The Darcy - Forchheimer and Stokes Coupled Problem
El Problema Acoplado de Darcy - Forchheimer y Stokes

Hugo Sebastián Díaz Norambuena.
Profesor Guía: Gabriel N. Gatica

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Hugo Sebastián Díaz Norambuena

Supervisors: Dr. Gabriel N. Gatica.

EVALUATION PANEL

Signature: ________________________________

Dr. Gabriel N. Gatica
Universidad de Concepción

Signature: ________________________________

Dr. Ricardo Oyarzúa
Universidad del Bío-Bío

Signature: ________________________________

Dr. Ricardo Ruiz-Baier
University of Lausanne

Signature: ________________________________

Dr. Manuel Solano
Universidad de Concepción

Dissertation date: ______________________________

Score: ______________________________

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Chapter 1

Introduction

In the last years the derivation of suitable mathematical and numerical models for the fluid movement which flows back and forth across a porous medium and a free fluid region has received a growing interest. To several applications in engineering and biology, to name a few. For example in filter design (cf. [17]) or in reservoir models (cf. [3]). Physically this is a coupled problem with two physical systems interacting across some interfaces. The most common mathematical formulation for this coupled problem is the Navier–Stokes–Darcy problem or simplifications of this, but the movement of a fluid in a porous medium is a complex phenomenon, which even in the standard case $R_e \approx 1$, not always the Darcy's law applies. In other cases the Darcy's law cannot be applied due to the *non-Darcy effects* as inertial effects. Then, the development of more complex models is needed, for example the Darcy-Forchheimer law which adds an additional term in the Darcy law in order to take into account the non-linear behavior of the pressure gradient and the velocity. On the other hand, for the free fluid region there are many models that approximate the fluid movement, one possibility is the Stokes model (Stokes flow) either linear or not, which is valid when the Reynolds number is low. The nonlinear Stokes model can be used in the modeling of the flow of quasi-Newtonian fluids. Finally, for the interface conditions it is very common to consider the Beavers-Joseph condition (cf. [5]) or some simplification of this, for example the Saffman or the Jones conditions (cf. [37] and [34]). Now, for the numerical model of this coupled problem, a widely used numerical technique is the finite element method (FEM). For example, the linear Stokes-Darcy problem with the Saffman condition on the interface and a fully-mixed formulation can be found in [25], whereas a nonlinear version of the Stokes-Darcy model with the Saffman condition and a primal-dual formulation is studied in [17]. The purpose of this thesis is to give a simple extension of the analysis from [24] (see also [36]) to the case of a fully-mixed formulation for the nonlinear model given by the Darcy - Forchheimer / Stokes coupled problem with the Saffman condition at the interface. Differently from the usual tools available in the literature, which are valid mainly for Hilbertian structures, our aproach is applicable to Banach spaces, which is precisely the case of the present nonlinear model. In addition, the appropriate choice of the discrete spaces allows, for example, the use of lowest order Raviart-Thomas interpolator directly to the velocity, thus avoiding the use of liftings to analyse the inf-sup conditions, and hence the hypothesis of quasiuniformity in a neighborhood of the interface on the porous medium side is not needed anymore. The rest of this work is organized as follows. In Chapter 2 we introduce the notation and the main aspects of the continuous Darcy - Forchheimer / Stokes coupled problem, which includes the weak formulation and the identification of the resulting system structure as a twofold saddle point problem. The analysis of a modified abstract theory for this kind of nonlinear operator equations, including the continuous and discrete setting, is analyzed in Chapter 3. The results of this chapter are then applied to our model problem in Chapter 4, and specific finite element subspaces satisfying the required conditions are defined. Next, in Chapter 5 we consider a more general problem which consists of a nonlinear version of the Stokes model in the free fluid part, and finally, some auxiliary results are detailed in the Appendices.

Chapter 2

The continuous problem

2.1 Preliminary notations

We begin by giving some definitions that we use throughout this thesis. As usual, $\mathbb{R}^{2\times2}$ is the space of square matrices of order two with real entries ; $\mathbf{I}$ is the identity matrix of $\mathbb{R}^{2\times2}$; and for any $\boldsymbol{\tau} = (\tau_{ij}), \boldsymbol{\sigma} = (\sigma_{ij}) \in \mathbb{R}^{2\times2}$, we write:

$$\boldsymbol{\tau}^t := (\tau_{ji}), \qquad \operatorname{tr}\boldsymbol{\tau} = \tau_{11} + \tau_{22} \;\text{ and }\; \boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\mathbf{I},$$

which corresponds, respectively, to the transpose, the trace, and the deviator of the tensor $\boldsymbol{\tau}$. We also define the inner tensor product (also called Frobenius inner product) between $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ given by:

$$(\boldsymbol{\tau}, \boldsymbol{\sigma}) \mapsto \boldsymbol{\tau} : \boldsymbol{\sigma} := \sum_{i,j=1}^{2} \tau_{ij}\sigma_{ij} = \operatorname{tr}\left(\boldsymbol{\tau}^t\boldsymbol{\sigma}\right).$$

Note that the Frobenius inner product between $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$ coincides with the sum of the entries of the Hadamard product $(\boldsymbol{\tau} \circ \boldsymbol{\sigma})$ between $\boldsymbol{\tau}$ and $\boldsymbol{\sigma}$, and it yields the $\mathbb{R}^{2\times2}$ orthogonal decompositions into the symmetric and the skew-symmetric tensors, or into the isotropic and non-isotropic tensors (see Apendix .1).

In what follows we utilize a simplified terminology for Sobolev spaces. In particular, if Ω is an open bounded polygon with Lipschitz continuous boundary Γ (cf. Def. 1.2.1.1 [30]), $\mathcal{S}$ is an open or closed Lipschitz curve and $\mathrm{X}(\Omega)$ (resp. $\mathrm{X}(\mathcal{S})$) a Sobolev space on Ω (resp. $\mathcal{S}$), we define

$$\mathbf{X}(\Omega) := \left[\mathrm{X}(\Omega)\right]^2, \;\; \mathbb{X}(\Omega) := \left[\mathrm{X}(\Omega)\right]^{2\times2} \quad\text{and}\quad \mathbf{X}(\mathcal{S}) := \left[\mathrm{X}(\mathcal{S})\right]^2.$$

In turn, $1 \le p < \infty$, we define the Soboblev spaces (cf. [1])

$$L^p(\Omega) := \left\{ q : q \text{ measurable }, \int_\Omega |q(x)|^p dx < \infty \right\},$$

$$W^{1,p}(\Omega) \equiv \left\{ u \in \mathrm{L}^p(\Omega) : \partial^\alpha u \in \mathrm{L}^p(\Omega) \text{ for } |\alpha| \le 1 \right\},$$

equipped with the norms

$$\|q\|_{0,p;\Omega} := \left(\int_\Omega |q(x)|^p dx \right)^{\frac{1}{p}} \quad \text{and} \quad \|u\|_{1,p;\Omega} := \sum_{0 \le |\alpha| \le 1} \|\partial^\alpha u\|_{0,p;\Omega}, \text{ respectively,}$$

where ∂^α is the distributional partial derivate (cf. [1, 1.57]). Also, we set

$$L^p_0(\Omega) := \left\{ q \in L^p(\Omega) : \int_\Omega q = 0 \right\} \quad \text{and} \quad [L^p(\Omega)]' \simeq L^q(\Omega) \ \text{ where } \frac{1}{p} + \frac{1}{q} = 1.$$

In addition, given $2 \le r < \infty$, we define the Sobolev space

$$\mathbf{W}^{0,r}(\mathrm{div}\,;\Omega) := \left\{ \mathbf{v} \in \mathbf{L}^r(\Omega) : \ \mathrm{div}\,\mathbf{v} \in L^r(\Omega) \right\}, \tag{2.1}$$

equipped with the norm $\|\mathbf{v}\|_{r,\mathrm{div}\,;\Omega} := \|\mathbf{v}\|_{0,r;\Omega} + \|\mathrm{div}\,\mathbf{v}\|_{0,r;\Omega}$, where the divergence operator div is understood in the sense of distributions, that is

$$\langle \mathrm{div}\,\mathbf{v}, \varphi \rangle_{\mathcal{D}(\Omega)' \times \mathcal{D}(\Omega)} := - \int_\Omega \mathbf{v} \cdot \nabla\varphi \ \mathrm{d}x \qquad \forall \varphi \in \mathcal{D}(\Omega). \tag{2.2}$$

In the particular case $r = 2$, we define $\mathbf{H}(\mathrm{div}\,;\Omega) := \mathbf{W}^{0,2}(\mathrm{div}\,;\Omega)$ and $\|\mathbf{v}\|_{\mathrm{div}\,;\Omega} := \|\mathbf{v}\|_{2,\mathrm{div}\,;\Omega}$.

The space of matrix valued functions whose rows belong to $\mathbf{H}(\mathrm{div}\,;\Omega)$ will be denoted $\mathbb{H}(\mathbf{div}\,;\Omega)$, where $\mathbf{div}$ stands for the action of div along each row of a tensor. The Sobolev norm of $\mathbb{H}(\mathbf{div}\,;\Omega)$ is denoted by $\|\cdot\|_{\mathbf{div}\,;\Omega}$. Finally, we employ Θ or $\mathbf{0}$ to denote a generic null vector (including the null functional and operator), and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take diferent values at diferent places.

2.2 The model problem

In what follows we will model the movement of the fluid flow both in a free flow region Ω_S and a porous medium Ω_D. We assume that these regions have a common interface Σ. The models that we use in both regions are simplifications of the model of Navier-Stokes. However note that the model that we use in the porous medium has an experimental origin, as happens with Darcy's law. Moreover, certain

More precisely, we seek a numerical approximation for the movement of an incompressible Newtonian or quasi-Newtonian viscous fluid in the free flow region, which flows towards and from the porous medium across the common interface, where the porous medium is saturated with the same fluid. Now we give a more precisely description of the flow domain. Let Ω_D and Ω_S strictly polygonal, connected and disjoint subsets of $\mathbb{R}^2$, such that $\Sigma := int(\partial\Omega_D \cap \partial\Omega_S)$ has a Lebesgue measure strictly positive; $\Gamma_S := \partial\Omega_S/\Sigma$, and $\Gamma_D := \partial\Omega_D/\Sigma$. Also we denote **n** and **t**, the generic unit outward normal vector and unit tangent vector, respectively, on Γ_S and Γ_D. On Σ the vectors **n** and **t** are chosen as the common ones of Ω_S (positive orientation, see Figure 2.1).

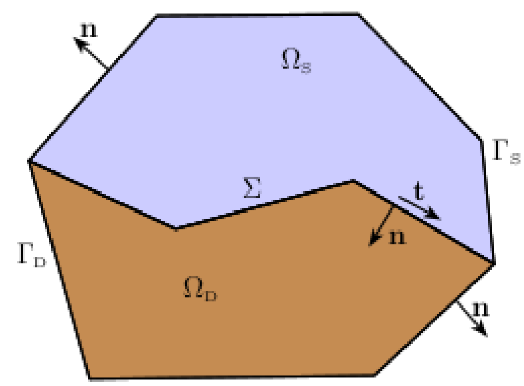

Figure 2.1: Sketch Domain

restrictions of both theoretical and experimental origin are imposed at the interface, in particularly a variation of the Breavers-Joseph condition (1967) (cf. [5]) done by Saffman (1971) (cf. [37]).

Here it is considered that the fluid is confined to $\Omega := \Omega_S \cup \Omega_D \cup \Sigma$, and that homogeneous boundary conditions are imposed on $\Gamma_S \cup \Gamma_D$, but making suitable small modifications is possible to establish more general boundary conditions (cf. [17]).

As the fluid here considered is a Newtonian or a quasi-Newtonian fluid, we consider a simplified version of Navier-Stokes model given by Stokes. Moreover in the porous medium we will use a generalization of Darcy's law called Darcy - Forchheimer Law.

On the other hand, for the condition at the interface we use the two well-accepted conditions given by the continuity of the normal forces and the continuity of the normal velocities. We remark that the experimental condition known as Beavers-Joseph-Saffman condition (cf. [37]), was validated by Jäger et al. in [33].

Further Notations

We now introduce additional notations to be used later. Given $\star \in \{S, D\}$ and $1 < p < \infty$, we define

$$
\begin{aligned}
(u,\ v)_\star &:= \int_{\Omega_\star} uv \qquad \forall u \in \mathrm{L}^p(\Omega_\star)\ \ \forall v \in \mathrm{L}^q(\Omega_\star)\,,\\
(\mathbf{u},\ \mathbf{v})_\star &:= \int_{\Omega_\star} \mathbf{u}\mathbf{v} \qquad \forall \mathbf{u} \in \mathbf{L}^p(\Omega_\star)\ \ \forall \mathbf{v} \in \mathbf{L}^q(\Omega_\star)\,,\\
[\mathbf{u},\ \mathbf{v}]_\star &:= \int_{\Omega_\star} \mathbf{u}\colon \mathbf{v} \qquad \forall \mathbf{u} \in \mathbb{L}^p(\Omega_\star)\ \ \forall \mathbf{v} \in \mathbb{L}^q(\Omega_\star)\,.
\end{aligned}
$$

On other hand, given $\Gamma_0 \subseteq \Gamma_\star$, we define

$$
\widetilde{W}^{\frac{1}{q},p}(\Gamma_0) := \Big\{ u \in W^{\frac{1}{q},p}(\Gamma_0) : \widetilde{u} \in W^{\frac{1}{q},p}(\Gamma)\,,\ \text{where } \widetilde{u} \text{ is the continuation by zero of } u \text{ to } \Gamma \setminus \Gamma_0 \Big\}.
$$

Below we show a model for the fluid flow in the free fluid region Ω_S. We consider the Stokes model which is valid for creeping flows, and for which inertial effects can be neglected. For example, high-viscosity fluids at low velocities. Equivalently, the Stokes model is valid when $Re \ll 1$ is satisfied.

Free fluid region Ω_S

Assuming a Newtonian or quasi-Newtonian fluid in Ω_S at low Reynolds numbers, with velocity $\mathbf{u}_S$, pressure p_S, and stress tensor $\boldsymbol{\sigma}_S$ associated with the flow, there holds

$$
\begin{aligned}
\mathbf{e}(\mathbf{u}_S) &:= \frac{1}{2}\Big(\nabla \mathbf{u}_S + (\nabla \mathbf{u}_S)^t\Big) && \big(\text{deformation rate tensor }\big),\\
\boldsymbol{\sigma}_S &:= -p_S \mathbf{I} + 2\mu\big(\|\mathbf{e}(\mathbf{u}_S)\|\big)\mathbf{e}(\mathbf{u}_S) && \big(\text{constitutive equation }\big),\\
\mathbf{div}\,\boldsymbol{\sigma}_S + \mathbf{f}_S &= \mathbf{0} && \big(\text{equilibrium state }\big),\\
\operatorname{div} \mathbf{u}_S &= \mathbf{0} && \big(\text{incompressible flow }\big),\\
\mathbf{u}_S &= \mathbf{0} \ \text{ on } \Gamma_S && \big(\text{no-slip condition on } \Gamma_S\big),
\end{aligned}
$$

where μ is the fluid viscosity and $\mathbf{f}_S$ is a source term that represents body forces such as gravity or electromagnetic forces. Noting that $\operatorname{tr}(\nabla \mathbf{u}_S) = \operatorname{tr}(\mathbf{e}(\mathbf{u}_S)) = \operatorname{div} \mathbf{u}_S \equiv \mathbf{0}$, the Stokes problem can be rewritten equivalently as:

$$\begin{aligned}
\frac{1}{2\mu\big(\|\mathbf{e}\,(\mathbf{u}_S)\|\big)}\boldsymbol{\sigma}_{\mathbf{S}}{}^d &= \mathbf{e}\,(\mathbf{u}_S) = \nabla\mathbf{u}_S - \boldsymbol{\gamma}_{\boldsymbol{S}}, && (2.3)\\
\text{div }\,\boldsymbol{\sigma}_S + \mathbf{f}_S &= \mathbf{0}, && (2.4)\\
p_S &= -\frac{1}{2}\text{tr}\,\boldsymbol{\sigma}_S, && (2.5)\\
\mathbf{u}_S &= \mathbf{0}\ \text{ on } \Gamma_S, && (2.6)\\
\boldsymbol{\gamma}_{\boldsymbol{S}} &:= \frac{1}{2}\Big(\nabla\mathbf{u}_S - (\nabla\mathbf{u}_S)^t\Big), && (2.7)
\end{aligned}$$

where $\boldsymbol{\gamma}_{\boldsymbol{S}}$ is called vorticity. In order to simplify our analysis, from now on we assume that the fluid is Newtonian i.e., μ is a constant function. It will be shown below that there is a way to solve the case of quasi-Newtonian fluids under the same hypothesis to be developed in what follows (see Section 5).

Now, in the porous medium we consider a nonlinear version of Darcy problem to approximate the pressure and the flow velocity. These types of models are needed when the fluid velocity is high, more precisely when the kinematic forces dominate over viscous forces, or equivalently when $R_e \geq 1$. The specific model we consider in what follows is the one given by the Darcy - Forchheimer law.

The porous medium Ω_D

When the kinematic effects are more important than viscous effects in the porous media Ω_D, the Darcy velocity $\mathbf{u}_D$ and the pressure gradient ∇p_D do not satisfy a linear relation. In this case, a better approximation is given by the Darcy - Forchheimer law, which, assuming Neumann boundary conditions, is stated as follows:

$$\begin{aligned}
\frac{\mu}{\rho}\mathbf{K}^{-\mathbf{1}}\mathbf{u}_D + \frac{\beta}{\rho}|\mathbf{u}_D|\mathbf{u}_D + \nabla p_D &= \mathbf{g}_D \quad \text{in } \Omega_D,\\
\text{div}\,\mathbf{u}_D &= f_D \quad \text{in } \Omega_D,\\
\mathbf{u}_D\cdot\mathbf{n} &= 0 \quad \text{on } \Gamma_D,
\end{aligned} \tag{2.8}$$

where ρ, μ and β are the density, viscosity and dynamic viscosity of the fluid, respectively, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_D)$ is a symmetric and uniformly elliptic tensor describing the permeability of the porous medium. In turn, f_D and $\mathbf{g}_D$ are given, and according to the compressibility conditions, the boundary conditions on $\mathbf{u}_D$ and $\mathbf{u}_S$, and the principle of mass conservation (cf. (2.13) below), there must hold

$$\int_{\Omega_D} f_D = 0.$$

Now, we define the nonlinear mapping

$$\begin{aligned}
\mathcal{A}_D :\ \mathbf{L}^3(\Omega_D) &\mapsto \mathbf{L}^{\frac{3}{2}}(\Omega_D)\\
\mathbf{u}_D &\mapsto \mathcal{A}_D(\mathbf{u}_D) := \frac{\mu}{\rho}\mathbf{K}^{-\mathbf{1}}\mathbf{u}_D + \frac{\beta}{\rho}|\mathbf{u}_D|\mathbf{u}_D
\end{aligned} \tag{2.9}$$

and set $\lambda_{min} := \min\Big\{\lambda : \lambda \text{ is a eigenvalue of } \mathbf{K}\Big\}$. Thus, from the assumptions on $\mathbf{K}$ it follows that there exists $\lambda_0 > 0$ such that

$$\lambda_{min}(\mathbf{x}) \geq \lambda_0 > 0 \quad \forall \mathbf{x} \in \overline{\Omega}_D. \tag{2.10}$$

To conclude with the set of equations, at the interface Σ we introduce three transmission conditions: the mass conservation (i.e., the fluid entering and exiting each region remains constant), balance of the normal forces, and the Beavers-Joseph-Saffman condition.

The interface $\boldsymbol{\Sigma}$

At the interface between the free and the porous media flow, the conservation of mass and balance of normal forces are well-accepted conditions (cf. [25],[17] and [41]). On the other hand, there exists experimental conditions that must be satisfied. An example of this kind of conditions are the B-J conditions (cf. [5]), which relate the jump of the velocity field between the free fluid and the fluid in the porous media with the traction (tangential component of the normal stress). On the other hand, in some cases it is possible to simplify the B-J conditions (cf. [37] [34]). In what follows we will consider the Beavers-Joseph-Saffman conditions (cf. [37]), which is obtained by neglecting the tangential velocity in the porous-medium at the B-J condition.

Summarizing, the following three conditions are imposed at the interface Σ

- Conservation of mass, i.e., continuity of the normal velocity

$$\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}.$$

- Balance of normal forces

$$(\boldsymbol{\sigma}_S \mathbf{n}) \cdot \mathbf{n} = -p_D. \tag{2.11}$$

- The Beavers-Joseph-Saffman condition which is another constraint on the traction $\boldsymbol{\sigma}_S \mathbf{n}$:

$$(\boldsymbol{\sigma}_S \mathbf{n}) \cdot \mathbf{t} = -\mu \kappa^{-1} (\mathbf{u}_S \cdot \mathbf{t}), \tag{2.12}$$

where κ is the friction coefficient. Considering that $(\mathbf{n}, \mathbf{t})$ is a local orthonormal basis on Σ, we can rewrite (2.11) and (2.12) as a single equation (cf. (2.14) bellow), so that our transmission conditions become:

$$\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n} \qquad \text{on } \Sigma, \tag{2.13}$$

$$\boldsymbol{\sigma}_S \mathbf{n} + \mu \kappa^{-1} (\mathbf{u}_S \cdot \mathbf{t}) \, \mathbf{t} = -p_D \mathbf{n} \qquad \text{on } \Sigma. \tag{2.14}$$

Now, we proceed to establish a weak form for the three sets of equations: Stokes, Darcy - Forchheimer and the transmission conditions. In order to do it, we will apply a very similar analysis to the one presented in [25], which introduces $\mathbf{u}_S|_\Sigma$ and $p_D|_\Sigma$ as additional unknowns of physical interest.

2.3 Weak formulations

We begin by deriving a weak formulation for the Stokes problem, for which we use a similar formulation to the one presented in [25]. We use the orthogonal decomposition of a tensor in its isotropic and non-isotropic parts (cf. Table .1), and as usual, the hypothesis of zero mean for the pressure p_S is also considered.

2.3.1 Stokes problem

As it is common for the Stokes problem, for the uniqueness of solution we assume that $p_S \in \mathrm{L}_0^2(\Omega_S)$. In addition, we can "drop" p_S from the unknowns, and hence we simply seek $\boldsymbol{\sigma}_S$ in $\mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ and $\mathbf{u}_S \in \mathbf{L}^2(\Omega_S)$, where $\mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ is the space of matrix-valued functions $\boldsymbol{\sigma}_S$ in $\mathbb{H}(\mathbf{div}\,;\Omega_S)$ such that $\displaystyle\int_{\Omega_S} \operatorname{tr} \boldsymbol{\sigma}_S = 0$.
Note that the last restriction comes from the fact that $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\operatorname{div}\,;\Omega_S) \Leftrightarrow p_S \in \mathrm{L}_0^2(\Omega_S)$, which follows from the identity $p_S = -\frac{1}{2}\operatorname{tr}\boldsymbol{\sigma}_S$. Next, as in [25], we define the linear operator

$$
\begin{aligned}
\mathcal{A}_S : \mathbb{H}_0(\mathbf{div}\,;\Omega_S) &\to \left[\mathbb{H}_0(\mathbf{div}\,;\Omega_S)\right]' \\
\boldsymbol{\sigma}_S &\to \mathcal{A}_S(\boldsymbol{\sigma}_S) \\
\Big[\mathcal{A}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S\Big]_S &:= \left[\frac{1}{2\mu}\boldsymbol{\sigma}_{\mathbf{S}}{}^d, \boldsymbol{\tau}_S\right]_S \qquad \forall \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S).
\end{aligned} \tag{2.15}
$$

Now, testing the equations (2.3) and (2.4) with $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div};\Omega_S)$ and $\mathbf{v}_S \in \mathbf{L}^2(\Omega_S)$, respectively, and imposing the symmetry of $\boldsymbol{\sigma}_S$ in a weak sense, we arrive to the following weak formulation for Stokes:

$$
(P_S)\begin{cases}
\text{find } (\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_{\boldsymbol{S}}, \boldsymbol{\varphi}) \in \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{L}^2(\Omega_S) \times \mathbb{L}^2_{skew}(\Omega_S) \times \widehat{\mathbf{H}}_{00}^{\frac{1}{2}}(\Sigma) \quad \text{such that} \\
[\mathcal{A}_S(\boldsymbol{\sigma}_{\mathbf{S}}), \boldsymbol{\tau}_S] + (\mathbf{div}\;\; \boldsymbol{\tau}_S, \mathbf{u}_S)_S + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi}\rangle_\Sigma + (\boldsymbol{\gamma}_{\boldsymbol{S}}, \boldsymbol{\tau}_S)_S \quad = 0, \\
(\mathbf{div}\, \boldsymbol{\sigma}_S, \mathbf{v}_S)_S \qquad = (-\mathbf{f}_S, \mathbf{v}_S)_S\,, \\
(\boldsymbol{\sigma}_S, \boldsymbol{\eta}_S)_S \qquad = 0, \\
\forall (\boldsymbol{\tau}_S, \mathbf{v}_S, \boldsymbol{\eta}_S) \in \mathbb{H}_0(\mathbf{div};\Omega_S) \times \mathbf{L}^2(\Omega_S) \times \mathbb{L}^2_{skew}(\Omega_S),
\end{cases}
$$

where $\mathbb{L}^2_{skew}(\Omega_S)$ is the space of skew-symmetric tensors set in $\mathbb{L}^2(\Omega_S)$, that is

$$\mathbb{L}^2_{skew}(\Omega_S) := \Big\{ \varrho \in \mathbb{L}^2(\Omega_S) : \ \varrho^t = -\varrho \ \Big\}, \tag{2.16}$$

and $\boldsymbol{\varphi} := -\mathbf{u}_S|_\Sigma$. If we assume that $\mathbf{u}_S \in \mathbf{W}^{1,2}_{\Gamma_S}(\Omega_S)$ then the natural search space for $\boldsymbol{\varphi}$ is $\widetilde{\mathbf{W}}^{\frac{1}{2},2}(\Sigma) =: \mathbf{H}^{\frac{1}{2}}_{00}(\Sigma)$, though more precisely

$$\boldsymbol{\varphi} \in \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma) = \Big\{ \mathbf{v} \in \mathbf{H}^{\frac{1}{2}}_{00}(\Sigma) : \langle \mathbf{v}\cdot \mathbf{n}, 1\rangle_\Sigma = 0 \Big\}, \tag{2.17}$$

which is a consequence of the no-slip condition (2.6) and the incompressibility constraint $\operatorname{div} \mathbf{u}_S = 0$.

In the next section, we provide a weak formulation for the Darcy - Forchheimer law by proceeding similarly as in [19] and [18].

2.3.2 Darcy - Forchheimer equations

The Darcy - Forchheimer problem with non-slip condition on Γ_D is given by:

$$\left(\widetilde{P}_{DF}\right) \begin{cases} \text{find } (\mathbf{u}_D, p_D) \in \mathbf{L}^3(\Omega_D) \times W^{1,\frac{3}{2}}(\Omega_D)/\mathbb{R} \quad \text{such that} \\ \mathcal{A}_D(\mathbf{u}_D) + \nabla p_D \;=\; \mathbf{g}_D \quad \text{in } \Omega_D, \\ \operatorname{div} \mathbf{u}_D \;=\; f_D \quad \text{in } \Omega_D, \\ \mathbf{u}_D\cdot \mathbf{n} \;=\; 0 \quad \text{on } \Gamma_D, \end{cases}$$

where $\mathbf{g}_D \in \mathbf{L}^{\frac{3}{2}}(\Omega_D)$ and $f_D \in L^3_0(\Omega_D)$ are given terms. Next we define the space:

$$\mathbf{W}^{0,3}_{\Gamma_D}(\operatorname{div};\Omega_D) := \Big\{ \mathbf{v} \in \mathbf{W}^{0,3}(\operatorname{div};\Omega_D) : \mathbf{v}\cdot \mathbf{n} = 0 \text{ on } \Gamma_D \Big\},$$

which is equipped with the norm $\|\cdot\|_{3,\operatorname{div};\Omega_D}$. The precise meaning of the statement $\mathbf{v}\cdot\mathbf{n} = 0$ on Γ_D is specified below. Now, testing the first and the second equation of $\left(\widetilde{P}_{DF}\right)$ with $s_D \in \mathbf{W}^{0,3}_{\Gamma_D}(\operatorname{div};\Omega_D)$ and $q \in L^{\frac{3}{2}}(\Omega_D) \setminus \mathbb{R}$ respectively, the problem becomes

$$(P_{DF}) \begin{cases} \text{find } (\mathbf{u}_D, p_D, \lambda) \in \mathbf{W}^{0,3}_{\Gamma_D}(\operatorname{div};\Omega_D) \times L^{\frac{3}{2}}(\Omega_D) \setminus \mathbb{R} \times W^{\frac{1}{3},\frac{3}{2}}(\Sigma) \;\text{ such that} \\ (\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D)_D - (\operatorname{div} \mathbf{v}_D, p_D)_D - \langle \mathbf{v}_D\cdot \mathbf{n}, \lambda\rangle_\Sigma \;=\; (\mathbf{g}_D, \mathbf{v}_D)_D\,, \\ -(\operatorname{div} \mathbf{u}_D, q_D)_D \;=\; -(f_D, q_D)_D\,, \\ \forall (\mathbf{v}_D, q_D) \in \mathbf{W}^{0,3}_{\Gamma_D}(\operatorname{div};\Omega_D) \times L^{\frac{3}{2}}(\Omega_D) \setminus \mathbb{R}, \end{cases}$$

where λ is the trace of p_D on Σ. In what follows we give a precise sense to the parity $\langle \mathbf{v}_D\cdot \mathbf{n}, \lambda\rangle_\Sigma$.

We begin by remarking that the parity $\langle \mathbf{v}\cdot \mathbf{n}, \xi\rangle_\Sigma$ is well defined $\forall(\mathbf{v},\xi) \in \mathbf{W}^{0,q}_{\Gamma_D}(\mathrm{div}\,;\Omega_D)\times W^{\frac{1}{q},p}(\Sigma)$, when $1 < p < 2$, and $\dfrac{1}{q}+\dfrac{1}{p} = 1$. In fact, we first note that $\mathbf{v}\cdot \mathbf{n} \in W^{-\frac{1}{q},q}(\partial\Omega_D) \quad \forall \mathbf{v} \in \mathbf{W}^{0,q}(\mathrm{div}\,;\Omega_D)$, and that the condition $\mathbf{v}\cdot \mathbf{n} = 0$ on Γ_D, is understood in the sense

$$\Big\langle \mathbf{v}\cdot \mathbf{n}, E^0_\Sigma(\xi)\Big\rangle_{\partial\Omega_D} = 0 \qquad \forall \xi \in \widetilde{W}^{\frac{1}{q},p}(\Gamma_D),$$

where $\langle\cdot\,,\cdot\rangle_{\partial\Omega_D}$ is the duality parity between $W^{-\frac{1}{q},q}(\partial\Omega_D)$ and $W^{\frac{1}{q},p}(\partial\Omega_D)$, and E^0_Σ is the extension by zero on Σ. Note here, according to Theorem .3.2 given below in Chapter 6, that $\widetilde{W}^{\frac{1}{q},p}(\Gamma_D)$ is identified with $W^{\frac{1}{q},p}(\Gamma_D)$, and therefore the previous condition is equivalent to

$$\Big\langle \mathbf{v}\cdot \mathbf{n}, E^0_\Sigma(\xi)\Big\rangle_{\partial\Omega_D} = 0 \qquad \forall \xi \in W^{\frac{1}{q},p}(\Gamma_D).$$

In this way, and thanks to Corollary .4.1, it suffices to consider

$$\langle \mathbf{v}\cdot \mathbf{n}, \xi\rangle_\Sigma := \Big\langle \mathbf{v}\cdot \mathbf{n}, E^0_{\Gamma_D}(\xi)\Big\rangle_{\partial\Omega_D} \qquad (\mathbf{v},\xi) \in \mathbf{W}^{0,q}_{\Gamma_D}(\mathrm{div}\,;\Omega_D)\times W^{\frac{1}{q},p}(\Sigma).$$

2.3.3 Transmission conditions

The transmission conditions considered here are the same as in [17] and [25], but the corresponding spaces are different. Therefore, it is necessary to clarify in what sense they will be imposed. More precisely, we begin by testing the conservation mass condition (2.13) with an arbitrary function $\xi \in W^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, and testing the constraints on the traction (2.14) with an arbitrary function $\boldsymbol{\psi} \in \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma)$, which yields:

find $(\boldsymbol{\varphi}, \mathbf{u}_D, \lambda, \boldsymbol{\sigma}_S) \in \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma)\times \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div}\,;\Omega_D)\times W^{\frac{1}{3},\frac{3}{2}}(\Sigma)\times \mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ such that:

$$\Big\langle \boldsymbol{\varphi}\cdot \mathbf{n},\ \xi\Big\rangle_\Sigma + \Big\langle \mathbf{u}_D\cdot \mathbf{n},\ \xi\Big\rangle_\Sigma = 0 \qquad \forall \xi \in W^{\frac{1}{3},\frac{3}{2}}(\Sigma),$$

$$\Big\langle \boldsymbol{\sigma}_S\mathbf{n},\ \boldsymbol{\psi}\Big\rangle_\Sigma + \Big\langle \boldsymbol{\psi}\cdot \mathbf{n},\ \lambda\Big\rangle_\Sigma - \mu k_f^{-1}\Big\langle \boldsymbol{\psi}\cdot \mathbf{t},\ \boldsymbol{\varphi}\cdot \mathbf{t}\Big\rangle_\Sigma = 0 \qquad \forall \boldsymbol{\psi} \in \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma).$$

We now observe that the duality pairing $\Big\langle \boldsymbol{\psi}\cdot \mathbf{n},\ \lambda\Big\rangle_\Sigma$ is well-defined $\forall(\boldsymbol{\psi},\lambda) \in \widetilde{\mathbf{W}}^{\frac{1}{2},2}(\Sigma)\times W^{\frac{1}{3},\frac{3}{2}}(\Sigma)$. Indeed, according to the trace theorem there exists $\widehat{\boldsymbol{\psi}} \in \mathbf{W}^{1,2}_{\Gamma_S}(\Omega_S)$ and $C > 0$ such that

$$\gamma_0\left(\widehat{\boldsymbol{\psi}}\right)\Big|_\Sigma = \boldsymbol{\psi} \quad \text{and} \quad \left\|\widehat{\boldsymbol{\psi}}\right\|_{1,\Omega_S} \le C\,\|\boldsymbol{\psi}\|_{\frac{1}{2},\Sigma}.$$

On the other hand, according to the continuous injection $i : \mathbf{W}^{1,2}(\Omega_S) \to \mathbf{L}^p(\Omega_S)$ for $p > 2$ (cf. [1, Theo. 5.4 (6)]), we have that $\widehat{\boldsymbol{\psi}} \in \mathbf{L}^p(\Omega_S)$, which together with the fact that $\mathrm{div}\,\widehat{\boldsymbol{\psi}} \in L^2(\Omega_S)$, yields similarly as in [32, Lemma 3.15.]

$$\begin{aligned}\left\langle \boldsymbol{\psi}\cdot \mathbf{n},\ \lambda\right\rangle_{\Sigma} := \int_{\Sigma} \widehat{\boldsymbol{\psi}}\cdot \mathbf{n}\lambda &\leq \widetilde{C}\left(\left\|\widehat{\boldsymbol{\psi}}\right\|_{0,p;\Omega_S} + \left\|\operatorname{div}\widehat{\boldsymbol{\psi}}\right\|_{0,2;\Omega_S}\right)\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma}\\ &\leq \widetilde{C}\left(c_p\left\|\widehat{\boldsymbol{\psi}}\right\|_{1,\Omega_S} + \left\|\operatorname{div}\widehat{\boldsymbol{\psi}}\right\|_{0,2;\Omega_S}\right)\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma}\\ &\leq \widehat{C}_p\left\|\widehat{\boldsymbol{\psi}}\right\|_{1,\Omega_S}\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma}\\ &\leq \widehat{C}_p C\,\|\boldsymbol{\psi}\|_{\frac{1}{2},\Sigma}\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma}.\end{aligned}$$

In order to apply a generalization of the Babuska-Brezzi theory to twofold saddle point problems, in what follows we rewrite the terms conveniently, similarly as in [23] and [25], thus obtaining a system with a penalty term.

2.4 Resulting system structure

Following [23] and [25], we order the equations from the weak forms of Stokes, Darcy - Forchheimer and the transmission conditions, in a nonlinear system with a twofold saddle point structure. We introduce the problem $(\mathbf{P}_\alpha)$: find $(\underline{\sigma},\underline{u},\underline{\eta}) \in X\times Y\times Z$ such that

$$\begin{array}{lllllll}\left[\mathbb{A}(\underline{\sigma}),\underline{\tau}\right] & + & \left[\mathbb{B}_1(\underline{\tau}),\underline{u}\right] & + & \left[\mathbb{B}(\underline{\tau}),\underline{\eta}\right] & = & \left[F,\underline{\tau}\right] \quad \forall \underline{\tau}\in X,\\ \left[\mathbb{B}_1(\underline{\sigma}),\underline{v}\right] & - & \left[\mathbb{C}(\underline{u}),\underline{v}\right] & & & = & \left[G,\underline{v}\right] \quad \forall \underline{v}\in Y,\\ \left[\mathbb{B}(\underline{\sigma}),\underline{\vartheta}\right] & & & & & = & \left[E,\underline{\vartheta}\right] \quad \forall \underline{\vartheta}\in Z,\end{array} \tag{2.18}$$

where the spaces are given by

$$\begin{aligned}X &:= \mathbb{H}_0(\mathbf{div}\,;\Omega_S)\times \mathbf{W}^{0,3}_{\Gamma_D}(\operatorname{div},\Omega_D),\\ Y &:= \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma)\times W^{\frac{1}{3},\frac{3}{2}}(\Sigma),\\ Z &:= \mathbf{L}^2(\Omega_S)\times L^{\frac{3}{2}}_0(\Omega_D)\times \mathbb{L}^2_{skew}(\Omega_S),\end{aligned}$$

the nonlinear operator $\mathbb{A} : X \to X'$ is defined as:

$$\left[\mathbb{A}(\underline{\sigma}),\underline{\tau}\right] := \left[\mathcal{A}_S\left(\boldsymbol{\sigma}_S\right),\boldsymbol{\tau}_S\right]_S + \left[\mathcal{A}_D(\mathbf{u}_D),\mathbf{v_D}\right]_D \quad \forall \underline{\sigma} := (\boldsymbol{\sigma}_S,\mathbf{u}_D), \underline{\tau} := (\boldsymbol{\tau}_S,\mathbf{v}_D)\in X,$$

and the linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$ and $\mathbb{C} : Y \to Y'$ are given as follows

$$\left[\mathbb{B}(\underline{\tau}),\underline{\eta}\right] := (\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{u}_S)_S + (\boldsymbol{\tau}_S,\boldsymbol{\gamma_S})_S - (\operatorname{div}\mathbf{v}_D,p_D)_D \quad \forall \underline{\eta} := (\mathbf{u}_S,p_D,\boldsymbol{\gamma_S})\in Z,$$

$$\left[\mathbb{B}_1(\underline{\boldsymbol{\sigma}}),\underline{v}\right] := \left\langle \boldsymbol{\sigma}_S n,\ \boldsymbol{\psi}\right\rangle_{\Sigma} - \left\langle \mathbf{u}_D\cdot n,\ \xi\right\rangle_{\Sigma} \quad \forall \underline{v} := (\boldsymbol{\psi},\xi)\in Y,$$

$$\left[\mathbb{C}(\underline{u}),\underline{v}\right] := \mu k_f^{-1}\left\langle \boldsymbol{\psi}\cdot t,\ \boldsymbol{\varphi}\cdot t\right\rangle_{\Sigma} - \left\langle \boldsymbol{\psi}\cdot n,\ \lambda\right\rangle_{\Sigma} + \left\langle \boldsymbol{\varphi}\cdot n,\ \xi\right\rangle_{\Sigma} \quad \forall \underline{u} := (\boldsymbol{\varphi},\lambda)\in Y.$$

Note that $\mathbb{B}$ and $\mathbb{B}_1$ show a diagonal structure, and that $\mathbb{C}$ is positive semi-definite. In the next chapter we provide an abstract theory that allows us to analyse the solvability of (2.18).

Chapter 3

A modified abstract theory for a twofold saddle point problem

In this chapter we follow the approach from [22] and [23], and develop a new abstract theory for a twofold saddle point problem having the structure of $(\mathbf{P}_\alpha)$.

3.1 The continuous setting

Let X, Y and Z be separable and reflexive Banach spaces with duals X', Y', Z' also separable reflexive Banach spaces. Additionally bounded linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$, $\mathbb{C} : Y \to Y'$, and a non-linear operator $\mathbb{A} : X \to X'$ are defined and we also assume $\mathbb{C}$ is positive semi-definite. Therefore, given $(F, G_1, G) \in X' \times Y' \times Z'$, we are interested in the following variational problem (P_1): find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in X \times Y \times Z$ such that:

$$\begin{aligned} \Big[\mathbb{A}(\mathbf{t}), \mathbf{s}\Big] \;+\; \Big[\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}\Big] \;+\; \Big[\mathbb{B}^*(\mathbf{u}), \mathbf{s}\Big] &= \Big[F, \mathbf{s}\Big], \\ \Big[\mathbb{B}_1(\mathbf{t}), \boldsymbol{\tau}\Big] \;-\; \Big[\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}\Big] &= \Big[G_1, \boldsymbol{\tau}\Big], \\ \Big[\mathbb{B}(\mathbf{t}), \mathbf{v}\Big] &= \Big[G, \mathbf{v}\Big], \\ & \forall (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in X \times Y \times Z. \end{aligned} \tag{3.1}$$

In what follows we will to adopt the analysis developed in [22] and [23] to derive sufficient conditions under which (P_1) is well-posed. We begin by observing that in order to guarantee the existence of a unique preimage $\mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B})$ such that $\mathbb{B}(\mathbf{t}_G) = G$, we require X to be uniformly convex (cf. [35, remark A.1]). Therefore, henceforth we assume:

i) $\mathbb{B}$ is surjective, which means that there exists $\beta > 0$ such that

$$\sup_{\mathbf{t}\in X;\mathbf{t}\neq\mathbf{0}} \frac{\Big[\mathbb{B}(\mathbf{t}), \mathbf{v}\Big]}{\|\mathbf{t}\|_X} \geq \beta\|\mathbf{v}\|_Z.$$

This condition is called inf-sup condition for $\mathbb{B}$. Note that it gives the upper bound $\frac{1}{\beta}$ for the norm of the pseudoinverse $\widetilde{\mathbb{B}}^{-1}$ of $\mathbb{B}$.

ii) X is uniformly convex.

As a consequence of these assumptions, we first observe that $\mathbb{B}$ has a continuous pseudoinverse $\widetilde{\mathbb{B}}^{-1}$ (cf. [39, Lemme 1.3 B.] and [35, Lemma A.1]). In addition, from the inf-sup condition for $\mathbb{B}$ we conclude that $\mathbb{B}^*$ is injective and hence bijective onto $\mathcal{R}(\mathbb{B}^*) = {}^\circ\mathcal{N}(\mathbb{B})$. Note that $\mathcal{N}(\mathbb{B})$ is also uniformly convex. Then, the third row of (P_1) is always satisfied, so it is possible to omit this hereinafter, and finally it is possible to reduce the first row of (P_1), in the sense that it is equivalent to solve a problem with one less variable, these results follow from the lemmas shown below.

Lemma 3.1.1 *Under the previous assumptions, the following problems are equivalent*

$$(P)\begin{cases} \textit{find } (\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X\times Y\times Z \textit{ such that:} \\ \Big[\mathbb{A}(\mathbf{t}), \mathbf{s}\Big] + \Big[\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}\Big] + \Big[\mathbb{B}^*(\mathbf{u}), \mathbf{s}\Big] = \Big[F, \mathbf{s}\Big], \\ \forall \mathbf{s} \in X, \end{cases} \qquad (\widetilde{P})\begin{cases} \textit{find } (\mathbf{t},\boldsymbol{\sigma}) \in X\times Y \textit{ such that} \\ \Big[\mathbb{A}(\mathbf{t}), \mathbf{s}_0\Big] + \Big[\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\Big] = \Big[F, \mathbf{s}_0\Big], \\ \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B}) = \mathcal{R}(\mathbb{B}^*)^\circ. \end{cases}$$

More precisely, if $(\mathbf{t},\boldsymbol{\sigma}) \in X\times Y$ *is a solution for* $(\widetilde{P})$*, and we define* $\mathbf{u}\in Z$ *as the unique solution of the following problem:*

$$(\widehat{P})\begin{cases} \textit{find } \mathbf{u} \in Z \textit{ such that:} \\ \Big[\mathbb{B}^*(\mathbf{u}), \mathbf{s}\Big] = \Big[F - \Big(\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})\Big), \mathbf{s}\Big], \\ \forall \mathbf{s} \in X, \end{cases}$$

then $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ *is a solution of* (P)*. Conversely, if* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in X\times Y\times Z$ *is a solution of* (P)*, then* $(\mathbf{t},\boldsymbol{\sigma})$ *is a solution of* $(\widetilde{P})$ *and* $\mathbf{u}$ *is solution of* $(\widehat{P})$*.*

Proof : Given $(\mathbf{t},\boldsymbol{\sigma}) \in X\times Y$ a solution of $(\widetilde{P})$, we first see that $(\widehat{P})$ has a unique solution. In fact, if $(\mathbf{t},\boldsymbol{\sigma})$ is a solution of $(\widetilde{P})$, then $F - \Big(\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})\Big) \in {}^\circ\mathcal{N}(\mathbb{B}) = \mathcal{R}(\mathbb{B}^*)$, and therefore there exists a unique $\mathbf{u} \in Z$ such that $\mathbb{B}^*(\mathbf{u}) = F - \Big(\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})\Big)$ which is equivalent to stating $\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma}) + \mathbb{B}^*(\mathbf{u}) = F$, i.e., $\mathbf{u}$ is a solution of $(\widehat{P})$ and $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ is solution of (P). The converse implication follows by taking as a particular case $s \in \mathcal{N}(\mathbb{B}) = \mathcal{R}(\mathbb{B}^*)^\circ$, and by using that given $(\mathbf{t}, \boldsymbol{\sigma}) \in X \times Y$ there is a unique $\mathbf{u} \in Z$ solution to $(\widehat{P})$.

Lemma 3.1.2 *Under the previous assumptions. The problem (P_1) (cf. 3.1) is equivalent to:*

$$(P_2)\begin{cases} \textit{Given } \mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B}) \textit{such that: } \mathbb{B}(\mathbf{t}_G) = G, \\ \textit{find } (\mathbf{t}_0, \boldsymbol{\sigma}) \in \mathcal{N}(\mathbb{B}) \times Y \textit{such that:} \\ \Big[\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{s}_0\Big] + \Big[\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\Big] = \Big[F, \mathbf{s}_0\Big], \\ \Big[\mathbb{B}_1(\mathbf{t}_0 + \mathbf{t}_G), \boldsymbol{\tau}\Big] - \Big[\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}\Big] = \Big[G_1, \boldsymbol{\tau}\Big], \\ \qquad \forall (\mathbf{s}_0, \boldsymbol{\tau}) \in \mathcal{N}(\mathbb{B}) \times Y. \end{cases}$$

More precisely, given $\mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B})$ such that $\mathbb{B}(\mathbf{t}_G) = G$ (which exist by the surjectivity of $\mathbb{B}$), and given $(\mathbf{t}_0, \boldsymbol{\sigma})$ a solution of (P_2), we have that $(\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\sigma}, \mathbf{u})$ is a solution of (P_1), where, according to Lemma 3.1.1, $\mathbf{u}$ is the unique element in Z such that

$$\Big[\mathbb{B}^*(\mathbf{u}), \mathbf{s}\Big] = \Big[F - \Big(\mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma})\Big), \mathbf{s}\ \Big] \quad \forall s \in X.$$

Conversely, given $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ a solution of (P_1), we let $\mathbf{t}_0 := \mathbf{t} - \mathbf{t}_G$, where is the unique vector in $X \setminus \mathcal{N}(\mathbb{B})$ such that $\mathbb{B}(\mathbf{t}_G) = G$, and then observe that $(\mathbf{t}_0, \boldsymbol{\sigma})$ is a solution of (P_2).

Proof: It follows from the previous analysis.

Corollary 3.1.3 *Under the previous assumptions, the problem (P_1) has unique solution if and only if the problem (P_2) has a unique solution.*

Proof: It follows from Lemma 3.1.2.

According to the above analysis, our next goal is to study the solvability of (P_2), for which we follow the approach from [23]. To this end, we first need to show the well-posedness of the problem

$$(\mathcal{Q})\begin{cases} \text{Given } \mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B}) \text{ and } \boldsymbol{\sigma} \in Y, \\ \text{find } \mathbf{t}_0 \in \mathcal{N}(\mathbb{B}) \text{ such that:} \\ \Big[\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{s}_0\Big] = \Big[F - \mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\Big], \\ \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B}). \end{cases}$$

Indeed, hereafter we assume:

(A_0) $\mathbb{A} : X \mapsto X^{'}$ is bounded for bounded subsets of X. More precisely, there exist constants $\gamma_1, \gamma_2 > 0$, $\varsigma_1, \varsigma_2 \geq 0$, $r_1, r_2 \geq 2$, depending only on the domain (and possibly on physical parameters involved), such that

$$\left\|\mathbb{A}(\mathbf{s}_1, \mathbf{s}_2) - \mathbb{A}(\mathbf{v}_1, \mathbf{v}_2)\right\|_{X'} \leq \sum_{j=1}^{2}\left\{\varsigma_j \left\|\mathbf{s}_j - \mathbf{v}_j\right\|_{X_j} + \gamma_j \left\|\mathbf{s}_j - \mathbf{v}_j\right\|_{X_j} \left(\left\|\mathbf{s}_j\right\|_{X_j} + \left\|\mathbf{v}_j\right\|_{X_j}\right)^{r_j-2}\right\}$$

for all $(\mathbf{s}_1, \mathbf{s}_2), (\mathbf{v}_1, \mathbf{v}_2) \in X := X_1 \times X_2$.

(A_1) $\mathbb{A}\left(\cdot + \mathbf{t}_G\right) : \mathcal{N}(\mathbb{B}) \mapsto \mathcal{N}(\mathbb{B})^{'}$ is a strictly monotone mapping. More precisely, there exists $\alpha > 0$, independent of $\mathbf{t}_G$, such that:

$$\Big[\mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{v} + \mathbf{t}_G), \mathbf{s} - \mathbf{v}\Big] \geq \alpha\Big(\|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1}^{r_1} + \|\mathbf{s}_2 - \mathbf{v}_2\|_{X_2}^{r_2}\Big)$$

for all $\mathbf{s} = (\mathbf{s}_1, \mathbf{s}_2), \mathbf{v} = (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{N}(\mathbb{B}) \subseteq X$.

(A_2) $\mathbb{A}(\cdot + \mathbf{t}_G)$ is hemi-continuous, i.e., given $\mathbf{t}, \mathbf{v} \in X \setminus \mathcal{N}(\mathbb{B})$,

$$\begin{aligned} G : \mathbb{R} &\mapsto \mathbb{R} \\ x &\mapsto G(x) := \Big\langle \mathbb{A}(\mathbf{t} + x\mathbf{v} + \mathbf{t}_G), \mathbf{v} \Big\rangle \end{aligned}$$

is a continuous map.

(A_3) X_1 and X_2 are uniformly convex and separable Banach spaces.

(A_4) there exists $\beta_1 > 0$ such that

$$\sup_{\mathbf{s}_0 \in \mathcal{N}(\mathbb{B}); \mathbf{s}_0 \neq \mathbf{0}} \frac{\Big[\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\Big]}{\|\mathbf{s}_0\|_X} \geq \beta_1 \|\boldsymbol{\sigma}\|_Y \qquad \forall \boldsymbol{\sigma} \in Y.$$

Corollary 3.1.4 *Under the hypotheses* $(A_0) - (A_2)$ *the problem* $(\mathcal{Q})$ *has a unique solution, i.e., given* $\mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B})$ *and* $\boldsymbol{\sigma} \in Y$*, there exists a unique* $\mathbf{t}_0 \in \mathcal{N}(\mathbb{B})$ *such that:*

$$\Big[\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{s}_0\Big] = \Big[F - \mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0\Big] \qquad \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B}).$$

Proof: According to [40], the hypotheses $(A_0) - (A_2)$ implies the biyectivity of $\mathbb{A}(\cdot + \mathbf{t}_G)$.

In what follows, given $\boldsymbol{\tau} \in Y$, we define $\mathbf{t}_0(\boldsymbol{\tau})$ as the unique element in $\mathcal{N}(\mathbb{B})$ such that:

$$\Big[\mathbb{A}\Big(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G\Big), \mathbf{s}_0\Big] = \Big[F - \mathbb{B}_1^*(\boldsymbol{\tau}), \mathbf{s}_0\Big] \qquad \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B}). \tag{3.2}$$

In light of the foregoing, problem (P_2) is equivalent to

$$(P_3)\begin{cases} \text{Given } \mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B}) \text{ such that } \ \mathbb{B}(\mathbf{t}_G) = G, \\ \quad \text{find } \boldsymbol{\sigma} \in Y \text{ such that:} \\ \quad \Big[\mathbb{T}(\boldsymbol{\sigma}), \boldsymbol{\tau}\Big] := \Big[-\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\sigma})), \boldsymbol{\tau}\Big] + \Big[\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}\Big] = \Big[\widetilde{G}_1, \boldsymbol{\tau}\Big] \ \ \forall \boldsymbol{\tau} \in Y, \\ \text{where } \Big[\widetilde{G}_1, \boldsymbol{\tau}\Big] := \Big[\mathbb{B}_1(\mathbf{t}_G) - G_1, \boldsymbol{\tau}\Big] \ \ \forall \boldsymbol{\tau} \in Y. \end{cases} \tag{3.3}$$

Therefore, we now focus on proving the injectivity and surjectivity of $\mathbb{T}$. To simplify the analysis note that we have the following identities

$$\Big[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{s}_0\Big] = \Big[\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s}_0\Big] \ \ \forall \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y, \ \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B}), \tag{3.4}$$

particularly

$$\Big[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\Big] = \Big[\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\Big] = \Big[\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1, \mathbb{B}_1\big(\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\tag{3.5}$$

The next identity gives a bound for $\|\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\|_X$ in terms of $\|\mathbb{B}_1^*(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2)\|$.

We now recall that $X := X_1 \times X_2$ and descompose $\mathbf{t}_0(\boldsymbol{\tau}_1)$ and $\mathbf{t}_0(\boldsymbol{\tau}_2)$ as:

$$\mathbf{t}_0(\boldsymbol{\tau}_1) = \mathbf{s} := (s_1, s_2), \ \ \mathbf{t}_0(\boldsymbol{\tau}_2) = \mathbf{v} := (v_1, v_2) \in X_1 \times X_2,$$

whence (3.5) can be rewritten as

$$\Big[\mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{v} + \mathbf{t}_G), \mathbf{s} - \mathbf{v}\Big] = \Big[\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s} - \mathbf{v}\Big] \leq \|\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\| \, \|\mathbf{s} - \mathbf{v}\| \,.$$

Then, according to the monotonicity of $\mathbb{A}$ (cf. (A_1)), we obtain

$$\alpha\Big\{\|s_1 - v_1\|_{X_1}^{r_1} + \|s_2 - v_2\|_{X_2}^{r_2}\Big\} \leq \Big[\mathbb{A}(\mathbf{s} + \mathbf{t}_G) - \mathbb{A}(\mathbf{v} + \mathbf{t}_G), \mathbf{s} - \mathbf{v}\Big] = \Big[\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1), \mathbf{s} - \mathbf{v}\Big] \leq \|\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\| \, \|\mathbf{s} - \mathbf{v}$$

that is

$$\alpha\Big\{\|s_1 - v_1\|_{X_1}^{r_1} + \|s_2 - v_2\|_{X_2}^{r_2}\Big\} \leq \|\mathbb{B}_1^*(\boldsymbol{\tau}_2 - \boldsymbol{\tau}_1)\| \, \Big(\|s_1 - v_1\|_{X_1} + \|s_2 - v_2\|_{X_2}\Big),$$

which yields

$$\|\mathbf{s}-\mathbf{v}\|_X = \|s_1 - v_1\|_{X_1} + \|s_2 - v_2\|_{X_2} \leq 2\max\left\{ \left(\frac{2}{\alpha}\,\|\mathbb{B}_1^*(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)\|\right)^{\frac{1}{r_1-1}}, \left(\frac{2}{\alpha}\,\|\mathbb{B}_1^*(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)\|\right)^{\frac{1}{r_2-1}} \right\}$$

or equivalently

$$\|\mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\|_X \leq 2\max\left\{ \left(\frac{2}{\alpha}\,\|\mathbb{B}_1^*(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)\|\right)^{\frac{1}{r_1-1}}, \left(\frac{2}{\alpha}\,\|\mathbb{B}_1^*(\boldsymbol{\tau}_2-\boldsymbol{\tau}_1)\|\right)^{\frac{1}{r_2-1}} \right\}. \tag{3.6}$$

Lemma 3.1.5 $\mathbb{T}$ *is injective.*

Proof: Let $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$ such that $\mathbb{T}(\boldsymbol{\tau}_1) = \mathbb{T}(\boldsymbol{\tau}_2)$. It follows that $\Big[\mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}\Big] = 0 \ \ \forall \boldsymbol{\tau} \in Y$, which, according to the definition of the operator $\mathbb{T}$ (cf. (3.3)), and taking in particular $\boldsymbol{\tau} = \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1$, gives

$$\Big[\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)), \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1\Big] + \Big[\mathbb{C}(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2), \boldsymbol{\tau}_2 - \boldsymbol{\tau}_1\Big] = 0.$$

The foregoing equation and the fact that $\mathbb{C}$ is positive semi-definite allow to deduce that

$$0 \leq \Big[\mathbb{B}_1(\mathbf{t}_0(\boldsymbol{\sigma}_2) - \mathbf{t}_0(\boldsymbol{\sigma}_1)), \boldsymbol{\sigma}_2 - \boldsymbol{\sigma}_1\Big],$$

which, thanks to the identity (3.5), yields

$$\Big[\mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}_1) - \mathbf{t}_0(\boldsymbol{\tau}_2)\Big] \leq 0. \tag{3.7}$$

Then, applying the strict monotonicity of $\mathbb{A}$ (cf. (A_1)), we deduce from (3.7) that $\mathbf{t}_0(\boldsymbol{\tau}_1) = \mathbf{t}_0(\boldsymbol{\tau}_2)$, and hence (3.2) implies $\Big[\mathbb{B}_1^*(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2), \mathbf{s}_0\Big] = 0 \ \ \forall \mathbf{s}_0 \in \mathcal{N}(\mathbb{B})$. Finally, the inf-sup condition given in (A_4) confirms that $\boldsymbol{\tau}_1 = \boldsymbol{\tau}_2$, which completes the proof.

Next, we show the surjectivity of $\mathbb{T}$ by applying clasical results from nonlinear functional analysis. More precisely, in what follows we show that under the hypotheses assumed for the solvability of (P_2), the operator $\mathbb{T}$ is continuous, monotone, bounded and coercive (the first two properties imply that $\mathbb{T}$ is of type M), and hence, thanks to [40, Corollary 2.2], $\mathbb{T}$ is surjective.

Lemma 3.1.6 $\mathbb{T}$ *is continuous.*

Proof: Let $\{\boldsymbol{\tau}_n\}_{n\in\mathbb{N}} \subseteq Y$ and $\boldsymbol{\tau} \in Y$ such that $\|\boldsymbol{\tau}_n - \boldsymbol{\tau}\|_Y \xrightarrow{n} 0$. Thus, from the definition of $\mathbb{T}$, we have

$$\begin{aligned}\Big\|\mathbb{T}(\boldsymbol{\tau}_n) - \mathbb{T}(\boldsymbol{\tau})\Big\|_{Y'} &= \Big\|\mathbb{B}_1\Big(\mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\boldsymbol{\tau}_n)\Big) + \mathbb{C}(\boldsymbol{\tau}_n - \boldsymbol{\tau})\Big\|_{Y'} \\ &\leq \|\mathbb{B}_1\|\|\mathbf{t}_0(\boldsymbol{\tau}_n) - \mathbf{t}_0(\boldsymbol{\tau})\|_X + \|\mathbb{C}\|\|\boldsymbol{\tau}_n - \boldsymbol{\tau}\|_Y,\end{aligned}$$

The foregoing inequality and the identity (3.6) imply that $\|\mathbb{T}(\boldsymbol{\tau}_n) - \mathbb{T}(\boldsymbol{\tau})\|_{Y'} \xrightarrow{n} 0$.

Lemma 3.1.7 $\mathbb{T}$ *is monotone.*

Proof: Given $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \in Y$, we have according to the definition of $\mathbb{T}$,

$$\frac{\Big\langle \mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|} = \frac{\Big\langle \mathbb{B}_1\Big(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)\Big), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|} + \frac{\Big\langle \mathbb{C}(\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|}.$$

Noting that $\mathbb{C}$ is positive semi-definite, we discard the last term in the above equation, getting

$$\frac{\Big\langle \mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|} \geq \frac{\Big\langle \mathbb{B}_1\Big(\mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1)\Big), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|},$$

which according to the identity (3.5) and the strict monotonicity of $\mathbb{A}$, yields

$$\frac{\Big\langle \mathbb{T}(\boldsymbol{\tau}_1) - \mathbb{T}(\boldsymbol{\tau}_2), \boldsymbol{\tau}_1 - \boldsymbol{\tau}_2 \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|} \geq \frac{\Big\langle \mathbb{A}\left(\mathbf{t}_0(\boldsymbol{\tau}_2) + \mathbf{t}_G\right) - \mathbb{A}\left(\mathbf{t}_0(\boldsymbol{\tau}_1) + \mathbf{t}_G\right), \mathbf{t}_0(\boldsymbol{\tau}_2) - \mathbf{t}_0(\boldsymbol{\tau}_1) \Big\rangle}{\|\boldsymbol{\tau}_1 - \boldsymbol{\tau}_2\|} \geq 0,$$

thus proving that $\mathbb{T}$ is monotone.

From the previous Lemmas, we have that $\mathbb{T}$ is of type M (cf. [40, Lemma 2.1]), hence, in order to conclude that $\mathbb{T}$ is bijective, it remains to show that $\mathbb{T}$ is bounded and coercive.

Lemma 3.1.8 $\mathbb{T}$ *is bounded.*

Proof: Let $\boldsymbol{\tau} \in Y$. According to the triangle inequality and the definition of $\mathbb{T}$ (cf. (3.3)), we obtain

$$\|\mathbb{T}(\boldsymbol{\tau})\|_{Y'} \leq \|\mathbb{T}(\boldsymbol{\tau}) - T(\mathbf{0})\|_{Y'} + \|\mathbb{T}(\mathbf{0})\|_{Y'} \leq \|\mathbb{B}_1\|\|\mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0})\|_X + \|\mathbb{C}\|\|\boldsymbol{\tau}\|_Y + \|\mathbb{B}_1(\mathbf{t}_0(\mathbf{0}))\|.$$

In turn, thanks to the identity (3.6), we have

$$\|\mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0})\| \leq 2\max\left(\left(\frac{2}{\alpha}\|\mathbb{B}_1^*(\boldsymbol{\tau})\|\right)^{\frac{1}{r_1-1}}, \left(\frac{2}{\alpha}\|\mathbb{B}_1^*(\boldsymbol{\tau})\|\right)^{\frac{1}{r_2-1}}\right),$$

and from the foregoing inequalities we conclude that $\mathbb{T}$ is bounded.

Lemma 3.1.9 $\mathbb{T}$ *is coercive.*

Proof: Let $\boldsymbol{\tau} \in Y$. Similarly as in Lemma 3.1.7, we have

$$\frac{\Big\langle \mathbb{T}(\boldsymbol{\tau}), \boldsymbol{\tau} \Big\rangle}{\|\boldsymbol{\tau}\|} \geq \frac{\Big\langle -\mathbb{B}_1(\boldsymbol{\tau}), \boldsymbol{\tau} \Big\rangle}{\|\boldsymbol{\tau}\|} = \frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - F, \mathbf{t}_0(\boldsymbol{\tau}) \Big\rangle}{\|\boldsymbol{\tau}\|} = \frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}) \Big\rangle}{\|\boldsymbol{\tau}\|}$$

$$= \frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0}) \Big\rangle}{\|\boldsymbol{\tau}\|} + \frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\mathbf{0}) \Big\rangle}{\|\boldsymbol{\tau}\|}. \tag{3.8}$$

Next, we show that (3.8) diverges when $\|\boldsymbol{\tau}\| \to \infty$. In fact, we prove that the left term on (3.8) diverges and its right term is bounded, when $\|\boldsymbol{\tau}\| \to \infty$. We begin by observing, thanks to the $\inf - \sup$ condition for $\mathbb{B}_1$ ($cf.$ (A_4)), and the identity (3.4), that

$$\beta_1 \|\boldsymbol{\tau}\|_Y \leq \Big\| \mathbb{A}\big(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G\big) - \mathbb{A}\big(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G\big) \Big\| \qquad \forall \boldsymbol{\tau} \in Y. \tag{3.9}$$

Now, we set $\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G =: (s_1, s_2)$ and $(v_1, v_2) =: \mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G$. Then, applying the boundedness property of $\mathbb{A}$ ($cf.$ (A_0)), the triangle inequality, and the fact that

$$(a+b)^p \leq 2^{p-1}\Big(a^p + b^p\Big) \qquad \forall a, b \geq 0,\ p \geq 1$$

(cf. [1, Lemma 2.24]), we get:

$$\beta_1 \|\boldsymbol{\tau}\| \leq \|\mathbb{A}(s_1, s_2) - \mathbb{A}(v_1, v_2)\|_{X'} \leq \sum_{j=1}^{2} \left\{ \varsigma_j \|s_j - v_j\|_{X_j} + \gamma_j \|s_j - v_j\|_{X_j} \Big(\|s_j\|_{X_j} + \|v_j\|_{X_j} \Big)^{r_j - 2} \right\}$$

$$\leq \sum_{j=1}^{2} \left\{ \varsigma_j \|s_j - v_j\|_{X_j} + 2^{r_j - 3} \gamma_j \|s_j - v_j\|_{X_j}^{r_j - 1} + 2^{r_j - 2} \gamma_j \|s_j - v_j\|_{X_j} \|v_j\|_{X_j}^{r_j - 2} \right\}. \tag{3.10}$$

Then, it follows that:

$$\|\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G\|_X = \|s_1\|_{X_1} + \|s_2\|_{X_2} \to \infty, \text{ when } \|\boldsymbol{\tau}\|_Y \to \infty.$$

Thus, rewriting the left term of (3.8) in terms of (s_1, s_2) and (v_1, v_2), and applying the strict monotonicity property of $\mathbb{A}$ (cf. (A_1)) and the inequality (3.10), we find that

$$\frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\tau}) - \mathbf{t}_0(\mathbf{0}) \Big\rangle}{\|\boldsymbol{\tau}\|} = \frac{\Big\langle \mathbb{A}((s_1, s_2)) - \mathbb{A}((v_1, v_2)), (s_1 - v_1, s_2 - v_2) \Big\rangle}{\|\boldsymbol{\tau}\|}$$

$$\geq \frac{\beta_1 \alpha \Big(\|s_1 - v_1\|_{X_1}^{r_1} + \|s_2 - v_2\|_{X_2}^{r_2} \Big)}{\sum_{j=1}^{2} \Big\{ \varsigma_j \|s_j - v_j\|_{X_j} + 2^{r_j - 3} \gamma_j \|s_j - v_j\|_{X_j}^{r_j - 1} + 2^{r_j - 2} \gamma_j \|s_j - v_j\|_{X_j} \|v_j\|_{X_j}^{r_j - 2} \Big\}},$$

which tends to infinity when $\|\mathbf{s}_1\|_{X_1} + \|\mathbf{s}_2\|_{X_2} \to \infty$, since (v_1, v_2) is fixed (not dependent of $\boldsymbol{\tau}$) and $r_1, r_2 \geq 2$. On the other hand, for the right term of (3.8) it suffices to observe

$$\frac{\Big\langle \mathbb{A}(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G) - \mathbb{A}(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G), \mathbf{t}_0(\mathbf{0}) \Big\rangle}{\|\boldsymbol{\tau}\|} \geq -\beta_1 \|\mathbf{t}_0(\mathbf{0})\| \,,$$

and the proof follows from (3.9). Thus, $\mathbb{T}$ is coercive.

According to the foregoing analysis, we conclude that $\mathbb{T}$ is injective and surjective, i.e., problem (P_3) has a unique solution $\boldsymbol{\tau}$. Thus, according to Corollary 3.1.4, $(\mathbf{t}_0(\boldsymbol{\tau}), \boldsymbol{\tau})$ is the unique solution of (P_2), and thanks to Lemma 3.1.1, there exists a unique $\mathbf{u} \in Z$ such that $(\mathbf{t}_0(\boldsymbol{\tau}) + \mathbf{t}_G, \boldsymbol{\tau}, \mathbf{u})$ is the unique solution to Problem (P_1).

Next, we show the a priori bound for the solution of (P_1) by establishing this result first for (P_2). Recall that the latter consists of

$$(P_2) \begin{cases} \text{Given } \mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B}) \text{ such that } \mathbb{B}(\mathbf{t}_G) = G, \\ \quad \text{find } (\mathbf{t}_0, \boldsymbol{\sigma}) \in \mathcal{N}(\mathbb{B}) \times Y \text{ such that:} \\ \qquad [\mathbb{A}(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{s}_0] \qquad + [\mathbb{B}_1^*(\boldsymbol{\sigma}), \mathbf{s}_0] \;=\; [F, \mathbf{s}_0]\,, \\ \qquad [\mathbb{B}_1(\mathbf{t}_0 + \mathbf{t}_G), \boldsymbol{\tau}] \qquad - [\mathbb{C}(\boldsymbol{\sigma}), \boldsymbol{\tau}] \;=\; [G_1, \boldsymbol{\tau}]\,, \\ \forall (\mathbf{s}_0, \boldsymbol{\tau}) \in \mathcal{N}(\mathbb{B}) \times Y. \end{cases}$$

We begin by observing, thanks to (3.6), that it suffices to bound $\|\boldsymbol{\sigma}\|_Y$. Proceeding similarly as in [23, Lemma 2.1-(2.6)], using the positive semi-definite hypothesis over $\mathbb{C}$, and according to the notation introduced in (3.2) and (3.3), we have

$$\Big[\mathbb{A}\Big(\mathbf{t}_0(\boldsymbol{\sigma}) + \mathbf{t}_G\Big) - \mathbb{A}\Big(\mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G\Big), \mathbf{t}_0(\boldsymbol{\sigma}) - \mathbf{t}_0(\mathbf{0})\Big] \leq [\mathbb{T}(\boldsymbol{\sigma}) - \mathbb{T}(\mathbf{0}), \boldsymbol{\sigma}] = [\widetilde{G}_1 - \mathbb{T}(\mathbf{0}), \boldsymbol{\sigma}].$$

Now, we set $(\mathbf{s}_1, \mathbf{s}_2) =: \mathbf{t}_0 + \mathbf{t}_G$ and $(\mathbf{v}_1, \mathbf{v}_2) =: \mathbf{t}_0(\mathbf{0}) + \mathbf{t}_G$. Then, according to the strict monotonicity of $\mathbb{A}$ (*cf.* (A_1)), we have

$$\alpha\Big\{\|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1}^{r_1} + \|\mathbf{s}_2 - \mathbf{v}_2\|_{X_2}^{r_2}\Big\} \leq \Big\|\widetilde{G}_1 - \mathbb{T}(\mathbf{0})\Big\| \, \|\boldsymbol{\sigma}\|_Y \,,$$

which implies that

$$\|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1} \leq \mathcal{M}^{1/r_1} \, \|\boldsymbol{\sigma}\|_Y^{1/r_1} \quad \text{and} \quad \|\mathbf{s}_2 - \mathbf{v}_2\|_{X_2} \leq \mathcal{M}^{1/r_2} \, \|\boldsymbol{\sigma}\|_Y^{1/r_2} \,, \tag{3.11}$$

where $\mathcal{M} := \frac{1}{\alpha}\left\|\widetilde{G}_1 - \mathbb{T}(\mathbf{0})\right\|_{Y'}$. On the other hand, using (3.9) and the boundedness of $\mathbb{A}$ (cf. (A_0)), we find that

$$\begin{aligned}\beta_1\|\boldsymbol{\sigma}\|_Y &\leq \|\mathbb{A}(\mathbf{t}_0+\mathbf{t}_G)-\mathbb{A}(\mathbf{t}_0(\mathbf{0})+\mathbf{t}_G)\| = \|\mathbb{A}(\mathbf{s}_1,\mathbf{s}_2)-\mathbb{A}(\mathbf{v}_1,\mathbf{v}_2)\| \\ &\leq \sum_{j=1}^{2}\left\{\varsigma_j\|\mathbf{s}_j-\mathbf{v}_j\|_{X_j} + 2^{r_j-3}\gamma_j\|\mathbf{s}_j-\mathbf{v}_j\|_{X_j}^{r_j-1} + 2^{r_j-2}\gamma_j\|\mathbf{s}_j-\mathbf{v}_j\|_{X_j}\|\mathbf{v}_j\|_{X_j}^{r_j-2}\right\},\end{aligned}$$

which, applying (3.11), gives

$$\beta_1\|\boldsymbol{\sigma}\|_Y \leq \sum_{j=1}^{2}\left\{\left(\varsigma_j\mathcal{M}^{\frac{1}{r_j}} + \gamma_j 2^{r_j-2}\|\mathbf{v}_j\|_{X_j}^{r_j-2}\right)\|\boldsymbol{\sigma}\|_Y^{\frac{1}{r_j}} + \left(\gamma_j 2^{r_j-3}\mathcal{M}^{\frac{r_j-1}{r_j}}\right)\|\boldsymbol{\sigma}\|_Y^{\frac{r_j-1}{r_j}}\right\},$$

and hence, by Young's inequality (cf. [1]), we conclude

$$\|\boldsymbol{\sigma}\|_Y \leq \sum_{j=1}^{2}\left\{c_j(r_j,\beta_1)\left(\varsigma_j\mathcal{M}^{\frac{1}{r_j}} + \gamma_j 2^{r_j-2}\|\mathbf{v}_j\|_{X_j}^{r_j-2}\right)^{r_j'} + \widehat{\gamma}_j(r_j,\beta_1)\mathcal{M}^{r_j-1}\right\},$$

which constitutes the a priori bound for (P_2).

Next, let $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})$ be the solution of (P_1). We known that $\mathbf{t}=\mathbf{t}_0+\mathbf{t}_G$, where $(\mathbf{t}_0,\boldsymbol{\sigma})$ is the solution of (P_2). Then according to the inf-sup condition for $\mathbb{B}$ (cf. (i)), we have

$$\|\mathbf{t}_G\|_X \leq \frac{1}{\beta}\|G\| \quad \text{and} \quad \|\mathbf{u}\|_Y \leq \frac{1}{\beta}\left\|F-\mathbb{A}(\mathbf{t})-\mathbb{B}_1^*(\boldsymbol{\sigma})\right\|_{X'},$$

and hence, the boundedness of the solution of (P_1) follows from the boundedness of the solution of (P_2).

We summarize the foregoing results in the following theorem.

Theorem 3.1.10 *Let X_1, X_2, Y and Z be separable and reflexive Banach spaces and let $X = X_1\times X_2$. In addition, we consider bounded linear operators $\mathbb{B}: X\to Z'$, $\mathbb{B}_1: X\to Y'$ and $\mathbb{C}: Y\to Y'$, and a non-linear operator $\mathbb{A}: X\to X'$. We assume that $\mathbb{C}$ is positive semi-definite. Then, given $(F,G_1,G)\in(X',Y',Z')$, we define the variational problem (P_1) as follows: find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\in X\times Y\times Z$ such that:*

$$\begin{array}{llllll} [\mathbb{A}(\mathbf{t}),\mathbf{s}] & +[\mathbb{B}_1^*(\boldsymbol{\sigma}),\mathbf{s}] & +[\mathbb{B}^*(\mathbf{u}),\mathbf{s}] & = & [F,\mathbf{s}] & \forall\mathbf{s}\in X,\\ [\mathbb{B}_1(\mathbf{t}),\boldsymbol{\tau}] & -[\mathbb{C}(\boldsymbol{\sigma}),\boldsymbol{\tau}] & & = & [G_1,\boldsymbol{\tau}] & \forall\boldsymbol{\tau}\in Y,\\ [\mathbb{B}(\mathbf{t}),q] & & & = & [G,q] & \forall q\in Z.\end{array} \tag{3.12}$$

Assume that

i) there exists $\beta > 0$ *such that*

$$\sup_{\mathbf{t}\in X;\mathbf{t}\neq\mathbf{0}} \frac{\left[\mathbb{B}^*(q),\mathbf{t}\right]}{\|\mathbf{t}\|_X} \geq \beta\|q\|_Z \qquad \forall q \in Z. \tag{3.13}$$

ii) X_1 *and* X_2 *are uniformly convex sets.*

iii) there exists $\beta_1 > 0$ *such that:*

$$\sup_{\mathbf{s}_0\in\mathcal{N}(\mathbb{B});\mathbf{s}_0\neq\mathbf{0}} \frac{\left[\mathbb{B}_1^*(\boldsymbol{\sigma}),\mathbf{s}_0\right]}{\|\mathbf{s}_0\|_X} \geq \beta_1\|\boldsymbol{\sigma}\|_Y \qquad \forall\boldsymbol{\sigma} \in Y. \tag{3.14}$$

iv) there exist constants $\gamma_1, \gamma_2 > 0$, $\varsigma_1, \varsigma_2 \geq 0$ *and* $r_1, r_2 \geq 2$, *depending only on the domain, such that*

$$\|\mathbb{A}(\mathbf{s}_1,\mathbf{s}_2) - \mathbb{A}(\mathbf{v}_1,\mathbf{v}_2)\|_{X'} \leq \sum_{j=1}^{2}\left\{\varsigma_j\,\|\mathbf{s}_j - \mathbf{v}_j\|_{X_j} + \gamma_j\,\|\mathbf{s}_j - \mathbf{v}_j\|_{X_j}\left(\|\mathbf{s}_j\|_{X_j} + \|\mathbf{v}_j\|_{X_j}\right)^{r_j-2}\right\}, \tag{3.15}$$

for all $(\mathbf{s}_1,\mathbf{s}_2)$, $(\mathbf{v}_1,\mathbf{v}_2) \in X := X_1 \times X_2$.

v) $\mathbb{A}\left(\cdot + \mathbf{t}_G\right) : \mathcal{N}(\mathbb{B}) \mapsto \mathcal{N}(\mathbb{B})'$ *is a strictly monotone mapping. More precisely, there exists* $\alpha > 0$, *independent of* $\mathbf{t}_G$, *such that*

$$\left[\mathbb{A}(\mathbf{s}+\mathbf{t}_G) - \mathbb{A}(\mathbf{v}+\mathbf{t}_G), \mathbf{s}-\mathbf{v}\right] \geq \alpha\Big(\|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1}^{r_1} + \|\mathbf{s}_2 - \mathbf{v}_2\|_{X_2}^{r_2}\Big), \tag{3.16}$$

for all $\mathbf{s} = (\mathbf{s}_1,\mathbf{s}_2), \mathbf{v} = (\mathbf{v}_1,\mathbf{v}_2) \in \mathcal{N}(\mathbb{B}) \subseteq X_1 \times X_2$, *for all* $\mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B})$.

vi) $\mathbb{A}(\cdot + \mathbf{t}_G)$ *is hemi-continuous, i.e., given* $\mathbf{t}, \mathbf{v} \in X \setminus \mathcal{N}(\mathbb{B})$

$$\begin{aligned} G : \mathbb{R} &\mapsto \mathbb{R} \\ x &\mapsto G(x) := \Big\langle \mathbb{A}(\mathbf{t} + x\mathbf{v} + \mathbf{t}_G), \mathbf{v}\Big\rangle \end{aligned} \tag{3.17}$$

is a continuous map, for all $\mathbf{t}_G \in X \setminus \mathcal{N}(\mathbb{B})$.

Then (P_1) *has a unique solution which is bounded in terms of the data.*

The proof is immediate from the recently exposed results.

3.2 The discrete setting

In what follows, we consider a conforming finite element method for (P_1) (cf. Sect. 3.1). As it is usual, we require certain restrictions on the finite dimensional spaces to be chosen.

Let X_1, X_2, Y and Z be separable and reflexive Banach spaces with duals X_1', X_2', Y', Z', and let $X = X_1 \times X_2$. Additionally, we consider bounded linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$, $\mathbb{C} : Y \to Y'$, with $\mathbb{C}$ positive semi-definite and a non-linear operator $\mathbb{A} : X \to X'$. Then, given $(F, G_1, G) \in (X', Y', Z')$, and let X_h, Y_h, and Z_h finite-dimensional subspaces of X, Y and Z, respectively, we now introduce the problem (P_{1h}) as follows:

find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in X_h \times Y_h \times Z_h$ such that:

$$\begin{array}{llll} [\mathbb{A}(\mathbf{t}_h), \mathbf{s}_h] + [\mathbb{B}_1^*(\boldsymbol{\sigma}_h), \mathbf{s}_h] + [\mathbb{B}^*(\mathbf{u}_h), \mathbf{s}_h] &=& [F, \mathbf{s}_h] & \forall \mathbf{s}_h \in X_h, \\ [\mathbb{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] - [\mathbb{C}(\boldsymbol{\sigma}_h), \boldsymbol{\tau}_h] &=& [G_1, \boldsymbol{\tau}_h] & \forall \boldsymbol{\tau}_h \in Y_h, \\ [\mathbb{B}(\mathbf{t}_h), q_h] &=& [G, q_h] & \forall q_h \in Z_h. \end{array} \tag{3.18}$$

Also, we define the discrete "kernel of $\mathbb{B}$" as

$$\mathbb{V}_h := \left\{ \mathbf{t}_h \in X_h : \Big[\mathbb{B}(\mathbf{t}_h), q_h\Big] = 0 \ \forall q_h \in Z_h \right\}, \tag{3.19}$$

and the orthogonal complement of $\mathbb{V}_h$

$$\mathbb{V}_h^{\perp} := \left\{ \mathbf{t}_h \in X_h : \Big\langle t_h, w_h \Big\rangle_{X_h} = 0 \ \forall w_h \in \mathbb{V}_h \right\}. \tag{3.20}$$

Note that, $X_h = X_{1h} \times X_{2h}$, where X_{1h}, X_{2h} are finite-dimensional subspaces of X_1 and X_2, respectively.

Theorem 3.2.1 *Assume that*

i) there exists $\beta_h > 0$ such that

$$\sup_{\mathbf{t}_h \in X_h; \mathbf{t}_h \neq \mathbf{0}} \frac{\Big[\mathbb{B}^*(q_h), \mathbf{t}_h\Big]}{\|\mathbf{t}_h\|_X} \geq \beta_h \|q_h\|_Z \qquad \forall q_h \in Z_h. \tag{3.21}$$

ii) there exists $\beta_{1h} > 0$ such that

$$\sup_{\mathbf{t}_{0,h} \in \mathbb{V}_h; \mathbf{t}_{0,h} \neq \mathbf{0}} \frac{\Big[\mathbb{B}_1^*(\boldsymbol{\tau}_h), \mathbf{t}_{0,h}\Big]}{\|\mathbf{t}_{0,h}\|_X} \geq \beta_{1h} \|\boldsymbol{\tau}_h\|_Y \qquad \forall \boldsymbol{\tau}_h \in Y_h. \tag{3.22}$$

iii) there are constants $\gamma_1, \gamma_2 > 0$, $\varsigma_1, \varsigma_2 \geq 0$ and $r_1, r_2 \geq 2$, depending only on the domain (and possibly on physical parameters involved), such that

$$\left\|\mathbb{A}(\mathbf{s}_h) - \mathbb{A}(\mathbf{v}_h)\right\|_{X'} \leq \sum_{j=1}^{2}\left\{\varsigma_j \left\|\mathbf{s}_{j,h} - \mathbf{v}_{j,h}\right\|_{X_j} + \gamma_j \left\|\mathbf{s}_{j,h} - \mathbf{v}_{j,h}\right\|_{X_j} \left(\left\|\mathbf{s}_{j,h}\right\|_{X_j} + \left\|\mathbf{v}_{j,h}\right\|_{X_j}\right)^{r_j - 2}\right\}$$

for all $\mathbf{s}_h := (\mathbf{s}_{1,h}, \mathbf{s}_{2,h})$, $\mathbf{v}_h := (\mathbf{v}_{1,h}, \mathbf{v}_{2,h}) \in X_h := X_{1h} \times X_{2h}$.

iv) given $\mathbf{t}_{hG} \in \mathbb{V}_h^{\perp}$,

iv−1) there exists $\alpha_h > 0$, independent of $\mathbf{t}_{hG}$, such that:

$$\left[\mathbb{A}(\mathbf{s}_h + \mathbf{t}_{hG}) - \mathbb{A}(\mathbf{v}_h + \mathbf{t}_{hG}), \mathbf{s}_h - \mathbf{v}_h\right] \geq \alpha_h \left\{\left\|\mathbf{s}_{1,h} - \mathbf{v}_{1,h}\right\|_{X_1}^{r_1} + \left\|\mathbf{s}_{2,h} - \mathbf{v}_{2,h}\right\|_{X_2}^{r_2}\right\}, \quad (3.23)$$

for all $\mathbf{s}_h = (\mathbf{s}_{1,h}, \mathbf{s}_{2,h})$, $\mathbf{v}_h = (\mathbf{v}_{1,h}, \mathbf{v}_{2,h}) \in \mathbb{V}_h \subseteq X_{1h} \times X_{2h}$.

iv−2) $\mathbb{A}(\cdot + \mathbf{t}_{hG})$ is hemi-continuous on $\mathbb{V}_h^{\perp}$, i.e., given $\mathbf{t}_h, \mathbf{v}_h \in \mathbb{V}_h^{\perp}$, the real function

$$\begin{aligned} G : \mathbb{R} &\mapsto \mathbb{R} \\ t &\mapsto G(t) := \left\langle \mathbb{A}(\mathbf{t}_h + \mathbf{t}_{hG} + t\mathbf{v}_h), \mathbf{v}_h \right\rangle \end{aligned} \quad (3.24)$$

is continuous.

Then the problem (P_{1h}) has a unique solution which is bounded in terms of the data.

Proof: It reduces to a simple application of Theorem 3.1.10 to the present discrete setting.

3.3 A priori error estimate

Let $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)$ and $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})$ be the solutions of the discrete and continuous problems (3.12) and (3.18), respectively. Then, at discrete level, we have:

$$\mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u}_h) = \mathbb{A}(\mathbf{t}) + \mathbb{B}_1^*(\boldsymbol{\sigma}) + \mathbb{B}^*(\mathbf{u}), \quad (3.25)$$

$$\mathbb{B}_1(\mathbf{t}_h) - \mathbb{C}(\boldsymbol{\sigma}_h) = \mathbb{B}_1(\mathbf{t}) - \mathbb{C}(\boldsymbol{\sigma}), \quad (3.26)$$

$$\mathbb{B}(\mathbf{t}_h) = \mathbb{B}(\mathbf{t}), \quad (3.27)$$

which means that the foregoing equations hold for $\mathbf{s}_h \in X_h, \boldsymbol{\tau}_h \in Y_h$, and $q_h \in Z_h$, respectively. Now, noting that $\mathbf{t}_h = \mathbf{t}_h^V + \mathbf{t}_h^{V^{\perp}}$, where $\mathbf{t}_h^V \in \mathbb{V}_h$ and $\mathbf{t}_h^{V^{\perp}} \in \mathbb{V}_h^{\perp}$ (cf. (3.19) and (3.20)), we see that (3.27) reduces to $\mathbb{B}\left(\mathbf{t}_h^{V^{\perp}}\right) = \mathbb{B}(\mathbf{t})$.

Lemma 3.3.1 *For each* $\mathbf{r}_h \in X_h$ *there holds*

$$\left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\| \leq \left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|, \tag{3.28}$$

where $\mathbf{r}_h = \mathbf{r}_h^V + \mathbf{r}_h^{V^\perp}$, *with* $\mathbf{r}_h^V \in \mathbb{V}_h$ *and* $\mathbf{r}_h^{V^\perp} \in \mathbb{V}_h^\perp$.

Proof: Let $\mathbf{r}_h = \mathbf{r}_h^V + \mathbf{r}_h^{V^\perp} \in X_h$ as indicated. Then, according to the triangle inequality, we have that

$$\left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\| \leq \left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{r}_h^{V^\perp}\right)\right\| + \left\|\mathbf{t}_h^{V^\perp}-\mathbf{r}_h^{V^\perp}\right\| = \|\mathbf{t}-\mathbf{r}_h\| + \left\|\mathbf{t}_h^{V^\perp}-\mathbf{r}_h^{V^\perp}\right\|$$

On the other hand, noting from (3.27) that at the discrete level there holds

$$\mathbb{B}\left(\mathbf{t}-\mathbf{r}_h\right) = \mathbb{B}\left(\mathbf{t}_h-\mathbf{r}_h\right) = \mathbb{B}\left(\mathbf{t}_h^{V^\perp}-\mathbf{r}_h^{V^\perp}\right),$$

and then employing the discrete inf-sup condition (cf. (3.21)), we find that

$$\beta_h\left\|\mathbf{t}_h^{V^\perp}-\mathbf{r}_h^{V^\perp}\right\| \leq \|\mathbb{B}(\mathbf{t}-\mathbf{r}_h)\| \leq \|\mathbb{B}\|\,\|\mathbf{t}-\mathbf{r}_h\|,$$

which concludes the proof.

Now, given $\mathbf{r}_h = \mathbf{r}_h^V + \mathbf{r}_h^{V^\perp} \in X_h$, we have from the triangle inequality

$$\|\mathbf{t}-\mathbf{t}_h\|_X = \left\|\mathbf{t}-\left(\mathbf{t}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\|_X \leq \left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\|_X + \left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|_X. \tag{3.29}$$

Then, according to the estimate provided by Lemma 3.3.1, it would suffice to bound the expression $\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|_X$ in terms of $\mathrm{dist}(\mathbf{t},X_h), \mathrm{dist}(\boldsymbol{\sigma},Y_h)$ and $\mathrm{dist}(\mathbf{u},Z_h)$. To this end, we first observe from the identity (3.25) that

$$\mathbb{A}\left(\mathbf{t}_h^V+\mathbf{t}_h^{V^\perp}\right) - \mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right) = \mathbb{A}\left(\mathbf{t}_h\right) - \mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)$$

$$= \mathbb{A}(\mathbf{t}) - \mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right) + \mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u}-\mathbf{u}_h). \tag{3.30}$$

Now using the monotonicity of $\mathbb{A}$ (cf. (3.23)) noting that $\mathbf{t}_h^V - \mathbf{r}_h^V \in \mathbb{V}_h \subseteq X_1 \times X_2$, and denoting $\mathbf{t}_h^V := (\mathbf{t}_1, \mathbf{t}_2)$ and $\mathbf{r}_h^V := (\mathbf{r}_1, \mathbf{r}_2)$, we deduce, employing also (3.30), that for each $\boldsymbol{\tau}_h \in Y_h$ there holds

$$\alpha^h\Big\{\|\mathbf{t}_1-\mathbf{r}_1\|_{X_1}^{r_1} + \|\mathbf{t}_2-\mathbf{r}_2\|_{X_2}^{r_2}\Big\} \leq \left[\mathbb{A}\left(\mathbf{t}_h^V+\mathbf{t}_h^{V^\perp}\right) - \mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right), \mathbf{t}_h^V-\mathbf{r}_h^V\right]$$

$$= \left[\mathbb{A}(\mathbf{t}) - \mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right) + \mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\tau}_h) + \mathbb{B}_1^*(\boldsymbol{\tau}_h-\boldsymbol{\sigma}_h) + \mathbb{B}^*(\mathbf{u}-\mathbf{u}_h),\ \mathbf{t}_h^V-\mathbf{r}_h^V\right]. \tag{3.31}$$

Next, we bound the first two terms on the right-hand side of (3.31). To this end, we first show the following Lemma.

Lemma 3.3.2 *Given* $\mathbf{t}, \mathbf{r} \in X$, *there exists* $C_A := C_A\big(\|\mathbf{t}\|, r_1, r_2\big) > 0$ *such that*

$$\|\mathbb{A}(\mathbf{t}) - \mathbb{A}\,(\mathbf{r}))\| \leq C_A\Bigg(\|\mathbf{t}-\mathbf{r}\| + \|\mathbf{t}-\mathbf{r}\|^{r_1-1} + \|\mathbf{t}-\mathbf{r}\|^{r_2-1}\Bigg),$$

where $r_1, r_2 \geq 2$ *are specified in the assumption* (A_0).

Proof: It follows from the boundedness property of $\mathbb{A}$ (cf. (A_0)) and the triangle inequality.

The foregoing estimate suggests to define the real function

$$\begin{aligned}\mathcal{F}_A &: \mathbb{R}^+ \mapsto \mathbb{R}^+\\ &x \mapsto \mathcal{F}_A(x) = C_A\Bigg(x + x^{r_1-1} + x^{r_2-1}\Bigg).\end{aligned}$$

Note that $\mathcal{F}_A$ is a strictly monotone mapping, and according to Lemma 3.3.2, there holds

$$\|\mathbb{A}(\mathbf{t}) - \mathbb{A}\,(\mathbf{r}))\| \leq \mathcal{F}_A\Big(\|\mathbf{t}-\mathbf{r}\|\Big) \qquad \forall \mathbf{t}, \mathbf{r} \in X,$$

which, thanks to Lemma 3.3.1 and the monotonicity of $\mathcal{F}_A$, satisfies

$$\mathcal{F}_A\bigg(\Big\|\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\Big\|\bigg) \leq \mathcal{F}_A\bigg(\bigg(1 + \frac{\|\mathbb{B}\|}{\beta_h}\bigg)\,\|\mathbf{t}-\mathbf{r}_h\|\bigg) \qquad \forall \mathbf{r}_h \in X_h.$$

Therefore, it is clear from the previous estimates that

$$\bigg[\mathbb{A}(\mathbf{t}) - \mathbb{A}\left(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\right),\ \mathbf{t}_h^V - \mathbf{r}_h^V\bigg] \leq \mathcal{F}_A\bigg(\bigg(1 + \frac{\|\mathbb{B}\|}{\beta_h}\bigg)\,\|\mathbf{t}-\mathbf{r}_h\|\bigg)\Big\|\mathbf{t}_h^V - \mathbf{r}_h^V\Big\| \qquad \forall \mathbf{r}_h \in X_h. \quad (3.32)$$

Next, we utilize (3.26) to bound the fourth term on the right-hand side of (3.31) as follows

$$\begin{aligned}\Big[\mathbb{B}_1^*(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h), \mathbf{t}_h^V - \mathbf{r}_h^V\Big] &= \bigg[\mathbb{B}_1^*(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h),\ \mathbf{t}_h - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\bigg]\\ &= \bigg[\mathbb{B}_1\bigg(\mathbf{t}_h - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\bigg),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\bigg]\\ &= \bigg[\mathbb{B}_1\bigg(\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\bigg),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\bigg] - \Big[\mathbb{C}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big]\end{aligned}$$

$$= \bigg[\mathbb{B}_1\bigg(\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\bigg),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\bigg] - \Big[\mathbb{C}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big] - \Big[\mathbb{C}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h),\ \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big]$$

Then, using the positive semi-definite property of $\mathbb{C}$ it follows that $\Big[\mathbb{C}(\boldsymbol{\tau}_h - \boldsymbol{\sigma}_h), \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big] \geq 0$, and hence

$$\begin{aligned}\Big[\mathbb{B}_1^*(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h), \mathbf{t}_h^V - \mathbf{r}_h^V\Big] &\leq \Big[\mathbb{B}_1\Big(\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\Big), \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big] - \Big[\mathbb{C}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h), \boldsymbol{\tau}_h - \boldsymbol{\sigma}_h\Big] \\ &\leq \left(\Big\|\mathbb{B}_1\Big(\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\Big)\Big\| + \Big\|\mathbb{C}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\Big\|\right) \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\| . \end{aligned} \tag{3.33}$$

Now, we utilize the definition of the discrete kernel of $\mathbb{B}$ (cf. (3.19)) to show that the fifth term on the right-hand side of (3.31) can be written and bounded as follows

$$\begin{aligned}\Big[\mathbb{B}^*(\mathbf{u} - \mathbf{u}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] &= \Big[\mathbb{B}^*(\mathbf{u} - \boldsymbol{z}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] + \Big[\mathbb{B}^*(\boldsymbol{z}_h - \mathbf{u}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] \\ &= \Big[\mathbb{B}^*(\mathbf{u} - \boldsymbol{z}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] \leq \|\mathbb{B}^*(\mathbf{u} - \boldsymbol{z}_h)\| \left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\| \qquad \forall \boldsymbol{z}_h \in Z_h. \end{aligned} \tag{3.34}$$

Note here that in the particular case that $\mathbb{V}_h \subseteq \mathcal{N}(\mathbb{B})$, i.e. the discrete kernel of $\mathbb{B}$ is contained in the continuous one, there holds

$$\Big[\mathbb{B}^*(\mathbf{u} - \mathbf{u}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] = 0. \tag{3.35}$$

Therefore, according to (3.32), (3.33) and (3.34) it is possible to bound the right hand side of (3.31) as follows

$$\begin{aligned}&\Big[\mathbb{A}(\mathbf{t}) - \mathbb{A}\left(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\right) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\tau}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{u}_h),\ \mathbf{t}_h^V - \mathbf{r}_h^V\Big] \\ &\leq \left(\mathcal{F}_A\left(\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\|\right) + \|\mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\| + \|\mathbb{B}^*(\mathbf{u} - \boldsymbol{z}_h)\|\right)\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\| \\ &+ \left(\Big\|\mathbb{B}_1\Big(\mathbf{t} - \Big(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\Big)\Big)\Big\| + \|\mathbb{C}(\boldsymbol{\sigma} - \boldsymbol{\tau}_h)\|\right)\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\| \qquad \forall \boldsymbol{z}_h \in Z_h, \forall \boldsymbol{\tau}_h \in Y_h. \end{aligned} \tag{3.36}$$

Thus, defining:

$$\begin{aligned}\mathcal{M}_1 &:= \mathcal{F}_A\left(\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\|\right) + \|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\| + \|\mathbb{B}^*\|\,\|\mathbf{u} - \boldsymbol{z}_h\| , \\ \mathcal{M}_2 &:= \|\mathbb{B}_1\|\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\| + \|\mathbb{C}\|\,\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\| , \end{aligned}$$

and utilizing (3.31) and (3.36), we find that

$$\alpha^h\Big\{\|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{r_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{r_2}\Big\} \leq \mathcal{M}_1\Big\{\|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}\Big\} + \mathcal{M}_2\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\| . \tag{3.37}$$

In turn, from (3.37) we have a bound for $\|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}$ in terms of $\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|$. In fact, applying Young's inequality $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$, where $a, b \geq 0$ and $1/p + 1/q = 1$ (cf. [1, Theo. 2.3]), we find that

$$\|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{r_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{r_2} \leq \left(\frac{1}{r_1'}\left(\frac{2}{\alpha^h}\right)^{r_1'+1}\right)\mathcal{M}_1^{r_1'} + \left(\frac{1}{r_2'}\left(\frac{2}{\alpha^h}\right)^{r_2'+1}\right)\mathcal{M}_1^{r_2'} + \left(\frac{2}{\alpha^h}\right)\mathcal{M}_2\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|$$

$$\leq C\left(\alpha^h, r_1, r_2\right)\left(\mathcal{M}_1^{r_1'} + \mathcal{M}_1^{r_2'} + \mathcal{M}_2\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|\right), \tag{3.38}$$

from which it is easy to see that

$$\|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2} \leq \widetilde{C}\left(\mathcal{M}_1^{\frac{r_1'}{r_1}} + \mathcal{M}_1^{\frac{r_1'}{r_2}} + \mathcal{M}_2^{\frac{r_2'}{r_1}} + \mathcal{M}_2^{\frac{r_2'}{r_1}} + \mathcal{M}_2^{\frac{1}{r_1}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^{\frac{1}{r_1}} + \mathcal{M}_2^{\frac{1}{r_2}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^{\frac{1}{r_2}}\right),$$

that is, recalling that $\mathbf{t}_h^V = (\mathbf{t}_1, \mathbf{t}_2)$ and $\mathbf{r}_h^V = (\mathbf{r}_1, \mathbf{r}_2)$ and denoting $\widetilde{\mathcal{M}} := \mathcal{M}_1^{\frac{r_1'}{r_1}} + \mathcal{M}_1^{\frac{r_1'}{r_2}} + \mathcal{M}_2^{\frac{r_2'}{r_1}} + \mathcal{M}_2^{\frac{r_2'}{r_1}}$, we can write

$$\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|_X \leq \widetilde{C}\left(\widetilde{\mathcal{M}} + \mathcal{M}_2^{\frac{1}{r_1}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^{\frac{1}{r_1}} + \mathcal{M}_2^{\frac{1}{r_2}}\,\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|^{\frac{1}{r_2}}\right). \tag{3.39}$$

Now, we give a bound for $\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|$ in terms of $\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|$, $\|\mathbf{t} - \mathbf{r}_h\|$ and $\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_X$ with $(\mathbf{r}_h, \boldsymbol{\tau}_h) \in X_h \times Z_h$.

Lemma 3.3.3 *There exists $C = C(\|\mathbf{t}\|_X, \|\mathbf{t}_h\|_X) > 0$ such that*

$$\beta_{1h}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq C\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|_X + C\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\| + \|\mathbb{B}_1^*\|\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\|\|\mathbf{u} - \boldsymbol{z}_h\|_Z$$

for all $(\mathbf{r}_h, \boldsymbol{z}_h) \in X_h \times Z_h$.

Proof: According to the identity (3.25) there holds

$$\mathbb{B}_1^*(\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h) = \mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h) + \mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\tau}_h) + \mathbb{B}^*(\mathbf{u} - \mathbf{u}_h).$$

Then, using the discrete inf-sup condition for $\mathbb{B}_1$ (cf. (3.22)), and proceeding similarly to (3.34), we obtain

$$\beta_{1h}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq \left\|\mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h)\right\|_X + \left\|\mathbb{B}_1^*\left(\boldsymbol{\sigma} - \boldsymbol{\tau}_h\right)\right\|_Y + \left\|\mathbb{B}^*\left(\mathbf{u} - \boldsymbol{z}_h\right)\right\|_Z \qquad \forall \boldsymbol{z}_h \in Z_h,$$

which, according to the boundedness properties of $\mathbb{A}, \mathbb{B}$ and $\mathbb{B}_1$, yields

$$\beta_{1h}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq C_A\|\mathbf{t} - \mathbf{t}_h\|_X + \|\mathbb{B}_1^*\|\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\|\|\mathbf{u} - \boldsymbol{z}_h\|_Z \qquad \forall \boldsymbol{z}_h \in Z_h,$$

where $C_A = C_A(\|\mathbf{t}\|_X, \|\mathbf{t}_h\|_X) > 0$. Then, adding and subtracting $\mathbf{r}_h^V + \mathbf{t}_h^V$ within $\|\mathbf{t} - \mathbf{t}_h\|_X$, we deduce that

$$\beta_{1h}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq C_A\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|_X + C_A\left\|\mathbf{t} - \left(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\right)\right\|_X + \|\mathbb{B}_1^*\|\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\|\|\mathbf{u} - \boldsymbol{z}_h\|_Z,$$

for all $(\mathbf{r}_h, \boldsymbol{z}_h) \in X_h \times Z_h$.
Thus, according to Lemma 3.3.1, $\left\|\mathbf{t} - \left(\mathbf{r}_h^V + \mathbf{t}_h^{V^\perp}\right)\right\|_X$ is bounded above by $\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\|$, and hence

$$\beta_{1h}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq C_A\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|_X + C_A\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\| + \|\mathbb{B}_1^*\|\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\|\|\mathbf{u} - \boldsymbol{z}_h\|_Z,$$

for all $(\mathbf{r}_h, \boldsymbol{z}_h) \in X_h \times Z_h$, which completes the proof.

Next, applying Lemma 3.3.3 and Young's inequality to the right-hand side of (3.39), we obtain

$$\begin{aligned}\|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y \leq \widehat{C}\Bigg(&\mathcal{M}_1^{\frac{r_1'}{r_1}} + \mathcal{M}_1^{\frac{r_1'}{r_2}} + \mathcal{M}_2^{\frac{r_2'}{r_1}} + \mathcal{M}_2^{\frac{r_2'}{r_1}} + \mathcal{M}_2^{\frac{r_1'}{r_1}} + \mathcal{M}_2^{\frac{r_2'}{r_2}}\Bigg)\\ + \ \widehat{C}\Bigg\{&\left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t} - \mathbf{r}_h\| + \|\mathbb{B}_1^*\|\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + \|\mathbb{B}^*\|\|\mathbf{u} - \boldsymbol{z}_h\|_Z\Bigg\} \qquad \forall (\mathbf{r}_h, \boldsymbol{z}_h) \in X_h \times Z_h. \end{aligned} \tag{3.40}$$

Finally, applying the triangle inequality and (3.40), we arrive at

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \leq \|\boldsymbol{\sigma}_h - \boldsymbol{\tau}_h\|_Y + \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y \leq \mathcal{F}\Big(\|\mathbf{t} - \mathbf{r}_h\|, \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y, \|\mathbf{u} - \boldsymbol{z}_h\|_Z\Big) \qquad \forall (\mathbf{r}_h, \boldsymbol{\tau}_h, \boldsymbol{z}_h) \in X_h \times Y_h \times Z_h,$$

where $\mathcal{F}$, a strictly monotone mapping in each one of its components, represents the right-hand side of (3.40). In this way, we conclude that

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \leq \mathcal{F}\Big(\operatorname{dist}(\mathbf{t}, X_h), \operatorname{dist}(\boldsymbol{\sigma}, Y_h), \operatorname{dist}(\mathbf{u}, Z_h)\Big). \tag{3.41}$$

Now, we proceed to bound $\left\|\mathbf{t}_h^V - \mathbf{r}_h^V\right\|$ in terms of $\|\mathbf{t} - \mathbf{r}_h\|, \operatorname{dist}(\mathbf{u}, Z_h)$ and $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|$, which, thanks to (3.29) and Lemma 3.3.1, will complete the bound for $\|\mathbf{t} - \mathbf{t}_h\|$. Indeed, given $(\mathbf{r}_h, \boldsymbol{z}_h) \in X_h \times Z_h$, and proceeding similarly to (3.31) and (3.34), we obtain

$$\alpha^h\Big\{\|\mathbf{t}_1-\mathbf{r}_1\|_{X_1}^{r_1}+\|\mathbf{t}_2-\mathbf{r}_2\|_{X_2}^{r_2}\Big\}\leq\left[\mathbb{A}\left(\mathbf{t}_h^V+\mathbf{t}_h^{V^\perp}\right)-\mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right),\mathbf{t}_h^V-\mathbf{r}_h^V\right]$$

$$=\left[\mathbb{A}(\mathbf{t})-\mathbb{A}\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)+\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\mathbb{B}^*(\mathbf{u}-\boldsymbol{z}_h),\mathbf{t}_h^V-\mathbf{r}_h^V\right]$$

$$\leq\left(\mathcal{F}_A\left(\left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\|_X\right)+\|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|+\|\mathbb{B}^*\|\,\|\mathbf{u}-\boldsymbol{z}_h\|\right)\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|$$

$$\leq\left(\mathcal{F}_A\left(\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|\right)+\|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|+\|\mathbb{B}^*\|\,\|\mathbf{u}-\boldsymbol{z}_h\|\right)\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|.$$

Moreover, defining $\mathcal{F}_0:=\|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|+\|\mathbb{B}^*\|\,\mathrm{dist}(\mathbf{u},Z_h)$, the foregoing inequality becomes

$$\alpha^h\Big\{\|\mathbf{t}_1-\mathbf{r}_1\|_{X_1}^{r_1}+\|\mathbf{t}_2-\mathbf{r}_2\|_{X_2}^{r_2}\Big\}\leq\left(\mathcal{F}_A\left(\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|\right)+\mathcal{F}_0\right)\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|,$$

which, using that $\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|=\|\mathbf{t}_1-\mathbf{r}_1\|_{X_1}+\|\mathbf{t}_2-\mathbf{r}_2\|_{X_2}$ and employing Young's inequality, reduces to

$$\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|\leq C(\alpha_h;r_1,r_2)\sum_{j=1}^{2}\left\{\mathcal{F}_A\left(\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|\right)+\mathcal{F}_0\right\}^{\frac{1}{r_j-1}}. \tag{3.42}$$

Now, according to Lemma 3.3.1, the bound (3.42) and the triangular inequality, we deduce that

$$\|\mathbf{t}-\mathbf{t}_h\|_X\leq\left\|\mathbf{t}-\left(\mathbf{r}_h^V+\mathbf{t}_h^{V^\perp}\right)\right\|_X+\left\|\mathbf{t}_h^V-\mathbf{r}_h^V\right\|_X$$

$$\leq\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|+C(\alpha_h,r_1,r_2)\sum_{j=1}^{2}\left\{\mathcal{F}_A\left(\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\|\mathbf{t}-\mathbf{r}_h\|\right)+\mathcal{F}_0\right\}^{\frac{1}{r_j-1}}\qquad\forall\mathbf{r}_h\in X_h,$$

which, using that $\mathcal{F}_A$ is strictly monotone, gives

$$\|\mathbf{t}-\mathbf{t}_h\|_X\leq\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\mathrm{dist}(\mathbf{t},X_h)+C(\alpha_h,r_1,r_2)\sum_{j=1}^{2}\left\{\mathcal{F}_A\left(\left(1+\frac{\|\mathbb{B}\|}{\beta_h}\right)\mathrm{dist}(\mathbf{t},X_h)\right)+\mathcal{F}_0\right\}^{\frac{1}{r_j-1}}. \tag{3.43}$$

Finally, we bound $\|\mathbf{u}-\mathbf{u}_h\|_Z$ in terms of $\|\mathbf{t}-\mathbf{t}_h\|_X$, $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_Y$ and $\mathrm{dist}(\mathbf{u},Z_h)$. To this end, and according to the identity (3.25), we first recall that

$$\mathbb{B}^*\left(\mathbf{u}_h-\boldsymbol{z}_h\right)=\mathbb{A}(\mathbf{t})-\mathbb{A}(\mathbf{t}_h)+\mathbb{B}_1^*(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)+\mathbb{B}^*(\mathbf{u}-\boldsymbol{z}_h)\qquad\forall\boldsymbol{z}_h\in Z_h,$$

which, using the discrete inf-sup condition for $\mathbb{B}$ (cf. (3.21)), yields

$$\begin{aligned}\beta_h\|\mathbf{u}_h - \boldsymbol{z}_h\|_Z \leq & \|\mathbb{A}(\mathbf{t}) - \mathbb{A}(\mathbf{t}_h)\| + \|\mathbb{B}_1^*(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\| + \|\mathbb{B}^*(\mathbf{u} - \boldsymbol{z}_h)\| \\ & \leq C_A\,\|\mathbf{t} - \mathbf{t}_h\| + \|\mathbb{B}_1^*\|\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \|\mathbb{B}\|\,\|\mathbf{u} - \boldsymbol{z}_h\| \quad \forall \boldsymbol{z}_h \in Z_h,\end{aligned}$$

and therefore

$$\|\mathbf{u} - \mathbf{u}_h\|_Z \leq \frac{C}{\beta_h}\,\|\mathbf{t} - \mathbf{t}_h\| + \frac{\|\mathbb{B}_1^*\|}{\beta_h}\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\| + \left(1 + \frac{\|\mathbb{B}\|}{\beta_h}\right)\operatorname{dist}(\mathbf{u}, Z_h). \tag{3.44}$$

We summarize the foregoing analysis in the following theorem.

Theorem 3.3.4 *There exist a strictly monotone mapping and continuous function $\mathcal{G} : \mathbb{R}^3 \mapsto \mathbb{R}$ in each one of its components, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y + \|\mathbf{t} - \mathbf{t}_h\|_X + \|\mathbf{u} - \mathbf{u}_h\|_Z \leq \mathcal{G}\Big(\operatorname{dist}(\mathbf{t}, X_h), \operatorname{dist}(\boldsymbol{\sigma}, Y_h), \operatorname{dist}(\mathbf{u}, Z_h)\Big).$$

Proof: It follows straightforwardly from (3.41), (3.43) and (3.44).

We end this chapter by remarking that for $r_1 = r_2 = 2$, the previous analysis leads to the estimate

$$\|\mathbf{t} - \mathbf{t}_h\| + \|\mathbf{u} - \mathbf{u}_h\| + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_Y \leq C_1\,\|\mathbf{t} - \mathbf{r}_h\| + C_2\|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_Y + C_3\|\mathbf{u} - \boldsymbol{z}_h\|_Z \qquad \forall (\mathbf{r}_h, \boldsymbol{\tau}_h, \boldsymbol{z}_h) \in X_h \times Y_h \times Z_h$$

where $C_1, C_2, C_3 > 0$ are known explicitly.

Chapter 4

Analysis of the coupled problem

4.1 Analysis of the continuous problem $(\mathbf{P}_\alpha)$

In what follow we analyse the properties of the spaces and operators associated with the problem $(\mathbf{P}_\alpha)$ (cf.(2.18)). In order to apply Theorem 3.1.10, we begin by showing that the general hypothesis on the spaces involved are satisfied. Recall from Section 2.4 that these spaces are given by

$$\begin{aligned} X_1 &:= \mathbb{H}_0(\mathbf{div}\,;\Omega_S), \\ X_2 &:= \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div}\,,\Omega_D), \\ X &:= X_1 \times X_2, \\ Z &:= \mathbf{L}^2(\Omega_S) \times L^{3/2}_0(\Omega_D) \times \mathbb{L}^2_{skew}(\Omega_S), \\ Y &:= \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma) \times W^{\frac{1}{3},\frac{3}{2}}(\Sigma). \end{aligned}$$

We first observe that X is a uniformly convex space. This property follows from Hanner's inequalities, which imply that $L^p(\Omega)$ is uniformly convex when $1 < p < \infty$, the fact that every closed subspace of a uniformly convex Banach space is uniformly convex, the continuity of the trace and normal trace operators, and the orthogonal decomposition theorem. In turn, using similar arguments, all the other spaces are uniformly convex and separable Banach spaces.

Next, in order to apply Theorem 3.1.10, we show that the assumptions $(3.13)-(3.17)$ are satisfied. More precisely, we take advantage of the diagonal structure shown by $\mathbb{B}$ and $\mathbb{B}_1$, and show that these inf-sup conditions can be reduced, equivalently, to four simpler inf-sup conditions.

Lemma 4.1.1 $\mathbb{B}$ *satisfies the inf-sup condition (3.13), that is, there exists $\beta > 0$ such that*

$$\sup_{\underline{\tau}\in X;\underline{\tau}\neq\mathbf{0}} \frac{\big[\mathbb{B}(\underline{\tau}),\underline{\eta}\,\big]}{\|\underline{\tau}\|_X} \geq \beta\|\underline{\eta}\|_Z \qquad \forall\underline{\eta}\in Z. \tag{4.1}$$

Proof: Let us first recall that

$$\big[\mathbb{B}(\underline{\tau}),\underline{\eta}\big] = (\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{u}_S)_S + (\boldsymbol{\tau}_S,\boldsymbol{\gamma}_{\boldsymbol{S}})_S - (\mathrm{div}\,\mathbf{v}_D,p_D)_D \quad \forall\underline{\tau}=(\boldsymbol{\tau}_S,\mathbf{v}_D)\in X,\ \forall\underline{\eta}=(\mathbf{u}_S,p_D,\boldsymbol{\gamma}_{\boldsymbol{S}})\in Z.$$

Then, we observe, according to the diagonal structure shown by $\mathbb{B}$, that (4.1) is equivalent to the following inf-sup conditions

$$\sup_{\boldsymbol{\tau}_S\in X_1;\boldsymbol{\tau}_S\neq\mathbf{0}} \frac{(\mathbf{div}\,\boldsymbol{\tau}_S,\mathbf{u}_S)_S + (\boldsymbol{\tau}_S,\boldsymbol{\gamma}_{\boldsymbol{S}})_S}{\|\boldsymbol{\tau}_S\|_{X_1}} \geq \beta\|(\mathbf{u}_S,\boldsymbol{\gamma}_{\boldsymbol{S}})\| \qquad \forall(\mathbf{u}_S,\boldsymbol{\gamma}_{\boldsymbol{S}})\in\mathbf{L}^2(\Omega_S)\times\mathbb{L}^2_{skew}(\Omega_S),$$

and

$$\sup_{\mathbf{v}_D\in X_2;\mathbf{v}_D\neq\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_D,p_D)_D}{\|\mathbf{v}_D\|_{X_2}} \geq \beta\|p_D\| \qquad \forall p_D\in L_0^{\frac{3}{2}}(\Omega_D).$$

Now, in order to prove the foregoing inf-sup conditions, we proceed as usual by introducing suitable auxiliary problems:

- Given $(\mathbf{u}_S,\boldsymbol{\gamma}_{\boldsymbol{S}})\in\mathbf{L}^2(\Omega_S)\times\mathbb{L}^2_{skew}(\Omega_S)$, find $z\in H^1(\Omega_S)$ such that

$$\begin{aligned} \mathbf{div}\,\mathbf{e}(z) &= -\mathbf{u}_S-\boldsymbol{\gamma}_{\boldsymbol{S}} \qquad \text{in } \Omega_S,\\ z &= 0 \qquad \text{on } \partial\Omega_S, \end{aligned} \tag{P_1}$$

where $\mathbf{e}(z)=\dfrac{1}{2}\Big(\nabla z+(\nabla z)^t\Big)$.

- Given $p_D\in L_0^{\frac{3}{2}}(\Omega_D)$, find $w\in \mathrm{W}^{1,3}(\Omega_D)$ such that

$$\begin{aligned} \mathrm{div}\,\Big(\nabla w\Big) &= \widetilde{p}_D \qquad \text{in } \Omega_D,\\ \gamma_\nu(\nabla w) &= 0 \qquad \text{in } \Gamma_D,\\ \int_{\Omega_D} w &= 0, \end{aligned} \tag{P_2}$$

where $\widetilde{p}_D := \widehat{p}_D - \dfrac{1}{|\Omega_D|}\displaystyle\int_{\Omega_D}\widehat{p}_D$, and $\widehat{p}_D(x)$ is given by

$$\widehat{p}_D(x) := \begin{cases} \dfrac{p_D(x)}{|p_D(x)|^{\frac{1}{2}}} & : \ p_D(x)\neq 0,\\[2mm] \theta & : \ p_D(x)=0. \end{cases}$$

The variational problem of (P_1) is given by (cf. [21, (2.54)]): find $z\in\mathbf{H}_0^1(\Omega_S)$ such that

$$\int_{\Omega_S}\mathbf{e}(z):\mathbf{e}(w) = \int_{\Omega_S}\mathbf{u}_S\cdot w - \int_{\Omega_S}\boldsymbol{\gamma}_{\boldsymbol{S}}:\nabla w \qquad \forall w\in\mathbf{H}_0^1(\Omega_S).$$

Thanks to Korn and Friedrich-Poincaré inequalities, the Lax-Milgram Lemma implies that the above problem has a unique solution $z \in \mathbf{H}_0^1(\Omega_S)$, which satisfies

$$|z|_{1,\Omega_S} \leq C_S \left\{ \|\mathbf{u}_S\|_{0,\Omega_S} + \|\boldsymbol{\gamma_S}\|_{0,\Omega_S} \right\}, \tag{4.2}$$

where $C_S > 0$ is a constant independent of the solution z. Then, defining $\boldsymbol{\sigma}_S := \mathbf{e}(z) + \boldsymbol{\gamma_S}$, it is straightforward to check that

$$\boldsymbol{\sigma}_S \in \mathbb{L}^2(\Omega_S),\ \mathbf{div}\,\boldsymbol{\sigma}_S = -\mathbf{u}_S,\quad \frac{\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_S{}^t}{2} = \boldsymbol{\gamma_S} \quad \text{and} \quad \int_{\Omega_S} \operatorname{tr} \boldsymbol{\sigma}_S = \int_{\Omega_S} \operatorname{div} z = \int_{\partial\Omega_S} z \cdot n = 0,$$

from which it follows that $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ and $\|\boldsymbol{\sigma}_S\|^2_{0;\Omega_S} = \|\mathbf{e}(z)\|^2_{0;\Omega_S} + \|\boldsymbol{\gamma_S}\|^2_{0;\Omega_S}$. In addition, by the continuous dependence property (4.2), we obtain

$$\|\mathbf{e}(z)\|^2_{0;\Omega_S} \leq \|\nabla z\|^2_{0;\Omega_S} \leq 2C_S^2 \left\{ \|\mathbf{u}_S\|^2_{0,\Omega_S} + \|\boldsymbol{\gamma_S}\|^2_{0,\Omega_S} \right\}.$$

Finally, we have $\|\mathbf{div}\,\boldsymbol{\sigma}_S\|^2_{0;\Omega_S} = \|\mathbf{u}_S\|^2_{0;\Omega_S}$, and therefore

$$\|\boldsymbol{\sigma}_S\|^2_{\mathbf{div}\,;\Omega_S} \leq 2(C_S^2 + 1) \left\{ \|\mathbf{u}_S\|^2_{0,\Omega_S} + \|\boldsymbol{\gamma_S}\|^2_{0,\Omega_S} \right\}.$$

On the other hand, according to [15], the auxiliary problem (P_2) has a unique solution $w \in \mathrm{W}^{1,3}(\Omega_D)$. Then, defining $\mathbf{u}_D := \nabla w$, we can prove (cf. Lemma .4.4) that

$$\Big(\operatorname{div} \mathbf{u}_D, p_D\Big)_D = \Big(\operatorname{div} \nabla w, p_D\Big)_D = \Big(\widetilde{p}_D, p_D\Big)_D = \Big(\widehat{p}_D, p_D\Big)_D = \ \Big\|p_D\Big\|^{\frac{3}{2}}_{0,\frac{3}{2};\Omega_D},$$

and thanks to the continuous dependence result for (P_2) there exists $C_D > 0$, independent of w, such that

$$\big\|\mathbf{u}_D\big\|_{0,3;\Omega_D} = \big\|\nabla w\big\|_{0,3;\Omega_D} \leq \big\|w\big\|_{1,3;\Omega_D} \leq C_D \big\|\widetilde{p}_D\big\|_{0,3;\Omega_D} \leq C_D \big\|\widehat{p}_D\big\|_{0,3;\Omega_D} = C_D \big\|p_D\big\|^{\frac{1}{2}}_{0,\frac{3}{2};\Omega_D},$$

where the last equality follows also from Lemma .4.4. In turn, according to the definition of $\mathbf{u}_D$, and using again Lemma .4.4, we find that

$$\big\|\operatorname{div} \mathbf{u}_D\big\|_{0,3;\Omega_D} = \big\|\widetilde{p}_D\big\|_{0,3;\Omega_D} \leq \big\|\widehat{p}_D\big\|_{0,3;\Omega_D} = \big\|p_D\big\|^{\frac{1}{2}}_{0,\frac{3}{2};\Omega_D},$$

from where

$$\big\|\mathbf{u}_D\big\|_{3,\operatorname{div}\,;\Omega_D} \leq (C_D + 1) \big\|p_D\big\|^{\frac{1}{2}}_{0,\frac{3}{2};\Omega_D}.$$

Finally, thanks to both auxiliary problems we can conclude that there exists $\beta = \beta\,(C_S, C_D) > 0$ such that

$$\sup_{\substack{(\boldsymbol{\tau}_S,\mathbf{v}_D)\in X\\(\boldsymbol{\tau}_S,\mathbf{v}_D)\neq\mathbf{0}}}\frac{\big[\mathbb{B}(\boldsymbol{\tau}_S,\mathbf{v}_D),(\mathbf{u}_S,p_D,\boldsymbol{\gamma_S})\big]}{\|(\boldsymbol{\tau}_S,\mathbf{v}_D)\|_X}\geq\frac{\Big[\mathbb{B}(\boldsymbol{\sigma}_S,\mathbf{u}_D),(\mathbf{u}_S,p_D,\boldsymbol{\gamma_S})\Big]}{\|(\boldsymbol{\sigma}_S,\mathbf{u}_D)\|_X}\geq\beta\Big\|(\mathbf{u}_S,p_D,\boldsymbol{\gamma_S})\Big\|_Z.$$

We now prove the continuous inf-sup condition for $\mathbb{B}_1$. Similarly to the previous Lemma we take advantage of the diagonal structure of $\mathbb{B}_1$.

Lemma 4.1.2 $\mathbb{B}_1$ *satisfies the inf-sup condition (3.14), that is, there exists* $\beta_1>0$ *such that*

$$\sup_{\substack{\underline{\tau}\in\mathcal{N}(\mathbb{B})\\\underline{\tau}\neq\Theta}}\frac{\Big[\mathbb{B}_1(\underline{\tau}),(\boldsymbol{\varphi},\lambda)\Big]}{\|\underline{\tau}\|_X}\geq\beta_1\|(\boldsymbol{\varphi},\lambda)\|_Y\qquad\forall(\boldsymbol{\varphi},\lambda)\in Y.$$

Proof: Let us first recall that

$$[\mathbb{B}_1(\underline{\tau}),(\boldsymbol{\varphi},\lambda)]=\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma-\langle\mathbf{v}_D\cdot\mathbf{n},\lambda\rangle_\Sigma\qquad\forall\underline{\tau}=(\boldsymbol{\tau}_S,\mathbf{v}_D)\in X,\quad\forall(\boldsymbol{\varphi},\lambda)\in Y,$$

and that

$$\mathcal{N}\,(\mathbb{B})=\Big\{(\boldsymbol{\tau},\mathbf{v})\in X_1\times X_2:\boldsymbol{\tau}=\boldsymbol{\tau}^t,\ \mathbf{div}\,\boldsymbol{\tau}=\mathbf{0}\quad\text{and}\quad\mathrm{div}\,\mathbf{v}\in\mathcal{P}_0(\Omega_D)\Big\}.\tag{4.3}$$

Then, according to the diagonal structure of $\mathbb{B}_1$, the required inf-sup condition is equivalent to the following two independent inf-sup conditions

$$\sup_{\boldsymbol{\tau}_S\in\widetilde{X}_1;\boldsymbol{\tau}_S\neq\mathbf{0}}\frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\varphi}\rangle_\Sigma}{\|\boldsymbol{\tau}_S\|_{X_1}}\geq\beta_1\|\boldsymbol{\varphi}\|\qquad\forall\boldsymbol{\varphi}\in\widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma),$$

$$\sup_{\mathbf{v}_D\in\widetilde{X}_2;\mathbf{v}_D\neq\mathbf{0}}\frac{\langle\mathbf{v}_D\cdot\mathbf{n},\lambda\rangle_\Sigma}{\|\mathbf{v}_D\|_{X_2}}\geq\beta_1\|\lambda\|\qquad\forall\lambda\in W^{\frac{1}{3},\frac{3}{2}}(\Sigma),$$

where

$$\widetilde{X}_1:=\Big\{\boldsymbol{\tau}\in X_1:\boldsymbol{\tau}=\boldsymbol{\tau}^t\ \text{and}\ \ \mathbf{div}\,\boldsymbol{\tau}=\mathbf{0}\Big\}\ \text{and}\ \ \widetilde{X}_2:=\Big\{\mathbf{v}\in X_2:\mathrm{div}\,\mathbf{v}\in\mathcal{P}_0(\Omega_D)\Big\}.\tag{4.4}$$

Now, in a similar way to [21] and [17], we introduce the auxiliary problems:

• Given $\boldsymbol{\varphi} \in \widehat{\mathbf{H}}_{00}^{\frac{1}{2}}(\Sigma)$, find $z \in H^1(\Omega_S)$ such that

$$\begin{aligned} \mathbf{div}\, \mathbf{e}(z) &= \mathbf{0} \qquad \text{in } \Omega_D, \\ \mathbf{e}(z) \cdot \mathbf{n} &= \mathcal{R}_{00}^{-1}(\boldsymbol{\varphi}) \qquad \text{on } \Sigma, \\ z &= 0 \qquad \text{on } \Gamma_D, \end{aligned} \tag{P_3}$$

where $\mathcal{R}_{00} : \left[\widehat{\mathbf{H}}_{00}^{\frac{1}{2}}(\Sigma)\right]' \to \widehat{\mathbf{H}}_{00}^{\frac{1}{2}}(\Sigma)$ is the Riesz application (cf. [21, Sect. 2.4.2]).

• Given $\lambda \in W^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, find $w \in \mathrm{W}^{1,3}(\Omega_D)$ such that

$$\begin{aligned} \operatorname{div}\left(\nabla w\right) &= \langle\, \widehat{\lambda},\ 1\rangle_{\partial\Omega_D}, \\ \nabla w \cdot \mathbf{n} &= \widehat{\lambda}, \\ \int_{\Omega_D} w &= 0, \end{aligned} \tag{P_4}$$

where $\widehat{\lambda} \in \mathrm{W}^{-\frac{1}{3},3}(\partial\Omega_D)$ is a functional depending on λ to be defined later on.

The variational formulation of (P_3) (cf. [21, (2.64)]) is given by: find $z \in \mathbf{H}^1_{\Gamma_D}(\Omega_S)$ such that

$$\int_{\Omega_S} \mathbf{e}(z) : \mathbf{e}(w) = \langle \mathcal{R}_{00}^{-1}(\boldsymbol{\varphi}), \gamma_0(w)\rangle_\Sigma \qquad \forall w \in \mathbf{H}^1_{\Gamma_D}(\Omega_S).$$

In this case, Korn inequality, trace theorem, and Lax-Milgram Lemma establish that the above problem has a unique solution $z \in \mathbf{H}^1_{\Gamma_D}(\Omega_S)$, and there exists $C_S > 0$ such that

$$|z|_{1,\Omega_S} \le C_S \|\boldsymbol{\varphi}\|_{\frac{1}{2},00,\Sigma}. \tag{4.5}$$

Then, defining $\boldsymbol{\sigma}_S := \mathbf{e}(z) - \left(\dfrac{1}{2|\Omega_S|}\displaystyle\int_\Sigma z \cdot \mathbf{n}\right)\mathbf{I}$, and employing the Gauss theorem and (2.17), we readily have

$$\int_{\Omega_S} \operatorname{tr} \boldsymbol{\sigma}_S = 0, \quad \mathbf{div}\, \boldsymbol{\sigma}_S = \mathbf{0}, \ \ \boldsymbol{\sigma}_S = \boldsymbol{\sigma}_S{}^t \text{ and } \left\langle \boldsymbol{\sigma}_S \cdot \mathbf{n}, \boldsymbol{\varphi}\right\rangle_\Sigma = \left\langle e(z) \cdot \mathbf{n}, \boldsymbol{\varphi}\right\rangle_\Sigma,$$

which implies that $\boldsymbol{\sigma}_S \in \widetilde{X}_1$ (cf. (4.4)), and hence, thanks to (4.5), we obtain

$$\|\underline{\boldsymbol{\sigma}_S}\|_{\mathbf{div}\,;\Omega_S} = \|\underline{\boldsymbol{\sigma}_S}\|_{0;\Omega_S} \le |z|_{1,\Omega_S} \le C_S \|\boldsymbol{\varphi}\|_{\frac{1}{2},00,\Sigma}.$$

In turn, for the analysis of the auxiliary problem (P_4) we apply very similar techniques to the ones presented in [17, Lemma 3.2.]. In particular, given $\lambda \in W^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, and according to the norm definition, there exists $\widetilde{\lambda} \in W^{-\frac{1}{3},3}(\Sigma)$ such that

$$\left\langle \widetilde{\lambda},\ \lambda \right\rangle_\Sigma \ge \frac{1}{2} \|\lambda\,\|_{\frac{1}{3},\frac{3}{2};\Sigma} \|\widetilde{\lambda}\,\|_{-\frac{1}{3},3;\Sigma}. \tag{4.6}$$

Next, we introduce a continuous extension of $\widetilde{\lambda}$ onto $W^{-\frac{1}{3},3}(\partial\Omega_D)$, denoted by $\widehat{\lambda}$, which is defined by

$$\left\langle \widehat{\lambda},\ \eta \right\rangle_{\partial\Omega_D} := \left\langle \widetilde{\lambda},\ \eta|_\Sigma \right\rangle_\Sigma \qquad \forall \eta \in W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_D), \tag{4.7}$$

and for which it follows that $\|\widehat{\lambda}\|_{-\frac{1}{3},3;\partial\Omega_D} \leq \|\widetilde{\lambda}\|_{-\frac{1}{3},3;\Sigma}$. In this way, we now consider the auxiliary problem: find $w \in \mathrm{W}^{1,3}(\Omega_D)$ such that

$$\begin{aligned} \mathrm{div}\,(\nabla w) &= \frac{1}{|\Omega_D|}\left\langle \widehat{\lambda},\ 1 \right\rangle_{\partial\Omega_D}, \\ \frac{\partial w}{\partial \nu} &= \widehat{\lambda}, \\ \int_{\Omega_D} w &= 0. \end{aligned}$$

We know from [20] that this problem has a unique solution $w \in \mathrm{W}^{1,3}(\Omega_D)$, which depends continuously on the data. Then, defining $\mathbf{u}_D := \nabla w$, and using this continuous dependence, we obtain

$$\|\mathbf{u}_D\|_{3,\mathrm{div}\,;\Omega_D} \leq \|\nabla w\|_{0,3;\Omega_D} \leq C_D \|\widehat{\lambda}\|_{-\frac{1}{3},3;\partial\Omega_D} \leq C_D \|\widetilde{\lambda}\|_{-\frac{1}{3},3;\Sigma}. \tag{4.8}$$

Moreover, according to Corollary .4.1, and using (4.6),(4.7) and (4.8), we obtain

$$\left\langle \mathbf{u}_D\cdot\mathbf{n},\ \lambda \right\rangle_\Sigma := \left\langle \mathbf{u}_D\cdot\mathbf{n}, E^0_{\Gamma_D}(\lambda) \right\rangle_{\partial\Omega_D} = \left\langle \widehat{\lambda},\ E^0_{\Gamma_D}(\lambda) \right\rangle_{\partial\Omega_D} = \left\langle \widetilde{\lambda},\ \lambda \right\rangle_\Sigma \geq \frac{1}{2C_D}\|\lambda\|_{\frac{1}{3},\frac{3}{2};\Sigma}\|\mathbf{u}_D\|_{3,\mathrm{div}\,;\Omega_D}.$$

Then, thanks to the previous results it is straightforward to see that there exists $\beta_1 = \beta_1(C_S, C_D) > 0$ such that

$$\sup_{\underline{\tau}\in\mathcal{N}(\mathbb{B});\underline{\tau}\neq\mathbf{0}} \frac{\left[\mathbb{B}_1(\underline{\tau}), (\boldsymbol{\varphi},\lambda)\right]}{\|\underline{\tau}\|_X} \geq \frac{\left[\mathbb{B}_1(\boldsymbol{\sigma}_S, \mathbf{u}_D), (\boldsymbol{\varphi},\lambda)\right]}{\|(\boldsymbol{\sigma}_S,\mathbf{u}_D)\|_X} \geq \beta_1\|(\boldsymbol{\varphi},\lambda)\|_Y.$$

In what follows we show that the operator $\mathbb{A}$ satisfies the hypothesis (3.15)-(3.17) of Theorem 3.1.10.

Lemma 4.1.3 *There exist constants $\gamma_1, \gamma_2 > 0$, $\varsigma_1, \varsigma_2 \geq 0$, and $r_1, r_2 \geq 2$, depending only on the domain such that*

$$\|\mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{v})\|_{X_1'\times X_2'} \leq \sum_{j=1}^{2}\left\{\varsigma_j\,\|\mathbf{s}_j - \mathbf{v}_j\|_{X_j} + \gamma_j\,\|\mathbf{s}_j - \mathbf{v}_j\|_{X_j}\left(\|\mathbf{s}_j\|_{X_j} + \|\mathbf{v}_j\|_{X_j}\right)^{r_j-2}\right\},$$

for all $\mathbf{s} := (\mathbf{s}_1, \mathbf{s}_2),\ \mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in X_1 \times X_2$.

Proof: According to the definition of $\mathbb{A}$ (cf. section 2.4), there holds

$$\|\mathbb{A}(\mathbf{s}) - \mathbb{A}(\mathbf{v})\|_{X'} = \left\|\mathcal{A}_S(\mathbf{s}_1) - \mathcal{A}_S(\mathbf{v}_1)\right\|_{X_1'} + \left\|\mathcal{A}_D(\mathbf{s}_2) - \mathcal{A}_D(\mathbf{v}_2)\right\|_{X_2'}, \tag{4.9}$$

where $\mathcal{A}_S$ and $\mathcal{A}_D$ are defined by (2.15) and (2.9), respectively. Next, in order to establish a bound for the first term on the right-hand side of (4.9), we use the definition of $\mathcal{A}_S$ to obtain

$$\|\mathcal{A}_S(\mathbf{s}_1) - \mathcal{A}_S(\mathbf{v}_1)\|_{X_1} \leq \frac{1}{2\mu} \|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1}.$$

Now, concerning the second term on the right-hand side of (4.9), it follows from to the definition of $\mathcal{A}_D$, the general hypotheses on $\mathbf{K}$, and the triangle inequality, that

$$\|\mathcal{A}_D(\mathbf{s}_2) - \mathcal{A}_D(\mathbf{v}_2)\|_{X_2'} \leq \frac{\mu}{\rho} \left\|\mathbf{K}^{-1}\right\|_\infty \|\mathbf{s}_2 - \mathbf{v}_2\|_{0,\frac{3}{2};\Omega_D} + \frac{\beta}{\rho} \|\mathbf{s}_2 - \mathbf{v}_2\|_{0,3;\Omega_D} \left\{\|\mathbf{s}_2\|_{0,3;\Omega_D} + \|\mathbf{v}_2\|_{0,3;\Omega_D}\right\}.$$

Now, according to Jensen's inequalities (or Holder inequality cf. [1]), we have

$$\|\mathbf{s}_2 - \mathbf{v}_2\|_{0,\frac{3}{2};\Omega_D} \leq \widetilde{C} \|\mathbf{s}_2 - \mathbf{v}_2\|_{0,3;\Omega_D} \leq \widetilde{C} \|\mathbf{s}_2 - \mathbf{v}_2\|_{3,\mathrm{div};\Omega_D},$$

and hence (3.15) is satisfied setting $r_1 = 2,\ r_2 = 3,\ \varsigma_1 = 0,\ \varsigma_2 = \widetilde{C}\frac{\mu}{\rho}\left\|\mathbf{K}^{-1}\right\|_\infty,\ \gamma_1 = \frac{1}{2\mu}$ and $\gamma_2 = \frac{\mu}{\rho}$.

Next, we proceed to show the monotonicity of $\mathbb{A}$, with $r_1 = 2$ and $r_2 = 3$.

Lemma 4.1.4 *$\mathbb{A}$ satisfies the hypothesis (3.16), which means that, given $\mathbf{t} \in X \setminus \mathcal{N}(\mathbb{B})$, the operator $\mathbb{A}(\cdot + \mathbf{t}) : \mathcal{N}(\mathbb{B}) \mapsto \mathcal{N}(\mathbb{B})'$ is a strictly monotone mapping. More precisely, there exists $\alpha > 0$, independent of*
$\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2) \in X_1 \times X_2$, such that

$$\Big[\mathbb{A}\big(\mathbf{s} + \mathbf{t}\big) - \mathbb{A}\big(\mathbf{v} + \mathbf{t}\big), \mathbf{s} - \mathbf{v}\Big] \geq \alpha\Big\{\|\mathbf{s}_1 - \mathbf{v}_1\|_{X_1}^2 + \|\mathbf{s}_2 - \mathbf{v}_2\|_{X_2}^3\Big\},$$

for all $\mathbf{s} := (\mathbf{s}_1, \mathbf{s}_2),\ \mathbf{v} := (\mathbf{v}_1, \mathbf{v}_2) \in \mathcal{N}(\mathbb{B}) \subseteq X_1 \times X_2$.

Proof: Similarly to Lemma 4.1.3, and according to the definition of $\mathbb{A}$, we have

$$\begin{aligned} &\Big[\mathbb{A}\big(\mathbf{s} + \mathbf{t}\big) - \mathbb{A}\big(\mathbf{v} + \mathbf{t}\big), \mathbf{s} - \mathbf{v}\Big] \\ &= \Big[\mathcal{A}_S\left(\mathbf{s}_1 + \mathbf{t}_1\right) - \mathcal{A}_S\left(\mathbf{v}_1 + \mathbf{t}_1\right), \mathbf{s}_1 - \mathbf{v}_1\Big]_S + \Big[\mathcal{A}_D\left(\mathbf{s}_2 + \mathbf{t}_2\right) - \mathcal{A}_D\left(\mathbf{v}_2 + \mathbf{t}_2\right), \mathbf{s}_2 - \mathbf{v}_2\Big]_D. \end{aligned} \tag{4.10}$$

Now, we proceed to give a lower bound for the first term on the right-hand side of (4.10). In fact, thanks to the definition of $\mathcal{A}_S$ (cf. (2.15)), there holds

$$\Big[\mathcal{A}_S\left(\mathbf{s}_1 + \mathbf{t}_1\right) - \mathcal{A}_S\left(\mathbf{v}_1 - \mathbf{t}_1\right), \mathbf{s}_1 - \mathbf{v}_1\Big]_S = \frac{1}{2\mu} \left\|\left(\mathbf{s}_1 - \mathbf{v}_1\right)^d\right\|_{0;\Omega_S}^2,$$

and then, noticing that $\mathbf{div}\,(\mathbf{s}_1 - \mathbf{v}_1) = \mathbf{0}$ and $\int_{\Omega_S} \mathrm{tr}\,(\mathbf{s}_1 - \mathbf{v}_1) = 0$, and applying [9, proposition 3.1, Ch. IV] we find that, there exists $C_S > 0$ such that

$$\left\|\left(\mathbf{s}_1 - \mathbf{v}_1\right)^d\right\|_{0;\Omega_S}^2 \geq C_S \|\mathbf{s}_1 - \mathbf{v}_1\|_{0;\Omega_S}^2 = C_S \|\mathbf{s}_1 - \mathbf{v}_1\|_{\mathbf{div};\Omega_S}^2,$$

which yields

$$\Big[\mathcal{A}_S\,(\mathbf{s}_1+\mathbf{t}_1)-\mathcal{A}_S\,(\mathbf{v}_1-\mathbf{t}_1)\,,\mathbf{s}_1-\mathbf{v}_1\Big]_S \geq \frac{C_S}{2\mu}\,\|\mathbf{s}_1-\mathbf{v}_1\|^2_{\mathbf{div}\,;\Omega_S}\,.$$

In turn, in order to give a lower bound for the second term on the right-hand side of (4.10), we first notice that

$$\begin{aligned}&\Big[\mathcal{A}_D\,(\mathbf{s}_2+\mathbf{t}_2)-\mathcal{A}_D\,(\mathbf{v}_2+\mathbf{t}_2)\,,\,\mathbf{s}_2-\mathbf{v}_2\Big]_D\\&=\frac{\mu}{\rho}\Big[\mathbf{K}^{-1}(\mathbf{s}_2-\mathbf{v}_2),\ \mathbf{s}_2-\mathbf{v}_2\Big]_D+\frac{\beta}{\rho}\Big[\big|\mathbf{s}_2+\mathbf{t}_2\big|(\mathbf{s}_2+\mathbf{t}_2)-\big|\mathbf{v}_2+\mathbf{t}_2\big|(\mathbf{v}_2+\mathbf{t}_2),\ \mathbf{s}_2-\mathbf{v}_2\Big]_D.\end{aligned}\tag{4.11}$$

Now, according to (2.10), the first term on the right-hand side of (4.11) can be bounded as follows

$$\Big[\mathbf{K}^{-1}(\mathbf{s}_2-\mathbf{v}_2),\ \mathbf{s}_2-\mathbf{v}_2\Big]_D \geq \lambda_o\|\mathbf{s}_2-\mathbf{v}_2\|^2_{L^2(\Omega_D)}.$$

In turn, due to [29, Lemme 5.1], for the second term on the right-hand side of (4.11) we have

$$\Big[\big|\mathbf{s}_2+\mathbf{t}_2\big|(\mathbf{s}_2+\mathbf{t}_2)-\big|\mathbf{v}_2+\mathbf{t}_2\big|(\mathbf{v}_2+\mathbf{t}_2),\ \mathbf{s}_2-\mathbf{v}_2\Big]_D \geq c\,\|\mathbf{s}_2-\mathbf{v}_2\|^3_{L^3(\Omega_D)}\,,$$

where c is a positive constant independent of $\mathbf{t}$. Finally, thanks to Lemma .4.5, there exists $C_D>0$ such that

$$\Big[\big|\mathbf{s}_2+\mathbf{t}_2\big|(\mathbf{s}_2+\mathbf{t}_2)-\big|\mathbf{v}_2+\mathbf{t}_2\big|(\mathbf{v}_2+\mathbf{t}_2),\ \mathbf{s}_2-\mathbf{v}_2\Big]_D \geq c\,\|\mathbf{s}_2-\mathbf{v}_2\|^3_{L^3(\Omega_D)} \geq C_D\,\|\mathbf{s}_2-\mathbf{v}_2\|^3_{3,\mathrm{div}\,;\Omega_D}\,.$$

Therefore, setting $\alpha=\min\Big\{\frac{C_S}{2\mu},\ C_D\Big\}$, the proof is completed.

Next, we prove the hemi-continuity property of $\mathbb{A}$.

Lemma 4.1.5 $\mathbb{A}$ *satisfies the hypothesis* (3.17) *i.e., given* $\mathbf{t}:=(\mathbf{t}_1,\mathbf{t}_2)\in X\setminus\mathcal{N}(\mathbb{B})$ *the map* $\mathbb{A}\,(\cdot+\mathbf{t}):\mathcal{N}(\mathbb{B})\mapsto\mathcal{N}(\mathbb{B})'$ *is hemi-continuous.*

Proof: Given $\mathbf{s}:=(\mathbf{s}_1,\mathbf{s}_2),\ \mathbf{v}:=(\mathbf{v}_1,\mathbf{v}_2)\in\mathcal{N}(\mathbb{B})$, we introduce the real function

$$\begin{aligned}G:\mathbb{R}&\mapsto\mathbb{R}\\ t&\mapsto G(t):=\Big\langle\mathbb{A}\,(\mathbf{s}+t\mathbf{v}+\mathbf{t})\,,\mathbf{v}\Big\rangle\\&\qquad=\Big\langle\mathcal{A}_S(\mathbf{s}_1+t\mathbf{v}_1+\mathbf{t}_1),\mathbf{v}_1\Big\rangle_S+\Big\langle\mathcal{A}_D(\mathbf{s}_2+t\mathbf{v}_2+\mathbf{t}_2),\mathbf{v}_2\Big\rangle_D.\end{aligned}$$

According to the definition $\mathcal{A}_S$, is clear that $\mathcal{A}_S$ is a Lipschitz continuous operator. Now, in order to show that the second term in right-hand side of the las equality is a continuous function, we refer to [27, Proposition 3].

4.2 The Galerkin scheme of $(\mathbf{P}_\alpha)$

In this section we introduce and analyse a Galerkin scheme for the problem $(\mathbf{P}_\alpha)$ (cf. (2.18)).

4.2.1 Preliminaries

Here we define the discrete system and establish suitable assumptions on the finite element subspaces ensuring later on that it becomes well posed. For this purpose, we first select two collections of discrete spaces

$$
\begin{aligned}
\mathbf{H}_h(\Omega_S) \subseteq \mathbf{H}(\mathrm{div}\,;\Omega_S),\ \mathbf{H}_h(\Omega_D) \subseteq \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div}\,;\Omega_D),\\
L_h(\Omega_S) \subseteq L^2(\Omega_S),\ L_h(\Omega_D) \subseteq L_0^{\frac{3}{2}}(\Omega_D),\\
\Lambda_h^S(\Sigma) \subseteq \widehat{\mathbf{H}}_{00}^{\frac{1}{2}}(\Sigma),\ \Lambda_h^D(\Sigma) \subseteq W^{\frac{1}{3},\frac{3}{2}}(\Sigma),\\
\mathbb{L}_h^2(\Omega_S) \subseteq \mathbb{L}^2_{skew}(\Omega_S).
\end{aligned}
\tag{4.12}
$$

Note that the spaces for the Stokes domain will have to be doubled. In particular, for the unknown $\boldsymbol{\sigma}_S$ we consider the space of matrix-valued functions whose rows belong to $\mathbf{H}_h(\Omega_S)$. According to this we set

$$
\begin{aligned}
\mathbf{L}_h(\Omega_S) &:= L_h(\Omega_S) \times L_h(\Omega_S), \quad \boldsymbol{\Lambda}_h^S(\Sigma) := \Lambda_h^S(\Sigma) \times \Lambda_h^S(\Sigma),\\
\mathbb{H}_h(\Omega_S) &:= \Big\{ \boldsymbol{\tau} : \Omega_S \mapsto \mathbb{R}^{2\times 2};\ \mathbf{c}^{\mathbf{t}}\cdot \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall \mathbf{c} \in \mathbb{R}^2 \Big\} \subseteq \mathbb{H}(\mathbf{div}\,;\Omega_S),\\
\mathbb{H}_{h,0}(\Omega_S) &:= \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}\,;\Omega_S),\\
X_{S,h} &:= \mathbb{H}_{h,0}(\Omega_S),\ X_{D,h} := \mathbf{H}_h(\Omega_D).
\end{aligned}
$$

In this way, we define the global finite element subespaces as

$$
\begin{aligned}
X_h &:= X_{S,h} \times X_{D,h},\\
Y_h &:= \boldsymbol{\Lambda}_h^S(\Sigma) \times \Lambda_h^D(\Sigma),\\
Z_h &:= \mathbf{L}_h(\Omega_S) \times L_h(\Omega_D) \times \mathbb{L}_h^2(\Omega_S),
\end{aligned}
$$

and the Galerkin scheme associated with $(\mathbf{P}_\alpha)$ is given by:

Find $(\underline{\sigma}_h, \underline{u}_h, \underline{\eta}_h) \in X_h \times Y_h \times Z_h$ such that:

$$
\begin{array}{lllll}
[\mathbb{A}(\underline{\sigma}_h), \underline{\tau}_h] & + & [\mathbb{B}_1(\underline{\tau}_h), \underline{u}_h] & + \left[\mathbb{B}(\underline{\tau}_h), \underline{\eta}_h\right] & = [F, \underline{\tau}_h] \quad \forall \underline{\tau}_h \in X_h, \\
[\mathbb{B}_1(\underline{\sigma}_h), \underline{v}_h] & - & [\mathbb{C}(\underline{u}_h), \underline{v}_h] & & = [G, \underline{v}_h] \quad \forall \underline{v}_h \in Y_h, \\
[\mathbb{B}(\underline{\sigma}_h), \underline{\vartheta}_h] & & & & = [E, \underline{\vartheta}] \quad \forall \underline{\vartheta}_h \in Z_h.
\end{array}
\tag{4.13}
$$

According to Theorem 3.2.1, we now aim to show the discrete inf-sup condition for $\mathbb{B}$. More precisely, due to the diagonal structure of $\mathbb{B}$, it is suffices to show that there exists $\widehat{\beta} > 0$, independent of h, such that

$$
\sup_{\boldsymbol{\tau}_{S,h} \mathbb{H}_{h,0}(\Omega_S)\backslash\Theta} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{S,h}, \mathbf{u}_{S,h})_S + \left(\boldsymbol{\tau}_{S,h}, \boldsymbol{\gamma}_{S,h}\right)_S}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}\,;\Omega_S}} \geq \widehat{\beta}\left\|\left(\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}\right)\right\| \qquad \forall \left(\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}\right) \in \mathbf{L}_h(\Omega_S) \times \mathbb{L}^2_h(\Omega_S),
$$

and

$$
\sup_{\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D)\backslash\mathbf{0}} \frac{(\operatorname{div}\mathbf{v}_{D,h}, p_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\operatorname{div};\Omega_D}} \geq \widehat{\beta}\|p_{D,h}\| \qquad \forall p_{D,h} \in L_h(\Omega_D).
$$

Now, in order to have a more explicit definition of the discrete kernel $\mathbb{V}_h$ of $\mathbb{B}$, we introduce the following assumptions:

$$
\operatorname{div}\mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S) \quad \text{and} \quad \operatorname{div}\mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D).
$$

Because of the diagonal form of $\mathbb{B}$, $\mathbb{V}_h$ can be writen as $\mathbb{V}_h = \widetilde{X}_{S,h} \times \widetilde{X}_{D,h}$, where

$$
\begin{aligned}
\widetilde{X}_{D,h} &:= \left\{\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D) : \ \operatorname{div}\mathbf{v}_{D,h} \in \mathbb{P}_0(\Omega_D)\right\}, \\
\widetilde{X}_{S,h} &:= \left\{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) : \quad \left(\boldsymbol{\tau}_{S,h}, \boldsymbol{\rho}_{S,h}\right)_S = 0 \ \ \forall \boldsymbol{\rho}_{S,h} \in \mathbb{L}^2_h(\Omega_S) \text{ and } \mathbf{div}\,\boldsymbol{\tau}_{S,h} = \mathbf{0}\right\}.
\end{aligned}
$$

Similarly, due to the diagonal structure of $\mathbb{B}_1$ as well, its discrete inf-sup condition as equivalent to proving that there exists $\widehat{\beta}_1 > 0$ such that

$$
\sup_{\boldsymbol{\tau}_{S,h} \in \widetilde{\mathbb{H}}_{h,0}(\Omega_S)\backslash\Theta} \frac{\left\langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\varphi}\right\rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div};\Omega_S}} \geq \widehat{\beta}_1\|\boldsymbol{\varphi}\|_{\frac{1}{2},\Sigma} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{\Lambda}^S_h(\Sigma),
$$

and

$$
\sup_{\mathbf{v}_{D,h} \in \widetilde{\mathbf{H}}_h(\Omega_D)\backslash\mathbf{0}} \frac{\left\langle \mathbf{v}_{D,h}\cdot \mathbf{n}, \lambda\right\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\operatorname{div};\Omega_S}} \geq \widehat{\beta}_1\|\lambda\|_{\frac{1}{2},\frac{3}{2},\Sigma} \qquad \forall \lambda \in \Lambda^D_h(\Sigma).
$$

Summarizing, we introduce the following assumptions

$(\mathbf{H}.\mathbf{1})$ there exists $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{S,h}\mathbb{H}_{h,0}(\Omega_S)\backslash\Theta} \frac{(\mathbf{div}\,\boldsymbol{\tau}_{S,h}, \mathbf{u}_{S,h})_S + \big(\boldsymbol{\tau}_{S,h}, \boldsymbol{\gamma}_{S,h}\big)_S}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}\,;\Omega_S}} \geq \widehat{\beta}\,\big\|\big(\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}\big)\big\| \qquad \forall\,\big(\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}\big) \in \mathbf{L}_h(\Omega_S) \times \mathbb{L}^2_h(\Omega_S),$$

and

$$\sup_{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)\backslash\mathbf{0}} \frac{(\mathrm{div}\,\mathbf{v}_{D,h}, p_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \geq \widehat{\beta}\|p_{D,h}\| \qquad \forall p_{D,h} \in L_h(\Omega_D).$$

$(\mathbf{H}.\mathbf{2})$ $\mathrm{div}\,\mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S)$ and $\mathrm{div}\,\mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D)$.

$(\mathbf{H}.\mathbf{3})$ there exists $\widehat{\beta}_1 > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{S,h}\in\widetilde{\mathbb{H}}_{h,0}(\Omega_S)\backslash\Theta} \frac{\Big\langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\varphi}\Big\rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div};\Omega_S}} \geq \widehat{\beta}_1\|\boldsymbol{\varphi}\|_{\frac{1}{2},\Sigma} \qquad \forall \boldsymbol{\varphi} \in \boldsymbol{\Lambda}^S_h(\Sigma),$$

and

$$\sup_{\mathbf{v}_{D,h}\in\widetilde{\mathbf{H}}_h(\Omega_D)\backslash\mathbf{0}} \frac{\Big\langle \mathbf{v}_{D,h}\cdot\mathbf{n}, \lambda\Big\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_S}} \geq \widehat{\beta}_1\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma} \qquad \forall \lambda \in \Lambda^D_h(\Sigma).$$

From now on we assume that the arbitrary finite element subespaces introduced in (4.12) satisfy the previous derived hypotheses $(\mathbf{H}.\mathbf{1}), (\mathbf{H}.\mathbf{2}), (\mathbf{H}.\mathbf{3})$. Hence, it remains to show that the assumptions of Theorem 3.2.1 are satisfied. Since (4.13) is a conforming method, it is suffices to show that the monotonicity (cf. (3.23)) and the hemicontinuity (cf. (3.24)) assumptions are satisfied. Thus, we have the following Lemmas

Lemma 4.2.1 *Given* $\mathbf{t}_h = (\mathbf{t}_{1,h}, \mathbf{t}_{2,h}) \in \mathbb{V}_h^\perp \equiv X_h \setminus \mathbb{V}_h = \Big(\mathbb{H}_{h,0}(\Omega_S) \setminus \widetilde{X}_{S,h}\Big) \times \Big(\mathbf{H}_h(\Omega_D) \setminus \widetilde{X}_{D,h}\Big)$, *the map* $\mathbb{A}(\cdot + \mathbf{t}_h) : \mathbb{V}_h^\perp \mapsto \big(\mathbb{V}_h^\perp\big)'$ *is a strictly monotone mapping. More precisely, there exists* $\alpha > 0$, *independent of* $\mathbf{t}_h$ *and* h, *such that*

$$\Big[\mathbb{A}(\mathbf{s}_h + \mathbf{t}_h) - \mathbb{A}(\mathbf{v}_h + \mathbf{t}_h), \mathbf{s}_h - \mathbf{v}_h\Big] \geq \alpha\Big\{\|\mathbf{s}_{1,h} - \mathbf{v}_{1,h}\|^2_{\mathbf{div}\,;\Omega_S} + \|\mathbf{s}_{2,h} - \mathbf{v}_{2,h}\|^3_{3;\mathrm{div}\,;\Omega_D}\Big\},$$

for all $\mathbf{s}_h := (\mathbf{s}_{1,h}, \mathbf{s}_{2,h})$, $\mathbf{v}_h = (\mathbf{v}_{1,h}, \mathbf{v}_{2,h}) \in \mathbb{V}_h := \widetilde{X}_{S,h} \times \widetilde{X}_{D,h}$.

Proof: It follows from Lemma 4.1.4, by noting that $\mathbf{div}\,(\mathbf{s}_{S,h} - \mathbf{v}_{S,h}) = \mathbf{0}$ and $\mathrm{div}\,(\mathbf{s}_{D,h} - \mathbf{v}_{D,h}) \in \mathcal{P}_0(\Omega_D)$.

Lemma 4.2.2 *The map* $\mathbb{A}(\cdot + \mathbf{t}_h) : \mathbb{V}_h^{\perp} \mapsto \left(\mathbb{V}_h^{\perp}\right)'$ *is hemi-continuous, i.e., given* $\mathbf{s}_h, \mathbf{v}_h \in \mathbb{V}_h^{\perp}$*, the function*

$$\begin{aligned} G_h :\ & \mathbb{R} \mapsto \mathbb{R} \\ & t \mapsto G_h(t) := \Big\langle \mathbb{A}(\mathbf{s}_h + t\mathbf{v}_h + \mathbf{t}_h), \mathbf{v}_h \Big\rangle \qquad \textit{is continuous.} \end{aligned}$$

Proof: It follows as in the proof of Lemma 4.1.5. We omit further details.

4.2.2 Particular choice of finite element subspaces

Now we specify a possible choice of finite elements for this problem. In order to do it, some preliminary definitions are necessary. Let $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be triangulations for Ω_S and Ω_D, respectively, both shape-regular in the sense of Ciarlet-Raviart (cf. [13, page 247]), and let us assume that they match on Σ, that is $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. Furthermore, given an integrer $k \geq 0$ and a subset $\mathrm{S} \subseteq \mathbb{R}^2$, we let $P_k(\mathrm{S})$ be the space of the polynomials defined on S of total degree at most k, and denote $\mathbf{P}_k(\mathrm{S})$ and $\mathbb{P}_k(\mathrm{S})$ as $\left[P_k(\mathrm{S})\right]^2$ and $\left[P_k(\mathrm{S})\right]^{2\times 2}$, respectively. In addition, let b_T be the element-bubble function defined as the unique polynomial in $P_3(T)$ that vanishies on ∂T with $\int_T b_T = 1$, and denote by $\mathbf{x} := (x_1, x_2)$ a generic vector of $\mathbb{R}^2$. Then, we define for each $\mathrm{T} \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ the local Raviart-Thomas and bubble spaces of order 0, respectively, by

$$\mathbf{RT}_0(\mathrm{T}) := \mathbf{P}_0(\mathrm{T}) \oplus P_0(\mathrm{T})\mathbf{x} \quad \text{and} \quad \mathbf{B}_0(\mathrm{T}) := P_0(\mathrm{T}) \left(\frac{\partial b_T}{\partial x_2}, -\frac{\partial b_T}{\partial x_1} \right).$$

PEERS for Stokes and Raviart-Thomas for Darcy - Forchheimer

$$\begin{aligned} \mathbf{H}_h(\Omega_S) &:= \left\{ \mathbf{u} \in \mathbf{H}(\mathrm{div}\,; \Omega_S) :\ \mathbf{u}|_T \in \mathbf{RT_0}(\mathrm{T}) \oplus \mathbf{B_0}(\mathrm{T}) \ \ \forall \mathrm{T} \in \mathcal{T}_{\mathbf{h}}^{\mathbf{S}} \right\}, \\ \mathbf{H}_h(\Omega_D) &:= \left\{ \mathbf{u} \in \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div}\,; \Omega_D) :\ \mathbf{u}|_{\mathrm{T}} \in \mathbf{RT_0}(\mathrm{T}) \ \ \forall \mathrm{T} \in \mathcal{T}_{\mathbf{h}}^{\mathbf{D}} \right\}, \\ L_h(\Omega_S) &:= \left\{ \mathbf{v} \in \mathbf{L}^2(\Omega_S) : \mathbf{v}|_T \in \mathcal{P}_0(\mathrm{T})^2 \ \ \forall \mathrm{T} \in \mathcal{T}_h^S \right\}, \\ L_h(\Omega_D) &:= \left\{ q \in L_0^{\frac{3}{2}}(\Omega_D) : q|_T \in \mathcal{P}_0(\mathrm{T}) \ \ \forall \mathrm{T} \in \mathcal{T}_h^D \right\}, \\ \mathbb{L}_h^2\left(\Omega_S\right) &:= \left\{ \eta \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} : \eta \in \mathcal{C}\left(\overline{\Omega}_S\right),\ \eta|_T \in \mathcal{P}_1(\mathrm{T}) \ \ \forall \mathrm{T} \in \mathcal{T}_h^S \right\} \subseteq \mathbb{L}^2_{skew}(\Omega_S). \end{aligned}$$

Note that the foregoing definitions mean that we are considering PEERS elements (cf. [2]) for Stokes, while for Darcy - Forchheimer the Raviart-Thomas elements (cf. [9, 3.12, Ch. III]) are employed.

Now for the interface Σ we assume, without loss of generality, that the number of edges on Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h, and denote the resulting edges still by e. Since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation (that is, the ratio of lengths of adjacent adges is bounded) and, therefore, so is Σ_{2h}. Hence, denoting by x_0 and x_1 the extreme points of Σ, we define

$$\begin{aligned}
\Lambda_h^S(\Sigma) &:= \Big\{ \psi \in \mathcal{C}(\Sigma): \ \psi|_e \in P_1(e) \ \forall e \in \Sigma_{2h}, \ \psi(x_0) = \psi(x_1) = 0 \Big\}, \\
\mathbf{\Lambda}_h^S(\Sigma) &:= \Big(\Lambda_h^S(\Sigma) \times \Lambda_h^S(\Sigma) \Big) \cap \Big\{ \mathbf{v} \in \mathbf{H}_{00}^{\frac{1}{2}}(\Sigma) : \langle \mathbf{v} \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \Big\}, \\
\Lambda_h^D(\Sigma) &:= \Big\{ \psi : \Sigma \mapsto \mathbb{R} : \ \psi|_e \in P_0(e) \ \ \forall e \in \Sigma_h \Big\}, \\
\Phi_h(\Sigma) &:= \Big\{ \xi_h : \Sigma \mapsto \mathbb{R} : \ \xi|_e \in P_0(e) \ \ \forall e \in \Sigma_h \Big\}, \\
\Phi_h^S(\Sigma) &= \Phi_h^D(\Sigma) := \Phi_h(\Sigma).
\end{aligned} \tag{4.14}$$

We now turn to state the assumptions under which it is possible to ensure the validity of (**H.1**), (**H.2**) and (**H.3**). Recall that they are given by:

(**H.1**) there exists $\widehat{\beta} > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{S,h} \mathbb{H}_{h,0}(\Omega_S) \setminus \Theta} \frac{(\mathbf{div}\, \boldsymbol{\tau}_{S,h}, \mathbf{u}_{S,h})_S + (\boldsymbol{\tau}_{S,h}, \boldsymbol{\gamma}_{S,h})_S}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}\,;\Omega_S}} \geq \widehat{\beta} \, \|(\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h})\| \qquad \forall \, (\mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}) \in \mathbf{L}_h(\Omega_S) \times \mathbb{L}_h^2(\Omega_S),$$

and

$$\sup_{\mathbf{v}_{D,h} \in \mathbf{H}_h(\Omega_D) \setminus \mathbf{0}} \frac{(\operatorname{div} \mathbf{v}_{D,h}, p_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\operatorname{div};\Omega_D}} \geq \widehat{\beta} \|p_{D,h}\|_{0,\frac{3}{2}} \qquad \forall p_{D,h} \in L_h(\Omega_D).$$

(**H.2**) $\operatorname{div} \mathbf{H}_h(\Omega_S) \subseteq L_h(\Omega_S)$ and $\operatorname{div} \mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D)$.

(H.3) there exists $\widehat{\beta}_1 > 0$, independent of h, such that

$$\sup_{\boldsymbol{\tau}_{S,h} \in \widetilde{\mathbb{H}}_{h,0}(\Omega_S) \setminus \Theta} \frac{\Big\langle \boldsymbol{\tau}_{S,h} \mathbf{n}, \boldsymbol{\varphi} \Big\rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div};\Omega_S}} \geq \widehat{\beta}_1 \|\boldsymbol{\varphi}\|_{\frac{1}{2},\Sigma} \qquad \forall \boldsymbol{\varphi} \in \mathbf{\Lambda}_h^S(\Sigma),$$

and

$$\sup_{\mathbf{v}_{D,h}\in\widetilde{\mathbf{H}}_h(\Omega_D)\backslash\mathbf{0}} \frac{\Big\langle \mathbf{v}_{D,h}\cdot\mathbf{n},\lambda\Big\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_S}} \geq \widehat{\beta}_1\|\lambda\|_{\frac{1}{3},\frac{3}{2},\Sigma} \qquad \forall\lambda\in\Lambda_h^D(\Sigma).$$

Note that **(H.2)** holds trivially from the definitions given by (4.14). Next, for the Stokes terms of **(H.1)** and **(H.3)**, and following [25], we need to introduce the hypothesis of quasiuniformity in a neighborhood of the interface Σ on the Ω_S-side, namely Ω_Σ^S. More precisely, we assume that Ω_Σ^S has Lipschitz continuous boundary and that there exists $c>0$, independent of h, such that

$$\max_{T\subseteq\Omega_\Sigma^S} h_T \leq c \min_{T\subseteq\Omega_\Sigma^S} h_T \qquad \forall h<h_0.$$

Under these new assumptions, it enough to prove the Darcy - Forchheimer part of (**H**.**1**) and (**H**.**3**). To this end, we need to introduce the Raviart-Thomas interpolation operator of lowest order in Ω_D.

Indeed, given a sufficiently smooth vector field $\mathbf{v}:\Omega_D\mapsto\mathbb{R}^2$, we define $\Pi_h^D(\mathbf{v})$ as the only element of $\mathbf{H}_h(\Omega_D)$ such that

$$\int_e \Pi_h^D(\mathbf{v})\cdot\mathbf{n} = \int_e \mathbf{v}\cdot\mathbf{n} \qquad \forall e\in\mathcal{E}_h^D, \tag{4.15}$$

where $\mathcal{E}_h^D$ is the set of edges of the triangulation $\mathcal{T}_h^D$. The main properties of this operator are collected in what follows.

(a) for each $p\in]2,+\infty[$ the interpolation operator Π_h^D is well defined in $W^{0,p}(\mathrm{div}\,;\Omega_D)$ (cf. [9, III.3.3]).

(b) for each $p\in]2,+\infty[$ there holds

$$\Big(\mathrm{div}\,\Pi_h^D(\mathbf{v}),q_h\Big)_D = (\mathrm{div}\,\mathbf{v},q_h)_D \qquad \forall q_h\in L_h(\Omega_D),\ \forall\mathbf{v}\in W^{0,p}(\mathrm{div}\,;\Omega_D).$$

(c) if $\mathbf{v}\cdot n\in\Phi_h(\Sigma)$, then $\Pi_h^D(\mathbf{v})\cdot n=\mathbf{v}\cdot n$.

(d) there exists $C_D>0$, independent of h, such that for each $p\in]2,+\infty[$ there holds

$$\|\Pi_h^D(\mathbf{v})\|_{0,p,\Omega_D} \leq C_D\Big\{\|\mathbf{v}\|_{0,p,\Omega_D}+\|\mathrm{div}\,\mathbf{v}\|_{0,p,\Omega_D}\Big\} \qquad \forall\mathbf{v}\in W^{0,p}(\mathrm{div}\,;\Omega_D).$$

- We notice that the foregoing estimate follows from Lemma .4.2, and in this case the regularity $H^\delta(\Omega_D)$ or $W^{\delta,p}(\Omega_D)$ is not necessary.

Now, we can prove **(H.1)** for the Darcy-Forchheimer part only.

Lemma 4.2.3 *There exists $\widehat{\beta} > 0$, independent of h, such that*

$$\sup_{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)\setminus\mathbf{0}} \frac{(\operatorname{div}\mathbf{v}_{D,h}, p_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\operatorname{div};\Omega_D}} \geq \widehat{\beta}\|p_{D,h}\|_{0,\frac{3}{2}} \qquad \forall p_{D,h}\in L_h(\Omega_D)\subseteq L_0^{\frac{3}{2}}(\Omega_D).$$

Proof: Similarly as [39, Exemple 3], we now proced locally on each triangle in $\mathcal{T}_h^D$. More precisely, we consider $p_{D,h}\in \mathrm{L}_h(\Omega_D)$ and define $\widehat{p}_h \in \mathrm{L}^3\left(\mathcal{T}_h^D\right)$ as

$$\widehat{p}_h|_T := \widehat{p_{D,h}|_T} \qquad \forall T\in\mathcal{T}_h^D,$$

which agrees with the notations provided in Lemma .4.4. Then we set $\widetilde{p}_h \in \mathrm{L}_0^3\left(\mathcal{T}_h^D\right)$ as

$$\widetilde{p}_h := \widehat{p}_h - \frac{1}{\Omega_D}\int_{\Omega_D}\widehat{p}_h.$$

Similarly to the proof of the continuous inf-sup condition for $\mathbb{B}$, Lemma 4.1.1, we now look for $\mathbf{u}_{D,h}\in\mathbf{H}_h(\Omega_D)$ such that $\operatorname{div}\mathbf{u}_{D,h} = \widetilde{p}_h$. To this end, we consider the problem

$$\begin{aligned} \operatorname{div}(\nabla z) &= \widetilde{p}_h \qquad \text{in } \Omega_D,\\ \gamma_\nu(\nabla z) &= 0 \qquad \text{on } \partial\Omega_D,\\ \int_{\Omega_D} z &= 0. \end{aligned} \tag{4.16}$$

Since $\widetilde{p}_h \in L_0^3(\Omega_D)$, we deduce from [15] that the foregoing problem has a unique solution $z\in W^{1+\delta,3}(\Omega_D)$ with $\delta\in\left(0,\ \frac{1}{3}\right)$. Note that $\nabla z\in\mathbf{W}^{0,3}(\operatorname{div};\Omega_D)$. Then, defining $\mathbf{u}_{D,h} = \Pi_h^D(\nabla z)$, we find that the continuous dependence of (4.16) and the properties of the Raviart-Thomas interpolator imply that

$$\|\mathbf{u}_{D,h}\|_{0,3;\Omega_D} \leq C_D\|\widehat{p}_h\|_{0,3;\Omega_D} \leq C_D\left\{\|p_{D,h}\|_{0,\frac{3}{2},\Omega_D}\right\}^{\frac{1}{2}},$$

and

$$\|\operatorname{div}\mathbf{u}_{D,h}\|_{0,3;\Omega_D} = \left\|\operatorname{div}\Pi_h^D(\nabla z)\right\|_{0,3;\Omega_D} \leq \|\widehat{p}_h\|_{0,3;\Omega_D} \leq \left\{\|p_{D,h}\|_{0,\frac{3}{2},\Omega_D}\right\}^{\frac{1}{2}}.$$

Therefore, bounding lowerly with $\mathbf{u}_{D,h}\in\mathbf{H}_h(\Omega_D)$, we deduce that

$$\sup_{\mathbf{v}_{D,h}\in\mathbf{H}_h(\Omega_D)\setminus\mathbf{0}} \frac{(\operatorname{div}\mathbf{v}_{D,h}, p_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{3,\operatorname{div};\Omega_D}} \geq \frac{1}{C_D+1}\|p_{D,h}\|_{0,\frac{3}{2};\Omega_D} \qquad \forall p_{D,h}\in L_h(\Omega_D),$$

which finishes the proof.

Lemma 4.2.4 *there exists $\widehat{\beta}_1 > 0$, independent of h, such that*

$$\sup_{\mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_D)\setminus \mathbf{0}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \lambda_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{3,\mathrm{div};\Omega_D}} \geq \widehat{\beta}_1 \|\lambda_h\|_{\frac{1}{3},\frac{3}{2},\Sigma} \qquad \forall \lambda_h \in \Lambda_h^D(\Sigma)$$

Proof: Similarly to the proof of the continuous inf-sup condition for $\mathbb{B}_1$ (cf. Lemma 4.1.2), and given $\lambda_h \in W^{\frac{1}{3},\frac{3}{2}}(\Sigma)$, we deduce that there exists $\widetilde{\lambda}_h \in W^{-\frac{1}{3},3}(\Sigma)$ such that

$$\left\langle \widetilde{\lambda}_h,\ \lambda_h \right\rangle_\Sigma \geq \frac{1}{2} \|\lambda_h\|_{\frac{1}{3},\frac{3}{2};\Sigma} \|\widetilde{\lambda}_h\|_{-\frac{1}{3},3;\Sigma}.$$

Then, we define $\widehat{\lambda}_h \in W^{-\frac{1}{3},3}(\partial\Omega_D)$ by

$$\left\langle \widehat{\lambda}_h,\ \eta \right\rangle_{\partial\Omega_D} := \left\langle \widetilde{\lambda}_h,\ \eta|_\Sigma \right\rangle_\Sigma \qquad \forall \eta \in W^{\frac{1}{3},\frac{3}{2}}(\partial\Omega_D),$$

and observe that $\|\widehat{\lambda}_h\|_{-\frac{1}{3},3;\partial\Omega_D} \leq \|\widetilde{\lambda}_h\|_{-\frac{1}{3},3;\Sigma}$. Moreover, we consider the problem: find $w \in \mathrm{W}^{1,3}(\Omega_D)$ such that

$$\begin{aligned} \mathrm{div}\left(\nabla w\right) &= \frac{1}{|\Omega_D|} \langle \widehat{\lambda}_h,\ 1 \rangle_{\partial\Omega_D} \quad \text{in } \Omega_D, \\ \frac{\partial w}{\partial \nu} &= \widehat{\lambda}_h \quad \text{on } \Gamma_D, \\ \int_{\Omega_D} w &= 0, \end{aligned}$$

which, according to [20], has a unique solution w. Thus, we let $\mathbf{u}_D := \nabla w \in \mathbf{W}^{0,3}(\mathrm{div}\,;\Omega_D)$, and notice from the corresponding continuous dependence bound that

$$\|\mathbf{u}_D\|_{3,\mathrm{div}\,;\Omega_D} \leq C \|\widehat{\lambda}_h\|_{-\frac{1}{3},3;\partial\Omega_D} \leq C \|\widetilde{\lambda}_h\|_{-\frac{1}{3},3;\Sigma}.$$

Furthermore, thanks to Corollary .4.1, we have

$$\left\langle \mathbf{u}_D \cdot n,\ \lambda_h \right\rangle_\Sigma := \left\langle \mathbf{u}_D \cdot n, E^0_{\Gamma_D}(\lambda_h) \right\rangle_{\partial\Omega_D} = \left\langle \widehat{\lambda_h},\ E^0_{\Gamma_D}(\lambda_h) \right\rangle_{\partial\Omega_D} = \left\langle \widetilde{\lambda},\ \lambda_h \right\rangle_\Sigma \geq \frac{1}{2C} \|\lambda_h\|_{\frac{1}{3},\frac{3}{2};\Sigma} \|\mathbf{u}_D\|_{3,\mathrm{div}\,;\Omega_D}.$$

Finally, according to the properties of Π_h^D, there holds

$$\left\langle \mathbf{u}_D \cdot n,\ \lambda_h \right\rangle_\Sigma = \int_\Sigma \Pi_h^D(\mathbf{u}_D) \cdot n\ \lambda_h,$$

and

$$\|\Pi_h^D(\mathbf{u}_D)\|_{0,3,\Omega_D} \leq C_D \|\mathbf{u}_D\|_{3,\mathrm{div}\,;\Omega},$$

and therefore, bounding lowerly with $\Pi_h^D(\mathbf{u}_D) \in \widetilde{\mathbf{H}}_h(\Omega_D)$, we conclude that

$$\sup_{\mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_D)\setminus \mathbf{0}} \frac{\langle \mathbf{v}_h \cdot \mathbf{n}, \lambda_h \rangle_\Sigma}{\|\mathbf{v}_h\|_{3,\mathrm{div};\Omega_D}} \geq \frac{\langle \Pi_h^D(\mathbf{u}_D) \cdot \mathbf{n},\ \lambda_h \rangle_\Sigma}{\|\Pi_h^D(\mathbf{u}_D)\|_{3,\mathrm{div};\Omega_D}} \geq \widetilde{C_D} \|\lambda_h\|_{\frac{1}{3},\frac{3}{2},\Sigma},$$

which ends the proof.

4.3 Numerical Results

In this section we present two numerical examples illustrating the performance of the Galerkin scheme (4.13), with the finite element subspaces introduced in Section 4.2.2. The first one is a 2D example for which the corresponding discrete analysis is available in Section 4.2, whereas the second one deals with a 3D model for which we claim that the aforementioned analysis can be extended with some minor changes in the discusion on the interfaces conditions.

We now give additional notations. The individual errors are denoted by:

$$\mathsf{e}(\mathbf{u}_D) := \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{3,\mathrm{div}\,;\Omega_D}\,, \quad \mathsf{e}(\boldsymbol{\sigma}_S) := \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div}\,;\Omega_S}\,,$$

$$\mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi}_S - \boldsymbol{\varphi}_{S,h}\|_{\frac{1}{2},\Sigma}\,, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_h\|_{\frac{1}{3},\frac{3}{2},\Sigma}\,, \quad \mathsf{e}(p_D) := \|p_D - p_{D,h}\|_{0,\frac{3}{2},\Omega_D}\,,$$

$$\mathsf{e}(\mathbf{u}_S) := \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,\Omega_S} \text{ and } \mathsf{e}(\boldsymbol{\gamma_S}) := \|\boldsymbol{\gamma_S} - \boldsymbol{\gamma}_{S,h}\|_{0,\Omega_S},$$

where $\underline{\sigma} := (\boldsymbol{\sigma}_S, \mathbf{u}_D) \in X,\ \underline{u} := (\boldsymbol{\varphi}_S, \lambda) \in Y$ and $\underline{\eta} := (\mathbf{u}_S, p_D, \boldsymbol{\gamma_S}) \in Z$ constitute the unique solution of (2.18), and $\underline{\sigma}_h := (\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{D,h}) \in X_h,\ \underline{u} := (\boldsymbol{\varphi}_{S,h}, \lambda_h) \in Y_h$ and $\underline{\eta} := (\mathbf{u}_{S,h}, p_{D,h}, \boldsymbol{\gamma_S}) \in Z_h$ is the solution of (4.13). Here, N stands for the number of degrees of freedom defining $\mathbb{X}_h \times \mathbb{Y}_h \times Z_h$. Furthermore, we let $r(\boldsymbol{\sigma}_S),\ r(\mathbf{u}_S),\ r(\mathbf{u}_D),\ r(p_D),\ r(\boldsymbol{\gamma_S})$ and $r(\boldsymbol{\varphi}_S)$ be the experimental rates of convergence given by

$$r(\bigstar) := \frac{\log(\mathsf{e}(\bigstar)/\mathsf{e}'(\bigstar))}{\log(h/h')}, \qquad \text{for each } \bigstar \in \{\boldsymbol{\sigma}_S, \mathbf{u}_D, \mathbf{u}_S, p_D, \boldsymbol{\gamma_S}, \boldsymbol{\varphi}_S\},$$

where h and h' denote two consecutive meshsizes with errors e and e', respectively.

For Example 1 we let $\Omega_S :=]-1,1[\ \times\ ]0,1[$ and $\Omega_D :=]-1,1[\ \times\ ]-1,0[$. Then we choose the data $\mathbf{f}_S$, $\mathbf{g}_D$ and f_D so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}_S(x_1,x_2) &= \operatorname{curl}\left(x_2^2\sin(\pi x_1)\right) \qquad \forall (x_1,x_2) \in \Omega_S,\\
p_S(x_1,x_2) &= x_1^3 + x_2^3 \qquad \forall (x_1,x_2) \in \Omega_S,\\
p_D(x_1,x_2) &= (x_1 - x_1^2)(x_2 - x_2^2) \qquad \forall (x_1,x_2) \in \Omega_D,
\end{aligned}$$

with the parameters $\rho = \beta = 1$, and $\mathbf{K} = \mathbf{I}$.

In turn, Example 2 considers the cubes $\Omega_S :=]0,1[^2\ \times\ ]1,2[$ and $\Omega_D :=]0,1[^3$, and choose the data so that the exact solution is given by

$$\mathbf{u}_S(x_1,x_2) = \nabla \times \begin{pmatrix} x_1^2(1-x_1)^2x_2^2(1-x_2)^2(2-x_3)^2\sin(\pi x_1) \\ x_1^2(1-x_1)^2x_2^2(1-x_2)^2(2-x_3)^2\sin(\pi x_2) \\ x_1^2(1-x_1)^2x_2^2(1-x_2)^2(2-x_3)^2\sin(\pi x_3) \end{pmatrix} \qquad \forall (x_1,x_2,x_3) \in \Omega_S,$$

$$p_S(x_1,x_2) = (x_1^3 + x_2^3)e^{x_3} \qquad \forall (x_1,x_2,x_3) \in \Omega_S,$$

$$p_D(x_1,x_2) = (x_1 - x_1^2)(x_2 - x_2^2)(x_3 - x_3^2) \qquad \forall (x_1,x_2,x_3) \in \Omega_D.$$

Next, in the following tables we present the convergence history for both examples. We observe there, according to the approximation properties of the finite element subspaces employed, that the numerical results suggest a linear behavior of the function $\mathcal{G}$ in Theorem 3.3.4.

h	N	$\mathsf{e}(\mathbf{u}_{\rm D})$	$r(\mathbf{u}_{\rm D})$	$\mathsf{e}(\boldsymbol{\sigma}_{\rm S})$	$r(\boldsymbol{\sigma}_{\rm S})$	$\mathsf{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathsf{e}(\lambda)$	$r(\lambda)$
1/8	897	4.955E−01	–	1.186E−00	–	5.155E−01	–	5.483E−02	–
1/10	1380	3.976E−01	0.985	9.424E−01	1.029	3.498E−01	1.739	3.199E−02	2.414
1/12	1967	3.321E−01	0.989	7.842E−01	1.008	2.561E−01	1.711	2.111E−02	2.282
1/14	2658	2.850E−01	0.992	6.729E−01	0.993	1.973E−01	1.690	1.521E−02	2.125
1/16	3453	2.496E−01	0.994	5.891E−01	0.996	1.580E−01	1.662	1.174E−02	1.937
1/18	4352	2.220E−01	0.995	5.236E−01	1.000	1.303E−01	1.638	9.268E−03	2.010
1/20	5355	1.999E−01	0.996	4.713E−01	0.999	1.099E−01	1.615	7.539E−03	1.961
1/22	6462	1.818E−01	0.997	4.285E−01	0.998	9.441E−02	1.598	6.281E−03	1.915
1/24	7673	1.668E−01	0.997	3.929E−01	0.999	8.225E−02	1.584	5.330E−03	1.886
1/26	8988	1.539E−01	0.998	3.627E−01	0.999	7.252E−02	1.573	4.594E−03	1.858
1/28	10407	1.429E−01	0.998	3.368E−01	0.999	6.458E−02	1.565	4.014E−03	1.820
1/30	11930	1.334E−01	0.998	3.144E−01	0.999	5.800E−02	1.557	3.548E−03	1.791
1/32	13557	1.251E−01	0.998	2.947E−01	0.999	5.247E−02	1.551	3.165E−03	1.770
1/34	15288	1.177E−01	0.999	2.774E−01	0.999	4.780E−02	1.538	2.848E−03	1.742
1/36	17123	1.112E−01	0.999	2.620E−01	0.999	4.377E−02	1.541	2.580E−03	1.725
1/40	21105	1.001E−01	0.999	2.358E−01	0.999	3.723E−02	1.536	2.157E−03	1.702
1/48	30317	8.342E−01	0.999	1.965E−01	1.000	2.818E−02	1.528	1.593E−03	1.661
1/56	41193	7.151E−02	0.999	1.685E−01	1.000	2.229E−02	1.521	1.241E−03	1.622
1/64	53733	6.257E−02	1.000	1.474E−01	1.000	1.820E−02	1.516	1.003E−03	1.594
1/80	83805	5.006E−02	1.000	1.179E−01	1.000	1.292E−02	1.536	7.041E−04	1.584
1/96	120533	4.172E−02	1.000	9.828E−02	1.000	9.797E−03	1.518	5.305E−04	1.553
1/112	163917	3.576E−02	1.000	8.424E−02	1.000	7.752E−03	1.519	4.183E−04	1.541
1/128	213957	3.129E−02	1.000	7.371E−02	1.000	6.327E−03	1.521	3.409E−04	1.534
1/144	270653	2.782E−02	1.000	6.552E−02	1.000	5.494E−03	**1.199**	2.928E−04	1.290
1/160	334005	2.503E−02	1.000	5.897E−02	1.000	4.724E−03	1.434	2.512E−04	1.453

Table 4.1: Example 1, Convergence history with PEERS

N	$\mathsf{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathsf{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathsf{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathsf{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$	$r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$
897	3.725E−03	–	3.683E−02	–	3.082E−01	–	1.474E−00	–
1380	2.805E−03	1.271	2.746E−02	1.316	2.217E−01	1.476	1.145E−00	1.134
1967	2.247E−03	1.218	2.235E−02	1.128	1.730E−01	1.361	9.394E−01	1.087
2658	1.889E−03	1.124	1.899E−02	1.059	1.413E−01	1.315	7.980E−01	1.058
3453	1.631E−03	1.103	1.654E−02	1.032	1.194E−01	1.258	6.939E−01	1.047
4352	1.435E−03	1.082	1.467E−02	1.019	1.034E−01	1.221	6.138E−01	1.041
5355	1.283E−03	1.066	1.319E−02	1.013	9.125E−02	1.188	5.505E−01	1.033
6462	1.160E−03	1.056	1.198E−02	1.009	8.165E−02	1.166	4.992E−01	1.028
7673	1.059E−03	1.047	1.097E−02	1.006	7.388E−02	1.149	4.566E−01	1.024
8988	9.745E−04	1.040	1.012E−02	1.005	6.745E−02	1.137	4.207E−01	1.022
10407	9.026E−04	1.034	9.399E−03	1.004	6.205E−02	1.126	3.901E−01	1.020
11930	8.407E−04	1.030	8.771E−03	1.003	5.745E−02	1.117	3.637E−01	1.018
13557	7.868E−04	1.027	8.221E−03	1.003	5.348E−02	1.109	3.406E−01	1.016
15288	7.395E−04	1.023	7.736E−03	1.002	5.002E−02	1.103	3.202E−01	1.015
17123	6.975E−04	1.021	7.306E−03	1.002	4.698E−02	1.097	3.022E−01	1.014
21105	6.266E−04	1.018	6.574E−03	1.002	4.189E−02	1.089	2.716E−01	1.013
30317	5.208E−04	1.014	5.477E−03	1.001	3.442E−02	1.077	2.259E−01	1.011
41193	4.457E−04	1.010	4.694E−03	1.001	2.921E−02	1.065	1.934E−01	1.009
53733	3.896E−04	1.008	4.107E−03	1.001	2.536E−02	1.057	1.690E−01	1.008
83805	3.113E−04	1.006	3.285E−03	1.000	2.008E−02	1.047	1.350E−01	1.007
120533	2.592E−04	1.004	2.738E−03	1.000	1.662E−02	1.039	1.124E−01	1.005
163917	2.221E−04	1.003	2.347E−03	1.000	1.417E−02	1.033	9.628E−02	1.004
213957	1.943E−04	1.002	2.053E−03	1.000	1.235E−02	1.028	8.420E−02	1.004
270653	1.727E−04	0.999	1.825E−03	1.000	1.094E−02	1.031	7.483E−02	1.002
334005	1.554E−04	1.001	1.643E−03	1.000	9.822E−03	1.023	6.733E−02	1.003

Table 4.2: Example 1, Convergence history with PEERS

h	N	$\mathtt{e}(\mathbf{u}_{\mathrm{D}})$	$r(\mathbf{u}_{\mathrm{D}})$	$\mathtt{e}(\boldsymbol{\sigma}_{\mathrm{S}})$	$r(\boldsymbol{\sigma}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$\mathtt{e}(\lambda)$	$r(\lambda)$
1/4	6086	2.239E−01	–	5.212E−01	–	1.970E−02	–	7.066E−03	–
1/8	46884	1.163E−01	0.945	2.617E−01	0.994	1.043E−02	0.917	3.471E−03	1.026
1/12	156386	7.848E−02	0.970	1.738E−01	1.009	5.616E−03	1.528	1.968E−03	1.400
1/16	368576	5.912E−02	0.985	1.299E−01	1.012	3.584E−03	1.562	1.269E−03	1.524
1/20	717438	4.740E−02	0.991	1.037E−01	1.011	2.527E−03	1.566	9.071E−04	1.505
1/24	1236956	3.954E−02	0.994	8.622E−02	1.010	1.898E−03	1.568	6.915E−04	1.489

Table 4.3: Example 2, Convergence history with PEERS

N	$\mathtt{e}(p_{\mathrm{D}})$	$r(p_{\mathrm{D}})$	$\mathtt{e}(\mathbf{u}_{\mathrm{S}})$	$r(\mathbf{u}_{\mathrm{S}})$	$\mathtt{e}(\boldsymbol{\gamma}_{\mathrm{S}})$	$r(\boldsymbol{\gamma}_{\mathrm{S}})$	$\mathtt{e}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$	$r(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}, \underline{p})$
6086	1.268E−03	–	8.682E−03	–	5.638E−02	–	5.866E−01	–
46884	6.133E−04	1.048	2.849E−03	1.608	2.101E−02	1.424	2.976E−01	0.979
156386	4.059E−04	1.018	1.403E−03	1.747	1.155E−02	1.476	1.985E−01	0.999
368576	3.033E−04	1.012	8.348E−04	1.804	7.472E−03	1.514	1.486E−01	1.005
717438	2.421E−04	1.010	5.536E−04	1.841	5.304E−03	1.535	1.187E−01	1.006
1236956	2.014E−04	1.009	3.943E−04	1.862	4.000E−03	1.548	9.885E−02	1.006

Table 4.4: Example 2, Convergence history with PEERS

Chapter 5

The case of a nonlinear Stokes problem

Most fluids are non-Newtonian. Examples of this include high polymers, blood and yogurt. In this case, the relation between extra stress tensor $\boldsymbol{\sigma}^E$ and $\mathbf{e}(\mathbf{u})$ is non-linear. In particular, in what follows we will consider quasi-Newtonian fluids, which satisfy the generic condition:

$$\boldsymbol{\sigma}^E = 2\mu_s\,(|\mathbf{e}(\mathbf{u})|)\,\mathbf{e}(\mathbf{u}),$$

where μ_s is the fluid viscosity. As usual the stress tensor can be written as $\boldsymbol{\sigma} := \boldsymbol{\sigma}^E - p\mathbf{I}$, and according to the incompressibility condition $\operatorname{div}\mathbf{u} = 0$, we obtain that the stress deviatoric tensor is given by

$$\boldsymbol{\sigma}^d = 2\mu_s\,(|\mathbf{e}(\mathbf{u})|)\,\mathbf{e}(\mathbf{u}). \tag{5.1}$$

Now, in order to derive a weak form of (5.1), we need to apply a suitable Green identity, which requires to isolate $\mathbf{e}(\mathbf{u})$ in terms of $\boldsymbol{\sigma}^d$. To this end, we first show that the relation (5.1) can be inverted, that is one can find a matrix function Φ such that

$$\mathbf{e}(\mathbf{u}) = \Phi\left(\boldsymbol{\sigma}^d\right). \tag{5.2}$$

One way of proving the existence of Φ is by the implicit function theorem. In this regard, we first show some viscosity function μ_S for which the relation between $\boldsymbol{\sigma}^d$ and $\mathbf{e}(\mathbf{u})$ can be inverted in the sense of (5.2) (cf. [35], [26] and [36]) There are as follows:

- the power-law or Ostwald de Waele model

$$\mu_S\Big(|\mathbf{e}(\mathbf{u})|\Big) = \mu_0\,|\mathbf{e}(\mathbf{u})|^{\beta-2}, \qquad \mu_0 > 0,\ \beta > 1,$$

- the Carreau viscosity equation

$$\mu_S\Big(|\mathbf{e}(\mathbf{u})|\Big) = \mu_0 + \mu_1\left(1 + |\mathbf{e}(\mathbf{u})|^2\right)^{\frac{\beta-2}{2}}, \qquad \mu_0 > 0,\ \beta \geq 1.$$

Other models (including these) can be found in [6] and [12], for example: the Cross viscosity equation and the Ellis fluid model.

In this work, we only consider the Carreau viscosity equation which is studied in the next section. In the case of the *power-law model*, others spaces are required for the Stokes part.

5.1 Coupled Problem considering the Carreau viscosity equation

We consider a fluid with viscosity in Ω_S given by the Carreau viscosity equation. Now, in order to apply directly the theory developped in Chapter 3, it is necessary to assume some restrictions on the parameters associated with this law. To this end, we introduce the following notation: Let μ_S a fluid viscosity law defined for $\boldsymbol{z} \geq 0$ and $r > 1$ by

$$\mu_S(\boldsymbol{z}\,;r) := \mu_0 + \psi\left(\hat{\delta} + \phi|\boldsymbol{z}|^2\right)^{\frac{r-2}{2}}, \tag{5.3}$$

where $\mu_0, \hat{\delta} \geq 0$ and $\psi, \phi > 0$. Note here that the Carreau viscosity equation is a particular case of (5.3). Now, following the analysis given in [38], we assume that $1 < r \leq 2$ and $\mu_0 > 0$. Then, according to [38, Lemme 3.1], we have the following result

Lemma 5.1.1 *Given $1 < r \leq 2$, we define the nonlinear mapping*

$$\begin{aligned} F_r \,:\, & \mathbb{R}^{2\times 2} \mapsto \mathbb{R}^{2\times 2} \\ & \boldsymbol{\tau} \mapsto F_r(\boldsymbol{\tau}) := \psi\Big(\hat{\delta} + \phi|\boldsymbol{\tau}|^2\Big)^{\frac{r-2}{2}}\boldsymbol{\tau} \qquad \forall \boldsymbol{\tau} \in \mathbb{R}^{2\times 2}. \end{aligned} \tag{5.4}$$

Then there hold:

1. *there exists $c_1 = c_1(r, \delta, \psi, \phi) > 0$ such that*

$$c_1 \frac{|\boldsymbol{\tau}|^2}{\Big(\hat{\delta} + |\boldsymbol{\sigma}|^{2-r} + |\boldsymbol{\sigma}+\boldsymbol{\tau}|^{2-r}\Big)} \leq \Big(F_r(\boldsymbol{\sigma}+\boldsymbol{\tau}) - F_r(\boldsymbol{\sigma}) : \boldsymbol{\tau}\Big) \qquad \forall \boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}.$$

2. *there exists $c_2 = c_2(r, \delta, \psi, \phi) > 0$ such that*

$$\Big|F_r(\boldsymbol{\sigma}+\boldsymbol{\tau}) - F_r(\boldsymbol{\sigma})\Big| \leq \; c_2 \frac{|\boldsymbol{\tau}|}{\Big(\hat{\delta} + |\boldsymbol{\sigma}|^{2-r} + |\boldsymbol{\sigma}+\boldsymbol{\tau}|^{2-r}\Big)} \qquad \forall \boldsymbol{\tau}, \boldsymbol{\sigma} \in \mathbb{R}^{2\times 2}.$$

Proof: See [38, Lemme 3.1].

In what follows we model the movement of a fluid with a viscosity law as in (5.3) in the free fluid region Ω_S. To this end, in a similar manner as the Newtonian case (cf. (2.15)), we redefine the operator $\mathcal{A}_S$ as the non-linear mapping, given by

$$\begin{aligned}
&\mathcal{A}_S : \mathbb{H}_0(\mathbf{div}\,;\Omega_S) \to \mathbb{H}_0(\mathbf{div}\,;\Omega_S)', \text{ where} \\
&\left[\mathcal{A}_S(\boldsymbol{\sigma}_S), \boldsymbol{\tau}_S\right]_S \equiv \left(\Phi\left(\boldsymbol{\sigma}_S^d\right) : \boldsymbol{\tau}_S\right)_S = \left(\frac{1}{2\mu\left(\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)\right|\right)}\boldsymbol{\sigma}_{\mathbf{S}}{}^d : \boldsymbol{\tau}_S^d\right)_S .
\end{aligned} \tag{5.5}$$

Note that this is well defined thanks to the particular choice of parameters (i.e., μ_S satisfies the hypotheses of the implicit function theorem) and to the fact that $\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)\right| = |\mathbf{e}(\mathbf{u}_S)|$ (cf. (5.2)). Note here that the choice of the space $\mathbb{H}_0(\mathbf{div}\,;\Omega_S)$ is according to [38, pag. 134] and the incompressibility equation $\operatorname{div}\mathbf{u}_S = 0$.

Then in order to verify that problem $(\mathbf{P}_\alpha)$ (cf. (2.18)) is also well-posed, with $\mathcal{A}_S$ defined as in (5.5) instead of (2.15), and according to Theorem 3.1.10, we only need to show that Lemmas 4.1.3 and 4.1.4 are satisfied. In fact, we have the followings results.

Lemma 5.1.2 *There exists $C = C(r, \delta, \psi, \phi, \Omega_S) > 0$, such that*

$$\left[\mathcal{A}_S(\boldsymbol{\sigma}_S) - \mathcal{A}_S(\boldsymbol{\tau}_S), \boldsymbol{\sigma}_S - \boldsymbol{\tau}_S\right] \geq C \left\|\boldsymbol{\sigma}_S{}^d - \boldsymbol{\tau}_S^d\right\|_{0,\Omega_S}^2 \qquad \forall \boldsymbol{\sigma}_S, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S).$$

Proof: Let $\boldsymbol{\sigma}_S, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}\,;\Omega_S)$. Then, due to (5.5), (5.1) and (5.2), we have

$$\left[\mathcal{A}_S(\boldsymbol{\sigma}_S) - \mathcal{A}_S(\boldsymbol{\tau}_S), \boldsymbol{\sigma}_S - \boldsymbol{\tau}_S\right]_S = 2\Big[\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi(\boldsymbol{\tau}_S^d),\ \mu_S\left(\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)\right|\right)\Phi\left(\boldsymbol{\sigma}_S^d\right) - \mu_S\left(\left|\Phi(\boldsymbol{\tau}_S^d)\right|\right)\Phi(\boldsymbol{\tau}_S^d)\Big]_S,$$

which, using the notation given in (5.4), yields

$$\left[\mathcal{A}_S(\boldsymbol{\sigma}_S) - \mathcal{A}_S(\boldsymbol{\tau}_S), \boldsymbol{\sigma}_S - \boldsymbol{\tau}_S\right] = 2\mu_0\left\|\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi\left(\boldsymbol{\tau}_S^d\right)\right\|_{0,2;\Omega_S}^2 + \left(\left(\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi\left(\boldsymbol{\tau}_S^d\right)\right) : F_r(\Phi\left(\boldsymbol{\sigma}_S^d\right)) - F_r(\Phi(\boldsymbol{\tau}_S^d$$

Then, according to Lemma 5.1.1, we have

$$\begin{aligned}
&\left[\mathcal{A}_S(\boldsymbol{\sigma}_S) - \mathcal{A}_S(\boldsymbol{\tau}_S), \boldsymbol{\sigma}_S - \boldsymbol{\tau}_S\right] = 2\mu_0\left\|\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi\left(\boldsymbol{\tau}_S^d\right)\right\|_{0,2;\Omega_S}^2 + \left(\left(\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi\left(\boldsymbol{\tau}_S^d\right)\right) : F_r(\Phi\left(\boldsymbol{\sigma}_S^d\right)) - F_r(\Phi(\boldsymbol{\tau}_S^d \\
&\geq\ 2\mu_0\left\|\Phi\left(\boldsymbol{\sigma}_S^d\right) - \Phi\left(\boldsymbol{\tau}_S^d\right)\right\|_{0,2;\Omega_S}^2 + \left(\frac{c_1 c_1^2}{\widehat{\delta}}\right)\left\|F_r(\Phi\left(\boldsymbol{\sigma}_S^d\right)) - F_r(\Phi(\boldsymbol{\tau}_S^d))\right\|_{0,2;\Omega_S}^2 .
\end{aligned} \tag{5.6}$$

On the other hand, thanks to (5.1) and (5.2), we have

$$\Phi\left(\boldsymbol{\sigma}_S^d\right) = \frac{1}{2\mu_S\left(\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)\right|\right)}\boldsymbol{\sigma}_S{}^d \quad \text{and} \quad \Phi\left(\boldsymbol{\tau}_S^d\right) = \frac{1}{2\mu_S\left(\left|\Phi(\boldsymbol{\tau}_S^d)\right|\right)}\boldsymbol{\tau}_S{}^d,$$

from where

$$2\mu_S\left(\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)\right|\right)\Phi\left(\boldsymbol{\sigma}_S^d\right)={\boldsymbol{\sigma}_S}^d \quad \text{and} \quad 2\mu_S\left(\left|\Phi(\boldsymbol{\tau}_S^d)\right|\right)\Phi\left(\boldsymbol{\tau}_S^d\right)={\boldsymbol{\tau}_S}^d.$$

Thus, subtracting both identities, we obtain

$$\begin{aligned}\left|{\boldsymbol{\sigma}_S}^d-\boldsymbol{\tau}_S^d\right| &= 2\left|\mu_0\left(\Phi\left(\boldsymbol{\sigma}_S^d\right)-\Phi\left(\boldsymbol{\tau}_S^d\right)\right)+\left(F_r(\Phi\left(\boldsymbol{\sigma}_S^d\right))-F_r(\Phi(\boldsymbol{\tau}_S^d))\right)\right|\\ &\leq 2\mu_0\left|\Phi\left(\boldsymbol{\sigma}_S^d\right)-\Phi\left(\boldsymbol{\tau}_S^d\right)\right|+2\left|F_r(\Phi\left(\boldsymbol{\sigma}_S^d\right))-F_r(\Phi(\boldsymbol{\tau}_S^d))\right|. \end{aligned} \tag{5.7}$$

Finally, the proof follows straightforwardly from (5.6) and (5.7).

Corollary 5.1.3 $\mathcal{A}_S$ *is strongly monotone in* $\mathcal{N}(\mathbb{B})$ *(cf. (4.3)).*

Proof: Let $\boldsymbol{\sigma}_S,\boldsymbol{\tau}_S\in\mathcal{N}(\mathbb{B})$. Then $\mathbf{div}\ (\boldsymbol{\sigma}_S-\boldsymbol{\tau}_S)=\mathbf{0}$, and according to [25, Lemma 3.3], there exists $\widetilde{C}>0$ such that

$$\left\|{\boldsymbol{\sigma}_S}^d-\boldsymbol{\tau}_S^d\right\|_{0,\Omega_S}\geq\widetilde{C}\,\|\boldsymbol{\sigma}_S-\boldsymbol{\tau}_S\|_{0,\Omega_S}=\widetilde{C}\,\|\boldsymbol{\sigma}_S-\boldsymbol{\tau}_S\|_{\mathbf{div},\Omega_S}.$$

Thus, according to Lemma 5.1.2, we have

$$\left[\mathcal{A}_S(\boldsymbol{\sigma}_S)-\mathcal{A}_S(\boldsymbol{\tau}_S),\boldsymbol{\sigma}_S-\boldsymbol{\tau}_S\right]\geq C\widetilde{C}^2\,\|\boldsymbol{\sigma}_S-\boldsymbol{\tau}_S\|^2_{\mathbf{div},\Omega_S}\qquad\forall\boldsymbol{\sigma}_S,\boldsymbol{\tau}_S\in\mathcal{N}(\mathbb{B}).$$

which ends the proof.

Lemma 5.1.4 $\mathcal{A}_S$ *is a Lipschitz continuous mapping.*

Proof: Let $\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}\,;\Omega_S)$. Then according to (5.1), (5.2), (5.3) and Lemma 5.1.1, we have

$$\left(2\mu_0+\frac{c_1}{\hat{\delta}+\left|\Phi(\boldsymbol{\sigma})\right|^{2-r}+\left|\Phi(\boldsymbol{\tau})\right|^{2-r}}\right)\left|\Phi(\boldsymbol{\sigma})-\Phi(\boldsymbol{\tau})\right|^2\leq\Big(\boldsymbol{\sigma}-\boldsymbol{\tau}:\ \Phi(\boldsymbol{\sigma})-\Phi(\boldsymbol{\tau})\Big),$$

and applying the Cauchy–Schwarz inequality we get,

$$\left\|\Phi(\boldsymbol{\sigma})-\Phi(\boldsymbol{\tau})\right\|_{0,2;\Omega_S}\leq\frac{1}{2\mu_0}\left\|\boldsymbol{\sigma}-\boldsymbol{\tau}\right\|_{0,2;\Omega_S}\qquad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}\,;\Omega_S),$$

whence

$$\left\|\mathcal{A}_S(\boldsymbol{\sigma})-\mathcal{A}_S(\boldsymbol{\tau})\right\|_{X_1'}\leq\left\|\Phi(\boldsymbol{\sigma})-\Phi(\boldsymbol{\tau})\right\|_{0,2;\Omega_S}\leq\frac{1}{2\mu_0}\left\|\boldsymbol{\sigma}-\boldsymbol{\tau}\right\|_{\mathbf{div}\,;\Omega_S}\qquad\forall\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}\,;\Omega_S),$$

which ends the proof.

Theorem 5.1.5 *The following problem has a unique solution:*

find $(\underline{\sigma}, \underline{u}, \underline{\eta}) \in X \times Y \times Z$ *such that*

$$\begin{array}{lllllll} \big[\mathbb{A}(\underline{\sigma}), \underline{\tau}\big] & + & \big[\mathbb{B}_1(\underline{\tau}), \underline{u}\big] & + & \big[\mathbb{B}(\underline{\tau}), \underline{\eta}\big] & = & \big[F, \underline{\tau}\big] \quad \forall \underline{\tau} \in X, \\ \big[\mathbb{B}_1(\underline{\sigma}), \underline{v}\big] & - & \big[\mathbb{C}(\underline{u}), \underline{v}\big] & & & = & \big[G, \underline{v}\big] \quad \forall \underline{v} \in Y, \\ \big[\mathbb{B}(\underline{\sigma}), \underline{\vartheta}\big] & & & & & = & \big[E, \underline{\vartheta}\big] \quad \forall \underline{\vartheta} \in Z, \end{array}$$

where the spaces are given by

$$\begin{aligned} X &:= \mathbb{H}_0(\mathbf{div}\,; \Omega_S) \times \mathbf{W}^{0,3}_{\Gamma_D}(\mathrm{div}\,, \Omega_D), \\ Y &:= \widehat{\mathbf{H}}^{\frac{1}{2}}_{00}(\Sigma) \times W^{\frac{1}{3}, \frac{3}{2}}(\Sigma), \\ Z &:= \mathbf{L}^2(\Omega_S) \times L^{\frac{3}{2}}_0(\Omega_D) \times \mathbb{L}^2_{skew}(\Omega_S), \end{aligned}$$

the nonlinear operator $\mathbb{A} : X \to X'$ *is defined as:*

$$\big[\mathbb{A}(\underline{\sigma}), \underline{\tau}\big] := \Big[\Phi\left(\boldsymbol{\sigma}_S{}^d\right), \boldsymbol{\tau}_S\Big]_S + [\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_\mathbf{D}]_D \quad \forall \underline{\sigma} := (\boldsymbol{\sigma}_S, \mathbf{u}_D), \underline{\tau} := (\boldsymbol{\tau}_S, \mathbf{v}_D) \in X,$$

and the linear operators $\mathbb{B} : X \to Z'$, $\mathbb{B}_1 : X \to Y'$ *and* $\mathbb{C} : Y \to Y'$ *are given as follows*

$$\begin{aligned} \big[\mathbb{B}(\underline{\tau}), \underline{\eta}\big] &:= (\mathbf{div}\,\boldsymbol{\tau}_S, \mathbf{u}_S)_S + (\boldsymbol{\tau}_S, \boldsymbol{\gamma_S})_S - (\mathrm{div}\,\mathbf{v}_D, p_D)_D \quad \forall \underline{\eta} := (\mathbf{u}_S, p_D, \boldsymbol{\gamma_S}) \in Z, \\ \big[\mathbb{B}_1(\underline{\boldsymbol{\sigma}}), \underline{v}\big] &:= \Big\langle \boldsymbol{\sigma}_S \mathbf{n},\ \boldsymbol{\psi} \Big\rangle_\Sigma - \Big\langle \mathbf{u}_D \cdot \mathbf{n},\ \xi \Big\rangle_\Sigma \quad \forall \underline{v} := (\boldsymbol{\psi}, \xi) \in Y, \\ \big[\mathbb{C}(\underline{u}), \underline{v}\big] &:= \mu k_f^{-1} \Big\langle \boldsymbol{\psi} \cdot \mathbf{t},\ \boldsymbol{\varphi} \cdot \mathbf{t} \Big\rangle_\Sigma - \Big\langle \boldsymbol{\psi} \cdot \mathbf{n},\ \lambda \Big\rangle_\Sigma + \Big\langle \boldsymbol{\varphi} \cdot \mathbf{n},\ \xi \Big\rangle_\Sigma \quad \forall \underline{u} := (\boldsymbol{\varphi}, \lambda) \in Y. \end{aligned}$$

Proof: It suffices to see, thanks to Lemmas 5.1.4 and 5.1.2, that this problem satisfies the hypotheses of Theorem 3.1.10.

Similarly, we can show that the discrete problem asociated with the one considered in Theorem 5.1.5 and the discrete spaces considered in Section 4.2.2 provide a stable Galeking scheme, which, however, is not necessarily computable. When the function Φ is unknown we propose a simple strategy to implement a numerical method, which replaces the term $\Phi(\boldsymbol{\sigma}^n_{S,h})$ by $\frac{1}{\mu_S\left(|\Phi(\boldsymbol{\sigma}^{n-1}_{S,h})|\right)}\boldsymbol{\sigma}^n_{S,h}$ in the iterative scheme considered to solve the discrete problem.

Appendices

.1 Orthogonal decompositions of $\mathbb{R}^{n\times n}$

.

$\boldsymbol{\sigma} \mapsto \begin{cases} \frac{1}{2}\left(\boldsymbol{\sigma}+\boldsymbol{\sigma}^t\right), \\ \frac{1}{2}\left(\boldsymbol{\sigma}-\boldsymbol{\sigma}^t\right). \end{cases}$	This is the decomposition of $\boldsymbol{\sigma} \in \mathbb{R}^{n\times n}$ in its symmetric and skew-symmetric parts, for example in Continuum Mechanics the infinitesimal rotation tensor is a skew-symmetric tensor, and the infinitesimal strain tensor is symetric.
$\boldsymbol{\sigma} \mapsto \begin{cases} \boldsymbol{\sigma}^d := \boldsymbol{\sigma} - \left(\frac{\operatorname{tr}\boldsymbol{\sigma}}{n}\right)\mathbf{I}, \\ \left(\frac{\operatorname{tr}\boldsymbol{\sigma}}{n}\right)\mathbf{I}. \end{cases}$	This is the decomposition of $\boldsymbol{\sigma}$ in its isotropic and no isotropic part, For example "In isotropic materials the deviatoric component of the stress tensor does not cause volume changes" [14].

Table: .1 Orthogonal decompositions $\mathbb{R}^{n\times n}$.

.2 Sobolev spaces in polygonal domains

Definition .2.1 *(cf. [30, Definition 1.3.2.1]) We denote by $W_p^s(\Omega)$ the space of all distributions* $\mathbf{u}$ *defined in Ω, such that*

a) $D^\alpha\mathbf{u} \in L^p(\Omega)$, for $|\alpha| \leq m$, when $s = m$ is a nonnegative integer,

b) $\mathbf{u} \in W_p^m(\Omega)$ and

$$\iint_{\Omega\times\Omega} \frac{|D^\alpha\mathbf{u}(x) - D^\alpha\mathbf{u}(y)|^p}{|x-y|^{n+\sigma p}} < \infty, \tag{8}$$

for $|\alpha| = m$, when $s = m + \sigma$ is nonnegative and is not an integer.

We define the Banach norm on $W_p^s(\Omega)$ by

$$\|\mathbf{u}\|_{m,p,\Omega} := \left\{ \sum_{|\alpha|\leq m} \int_\Omega |D^\alpha\mathbf{u}|^p \, dx \right\}^{\frac{1}{p}}, \tag{9}$$

in case (a), and by

$$\|\mathbf{u}\|_{s,p,\Omega} := \left\{ \|\mathbf{u}\|_{m,p,\Omega}^p + \sum_{|\alpha|=m} \iint_{\Omega\times\Omega} \frac{|D^\alpha\mathbf{u}(x) - D^\alpha\mathbf{u}(y)|^p}{|x-y|^{n+\sigma p}} \right\}^{\frac{1}{p}}, \tag{10}$$

in case (b).

Definition .2.2 *(cf. [30, Definition 1.3.2.2]) For* $s > 0$, *we denote by* $\mathring{W}_p^s(\Omega)$ *the closure of* $\mathcal{D}(\Omega)$ *in* $W_p^s(\Omega)$.
Equivalently, it is the clousure in $W_p^s(\Omega)$ *of all distributions with compact support in* Ω *which belong to* $W_p^s(\Omega)$.

Definition .2.3 *(cf. [30, Definition 1.3.2.3]) For* $s < 0$, *we denote by* $W_p^s(\Omega)$ *the dual space of* $\mathring{W}_q^{-s}(\Omega)$, *where* $\dfrac{1}{p} + \dfrac{1}{q} = 1$.

Definition .2.4 *(cf. [30, Definition 1.3.2.5]) For every positive* s, *we denote by* $\widetilde{W}_p^s(\Omega)$, *the space of all* $\mathbf{u} \in W_p^s(\Omega)$ *such that* $\widetilde{\mathbf{u}}$, *the continuation of* $\mathbf{u}$ *by zero outside* Ω, *belongs to* $\widetilde{W}_p^s(\mathbb{R}^n)$.

Lemma .2.5 *(cf. [30, Lemma 1,3,2,12]) Let* Ω *be bounded with Lipschitz boundary* Γ. *Then, there exist two constants* C_1, C_2 *with* $0 < C_1 \leq C_2$ *such that*

$$
\begin{aligned}
C_1 d(x,\Gamma)^{-\sigma p} \leq \rho_{\sigma,p}(x) \leq C_2 d(x,\Gamma)^{-\sigma p}, \quad 0 < \sigma < 1, p \geq 1,\\
\rho_{\sigma,p}(x) := 2 \int_{C\Omega} \frac{dy}{|x-y|^{n+\sigma p}},
\end{aligned}
$$

where

$d(x,\Gamma)$ *denotes the distance from* x *to* Γ.

The same inequalities hold when Ω *is a uniform Lipschitz epigraph.*

Definition .2.6 *(cf. [30, Definition 1.3.3.2.]) Let* Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a boundary* Γ *of class* $\mathcal{C}^{k,1}$, *where* k *is a nonnegative integer. Let* Γ_0 *be an open subset of* Γ. *A distribution* $\mathbf{u}$ *on* Γ_0 *belongs to* $W_p^s(\Gamma_0)$ *with* $|s| \leq k+1$ *if* $\mathbf{u} \circ \phi \in W_p^s(V' \cap \phi^{-1}(\Gamma_0 \cap V))$ *for all* V *and* ϕ *fulfilling some assumptions (cf. [30, Definition 1.2.1.1]). In the particular case* $s \in (0,1)$, *one can define the norm*

$$
\mathbf{u} \mapsto \left\{ \int_{\Gamma_0} |u|^p d\sigma + \iint_{\Gamma_0 \times \Gamma_0} \frac{|u(x)-u(y)|^p}{|x-y|^{n-1+sp}} d\sigma(x) d\sigma(y) \right\}^{\frac{1}{p}}.
$$

Theorem .2.7 *(cf. [30, Theo. 1.4.5.2]) Let* Ω *be a open and bounded subset of* $\mathbb{R}^2$ *whose boundary* Γ *is a curvilinear polygon. Then we have the following inclusions and identities:*

(a) $\widetilde{W}_p^s(\Omega) \subseteq \mathring{W}_p^s(\Omega) \subseteq W_p^s(\overline{\Omega}) = W_p^s(\Omega)$ *for* $s > 0$.

(b) $\widetilde{W}_p^s(\Omega) = \mathring{W}_p^s(\Omega)$ *for* $s - 1/p$ *non-integer,*

(c) $\mathring{W}_p^s(\Omega) = W_p^s(\Omega)$ *for* $s < 1/p$,

(d) $\widetilde{W}_p^s(\Omega) = \left\{\mathbf{u} \in \mathring{W}_p^s(\Omega) : \frac{D^\alpha \mathbf{u}}{\rho^\sigma} \in L^p(\Omega),\ |\alpha| = m\right\}$,
for $s = m + \sigma$, m *a nonnegative integer.*

.3 Traces theorems in polygonal domains

Theorem .3.1 *Traces theorem. (cf. [30, Theo. 1.5.1.3.])*

Let Ω *be a bounded open subset of* $\mathbb{R}^n$ *with a Lipschitz boundary* Γ. *Then the mapping* $\mathbf{u} \mapsto \gamma\mathbf{u}$ *which is defined for* $\mathbf{u} \in \mathcal{C}^{0,1}(\overline{\Omega})$, *has a unique continuous extension as an operator from* $W^{1,p}(\Omega)$ *onto* $W^{1-\frac{1}{p},p}(\Gamma)$. *This operator has a continuous inverse independent of* p.

Theorem .3.2 *Traces of* $W_p^1(\Omega)$ *(cf. [30, Theo. 1.5.2.3])*
Let Ω *be a bounded open subset of* $\mathbb{R}^2$ *whose boundary* Γ *is a curvilinear polygon of class* $\mathcal{C}^1$. *Then the mapping from* $W_p^1(\Omega)$ $\mathbf{u} \mapsto \{f_j\}_{j=1}^N$, *where* $f_j = \gamma_j \mathbf{u}$, *is a linear continuous mapping onto the subspace of* $\prod_{j=1}^N W_p^{1-\frac{1}{p}}(\Gamma_j)$ *defined by:*

(a) *no extra condition when* $1 < p < 2$,

(b) $f_j(S_j) = f_{j+1}(S_{j+1}),\ 1 \le j \le N$ *When* $2 < p < \infty$ *(*S_{j-1}, S_j *are the endpoints of* $\overline{\Gamma}_j$*),*

(c) $\displaystyle\int_0^{\delta_j} \frac{|f_{j+1}(x_j(\sigma)) - f_j(x_j(-\sigma))|^2}{\sigma} d\sigma < \infty,\ 1 \le j \le N$, *when* $p = 2$.

.4 Additional results in polygonal domains

Corollary .4.1 *Let* Ω *be a bounded open subset of* $\mathbb{R}^2$, *whose boundary* Γ *is a curvilinear polygon of class* $\mathcal{C}^1$, Γ_0 *be an open subset of* Γ *and* $p \in (1,2)$. *Then there exists* $C > 0$ *such that*

$$\|E_{\Gamma_0}(\xi)\|_\Gamma \le C\|\xi\|_{\Gamma_0} \qquad \forall \xi \in W^{\frac{1}{p'},p}(\Gamma_0),$$

where $E_{\Gamma_0}(\xi)$ *is the continuation by zero of* ξ *on* $\Gamma \setminus \Gamma_0$, *and*

$$\|u\|_S = \left\{\int_S |u|^p d\sigma + \iint_{S\times S} \frac{|u(x) - u(y)|^p}{|x-y|^{n-1+sp}} d\sigma(x) d\sigma(y)\right\}^{\frac{1}{p}} \quad \forall u \in W^{\frac{1}{q},p}(S), \quad S \in \{\Gamma, \Gamma_0\}.$$

for $\frac{1}{q} = 1 - \frac{1}{p}$.

Proof: According to Theorem .3.2, we have $E_{\Gamma_0}(\xi) \in W^{\frac{1}{q},p}(\Gamma)$, $\forall \xi \in W^{\frac{1}{q},p}(\Gamma_0)$ i.e., $\widetilde{W}^{\frac{1}{q},p}(\Gamma_0) = W^{\frac{1}{q},p}(\Gamma_0) =: X$. On the other hand, since

$$\|\xi\|_{\Gamma_0} \leq \|E_{\Gamma_0}(\xi)\|_{\Gamma} \qquad \forall \xi \in X,$$

the operator

$$i : (X, \|\cdot\|_{\Gamma}) \mapsto (X, \|\cdot\|_{\Gamma_0})$$
$$x \mapsto i(x) = x$$

is a linear, bounded and injective operator, and therefore from the bounded inverse theorem, there exists $C > 0$ such that

$$\|E_{\Gamma_0}(\xi)\|_{\Gamma} \leq C\|\xi\|_{\Gamma_0} \qquad \forall \xi \in W^{\frac{1}{q},p}(\Gamma_0).$$

Lemma .4.2 (Continuity of edge moments) *(cf. [32, Lemma 3.15] and [32, Lemma 3.13]) For a face $\widehat{F}$ of the reference element $\widehat{T}$ and $\varphi \in W_q^{1-\frac{1}{q}}(\widehat{F})$ with $1 < q < 2$, we have*

$$\left| \int_{\widehat{F}} \varphi \mathbf{u} \cdot n dS \right| \leq C \left(\|\mathbf{u}\|_{0,p;\widehat{T}} + \|\operatorname{div} \mathbf{u}\|_{0,p;\widehat{T}} \right) \|\varphi\|_{W_q^{1/p}(\widehat{F})}.$$

Theorem .4.3 (Green formula) *(cf. [30, Theo. 1.5.2.1])*
Let Ω be a bounded open subset of $\mathbb{R}^n$ with a Lipschitz boundary Γ. Then for every $\mathbf{u} \in W_p^1(\Omega)$ and $\mathbf{v} \in W_q^1(\Omega)$, with $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$\int_{\Omega} D_i \mathbf{u}\mathbf{v} dx + \int_{\Omega} \mathbf{u} D_i \mathbf{v} dx = \int_{\Gamma} \gamma \mathbf{u} \gamma \mathbf{v} \nu^i d\sigma.$$

Lemma .4.4 *Given $1 < p < \infty$, and $f \in L^p(\Omega)$, we define the sets*

$$\Omega_0 := \left\{ x \in \Omega : f(x) = 0 \right\} \quad \textit{and} \quad \Omega_1 := \Omega \setminus \Omega_0, \textit{ and the function } \widehat{f} \textit{ given by:}$$

$$\widehat{f} := \begin{cases} |f|^{p-2} f & : x \in \Omega_1, \\ 0 & : x \in \Omega_0. \end{cases}$$

Then

$$\widehat{f} \in L^{p'}(\Omega), when \quad 1/p + 1/q = 1,$$
$$f = \left| \widehat{f} \right|^{p'-2} \widehat{f}, \ in\ \Omega_1 \quad a.e.,$$
$$\int_{\Omega} f\widehat{f} = \int_{\Omega_1} f\widehat{f} = \left\| f \right\|_{0,p;\Omega}^{p} = \left\| \widehat{f} \right\|_{0,q;\Omega}^{q} = \left\| f \right\|_{0,p;\Omega} \left\| \widehat{f} \right\|_{0,q;\Omega}.$$

Proof: It is direct from the definition of $\widehat{f}$.

Lemma .4.5 *For each* $1 < p < \infty$ *there exists* $\widehat{C} = \widehat{C}(\Omega_D, p) > 0$ *such that*

$$\|\mathbf{v}\|_{p,\mathrm{div};\Omega_D} = \|\mathbf{v}\|_{0,p;\Omega_D} + \|\mathrm{div}\,\mathbf{v}\|_{0,p;\Omega_D} \leq \widehat{C}\|\mathbf{v}\|_{0,p;\Omega_D} \qquad \forall \mathbf{v} \in \Big\{\mathrm{W}^{0,p}\,(\mathrm{div}\,;\Omega_D):\ \mathrm{div}\,(\mathbf{v}) \in \mathcal{P}_0(\Omega_D)\Big\}.$$

More precisely, there exists $C = C(\Omega_D, p) > 0$ *such that*

$$\|\mathrm{div}\,\mathbf{v}\|_{0,p;\Omega_D} \leq C\|\mathbf{v}\|_{0,p;\Omega_D} \qquad \forall \mathbf{v} \in \Big\{\mathrm{W}^{0,p}\,(\mathrm{div}\,;\Omega_D):\ \mathrm{div}\,(\mathbf{v}) \in \mathcal{P}_0(\Omega_D)\Big\}.$$

Proof: According to the Green's formula, we have

$$\begin{aligned}\Big\langle \gamma_n(\mathbf{v}), \gamma_0(w)\Big\rangle_{W_p^{-\frac{1}{p}}(\partial\Omega_D), W_q^{\frac{1}{p}}(\partial\Omega_D)} &= \Big(\mathbf{v}, \nabla w\Big)_D + \Big(\mathrm{div}\,\mathbf{v}, w\Big)_D \\ &= \Big(\mathbf{v}, \nabla w\Big)_D + \mathrm{div}\,\mathbf{v}\Big(1, w\Big)_D \\ &\quad \forall \mathbf{v} \in \Big\{\mathrm{W}^{0,p}\,(\mathrm{div}\,;\Omega_D):\ \mathrm{div}\,(\mathbf{v}) \in \mathcal{P}_0(\Omega_D)\Big\}, \forall w \in \mathrm{W}_q^1(\Omega_D).\end{aligned}$$

In particular, taking $w \in \mathcal{C}_0^\infty(\Omega_D)$, we obtain

$$|\mathrm{div}\,\mathbf{v}|\,\Big|\Big(1, w\Big)_D\Big| = \Big|\Big(\mathbf{v}, \nabla w\Big)_D\Big| \leq \Big\|\nabla w\Big\|_{0,q}\|\mathbf{v}\|_{0,p} \qquad \forall \mathbf{v} \in \Big\{\mathrm{W}^{0,p}\,(\mathrm{div}\,;\Omega_D):\ \mathrm{div}\,(\mathbf{v}) \in \mathcal{P}_0(\Omega_D)\Big\}.$$

Thus, taking $\widetilde{w} \in \mathcal{C}_0^\infty(\Omega_D)$ such that $\int_{\Omega_D} \widetilde{w} = 1$ (for example a cut-off function), we obtain

$$\|\mathrm{div}\,\mathbf{v}\|_{0,p;\Omega_D} = |\mathrm{div}\,\mathbf{v}||\Omega_D|^{\frac{1}{p}} \leq \Big(|\Omega_D|^{\frac{1}{p}}\|\nabla\widetilde{w}\|_{0,q}\Big)\,\|\mathbf{v}\|_{0,p;\Omega_D} \qquad \forall \mathbf{v} \in \Big\{\mathrm{W}^{0,p}\,(\mathrm{div}\,;\Omega_D):\ \mathrm{div}\,(\mathbf{v}) \in \mathcal{P}_0(\Omega_D)\Big\},$$

and therefore, setting $C = |\Omega_D|^{\frac{1}{p}}\|\nabla\widetilde{w}\|_{0,q}$ the proof is concluded.

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