{I}$ (resp. $\mathbf{0} = \boldsymbol{\varepsilon}(\mathbf{u}_h)^{\mathtt{d}} - \frac{1}{2\mu}\boldsymbol{\sigma}_h^{\mathtt{d}}$) (cf. (4.2) (resp. (4.4))). $\square$

LEMMA 9.7. *There exist $C_7, C_8 \geq 0$, independent of h, such that for any $T \in \mathcal{T}_h$*

$$h_T^2\|\mathbf{div}\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\|_{[L^2(T)]^2}^2 \leq C_7\left\{|\mathbf{u} - \mathbf{u}_h|_{[H^1(T)]^2}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}}^2\right\} \tag{9.14}$$

and

$$h_T^2\|\mathbf{div}\Big(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\Big)\|_{[L^2(T)]^2}^2 \leq C_8\left\{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(T)]^{2\times 2}}^2 + |\mathbf{u} - \mathbf{u}_h|_{[H^1(T)]^2}^2\right\}. \tag{9.15}$$

PROOF. The estimate (9.14) follows from Lemma 9.3 defining $\boldsymbol{\rho}_h := \boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}$, introducing $\mathbf{0} = \boldsymbol{\varepsilon}(\mathbf{u}) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathtt{t}})^{\mathtt{d}}$ (cf. (4.4)), and then using the triangle inequality and the continuity of the operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^{\mathtt{d}}$. Similarly, applying Lemma 9.3 with $\boldsymbol{\rho}_h := \boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)$ and introducing $\mathbf{0} = \boldsymbol{\gamma} - \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)$ yields (9.15). $\square$

LEMMA 9.8. *There exist $C_9, C_{10} > 0$, independent of h, such that for any $e \in \mathcal{E}_h$*

$$h_e\|J\big[\Big(\boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}\Big)\boldsymbol{\nu}_T\big]\|_{[L^2(e)]^2}^2 \leq C_9\left\{|\mathbf{u} - \mathbf{u}_h|_{[H^1(T)]^2}^2 + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{[L^2(T)]^{2\times 2}}^2\right\} \tag{9.16}$$

and

$$h_e\|J\big[\Big(\boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)\Big)\boldsymbol{\nu}_T\big]\|_{[L^2(e)]^2}^2 \leq C_{10}\left\{\|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(T)]^{2\times 2}}^2 + |\mathbf{u} - \mathbf{u}_h|_{[H^1(T)]^2}^2\right\}. \tag{9.17}$$

PROOF. The estimate (9.16) follows from Lemma 9.4 with $\boldsymbol{\rho}_h := \boldsymbol{\varepsilon}(\mathbf{u}_h) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathtt{t}})^{\mathtt{d}}$, introducing $\mathbf{0} = \boldsymbol{\varepsilon}(\mathbf{u}) - \frac{1}{2\mu}\frac{1}{2}(\boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathtt{t}})^{\mathtt{d}}$ (cf. (4.4)) and then employing again the triangle inequality and the continuity of the operators $\boldsymbol{\varepsilon}$ and $\boldsymbol{\tau} \longrightarrow \boldsymbol{\tau}^{\mathtt{d}}$. Analogously, the estimate (9.17) follows from Lemma 9.4 defining $\boldsymbol{\rho}_h := \boldsymbol{\gamma}_h - \frac{1}{2}\left(\nabla\mathbf{u}_h - (\nabla\mathbf{u}_h)^{\mathtt{t}}\right)$ and then introducing $\mathbf{0} = \boldsymbol{\gamma} - \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)$. $\square$

Thus, the efficiency of $\boldsymbol{\theta}$ follows straightforwardly from the estimates (9.1) - (9.17) after summing over all $T \in \mathcal{T}_h$ and using that the number or triangles on each domain ω_e is bounded by two.

10. Numerical results for both uniformly and adaptively refined meshes

In this section we present several numerical results illustrating the performance of the augmented finite element scheme (6.1) and the a posteriori error estimator $\boldsymbol{\theta}$ analyzed in Section 7, using the specific finite element subspace $\tilde{\mathbf{H}}_h$ (cf. (6.10)). We recall that in this case the local indicator θ_T^2 reduces to (8.28). Now, in order to implement the zero integral mean condition for functions of the space $\tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ (cf. (6.5)), we introduce, as described in [**27**], a Lagrange multiplier $\varphi_h \in \mathbb{R}$. That is, instead of (6.1) with $\mathbf{H}_h = \tilde{\mathbf{H}}_h$, we consider the equivalent problem: Find $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \varphi_h) \in \tilde{H}_h^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ such that

$$\begin{aligned} A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) + \varphi_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) &= F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h), \\ \psi_h \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) &= 0, \end{aligned} \tag{10.1}$$

for all $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h, \psi_h) \in \tilde{H}_h^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}} \times \mathbb{R}$. We state the equivalence between (6.1) and (10.1) through the application of the following Theorem, adapted from Theorem 4.3 in [**27**].

THEOREM 10.1.

i. *Let* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \tilde{\mathbf{H}}_h$ *be the solution of* (6.1). *Then* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, 0)$ *is a solution of* (10.1).
ii. *Let* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \varphi_h) \in \tilde{H}_h^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ *be a solution of* (10.1). *Then* $\varphi_h = 0$ *and* $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ *is the solution of* (6.1).

PROOF. We first observe, according to the definition of A (*cf.* (5.5)), that for each $(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) \in H(\mathbf{div};\Omega) \times L^2(\Omega) \times [H_0^1(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ there holds

$$A((\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}), (\mathbf{I}, -1, \mathbf{0}, \mathbf{0})) = 0. \tag{10.2}$$

Now, let $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ be the solution of (6.1), and let $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \tilde{H}_h^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}}$. We write $\boldsymbol{\tau}_h = \boldsymbol{\tau}_{0,h} + d_h \mathbf{I}$, with $\boldsymbol{\tau}_{0,h} \in \tilde{H}_{0,h}^{\boldsymbol{\sigma}}$ and $d_h \in \mathbb{R}$ and observe that $(\boldsymbol{\tau}_{0,h}, q_h + d_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \tilde{\mathbf{H}}_h$, whence (5.6), (6.1) and (10.2) yield

$$\begin{aligned} F(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) = F(\boldsymbol{\tau}_{0,h}, q_h + d_h, \mathbf{v}_h, \boldsymbol{\eta}_h) &= A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_{0,h}, q_h + d_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) \\ &= A((\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h)). \end{aligned}$$

This identity and the fact that $\boldsymbol{\sigma}_h$ clearly satisfies the second equation of (10.1), show that $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, 0)$ is indeed a solution of (10.1).

Conversely, let $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \varphi_h) \in \tilde{H}_h^{\boldsymbol{\sigma}} \times \tilde{H}_h^p \times \tilde{H}_{0,h}^{\mathbf{u}} \times \tilde{H}_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ be a solution of (10.1). Then, taking $(\boldsymbol{\tau}_h, q_h, \mathbf{v}_h, \boldsymbol{\eta}_h) = (\mathbf{I}, -1, \mathbf{0}, \mathbf{0})$ in the first equation of (10.1) and using (5.6) and (10.2), we find that $\varphi_h = 0$, whence $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ becomes the solution of (6.1). □

In what follows, N stands for the total number of degrees of freedom (unknowns) of (10.1), which, at least for uniform refinements, behaves asymptotically as six times the numbers of elements of each triangulation. Also, the individual and total errors are denoted by

$$\begin{aligned} e(\boldsymbol{\sigma}) &:= \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{H(\mathbf{div};\Omega)}, \quad & e(p) &:= \|p - p_h\|_{L^2(\Omega)}, \\ e(\mathbf{u}) &:= |\mathbf{u} - \mathbf{u}_h|_{[H^1(\Omega)]^2}, \quad & e(\boldsymbol{\gamma}) &:= \|\boldsymbol{\gamma} - \boldsymbol{\gamma}_h\|_{[L^2(\Omega)]^{2\times 2}}, \end{aligned}$$

and

$$e := \left\{ [e(\boldsymbol{\sigma})]^2 + [e(p)]^2 + [e(\mathbf{u})]^2 + [e(\boldsymbol{\gamma})]^2 \right\}^{1/2},$$

respectively, whereas the effectivity index with respect to $\boldsymbol{\theta}$ is defined by $e/\boldsymbol{\theta}$.

Since the augmented method (for the compressible case) was shown in [**27**] to be robust with respect to the parameters κ_1, κ_2, and κ_3, we simply consider for all the examples $(\kappa_1, \kappa_2, \kappa_3) = \left(\mu, \frac{1}{2\mu}, \frac{\mu}{2}\right)$, which satisfy the assumptions of Theorem 6.4. In addition, since the choice of κ_0 in (6.1) depends on the unknown constant c_1 from Lemma 4.4, we simply take here $\kappa_0 = \mu$. As we will see below, this choice works out well in all the examples

We now specify the data of the three examples to be presented here. We take Ω as either the square $]0,1[^2$ or the triangle $\hat{T} := \{(x_1, x_2) : x_1, x_2 > 0 \text{ and } x_1 + x_2 < 1\}$, and choose the datum $\mathbf{f}$ so that the exact solution $\mathbf{u}(x_1,x_2) := (u_1(x_1,x_2), u_2(x_1,x_2))^{\mathtt{t}}$ and $p(x_1,x_2)$ are given in the table below. Actually, according to (4.1) we have $\boldsymbol{\sigma} = 2\mu\boldsymbol{\varepsilon}(\mathbf{u}) - p\mathbf{I}$, and hence simple computations show that $\mathbf{f} := -\mathbf{div}(\boldsymbol{\sigma}) = -\mu\Delta\mathbf{u} - \mu\nabla(\mathbf{div}\ \mathbf{u}) + \nabla p = -\mu\Delta\mathbf{u} + \nabla p$. We also recall that the rotation γ is defined by $\frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)$. In all the examples we take $\mu = 1.0$.

We emphasize that from (4.1) an admissible solution $\mathbf{u}$ must satisfy both $\mathbf{u} = \mathbf{0}$ on Γ and $\mathrm{div}(\mathbf{u}) = \mathbf{0}$ in Ω, and from (4.2) and the fact that $\boldsymbol{\sigma} \in H_0$ (cf. (4.6)) an admissible solution p must satisfy $\displaystyle\int_\Omega p = 0$.

EXAMPLE	Ω	$\mathbf{u}(x_1,x_2)$	$p(x_1,x_2)$
1	$]0,1[^2$	$\mathbf{curl}(x_1^2x_2^2(x_1-1)^2(x_2-1)^2)$	$x_1^2 + x_2^2 - \frac{2}{3}$
2	$\hat{T}$	$10^2\,\mathbf{curl}\left(x_1^2x_2^2(1-x_1-x_2)^2(x_1^2+x_2^2)^{-3/4}\right)$	$x_1^2 + x_2^2 - \frac{1}{3}$
3	$]0,1[^2$	$\mathbf{curl}\left(\dfrac{900\,x_1^2x_2^2(1-x_1)^2(1-x_2)^2}{(300\,x_1-100)^2+(300\,x_2-100)^2+90}\right)$	$\left(\frac{x_1}{100}\right)^2 + \left(\frac{x_2}{100}\right)^2 - \frac{2}{3}\times 10^{-4}$

We observe that the solution of Example 2 is singular at the boundary point $(0,0)$. Thus, according to Theorem 6.5 we expect a rate of convergence lower than 1 for the uniform refinement. On the other hand, the solution of Example 3 shows a large stress region in the vicinity of the interior point $(1/3, 1/3)$.

The numerical results shown below were obtained in a *Pentium Xeon computer with dual processors* using a Fortran Code and the TRIANGLE mesh generator [**37**]. The linear system arising from (10.1) is solved with the sequential LU package [**18**]. Individual errors are computed on each triangle using a Gaussian quadrature rule.

We first utilize the Example 1 to illustrate the good behaviour of the a posteriori error estimator $\boldsymbol{\theta}$ in a sequence of quasi-uniform meshes. In Table 1 we present the individual and total errors, the a posteriori estimators, and the effectivity indexes for this example with this sequence of quasi-uniform meshes. The index always remains in a neighborhood of 0.600 in this example, which confirms the reliability and efficiency of $\boldsymbol{\theta}$. For this example surface plots of the solution are shown in Figures 3.5, 3.6, and 3.7.

Next we consider Examples 2 and 3 to illustrate the performance of the following adaptive algorithm based on $\boldsymbol{\theta}$ for the computation of solutions of (10.1):

1. Start with a coarse mesh $\mathcal{T}_h$.
2. Solve the Galerkin scheme (10.1) for the current mesh $\mathcal{T}_h$.
3. Compute θ_T for each triangle $T \in \mathcal{T}_h$.
4. Consider stopping criterion and decide to finish or go to next step.
5. Instruct the mesh generator to ensure that in the next mesh the region enclosed by each element $T' \in \mathcal{T}_h$ of the current mesh whose local indicator $\theta_{T'}$ satisfies $\theta_{T'} \geq \frac{1}{2}\max\{\theta_T : T \in \mathcal{T}_h\}$ encompasses no triangle with area larger than $\frac{|T'|}{4}$.
6. Generate the next mesh, store it as $\mathcal{T}_h$ and go to step 2.

At this point we introduce the experimental rate of convergence, which, given two consecutive triangulations with degrees of freedom N and N' and corresponding errors e and e', is defined by

$$r(e) := -2\frac{\log(e/e')}{\log(N/N')}.$$

In Tables 2 through 5 we provide the individual and total errors, the experimental rates of convergence, the a posteriori error estimators and the effectivity indexes for the uniform and adaptive refinements as applied to Examples 2 and 3. In this case the quasi-uniform sequences

TABLE 1. Mesh sizes, individual and total errors, a posteriori error estimators, and effectivity indexes for a sequence of quasi-uniform meshes (Example 1).

N	h	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	e	$\boldsymbol{\theta}$	$e/\boldsymbol{\theta}$
99	0.500	0.681E-00	0.151E-00	0.130E-00	0.587E-01	0.712E-00	0.923E-00	0.772
165	0.500	0.557E-00	0.126E-00	0.844E-01	0.453E-01	0.579E-00	0.794E-00	0.729
207	0.500	0.528E-00	0.115E-00	0.818E-01	0.428E-01	0.548E-00	0.738E-00	0.743
363	0.288	0.374E-00	0.791E-01	0.609E-01	0.367E-01	0.389E-00	0.589E-00	0.660
435	0.271	0.345E-00	0.756E-01	0.584E-01	0.315E-01	0.359E-00	0.523E-00	0.687
627	0.257	0.282E-00	0.601E-01	0.463E-01	0.271E-01	0.293E-00	0.452E-00	0.648
849	0.250	0.253E-00	0.555E-01	0.420E-01	0.273E-01	0.264E-00	0.412E-00	0.639
1245	0.250	0.204E-00	0.485E-01	0.358E-01	0.220E-01	0.214E-00	0.335E-00	0.638
1707	0.147	0.181E-00	0.388E-01	0.305E-01	0.198E-01	0.188E-00	0.303E-00	0.622
2433	0.125	0.148E-00	0.323E-01	0.254E-01	0.154E-01	0.155E-00	0.243E-00	0.635
3369	0.125	0.128E-00	0.287E-01	0.218E-01	0.135E-01	0.133E-00	0.211E-00	0.632
4833	0.125	0.103E-00	0.229E-01	0.185E-01	0.120E-01	0.108E-00	0.180E-00	0.603
6927	0.077	0.880E-01	0.188E-01	0.154E-01	0.961E-02	0.918E-01	0.149E-00	0.615
9681	0.065	0.743E-01	0.159E-01	0.131E-01	0.851E-02	0.776E-01	0.129E-00	0.601
13563	0.062	0.632E-01	0.137E-01	0.112E-01	0.736E-02	0.661E-01	0.111E-00	0.595

TABLE 2. Individual and total errors, experimental rates of convergence, a posteriori error estimators, and effectivity indexes for a sequence of quasi-uniform meshes (Example 2).

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	e	$r(e)$	$\boldsymbol{\theta}$	$e/\boldsymbol{\theta}$
159	0.159E+03	0.626E+01	0.103E+02	0.527E+01	0.160E+03	——	0.166E+03	0.965
633	0.122E+03	0.363E+01	0.677E+01	0.366E+01	0.123E+03	0.383	0.127E+03	0.965
2367	0.941E+02	0.202E+01	0.365E+01	0.198E+01	0.942E+02	0.403	0.961E+02	0.979
9591	0.725E+02	0.107E+01	0.188E+01	0.106E+01	0.725E+02	0.373	0.733E+02	0.989

of meshes are generated by instructing the mesh generator to provide only triangles with area below a decreasing threshold, subject to a minimum angle constraint. We observe from these tables that the errors of the adaptive procedure decrease much faster than those obtained by the quasi-uniform one, which is confirmed by the experimental rates of convergence provided there. This fact can also be seen in Figures 1 and 2 where we display the total error e vs. the degrees of freedom N for both refinements. As shown by the values of $r(e)$, particularly in Example 2 (where $r(e)$ approaches 0.38 for the quasi-uniform refinement), the adaptive method is able to recover, at least approximately, the quasi-optimal rate of convergence $\mathcal{O}(h)$ for the total error. Furthermore, the effectivity indexes remain again bounded from above and below, which confirms the reliability and efficiency of $\boldsymbol{\theta}$ for the adaptive algorithm. On the other hand, some intermediate meshes obtained with the adaptive refinement are displayed in Figures 3 and 4. Note that the method is able to recognize the singularities and large stress regions of the solutions. In particular, this fact is observed in Example 2 (see Figure 3) where adapted meshes are highly refined around the singular point $(0,0)$. Similarly, the adapted meshes obtained in Example 3 (see Figure 4) concentrate the refinement around the interior point $(1/3,1/3)$, where the largest stress occur.

Summarizing, the numerical results presented in this section exhibit, on one hand, the expected $\mathcal{O}(h)$ behaviour of this augmented method for smooth problems and, on the other hand, underline the reliability and efficiency of $\boldsymbol{\theta}$. In addition, they strongly demonstrate that the associated adaptive algorithm is much more suitable than a uniform discretization procedure when solving problems with non-smooth solutions.

Table 3. Individual and total errors, experimental rates of convergence, a posteriori error estimators, and effectivity indexes for the adaptive refinement (Example 2).

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	e	$r(e)$	$\boldsymbol{\theta}$	$e/\boldsymbol{\theta}$
159	0.159E+03	0.626E+01	0.103E+02	0.527E+01	0.160E+03	——	0.166E+03	0.965
249	0.119E+03	0.569E+01	0.892E+01	0.489E+01	0.119E+03	1.301	0.128E+03	0.929
345	0.109E+03	0.542E+01	0.837E+01	0.454E+01	0.110E+03	0.508	0.118E+03	0.926
417	0.993E+02	0.541E+01	0.836E+01	0.455E+01	0.999E+02	1.021	0.109E+03	0.912
531	0.910E+02	0.542E+01	0.825E+01	0.451E+01	0.916E+02	0.716	0.101E+03	0.899
627	0.841E+02	0.485E+01	0.809E+01	0.422E+01	0.847E+02	0.944	0.943E+02	0.898
981	0.711E+02	0.379E+01	0.622E+01	0.314E+01	0.715E+02	0.758	0.781E+02	0.915
1545	0.578E+02	0.313E+01	0.534E+01	0.273E+01	0.582E+02	0.906	0.642E+02	0.906
1899	0.560E+02	0.267E+01	0.441E+01	0.233E+01	0.563E+02	0.324	0.610E+02	0.922
2571	0.499E+02	0.255E+01	0.410E+01	0.210E+01	0.501E+02	0.759	0.544E+02	0.922
3651	0.413E+02	0.224E+01	0.353E+01	0.187E+01	0.416E+02	1.068	0.456E+02	0.912
5187	0.355E+02	0.202E+01	0.325E+01	0.162E+01	0.357E+02	0.867	0.390E+02	0.915
6957	0.310E+02	0.184E+01	0.297E+01	0.149E+01	0.312E+02	0.910	0.344E+02	0.906
9843	0.253E+02	0.133E+01	0.216E+01	0.115E+01	0.254E+02	1.179	0.280E+02	0.909
13707	0.214E+02	0.114E+01	0.194E+01	0.102E+01	0.215E+02	1.014	0.238E+02	0.904

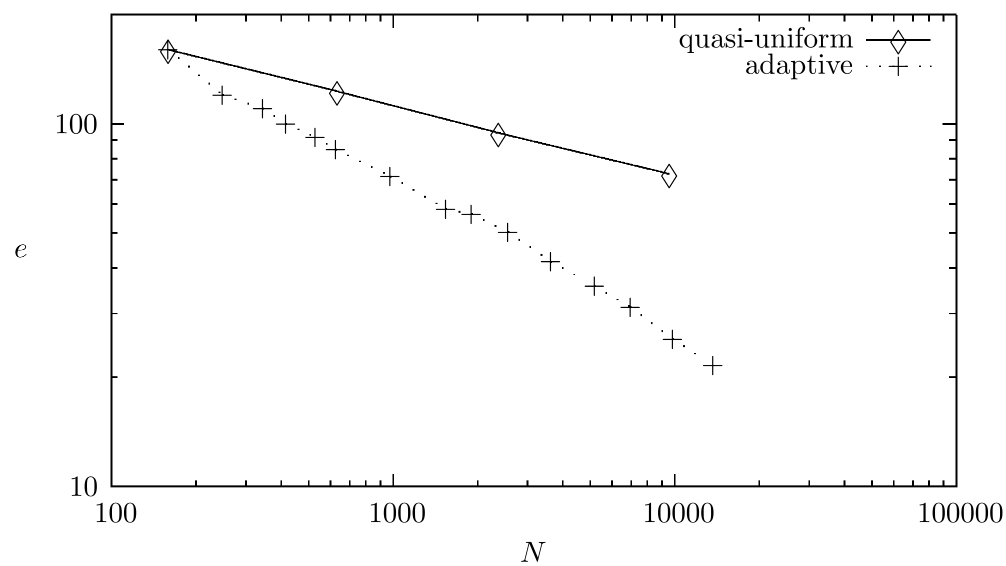

Figure 3.1. Total errors e vs. degrees of freedom N for the quasi-uniform and adaptive refinements (Example 2).

Table 4. Individual and total errors, experimental rates of convergence, a posteriori error estimators, and effectivity indexes for a sequence of quasi-uniform meshes (Example 3).

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	e	$r(e)$	$\boldsymbol{\theta}$	$e/\boldsymbol{\theta}$
435	0.228E+03	0.691E+00	0.210E+01	0.106E+01	0.228E+03	——	0.228E+03	1.000
1245	0.195E+03	0.738E+01	0.562E+01	0.248E+01	0.195E+03	0.294	0.196E+03	0.999
3369	0.204E+03	0.613E+01	0.354E+01	0.177E+01	0.204E+03	——	0.204E+03	1.000
9681	0.133E+03	0.126E+01	0.183E+01	0.102E+01	0.133E+03	0.815	0.133E+03	0.998

TABLE 5. Individual and total errors, experimental rates of convergence, a posteriori error estimators, and effectivity indexes for the adaptive refinement (Example 3).

N	$e(\boldsymbol{\sigma})$	$e(p)$	$e(\mathbf{u})$	$e(\boldsymbol{\gamma})$	e	$r(e)$	$\boldsymbol{\theta}$	$e/\boldsymbol{\theta}$
435	0.228E+03	0.691E+00	0.210E+01	0.106E+01	0.228E+03	—	0.228E+03	1.000
555	0.221E+03	0.681E+01	0.542E+01	0.299E+01	0.221E+03	0.270	0.222E+03	0.995
651	0.178E+03	0.369E+01	0.300E+01	0.167E+01	0.178E+03	2.702	0.178E+03	0.998
819	0.119E+03	0.119E+01	0.155E+01	0.778E+00	0.119E+03	3.473	0.119E+03	0.997
1083	0.752E+02	0.919E+00	0.115E+01	0.591E+00	0.752E+02	3.320	0.756E+02	0.995
1431	0.500E+02	0.733E+00	0.914E+00	0.544E+00	0.500E+02	2.933	0.504E+02	0.992
2139	0.366E+02	0.453E+00	0.584E+00	0.391E+00	0.366E+02	1.552	0.368E+02	0.992
2775	0.303E+02	0.417E+00	0.488E+00	0.302E+00	0.303E+02	1.453	0.305E+02	0.993
3471	0.261E+02	0.369E+00	0.430E+00	0.262E+00	0.261E+02	1.333	0.262E+02	0.993
4707	0.211E+02	0.314E+00	0.367E+00	0.226E+00	0.211E+02	1.383	0.213E+02	0.992
6399	0.177E+02	0.284E+00	0.323E+00	0.197E+00	0.177E+02	1.123	0.179E+02	0.991
8667	0.151E+02	0.252E+00	0.283E+00	0.172E+00	0.151E+02	1.050	0.153E+02	0.991
12147	0.122E+02	0.220E+00	0.241E+00	0.146E+00	0.122E+02	1.271	0.123E+02	0.990

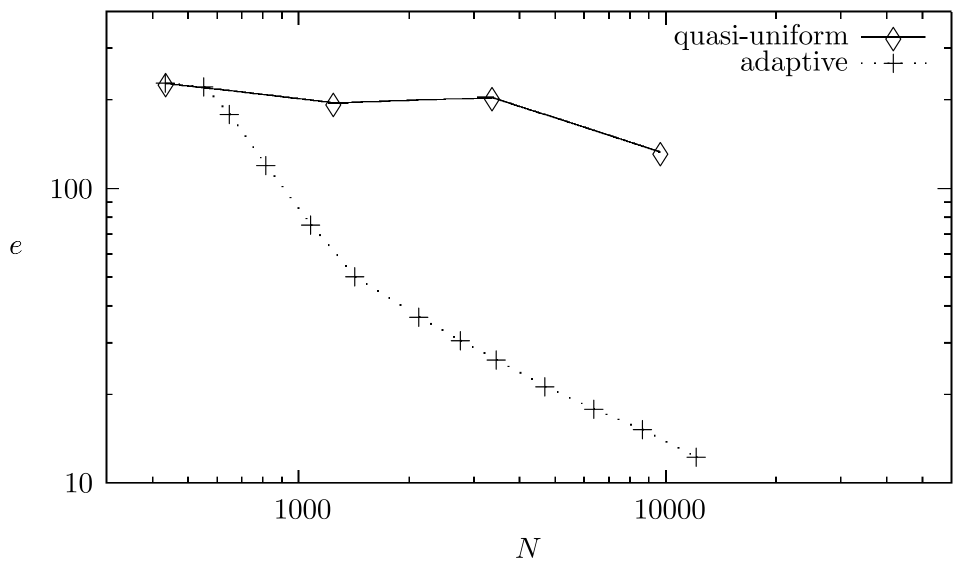

FIGURE 3.2. Total errors e vs. degrees of freedom N for the quasi-uniform and adaptive refinements (Example 3).

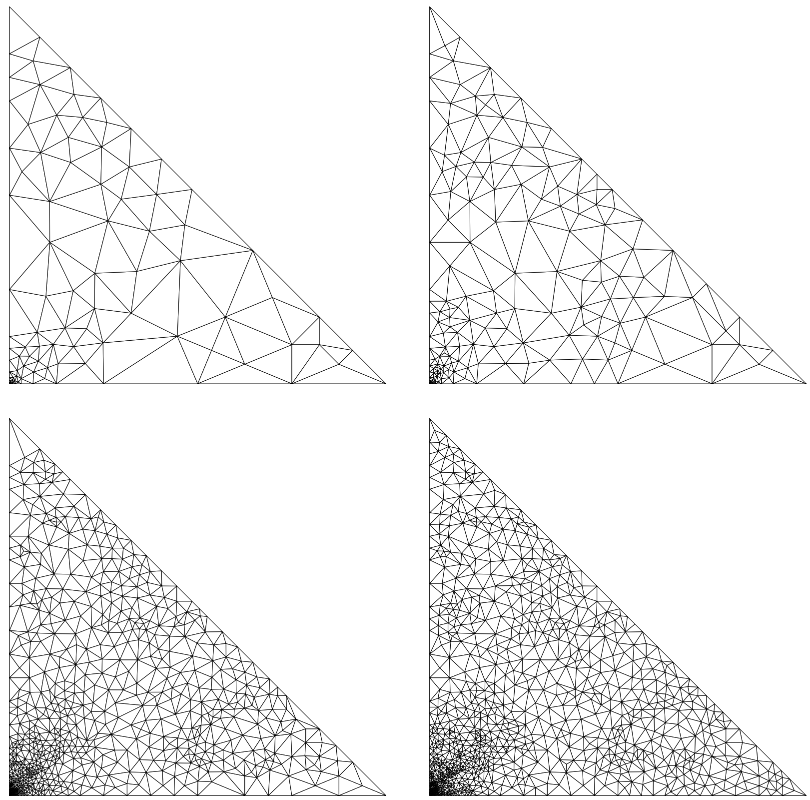

FIGURE 3.3. Adapted intermediate meshes with 981, 1899, 9843, and 13707 degrees of freedom (Example 2).

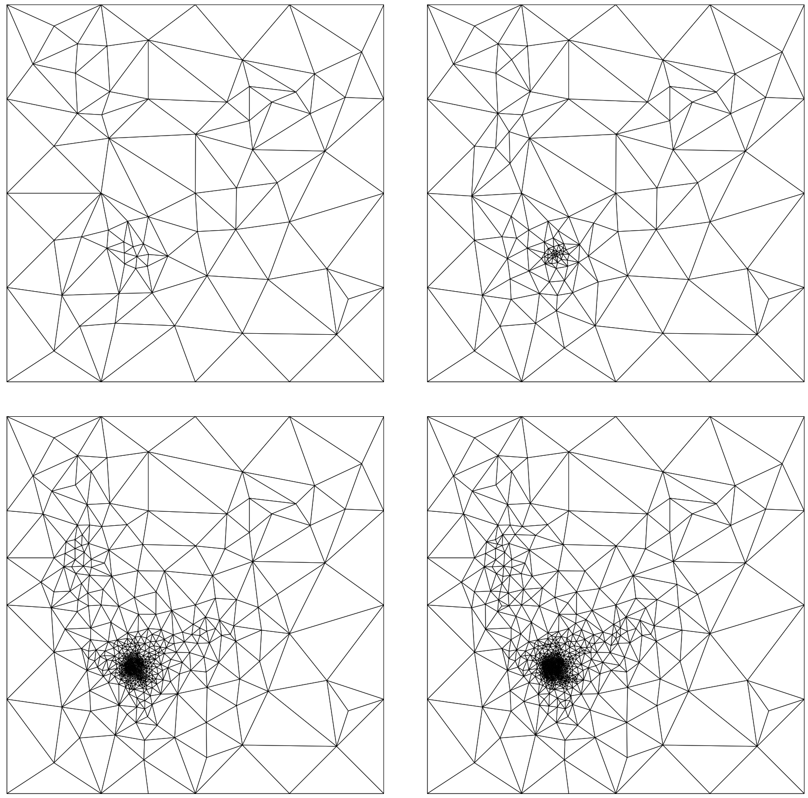

FIGURE 3.4. Adapted intermediate meshes with 819, 1431, 8667, and 12147 degrees of freedom (Example 3).

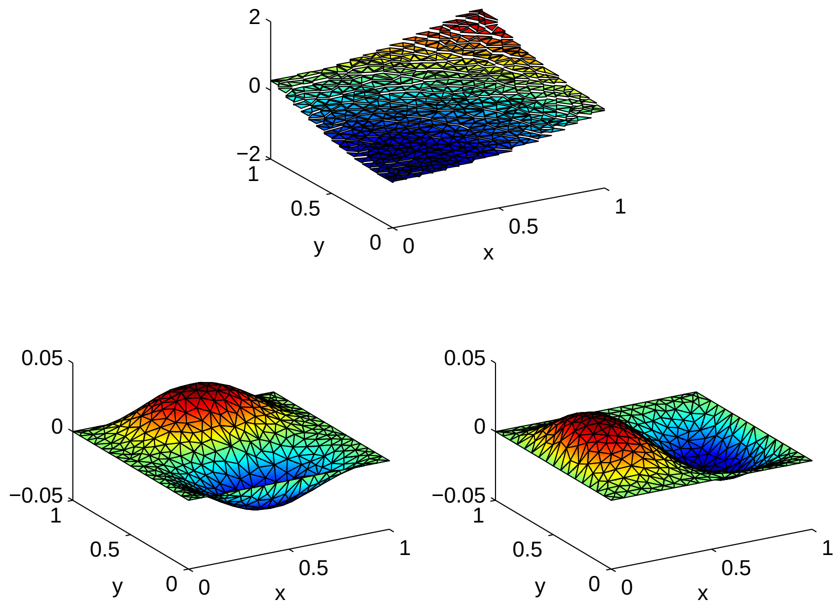

FIGURE 3.5. Pressure field and the two components of displacement for the Example 1 with 4833 degrees of freedom

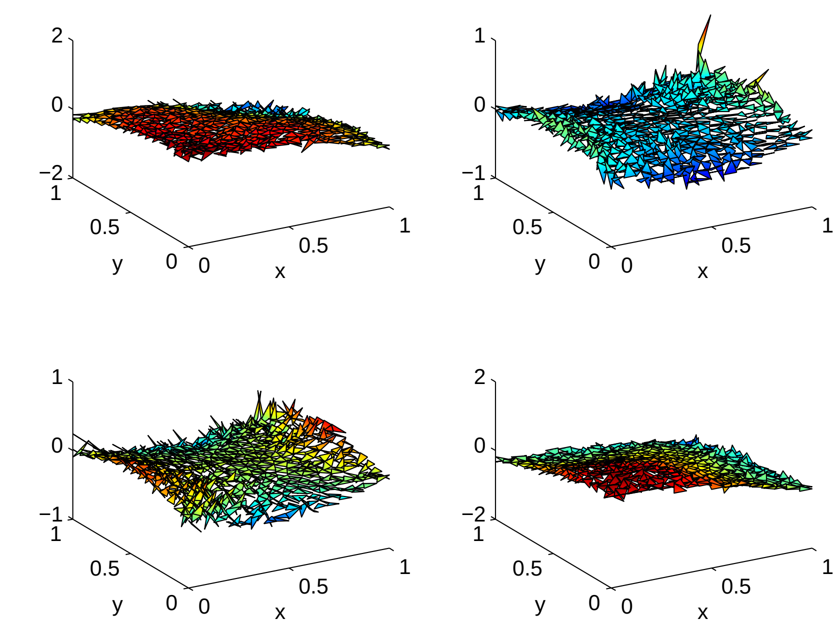

FIGURE 3.6. The four components of the stress tensor for the Example 1 with 4833 degrees of freedom

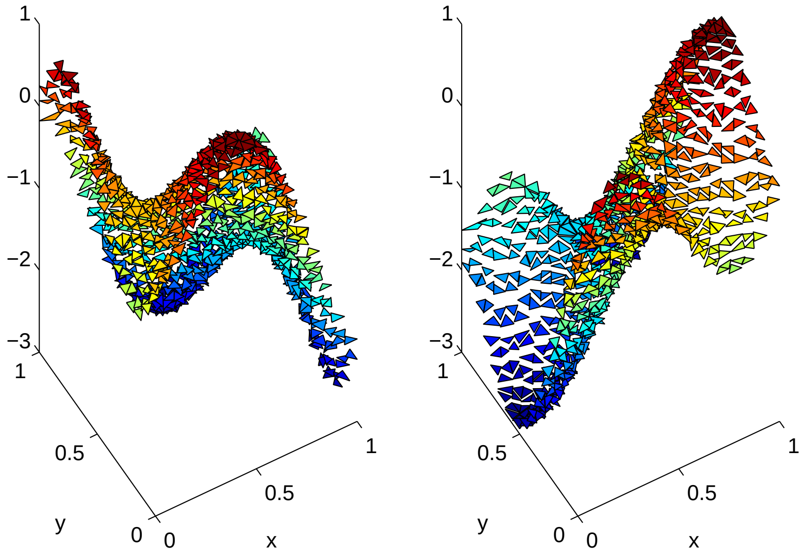

FIGURE 3.7. The divergence of the stress tensor for Example 1 with 4833 degrees of freedom

CHAPTER 4

GMRES iteration for augmented linear elasticity

11. The augmented mixed finite element method for linear elasticity

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with Lipschitz-continuous boundary Γ. Our goal is to determine the displacement $\mathbf{u}$ and stress tensor $\boldsymbol{\sigma}$ of a linear elastic material occupying the region Ω. In other words, given a volume force $\mathbf{f} \in [L^2(\Omega)]^2$, we seek a symmetric tensor field $\boldsymbol{\sigma}$ and a vector field $\mathbf{u}$ such that

$$\boldsymbol{\sigma} = \mathcal{C}\boldsymbol{\varepsilon}(\mathbf{u}), \quad \mathbf{div}(\boldsymbol{\sigma}) = -\mathbf{f} \text{ in } \Omega, \text{ and } \mathbf{u} = 0 \text{ on } \Gamma, \tag{11.1}$$

where $\boldsymbol{\varepsilon}(\mathbf{u}) := \frac{1}{2}\left(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathtt{t}}\right)$ is the linearized strain tensor and $\mathcal{C}$ is the elasticity tensor determined by Hooke's law, that is

$$\mathcal{C}\boldsymbol{\zeta} := \lambda \operatorname{tr}(\boldsymbol{\zeta})\mathbf{I} + 2\mu\boldsymbol{\zeta} \quad \forall\, \boldsymbol{\zeta} \in [L^2(\Omega)]^{2\times 2}, \tag{11.2}$$

where $\lambda, \mu > 0$ denote the corresponding Lamé constants. It is easy to see from 11.2 that the inverse tensor $\mathcal{C}^{-1}$ reduces to

$$\mathcal{C}^{-1}\boldsymbol{\zeta} := \frac{1}{2\mu}\boldsymbol{\zeta} - \frac{\lambda}{4\mu(\lambda+\mu)}\operatorname{tr}(\boldsymbol{\zeta})\mathbf{I} \quad \forall\, \boldsymbol{\zeta} \in [L^2(\Omega)]^{2\times 2}. \tag{11.3}$$

We now define the spaces $H := H(\mathbf{div};\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}$, $H_0 := \{\boldsymbol{\tau} \in H : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\}$, and note that $H = H_0 \oplus \mathbb{R}\mathbf{I}$. In addition, we define the space of skew-symmetric tensors $[L^2(\Omega)]^{2\times 2}_{\mathtt{asym}} := \{\boldsymbol{\eta} \in [L^2(\Omega)]^{2\times 2} : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\}$ and introduce the rotation $\boldsymbol{\gamma} := \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right) \in [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ as an auxiliary unknown. Then, given positive parameters κ_1, κ_2 and κ_3 we consider from [**27**] the following augmented variational formulation for (11.1): Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0 := H_0 \times [H^1_0(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}}$ such that

$$A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) = F(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \quad \forall\, (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0, \tag{11.4}$$

where the bilinear form $A : \mathbf{H}_0 \times \mathbf{H}_0 \longrightarrow \mathbb{R}$ and the functional $F : \mathbf{H}_0 \longrightarrow \mathbb{R}$ are defined by

$$\begin{aligned} A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) &:= \int_\Omega \mathcal{C}^{-1}\boldsymbol{\sigma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} - \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \\ &- \int_\Omega \boldsymbol{\eta} : \boldsymbol{\sigma} + \kappa_1 \int_\Omega \left(\boldsymbol{\varepsilon}(\mathbf{u}) - \mathcal{C}^{-1}\boldsymbol{\sigma}\right) : \left(\boldsymbol{\varepsilon}(\mathbf{v}) + \mathcal{C}^{-1}\boldsymbol{\tau}\right) + \kappa_2 \int_\Omega \mathbf{div}(\boldsymbol{\sigma})\cdot\mathbf{div}(\boldsymbol{\tau}) \\ &+ \kappa_3 \int_\Omega \left(\boldsymbol{\gamma} - \frac{1}{2}\left(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathtt{t}}\right)\right) : \left(\boldsymbol{\eta} + \frac{1}{2}\left(\nabla\mathbf{v} - (\nabla\mathbf{v})^{\mathtt{t}}\right)\right), \end{aligned} \tag{11.5}$$

and

$$F(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) := \int_\Omega \mathbf{f}\cdot\left(\mathbf{v} - \kappa_2\mathbf{div}(\boldsymbol{\tau})\right). \tag{11.6}$$

The well posedness of (11.4) was proved in [**27**]. More precisely, we have the following result.

Theorem 11.1. *Assume that $(\kappa_1, \kappa_2, \kappa_3)$ is independent of λ and such that $0 < \kappa_1 < 2\mu$, $0 < \kappa_2$, and $0 < \kappa_3 < \kappa_1$. Then, there exist positive constants M, α, independent of λ, such that*

$$|A((\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}))| \leq M \, \|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})\|_{\mathbf{H}_0} \, \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|_{\mathbf{H}_0} \tag{11.7}$$

and

$$A((\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}), (\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})) \geq \alpha \, \|(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta})\|^2_{\mathbf{H}_0} \tag{11.8}$$

for all $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma})$, $(\boldsymbol{\tau}, q, \mathbf{v}, \boldsymbol{\eta}) \in \mathbf{H}_0$. In particular, taking

$$\kappa_1 = \tilde{C}_1 \mu, \qquad \kappa_2 = \frac{1}{\mu}\left(1 - \frac{\kappa_1}{2\mu}\right), \qquad \text{and } \kappa_3 = \tilde{C}_3 \kappa_1, \tag{11.9}$$

with any $\tilde{C}_1 \in]0,2[$ and any $\tilde{C} \in]0,1[$, yields M and α depending only on μ, $\frac{1}{\mu}$, and Ω. Therefore, the augmented variational formulation (11.4) has a unique solution $(\boldsymbol{\sigma}, p, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$, and there exists a positive constant C, independent of λ, such that

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma})\|_{\mathbf{H}_0} \leq C \, \|F\|_{\mathbf{H}_0'} \leq \|\mathbf{f}\|_{[L^2(\Omega)]^2} \, . \tag{11.10}$$

Proof. See T'heorems 3.1 and 3.2 in [**27**]. □

In what follows we consider specific finite element subspaces and define the associated Galerkin scheme. For simplicity, from now on we assume that the boundary Γ of Ω is a polygonal curve.

Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangulations of the polygonal region $\bar{\Omega}$ by triangles T of diameter h_T such that $\bar{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $\ell \geq 0$ and a subset S of $\mathbb{R}^2$, we denote by $\mathbb{P}_\ell(S)$ the space of polynomials of total degree at most ℓ defined on S and for each $T \in \mathcal{T}_h$ we define the local Raviart-Thomas space of order zero

$$\mathbb{RT}_0(T) := \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\} \subseteq [\mathbb{P}_1(T)]^2,$$

where $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in T$. Their respective global counterparts are defined by

$$\mathbb{P}_\ell(\mathcal{T}_h) := \left\{ p \in L^2(\Omega) : p|_T \in \mathbb{P}_\ell(T) \quad \forall T \in \mathcal{T}_h \right\}$$

and

$$\mathbb{RT}_0(\mathcal{T}_h) := \left\{ \tau \in [L^2(\Omega)]^2 : \tau|_T \in \mathbb{RT}_0(T) \quad \forall T \in \mathcal{T}_h \right\}$$

We define the normal-component-continuous piecewise Raviart Thomas functions

$$H_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h \in H(\mathbf{div}; \Omega) : \boldsymbol{\tau}_h|_T \in [\mathbb{RT}_0(T)^{\mathtt{t}}]^2 \quad \forall T \in \mathcal{T}_h \right\} = H(\mathbf{div}; \Omega) \cap [\mathbb{RT}_0(\mathcal{T}_h)^{\mathtt{t}}]^2,$$

and continuous piecewise affine functions

$$H_h^{\mathbf{u}} := \left\{ \mathbf{v}_h \in [C(\bar{\Omega})]^2 : \mathbf{v}_h|_T \in [\mathbb{P}_1(T)]^2 \quad \forall T \in \mathcal{T}_h \right\} = [C(\bar{\Omega})]^2 \cap [\mathbb{P}_1(\mathcal{T}_h)]^2$$

and take

$$\mathbf{H}_{0,h} := H_{0,h}^{\boldsymbol{\sigma}} \times H_{0,h}^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}}, \tag{11.11}$$

where

$$\begin{aligned} H_{0,h}^{\boldsymbol{\sigma}} &:= \left\{ \boldsymbol{\tau}_h \in H_h^{\boldsymbol{\sigma}} : \int_\Omega \text{tr}(\boldsymbol{\tau}_h) = 0 \right\}, \\ H_{0,h}^{\mathbf{u}} &:= \left\{ \mathbf{v}_h \in H_h^{\mathbf{u}} : \mathbf{v}_h = 0 \text{ on } \partial\Omega \right\}, \\ H_h^{\boldsymbol{\gamma}} &:= \left\{ \boldsymbol{\eta}_h \in [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}} : \boldsymbol{\eta}_h|_T \in [\mathbb{P}_0(T)]^{2\times 2} \quad \forall T \in \mathcal{T}_h \right\}. \end{aligned}$$

In this way, the Galerkin scheme associated with the continuous problem (11.4) reads as follows: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h} \subseteq \mathbf{H}_0$ such that

$$A((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) = F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \quad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h}, \tag{11.12}$$

where the parameters κ_1, κ_2 and κ_3 in A satisfy the assumptions of Theorem 11.1.

The unique solvability of (11.12), the corresponding error estimate, and an error decay rate are stated in the two following theorems, taken from [**27**].

THEOREM 11.2. *Assume that the parameters κ_1, κ_2 and κ_3 satisfy the assumptions of Theorem 11.1. Then, the Galerkin scheme* (11.12) *has a unique solution $(\boldsymbol{\sigma}_h, p_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h}$, and there exist positive constants C, $\tilde{C}$, independent of h, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \leq C \sup_{\substack{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h} \\ (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \neq \mathbf{0}}} \frac{|F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)|}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0}} \leq C\, \|\mathbf{f}\|_{[L^2(\Omega)]^2}\,,$$

and

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\| \leq \tilde{C} \inf_{(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h) \in \mathbf{H}_{0,h}} \|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)\|_{\mathbf{H}_0}, \tag{11.13}$$

PROOF. This follows from Theorem 11.1, the Lax-Milgram Lemma, and Cea's estimate. □

THEOREM 11.3. *Let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) \in \mathbf{H}_0$ and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h} := H^{\boldsymbol{\sigma}}_{0,h} \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_h$ be the unique solutions of the continuous and discrete augmented formulations* (11.4) *and* (11.12), *respectively. Assume that $\boldsymbol{\sigma} \in [H^r(\Omega)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^r(\Omega)]^2$, $\mathbf{u} \in [H^{r+1}(\Omega)]^2$, and $\boldsymbol{\gamma} \in [H^r(\Omega)]^{2\times 2}$, for some $r \in (0,1]$. Then there exists $C > 0$, independent of h, such that*

$$\|(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)\|_{\mathbf{H}_0} \leq$$
$$C h^r \left\{ \|\boldsymbol{\sigma}\|_{[H^r(\Omega)]^{2\times 2}} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{[H^r(\Omega)]^2} + \|\mathbf{u}\|_{[H^{r+1}(\Omega)]^2} + \|\boldsymbol{\gamma}\|_{[H^r(\Omega)]^{2\times 2}} \right\}.$$

PROOF. It is a consequence of Cea's estimate, approximation properties for the component discrete spaces and suitable interpolation theorems in the corresponding function spaces. □

The null mean value condition required by the traces of the elements in $H^{\boldsymbol{\sigma}}_{0,h}$ is not very convenient for the numerical implementation of (11.12). The usual way to obtain a basis for $H^{\boldsymbol{\sigma}}_{0,h}$ is to start with one from the given subspace $H^{\boldsymbol{\sigma}}_h$, and then take the $H^{\boldsymbol{\sigma}}_{0,h}$-components of the latter according to the decomposition $H^{\boldsymbol{\sigma}}_h = H^{\boldsymbol{\sigma}}_{0,h} \oplus \mathbb{R}\mathbf{I}$. However, it is easy to see that this procedure yields basis functions of $H^{\boldsymbol{\sigma}}_{0,h}$ with support Ω, and hence the corresponding block in the global stiffness matrix will likely become full.

In order to overcome the above difficulty we consider, instead of (11.12), the modified discrete scheme: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \varphi_h) \in H^{\boldsymbol{\sigma}}_h \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_h \times \mathbb{R}$ such that

$$\begin{aligned} A((\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)) \quad + \varphi_h \int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) \;&=\; F(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h), \\ + \psi_h \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) \;&=\; 0, \end{aligned} \tag{11.14}$$

for all $(\boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h, \psi_h) \in H^{\boldsymbol{\sigma}}_h \times H^{\mathbf{u}}_{0,h} \times H^{\boldsymbol{\gamma}}_h \times \mathbb{R}$. In this way, the Lagrange multiplier $\varphi_h \in \mathbb{R}$ and the corresponding test constants $\psi_h \in \mathbb{R}$ take care of the above mentioned mean value condition, whence (11.12) and (11.14) become equivalent, as it is established in the following theorem:

THEOREM 11.4.

1. *Let* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h) \in \mathbf{H}_{0,h}$ *be the solution of* (11.12). *Then* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, 0)$ *is a solution of* (11.14).
2. *Let* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h, \varphi_h) \in H_h^{\boldsymbol{\sigma}} \times H_{0,h}^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}} \times \mathbb{R}$ *be a solution of* (11.14). *Then* $\varphi_h = 0$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ *is the solution of* (11.12).

PROOF. See Theorem 2.4 in [**27**]. □

12. Spectral properties of the stiffness matrix

Let $\{\ell_i\}_{i=1}^{\bar{N}}$, $\{\mathbf{x}_i\}_{i=1}^{\bar{n}}$, and $\{T_i\}_{i=1}^{\bar{m}}$ be the edges, interior nodes and triangles of the triangulation $\mathcal{T}_h$, respectively. Further, let $\{\tau_i\}_{i=1}^{\bar{N}}$, $\{v_i\}_{i=1}^{\bar{n}}$, and $\{\eta_i\}_{i=1}^{\bar{m}}$ be the canonical basis of $H(\mathrm{div};\Omega) \cap \mathbb{RT}_0(\mathcal{T}_h)$, $C_0(\bar{\Omega}) \cap \mathbb{P}_1(\mathcal{T}_h)$, and $\mathbb{P}_0(\mathcal{T}_h)$, respectively. That is, $\tau_i \cdot \nu_j|_{\ell_j} \in \{-\delta_{i,j}, \delta_{i,j}\}$, $v_i(\mathbf{x}_j) = \delta_{i,j}$, and $\eta_i|_{T_j} = \delta_{i,j}$. We define, for $1 \le i \le \bar{N}$, $1 \le j \le \bar{n}$, $1 \le k \le \bar{m}$

$$
\begin{aligned}
\boldsymbol{\tau}_i := \begin{pmatrix} \tau_{i1} & \tau_{i2} \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{\tau}_{i+\bar{N}} := \begin{pmatrix} 0 & 0 \\ \tau_{i1} & \tau_{i2} \end{pmatrix}, \\
\mathbf{v}_j := \begin{pmatrix} v_j \\ 0 \end{pmatrix}, \quad \mathbf{v}_{j+\bar{n}} := \begin{pmatrix} 0 \\ v_j \end{pmatrix}, \quad \boldsymbol{\eta}_k := \begin{pmatrix} 0 & \eta_k \\ -\eta_k & 0 \end{pmatrix}.
\end{aligned}
\tag{12.1}
$$

It follows that the corresponding canonical bases for the spaces $H_h^{\boldsymbol{\sigma}}$, $H_{0,h}^{\mathbf{u}}$, and $H_h^{\boldsymbol{\gamma}}$ are given, respectively, by the sets $\{\boldsymbol{\tau}_i\}_{i=1}^{2\bar{N}}$, $\{\mathbf{v}_i\}_{i=1}^{2\bar{n}}$, $\{\boldsymbol{\eta}_i\}_{i=1}^{\bar{k}}$.

The linear system associated to (11.14) and (12.1) takes the form

$$
\tilde{\mathcal{A}} \begin{pmatrix} \vec{\boldsymbol{\sigma}} \\ \vec{\mathbf{u}} \\ \vec{\gamma} \\ \varphi_h \end{pmatrix} := \begin{pmatrix} \mathbf{A} & -\mathbf{D}^{\mathrm{t}} & -\mathbf{P}^{\mathrm{t}} & \mathbf{V} \\ \mathbf{D} & \mathbf{J} & -\mathbf{S}^{\mathrm{t}} & \mathbf{0} \\ \mathbf{P} & \mathbf{S} & \mathbf{U} & \mathbf{0} \\ \mathbf{V}^{\mathrm{t}} & \mathbf{0} & \mathbf{0} & 0 \end{pmatrix} \begin{pmatrix} \vec{\boldsymbol{\sigma}} \\ \vec{\mathbf{u}} \\ \vec{\gamma} \\ \varphi_h \end{pmatrix} = \begin{pmatrix} \vec{F}_\sigma \\ \vec{F}_u \\ \vec{0} \\ 0 \end{pmatrix}. \tag{12.2}
$$

The blocks $\mathbf{A}, \mathbf{D}$, $\mathbf{P}$, $\mathbf{J}$, $\mathbf{S}$, and $\mathbf{U}$ correspond to evaluations of the bilinear form A of (11.5) and the vector $\mathbf{V} \in \mathbb{R}^{2\bar{N}}$ arises from the terms in (11.14) involving the Lagrange multiplier φ_h.

Since we are interested in using the GMRES method of Saad and Schultz [**36**], we need to make sure that the symmetric part of the stiffness matrix is positive definite. In order to accomplish that we solve, instead of (11.14), the reduced system

$$
\mathcal{A} \begin{pmatrix} \vec{\boldsymbol{\sigma}} \\ \vec{\mathbf{u}} \\ \vec{\gamma} \end{pmatrix} := \begin{pmatrix} \mathbf{A} & -\mathbf{D}^{\mathrm{t}} & -\mathbf{P}^{\mathrm{t}} \\ \mathbf{D} & \mathbf{J} & -\mathbf{S}^{\mathrm{t}} \\ \mathbf{P} & \mathbf{S} & \mathbf{U} \end{pmatrix} \begin{pmatrix} \vec{\sigma} \\ \vec{\mathbf{u}} \\ \vec{\gamma} \end{pmatrix} = \begin{pmatrix} \vec{F}_{\boldsymbol{\sigma}} \\ \vec{F}_u \\ \vec{0} \end{pmatrix}. \tag{12.3}
$$

This means that we will not make use of Lagrange multipliers to enforce the null mean trace condition and insted we will let the system suited for GMRES in another way.

LEMMA 12.1. *The systems* (12.2) *and* (12.3) *have one and the same solution.*

PROOF. From Theorem 11.4 we know that (12.2) has only one solution, which happens to satisfy $\varphi_h = 0$. It is clear that the solution of (12.2) is turned into a solution of by just dropping the φ_0. a solution of (12.3). On the other hand, (12.3) cannot have any more solutions. In fact, given $(\boldsymbol{\tau}, \mathbf{v}, \boldsymbol{\eta}) \in H_h^{\boldsymbol{\sigma}} \times H_{0,h}^{\mathbf{u}} \times H_h^{\boldsymbol{\gamma}}$, we can decompose $\boldsymbol{\tau}$ into $\boldsymbol{\tau}_0 + \rho\mathbf{I}$, where

$\boldsymbol{\tau}_0 \in H^{\boldsymbol{\sigma}}_{0,h}$ and $\rho = \frac{1}{2|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\tau})$. Using this fact, the definition of A (11.5), the ranges of κ_1, μ, and λ, and (11.3) we have

$$\begin{aligned}
&\begin{pmatrix}\vec{\boldsymbol{\tau}}\\ \vec{\mathbf{v}}\\ \vec{\boldsymbol{\eta}}\end{pmatrix}^{\mathtt{t}} \mathcal{A} \begin{pmatrix}\vec{\boldsymbol{\tau}}\\ \vec{\mathbf{v}}\\ \vec{\boldsymbol{\eta}}\end{pmatrix} = A((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}))\\
&= A((\boldsymbol{\tau}_0,\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau}_0,\mathbf{v},\boldsymbol{\eta})) + A((\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}),(\rho\mathbf{I},\mathbf{0},\mathbf{0})) + A((\rho\mathbf{I},\mathbf{0},\mathbf{0}),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}))\\
&\qquad + A((\rho\mathbf{I},\mathbf{0},\mathbf{0}),(\rho\mathbf{I},\mathbf{0},\mathbf{0}))\\
&= A((\boldsymbol{\tau}_0,\mathbf{v},\boldsymbol{\eta}),(\boldsymbol{\tau}_0,\mathbf{v},\boldsymbol{\eta})) + \frac{\rho^2}{\mu}\int_\Omega \left(1-\frac{\lambda}{\lambda+\mu}\right)\left(1-\frac{\kappa_1}{2\mu}\left(1-\frac{\lambda}{\lambda+\mu}\right)\right)\\
&\geq \alpha\,\|(\boldsymbol{\tau}_0,\mathbf{v},\boldsymbol{\eta})\|^2_{\mathbf{H}_0} + C\rho^2\\
&\geq \hat{C}\left((\|\boldsymbol{\tau}\|^2_{H(\mathbf{div};\Omega)} + |\mathbf{u}|^2_{[H^1(\Omega)]^2} + \|\boldsymbol{\gamma}\|^2_{[L^2(\Omega)]^{2\times 2}}\right).
\end{aligned} \tag{12.4}$$

Thus, $\mathcal{A}$ is nonsingular and positive definite. Besides, we have just shown the ellipticity of A over the whole of $H(\mathbf{div};\Omega) \times [H^1_0(\Omega)]^2 \times [L^2(\Omega)]^{2\times 2}_{\mathtt{asym}} \supset \mathbf{H}_0$. □

Beacuse of this result we now focus on (12.3) instead of (12.2). Let us now consider a symmetric positive-definite preconditioner (or just a scaling) of the general block form

$$\mathcal{P} := \begin{pmatrix} \mathbf{P}_1 & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{P}_2 & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{P}_3\end{pmatrix}, \tag{12.5}$$

where $\mathbf{P}_1 \in \mathbb{R}^{\bar{N}\times\bar{N}}$, $\mathbf{P}_2 \in \mathbb{R}^{\bar{n}\times\bar{n}}$, and $\mathbf{P}_3 \in \mathbb{R}^{\bar{m}\times\bar{m}}$. Then, the preconditioned stiffness matrix, $\mathcal{A}_{\mathcal{P}}$, has the form

$$\mathcal{A}_{\mathcal{P}} := \mathcal{P}^{-1/2}\mathcal{A}\mathcal{P}^{-1/2} = \begin{pmatrix} \mathbf{P}_1^{-1/2}\mathbf{A}\mathbf{P}_1^{-1/2} & -\mathbf{P}_1^{-1/2}\mathbf{D}^{\mathtt{t}}\mathbf{P}_2^{-1/2} & -\mathbf{P}_1^{-1/2}\mathbf{P}^{\mathtt{t}}\mathbf{P}_3^{-1/2}\\ \mathbf{P}_2^{-1/2}\mathbf{D}\mathbf{P}_1^{-1/2} & \mathbf{P}_2^{-1/2}\mathbf{J}\mathbf{P}_2^{-1/2} & -\mathbf{P}_2^{-1/2}\mathbf{S}^{\mathtt{t}}\mathbf{P}_3^{-1/2}\\ \mathbf{P}_3^{-1/2}\mathbf{P}\mathbf{P}_1^{-1/2} & \mathbf{P}_3^{-1/2}\mathbf{S}\mathbf{P}_2^{-1/2} & \mathbf{P}_3^{-1/2}\mathbf{U}\mathbf{P}_3^{-1/2}\end{pmatrix}. \tag{12.6}$$

Following [**20**] the relative reduction of the residual after i steps of the preconditioned GMRES method is bounded by $(1-\frac{c_0^2}{C_0})^{i/2}$, where

$$c_0 := \inf_{\vec{x}\in\mathbb{R}^d\setminus\{0\}} \frac{\vec{x}^{\mathtt{t}}\mathcal{A}_{\mathcal{P}}^{\mathtt{sym}}\vec{x}}{\vec{x}^{\mathtt{t}}\vec{x}}, \tag{12.7}$$

and

$$C_0 := \sup_{\vec{x}\in\mathbb{R}^d\setminus\{0\}} \frac{\vec{x}^{\mathtt{t}}\mathcal{A}_{\mathcal{P}}^{\mathtt{t}}\mathcal{A}_{\mathcal{P}}\vec{x}}{\vec{x}^{\mathtt{t}}\vec{x}}, \tag{12.8}$$

and $d := 2\bar{N} + 2\bar{n} + \bar{m}$. Let c_i and C_i, $i = 1,2,3$, be positive numbers, possibly dependent on h, such that, for all $(\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$ in $H^{\boldsymbol{\sigma}}_h \times H^{\mathbf{u}}_{0,h} \times H^{\gamma}_h$,

$$\begin{aligned}
c_1\vec{\boldsymbol{\sigma}}^{\mathtt{t}}\mathbf{P}_1\vec{\boldsymbol{\sigma}} &\leq A(\boldsymbol{\sigma},\mathbf{0},\mathbf{0}),(\boldsymbol{\sigma},\mathbf{0},\mathbf{0}) \leq C_1\vec{\boldsymbol{\sigma}}^{\mathtt{t}}\mathbf{P}_1\vec{\boldsymbol{\sigma}},\\
c_2\vec{\mathbf{u}}^{\mathtt{t}}\mathbf{P}_2\vec{\mathbf{u}} &\leq A(\mathbf{0},\mathbf{u},\mathbf{0}),(\mathbf{0},\mathbf{u},\mathbf{0}) \leq C_2\vec{\mathbf{u}}^{\mathtt{t}}\mathbf{P}_2\vec{\mathbf{u}},\\
c_3\vec{\boldsymbol{\gamma}}^{\mathtt{t}}\mathbf{P}_3\vec{\boldsymbol{\gamma}} &\leq A(\mathbf{0},\mathbf{0},\boldsymbol{\gamma}),(\mathbf{0},\mathbf{0},\boldsymbol{\gamma}) \leq C_3\vec{\boldsymbol{\gamma}}^{\mathtt{t}}\mathbf{P}_3\vec{\boldsymbol{\gamma}}.
\end{aligned} \tag{12.9}$$

It holds that

$$\inf_{\vec{\boldsymbol{\tau}}\in\mathbb{R}^{2\bar{N}}\setminus\{\mathbf{0}\}} \frac{\vec{\boldsymbol{\tau}}^{\mathtt{t}}\mathbf{P}_1^{-1/2}\mathbf{A}\mathbf{P}_1^{-1/2}\vec{\boldsymbol{\tau}}}{\vec{\boldsymbol{\tau}}^{\mathtt{t}}\vec{\boldsymbol{\tau}}} = \inf_{\vec{\boldsymbol{\sigma}}\in\mathbb{R}^{2\bar{N}}\setminus\{\mathbf{0}\}} \frac{\vec{\boldsymbol{\sigma}}^{\mathtt{t}}A\vec{\boldsymbol{\sigma}}}{\vec{\boldsymbol{\sigma}}^{\mathtt{t}}\mathbf{P}_1\vec{\boldsymbol{\sigma}}} \geq c_1 \tag{12.10}$$

TABLE 1. The number of GMRES iterations to reach the desired reduction in a 1e-5 proportion of the original is shown.

N	c_0^2	C_0	# of iterations
12	0.03516	1.42325	2
42	0.03516	1.42325	10
92	0.03516	1.42325	15
162	0.03516	1.42325	22
252	0.03516	1.42325	25
362	0.03516	1.42325	26
492	0.03516	1.42325	27
642	0.03516	1.42325	27
812	0.03516	1.42325	29
1002	0.03516	1.42325	29

Analogous relations are drawn between the smallest eigenvalue of the second and third diagonal block of the symmetric part of $\mathcal{A}_\mathcal{P}$ and c_2 and c_3, respectively, and then

$$c_0 \geq \min\{c_1, c_2, c_3\}\,. \tag{12.11}$$

On the other side,

$$C_0 = \sup_{\vec{x}\in\mathbb{R}^d\setminus\{0\}} \frac{\vec{x}^{\mathsf{t}}\mathcal{A}_\mathcal{P}^{\mathsf{t}}\mathcal{A}_\mathcal{P}\vec{x}}{\vec{x}^{\mathsf{t}}\vec{x}} = \sup_{\vec{x},\vec{y}\in\mathbb{R}^d\setminus\{0\}} \frac{(\vec{x}^{\mathsf{t}}\mathcal{A}_\mathcal{P}\vec{y})^2}{\|\vec{x}\|_2^2\,\|\vec{y}\|_2^2} \tag{12.12}$$

If we let $(\vec{\boldsymbol{\sigma}}^{\mathsf{t}}\ \vec{\mathbf{u}}^{\mathsf{t}}\ \vec{\gamma}^{\mathsf{t}})^{\mathsf{t}} = \mathcal{P}^{-1/2}\vec{x}$ and $(\vec{\boldsymbol{\tau}}^{\mathsf{t}}\ \vec{\mathbf{v}}^{\mathsf{t}}\ \vec{\boldsymbol{\eta}}^{\mathsf{t}})^{\mathsf{t}} = \mathcal{P}^{-1/2}\vec{y}$ we get from (12.9)

$$\begin{aligned} C_0 &\leq \sup_{\vec{x},\vec{y}\in\mathbb{R}^d\setminus\{0\}} \frac{A((\boldsymbol{\sigma},\mathbf{u},\gamma),(\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta}))^2}{(\vec{\boldsymbol{\sigma}}^{\mathsf{t}}\mathbf{P}_1\vec{\boldsymbol{\sigma}} + \vec{\mathbf{u}}^{\mathsf{t}}\mathbf{P}_2\vec{\mathbf{u}} + \vec{\gamma}^{\mathsf{t}}\mathbf{P}_3\vec{\gamma})(\vec{\boldsymbol{\tau}}^{\mathsf{t}}\mathbf{P}_1\vec{\boldsymbol{\tau}} + \vec{\mathbf{v}}^{\mathsf{t}}\mathbf{P}_2\vec{\mathbf{v}} + \vec{\boldsymbol{\eta}}^{\mathsf{t}}\mathbf{P}_3\vec{\boldsymbol{\eta}})} \\ &\leq \frac{M^2}{\alpha^2}\max\{C_1, C_2, C_3\}^2\,, \end{aligned} \tag{12.13}$$

where M and α are the continuity and ellipticity constants of the bilinear form A.

We have the following main result.

THEOREM 12.2. *If the block preconditioners are spectrally equivalent with the corresponding inner products, then the GMRES method will take a bounded amount of iterations to converge to a prescribed tolerance.*

PROOF. It is readily seen from (12.9), (12.11) and (12.13). □

A blunt approach to obtaining such spectrally equivalent block preconditioners is to store the inner product Gram matrix of the ansatz spaces. 'True' preconditioners for the $H(\mathbf{div};\Omega)$ and the $H^1(\Omega)$ blocks can be found in [**5**] and [**10**], respectively, and references therein.

13. Numerical results

We used GMRES for the system (12.3) with the inner-product based preconditioner and a sequence of regular meshes. The results are summarized in Table 1 and Figure 4.1.

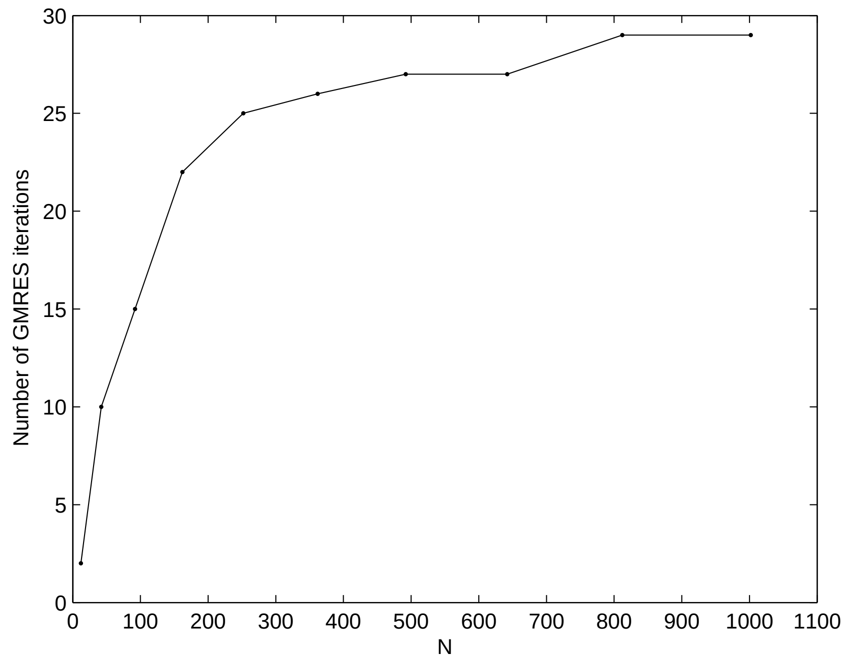

FIGURE 4.1. N versus the number of GMRES iterations taken to reduce the inital residual to 1e-5 of the original.

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Whatever you do, work at it with all your heart, as working for the Lord, not for men. Colossians 3:23 (NIV)

I am a bit wary about how much I have worked as if for the Lord, needless to say.

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Bibliography

[1] Ainsworth, M., McLean, W. and Tran, T., *The conditioning of boundary element equations on locally refined meshes and preconditioning by diagonal scaling*. SIAM Journal on Numerical Analysis, vol. 36, 6, pp. 1901-1932, (1999).

[2] Arnold, D.N., Brezzi, F. and Douglas, J., *PEERS: A new mixed finite element method for plane elasticity*. Japan Journal of Applied Mathematics, vol. 1, pp. 347-367, (1984).

[3] Arnold, D.N., Brezzi, F. and Fortin, M., *A stable finite element method for the Stokes equations*. Calcolo, vol. 21, pp. 337-344, (1984).

[4] Arnold, D.N., Douglas, J. and Gupta, Ch.P., *A family of higher order mixed finite element methods for plane elasticity*. Numerische Mathematik, vol. 45, pp. 1-22, (1984).

[5] Arnold, D.N., Falk, R.S. and Winther, R., *Precontitioning in $H(\mathrm{div})$ and applications*. Mathematics of computation, vol. 66, pp. 957-984, (1997).

[6] Bank, R.E. and Scott, L.R., *On the conditioning of Finite Element Equations with Highly Refined Meshes*. SIAM Journal on Numerical Analysis, vol. 26, 6, pp. 1383-1394, (1989).

[7] Barrientos, M., Gatica, G.N. and Stephan, E.P., *A mixed finite element method for nonlinear elasticity: two-fold saddle point approach and a-posteriori error estimate*. Numerische Mathematik, vol. 91, 2, pp. 197-222, (2002).

[8] Barrios, T.P., Gatica, G.N., González, M. and Heuer, N., *A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity*. Mathematical Modelling and Numerical Analysis, vol. 40, 5, (2006), to appear.

[9] Benzi, M., Golub, G.H. and Liesen, J., *Numerical solution of saddle point problems*. Acta Numerica, vol. 14, pp. 1-137, (2005).

[10] Brenner, S.C. and Scott, L.R., *The Mathematical Theory of Finite Element Methods*. Springer Verlag, (2002).

[11] Brezzi, F. and Douglas, J., *Stabilized mixed methods for the Stokes problem*. Numerische Mathematik, vol. 53, 1&2, pp. 225-235, (1988).

[12] Brezzi, F. and Fortin, M., Mixed and Hybrid Finite Element Methods. Springer Verlag, 1991.

[13] Brezzi, F. and Fortin, M., *A minimal stabilisation procedure for mixed finite element methods*. Numerische Mathematik, vol. 89, 3, pp. 457-491, (2001).

[14] Brezzi, F., Fortin, M. and Marini, L.D., *Mixed finite element methods with continuous stresses*. Mathematical Models and Methods in Applied Sciences, vol. 3, 2, pp. 275-287, (1993).

[15] Carstensen, C. and Dolzmann, G., *A posteriori error estimates for mixed FEM in elasticity*. Numerische Mathematik, vol. 81, pp. 187-209, (1998).

[16] Ciarlet, P.G., The Finite Element Method for Elliptic Problems. North-Holland, 1978.

[17] Clément, P., *Approximation by finite element functions using local regularisation*. RAIRO Modélisation Mathématique et Analyse Numérique, vol. 9, pp. 77-84, (1975).

[18] Demmel, J.W., Eisenstat, S.C., Gilbert, J.R., Li, X.S. and Liu, J.W.H., *A supernodal approach to sparse partial pivoting*. SIAM Journal on Matrix Analysis and Applications, vol. 20, 3, pp. 720-755, (1999).

[19] Douglas, J. and Wan, J., *An absolutely stabilized finite element method for the Stokes problem*. Mathematics of Computation, vol. 52, 186, pp. 495-508, (1989).

[20] Eisenstat, S.C., Elman, H.C. and Schultz, M.H., *Variational iterative methods for non-symmetric systems of linear equations*. SIAM Journal on Numerical Analysis, vol. 20, 2, pp. 345-357, (1983).

[21] Franca, L.P., New Mixed Finite Element Methods. Ph.D. Thesis, Stanford University, 1987.

[22] Franca, L.P. and Hughes, T.J.R., *Two classes of finite element methods*. Computer Methods in Applied Mechanics and Engineering, vol. 69, pp. 89-129, (1988).

[23] Franca, L.P. and Russo, A., *Unlocking with residual-free bubbles*. Computer Methods in Applied Mechanics and Engineering, vol. 142, 3&4, pp. 361-364, (1997).

[24] Franca, L.P. and Stenberg, R., *Error analysis of Galerkin least squares methods for the elasticity equations*. SIAM Journal on Numerical Analysis, vol. 28, 6, pp. 1680-1697, (1991).

[25] Gatica, G.N. and Heuer, N., *Minimum residual iteration for a dual-dual mixed formulation of exterior transmission problems.* Numerical Linear Algebra with Applications, vol. 8, 3, pp. 147-164, (2001).

[26] Gatica, G.N. and Heuer, N., *An expanded mixed finite element approach via a dual-dual formulation and the minimum residual method.* Journal of Computational and Applied Mathematics, vol. 132, 2, pp. 371-385, (2001).

[27] Gatica, G.N., *Analysis of a new augmented mixed finite element method for linear elasticity allowing* $\mathbb{RT}_0 - \mathbb{P}_1 - \mathbb{P}_0$ *approximations.* Mathematical Modelling and Numerical Analysis, vol. 40, 1, pp. 1-28, (2006).

[28] Gatica, G.N., *Further remarks on an augmented mixed finite element method for the linear elasticity problem.* Preprint 2006-19, Departamento de Ingeniería Matemática, Universidad de Concepción, Chile, (2006).

[29] Gurtin, M. *The Linear Theory of Elasticity.* In Mechanics of Solids, Truesdell, C. (ed.), Springer Verlag, vol. 2, pp. 1-273, (1984)

[30] Heuer, N. and Stephan, E., *Preconditioners for the p-version of the Galerkin Method for a Coupled Finite Element/Boundary Element System.* Numerical Methods for Partial Differential Equations, vol. 14, 1, pp. 47-61, (1998).

[31] Kechkar, N. and Silvester, D., *Analysis of locally stabilized mixed finite element methods for the Stokes problem.* Mathematics of Computation, vol. 58, 197, pp. 1-10, (1992).

[32] Lube, G. and Auge, A., *Stabilized mixed finite element approximations of incompressible flow problems.* Zeitschrift für Angewandte Mathematik und Mechanik, vol. 72, 6, T483-T486, (1992).

[33] McLean, W., Strongly Elliptic Systems and Boundary Integral Equations. Cambridge University Press, 2000.

[34] Masud, A. and Hughes, T.J.R., *A stabilized mixed finite element method for Darcy flow.* Computer Methods in Applied Mechanics and Engineering, vol. 191, 39&40, pp. 4341-4370, (2002).

[35] Rusten, T. and Winther, R., *A preconditioned iterative method for saddlepoint problems*, SIAM Journal on Matrix Analysis and Applications, vol. 13, 3, pp. 887-904, (1992).

[36] Saad, Y. and Schultz, M.H., *GMRES: A generalized minimal residual algotithm for solving nonsymmetric linear systems.* SIAM Journal on Scientific and Statistical Computing, vol. 7, 3, pp. 856-869, (1986).

[37] Shewchuk, J.R., *Triangle: Engineering a 2D quality mesh generator and Delaunay triangulator.* In Applied Computational Geometry Towards Geometric Engineering, Lin, M.C., Manocha, D. (eds.), Springer Verlag, pp. 203-222, (1996).

[38] Silvester, D., Elman, H., Kay, D. and Wathen, A. *Efficient preconditioning of the linearized Navier-Stokes equations for incompressible flow.* Journal of Computational and Applied Mathematics, vol. 128, pp. 261-279, (2001).

[39] Stenberg, R., *A family of mixed finite elements for the elasticity problem.* Numerische Mathematik, vol. 53, 5, pp. 513-538, (1988).

[40] Verfürth, R., A Review of A Posteriori Error Estimation and Adaptive Mesh-Refinement Techniques. Wiley-Teubner (Chichester), 1996.

[41] Wathen, A.J. and Silvester, D.J., *Fast iterative solution of stabilised Stokes systems, part I: Using simple diagonal preconditioners.* SIAM Journal on Numerical Analysis, vol. 30, 3, pp. 630-649 (1993).

[42] Wathen, A.J., Fischer B. and Silvester, D.J., *The convergence rate of the minimum residual method for the Stokes problem.* Numerische Mathematik, vol. 71, pp. 121-134, (1995).