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Mixed Virtual Element Methods. Applications in Fluid Mechanics.

Métodos de Elementos Virtuales Mixtos. Aplicaciones en Mecánica de Fluidos.

Memoria para optar al título de
Ingeniero Civil Matemático

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January 27, 2015

CONCEPCIÓN, CHILE

Acknowledgements

To my mother and sisters, as well as to my other relatives. Their support throughout these years has been fundamental.

To my advisor, for all his dedication and support throughout the development of this work, as well as in the courses in which he was the professor. I have learned a lot from him.

To all the professors who have formed part of this process. I would like to give special thanks to G. Avello and F. Paiva.

To all the classmates and friends who were always supporting me. Particularly, to Sebastián (GCB), Patrick, Belén, Carota, Franco, Cristian, Kleme, among several other ones.

To project Anillo ACT1118 (ANANUM) through Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, for supporting this thesis work.

ABSTRACT

In this thesis work we introduce and analyze a virtual element method (VEM) for a variational formulation of the Poisson problem, and a mixed virtual element method (MVEM) for a mixed variational formulation of the Darcy problem and Stokes problem, respectively. The novelty here constitutes the analysis of a variational formulation of the Stokes problem, since the other two problems were already analyzed on [3] and [9]. However, further details on the construction of the method and the proofs of the associated results of such problems are provided. Therefore, we consider a non–standard mixed approach for the Stokes problem in which the velocity, the pressure and the pseudostress tensor are the main unknowns. However, the pressure shall be eliminated from the original equations, thus yielding an equivalent formulation in which the velocity and the pseudostress tensor are the only unknowns. We then define the virtual finite element subspaces to be employed, introduce the associated interpolation operators, and provide the respective approximation properties. In particular, the latter includes the estimation of the interpolation error for the pseudostress variable measured in the $\mathbb{H}(\mathbf{div})$–norm. Next, and in order to define calculable discrete bilinear forms, we propose a new local projector onto a suitable space of polynomials, which takes into account the main features of the continuous solution and allows the explicit integration of the terms involving the deviatoric tensors. The uniform boundedness of the resulting family of local projectors and its approximation properties are also established. In addition, we show that the global discrete bilinear forms satisfy all the hypotheses required by the Babŭska–Brezzi theory. In this way, we conclude the well-posedness of the actual Galerkin scheme and derive the associated a priori error estimates for the virtual solution as well as for the fully computable projection of it. Finally, several numerical examples illustrating the good performance of the method and confirming the theoretical rates of convergence are presented.

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Chapter 1

Introduction

The virtual element method (VEM), which arose as a natural consequence of new developments and interpretations of the mimetic finite difference method (MFDM) (see, e.g. [8]), was first introduced and analyzed in [3] by employing the Poisson problem as a model. The VEM approach can be viewed as an extension of the classical finite element technique to general polygonal and polyhedral meshes, and also as a generalization of the MFDM to arbitrary degrees of accuracy and arbitrary continuity properties. Its basic idea consists of the utilization of one or more virtual discrete spaces defined on meshes made of polygonal or polyhedral elements, and the incorporation of approximated bilinear forms that mimic the original ones and that still provide consistence and stability of the resulting discrete scheme. The concept *virtual* when referring to a discrete space means that the corresponding basis functions do not need to be known explicitly in order to implement the method, but only the degrees of freedom defining them uniquely on each element are required. As remarked in [5], the main advantages of VEM, when compared with finite volume methods, MMFD, and related techniques, are given by its solid mathematical grounding, the simplicity of the respective coding, and the quality of the numerical results provided. In addition, the computational domain can be decomposed into nonoverlapping convex or nonconvex polygonal elements that can be of very general shape. Further developments of VEM can be found in [1], [4], [6], and [11]. In particular, VEM is utilized in [4] to solve two-dimensional linear elasticity problems, including the compressible and nearly incompressible cases. Moreover, the related application to the linear plate bending problem, in the Kirchhoff-Love formulation, is given in [11]. Also, the eventual incorporation of further global regularity into the discrete solution is investigated in [6]. The main motivation here is the derivation of highly regular methods that could lead to less complicated discretizations of higher-order problems, and also to more direct computations of other variables of physical interest, such as stresses, rotations, and vorticities. Other recent contributions include [5], [22], [33], and [9], which refer, respectively, to practical aspects for the computational implementation of VEM, the application of VEM to three-dimensional linear elasticity problems, the numerical analysis of the two-dimensional Steklov eigenvalue problem by using VEM, and the extension of VEM to the discretization of H(div)-conforming vector fields. Up to the authors' knowledge, [9] is the only work available in the literature that deals with mixed virtual element methods.

According to the above, in the present thesis we are interested in continuing the research line drawn by [9], and aim to develop a mixed-VEM for the Stokes problem. In order to do this, we first review most of the existing results on VEM. In Chapter 2, we use [7] and provide upper bounds for the polynomial approximation error of functions belonging to a Sobolev space. Also, and following [19], an extension to the fractional order case will be presented. In Chapter 3, we use [3] and present VEM on the simplest possible case. Also, and following [5], we present and detail the computational implementation of the method. In Chapter 4, we follow [9] and present the basic features of Mixed Virtual Element Method, as well as the computational implementation of it using, again, [5]. Finally, in Chapter 5, we present a Mixed Virtual Element Method for the Stokes problem and the corresponding computational implementation. Though some of the proofs of the corresponding results are sketched, for sake of clearness and completeness, we try to give as much detail as possible in all the chapters below.

We end this chapter by introducing some notations to be used in all the chapters below. In what follows, $\mathbb{I}$ is the identity matrix of $\mathrm{R}^{2\times 2}$, and given $\boldsymbol{\tau} := (\tau_{ij})$ and $\boldsymbol{\zeta} := (\zeta_{ij})$, we write as usual

$$\boldsymbol{\tau}^t := (\tau_{ji}), \qquad \operatorname{tr}\boldsymbol{\tau} := \sum_{j=1}^{2} \tau_{jj}, \qquad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{2}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}, \qquad \text{and} \qquad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\zeta_{ij},$$

which correspond, respectively, to the transpose, the trace, the deviator tensor of $\boldsymbol{\tau}$, and to the tensorial product between $\boldsymbol{\tau}$ and $\boldsymbol{\zeta}$. In addition, we utilize standard simplified terminology for Sobolev spaces and norms. In particular, if $\mathcal{O} \subset \mathrm{R}^2$ is a domain, $\mathcal{S} \subset \mathrm{R}^2$ is an open or closed Lipschitz curve, and $r \in \mathrm{R}$, we define

$$\mathbf{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^2, \qquad \mathbb{H}^r(\mathcal{O}) := [H^r(\mathcal{O})]^{2\times 2}, \qquad \text{and} \qquad \mathbf{H}^r(\mathcal{S}) := [H^r(\mathcal{S})]^2.$$

However, when $r = 0$ we usually write $\mathbf{L}^2(\mathcal{O})$, $\mathbb{L}^2(\mathcal{O})$, and $\mathbf{L}^2(\mathcal{S})$ instead of $\mathbf{H}^0(\mathcal{O})$, $\mathbb{H}^0(\mathcal{O})$, and $\mathbf{H}^0(\mathcal{S})$, respectively. The corresponding inner products, norms, and semi-norms are denoted, respectively, by $\langle \cdot, \cdot \rangle_{r,\mathcal{O}}$, $\| \cdot \|_{r,\mathcal{O}}$, and $| \cdot |_{r,\mathcal{O}}$ (for $H^r(\mathcal{O})$, $\mathbf{H}^r(\mathcal{O})$, and $\mathbb{H}^r(\mathcal{O})$), and $\langle \cdot, \cdot \rangle_{r,\mathcal{S}}$, $\| \cdot \|_{r,\mathcal{S}}$, and $| \cdot |_{r,\mathcal{S}}$ (for $H^r(\mathcal{S})$ and $\mathbf{H}^r(\mathcal{S})$). In general, given any Hilbert space H, we use $\mathbf{H}$ and $\mathbb{H}$ to denote H^2 and $H^{2\times 2}$, respectively. In turn, $\langle \cdot, \cdot \rangle_{\mathcal{S}}$ stands for the usual duality pairing between $H^{-1/2}(\mathcal{S})$ and $H^{1/2}(\mathcal{S})$, and $\mathbf{H}^{-1/2}(\mathcal{S})$ and $\mathbf{H}^{1/2}(\mathcal{S})$. However, when no confusion arises, we write $\langle \cdot, \cdot \rangle$ instead of $\langle \cdot, \cdot \rangle_{\mathcal{S}}$. Furthermore, with div denoting the usual divergence operator, the Hilbert space

$$\mathbf{H}(\operatorname{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \quad \operatorname{div}(\mathbf{w}) \in L^2(\mathcal{O}) \right\},$$

is standard in the realm of mixed problems (see [10], [30]). The space of matrix valued functions whose rows belong to $\mathbf{H}(\operatorname{div};\mathcal{O})$ will be denoted $\mathbb{H}(\mathbf{div};\mathcal{O})$, where $\mathbf{div}$ stands for the action of div along each row of a tensor. The Hilbert norms of $\mathbf{H}(\operatorname{div};\mathcal{O})$ and $\mathbb{H}(\mathbf{div};\mathcal{O})$ are denoted by $\|\cdot\|_{\operatorname{div};\mathcal{O}}$ and $\|\cdot\|_{\mathbf{div};\mathcal{O}}$, respectively. Note that if $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\mathcal{O})$, then $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\mathcal{O})$ and also $\boldsymbol{\tau}\mathbf{n} \in \mathbf{H}^{-1/2}(\partial\mathcal{O})$, where $\mathbf{n}$ is the unit outward normal at the boundary $\partial\mathcal{O}$. Finally, we employ $\mathbf{0}$ to denote a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

Chapter 2

Polynomial approximation of Functions in Sobolev Spaces

2.1 Introduction

In this chapter, we follow [7, Ch. 4] and develop an appropiate approximation theory for the interpolators and projectors to be defined in chapters 3, 4 and 5. In Section 2.2, we note that we will be dealing with functions belonging to a Sobolev space, and therefore an averaged version of the well–known Taylor polynomial will be defined in order to deal with such an issue. In Section 2.3, we define the remainder term of the aforementioned Averaged Taylor Polynomial and from which the error estimate will be based on. In Section 2.4, we derive an upper bound for the Averaged Taylor Polynomial approximation of a function which belongs to a Sobolev space. Finally, in Section 2.5, we follow [19, Section 6] and extend the obtained result in Section 2.4 to the case of fractional order Sobolev spaces.

2.2 Averaged Taylor polynomials

Given $n \in \mathbb{N}$, let $B := \{x \in \mathrm{R}^n : |x - x_0| < \rho\}$ be a ball of radius ρ, centered at x_0. A function $\phi \in C_0^\infty(\mathrm{R}^n)$ with the properties

$$(a)\,\mathrm{supp}\,\phi = \overline{B}, \qquad \text{and} \qquad (b) \int_{\mathrm{R}^n} \phi = 1,$$

will be called a *cut-off* function. In particular, if

$$\psi(x) := \begin{cases} e^{-(1-(|x-x_0|/\rho)^2)^{-1}}, & |x - x_0| < \rho \\ 0, & |x - x_0| \geq \rho \end{cases},$$

and $c := \int_{\mathrm{R}^n} \psi$, then $c > 0$, and $\phi(x) := \psi(x)/c$ satisfies properties (a) and (b). Also, $\max|\phi| \leq C\rho^{-n}$. Now, let $u \in C^{m-1}(\mathrm{R}^n)$.

Definition 2.2.1. The Taylor polynomial of order m of u evaluated at $y \in \mathrm{R}^n$ is given by

$$T_y^m u(x) := \sum_{|\alpha|<m} \frac{D^\alpha u(y)}{\alpha!}(x - y)^\alpha. \tag{2.2.1}$$

In general, and remembering that we will be dealing with functions from a Sobolev space, $D^\alpha u$ may not exist in the pointwise sense. In order to fix this, we introduce a new Taylor polynomial function, given by the following

Definition 2.2.2. Suppose that u has weak derivatives of order strictly less than m in a region Ω such that $B \subset \Omega$. The Taylor polynomial of order m for u, averaged over B, is defined as

$$Q^m u(x) := \int_B T_y^m u(x)\phi(y)\,dy, \tag{2.2.2}$$

where T_y^m is defined as in (2.2.1), B is the ball centered at $x_0 \in B$ with radius ρ, and ϕ is the cut-off function defined above.

If $u \in H^{m-1}(\Omega)$, then a typical term of $Q^m u$ takes the form

$$\int_B \frac{D^\alpha u(y)}{\alpha!}(x - y)^\alpha \phi(y)\,dy,$$

which exists since $D^\alpha u \in L^1_{loc}(\Omega)$ if $|\alpha| < m$. Moreover, we can write

$$(x-y)^\alpha = \prod_{i=1}^{n}(x_i - y_i)^{\alpha_i} = \sum_{\gamma+\beta=\alpha} a_{(\gamma,\beta)} x^\gamma y^\beta,$$

where β, γ are multi-indices and $a_{(\gamma,\beta)}$ are constants. Therefore,

$$Q^m u(x) = \sum_{|\alpha|<m} \sum_{\gamma+\beta=\alpha} \frac{a_{(\gamma,\beta)} x^\gamma}{\alpha!} \int_B D^\alpha u(y) y^\beta \phi(y)\, dy. \tag{2.2.3}$$

Thus, we have the following result, whose proof is given directly by (2.2.3).

Proposition 2.2.1. $Q^m u$ is a polynomial of degree less than m in x.

Now, we note that $Q^m u$ can be defined in terms of functions belonging to $L^1(B)$. in order to do this, we just need to rewrite (2.2.3) by integrating by parts:

$$Q^m u(x) = \sum_{|\alpha|<m} \sum_{\gamma+\beta=\alpha} \frac{(-1)^{|\alpha|}}{\alpha!} a_{(\gamma,\beta)} x^\gamma \int_B u(y) D^\alpha (y^\beta \phi(y))\, dy. \tag{2.2.4}$$

Remark 2.2.1. Note that if u has weak derivatives of all orders less than m in Ω, then (2.2.3) is equivalent to (2.2.4) by integrating by parts.

Proposition 2.2.2. $Q^m u$ is defined for all $u \in L^1(B)$, and

$$Q^m u(x) := \sum_{|\lambda|<m} x^\lambda \int_B \psi_\lambda(y) u(y)\, dy, \tag{2.2.5}$$

where $\psi_\lambda \in C_0^\infty(\mathrm{R}^n)$ and $\operatorname{supp} \psi_\lambda \subset \overline{B}$.

Proof. It follows directly from (2.2.4) by defining

$$\psi_\lambda(y) := \sum_{\lambda\le\alpha, |\alpha|<m} \frac{(-1)^{|\alpha|}}{\alpha!} a_{(\lambda,\alpha-\lambda)} D^\alpha (y^{\alpha-\lambda} \phi(y)). \tag{2.2.6}$$

$\square$

Proposition 2.2.3. For any α such that $|\alpha| \le m-1$,

$$D^\alpha Q^m u = Q^{m-|\alpha|} D^\alpha u \;\; \forall u \in W_1^{|\alpha|}(B). \tag{2.2.7}$$

Proof. We just need to prove the statement for a function $u \in C^\infty(\Omega)$, hence the general case follows from a density argument. If $|\alpha| \le m-1$, then

$$D^\alpha T_y^m u(x) = T_y^{m-|\alpha|} D^\alpha u(x). \tag{2.2.8}$$

Indeed, it is clear that (2.2.8) holds when $|\alpha| = 0$. Supposing now that $D^\beta T_y^m u(x) = T_y^{m-|\beta|} D^\beta u(x)$, for each β, $|\beta| < |\alpha| \le m-1$, then we can write $\alpha = \beta + \gamma$, for some β, γ, such that $|\beta| = |\alpha| - 1$, $|\gamma| = 1$. Hence, there exists $j \in \{1, \dots, n\}$ such that $D^\gamma u = \dfrac{\partial u}{\partial x_j}$ and thus

$$\begin{aligned}
D^\alpha T_y^m u(x) &= D^\gamma D^\beta T_y^m u(x) = \frac{\partial}{\partial x_j} T_y^{m-|\beta|} D^\beta u(x) = \frac{\partial}{\partial x_j} T_y^{m-|\alpha|+1} D^{\alpha-\gamma} u(x) \\
&= \frac{\partial}{\partial x_j} \sum_{|\eta| < m-|\alpha|+1} \frac{D^\eta D^{\alpha-\gamma} u(y)}{\eta!} (x-y)^\eta = \sum_{|\eta| < m-|\alpha|+1} \eta_j \frac{D^{\eta-\gamma} D^\alpha u(y)}{(\eta-\gamma)! \eta_j} (x-y)^{\eta-\gamma} \\
&= \sum_{|\tilde{\eta}| < m-|\alpha|} \frac{D^{\tilde{\eta}} D^\alpha u(y)}{\tilde{\eta}!} (x-y)^{\tilde{\eta}} = T_y^{m-|\alpha|} D^\alpha u(x),
\end{aligned}$$

which proves that $D^\alpha T_y^m u(x) = T_y^{m-|\alpha|} D^\alpha u(x)$. Therefore,

$$D^\alpha Q^m u(x) = \int_B D^\alpha T_y^m(x) \phi(y)\, dy = \int_B T_y^{m-|\alpha|} D^\alpha u(x) \phi(y)\, dy = Q^{m-|\alpha|} D^\alpha u(x),$$

which finishes the proof. $\square$

2.3 Error Representation

We first remember Taylor's formula with integral remainder for a function $f \in C^m([0,1])$:

$$f(1) = \sum_{k=0}^{m-1} \frac{f^{(k)}(0)}{k!} + m \int_0^1 \frac{s^{m-1}}{m!} f^{(m)}(1-s)\, ds. \tag{2.3.1}$$

Definition 2.3.1. Ω is star-shaped with respect to a ball $B \subset \Omega$ if, for all $x_0 \in B$, x_0 is a center for Ω, or, for all $x_0 \in \Omega$, the closed convex hull of $\{x\} \cup B$ is a subset of Ω.

From now on, we assume that Ω is star-shaped with respect to B. Now, let $u \in C^m(\Omega)$. Given $x \in \Omega$ and $y \in B$, we define $f(s) := u(y + s(x-y))$ and we first notice that f is a well-defined function on $[0,1]$ since Ω is star-shaped respect to B. Also $f \in C^1([0,1])$, and using the chain rule, we obtain

$$\frac{f^{(k)}(s)}{k!} = \sum_{|\alpha|=k} \frac{D^\alpha u(y+s(x-y))}{\alpha!}(x-y)^\alpha, \tag{2.3.2}$$

whence, combining (2.3.1) and (2.3.2), we find that

$$\begin{aligned} u(x) &= \sum_{|\alpha|<m} \frac{D^\alpha u(y)}{\alpha!}(x-y)^\alpha + \sum_{|\alpha|=m}(x-y)^\alpha \int_0^1 \frac{m}{\alpha!}s^{m-1}D^\alpha u(x+s(y-x))\,ds \\ &= T_y^m u(x) + m\sum_{|\alpha|=m}(x-y)^\alpha \int_0^1 \frac{s^{m-1}}{\alpha!}D^\alpha u(x+s(y-x))\,ds. \end{aligned} \tag{2.3.3}$$

Definition 2.3.2. The mth-order remainder term of u is given by

$$R^m u(x) := u(x) - Q^m u(x).$$

Now, using the above definition, property (b) of the definition of a cut-off function, (2.2.2) and (2.3.3), we have that

$$\begin{aligned} R^m u(x) &= u(x)\int_B \phi(y)\,dy - \int_B T_y^m u(x)\phi(y)\,dy \\ &= \int_B \{u(x) - T_y^m(x)\}\phi(y)\,dy \\ &= m\int_B \phi(y)\left\{\sum_{|\alpha|=m}(x-y)^\alpha \int_0^1 \frac{s^{m-1}}{\alpha!}D^\alpha u(x+s(y-x))\,ds\right\}dy. \end{aligned} \tag{2.3.4}$$

Now, denoting the convex hull of $\{x\}\cup B$ by C_x, we have the following

Proposition 2.3.1. The remainder $R^m u$ satisfies

$$R^m u(x) = m\sum_{|\alpha|=m}\int_{C_x} k_\alpha(x,z)D^\alpha u(z)\,dz, \tag{2.3.5}$$

where $z = x + s(y-x)$, $k_\alpha(x,z) := \dfrac{(x-z)^\alpha}{\alpha!}k(x,z)$ and k is a function to be defined later on and which satisfies

$$|k(x,z)| \le C\left(1+\frac{|x-x_0|}{\rho}\right)^n |x-z|^{-n}. \tag{2.3.6}$$

Proof. We first make a change of variables from the (y, s)-space onto the (z, s)-space, where $z = x + s(y - x)$ and $ds\, dy = s^{-n} ds\, dz$. Now, the domain of integration in the original space is $B \times (0, 1]$, and the corresponding domain in the (z, s)-space is

$$A := \left\{ (z, s) : s \in (0, 1], \left| s^{-1}(z - x) + x - x_0 \right| < \rho \right\}.$$

We now notice that if $(z, s) \in A$, then $\dfrac{|z - x|}{|x - x_0| + \rho} < s$. Also $(x - y)^\alpha = s^{-m}(x - z)^\alpha$ if $|\alpha| = m$. If χ_A is the characteristic function of A, from (2.3.4) we find that

$$R^m u(x) = m \sum_{|\alpha|=m} \iint \chi_A(z, s) \phi(x + s^{-1}(z - x)) s^{-n-1} \frac{D^\alpha u(z)}{\alpha!} (x - z)^\alpha \, ds\, dz.$$

Now, the projection of A onto the z-space is C_x. Therefore, using Fubini's theorem, we find that

$$R^m u(x) = m \sum_{|\alpha|=m} \int_{C_x} k_\alpha(x, z) D^\alpha u(z)\, dz,$$

where

$$k(x, z) := \int_0^1 \phi(x + s^{-1}(z - x)) \chi_A(z, s) s^{-n-1}\, ds, \text{ and}$$
$$k_\alpha(x, z) := \frac{(x - z)^\alpha}{\alpha!} k(x, z).$$

Now, if $t = (|x - x_0| + \rho)^{-1} |z - x|$, then

$$\begin{aligned} |k(x, z)| &= \left| \int_0^1 \chi_A(z, s) \phi(x + s^{-1}(z - x)) s^{-n-1}\, ds \right| \le \int_t^1 \left| \phi(x + s^{-1}(z - x)) \right| s^{-n-1}\, ds \\ &\le \|\phi\|_{L^\infty(B)} \left. \frac{s^{-n}}{n} \right|_1^t \le \|\phi\|_{L^\infty(B)} \frac{t^{-n}}{n} = \frac{(\rho + |x - x_0|)^n}{n} \|\phi\|_{L^\infty(B)} |z - x|^{-n} \\ &\le C(1 + \rho^{-1}|x - x_0|)^n |z - x|^{-n}. \end{aligned}$$

We note that the use of Fubini's theorem above is justified by the following calculation:

$$\begin{aligned} &\int_{C_x} \int_0^1 \left| \phi(x + s^{-1}(z - x)) \right| \chi_A(z, s) s^{-n-1} \frac{|D^\alpha u(x)|}{\alpha!} |x - z|^m \, ds\, dz \\ &\le \int_{C_x} \frac{|D^\alpha u(z)|}{\alpha!} |x - z|^{m-n} C(1 + \rho^{-1} |x - x_0|^n)\, dz < \infty. \end{aligned}$$

$\square$

Definition 2.3.3. Suppose that Ω has diameter d and is star-shaped with respect to a ball B. Let $\rho_{\max} := \sup\Big\{\rho : \Omega \text{ is star-shaped with respect to a ball of radius } \rho\Big\}$. Then, the chunkiness parameter of Ω is defined by

$$\gamma := \frac{d}{\rho_{\max}} \tag{2.3.7}$$

Corollary 2.3.1. The ball B can be chosen so that the function $k(x,z)$ defined above satisfies the following estimate

$$|k(x,z)| \leq C(\gamma+1)^n \, |x-z|^{-n} \quad \forall x \in \Omega, \tag{2.3.8}$$

where γ is the chunkiness parameter of Ω.

Proof. We choose a ball B such that Ω is star shaped with respect to B and such that its radius satisfies $2\rho > \rho_{\max}$. Then,

$$\begin{aligned}|k(x,z)| &\leq C(1+\rho^{-1}|x-x_0|)^n \, |z-x|^{-n} \leq C\left(1+\frac{2d}{\rho_{\max}}\right)^n |z-x|^{-n} \\ &\leq C\, 2^n \, (1+\gamma)^n \, |z-x|^{-n}.\end{aligned}$$

□

2.4 Bounds for Riesz Potentials

Lemma 2.4.1. If $f \in L^2(\Omega)$ and $m > n/2$, then

$$\int_\Omega |x-z|^{m-n} \, |f(z)| \; dz \leq C(m,n) d^{m-n/2} \, \|f\|_{0,\Omega} \quad \forall x \in \Omega.$$

Proof. Using Cauchy-Schwarz' inequality and generalized polar coordinates, that is, for

$x := (x_1, \ldots, x_n)^{\mathtt{t}}$ and $z := (z_1, \ldots, z_n)^{\mathtt{t}}$,

$$\begin{aligned}
z_1 &= x_1 + r\cos\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \\
z_2 &= x_2 + r\sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-1} \\
z_3 &= x_3 + r\cos\theta_2 \sin\theta_3 \cdots \sin\theta_{n-1} \\
z_4 &= x_4 + r\sin\theta_2 \sin\theta_3 \cdots \sin\theta_{n-1} \\
\vdots &= \vdots \\
z_n &= x_n + r\cos\theta_{n-1},
\end{aligned}$$

and using that $\Omega \subset B(x,d)$, for all $x \in \Omega$, we find that

$$\begin{aligned}
\int_\Omega |x-z|^{m-n} |f(z)| \, dz &\le \left(\int_\Omega |x-z|^{2(m-n)} \, dz \right)^{1/2} \|f\|_{0,\Omega} \\
&\le C(n) \left(\int_0^d r^{2(m-n)+n-1} \, dr \right)^{1/2} \|f\|_{0,\Omega} = C(m,n) d^{m-n/2} \|f\|_{0,\Omega} ,
\end{aligned}$$

where $C(n)$ is the square root of the content of the $(n-1)$-unit ball. This finishes the proof.

□

Lemma 2.4.2. Let $f \in L^2(\Omega)$ and $m \ge 1$, and let

$$g(x) := \int_\Omega |x-z|^{m-n} |f(z)| \, dz \;\; \forall x \in \Omega.$$

Then,

$$\|g\|_{0,\Omega} \le C(m,n) d^m \|f\|_{0,\Omega} . \tag{2.4.1}$$

Proof. Using Lemma 2.4.1 and the fact that $|\Omega| \le C(n) d^n$, we find that

$$\begin{aligned}
\|g\|_{0,\Omega}^2 &= \int_\Omega |g(x)|^2 \, dx = \int_\Omega \left(\int_\Omega |x-z|^{m-n} |f(z)| \, dz \right)^2 dx \\
&\le \int_\Omega \left\{ \left(\int_\Omega |f(z)|^2 |x-z|^{m-n} \, dz \right) \left(\int_\Omega |x-z|^{m-n} \, dz \right) \right\} dx \\
&\le C(m,n) \|1\|_{0,\Omega} d^{m-n/2} \int_\Omega \int_\Omega |f(z)|^2 |x-z|^{m-n} \, dz \, dx \\
&\le C(m,n) d^m \int_\Omega |f(z)|^2 \left\{ \int_\Omega |x-z|^{m-n} \, dx \right\} dz \le C(m,n) d^{2m} \|f\|_{0,\Omega}^2 ,
\end{aligned}$$

so then, $\|g\|_{0,\Omega} \leq C(m,n)d^m \|f\|_{0,\Omega}$, which finishes the proof. $\square$

Now, we finally arrive at the main result, precised in the following

Lemma 2.4.3. Let B be a ball in a domain Ω such that Ω is star-shaped with respect to B, and such that its radius verifies $2\rho > \rho_{\max}$. Also, suppose that $\operatorname{diam}(\Omega) = 1$ and let $Q^m u$ be the Taylor polynomial of order m of u averaged over B, where $u \in H^m(\Omega)$. Then,

$$|u - Q^m u|_{k,\Omega} \leq C(m,n,\gamma)\,|u|_{m,\Omega}\,, \quad k \in \{0,\dots,m\}\,. \tag{2.4.2}$$

Proof. We first note that if $|\alpha| = m$, then $D^\alpha Q^m u \equiv 0$ by using Proposition 2.2.1. Then, $|u - Q^m u|_{m,\Omega} = |u|_{m,\Omega}$ and so the bound is verified for $k = m$. For $k = 0$, using (2.3.5), (2.3.6) and Lemma (2.4.2) with $f := D^\alpha u$, we obtain that

$$\begin{aligned}\|u - Q^m u\|_{0,\Omega} = \|R^m u\|_{0,\Omega} &\leq m \sum_{|\alpha|=m} \left\| \int_\Omega k_\alpha(x,z) D^\alpha u(z)\,dz \right\|_{0,\Omega} \\ &\leq C(m,n)(1+1/\rho)^n \sum_{|\alpha|=m} \left\| \int_\Omega |x-z|^{m-n} |D^\alpha u(z)|\,dz \right\|_{0,\Omega} \leq C(m,n,\gamma)\,|u|_{m,\Omega}\,.\end{aligned}$$

Finally, if $0 < k < m$ and using (2.2.7) and the case $k = 0$ already proved, we find that

$$\begin{aligned}|u - Q^m u|_{k,\Omega} = |R^m u|_{k,\Omega} &= \left(\sum_{|\alpha|=k} \left\| R^{m-k} D^\alpha u \right\|_{0,\Omega}^2 \right)^{1/2} \\ &\leq C(m,n,\gamma) \left(\sum_{|\alpha|=k} |D^\alpha u|_{m-k,\Omega}^2 \right)^{1/2} \leq C(m,n,\gamma)\,|u|_{m,\Omega}\,,\end{aligned}$$

which finishes the proof. $\square$

2.5 Polynomial approximation in fractional Sobolev spaces

Given $m \geq 0$, we can write $m = \bar{m} + \theta$, where $\bar{m}$ is the integer part of m, that is, $\bar{m}$ is a nonnegative integer and $0 \leq \theta < 1$. If $m \geq 0$ is not an integer, we define the fractional order semi-norm

$$|f|_{m,\Omega} := \left\{ \sum_{|\alpha|=\bar{m}} \int_{\Omega\times\Omega} \frac{|f^{(\alpha)}(x) - f^{(\alpha)}(y)|^2}{|x-y|^{n+2\theta}} dx\, dy \right\}^{1/2}, \tag{2.5.1}$$

also, we denote by $H^m(\Omega)$ the set of all the functions f belonging to $H^{\bar{m}}(\Omega)$ such that $|f|_{m,\Omega} < \infty$. Its corresponding norm is given by

$$\|f\|_{m,\Omega} := \|f\|_{\bar{m},\Omega} + |f|_{m,\Omega}.$$

Lemma 2.5.1. Suppose that $m = \bar{m}+\theta$, where $\bar{m}$ is a nonnegative integer and $0 < \theta < 1$. Suppose further that $\ell := \bar{m}+1$. Then, there exists $C = C(n, \phi, d, m)$ such that for every $f \in H^m(\Omega)$,

$$|f - Q^\ell f|_{k,\Omega} \le C|f|_{m,\Omega}, \quad k \in \{1, \dots, \bar{m}\}. \tag{2.5.2}$$

Proof. First, we can assume without loss of generality that $f \in C^\infty(\mathrm{R}^n)$ (See [31]). We now suppose that α is a multi-index such that $|\alpha| = \ell$, and let j such that $\alpha = \beta + \delta^j$, where β is some multi–index. Now, let

$$R^\alpha(x) := \int_\Omega f^{(\alpha)}(y) k_\alpha(x,y)\, dy \quad \forall x \in \Omega, \tag{2.5.3}$$

where k_α was defined on Proposition 2.3.1. Then, integrating by parts,

$$\begin{aligned} R^\alpha(x) &= \int_\Omega \frac{\partial}{\partial y_j}[f^{(\beta)}(y) - f^{(\beta)}(x)] k_\alpha(x,y)\, dy \\ &= \lim_{\epsilon\to 0} \left\{ - \int_{\{y\in\Omega: |x-y|>\epsilon\}} [f^{(\beta)}(y) - f^{(\beta)}(x)] \frac{\partial k_\alpha}{\partial y_j}(x,y)\, dy \right. \\ &\quad \left. + \int_{|x-y|=\epsilon} [f^{(\beta)}(y) - f^{(\beta)}(x)] k_\alpha(x,y)(x_j - y_j)\epsilon^{-1}\, ds \right\}, \end{aligned}$$

where ds is the surface measure. The above surface integral tends to 0 when $\epsilon \to 0^+$ since if $|x-y| = \epsilon$, (2.3.6) implies that $|k_\alpha(x,y)| \le C\epsilon^{1-n}$, also

$$|f^{(\beta)}(y) - f^{(\beta)}(x)| \le C\epsilon, \text{ and}$$

$$\int_{|x-y|=\epsilon} 1\, ds = C\epsilon^{n-1}.$$

Using (2.3.6) again, it follows that

$$|R^\alpha(x)| \le C_0 \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|}{|x-y|^n},$$

where C_0 depends on d. In turn, the expression within the integral is bounded by $C|x-y|^{1-n}$ and consequently R^α belongs to $L^1(\Omega)$. On the other side, using Cauchy-Schwarz' inequality and generalized polar coordinates, it follows that

$$\begin{aligned}
|R^\alpha(x)|^2 &\le C_0^2 \left(\int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|}{|x-y|^{\frac{n}{2}+\theta}|x-y|^{\frac{n}{2}-\theta}} \right)^2 \\
&\le C_0^2 \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|^2}{|x-y|^{n+2\theta}}\, dy \left(\int_\Omega |x-y|^{-n+2\theta}\, dy \right) \\
&\le C_0^2 \left(\int_{\partial B(x,d)} \int_0^d r^{2\theta-1}\, dr\, ds \right) \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|^2}{|x-y|^{n+2\theta}}\, dy \\
&\le C_0^2 \frac{C(n-1)}{2\theta} d^{2\theta} \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|^2}{|x-y|^{n+2\theta}}\, dy \\
&\le \left\{ C_0 \sqrt{\frac{C(n-1)}{2\theta}} d^\theta \right\}^2 \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|^2}{|x-y|^{n+2\theta}}\, dy \\
&= C(n,d) \int_\Omega \frac{|f^{(\beta)}(y) - f^{(\beta)}(x)|^2}{|x-y|^{n+2\theta}}\, dy,
\end{aligned}$$

now, integrating with respect to x and summing on all the indices α such that $|\alpha| = \ell$, we find that

$$\|f - Q^\ell f\|_{0,\Omega} \le \ell \sum_{|\alpha|=\ell} \|R^\alpha f\|_{0,\Omega} \le C|f|_{m,\Omega}.$$

Now, if $k \in \{0, \ldots, \bar{m}\}$, we use that $D^\alpha R^\ell f = R^{\ell-|\alpha|} D^\alpha f$ if $|\alpha| = k$ and so

$$|f - Q^\ell f|_{k,\Omega} \le \sum_{|\alpha|=k} \|R^{\ell-|\alpha|} D^\alpha f\|_{0,\Omega} \le C \sum_{|\alpha|=k} |D^\alpha f|_{m-|\alpha|,\Omega} \le C|f|_{m,\Omega}.$$

$\square$

The foregoing Proposition, combined with the main Lemma on the previous section (cf. Lemma 2.4.3), provides the following result, in which we omit further details.

Lemma 2.5.2. Suppose that $m = \bar{m} + \theta$, where $0 \leq \theta \leq 1$, and that $\bar{m}$ is a nonnegative integer. In turn, let $\ell := \bar{m} + 1$ and let Q^ℓ as defined above. Then, there exists a constant $C = C(n, \phi, d, m)$ such that

$$\|f - Q^\ell f\|_{m,\Omega} \leq C|f|_{m,\Omega} \quad \forall f \in H^m(\Omega) \tag{2.5.4}$$

Chapter 3

Virtual Elements Method for the Poisson problem

3.1 Introduction

In this chapter, we use [3] and present a Virtual Element Method for the Poisson problem in the simplest possible way, generalizing the result by considering now a non-homogeneus Dirichlet boundary condition. In Section 3.2 we introduce the boundary value problem of interest and the associated well–posedness result. Then, in Section 3.3 we introduce the virtual element subspaces to be employed, show the respective unisolvency, define the associated interpolation operators and provide their approximation properties. In particular, a Bramble–Hilbert type theorem for averaged Taylor polynomials (cf. Chapter 2) plays a key role in our analysis. Next, a fully calculable discrete bilinear form is introduced in Section 3.3.4 and its boundedness and related properties are established. To this end, a new local projector onto a suitable space of polynomials is proposed here. This operator responds to the need of integrate the terms associated to a bilinear form that involves first order derivatives of basis functions belonging to the discrete space on an explicit way. The family of local projectors is shown to be uniformly bounded, and the aforementioned compactness theorem is applied to derive their approximation properties.

An approximation of the right-hand side of the variational formulation is also needed and introduced in 3.3.5 to explicitly integrate the corresponding terms. Also, standard results on polynomial projectors are used to derive their approximation properties. The actual virtual element method is then introduced and analyzed in Section 3.3.6. The Lax–Milgram Lemma is applied to deduce the well–posedness of this scheme, and then suitable bounds and identities satisfied by the discrete bilinear form and the associated projector and interpolator involved will allow us to derive the a priori error estimates and the corresponding rates of convergence for the virtual solution, as well as for the projection of it. On the other hand, in Section 3.4 we use [5] and give details on the computational implementation of VEM, explaining how to assemble the global stiffness matrix and how to impose the boundary condition on the approximated solution. Finally, several numerical examples showing the good performance of the method, confirming the rates of convergence for regular and singular solutions, and illustrating the accuracy obtained with the approximate solutions, are reported in Section 3.4.

3.2 The continuous problem and its primal formulation

Given a polygonal domain Ω in R^2 with boundary Γ, and given data $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we consider the problem: Find u such that

$$\begin{aligned} -\Delta u &= f \quad \text{in} \quad \Omega, \\ u &= g \quad \text{on} \quad \Gamma = \partial\Omega. \end{aligned} \tag{3.2.1}$$

As usual, testing with $v \in H_0^1(\Omega)$ and integrating by parts, the primal formulation of this problem reads: *Find* $u \in V := H_0^1(\Omega)$ *such that*

$$a(u,v) = (f,v)_{0,\Omega} + (\nabla u_g, \nabla v)_{0,\Omega} \quad \forall v \in V, \tag{3.2.2}$$

where we have chosen u_g as any element belonging to $H^1(\Omega)$ such that $\gamma_0(u_g) = g$ and $\|u_g\|_{1,\Omega} \leq C\|g\|_{1/2,\Gamma}$, for some $C > 0$ depending only on Ω, and defined

$$a(u,v) := (\nabla u, \nabla v)_{0,\Omega} \quad \forall u, v \in V.$$

Using standard arguments, we can ensure the existence of a unique $u \in V$ such that (3.2.2) holds. This statement is established as follows.

Theorem 3.2.1. *There exists a unique $u \in V$ solution of* (3.2.2)*. Moreover, there exists $C > 0$, depending only on Ω, such that*

$$\|u\|_{1,\Omega} \leq C\Big\{\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}\Big\}. \tag{3.2.3}$$

Proof. We first note that $a : V \times V \to \mathrm{R}$ satisfies

$$|a(u,v)| \leq |u|_{1,\Omega}|v|_{1,\Omega} \leq \|u\|_{1,\Omega}\|v\|_{1,\Omega} \qquad \forall u, v \in V,$$

that is, a is a bounded bilinear form. In turn, since $V = H_0^1(\Omega)$, a direct application of Friedrichs–Poincaré's inequality ensures the existence of a constant $C > 0$, depending on Ω, such that $|u|_{1,\Omega} \geq C\|u\|_{1,\Omega}$, for each $u \in V$. That is, $a(u,u) = |u|_{1,\Omega}^2 \geq C\|u\|_{1,\Omega}$, for each $u \in V$, which shows particularly that a is V-elliptic. Therefore, an application of the Lax–Milgram Lemma (see for instance [7, p. 62]) shows that (3.2.2) has a unique solution $u \in V$. Moreover, there exists $C > 0$, depending on Ω, such that

$$\|u\|_{1,\Omega} \leq C\Big\{\|f\|_{0,\Omega} + \|u_g\|_{1,\Omega}\Big\},$$

and since $\|u_g\|_{1,\Omega} \leq C\|g\|_{1/2,\Gamma}$, it follows easily that

$$\|u\|_{1,\Omega} \leq C\Big\{\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}\Big\}.$$

$\square$

3.3 The discrete problem

3.3.1 The virtual element subspaces

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements. Given $K \in \mathcal{T}_h$, we denote its diameter by h_K, and define, as usual, $h := \max\{h_K : K \in \mathcal{T}_h\}$. In turn, given an integer $k \geq 0$, we let $\mathrm{P}_k(K)$ be the space of polynomials on K of total degree up to k. Then, given an integer $k \geq 1$, we first consider the following finite-dimensional subspace of $C(\partial K)$:

$$\mathbb{B}(\partial K) := \Big\{ v \in C(\partial K) \,:\, v|_e \in \mathbb{P}_k(e) \quad \forall \text{ edge } e \subset \partial K \Big\}. \tag{3.3.1}$$

We can easily check that if d_K is the number of edges of K, then $\dim(\mathbb{B}(\partial K)) = d_K k$, since a continuous function on ∂K which is a polynomial function of degree up to k on each edge of K is uniquely determined by its values at the d_K vertices of K, and its values at $k-1$ distributed points on each face as well. Now, we define the following finite-dimensional subspace of V:

$$V_h := \Big\{ v \in V \,:\, v|_{\partial K} \in \mathbb{B}_k(\partial K)\,, \quad \Delta v \in \mathrm{P}_{k-2}(K) \quad \forall K \in \mathcal{T}_h \Big\}. \tag{3.3.2}$$

Then, the Galerkin scheme associated with (3.2.2) would read: Find $u_h \in V_h$ such that

$$a(u_h, v_h) = (f, v_h)_{0,\Omega} + (\nabla u_g, \nabla v_h)_{0,\Omega} \quad \forall v_h \in V_h. \tag{3.3.3}$$

Nevertheless, we will observe later on that $a(u_h, v_h)$ can not be computed explicitly when u_h, v_h belongs to V_h, and hence a suitable approximation of a, namely a_h, will be introduced in Section 3.3.4 to redefine (3.3.3).

3.3.2 Unisolvency of the virtual elements subspaces

In what follows, we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$, that is, for each $x_0 \in B$, all the line segments joining x_0 with any $x \in K$ are contained in K, or, equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in K.

As a consequence of the above hypotheses, one can show that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only con $C_{\mathcal{T}}$), such that the number of edges of each $K \in \mathcal{T}_h$ is bounded above by $N_{\mathcal{T}}$.

Next, in order to choose the degrees of freedom of V_h, given an element $K \in \mathcal{T}_h$ with barycenter $\mathbf{x}_K$, and given an integer $\ell \geq 0$, we define the following set of $(\ell+1)(\ell+2)/2$ normalized monomials

$$\mathcal{B}_\ell(K) := \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^\alpha : 0 \leq |\alpha| \leq \ell \right\}, \tag{3.3.4}$$

which certainly constitutes a basis of $\mathrm{P}_\ell(K)$. Note that (3.3.4) makes use of the multi–index notation where, given $\mathbf{x} := (x_1, x_2)^t \in \mathrm{R}^2$ and $\alpha := (\alpha_1, \alpha_2)^t$, with nonnegative integers α_1, α_2, we set $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\alpha| := \alpha_1 + \alpha_2$. According to the above and the definition of V_h (cf. (3.3.2)), we propose the following degrees of freedom for a given $v \in V_h$ and a given $K \in \mathcal{T}_h$:

a) The values of v at the vertices of K,

b) If $k > 1$, the values of v at the $k-1$ Gauss-Lobatto quadrature points at each edge of K, (3.3.5)

c) If $k > 1$, the moments $\displaystyle\int_K p(\mathbf{x}) v(\mathbf{x})\, d\mathbf{x} \quad \forall p \in \mathcal{B}_{k-2}(K) \quad \forall K \in \mathcal{T}_h$.

We now observe, according to the cardinality of $\mathcal{B}_{k-2}(K)$, that the amount of local degrees of freedom, that is those related to a given $K \in \mathcal{T}_h$, is given by

$$n_k^K := d_K k + \frac{k(k-1)}{2},$$

where we recall that d_K is the number of vertices of K. Moreover, we have the following local unisolvence result.

Lemma 3.3.1. Given an integer $k \geq 1$, we define the local space

$$V_h^K := \left\{ v \in H^1(K) : v|_{\partial K} \in \mathbb{B}_k(\partial K), \Delta v|_K \in \mathrm{P}_{k-2}(K) \right\}. \tag{3.3.6}$$

Then, the n_k^K degrees of freedom arising from (3.3.5) are unisolvent in V_k^K.

Proof. Let $v \in V_h$ such that all its local degrees of freedom (that is, those related to K) arising from (3.3.5) are zero. Using *a)* and *b)* from (3.3.5), and given that $v|_{\partial K} \in \mathrm{P}_k(\partial K)$, it follows that $v = 0$ in ∂K. In turn, if $\mathcal{P}_{k-2}^K : L^2(K) \to \mathrm{P}_{k-2}(K)$ is the orthogonal projector, then $\mathcal{P}_{k-2}^K v = 0$ in K, since all the corresponding internal moments of v are zero. Hence, we just need to prove now that $v = 0$ in K. In order to do this, given $q \in \mathrm{P}_{k-2}(K)$, we consider the auxiliary problem: *Find* $w \in H_0^1(K)$ *such that*

$$a^K(w, v) := \int_K \nabla w \cdot \nabla v = (q, v)_{0,\Omega} \quad \forall v \in H_0^1(K), \tag{3.3.7}$$

which can be read, equivalently, as

$$-\Delta w = q \text{ in } K, \;\; w = 0 \text{ in } \partial K, \tag{3.3.8}$$

or, using existence and uniqueness properties, $w := -\Delta_{0,K}^{-1}(q)$, where given $q \in L^2(K)$, $\Delta_{0,K}^{-1}(q) \in H_0^1(K)$ is the unique solution of (3.3.7). Next, we consider the operator $R : \mathrm{P}_{k-2}(K) \to \mathrm{P}_{k-2}(K)$ defined by $R(q) := \mathcal{P}_{k-2}^K\left(-\Delta_{0,K}^{-1}(q)\right) = \mathcal{P}_{k-2}^K(w) \;\; \forall q \in \mathrm{P}_{k-2}(K)$. Therefore, if $q \in \mathrm{P}_{k-2}(K)$, using (3.3.7) and the orthogonality condition, it follows that

$$(R(q), q)_{0,K} = \left(\mathcal{P}_{k-2}^K w, q\right)_{0,K} = (q, w)_{0,K} = a^K(w, w),$$

whence, using that $|\cdot|_{1,K}$ and $\|\cdot\|_{1,K}$ are equivalent norms in $H_0^1(K)$, we see that $R(q) = \theta \Leftrightarrow a^K(w, w) = 0 \Leftrightarrow w = 0 \Leftrightarrow q = 0$. Hence, $\mathcal{P}_{k-2}^K v = R(-\Delta v)$ and thus $\Delta v = 0$ in K, if $v = 0$ on ∂K and $\mathcal{P}_{k-2}^K v = 0$ in K. Consequently, $v = 0$ in K. $\square$

3.3.3 Interpolation on the discrete subspace

In this section we define a suitable interpolation operator on our virtual element subspace and establish its corresponding approximation property. To this end, we need some preliminary notations and technical results. For each element $K \in \mathcal{T}_h$ we let $\widetilde{K} := T_K(K)$, where $T_K : \mathrm{R}^2 \longrightarrow \mathrm{R}^2$ is the bijective affine mapping defined by $T_K(\mathbf{x}) := \dfrac{\mathbf{x} - \mathbf{x}_B}{h_K}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$. Note that the diameter $h_{\widetilde{K}}$ of $\widetilde{K}$ is 1, and, according to the assumptions a) and b), it is easy to see that the shortest edge of $\widetilde{K}$ is bigger than $C_{\mathcal{T}}$, and that $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. Recall here that $\mathbf{x}_B$ is the center of the ball B with respect to which K is star-shaped. Then, by connecting each vertex of $\widetilde{K}$ to the center of $\widetilde{B}$, that is to the origin, we generate a partition of $\widetilde{K}$ into $d_{\widetilde{K}}$ triangles $\widetilde{\Delta}_i$, $i \in \{1, 2, \dots, d_{\widetilde{K}}\}$, where $d_{\widetilde{K}} \le N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\dfrac{\widetilde{h}_i}{\widetilde{\rho}_i} \le c_{\mathcal{T}} \;\; \forall\, i \in \{1, 2, \dots, d_{\widetilde{K}}\}$, where $\widetilde{h}_i$ is the diameter of $\widetilde{\Delta}_i$ and $\widetilde{\rho}_i$ is the diameter of the largest ball contained in $\widetilde{\Delta}_i$. We also let $\widehat{\Delta}$ be the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$, and for each $i \in \{1, 2, \dots, d_{\widetilde{K}}\}$ we let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\, \mathbf{x}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \widetilde{\Delta}_i$. We remark that the fact that the origin is a vertex of each triangle $\widetilde{\Delta}_i$ allows to choose F_i as indicated.

The next result taken from Lemma 2.4.3 is required in what follows as well.

Lemma 3.3.2. Let $\mathcal{O}$ be a domain of R^2 with diameter 1, such that it is star-shaped with respect to a ball B of radius $> \frac{1}{2}\, \rho_{\mathtt{max}}$, where

$$\rho_{\mathtt{max}} := \sup\Big\{\, \rho : \quad \mathcal{O} \text{ is star-shaped with respect to a ball of radius } \rho \Big\}.$$

In addition, given an integer $m \ge 1$ and $v \in H^m(\mathcal{O})$, we let $\mathrm{T}^m(v) \in \mathrm{P}_{m-1}(\mathcal{O})$ be the Taylor polynomial of order m of v averaged over B. Then, there exists $C > 0$, depending only on m and $\rho_{\mathtt{max}}$, such that

$$|v - \mathrm{T}^m(v)|_{\ell,\mathcal{O}} \le C\, |v|_{m,\mathcal{O}} \qquad \forall\, \ell \in \{0, 1, \dots, m\}\,.$$

We now proceed to define our interpolation operator $\Pi_k^h : H^1(\Omega) \to V_h$. Indeed, given $K \in \mathcal{T}_h$, denoting $\mathcal{V}_0(K)$ and $\mathcal{V}_1(K)$ the set of vertices of K and the set of internal Gauss–Lobatto points at the edges of K, respectively, and given $v \in H^1(\Omega)$, $\Pi_k^h(v)$ is the unique element in V_h such that

$$
\begin{aligned}
0 &= (\Pi_k^h(v) - v)(\mathbf{x}) && \forall\, \mathbf{x} \in \mathcal{V}_0(K) && \forall K \in \mathcal{T}_h\,, \\
0 &= (\Pi_k^h(v) - v)(\mathbf{x}) && \forall\, \mathbf{x} \in \mathcal{V}_1(K) && \forall K \in \mathcal{T}_h\,, \\
0 &= \int_K (\Pi_k^h(v) - v)\, q && \forall\, q \in \mathcal{B}_{k-2}(K) && \forall\, K \in \mathcal{T}_h\,.
\end{aligned} \tag{3.3.9}
$$

Note that the uniqueness of Π_k^h is guaranteed by Lemma 3.3.1. Now, given $K \in \mathcal{T}_h$, we define the local restriction of the interpolator operator as $\Pi_k^K(v) := \Pi_k^h(v)\big|_K \in V_h^K$. Note that since $V_h \subset V$, it follows that $\Pi_h^k(v) = 0$ in Γ, for every $v \in V$. Also, $\Pi_k^K(p) = p \quad \forall p \in \mathrm{P}_k(K)$. Then, we define the functionals:

$$
\begin{aligned}
m_{\mathcal{V}_0(K)}(v) &:= v(\mathbf{x}) && \forall\, \mathbf{x} \in \mathcal{V}_0(K)\,, \\
m_{\mathcal{V}_1(K)}(v) &:= v(\mathbf{x}) && \forall\, \mathbf{x} \in \mathcal{V}_1(K)\,, \\
m_q^K(v) &:= \int_K q\, v && \forall\, q \in \mathcal{B}_{k-2}(K)\,,
\end{aligned} \tag{3.3.10}
$$

and gather all the above in the set $\Big\{m_{j,K}(v)\Big\}_{j=1}^{n_k^K}$. Then, we let $\{\varphi_{j,K}\}_{j=1}^{n_k^K}$ be the canonical basis of V_h^K, that is, given $i \in \{1, 2, \ldots, n_k^K\}$, $\varphi_{i,K}$ is the unique element in V_h^K such that

$$
m_{j,K}(\varphi_{i,K}) = \delta_{ij} \quad \forall j \in \{1, 2, \ldots, n_k^K\}.
$$

Therefore,

$$
\Pi_k^K(v) = \sum_{j=1}^{n_k^K} m_{j,K}(v)\varphi_{j,K}, \tag{3.3.11}
$$

or, equivalently, $\Pi_k^K(v)$ is the unique element in V_h^K such that

$$
m_{j,K}(\Pi_k^K(v)) = m_{j,K}(v) \quad \forall j \in \{1, 2, \ldots, n_k^K\}.
$$

Then, we have the following result.

Lemma 3.3.3. Given integers k, ℓ such that $k + 1, \ell \geq 2$ and $K \in \mathcal{T}_h$, there holds $\widetilde{\Pi_k^K(v)} = \Pi_k^{\widetilde{K}}(\widetilde{v})$ for all $v \in H^k(K)$, and $\Pi_k^{\widetilde{K}} \in \mathcal{L}(H^\ell(\widetilde{K}), H^1(\widetilde{K}))$ with $\|\Pi_k^K\|_{\mathcal{L}(H^\ell(\widetilde{K}), H^1(\widetilde{K}))}$ independent of $\widetilde{K}$, namely depending only on $k, \ell, N_{\mathcal{T}}$ and $C_{\mathcal{T}}$.

Proof. Note that since $n = 2$ and $\ell \geq 2 > \frac{n}{2}$, it follows that $i : H^\ell(K) \to C(K)$ is a continuous injection. Then, it follows directly, by using the chain rule and the definition of T_K, that $\varphi \in V_h^K$ if and only if $\widetilde{\varphi} \in V_h^{\widetilde{K}}$. In particular, given $v \in H^k(K)$, there holds $\widetilde{\Pi_k^K(v)} \in H_h^{\widetilde{K}}$, and hence the required identity holds if and only if the values of the functionals defined on (3.3.10) coincide. Indeed, defining $\{\mathbf{x}_1, \ldots, \mathbf{x}_J\} := \mathcal{V}_0(K) \cup \mathcal{V}_1(K)$, and if $m_{1,K}, \ldots, m_{J,K}$ are the corresponding nodal values of v on K, that is, any of the two first functionals of (3.3.10), we have that

$$T_K\left(\sum_{j=1}^J m_{j,K}(v)\varphi_{j,K}\right) = \sum_{j=1}^J v(\mathbf{x}_j)\widetilde{\varphi}_{j,K} = \sum_{j=1}^J v \circ T_K^{-1}(\widetilde{\mathbf{x}}_j)\widetilde{\varphi}_{j,K} = \sum_{j=1}^J \widetilde{v}(\widetilde{\mathbf{x}}_j)\widetilde{\varphi}_{j,K},$$

that is,

$$T_K\left(\sum_{j=1}^J m_{j,K}(v)\varphi_{j,K}\right) = \sum_{j=1}^J m_{j,K}(\widetilde{v})\widetilde{\varphi}_{j,K}. \tag{3.3.12}$$

On the other hand, for each $q \in \mathcal{B}_{k-2}(K)$, and letting $\widetilde{q} := q \circ T_K^{-1} \in \mathrm{P}_{k-2}(\widetilde{K})$, we find that

$$\int_{\widetilde{K}} \widetilde{\Pi_k^K(v)}\widetilde{q} = h_K^{-2}\int_K \Pi_k^K(v)\, q = h_K^{-2}\int_K qv = \int_{\widetilde{K}} \widetilde{q}\widetilde{v}. \tag{3.3.13}$$

Therefore, putting (3.3.12) and (3.3.13) together, we have that

$$\widetilde{\Pi_k^K(v)} = \sum_{j=1}^{n_k^K} m_{j,\widetilde{K}}(\widetilde{v})\widetilde{\varphi}_{j,K}$$

as required. It remains to show that $\Pi_k^{\widetilde{K}} \in \mathcal{L}(H^\ell(\widetilde{K}), H^1(\widetilde{K}))$, where $\|\Pi_k^{\widetilde{K}}\|_{\mathcal{L}\left(H^\ell(\widetilde{K}),H^1(\widetilde{K})\right)}$ independent of $\widetilde{K}$. For this purpose, we observe from (3.3.11) that

$$\|\Pi_k^{\widetilde{K}}(v)\|_{1,\widetilde{K}} \leq \sum_{j=1}^{n_k^{\widetilde{K}}} |m_{j,\widetilde{K}}(v)|\|\varphi_{j,\widetilde{K}}\|_{1,\widetilde{K}},$$

where each $m_{j,\widetilde{K}}$ is defined according to (3.3.10) with $\mathcal{V}_0(\widetilde{K})$, $\mathcal{V}_1(\widetilde{K})$, $\mathcal{B}_{k-2}(\widetilde{K})$, and $\left\{\varphi_{j,\widetilde{K}}\right\}_{j=1}^{n_k^{\widetilde{K}}}$ is the canonical basis of $V_h^{\widetilde{K}}$. Next, we proceed to bound the functionals defined on (3.3.10) in terms of $\|v\|_{\ell,\widetilde{K}}$. In fact, given $\mathbf{x} \in \mathcal{V}_0(\widetilde{K})$,

$$|m_{\mathcal{V}_0(\widetilde{K})}(v)| = |v(\mathbf{x})| \leq \|v\|_{\infty,\widetilde{K}} \leq \|v\|_{\ell,\widetilde{K}},$$

and, similarly,

$$|m_{\mathcal{V}_1(\widetilde{K})}(v)| \le \|v\|_{\ell,\widetilde{K}}.$$

On the other hand, bearing in mind that $|\widetilde{K}| \le \widetilde{C}$, where $\widetilde{C}$ is independent of h, it follows for a given $q \in \mathcal{B}_{k-2}(\widetilde{K})$ that $\left\| \dfrac{\mathbf{x} - \mathbf{x}_{\widetilde{K}}}{h_{\widetilde{K}}} \right\|_{\infty,\widetilde{K}} \le 1$ and hence

$$|m_q^{\widetilde{K}}(v)| \le \|q\|_{0,\widetilde{K}} \|v\|_{0,\widetilde{K}} \le |K|^{1/2} \|v\|_{0,\widetilde{K}} \le \widetilde{C} \|v\|_{\ell,\widetilde{K}}.$$

Finally, we observe, thanks to the assumptions $a)$ and $b)$ (cf. beginning of Section 3.3.2) and the choice of the normalized monomials given by $\mathcal{B}_{k-2}(\widetilde{K})$ (cf. 3.3.4), there holds $\|\varphi_{j,\widetilde{K}}\|_{1,\widetilde{K}} = O(1)$, which completes the boundedness of $\Pi_k^{\widetilde{K}}$ and hence the proof. □

Lemma 3.3.4. Let k, ℓ and r be integers such that $k \ge 1$, $\ell \in \{0,1\}$ and $2 \le r \le k+1$. Then, there exists a constant $C > 0$, depending only on $k, \ell, r, c_{\mathcal{T}}$, $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$, there holds

$$|v - \Pi_k^K(v)|_{\ell,K} \le C h_K^{r-l} |v|_{r,K} \qquad \forall v \in H^r(K). \tag{3.3.14}$$

Proof. Given $v \in H^r(K)$, we let $\widetilde{\mathrm{T}}^r(\widetilde{v}) \in \mathrm{P}_{r-1}(\widetilde{K})$ be the Taylor polynomial of order r of $\widetilde{v}$ (cf. Lemma 3.3.2), and observe, since $r - 1 \le k$, that $\Pi_k^{\widetilde{K}}\big(\widetilde{\mathrm{T}}^r(\widetilde{v})\big) = \widetilde{\mathrm{T}}^r(\widetilde{v})$. It follows, using Lemmas 3.3.2 and 3.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned} |v - \Pi_k^K(v)|_{\ell,K} &= h_K^{-\ell-1} |\widetilde{v} - \widetilde{\Pi_k^K(v)}|_{\ell,\widetilde{K}} = h_K^{-\ell-1} |\widetilde{v} - \Pi_k^{\widetilde{K}}(\widetilde{v})|_{\ell,\widetilde{K}} \\ &= h_K^{-\ell-1} |(\mathrm{I} - \Pi_k^K)(\widetilde{v} - \widetilde{\mathrm{T}}^r(\widetilde{v}))|_{\ell,\widetilde{K}} \\ &\le h_K^{-\ell-1} \|\mathrm{I} - \Pi_k^{\widetilde{K}}\|_{\mathcal{L}(H^r(\widetilde{K}),H^1(\widetilde{K}))} |\widetilde{v} - \widetilde{\mathrm{T}}^r(\widetilde{v})|_{\ell,\widetilde{K}} \\ &\le C h_K^{-\ell-1} |\widetilde{v}|_{r,\widetilde{K}} = C h_K^{r-\ell} |v|_{r,K} \end{aligned}$$

which finishes the proof. □

3.3.4 The discrete bilinear form

The purpose of this section is to define a computable version $a_h : V_h \times V_h \to \mathrm{R}$ of the bilinear form a defined above. To this end, we observe that given $u, v \in V_h$, the expression

$$a(u,v) = \int_\Omega \nabla u \cdot \nabla v$$

is not explicitly calculable since in general u, v are not known on each $K \in \mathcal{T}_h$. In order to overcome this difficulty, we now proceed to introduce suitable spaces on which the elements of V_h will be projected later on, and for which the bilinear form a is computable. Indeed, let us first consider a particular choice of v given by $v := p \in \mathrm{P}_k(K)$. It follows that for each $u \in V_h$ there holds

$$\int_K \nabla u \cdot \nabla v = -\int_K u\,\Delta p + \int_{\partial K} u\,\frac{\partial p}{\partial \mathbf{n}}, \tag{3.3.15}$$

which, noting that $\Delta p \in \mathrm{P}_{k-2}(K)$ and $u\dfrac{\partial p}{\partial \mathbf{n}} \in \mathrm{P}_{2k-1}(\partial K)$, it follows from (3.3.10) and the fact that the second term of (3.3.15) is integrated exactly by using the Gauss–Lobatto quadrature rule, that $a(u,v)$ is in fact calculable in this case. We now introduce a projection operator $\widehat{\Pi}_k^K : H^1(K) \to \mathrm{P}_k(K)$. To this end, we set for each $K \in \mathcal{T}_h$ the local bilinear form

$$a^K(u,v) := \int_K \nabla u \cdot \nabla v \quad \forall u, v \in H^1(K).$$

Then, we let $\widehat{\mathrm{P}}_k(K) := \mathrm{span}\{\mathbf{x}^\alpha : 1 \le |\alpha| \le k\}$ and given $u \in H^1(K)$, we consider the fact that $\mathrm{P}_k(K) = \widehat{\mathrm{P}}_k(K) \oplus \mathrm{R}$, so that we can define $\widehat{u} := \widehat{\Pi}_k^K(u)$ in terms of the decomposition

$$\widehat{u} = q_u + c_u, \tag{3.3.16}$$

where the components $q_u \in \widehat{\mathrm{P}}_k^K$ and $c_u \in \mathrm{R}$ are computed according to the following sequentially connected problems:

- Find $q_u \in \widehat{\mathrm{P}}_k^K$ such that

$$a^K(q_u, v) = a^K(u,v) \quad \forall v \in \widehat{\mathrm{P}}_k(K)\,, \tag{3.3.17}$$

- Find $c_u \in \mathrm{R}$ such that

$$\overline{\widehat{u}} = \overline{u}\,, \tag{3.3.18}$$

where $\overline{v} := \dfrac{1}{n} \displaystyle\sum_{\mathbf{x} \in \mathcal{V}_0(K)} v(\mathbf{x})$. We remark that the unique solvability of (3.3.17) is guaranteed by the identity

$$a^K(q,q) = |q|^2_{1,K} > 0 \quad \forall q \in \widehat{\mathrm{P}}_k(K) \backslash \{0\}.$$

In this way, having computed $q_u \in \widehat{\mathrm{P}}_k(K)$, we replace it into (3.3.18), which yields

$$c_u = \overline{u - q_u}. \tag{3.3.19}$$

Let us now check that the right hand sides of (3.3.17) and (3.3.18) are indeed calculable when $u, v \in V_h^K \subseteq H^1(K)$ (cf. (3.3.6)). Given that $\Delta v \in \mathrm{P}_{k-2}(K)$ and $\dfrac{\partial v}{\partial \mathbf{n}} \in \mathrm{P}_{k-1}(\partial K)$, it follows that

$$\int_K \nabla u \cdot \nabla v = -\int_K u\,\Delta v + \int_{\partial K} u\,\frac{\partial v}{\partial \mathbf{n}} \tag{3.3.20}$$

is explicitly calculable due to the explicit knowledge of the degrees of freedom for u. On the other hand, and by using again the degrees of freedom of u, we have that the value of $u(\mathbf{x})$ is known whenever $\mathbf{x} \in \mathcal{V}_0(K)$. This implies particularly that $\overline{u}$ is explicitly computable. Finally, it is straightforward to check from (3.3.17) and (3.3.18) that $\widehat{\Pi}_k^K(p) = p \quad \forall p \in \mathrm{P}_k(K)$. Furthermore, the following result establishes the uniform boundedness of the family $\left\{\widehat{\Pi}_k^K\right\}_{K\in\mathcal{T}_h} \subseteq \left\{\mathcal{L}(H^1(K), H^1(K))\right\}_{K\in\mathcal{T}_h}$.

Lemma 3.3.5. There exists a constant $C > 0$, depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$ and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\widehat{\Pi}_k^K(v)\|_{1,K} \leq C\Big\{\|v\|_{0,K} + h_K |v|_{1,K}\Big\} \quad \forall v \in H^1(K). \tag{3.3.21}$$

Proof. Given $v \in H^1(K)$, we utilize again the decomposition (3.3.16) and set

$$\widehat{v} := \widehat{\Pi}_k^K(v) = q_v + c_v,$$

with $q_v \in \widehat{\mathrm{P}}_k(K)$ and $c_v \in \mathrm{R}$. Then, it follows straightforwardly from (3.3.17) and (3.3.19) that $|q_v| \leq |v|_{1,K}$ and $\|c_v\|_{1,K} = \|c_v\|_{0,K} \leq \|q_v\|_{0,K} + \|v\|_{0,K}$. In what follows we bound

$\|q_v\|_{0,K}$ in terms of $|q_v|_{1,K}$. For this porpuse we assume, without loss of generality, that K is star-shaped with respect to a ball B centered at the origin. Otherwise, instead of K we consider the shifted region $\bar{K} := \bar{T}_K(K)$, where $\bar{T}_K(\mathbf{x}) := \mathbf{x} - \mathbf{x}_B \;\; \forall\, \mathbf{x} \in K$, for which there holds $h_K = h_{\bar{K}}$. Then, analogously as described for $\widetilde{K}$ at the beginning of Section 3.3.3, we now let $\Big\{ \Delta_i : \quad i \in \{1, 2, ..., d_K\} \Big\}$ be the partition of K obtained by connecting each vertex of this element to the origin. In addition, for each $i \in \{1, 2, \ldots, d_K\}$ we let h_i and ρ_i be the geometric parameters of Δ_i, and let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\,\mathbf{x} \;\; \forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \Delta_i$. Recall that $\widehat{\Delta}$ is the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$. Hence, we can write

$$\|q_v\|^2_{0,K} \;=\; \sum_{i=1}^{d_K} \|q_v\|^2_{0,\Delta_i} \;=\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\; \|\widehat{q}_{v,i}\|^2_{0,\widehat{\Delta}}\,, \tag{3.3.22}$$

where $\widehat{q}_{v,i} \;:=\; q_v|_{\Delta_i} \circ F_i \;\in\; \widehat{\mathrm{P}}_k(\widehat{\Delta})$. We emphasize here that the fact that the origin is a vertex of each one of the triangles Δ_i has allowed to choose a linear (not affine) transformation F_i mapping $\widehat{\Delta}$ onto Δ_i, which, given that $q_v|_{\Delta_i} \in \widehat{\mathrm{P}}_k(\Delta_i)$, ensures that $\widehat{q}_{v,i}$ does belong to $\widehat{\mathrm{P}}_k(\widehat{\Delta})$. Moreover, the importance of it lies on the fact that $|\cdot|_{1,\widehat{\Delta}}$ is a norm on $\widehat{\mathrm{P}}_k(\widehat{\Delta})$, and therefore there exists $\widehat{c} > 0$, depending only on k and $\widehat{\Delta}$, such that, in particular, $\|\widehat{q}_{v,i}\|^2_{0,\widehat{\Delta}} \;\le\; \widehat{c}\,|\widehat{q}_{v,i}|^2_{1,\widehat{\Delta}}$. In this way, applying once more the scaling properties between Sobolev seminorms, we obtain from (3.3.22) that

$$\|q_v\|^2_{0,K} \;\le\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\,\widehat{c}\,|\widehat{q}_{v,i}|^2_{1,\widehat{\Delta}} \;\le\; \widehat{c} \sum_{i=1}^{d_K} h_i^2\,\widehat{\rho}^{\,-2}\,|q_{v,i}|^2_{1,\Delta_i} \;\le\; \widehat{C}\,h_K^2\,|q_v|^2_{1,K} \le \widehat{C}\,h_K^2\,|v|^2_{1,K}\,.$$

Therefore, $\|\widehat{v}\|_{1,K} \le C\Big\{ \|v\|_{0,K} + h_K |v|_{1,K} \Big\}$ as required. $\square$

The analogue of Lemma 3.3.3 is provided next.

Lemma 3.3.6. Given integers k, $\ell \ge 1$, and $K \in \mathcal{T}_h$, there holds $\widetilde{\widehat{\Pi}^K_k(v)} \;=\; \widehat{\Pi}^{\widetilde{K}}_k(\widetilde{v})$ for all $v \in H^k(K)$, and $\widehat{\Pi}^{\widetilde{K}}_k \in \mathcal{L}\big(H^\ell(\widetilde{K}), H^1(\widetilde{K})\big)$ with $\|\widehat{\Pi}^{\widetilde{K}}_k\|_{\mathcal{L}(H^\ell(\widetilde{K}),H^1(\widetilde{K}))}$ independent of $\widetilde{K}$, namely depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$.

Proof. Similarly as for Lemma 3.3.3, we first observe that $v \in \mathrm{P}_k(K)$ if and only $\widetilde{v} := v \circ T_K^{-1} \;\in\; \mathrm{P}_k(\widetilde{K})$. In particular, given $v \in H^1(K)$, there holds $\widetilde{\widehat{\Pi}^K_k(v)} \in \mathrm{P}_k(\widetilde{K})$, and hence, in order to obtain the required identity, it suffices to show that $\widetilde{\widehat{\Pi}_k^K(v)}$ solves the same problem as $\widehat{\Pi}_k^{\widetilde{K}}(\widetilde{v})$, namely (3.3.17) and (3.3.18), with $K = \widetilde{K}$ and $v = \widetilde{v}$. In fact, setting as before $\widehat{\Pi}_k^K(v) = q_v + c_v$, where $q_v \in \widehat{\mathrm{P}}_k(K)$ and $c_v \in \mathrm{R}$, we find, according to (3.3.17), that for each $q \in \widehat{\mathrm{P}}_k(K)$ there holds

$$a^{\widetilde{K}}(\widetilde{q_v}, \widetilde{q}) \;=\; h_K^{-4}\, a^K(q_v, q) \;=\; h_K^{-4}\, a^K(v, q) \;=\; a^{\widetilde{K}}(\widetilde{v}, \widetilde{q})\,. \tag{3.3.23}$$

Now, it is quite clear that $\widetilde{\widehat{\Pi}_k^K(v)}$ solves (3.3.18) with $K = \widetilde{K}$ and $v = \widetilde{v}$ and so it solves (3.3.17) and (3.3.18). Finally, a direct application of Lemma 3.3.5 implies the existence of a constant $C > 0$, independent of $\widetilde{K}$, such that

$$\|\widehat{\Pi}_k^{\widetilde{K}}(v)\|_{1,K} \le C\|v\|_{1,K} \le C\|v\|_{\ell,K} \quad \forall v \in H^\ell(\widetilde{K}),$$

which completes the proof. $\square$

Lemma 3.3.7. Let k, ℓ and r be integers such that $k \ge 1$, $\ell \in \{0, 1\}$ and $1 \le r \le k+1$. Then, there exists a constant $C > 0$, depending only on k, r, ℓ, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$|v \,-\, \widehat{\Pi}_k^K(v)|_{\ell,K} \;\le\; C\, h_K^{r-\ell}\, |v|_{r,K} \qquad \forall\, v \in H^r(K)\,.$$

Proof. It is completely analogous to the proof of Lemma 3.3.4 $\square$

We now let $a_h^K : V_h^K \times V_h^K \longrightarrow \mathrm{R}$ be the local discrete bilinear form given by

$$a_h^K(u, v) \;:=\; a^K\big(\widehat{\Pi}_k^K(u), \widehat{\Pi}_k^K(v)\big) + \mathcal{S}^K\big(u - \widehat{\Pi}_k^K(u), v - \widehat{\Pi}_k^K(v)\big) \qquad \forall\, u,\, v \,\in\, V_h^K\,, \tag{3.3.24}$$

where $\mathcal{S}^K : V_h^K \times V_h^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_k^K \times n_k^K}$ with respect to the basis $\{\varphi_{j,K}\}_{j=1}^{n_k^K}$ of V_k^K (cf. (3.3.10) - (3.3.11)), that is

$$\mathcal{S}^K(u, v) \;:=\; \sum_{i=1}^{n_k^K} m_{i,K}(u)\, m_{i,K}(v) \qquad \forall\, u,\, v \,\in\, V_h^K\,. \tag{3.3.25}$$

Next, as suggested by (3.3.24), we define the global discrete bilinear form $a_h : V_h \times V_h \longrightarrow \mathrm{R}$

$$a_h(u, v) \;:=\; \sum_{K \in \mathcal{T}_h} a_h^K(u, v) \qquad \forall\, u,\, v \,\in\, V_h\,. \tag{3.3.26}$$

Lemma 3.3.8. There exist c_0, $c_1 > 0$, depending only on $C_{\mathcal{T}}$, such that

$$c_0 \, |v|_{1,K}^2 \;\leq\; \mathcal{S}^K(v,v) \;\leq\; c_1 \, |v|_{1,K}^2 \qquad \forall \, K \in \mathcal{T}_h, \quad \forall \, v \in v_h^K \, . \tag{3.3.27}$$

Proof. See [9, eqn. 35]. □

The following result is a consequence of the properties of the projector $\widehat{\Pi}_k^K$ and the previous lemma.

Lemma 3.3.9. For each $K \in \mathcal{T}_h$, there holds

$$a_h^K(u,v) = a^K(u,v) \quad \forall u \in \mathrm{P}_k(K), \quad \forall v \in V_h^K, \tag{3.3.28}$$

and there exist constants $\alpha, \alpha_1, \alpha_2$, independent of h and K, such that

$$|a_h^K(u,v)| \leq \alpha \Big\{ \|u\|_{1,K} \|v\|_{1,K} + \|u - \widehat{\Pi}_k^K(u)\|_{1,K} \|v - \widehat{\Pi}_k^K(v)\|_{1,K} \Big\} \quad \forall K \in \mathcal{T}_h, \quad \forall u, v \in V_h^K, \tag{3.3.29}$$

and

$$\alpha_1 a^K(v,v) \leq a_h^K(v,v) \leq \alpha_2 a^K(v,v) \quad \forall K \in \mathcal{T}_h, \quad \forall v \in V_h^K. \tag{3.3.30}$$

Proof. Given $u \in \mathrm{P}_k^K$, we certainly have $\widehat{\Pi}_k^K(u) = u$, and bearing in mind problem (3.3.17) we deduce, starting from (3.3.24), that given $v \in V_h^K$ there holds

$$a_h^K(u,v) = a^K\Big(\widehat{\Pi}_k^K(u), \widehat{\Pi}_k^K(v)\Big) = a^K\Big(\widehat{\Pi}_k^K(v), u\Big) = a^K(v,u) = a^K(u,v),$$

which proves (3.3.28). Next, for the boundedness of a_h^K we apply the Cauchy-Schwarz inequality, (3.3.21) with $C = \widehat{C}$ and the upper bound in (3.3.27) (cf. Lemma 3.3.8), to obtain

$$\begin{aligned} |a_h^K(u,v)| &\leq \|\widehat{\Pi}_k^K(u)\|_{1,K}^2 \|\widehat{\Pi}_k^K(v)\|_{1,K}^2 \\ &\quad + \Big\{ S^K(u - \widehat{\Pi}_k^K(u), u - \widehat{\Pi}_k^K(u)) \Big\}^{1/2} \Big\{ S^K(v - \widehat{\Pi}_k^K(v), v - \widehat{\Pi}_k^K(v)) \Big\}^{1/2} \\ &\leq \widehat{C}^2 \|u\|_{1,K} \|v\|_{1,K} + c_1 \|u - \widehat{\Pi}_k^K(u)\|_{1,K} \|v - \widehat{\Pi}_k^K(v)\|_{1,K} \quad \forall u, v \in V_h^K, \end{aligned}$$

which gives (3.3.29) with $\alpha := \max\{\widehat{C}^2, c_1\}$. Finally, concerning (3.3.30), we see that

$$a_h^K(v,v) = a^K\Big(\widehat{\Pi}_k^K(v), \widehat{\Pi}_k^K(v)\Big) + S^K\Big(v - \widehat{\Pi}_k^K(v), v - \widehat{\Pi}_k^K(v)\Big) \geq \min\{1, c_0\} a^K(v,v),$$

and, similarly,

$$a_h^K(v,v) = a^K\left(\widehat{\Pi}_k^K(v), \widehat{\Pi}_k^K(v)\right) + S^K\left(v - \widehat{\Pi}_k^K(v), v - \widehat{\Pi}_k^K(v)\right) \leq \max\{1, c_0\} a^K(v,v),$$

which finishes the proof, with $\alpha_1 := \min\{1, c_1\}$ and $\alpha_2 := \max\{1, c_2\}$ $\square$

3.3.5 The right hand side

The purpose of this section is to define a computable version $F_h : V_h \to \mathrm{R}$ of the right-hand side defined above. To this end, we observe that given $v \in V_h$ and $f \in L^2(\Omega)$, the expression $(f, v)_{0,\Omega}$ is not explicitly computable since in general v is not known on each $K \in \mathcal{T}_h$. In order to overcome this difficulty, we now proceed to introduce a suitable approximation of f for which the associated right-hand side, say f_h, is computable. We consider first the case $k \geq 2$. In such a case, we define f_h on each element K as the $L^2(K)$ projection of f onto $\mathrm{P}_{k-2}(K)$, that is, given $K \in \mathcal{T}_h$,

$$f_h := \mathcal{P}_{k-2}^K(f) \text{ in } K. \tag{3.3.31}$$

Consequently, we define $F_h : V_h \to \mathrm{R}$ as

$$F_h(v_h) := \sum_{K \in \mathcal{T}_h} \int_K f_h v_h \quad \forall v_h \in V_h. \tag{3.3.32}$$

Now, if $v_h \in V_h$, we note that using the orthogonality condition,

$$F_h(v_h) = \sum_{K \in \mathcal{T}_h} \int_K v_h \, \mathcal{P}_{k-2}^K(f) = \sum_{K \in \mathcal{T}_h} \int_K f \, \mathcal{P}_{k-2}^K(v_h),$$

which, using (3.3.10) and (3.3.11), is explicitly computable from the corresponding degrees of freedom of v_h.

Lemma 3.3.10. Let $k \geq 2$ and let $F, F_h : V_h \to \mathrm{R}$ be given as $F(v_h) := (f, v_h)_{0,\Omega}$, and F_h as defined in (3.3.32). Suppose further that $f \in H^{k-1}(\Omega)$. Then, there exists $C > 0$, independent of h, such that

$$\|F - F_h\|_{V_h'} \leq C h^k |f|_{k-1,\Omega}.$$

Proof. Using standard approximation properties of $\mathcal{P}_{k-2}^K$ and the fact that for every $K \in \mathcal{T}_h$ we have that $(\mathcal{P}_{k-2}^K(f) - f, \mathcal{P}_0^K(v_h))_{0,K} = 0$, it follows that given $v_h \in V_h$,

$$\begin{aligned}\sum_{K\in\mathcal{T}_h}(f_h - f, v_h)_{0,K} &= \sum_{K\in\mathcal{T}_h}\int_K (\mathcal{P}_{k-2}^K(f) - f)(v_h - \mathcal{P}_0^K(v_h))\\ &\le C\sum_{K\in\mathcal{T}_h} h_K^{k-1}|f|_{k-1,K}h_K|v_h|_{1,K} \le Ch^k\left(\sum_{K\in\mathcal{T}_h}|v_h|_{1,K}^2\right)^{1/2}|f|_{k-1,\Omega},\end{aligned}$$

which implies that $\|F - F_h\|_{V_h'} \le Ch^k|f|_{k-1,\Omega}$. □

We finalize this section by considering $k = 1$. In such a case, we define f_h on every $K \in \mathcal{T}_h$ as $f_h := \mathcal{P}_0^K(f)$, and define thus $F_h : V_h \to \mathrm{R}$ as

$$F_h(v_h) := \sum_{K\in\mathcal{T}_h}\int_K \mathcal{P}_0^K(f)\,\overline{v_h} = \sum_{K\in\mathcal{T}_h}|K|\,\overline{v_h}\,\mathcal{P}_0^K(f). \tag{3.3.33}$$

On a similar way, the functional defined above is explicitly calculable for every $v_h \in V_h$ by using (3.3.10) and (3.3.11). The analogue of Lemma 3.3.10 is provided next.

Lemma 3.3.11. Let $F_h : V_h \to \mathrm{R}$ be defined as in (3.3.33) and suppose that $f \in H^1(\Omega)$. Then, there exists $C > 0$, independent of h, such that

$$\|F - F_h\|_{V_h'} \le Ch\|f\|_{1,\Omega}.$$

Proof. Given $v_h \in V_h$, using the standard approximation properties of $\mathcal{P}_0^K$ on each $K \in \mathcal{T}_h$, we find that

$$\begin{aligned}&F_h(v_h) - \sum_{K\in\mathcal{T}_h}(f, v_h)_{0,K} = \sum_{K\in\mathcal{T}_h}\int_K \Big((\mathcal{P}_0^K f - f)\,\overline{v_h} + (\overline{v_h} - v_h)\,f\Big)dx\\ &\le C\sum_{K\in\mathcal{T}_h}\Big\{h_K|f|_{1,K}\|v_h\|_{0,K} + h_K|v_h|_{1,K}\|f\|_{0,K}\Big\} \le Ch\left(\sum_{K\in\mathcal{T}_h}\|v_h\|_{1,K}\right)^{1/2}\|f\|_{1,\Omega}.\end{aligned}$$

Therefore, $\|F - F_h\|_{V_h'} \le Ch\|f\|_{1,\Omega}$ as required. □

3.3.6 The virtual element scheme

According to the analysis from the foregoing section, we reformulate the Galerkin scheme associated with (3.2.2) as: Find $u_h \in V_h$ such that

$$a_h(u_h, v_h) = F_h(v_h) + (\nabla u_g, \nabla v_h) \quad \forall v_h \in V_h. \tag{3.3.34}$$

Theorem 3.3.1. *There exists a unique $u_h \in V_h$ solution of* (3.3.34)*, and there exists a positive constant $C > 0$, independent of h, such that*

$$\|u_h\|_V \leq C\Big\{\|f\|_{0,\Omega} + \|g\|_{1/2,\Gamma}\Big\}.$$

Proof. The boundedness of a_h respect to the norm $\|\cdot\|_{1,\Omega}$ of V follows easily from (3.3.29) and (3.3.21) (cf. Lemma 3.3.5). Also, thanks to the lower bound of (3.3.30) (cf. Lemma 3.3.8) and the fact that $v_h = 0$ in Γ, a direct application of the Lax–Milgram Lemma completes the proof. $\square$

We now aim to provide the corresponding a priori estimates. To this end, and just for sake of clearness in what follows, we recall that $\Pi_k^h : V \to V_h$ is the interpolator defined on (3.3.9), whose associated local operator are denoted by Π_k^K. In turn, given our local projector $\widehat{\Pi}_k^K$ defined by (3.3.17) and (3.3.18), we denote by $\widehat{\Pi}_k^h$ its global counterpart, that is, given $v \in H^1(\Omega)$, we let

$$\widehat{\Pi}_k^h(v)|_K := \widehat{\Pi}_k^K(v|_K) \quad \forall K \in \mathcal{T}_h. \tag{3.3.35}$$

Then, we have the following main result.

Theorem 3.3.2. *Let u and u_h be the unique solutions of the continuous and discrete schemes* (3.2.2) *and* (3.3.34)*, respectively. Then, there exists $C > 0$, independent of h, such that*

$$\|u - u_h\|_{0,\Omega} \leq Ch\Big\{|u - \Pi_k^h(u)|_{1,\Omega} + |u - \widehat{\Pi}_k^h(u)|_{1,\Omega} + |u - \mathcal{P}_k^h(u)|_{1,\Omega} + \|F - F_h\|_{V_h'}\Big\}, \tag{3.3.36}$$

and

$$|u-u_h|_{1,\Omega} \le C\Big\{|u-\Pi_k^h(u)|_{1,\Omega} + |u-\widehat{\Pi}_k^h(u)|_{1,\Omega} + \|F-F_h\|_{V_h'}\Big\}. \tag{3.3.37}$$

Proof. We first note that

$$|u-u_h|_{1,\Omega} \le |u-\Pi_k^h(u)|_{1,\Omega} + |\Pi_k^h(u)-u_h|_{1,\Omega}. \tag{3.3.38}$$

Now, we let $\delta_h := u_h - \Pi_k^h(u)$ and we see that,

$$\begin{aligned}
\alpha_*|\delta_h|_{1,\Omega}^2 &= \alpha_* a(\delta_h,\delta_h) \le a_h(\delta_h,\delta_h)\\
&= a_h(u_h,\delta_h) - a_h(\Pi_k^h(u),\delta_h)\\
&= \sum_{K\in\mathcal{T}_h} \left\{F_h(\delta_h) + (\nabla u_g, \nabla\delta_h)_{0,\Omega} - a_h^K(\Pi_k^h(u),\delta_h)\right\}\\
&= \sum_{K\in\mathcal{T}_h} \left\{F_h(\delta_h) + (\nabla u_g, \nabla\delta_h)_{0,\Omega} - a_h^K(\Pi_k^h(u)-\widehat{\Pi}_k^h(u),\delta_h) - a_h(\widehat{\Pi}_k^h(u),\delta_h)\right\}\\
&= \sum_{K\in\mathcal{T}_h} \left\{F_h(\delta_h) - (f,\delta_h)_{0,K} - a_h(\Pi_k^h(u)-\widehat{\Pi}_k^h(u),\delta_h) - a^K(\widehat{\Pi}_k^h(u)-u,\delta_h)\right\}.
\end{aligned}$$

Then,

$$|\delta_h|_{1,\Omega}^2 \le C\left\{\|F-F_h\|_{V_h'} + |\Pi_k^h(u)-\widehat{\Pi}_k^h(u)|_{1,\Omega} + |u-\widehat{\Pi}_k^h(u)|_{1,\Omega}\right\}|\delta_h|_{1,\Omega},$$

and so

$$|\delta_h|_{1,\Omega} \le C\left\{\|F-F_h\|_{V_h'} + |\Pi_k^h(u)-\widehat{\Pi}_k^h(u)|_{1,\Omega} + |u-\widehat{\Pi}_k^h(u)|_{1,\Omega}\right\}.$$

Therefore, (3.3.37) is obtained by using (3.3.38). In order to prove (3.3.36), we consider a bounded and convex domain B which contains Ω and the boundary value problem: Find $\psi \in H_0^1(B)$ such that $-\Delta\psi = h$ in B and $\psi = 0$ in ∂B, where

$$h = \begin{cases} u-u_h, & \text{in } \Omega,\\ 0, & \text{otherwise}\end{cases}.$$

It follows, by elliptic regularity, that ψ is unique and $\|\psi\|_{2,\Omega} \le C\|h\|_{0,B} = C\|u-u_h\|_{0,\Omega}$, where C is a constant depending only on B and consequently on Ω. Therefore,

$$\|\psi - \Pi_k^h(\psi)\|_{1,\Omega} \le Ch|\psi|_{2,\Omega} \le Ch\|u-u_h\|_{0,\Omega}.$$

Thus,

$$
\begin{aligned}
\|u-u_h\|_{0,\Omega}^2 &= (u-u_h,-\Delta\psi)_{0,\Omega} \\
&= a(u-u_h,\psi) = a(u-u_h,\psi-\Pi_k^h(\psi)) + a(u-u_h,\Pi_k^h(\psi)) \\
&= a(u-u_h,\psi-\Pi_k^h(\psi)) + (f,\Pi_k^h(\psi))_{0,\Omega} - a(u_h,\Pi_k^h(\psi)) \\
&= a(u-u_h,\psi-\Pi_k^h(\psi)) + (f-f_h,\Pi_k^h(\psi))_{0,\Omega} + a_h(u_h,\Pi_k^h(\psi)) - a(u_h,\Pi_k^h(\psi))
\end{aligned}
$$

Now,

$$
a(u-u_h,\psi-\Pi_k^h(\psi)) \leq |u-u_h|_{1,\Omega}|\psi-\Pi_k^h(\psi)|_{1,\Omega} \leq Ch\|u-u_h\|_{0,\Omega}|u-u_h|_{1,\Omega}.
$$

On the other hand, defining $l := \max\{0,k-2\}$, we have that $f_h|_K \in \mathrm{P}_l(K)$, for every $K \in \mathcal{T}_h$, and so

$$
(f,\Pi_k^h(\psi))_{0,\Omega} - F_h(\Pi_k^h(\psi)) = 0 \quad \forall q \in \mathrm{P}_l(K).
$$

Therefore, denoting $\mathcal{P}_0^h : L^2(\Omega) \to \mathrm{P}_0(\mathcal{T}_h)$ the orthogonal projector,

$$
\begin{aligned}
&(f,\Pi_k^h(\psi))_{0,\Omega} - F_h(\Pi_k^h(\psi)) = (f,\Pi_k^h(\psi) - \mathcal{P}_0^K(\Pi_k^h(\psi)))_{0,\Omega} - F_h(\Pi_k^h(\psi) - \mathcal{P}_0^K(\Pi_k^h(\psi))) \\
&\quad \leq C \sum_{K\in\mathcal{T}_h} \|F-F_h\|_{V_h'} h_K |\Pi_k^h(\psi)|_{1,K} \leq Ch\|F-F_h\|_{V_h'}|\Pi_k^h(\psi)|_{1,K} \\
&\quad \leq Ch\|F-F_h\|_{V_h'}\|\psi\|_{1,K} \leq Ch\|F-F_h\|_{V_h'}\|\psi\|_{2,K} \leq Ch\|F-F_h\|_{V_h'}\|u-u_h\|_{0,\Omega}.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&a_h(u_h,\Pi_k^h(\psi)) - a_h(u_h,\Pi_k^h(\psi)) \\
&\quad = \sum_{K\in\mathcal{T}_h} a_h^K(u_h - \mathcal{P}_k^K(u),\Pi_k^K(\psi)) - a^K(u_h - \mathcal{P}_k^K(u),\Pi_k^K(\psi)) \\
&\quad = \sum_{K\in\mathcal{T}_h} a_h^K(u_h - \mathcal{P}_k^K(u),\Pi_k^K(\psi) - \mathcal{P}_1^K(\psi)) - a^K(u_h - \mathcal{P}_k^K(u),\Pi_k^K(\psi) - \mathcal{P}_1^K(\psi)),
\end{aligned}
$$

and we note that given $K \in \mathcal{T}_h$, we have that

$$
\begin{aligned}
|\Pi_k^K(\psi) - \mathcal{P}_1^K(\psi)|_{1,K} &\leq |\psi - \Pi_k^K(\psi)|_{1,K} + |\psi - \mathcal{P}_1^K(\psi)|_{1,K} \\
&\leq Ch_K|\psi|_{2,K} \leq Ch_K\|u-u_h\|_{0,\Omega},
\end{aligned}
$$

obtaining then that

$$a_h(u_h, \Pi_k^h(\psi)) - a(u_h, \Pi_k^h(\psi)) \leq Ch\|u - u_h\|_{0,\Omega} \left(|u - u_h|_{1,\Omega} + |u - \mathcal{P}_k^h u|_{1,\Omega}\right).$$

Putting all together, we find that

$$\|u - u_h\|_{0,\Omega}^2 \leq Ch\|u - u_h\|_{0,\Omega}\Big(|u - u_h|_{1,\Omega} + |u - \mathcal{P}_k^h u|_{1,\Omega} + \|F - F_h\|_{V_h'}\Big),$$

thus

$$\|u - u_h\|_{0,\Omega} \leq Ch\Big(|u - \Pi_k^h(u)|_{1,\Omega} + |u - \widehat{\Pi}_k^h(u)|_{1,\Omega} + |u - \mathcal{P}_k^h(u)|_{1,\Omega} + \|F - F_h\|_{V_h'}\Big),$$

which finishes the proof. □

Having established the a priori error estimates for our unknowns, we now provide the corresponding rates of convergence.

Theorem 3.3.3. *Let u and u_h be the unique solutions of the continuous and discrete schemes* (3.2.2) *and* (3.3.34)*, respectively. Assume that $\ell := \max\{0, k-2\}$, $t \in \{0,1\}$, and that for some $r \in [1, k+1]$ and $s \in [1, \ell+1]$ there hold $u|_K \in H^r(K)$ and $f|_K \in H^s(K)$, for each $K \in \mathcal{T}_h$. Then, there exists $C > 0$, independent of h, such that*

$$|u - u_h|_{s,\Omega} \leq Ch^{r-t}\left\{\sum_{K\in\mathcal{T}_h} \|u\|_{r,K}^2\right\}^{1/2} + Ch^{s-t}\left\{\sum_{K\in\mathcal{T}_h} \|f\|_{s,K}^2\right\}^{1/2} \tag{3.3.39}$$

Proof. The case of integers $r \in [1, k+1]$ and $s \in [1, \ell+1]$ follows from straigthforward application of the approximation properties provided by Lemmas 3.3.4, 3.3.7, and by Lemmas 3.3.10 or 3.3.11, depending on $k \geq 1$. In turn, the usual interpolation estimates of Sobolev spaces allow us to conclude for the remaning real values of r and s. We omit further details. □

We notice that if the assumed regularities in the foregoing theorem are global, then the estimate (3.3.39) becomes

$$|u - u_h|_{s,\Omega} \leq Ch^{r-t}\|u\|_{r,\Omega} + Ch^{s-t}\|f\|_{s,\Omega}.$$

In turn, it is also clear from the range of variability of the integers r and s that the highest possible rate of convergence for u in the norm $\|\cdot\|_{t,\Omega}$ is h^{k+1-t}, $t \in \{0,1\}$.
We now introduce the fully computable approximation of u given by

$$\widehat{u}_h := \widehat{\Pi}_k^K(u_h), \tag{3.3.40}$$

and establish next the corresponding a priori error estimate.

Theorem 3.3.4. *If $t \in \{0,1\}$, then there exist a positive constant $C_1 > 0$, independent of h, such that*

$$|u-\widehat{u}_h|_{t,\Omega} \leq C_1 h^{1-t}\Big\{|u-\Pi_k^h(u)|_{1,\Omega}+|u-\widehat{\Pi}_k^h(u)|_{1,\Omega}+|u-\mathcal{P}_k^h(u)|_{1,\Omega}+\|F-F_h\|_{V_h'}\Big\} \tag{3.3.41}$$

Proof. Note that

$$|u-\widehat{u}_h|_{t,\Omega} \leq |u-\widehat{\Pi}_k^h(u)|_{t,\Omega} + |\widehat{\Pi}_k^h(u-u_h)|_{t,\Omega}.$$

Thus, an application of Lemma 3.3.7 for $\ell = t$ and $r = 1$ yields

$$|u-\widehat{\Pi}_k^h(u)|_{t,\Omega} = |(\mathrm{I}-\widehat{\Pi}_k^h)(u-\widehat{\Pi}_k^h(u))|_{t,\Omega} \leq C h^{1-t}|u-\widehat{\Pi}_k^h(u)|_{1,\Omega}.$$

In turn, it follows by using (3.3.21) that

$$|\widehat{\Pi}_k^h(u-u_h)|_{t,\Omega} \leq \|\widehat{\Pi}_k^h(u-u_h)\|_{1,\Omega} \leq C\Big\{\|u-u_h\|_{0,\Omega} + h|u-u_h|_{1,\Omega}\Big\}.$$

Therefore, using (3.3.36) and (3.3.37), we obtain

$$\begin{aligned}&\|u-u_h\|_{0,\Omega}+h|u-u_h|_{1,\Omega}\\ &\qquad \leq Ch\Big\{|u-\Pi_k^h(u)|_{1,\Omega}+|u-\widehat{\Pi}_k^h(u)|_{1,\Omega}+|u-\mathcal{P}_k^h(u)|_{1,\Omega}+\|F-F_h\|_{V_h'}\Big\}.\end{aligned}$$

Finally, the proof is completed by putting the obtained estimates for $|u-\widehat{\Pi}_k^h(u)|_{t,\Omega}$ and $|\widehat{\Pi}_k^h(u-u_h)|_{t,\Omega}$ together. $\square$

We end this section by remarking, according to the upper bounds provided by (3.3.36) and (3.3.37) (cf. Theorem 3.3.2) and (3.3.41) (cf. Theorem 3.3.4), that u_h and $\widehat{u}_h$ share exactly the same rates of convergence given by Theorem 3.3.3.

3.4 Computational implementation

3.4.1 Introduction

In what follows, we consider the same notations as in previous chapters, and given a decomposition $\mathcal{T}_h$ of Ω into polygons, we consider the discretized problem: Find u_h in V_h such that

$$a_h(u_h, v_h) = F_h(v_h) \quad \forall v_h \in V_h. \tag{3.4.1}$$

where V_h was defined on (3.3.2) and F_h was defined on (3.3.32) or (3.3.33), when it corresponds. Using Theorem 3.3.1, we have that (3.4.1) has a unique solution for every pair of data f and g. Now, let $\{\varphi_j\}_{j=1}^{N}$ be a basis of V_h. If, $u_h := \sum_{j=1}^{N} u_j \varphi_j$, then (3.4.1) is turned into the linear system of equations

$$\sum_{i=1}^{N} u_i a_h(\varphi_i, \varphi_j) = \int_\Omega f \varphi_j, \tag{3.4.2}$$

for each $j \in \{1, \ldots, N\}$. Equivalently,

$$A_h x = b_h,$$

where $x = (x_1, \ldots, x_N)^t$, $A_h = (a_{ij})$ and $b_h = (b_{ij})$, where

$$\begin{aligned} A_{ij} &:= a_h(\varphi_{i,K}, \varphi_{j,K}) \quad i, j \in \{1, \ldots, N\}, \\ b_j &:= (f, \varphi_{j,K})_{0,\Omega} \quad j \in \{1, \ldots N\}. \end{aligned} \tag{3.4.3}$$

Furthermore, taking into account that

$$a_h(\varphi_i, \varphi_j) = \sum_{K \in \mathcal{T}_h} a_h^K(\varphi_i, \varphi_j),$$

and doing the same for F_h, we can then localize the calculations for A_h and b_h.

3.4.1.1 Notations

As in the above sections, given an element $K \in \mathcal{T}_h$, we will denote by $\mathbf{x}_K$, h_K and $|K|$ the barycenter, the diameter, and the area of K, respectively. The L^2 inner product of

$u, v \in L^2(K)$ will be denoted as $(u, v)_{0,K}$ as well. If $D \subset \mathrm{R}^2$, we define

$$n_k := \dim(\mathrm{P}_k(D)) = \frac{(k+1)(k+2)}{2}.$$

If $\alpha = (\alpha_1, \alpha_2)$ is a multi–index and $\mathbf{x} = (x_1, x_2)$, then, as usual, $\mathbf{x}^\alpha = x_1^{\alpha} x_2^{\alpha_2}$, and we will denote by m_α the scaled monomial of degree equal to $|\alpha|$ defined by

$$m_\alpha(\mathbf{x}) := \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K}\right)^\alpha \quad \forall \mathbf{x} \in K. \tag{3.4.4}$$

Clearly, the set of scaled monomials of degree less or equal than k

$$\mathcal{B}_k\,(D) := \{m_\alpha : 0 \le |\alpha| \le k\}$$

is a basis of $\mathrm{P}_k(D)$. We will also consider the identification

$$1 \leftrightarrow (0,0), \qquad 2 \leftrightarrow (1,0), \qquad 3 \leftrightarrow (0,1), \qquad 4 \leftrightarrow (2,0), \qquad \ldots \tag{3.4.5}$$

and so on. Therefore, we will write $m_1 \equiv 1$ instead of $m_{(0,0)}$ and so on. For each polygon $K \in \mathcal{T}_h$, we will denote d_K to the number of its vertices K, V_i, $i = 1, \ldots, N^K$ its vertices ordered counterclockwise, and by e_i the edge connecting V_i with V_{i+1}. All of the above is shown in the figure next.

We defined in (3.3.6) a local finite element space V_k^K. Its dimension will be denoted as n_k^K and is given by

$$n_k^K := d_K k + \frac{k(k-1)}{2}.$$

We also recall from (3.3.10) that

$$m_{i,K}(v_h) := i\text{-th degree of freedom of } v_h, \quad i, j \in \{1, \ldots, n_k^K\}.$$

and also

$$m_{i,K}(\varphi_{j,K}) = \delta_{ij} \quad i, j \in \{1, \ldots, n_k^K\},$$

so that, as in (3.3.11), we have a Lagrange-type interpolation identity

$$v_h = \sum_{i=1}^{n_k^K} m_{i,K}(v_h)\varphi_{i,K} \;\; \forall\, v_h \in V_k^K. \tag{3.4.6}$$

3.4.2 Calculating the local matrices

We want to compute the local stiffness matrix A_h^K of a_h in the polygon K, i.e.

$$\begin{aligned}(A_h^K)_{ij} &:= a_h^K(\varphi_i,\varphi_j)_{0,K} \\ &= (\hat{\Pi}_k^K\varphi_i,\hat{\Pi}_k^K\varphi_j)_{0,K} + \mathcal{S}^K((I-\hat{\Pi}_k^K)\varphi_i,(I-\hat{\Pi}_k^K)\varphi_j) \quad \forall i,j\in\{1,\dots,n_k^K\}\end{aligned} \tag{3.4.7}$$

where a_h^K, $\widehat{\Pi}_k^K$ and $\mathcal{S}^K$ were defined on (3.3.24), (3.3.17)-(3.3.18) and (3.3.25), respectively. In order to compute such a matrix, we take care of each part of the foregoing equation separately. Therefore, we have to compute first the projector $\widehat{\Pi}_k^K$. As we have seen before, given $v\in V_k^K$, $\widehat{\Pi}_k^K(v)$ can be written as

$$\widehat{\Pi}_k^K(v) := q_v + c_v,$$

where $q_v\in\widehat{\mathrm{P}}_k(K)$, $c_v\in\mathrm{R}$. Also, q_v and c_v are obtained by

$$(\nabla q_v,\nabla p)_{0,K} = (\nabla v,\nabla p)_{0,K} \quad \forall p\in\mathcal{B}_k(K),$$

and, to take care of the constant part, we can choose

$$\begin{aligned}c_v &= \overline{v-q_v}, \text{ if } k=1, \text{ or}\\ c_v &= \frac{1}{|K|}\int_K (v-q_v), \text{ if } k\geq 2.\end{aligned}$$

We remark at this point that the choice of c_v is inspired mainly from the degrees of freedom of v. Now, since $\hat{\Pi}_k^K(v)\in\mathrm{P}_k(K)$, we can write $\hat{\Pi}_k^K(v) = \sum_{j=1}^{n_k} s_j m_j$, and so the equations above are turned into the system

$$\begin{bmatrix} P_0(m_1) & P_0(m_2) & \cdots & P_0(m_{n_k}) \\ 0 & (\nabla m_2,\nabla m_2)_{0,K} & \cdots & (\nabla m_2,\nabla m_{n_k})_{0,K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (\nabla m_{n_k},\nabla m_2)_{0,K} & \cdots & (\nabla m_{n_k},\nabla m_{n_k})_{0,K}\end{bmatrix}\begin{bmatrix} s_1\\ s_2\\ \vdots\\ s_{n_k}\end{bmatrix} = \begin{bmatrix} P_0(v)\\ (\nabla m_2,\nabla v)_{0,K}\\ \vdots \\ (\nabla m_{n_k},\nabla v)_{0,K}\end{bmatrix},$$

where $P_0(v) := \overline{v}$ if $k=1$ and $P_0(v) := (v,1)_{0,K}$ if $k\geq 2$. The foregoing system can be written in the more compact form

$$\mathbf{Gs} = \mathbf{b}. \tag{3.4.8}$$

Now, each element of $\mathbf{G}$ can be computed exactly since we can compute integrals of polynomials in K, that is, via Gauss theorem and 1D integration rules on the edges. On the other hand, each element of $\mathbf{b}$ can be computed exactly: $P_0(v)$ uses the degrees of freedom of v on its calculations. Also, we consider that

$$(\nabla m_\alpha, \nabla v)_{0,K} = -\int_K v\Delta m_\alpha + \int_{\partial K} v \frac{\partial m_\alpha}{\partial \mathbf{n}}, \tag{3.4.9}$$

and we examine separately the two terms of (3.4.9). Since $\Delta m_\alpha \in \mathrm{P}_{k-2}(K)$, we can write

$$\Delta m_\alpha = \sum_{j=1}^{n_{k-2}} d_j^\alpha m_j,$$

and so

$$-\int_K \Delta m_\alpha v = -\sum_{j=1}^{n_{k-2}} d_j^\alpha \int_K m_j v = -\sum_{j=1}^{n_{k-2}} d_j^\alpha m_{kN^K+j,K}(v).$$

Now, the second term of (3.4.9) is a polynomial of degree $(k-1)+k = 2k-1$ on each edge e so it can be integrated exactly by using the Gauss-Lobatto quadrature points and the degrees of freedom of v.

3.4.2.1 Computation of $\widehat{\Pi}_k^K(\varphi_{i,K})$

For each basis function $\varphi_{i,K} \in V_k^K$, we define s_i^j, $1 \le j \le n_k$ as the coefficients of $\widehat{\Pi}_k^K(\varphi_{i,K})$ in the basis of $\mathcal{B}_k(K)$:

$$\widehat{\Pi}_k^K(\varphi_{i,K}) = \sum_{j=1}^{n_k} s_i^j m_j \quad i \in \{1, \ldots, n_k^K\}. \tag{3.4.10}$$

The coefficients s_i^j are the coefficients obtained from the system (3.4.8) by putting $\varphi_{i,K}$ instead of v in the corresponding right-hand side. That is:

$$\begin{bmatrix} P_0(m_1) & P_0(m_2) & \cdots & P_0(m_{n_k}) \\ 0 & (\nabla m_2, \nabla m_2)_{0,K} & \cdots & (\nabla m_2, \nabla m_{n_k})_{0,K} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & (\nabla m_{n_k}, \nabla m_2)_{0,K} & \cdots & (\nabla m_{n_k}, \nabla m_{n_k})_{0,K} \end{bmatrix} \begin{bmatrix} s_i^1 \\ s_i^2 \\ \vdots \\ s_i^{n_k} \end{bmatrix} = \begin{bmatrix} P_0(\varphi_{i,K}) \\ (\nabla m_2, \nabla \varphi_{i,K})_{0,K} \\ \vdots \\ (\nabla m_{n_k}, \nabla \varphi_{i,K})_{0,K} \end{bmatrix},$$

or, in a more compact form,

$$\mathbf{s}^{(i)} = \mathbf{G}^{-1}\mathbf{b}^{(i)}.$$

Moreover, denoting by $\mathbf{B}$ to the matrix in $\mathrm{R}^{n_k \times n_k^K}$ given by

$$\mathbf{B} := [\mathbf{b}^{(1)}\ \mathbf{b}^{(2)}\ \cdots\ \mathbf{b}^{(n_k^K)}] = \begin{bmatrix} P_0(\varphi_{1,K}) & \cdots & P_0(\varphi_{n_k^K,K}) \\ (\nabla m_2, \nabla \varphi_{1,K})_{0,K} & \cdots & (\nabla m_2, \nabla \varphi_{n_k^K,K})_{0,K} \\ \vdots & \ddots & \vdots \\ (\nabla m_{n_k}, \nabla \varphi_{1,K})_{0,K} & \cdots & (\nabla m_{n_k}, \nabla \varphi_{n_k^K,K})_{0,K} \end{bmatrix}, \tag{3.4.11}$$

we have that the matrix representation $\widehat{\mathbf{\Pi}}^*$ of the operator $\widehat{\Pi}_k^K$ acting from V_k^K to $\mathrm{P}_k(K)$ in the basis of $\mathcal{B}_k(K)$ is given by $(\widehat{\mathbf{\Pi}}^*)_{ij} = s_j^i$, that is,

$$\widehat{\mathbf{\Pi}}^* = \mathbf{G}^{-1}\mathbf{B}. \tag{3.4.12}$$

Now, in order to deal properly with the second term of (3.4.7), we need the matrix representation of $\widehat{\Pi}_k^K$, this time defined from V_k^K into itself. Thus, let

$$\widehat{\Pi}_k^K(\varphi_{i,K}) = \sum_{j=1}^{n_k^K} \pi_i^j \varphi_{j,K} \quad i \in \{1, \ldots, n_k^K\},$$

where $\pi_i^j := m_{j,K}(\widehat{\Pi}_k^K \varphi_{i,K})$. From (3.4.10) and (3.4.6), we have that

$$\widehat{\Pi}_k^K(\varphi_{i,K}) = \sum_{\alpha=1}^{n_k} s_i^\alpha m_\alpha = \sum_{\alpha=1}^{n_k} s_i^\alpha \sum_{j=1}^{n_k^K} m_{j,K}(m_\alpha)\varphi_{j,K},$$

thus,

$$\pi_i^j = \sum_{\alpha=1}^{n_k} s_i^\alpha m_{j,K}(m_\alpha). \tag{3.4.13}$$

Now, in order to express (3.4.13), we define $\mathbf{D} = (\mathbf{D}_{ij}) \in \mathrm{R}^{n_k^K \times n_k}$ as

$$\mathbf{D}_{ij} := m_{i,K}(m_j) \quad i \in \{1, \ldots, n_k^K\} \quad j \in \{1, \ldots, n_k\}. \tag{3.4.14}$$

Then, (3.4.13) becomes

$$\pi_i^j = \sum_{\alpha=1}^{n_k} (\mathbf{G}^{-1}\mathbf{B})_{\alpha i} \mathbf{D}_{j\alpha} = (\mathbf{D}\mathbf{G}^{-1}\mathbf{B})_{ji}.$$

Therefore, the matrix representation of $\widehat{\Pi}_k^K$, $\widehat{\mathbf{\Pi}} : V_k^K \to V_k^K$, in the canonical basis of V_k^K, is given by

$$\widehat{\mathbf{\Pi}} = \mathbf{D}\widehat{\mathbf{\Pi}}^*. \tag{3.4.15}$$

Proposition 3.4.1. If $\mathbf{G}$, $\mathbf{B}$ and $\mathbf{D}$ are given by (3.4.8), (3.4.11) and (3.4.14), respectively, then

$$\mathbf{G} = \mathbf{B}\mathbf{D}. \tag{3.4.16}$$

Proof. Let us first consider the case $\alpha = 1$ and $\beta \in \{1, \ldots, n_k\}$. Then,

$$\begin{aligned}(\mathbf{B}\mathbf{D})_{1\beta} &= \sum_{i=1}^{n_k^K} \mathbf{B}_{1i}\mathbf{D}_{i\beta} = \sum_{i=1}^{n_k^K} P_0(\varphi_{i,K}) m_{i,K}(m_\beta) \\ &= P_0\left(\sum_{i=1}^{n_k^K} m_{i,K}(m_\beta)\varphi_{i,K}\right) = P_0(m_\beta) = \mathbf{G}_{1\beta}.\end{aligned}$$

Finally, if $\alpha \geq 2$,

$$\begin{aligned}(\mathbf{B}\mathbf{D})_{\alpha\beta} &= \sum_{i=1}^{n_k^K} \mathbf{B}_{\alpha i}\mathbf{D}_{i\beta} = \sum_{i=1}^{n_k^K} (\nabla m_\alpha, \nabla \varphi_i)_{0,K} m_{i,K}(m_\beta) = \left(\nabla m_\alpha, \sum_{i=1}^{n_k^K} m_{i,K}(m_\beta)\varphi_{i,K}\right)_{0,K} \\ &= (\nabla m_\alpha, \nabla m_\beta)_{0,K} = \mathbf{G}_{\alpha\beta}.\end{aligned}$$

$\square$

Using the above proposition, we can improve the code in terms of speed. We can use (3.4.16) instead of calculating $\mathbf{G}$ by using (3.4.8). Now, we are ready to construct the local stiffness matrix. We first recall from (3.4.7) that

$$(A_h^K)_{ij} = (\widehat{\Pi}_k^K \varphi_i, \widehat{\Pi}_k^K \varphi_j)_{0,K} + \mathcal{S}^K((I - \widehat{\Pi}_k^K)\varphi_i, (I - \widehat{\Pi}_k^K)\varphi_j) \quad \forall i, j \in \{1, \ldots, n_k^K\}.$$

From (3.4.10), we have for the first term of the above equation that

$$\begin{aligned}&(\widehat{\Pi}_k^K \varphi_i, \widehat{\Pi}_k^K \varphi_j)_{0,K} \\ &\quad = \sum_{\alpha,\beta=1}^{n_k} s_i^\alpha s_j^\beta (\nabla m_\alpha, \nabla m_\beta)_{0,K} = \sum_{\alpha,\beta=1}^{n_k} (\widehat{\mathbf{\Pi}}^*)_{\alpha i} (\widehat{\mathbf{\Pi}}^*)_{\beta j} \widetilde{\mathbf{G}}_{\alpha\beta} = [(\widehat{\mathbf{\Pi}}^*)^T \widetilde{\mathbf{G}} (\widehat{\mathbf{\Pi}}^*)]_{ij},\end{aligned}$$

where $\widetilde{\mathbf{G}}$ is the matrix that is equal to $\mathbf{G}$, except for the first row which is equal to zero.

Also,

$$\begin{aligned}\mathcal{S}^K((\mathrm{I}-\widehat{\Pi}_k^K)\varphi_{i,K},(\mathrm{I}-\widehat{\Pi}_k^K)\varphi_{j,K}) &= \sum_{r=1}^{n_k^K} m_{r,K}((\mathrm{I}-\widehat{\Pi}_k^K)\varphi_{i,K})m_{r,K}((\mathrm{I}-\widehat{\Pi}_k^K)\varphi_{j,K})\\ &= \sum_{r=1}^{n_k^K}[(\mathrm{I}-\widehat{\mathbf{\Pi}})^T]_{ir}[(\mathrm{I}-\widehat{\mathbf{\Pi}})]_{rj} = [(\mathbf{I}-\widehat{\mathbf{\Pi}})^T(\mathrm{I}-\widehat{\mathbf{\Pi}})]_{ij}.\end{aligned}$$

Finally,

$$A_h^K = (\widehat{\mathbf{\Pi}}^*)^T\widetilde{\mathbf{G}}(\widehat{\mathbf{\Pi}}^*) + (\mathrm{I}-\widehat{\mathbf{\Pi}})^T(\mathrm{I}-\widehat{\mathbf{\Pi}}) \tag{3.4.17}$$

is the formula to compute the local stiffness matrix A_h^K.

Now, in order to calculate b_h^K we distinguish two cases. If $k=1$, we follow (3.3.33) and define

$$(b_h^K)_j := |K|f(\mathbf{x}_K)\overline{\varphi_j} = \frac{|K|f(\mathbf{x}_K)}{n}, \quad 1\le j\le n_1^K.$$

If $k\ge 2$, we follow (3.3.31) and first define f_h as the $L^2(K)$ projection onto $\mathrm{P}_{k-2}(K)$, that is,

$$f_h := \sum_{j=1}^{n_{k-2}} f_j m_j \tag{3.4.18}$$

where $f_1, f_2, \ldots, f_{n_{k-2}}$ are obtained from the Gram system

$$\begin{bmatrix} (m_1,m_1)_{0,K} & (m_1,m_2)_{0,K} & \cdots & (m_1,m_{n_{k-2}})_{0,K}\\ (m_2,m_1)_{0,K} & (m_2,m_2)_{0,K} & \cdots & (m_2,m_{n_{k-2}})_{0,K}\\ \vdots & \vdots & \ddots & \vdots\\ (m_{n_{k-2}},m_1)_{0,K} & (m_{n_{k-2}},m_2)_{0,K} & \cdots & (m_{n_{k-2}},m_{n_{k-2}})\end{bmatrix}\begin{bmatrix} f_1\\ f_2\\ \vdots\\ f_{n_{k-2}}\end{bmatrix} = \begin{bmatrix}(f,m_1)_{0,K}\\ (f,m_2)_{0,K}\\ \vdots\\ (f,m_{n_{k-2}})_{0,K}\end{bmatrix},$$

and therefore we define b_h^K following (3.3.32), that is

$$(b_h^K)_j := \int_K f_h\varphi_{j,K} = \sum_{i=1}^{n_{k-2}} f_i\int_K \varphi_{j,K}m_i = f_j\sum_{i=1}^{n_{k-2}}\delta_{(kN^K+i,j)}. \tag{3.4.19}$$

In summary, we have to do the following procedure on each element K:

1. Following (3.4.11) and (3.4.14) respectively, calculate $\mathbf{B}$ and $\mathbf{D}$.

2. Calculate $\mathbf{G}=\mathbf{BD}$ by following Proposition 3.4.1.

3. Calculate A_h^K by using (3.4.17).

4. Calculate F_h using (3.4.18) if $k = 1$ and (3.4.19) otherwise.

3.4.3 Assembling the global matrix

We have used above that a_h and F_h can be both split into an element by element sum. Thus, in order to assemble the global stiffness matrix, we just need to sum A_h^K and b_h by all over the elements of $\mathcal{T}_h$. Then, we will obtain a global matrix A_h and a global right-hand side b_h. Therefore, the coefficients $u_1, u_2, \ldots, u_N$ of u_h are given by the unique solution of $A_h x = b_h$. Is important to know that what one is actually calculating is $u_h \in V_h$. Such a solution is not manipulable, though. In order to have a manipulable solution fit for making graphs or producing error estimates, we consider the piecewise polynomial approximation $\widehat{u}_h$ as defined in (3.3.40). Furthermore, if $K \in \mathcal{T}_h$ and $\mathbf{u} := (u_1, u_2, \ldots, u_{n_k^K})$, where

$$u_h|_K = \sum_{j=1}^{n_k^K} u_j \varphi_{j,K},$$

then the coefficients of $\widehat{u}_h$ are given on each element by $\mathbf{\Pi}^* \mathbf{u}$, since

$$\begin{aligned}
\widehat{\Pi}_k^K(u_h|_K) = \widehat{\Pi}_k^K\Big(\sum_{j=1}^{n_k^K} u_j \varphi_{j,K}\Big) &= \sum_{j=1}^{n_k^K} u_j \widehat{\Pi}_k^K(\varphi_{j,K}) = \sum_{j=1}^{n_k^K} u_j \sum_{\alpha=1}^{n_k} (\mathbf{\Pi}^*)_{\alpha j} m_\alpha \\
&= \sum_{\alpha=1}^{n_k} \left(\sum_{j=1}^{n_k^K} (\mathbf{\Pi}^*)_{\alpha j} u_j \right) m_\alpha = \sum_{\alpha=1}^{n_k} (\mathbf{\Pi}^* \mathbf{u})_\alpha m_\alpha
\end{aligned}$$

3.4.4 Numerical results

In this section we present three numerical examples illustrating the good performance of the virtual element scheme (3.3.34), and confirming the rates of convergence predicted by Theorem 3.3.3. For all the computations we consider the virtual element subspace V_h given by (3.3.2), with $k = 1$. For each example, we assume first decompositions of Ω made of triangles. In addition, in Example 1 we also consider straight squares and hexagons, whereas Examples 2 and 3 make use of general quadrilateral elements as well.

We begin by introducing additional notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (3.3.34), that is, $N = \dim V_h$. Mote precisely, according to (3.3.5), and bearing in mind that $\dim \mathrm{P}_k(e) = k+1 \quad \forall$ edge $e \in \mathcal{T}_h$, and $\dim \mathrm{P}_{k-2}(K) = k(k-1)/2 \quad \forall K \in \mathcal{T}_h$, we find that in general

$$N = \text{number of vertices } V \in \mathcal{T}_h +$$
$$(k-1) \times \text{ number of edges } e \in \mathcal{T}_h \; + \; \frac{k(k-1)}{2} \times \text{ number of } K \in \mathcal{T}_h,$$

which, in the case $k = 1$, becomes

$$N = \text{number of vertices } V \in \mathcal{T}_h.$$

Also, the individual errors are defined by

$$\mathtt{e}_0(u) \,:=\, \|u - \widehat{u}_h\|_{0,\Omega}\,, \quad \mathtt{e}_1(u) \,:=\, |u - \widehat{u}_h|_{1,\Omega}\,, \quad \text{and} \quad \mathtt{e}_2(u) \,:=\, \|u - \widehat{u}_h\|_{1,\Omega}\,,$$

where $\widehat{u}_h$ is computed according to (3.3.40). In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}_j(u) \,:=\, \frac{\log\big(\mathtt{e}_j(u)/\mathtt{e}'_j(u)\big)}{\log(h/h')}\,, \quad j \in \{0,1,2\}$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the errors for two consecutive meshes with sizes h and h', respectively. The numerical results presented below were obtained using a `MATLAB` code. The corresponding linear systems were solved using the Conjugate Gradient method as main solver, and applying a stopping criterion determined by a relative tolerance of 10^{-10}. The specific examples to be considered are described next.

In Example 1 we consider $\Omega =]0,1[^2$, and choose the data f and g so that the exact solution of (3.2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$u(\mathbf{x}) \,:=\, \sin(\pi x_1)\,\cos(\pi x_2).$$

In Example 2 we consider the L-shaped domain $\Omega := \,]-1,1\,[^2 \setminus [0,1]^2$, and choose the data f and g so that the exact solution of (3.2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$u(\mathbf{x}) \,:=\, \frac{1}{2}\,\log(x_1^2 + x_2^2 + 1).$$

Finally, in Example 3 we consider the same geometry of Example 1, that is $\Omega =]0,1[^2$, and choose the data f and g so that the exact solution of (3.2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$u(\mathbf{x}) \,:=\, -\,(x_1^2 + x_2^2)^{1/3}.$$

Note in this example that the partial derivatives of u, and hence, in particular Δu, is singular at the origin. Moreover, because of the power $1/3$, there holds $u \in H^{5/3-\epsilon}(\Omega)$ and $\nabla u \in \mathbf{H}^{2/3-\epsilon}(\Omega)$ for each $\epsilon > 0$, which, applying Theorem 3.3.3 with $r = 5/3 - \epsilon$, should yield a rate of convergence very close to $O(h^{5/3})$ for u.

In Tables 3.1 up to 3.5 we summarize the convergence history of the virtual element scheme (3.3.34) as applied to Examples 1 and 2, for sequences of quasi-uniform refinements of each domain. We notice there that the rates of convergences $O(h^{k+1}) = O(h^2)$ and $O(h^k) = O(h)$ predicted by Theorem 3.3.3 (when $r = k+1$ and $s = k+1$) are attained by u, for triangular as well as for quadrilateral meshes. In turn, in Tables 3.6 up to 3.8 we display the corresponding convergence history of Example 3. As predicted in advance, and due to the limited regularity of u in this case, we observe that the orders $O(h^{1+\frac{2}{3}}) = O(h^{5/3})$ are attained by u. Finally, in order to illustrate the accurateness of the discrete scheme, in Figures 3.1 up to 3.8 we display several components of the approximate and exact solution for each example.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
9	0.707	$2.838E-01$	$-$	$1.541E+00$	$-$	$1.567E+00$	$-$
25	0.354	$9.097E-02$	1.642	$8.438E-01$	0.869	$8.487E-01$	0.884
81	0.177	$2.438E-02$	1.900	$4.325E-01$	0.964	$4.332E-01$	0.970
289	0.088	$6.207E-03$	1.974	$2.176E-01$	0.991	$2.177E-01$	0.992
1089	0.044	$1.559E-03$	1.993	$1.090E-01$	0.998	$1.090E-01$	0.998
4225	0.022	$3.902E-04$	1.998	$5.452E-02$	0.999	$5.452E-02$	1.000
16641	0.011	$9.758E-05$	2.000	$2.726E-02$	1.000	$2.726E-02$	1.000
66049	0.006	$2.440E-05$	2.000	$1.363E-02$	1.000	$1.363E-02$	1.000
263169	0.003	$6.099E-06$	2.000	$6.815E-03$	1.000	$6.815E-03$	1.000
1050625	0.001	$1.524E-06$	2.000	$3.408E-03$	1.000	$3.408E-03$	1.000

Table 3.1: Example 1, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
9	0.707	$2.420E-01$	$-$	$1.458E+00$	$-$	$1.478E+00$	$-$
25	0.354	$6.580E-02$	1.879	$7.186E-01$	1.021	$7.216E-01$	1.034
81	0.177	$1.673E-02$	1.976	$3.570E-01$	1.009	$3.573E-01$	1.014
289	0.088	$4.199E-03$	1.994	$1.782E-01$	1.002	$1.782E-01$	1.004
1089	0.044	$1.051E-03$	1.999	$8.905E-02$	1.001	$8.905E-02$	1.001
4225	0.022	$2.627E-04$	2.000	$4.452E-02$	1.000	$4.452E-02$	1.000
16641	0.011	$6.569E-05$	2.000	$2.226E-02$	1.000	$2.226E-02$	1.000
66049	0.006	$1.642E-05$	2.000	$1.113E-02$	1.000	$1.113E-02$	1.000
263169	0.003	$4.106E-06$	2.000	$5.565E-03$	1.000	$5.565E-03$	1.000
1050625	0.001	$1.026E-06$	2.000	$2.782E-03$	1.000	$2.782E-03$	1.000

Table 3.2: Example 1, quasi-uniform refinement with straight squares.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
190	0.139	$1.011E-02$	$-$	$3.036E-01$	$-$	$3.037E-01$	$-$
780	0.065	$2.408E-03$	1.903	$1.452E-01$	0.978	$1.453E-01$	0.978
1646	0.044	$1.128E-03$	1.958	$9.914E-02$	0.986	$9.915E-02$	0.987
2832	0.034	$6.519E-04$	1.973	$7.527E-02$	0.992	$7.527E-02$	0.992
4542	0.026	$4.070E-04$	1.959	$5.926E-02$	0.994	$5.926E-02$	0.994
6408	0.022	$2.875E-04$	1.988	$4.981E-02$	0.994	$4.981E-02$	0.994
8594	0.019	$2.139E-04$	1.991	$4.296E-02$	0.996	$4.296E-02$	0.996
11424	0.017	$1.612E-04$	1.968	$3.722E-02$	0.997	$3.722E-02$	0.997

Table 3.3: Example 1, quasi-uniform refinement with hexagons.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
21	0.707	$3.860E-02$	–	$2.054E-01$	–	$2.090E-01$	–
65	0.354	$1.028E-02$	1.909	$1.035E-01$	0.989	$1.040E-01$	1.007
225	0.177	$2.619E-03$	1.973	$5.183E-02$	0.997	$5.190E-02$	1.002
833	0.088	$6.585E-04$	1.992	$2.593E-02$	0.999	$2.594E-02$	1.001
3201	0.044	$1.649E-04$	1.997	$1.297E-02$	1.000	$1.297E-02$	1.000
12545	0.022	$4.125E-05$	1.999	$6.483E-03$	1.000	$6.483E-03$	1.000
49665	0.011	$1.031E-05$	2.000	$3.242E-03$	1.000	$3.242E-03$	1.000
197633	0.006	$2.579E-06$	2.000	$1.621E-03$	1.000	$1.621E-03$	1.000
788481	0.003	$6.447E-07$	2.000	$8.104E-04$	1.000	$8.104E-04$	1.000

Table 3.4: Example 2, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
21	0.800	$3.627E-02$	–	$1.849E-01$	–	$1.885E-01$	–
65	0.431	$9.617E-03$	2.145	$9.492E-02$	1.078	$9.540E-02$	1.100
225	0.215	$2.433E-03$	1.974	$4.830E-02$	0.970	$4.836E-02$	0.976
833	0.110	$6.147E-04$	2.048	$2.416E-02$	1.031	$2.417E-02$	1.032
3201	0.055	$1.526E-04$	2.002	$1.202E-02$	1.003	$1.202E-02$	1.003
12545	0.028	$3.798E-05$	2.074	$5.999E-03$	1.037	$5.999E-03$	1.037
49665	0.014	$9.482E-06$	2.017	$2.998E-03$	1.008	$2.998E-03$	1.008
197633	0.007	$2.369E-06$	2.003	$1.499E-03$	1.001	$1.499E-03$	1.001
788481	0.004	$5.923E-07$	2.002	$7.496E-04$	1.001	$7.496E-04$	1.001

Table 3.5: Example 2, quasi-uniform refinement with quadrilaterals.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
9	0.707	$2.866E-02$	–	$2.586E-01$	–	$2.601E-01$	–
25	0.354	$9.205E-03$	1.639	$1.713E-01$	0.594	$1.716E-01$	0.601
81	0.177	$2.925E-03$	1.654	$1.112E-01$	0.623	$1.113E-01$	0.625
289	0.088	$9.260E-04$	1.660	$7.137E-02$	0.640	$7.138E-02$	0.641
1089	0.044	$2.925E-04$	1.662	$4.547E-02$	0.650	$4.547E-02$	0.651
4225	0.022	$9.231E-05$	1.664	$2.884E-02$	0.657	$2.884E-02$	0.657
16641	0.011	$2.911E-05$	1.665	$1.825E-02$	0.660	$1.825E-02$	0.660
66049	0.006	$9.175E-06$	1.666	$1.153E-02$	0.663	$1.153E-02$	0.663
263169	0.003	$2.891E-06$	1.666	$7.274E-03$	0.664	$7.274E-03$	0.664
1050625	0.001	$9.110E-07$	1.666	$4.587E-03$	0.665	$4.587E-03$	0.665

Table 3.6: Example 3, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
9	0.707	$2.306E-02$	–	$2.482E-01$	–	$2.492E-01$	–
25	0.495	$9.348E-03$	2.531	$1.615E-01$	1.205	$1.618E-01$	1.212
81	0.277	$3.345E-03$	1.768	$1.102E-01$	0.658	$1.102E-01$	0.660
289	0.143	$1.066E-03$	1.722	$7.302E-02$	0.620	$7.303E-02$	0.620
1089	0.072	$3.202E-04$	1.755	$4.672E-02$	0.651	$4.672E-02$	0.651
4225	0.036	$9.527E-05$	1.754	$2.943E-02$	0.669	$2.943E-02$	0.669
16641	0.018	$2.860E-05$	1.737	$1.844E-02$	0.674	$1.844E-02$	0.674
66049	0.009	$8.689E-06$	1.719	$1.156E-02$	0.675	$1.156E-02$	0.675
263169	0.004	$2.668E-06$	1.704	$7.246E-03$	0.673	$7.246E-03$	0.673
1050625	0.002	$8.256E-07$	1.692	$4.548E-03$	0.672	$4.548E-03$	0.672

Table 3.7: Example 3, quasi-uniform refinement with distorted squares.

N	h	$\mathtt{e}_0(u)$	$\mathtt{r}_0(u)$	$\mathtt{e}_1(u)$	$\mathtt{r}_1(u)$	$\mathtt{e}_2(u)$	$\mathtt{r}_2(u)$
190	0.139	$1.364E-03$	–	$9.184E-02$	–	$9.185E-02$	–
780	0.065	$3.945E-04$	1.646	$5.546E-02$	0.669	$5.547E-02$	0.669
1646	0.044	$2.137E-04$	1.584	$4.358E-02$	0.623	$4.358E-02$	0.623
2832	0.034	$1.369E-04$	1.602	$3.654E-02$	0.634	$3.654E-02$	0.634
4542	0.026	$9.158E-05$	1.672	$3.101E-02$	0.683	$3.101E-02$	0.683
6408	0.022	$6.915E-05$	1.607	$2.775E-02$	0.635	$2.775E-02$	0.635
8594	0.019	$5.440E-05$	1.614	$2.523E-02$	0.639	$2.523E-02$	0.639
11424	0.017	$4.271E-05$	1.682	$2.286E-02$	0.687	$2.286E-02$	0.687

Table 3.8: Example 3, quasi-uniform refinement with hexagons.

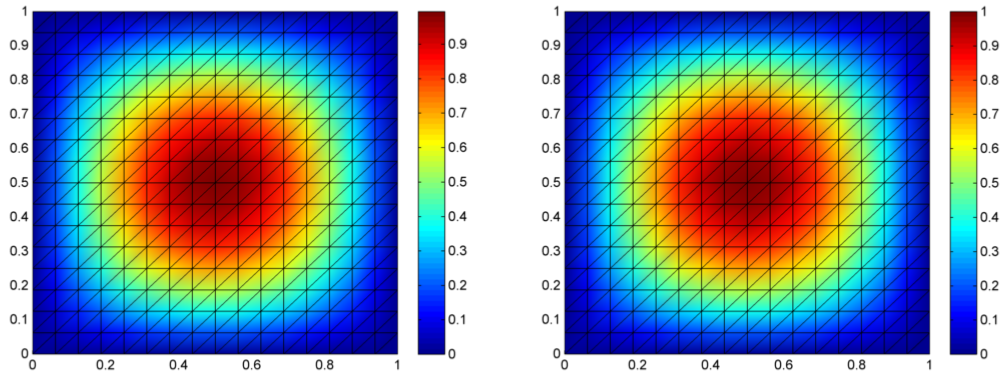

Figure 3.1: Example 1, $\widehat{u}_h$ and u for a mesh with triangles ($N = 289$).

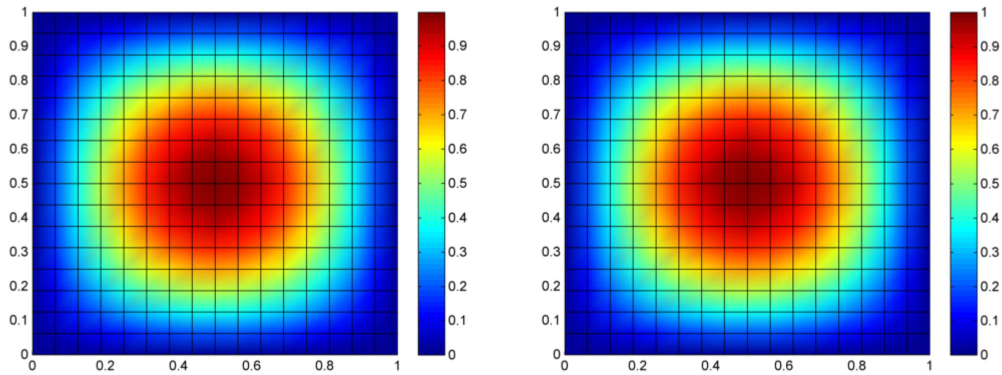

Figure 3.2: Example 1, $\widehat{u}_h$ and u for a mesh with straight squares ($N = 289$).

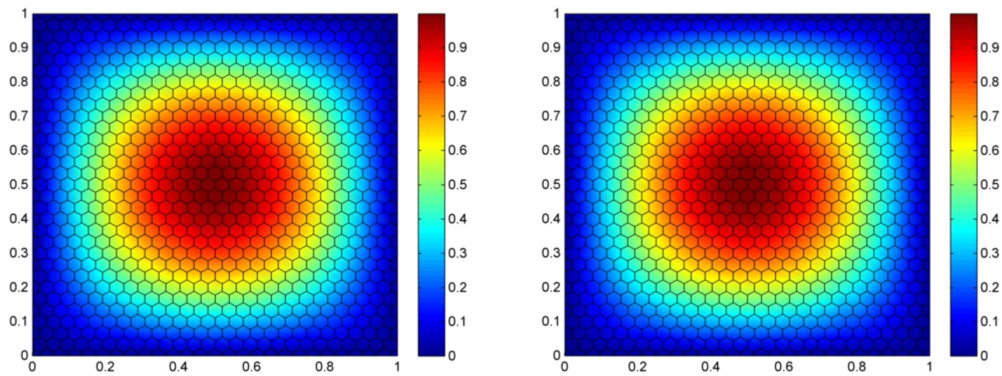

Figure 3.3: Example 1, $\widehat{u}_h$ and u for a mesh with hexagons ($N = 1646$).

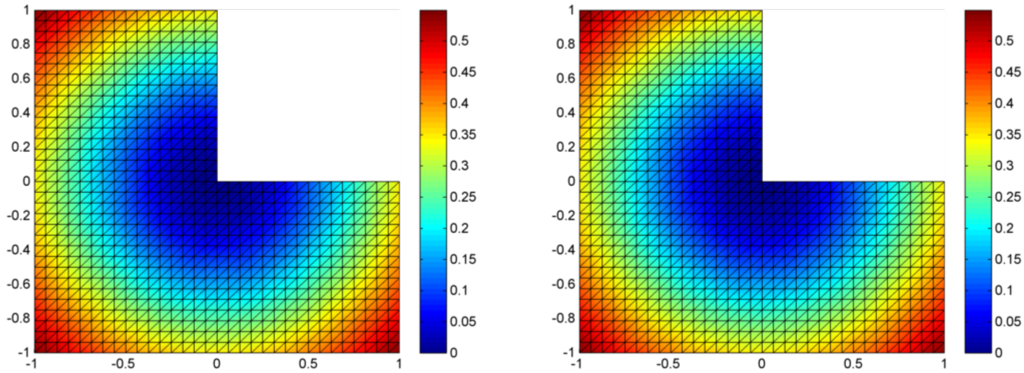

Figure 3.4: Example 2, $\widehat{u}_h$ and u for a mesh with triangles ($N = 12545$).

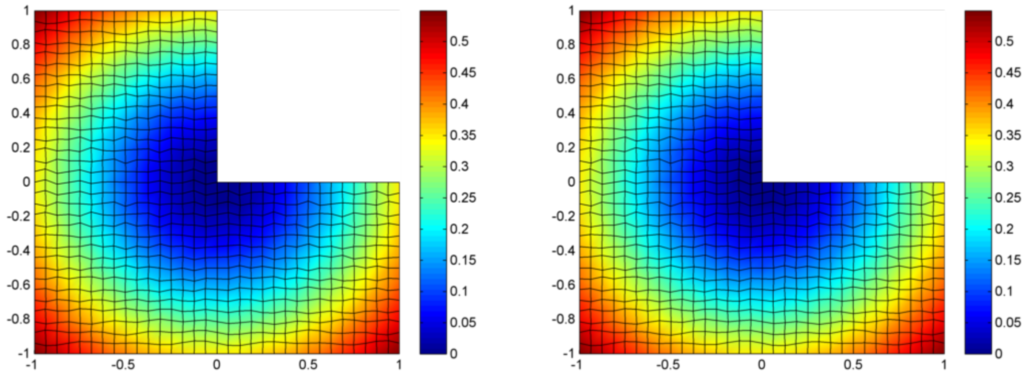

Figure 3.5: Example 2, $\widehat{u}_h$ and u for a mesh with quadrilaterals ($N = 12545$).

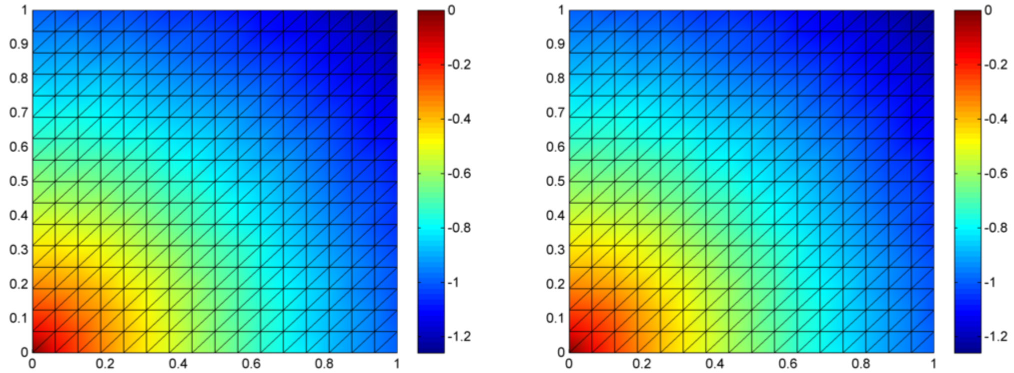

Figure 3.6: Example 3, $\widehat{u}_h$ and u for a mesh with triangles ($N = 289$).

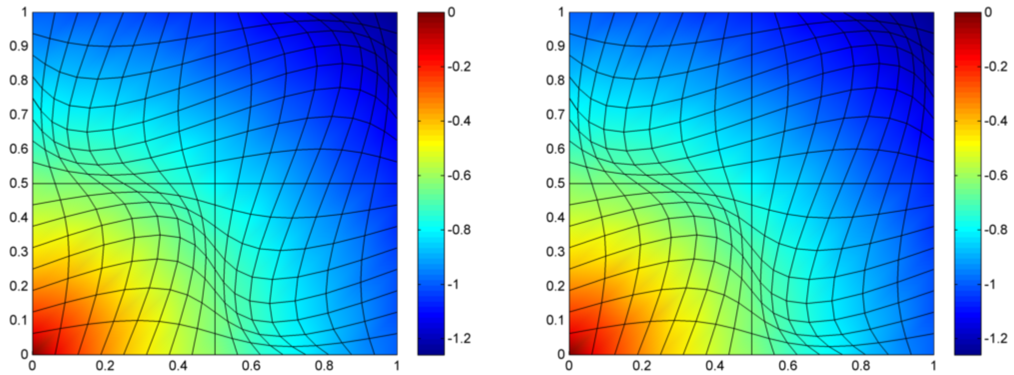

Figure 3.7: Example 3, $\widehat{u}_h$ and u for a mesh with distorted squares ($N = 289$).

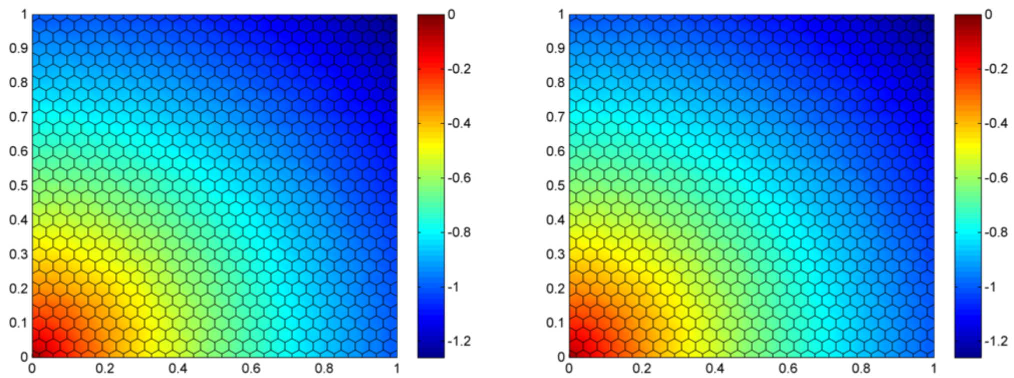

Figure 3.8: Example 3, $\widehat{u}_h$ and u for a mesh with hexagons ($N = 1646$).

Chapter 4

Mixed Virtual Elements Method for the Darcy problem

4.1 Introduction

In this chapter, we use [9] and present a mixed Virtual Element Method for the Darcy problem, generalizing the corresponding result by considering now a non–homogeneus Neumann boundary condition. In Section 4.2 we introduce the boundary value problem of interest and derive the associated mixed formulation, as well as the corresponding well-posedness result. Then, in Section 4.3 we follow [9] to introduce the virtual element subspaces that will be employed, and then show the respective unisolvency, define the associated interpolation operators, and provide their approximation properties. Though some of the proofs of these results are sketched in [9], for sake of clearness and completeness, in the present chapter we try to give as much detail as possible in some of them. In particular, a Bramble–Hilbert type theorem for averaged Taylor polynomials (cf. Chapter 2) plays a key role in our analysis. Next, fully calculable discrete bilinear forms are introduced in Section 4.3.4 and their boundedness and related properties are established. To this end, a new local projector onto a suitable space of polynomials is proposed here. This operator is somehow suggested by the main features of the continuous solution of the Darcy problem, and it also responds to the need of explicitly integrating the terms of the bilinear form that involves L^2 inner products. The family of local projectors is shown to be uniformly bounded, and the aforementioned compactness theorem is applied to derive its approximation properties. The actual mixed virtual element method is then introduced and analyzed in Section 4.3.5. The classical discrete Babuška–Brezzi theory is applied to deduce the well-posedness of this scheme, and then suitable bounds and identities satisfied by the bilinear forms and the projectors and interpolators involved, allow to derive the a priori error estimates and corresponding rates of convergence for the virtual solution as well as for the projection of it. On the other hand, in Section 4.4 we use [5] to provide details on the computational implementation of MVEM, explaining how to assemble the global stiffness matrix and how to impose the associated extra condition (cf. Section 4.2) on the solution. Finally, several numerical examples showing the good performance of the method, confirming the rates of convergence for regular and singular solutions, and illustrating the accuracy obtained with the approximate solutions, are reported in Section 4.4.4 .

4.2 The continuous problem and its mixed formulation

Let Ω be a simply connected polygonal domain in R^2 with boundary Γ. Our aim is to find the velocity u and the pressure p of a steady flow occupying Ω, under the action of external forces. More precisely, given a volume force $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we seek a scalar function $p \in H^1(\Omega)$ such that

$$-\operatorname{div}(\mathbb{K}\nabla p) = f \text{ in } \Omega \quad \text{and} \quad (\mathbb{K}\nabla p)\cdot \mathbf{n} = g \text{ in } \Gamma, \tag{4.2.1}$$

where $\mathbb{K}$ is a symmetric and positive definite tensor, which represents the permeability of the medium. We assume for simplicity that $\mathbb{K}$ is constant (or piecewise constant). Also, we will denote by $\|\mathbb{K}\|$ the Frobenius norm of $\mathbb{K}$, and we assume that the given data f, g

verify the compatibility condition

$$\int_\Omega f = \int_\Gamma g, \tag{4.2.2}$$

which is obtained by integrating each equation of (4.2.1) and by using the Gauss theorem. Also, we note that if p is a solution of (4.2.1), then $p+c$, $c \in \mathrm{R}$, is also a solution of (4.2.1). Then, in order to ensure uniqueness of solution of (4.2.1), we will require that $(p,1)_{0,\Omega} = 0$. In order to find a mixed variational formulation of (4.2.1), we define $\mathbf{u} := -\mathbb{K}\nabla p$, and consequently (4.2.1) can be re-written as

$$\mathbf{u} \;=\; -\mathbb{K}\nabla p \text{ in } \Omega, \qquad \operatorname{div}\mathbf{u} \;=\; f \text{ in } \Omega \quad \text{and} \quad \mathbf{u}\cdot\mathbf{n} \;=\; g \text{ in } \Gamma. \tag{4.2.3}$$

Taking the inner product of the first pair of equations of (4.2.3) with $\mathbf{v} \in H(\operatorname{div};\Omega)$ and integrating by parts the first resulting equation, we arrive at the following mixed variational formulation: Find $(\mathbf{u}, p) \in H_g \times Q$ such that

$$\begin{array}{lcllll} a(\mathbf{u},\mathbf{v}) & + & b(\mathbf{v},p) & = & 0 & \forall\, \mathbf{v} \in H := H_0, \\ b(\mathbf{u},q) & & & = & -\displaystyle\int_\Omega f\, q & \forall\, q \in Q, \end{array} \tag{4.2.4}$$

where

$$H_g \;:=\; \Big\{ \mathbf{v} \in H(\operatorname{div};\Omega) : \quad \mathbf{v}\cdot\mathbf{n} \;=\; g \quad \text{in} \quad \Gamma \Big\}, \qquad Q \;:=\; \mathbf{L}^2(\Omega)/\mathrm{R} \;=\; L^2_0(\Omega), \tag{4.2.5}$$

and H is endowed with the usual norm $\|\cdot\|_{\operatorname{div};\Omega}$ of $H(\operatorname{div};\Omega)$. In turn, $a: H \times H \to \mathrm{R}$, $b: H \times Q \to \mathrm{R}$ are the bilinear forms defined by

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \mathbb{K}^{-1}\,\mathbf{u}\cdot\mathbf{v},$$

and

$$b(\mathbf{v},q) := \int_\Omega q \operatorname{div}\mathbf{v}, \tag{4.2.6}$$

for each $\mathbf{u},\mathbf{v} \in H$ and for all $q \in Q$. The unique solvability of (4.2.4) is established as follows.

Theorem 4.2.1. *There exists a unique* $(\mathbf{u}, p) \in H \times Q$ *solution of* (4.2.4). *Moreover, there exists* $C > 0$, *depending only on* Ω, *such that*

$$\|(\mathbf{u}, p)\|_{H\times Q} \leq C\Big\{\|f\|_{0,\Omega} + \|g\|_{-1/2,\Gamma}\Big\}.$$

Proof. In order to use the Babǔska–Brezzi theory, we first note that a, b are bounded bilinear forms since $|a(\mathbf{u}, \mathbf{v})| \leq \|\mathbb{K}^{-1}\|\|\mathbf{u}\|_H\|\mathbf{v}\|_H$ and $|b(\mathbf{v}, q)| \leq \|\mathbf{v}\|_H\|q\|_Q$, for every $\mathbf{u}, \mathbf{v} \in H$ and $q \in Q$. Also, b satisfies the inf-sup condition. Indeed, given $q \in Q$, we consider the auxiliary problem: Find $z \in H_0^1(\Omega)$ such that $\nabla z = q$ in Ω and $z = 0$ in Γ. It is clear that such a z exists and is unique. Also, there exists $C > 0$, depending only on Ω, such that $\|z\|_{1,\Omega} \leq C\|q\|_{0,\Omega}$. Now, $\mathbf{u} := \nabla z \in H\,(\mathrm{div}\,;\Omega)$ since $q \in L^2\,(\Omega)$, and so

$$\sup_{\substack{\mathbf{v}\in H\\ \mathbf{v}\neq \mathbf{0}}} \frac{\displaystyle\int_\Omega q\,\mathrm{div}\,\mathbf{v}}{\|\mathbf{v}\|_{\mathrm{div}\,,\Omega}} \geq \frac{\displaystyle\int_\Omega q\,\mathrm{div}\,\mathbf{u}}{\|\mathbf{u}\|_H} \geq \widetilde{C}\|q\|_{0,\Omega},$$

where $\widetilde{C} := (1 + C^2)^{-1/2}$. We consider now the corresponding kernel

$$V := \Big\{\mathbf{v} \in H : b(\mathbf{v}, q) = 0 \ \ \forall q \in Q\Big\} = \Big\{\mathbf{v} \in H : \mathrm{div}\,\mathbf{v} = 0 \text{ in } \Omega\Big\},$$

and we have that $a(\mathbf{u}, \mathbf{u}) = (\mathbb{K}^{-1}\mathbf{u}, \mathbf{u})_{0,\Omega} \geq \alpha\|\mathbf{u}\|_{0,\Omega} = \alpha\|\mathbf{u}\|_H^2$, where α is a constant that arises from the fact that $\mathbb{K}$ is positive definite. By using the Babǔska–Brezzi theorem, we find that (4.2.4) has a unique solution $(\mathbf{u}, p) \in H \times Q$ which depends continuously on the data f, g. The later means that there exists $C > 0$, depending only on Ω, such that

$$\|(\mathbf{u}, p)\|_{H\times Q} \leq C\Big\{\|f\|_{0,\Omega} + \|g\|_{-1/2,\Gamma}\Big\},$$

as required. □

4.3 The discrete problem

4.3.1 Virtual elements subspaces

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements. For each $K \in \mathcal{T}_h$ we denote its diameter by h_K, and define, as usual, $h := \max\Big\{h_K : \quad K \in \mathcal{T}_h\Big\}$. Now, given an integer $k \geq 0$, we let $\mathrm{P}_k\,(K)$ be the space of polynomials on K of total degree up to k. Then, given an integer $k \geq 1$, we consider the following virtual element subspaces of H and Q, respectively:

$$\begin{aligned} H_h \;:=\; \Big\{ \mathbf{v} \in H : \quad \mathbf{v}\cdot\mathbf{n}\Big|_e \in \mathrm{P}_k(e) \quad \forall \text{ edge } e \in \mathcal{T}_h, \quad \operatorname{div}\mathbf{v}\Big|_K \in \mathrm{P}_{k-1}(K)\,, \\ \operatorname{rot}\mathbf{v}\Big|_K \in \mathrm{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \end{aligned} \tag{4.3.1}$$

and

$$Q_h \;:=\; \Big\{ q \in Q : \quad q\,\Big|_K \in \mathrm{P}_{k-1}(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \tag{4.3.2}$$

where

$$\operatorname{rot}\mathbf{v} := \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2} \quad \forall\,\mathbf{v} := (v_1, v_2) \in H.$$

Then, the Galerkin scheme associated with (4.2.4) would read: Find $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a\,(\mathbf{u}_h, \mathbf{v}_h) \,+\, b\,(\mathbf{v}_h, p_h) \;&=\; 0 \qquad && \forall\,\mathbf{v}_h \in H_h\,, \\ b\,(\mathbf{u}_h, q_h) \;&=\; -\int_\Omega f\, q_h \qquad && \forall\, q_h \in Q_h\,. \end{aligned} \tag{4.3.3}$$

Nevertheless, we will observe later on that $a(\mathbf{u}_h, \mathbf{v}_h)$ cannot be computed explicitly when both $\mathbf{u}_h$, $\mathbf{v}_h$ belong to H_h, and hence a suitable approximation of this bilinear form, namely a_h, will be introduced in Section 4.3.4 to redefine (4.3.3).

4.3.2 Unisolvency of the virtual element subspaces

In what follows we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}}\, h_K$ and center $\mathbf{x}_B \in K$, that is, for each $x_0 \in B$, all the line segments joining x_0 with any $x \in K$ are contained in K, or, equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in K.

As a consequence of the above hypotheses, one can show that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that the number of edges of each $K \in \mathcal{T}_h$ is bounded above by $N_{\mathcal{T}}$.

Next, in order to choose the degrees of freedom of H_h, given an edge $e \in \mathcal{T}_h$ with medium point x_e and length h_e, and given an integer $\ell \geq 0$, we first introduce the following set of $\ell + 1$ normalized monomials on e

$$\mathcal{B}_\ell(e) := \left\{ \left(\frac{x - x_e}{h_e} \right)^j \right\}_{0 \leq j \leq \ell}, \tag{4.3.4}$$

which certainly constitutes a basis of $\mathrm{P}_\ell(e)$. Similarly, given an element $K \in \mathcal{T}_h$ with barycenter $\mathbf{x}_K$, and given an integer $\ell \geq 0$, we define the following set of $\dfrac{(\ell+1)(\ell+2)}{2}$ normalized monomials

$$\mathcal{B}_\ell(K) := \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^\alpha \right\}_{0 \leq |\alpha| \leq \ell}, \tag{4.3.5}$$

which is a basis of $\mathrm{P}_\ell(K)$. Note that (4.3.5) makes use of the multi-index notation where, given $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathrm{R}^2$ and $\alpha := (\alpha_1, \alpha_2)^{\mathtt{t}}$, with nonnegative integers α_1, α_2, we set $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\alpha| := \alpha_1 + \alpha_2$. According to the above and the definition of H_h (cf. (4.3.1)), we propose the following degrees of freedom for a given $\mathbf{v} \in H_h$:

$$\begin{array}{llll} \text{a)} & \displaystyle\int_e q\, \mathbf{v} \cdot \mathbf{n} & \forall\, q \in \mathcal{B}_k(e) & \forall \text{ edge } e \in \mathcal{T}_h\,, \\ \text{b)} & \displaystyle\int_K \mathbf{v} \cdot \nabla q & \forall\, q \in \mathcal{B}_{k-1}(K) \setminus \{1\} & \forall\, K \in \mathcal{T}_h\,, \\ \text{c)} & \displaystyle\int_K q \operatorname{rot} \mathbf{v} & \forall\, q \in \mathcal{B}_{k-1}(K) & \forall\, K \in \mathcal{T}_h\,. \end{array} \tag{4.3.6}$$

We now observe, according to the cardinalities of $\mathcal{B}_k(e)$ and $\mathcal{B}_{k-1}(K)$, that the number of local degrees of freedom, that is those related to a given $K \in \mathcal{T}_h$, is given by

$$n_k^K := (k+1)\, d_K + \left\{ \frac{k(k+1)}{2} - 1 \right\} + \frac{k(k+1)}{2} = (k+1)\,(d_K + k) - 1\,, \tag{4.3.7}$$

where d_K is the number of edges of K. Moreover, we have the following local unisolvence result.

Lemma 4.3.1. Given an integer $k \geq 1$, we define for each $K \in \mathcal{T}_h$ the local space

$$H_h^K := \Big\{ \mathbf{v} \in H(\mathrm{div}\,; K) \cap H(\mathrm{rot}; K) : \quad \mathbf{v} \cdot \mathbf{n}\Big|_e \in \mathrm{P}_k(e) \quad \forall \text{ edge } e \subseteq \partial K\,, \\ \mathrm{div}\,\mathbf{v} \in \mathrm{P}_{k-1}(K)\,, \quad \mathrm{rot}\,\mathbf{v} \in \mathrm{P}_{k-1}(K) \Big\}\,. \tag{4.3.8}$$

Then, the n_k^K local degrees of freedom arising from (4.3.6) are unisolvent in H_h^K.

Proof. Let $\mathbf{v} \in H_h^K$ such that

$$\begin{aligned} \int_e q\, \mathbf{v} \cdot \mathbf{n} &= 0 \qquad \forall\, q \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,, \\ \int_K \mathbf{v} \cdot \nabla q &= 0 \qquad \forall\, q \in \mathcal{B}_{k-1}(K)\,, \\ \int_K q \,\mathrm{rot}\,\mathbf{v} &= 0 \qquad \forall\, q \in \mathcal{B}_{k-1}(K)\,. \end{aligned} \tag{4.3.9}$$

It follows easily from the definition (4.3.8) together with the first and third equations of (4.3.9) that

$$\mathrm{rot}\,\mathbf{v} = \mathbf{0} \quad \text{in} \quad K\,, \quad \text{and} \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{in} \quad \partial K\,. \tag{4.3.10}$$

In turn, integrating by parts the second equation in (4.3.9), we find that

$$0 = \int_K \mathbf{v} \cdot \nabla q = -\int_K q\,\mathrm{div}\,\mathbf{v} + \int_{\partial K} q\,\mathbf{v} \cdot \mathbf{n} = -\int_K q\,\mathrm{div}\,\mathbf{v} \qquad \forall\, q \in \mathcal{B}_{k-1}(K)\,,$$

which yields $\mathrm{div}\,\mathbf{v} = \mathbf{0}$ in K. Now, since K is simply connected, we know from the second identity in (4.3.10) and [30, Chapter I, Theorem 2.9] that there exists $\phi \in H^1(K)$ such that $\mathbf{v} = \nabla\phi$ in K. In this way, the free divergence property of $\mathbf{v}$, and the fact that its normal component is the null vector on ∂K, can be rewritten as

$$\Delta\phi = 0 \quad \text{in} \quad K\,, \quad \nabla\phi \cdot \mathbf{n} = 0 \quad \text{in} \quad \partial K\,.$$

Thus, the classical solvability analysis of this Neumann problem implies that ϕ is a constant vector, and hence $\mathbf{v}$ vanishes in K, which completes the proof. □

4.3.3 Interpolation on H_h and Q_h

In this section we define suitable interpolation operators on our virtual element subspaces and establish their corresponding approximation properties. To this end, we need some preliminary notations and technical results. For each element $K \in \mathcal{T}_h$ we let $\widetilde{K} := T_K(K)$, where $T_K : \mathrm{R}^2 \longrightarrow \mathrm{R}^2$ is the bijective affine mapping defined by $T_K(\mathbf{x}) := \dfrac{\mathbf{x} - \mathbf{x}_B}{h_K}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$. Note that the diameter $h_{\widetilde{K}}$ of $\widetilde{K}$ is 1, and, according to the assumptions a) and b), it is easy to see that the shortest edge of $\widetilde{K}$ is bigger than $C_{\mathcal{T}}$, and that $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. Recall here that $\mathbf{x}_B$ is the center of the ball B with respect to which K is star-shaped. Then, by connecting each vertex of $\widetilde{K}$ to the center of $\widetilde{B}$, that is to the origin, we generate a partition of $\widetilde{K}$ into $d_{\widetilde{K}}$ triangles $\widetilde{\Delta}_i$, $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $d_{\widetilde{K}} \leq N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\dfrac{\widetilde{h}_i}{\widetilde{\rho}_i} \leq c_{\mathcal{T}}$ $\forall\, i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$, where $\widetilde{h}_i$ is the diameter of $\widetilde{\Delta}_i$ and $\widetilde{\rho}_i$ is the diameter of the largest ball contained in $\widetilde{\Delta}_i$. We also let $\widehat{\Delta}$ be the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$, and for each $i \in \{1, 2, \ldots, d_{\widetilde{K}}\}$ we let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\, \mathbf{x}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \widetilde{\Delta}_i$. We remark that the fact that the origin is a vertex of each triangle $\widetilde{\Delta}_i$ allows to choose F_i as indicated. In what follows, given $q \in L^2(K)$, we let $\widetilde{q} := q \circ T_K^{-1} \in L^2(\widetilde{K})$. Then, we have the following result.

Lemma 4.3.2. Given an integer $\ell \geq 0$ and an element $K \in \mathcal{T}_h$, we let $\mathcal{P}_\ell^K : L^2(K) \to P_\ell(K)$ and $\mathcal{P}_\ell^{\widetilde{K}} : L^2(\widetilde{K}) \to P_\ell(\widetilde{K})$ be the corresponding orthogonal projectors. Then $\widetilde{\mathcal{P}_\ell^K(q)} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{q})$ for all $q \in L^2(K)$, and for any pair of nonnegative integers r and s there holds $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(H^r(\widetilde{K}), H^s(\widetilde{K}))$, with $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(H^r(\widetilde{K}), H^s(\widetilde{K}))}$ independent of $\widetilde{K}$, namely depending only on ℓ, s, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$.

Proof. Denoting $N_\ell := \dfrac{(\ell+1)(\ell+2)}{2}$, we let $\Big\{\varphi_1, \varphi_2, \ldots, \varphi_{N_\ell}\Big\}$ be a basis of $\mathrm{P}_\ell(K)$, in particular $\mathcal{B}_\ell(K)$ (cf. (4.3.5)), and observe that $\Big\{\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_{N_\ell}\Big\}$ becomes a basis of $\mathrm{P}_\ell(\widetilde{K})$. Hence, given $q \in \mathbf{L}^2(K)$, and bearing in mind that the Jacobian of T_K is h_K^{-2}, we find that for each $j \in \{1, 2, \ldots, N_\ell\}$ there holds

$$\int_{\widetilde{K}} \widetilde{\mathcal{P}_\ell^K(q)}\, \widetilde{\varphi}_j \;=\; h_K^{-2} \int_K \mathcal{P}_\ell^K(q)\, \varphi_j \;=\; h_K^{-2} \int_K q\, \varphi_j \;=\; \int_{\widetilde{K}} \widetilde{q}\, \widetilde{\varphi}_j \;=\; \int_{\widetilde{K}} \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{q})\, \widetilde{\varphi}_j\,,$$

which shows that $\widetilde{\mathcal{P}_\ell^K(q)} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{q})$. Throughout the rest of the proof we assume for simplicity that $\left\{\widetilde{\varphi}_1, \widetilde{\varphi}_2, \ldots, \widetilde{\varphi}_{N_\ell}\right\}$ is orthonormal, which yields $\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{q}) \,=\, \sum_{j=1}^{N_\ell} \langle \widetilde{q}, \widetilde{\varphi}_j \rangle_{0,\widetilde{K}}\, \widetilde{\varphi}_j$. Then, employing the Cauchy-Schwarz inequality, we obtain

$$\|\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{q})\|_{s,\widetilde{K}} \,\le\, \left\{\sum_{j=1}^{N_\ell} \|\widetilde{\varphi}_j\|_{s,\widetilde{K}}\right\} \|\widetilde{q}\|_{0,\widetilde{K}} \,\le\, \left\{\sum_{j=1}^{N_\ell} \|\widetilde{\varphi}_j\|_{s,\widetilde{K}}\right\} \|\widetilde{q}\|_{r,\widetilde{K}}\,,$$

which proves that $\mathcal{P}_\ell^{\widetilde{K}} \,\in\, \mathcal{L}(H^r(\widetilde{K}), H^s(\widetilde{K}))$, with

$$\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(H^r(\widetilde{K}),H^s(\widetilde{K}))} \,\le\, \sum_{j=1}^{N_\ell} \|\widetilde{\varphi}_j\|_{s,\widetilde{K}} \,=\, \sum_{j=1}^{N_\ell} \left\{\sum_{i=1}^{d_{\widetilde{K}}} \|\widetilde{\varphi}_j\|^2_{s,\widetilde{\Delta}_i}\right\}^{1/2}, \tag{4.3.11}$$

where the last equality makes use of the aforementioned decomposition of $\widetilde{K}$. We now apply the usual scaling properties connecting the Sobolev integer seminorms in each $\widetilde{\Delta}_i$ with those in $\widehat{\Delta}$. In this way, denoting $\widehat{\varphi}_{j,i} \,:=\, \widetilde{\varphi}_j|_{\widetilde{\Delta}_i} \circ F_i \in \mathrm{P}_\ell(\widehat{\Delta})$, using the equivalence of norms in $\mathrm{P}_\ell(\widehat{\Delta})$, and noting that $\widetilde{\rho}_i^{-1} \,\le\, c_{\mathcal{T}}\, \widetilde{h}_i^{-1} \,\le\, c_{\mathcal{T}}\, C_{\mathcal{T}}^{-1}$, we deduce that for each integer $t \ge 0$ there holds

$$\begin{aligned} |\widetilde{\varphi}_j|_{t,\widetilde{\Delta}_i} \,&\le\, C_t\, \widehat{h}^t\, \widetilde{\rho}_i^{-t}\, |\det B_i|^{1/2}\, \|\widehat{\varphi}_{j,i}\|_{t,\widehat{\Delta}} \,\le\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, |\det B_i|^{1/2}\, \widehat{c}\, \|\widehat{\varphi}_{j,i}\|_{0,\widehat{\Delta}} \\ &=\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}\, \|\widetilde{\varphi}_j\|_{0,\widetilde{\Delta}_i} \,\le\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}\, \|\widetilde{\varphi}_j\|_{0,\widetilde{K}} \,=\, C\,, \end{aligned}$$

where C_t depends on t, whereas $\widehat{c}$ depends on $\mathrm{P}_\ell(\widehat{\Delta})$ and t, and $C \,=\, C_t\, \widehat{h}^t\, c_{\mathcal{T}}^t\, C_{\mathcal{T}}^{-t}\, \widehat{c}$. The foregoing inequality and (4.3.11) give the announced independence of $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(H^r(\widetilde{K}),H^s(\widetilde{K}))}$, which ends the proof.

□

The next result taken from (2.4.2) (see also [7, Lemma 4.3.8] or [19]) is required in what follows as well.

Lemma 4.3.3. Let $\mathcal{O}$ be a domain of R^2 with diameter 1, such that it is star-shaped with respect to a ball B of radius $> \frac{1}{2}\, \rho_{\mathtt{max}}$, where

$$\rho_{\mathtt{max}} \,:=\, \sup\left\{\, \rho : \quad \mathcal{O} \text{ is star-shaped with respect to a ball of radius } \rho \right\}.$$

In addition, given an integer $m \ge 1$ and $q \in H^m(\mathcal{O})$, we let $\mathrm{T}^m(\mathbf{v}) \,\in\, \mathrm{P}_{m-1}(\mathcal{O})$ be the Taylor polynomial of order m of q averaged over B. Then, there exists $C > 0$, depending only on m and $\rho_{\mathtt{max}}$, such that

$$|q - \mathrm{T}^m(q)|_{\ell,\mathcal{O}} \,\leq\, C\,|q|_{m,\mathcal{O}} \qquad \forall\,\ell \in \{0,1,\ldots,m\}\,.$$

We now proceed to define our interpolation operators. We begin by letting $\mathcal{P}_{k-1}^h : L^2(\Omega) \longrightarrow Q_h$ be the orthogonal projector, that is, given $p \in Q := L^2(\Omega)$, $\mathcal{P}_{k-1}^h(p)$ is characterized by

$$\int_K \big(p \,-\, \mathcal{P}_{k-1}^h(p)\big)\; q \,=\, 0 \qquad \forall\, K \in \mathcal{T}_h\,, \quad \forall\, q \in \mathrm{P}_{k-1}(K)\,, \tag{4.3.12}$$

which means, equivalently, that

$$\mathcal{P}_{k-1}^h(p)\big|_K \,=\, \mathcal{P}_{k-1}^K(p|_K)\,,$$

where, as indicated in Lemma 4.3.2, $\mathcal{P}_{k-1}^K : L^2(K) \to \mathrm{P}_{k-1}(K)$ is the local orthogonal projector. The following lemma establishes the approximation properties of this operator.

Lemma 4.3.4. Let k, ℓ and r be integers such that $1 \leq r \leq k$ and $0 \leq \ell \leq r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$|q \,-\, \mathcal{P}_{k-1}^K(q)|_{\ell,K} \,\leq\, C\,h_K^{r-\ell}\,|q|_{r,K} \qquad \forall\, q \in H^r(K)\,.$$

Proof. Given integers k, ℓ and r as stated, $K \in \mathcal{T}_h$, and $q \in H^r(K)$, we first observe that there hold

$$|\widetilde{q}|_{\ell,\widetilde{K}} \,=\, h_K^{\ell+1}\,|q|_{\ell,K} \qquad \text{and} \qquad \mathcal{P}_{k-1}^{\tilde{K}}\big(\widetilde{\mathrm{T}}^r(\tilde{q})\big) \,=\, \widetilde{\mathrm{T}}^r(\tilde{q})\,,$$

where $\widetilde{\mathrm{T}}^r(\tilde{q}) \,\in\, \mathrm{P}_{r-1}(\widetilde{K})$ is the Taylor polynomial of order r of $\widetilde{q}$ averaged over a ball of radius $> \frac{1}{2}\,\widetilde{\rho}_{\mathtt{max}}$, where

$$\widetilde{\rho}_{\mathtt{max}} \,:=\, \sup\Big\{\,\rho : \quad \widetilde{K} \text{ is star-shaped with respect to a ball of radius } \rho\Big\}.$$

Recall here that $\widetilde{K}$ has diameter 1 and is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. It follows, using Lemmas 4.3.2 and 4.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$
\begin{aligned}
|q - \mathcal{P}_{k-1}^K(q)|_{\ell,K} \,&=\, h_K^{-\ell-1}\, |\widetilde{q} - \widetilde{\mathcal{P}_{k-1}^K(q)}|_{\ell,\widetilde{K}} \,=\, h_K^{-\ell-1}\, |\widetilde{q} - \mathcal{P}_{k-1}^{\widetilde{K}}(\widetilde{q})|_{\ell,\widetilde{K}} \\
&=\, h_K^{-\ell-1}\, \big|\big(\mathrm{I} - \mathcal{P}_{k-1}^{\widetilde{K}}\big)\big(\widetilde{q} - \widetilde{\mathbf{T}}^r(\widetilde{q})\big)\big|_{\ell,\widetilde{K}} \,\le\, h_K^{-\ell-1}\, \|\mathrm{I} - \mathcal{P}_{k-1}^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^\ell(\widetilde{K}))}\, \|\widetilde{q} - \widetilde{\mathbf{T}}^r(\widetilde{q})\|_{r,\widetilde{K}} \\
&\le\, C\, h_K^{-\ell-1}\, |\widetilde{q}|_{r,\widetilde{K}} \,=\, C\, h_K^{r-\ell}\, |q|_{r,K}\,,
\end{aligned}
$$

which finishes the proof. □

We now let

$$
\widetilde{H} \,:=\, \Big\{\mathbf{v} \in H : \quad \mathbf{v}|_K \in \mathbf{L}^s(K) \text{ (for some } s > 2) \text{ and } \operatorname{rot}\mathbf{v}|_K \in \mathrm{L}^1(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \tag{4.3.13}
$$

and introduce an interpolation operator $\Pi_k^h : \widetilde{H} \longrightarrow H_h$. Indeed, given $\mathbf{v} \in \widetilde{H}$, we let $\Pi_k^h(\mathbf{v})$ be the unique element in H_h such that

$$
\begin{aligned}
0 \,&=\, \int_e q\,\big(\mathbf{v} - \Pi_k^h(\mathbf{v})\big)\cdot\mathbf{n} \qquad &&\forall\, q \in \mathcal{B}_k(e) \qquad &&\forall \text{ edge } e \in \mathcal{T}_h\,, \\
0 \,&=\, \int_K \big(\mathbf{v} - \Pi_k^h(\mathbf{v})\big)\cdot\nabla q \qquad &&\forall\, q \in \mathcal{B}_{k-1}(K)\setminus\{1\} \qquad &&\forall\, K \in \mathcal{T}_h\,, \\
0 \,&=\, \int_K q \operatorname{rot}\big(\mathbf{v} - \Pi_k^h(\mathbf{v})\big) \qquad &&\forall\, q \in \mathcal{B}_{k-1}(K) \qquad &&\forall\, K \in \mathcal{T}_h\,.
\end{aligned} \tag{4.3.14}
$$

Note here that the extra local regularities on $\mathbf{v}$ and $\operatorname{rot}\mathbf{v}$ allow for defining normal traces of $\mathbf{v}$ on the edges of $\mathcal{T}_h$ and the moments involving $\operatorname{rot}\mathbf{v}$ in each $K \in \mathcal{T}_h$, respectively. In addition, the uniqueness of $\Pi_k^h(\mathbf{v})$ is guaranteed by Lemma 4.3.1. Next, we define the local restriction of the interpolation operator as $\Pi_k^K(\mathbf{v}) \,:=\, \Pi_k^h(\mathbf{v})|_K \,\in\, H_h^K$. It follows that for each $q \in \mathrm{P}_{k-1}(K)$ there holds

$$
\int_K \operatorname{div}\big(\mathbf{v} \,-\, \Pi_k^K(\mathbf{v})\big)\,q \,=\, -\int_K \big(\mathbf{v} - \Pi_k^K(\mathbf{v})\big)\cdot\nabla q \,+\, \int_{\partial K} q\,\big(\mathbf{v} - \Pi_k^K(\mathbf{v})\big)\cdot\mathbf{n} \,=\, 0\,,
$$

which, together with the fact that $\operatorname{div}\Pi_k^K(\mathbf{v}) \,\in\, \mathrm{P}_{k-1}(K)$, implies that

$$
\operatorname{div}\Pi_k^K(\mathbf{v}) \,=\, \mathcal{P}_{k-1}^K(\operatorname{div}\mathbf{v})\,. \tag{4.3.15}
$$

This identity implies the following result.

Lemma 4.3.5. Let k, ℓ and r be integers satisfying $1 \leq r \leq k$ and $0 \leq \ell \leq r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ and for any $\mathbf{v}$ verifying additionally that $\operatorname{div} \mathbf{v}|_K \in H^r(K)$ there holds

$$|\operatorname{div} \mathbf{v} - \operatorname{div} \Pi_k^K(\mathbf{v})|_{l,K} \leq C\, h_K^{r-l}\, |\operatorname{div} \mathbf{v}|_{r,K} . \tag{4.3.16}$$

Proof. It follows from a straightforward application of Lemma 4.3.4. □

We now consider $K \in \mathcal{T}_h$ and set the local moments defining H_h^K. Indeed, given $\mathbf{v}$ as required by (4.3.14), we define the K-moments:

$$\begin{aligned} m_{q,e}^{\mathbf{n}}(\mathbf{v}) \;&:=\; \int_e q\, \mathbf{v} \cdot \mathbf{n} && \forall\, q \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,, \\ m_{q,K}^{\operatorname{div}}(\mathbf{v}) \;&:=\; \int_K \mathbf{v} \cdot \nabla q && \forall\, q \in \mathcal{B}_{k-1}(K) \setminus \{1\} \\ m_{q,K}^{\operatorname{rot}}(\mathbf{v}) \;&:=\; \int_K q \operatorname{rot} \mathbf{v} && \forall\, q \in \mathcal{B}_{k-1}(K)\,, \end{aligned} \tag{4.3.17}$$

and gather all of the above in the set $\Big\{ m_{j,K}(\mathbf{v}) \Big\}_{j=1}^{n_k^K}$. Then, we let $\{\varphi_{j,K}\}_{j=1}^{n_k^K}$ be the canonical basis of H_h^K, that is, given $i \in \{1, 2, \ldots, n_k^K\}$, $\varphi_{i,K}$ is the unique element in H_h^K such that

$$m_{j,K}(\varphi_{i,K}) \;=\; \delta_{ij} \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,.$$

It follows easily that

$$\Pi_k^K(\mathbf{v}) \;:=\; \sum_{j=1}^{n_k^K} m_{j,K}(\mathbf{v})\, \varphi_{j,K}\,, \tag{4.3.18}$$

or, equivalently, $\Pi_k^K(\mathbf{v})$ is the unique element in H_h^K such that

$$m_{j,K}(\Pi_k^K(\mathbf{v})) \;=\; m_{j,K}(\mathbf{v}) \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,. \tag{4.3.19}$$

We now provide the analogue of Lemmas 4.3.2 and 4.3.4 for the foregoing local operator Π_k^K.

Lemma 4.3.6. Given integers k, $\ell \geq 1$, and $K \in \mathcal{T}_h$, there holds $\widetilde{\Pi_k^K(\mathbf{v})} \;=\; \Pi_k^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v} \in \mathbf{H}^k(K)$, and $\Pi_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbf{H}^\ell(\widetilde{K}), \mathbf{L}^2(\widetilde{K})\big)$ with $\|\Pi_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbf{H}^\ell(\widetilde{K}), \mathbf{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$, namely depending only on k, ℓ, $N_{\mathcal{T}}$, and $C_{\mathcal{T}}$.

Proof. We first note that clearly $\varphi \in H_h^K$ if and only if $\widetilde{\varphi} := \varphi \circ T_K^{-1} \in H_h^{\widetilde{K}}$. In particular, given $\mathbf{v} \in \mathbf{H}^k(K)$, there holds $\widetilde{\Pi_k^K(\mathbf{v})} \in H_h^{\widetilde{K}}$, and hence the required identity holds if and only if the $\widetilde{K}$-moments of $\widetilde{\Pi_k^K(\mathbf{v})}$ and $\Pi_k^{\widetilde{K}}(\widetilde{\mathbf{v}})$ coincide. Indeed, let $\widetilde{e}$ be an edge of $\partial\widetilde{K}$, and let $e := T_K^{-1}(\widetilde{e})$ be the corresponding edge of ∂K. Then, for each $q \in \mathcal{B}_k(e)$, and letting $\widetilde{q} := q \circ T_K^{-1} \in \mathrm{P}_k(\widetilde{e})$, we find, integrating by parts and using (4.3.19), that

$$\begin{aligned}
\int_{\widetilde{e}} q\, \widetilde{\Pi_k^K(\mathbf{v})} \cdot \widetilde{\mathbf{n}} \,&=\, \int_{\widetilde{K}} \widetilde{q}\, \mathrm{div}\, \widetilde{\Pi_k^K(\mathbf{v})} \,+\, \int_{\widetilde{K}} \widetilde{\Pi_k^K(\mathbf{v})} \cdot \nabla\widetilde{q} \\
&=\, h_K^{-3} \int_K q\, \mathrm{div}\, \Pi_k^K(\mathbf{v}) \,+\, h_K^{-3} \int_K \Pi_k^K(\mathbf{v}) : \nabla q \,=\, h_K^{-3} \int_e q\, \Pi_k^K(\mathbf{v}) \cdot \mathbf{n} \\
&=\, h_K^{-3} \int_e q\, \mathbf{v} \cdot \mathbf{n} \,=\, h_K^{-3} \int_K q\, \mathrm{div}\, \mathbf{v} \,+\, h_K^{-3} \int_K \mathbf{v} \cdot \nabla q \\
&=\, \int_K \widetilde{q}\, \mathrm{div}\, \widetilde{\mathbf{v}} \,+\, \int_K \widetilde{\mathbf{v}} \cdot \nabla\widetilde{q} \,=\, \int_{\widetilde{e}} \widetilde{q}\, \widetilde{\mathbf{v}} \cdot \widetilde{\mathbf{n}}\,.
\end{aligned} \tag{4.3.20}$$

Next, if $k \geq 2$ and $q \in \mathcal{B}_{k-1}(K)$, we easily obtain, using again (4.3.19), that

$$\int_{\widetilde{K}} \widetilde{\Pi_k^K(\mathbf{v})} \cdot \nabla\widetilde{q} \,=\, h_K^{-3} \int_K \Pi_k^K(\mathbf{v}) \cdot \nabla q \,=\, h_K^{-3} \int_K \mathbf{v} \cdot \nabla q \,=\, \int_{\widetilde{K}} \widetilde{\mathbf{v}} \cdot \nabla\widetilde{q}\,, \tag{4.3.21}$$

and

$$\int_{\widetilde{K}} \widetilde{q}\, \mathrm{rot}\, \widetilde{\Pi_k^K(\mathbf{v})} \,=\, h_K^{-3} \int_K q\, \mathrm{rot}\, \Pi_k^K(\mathbf{v}) \,=\, h_K^{-3} \int_K q\, \mathrm{rot}\, \mathbf{v} \,=\, \int_{\widetilde{K}} \widetilde{q}\, \mathrm{rot}\, \widetilde{\mathbf{v}}\,. \tag{4.3.22}$$

In this way, (4.3.17) together with (4.3.20), (4.3.21), and (4.3.22) confirm that $\widetilde{\Pi_k^K(\mathbf{v})}$ and $\Pi_k^{\widetilde{K}}(\widetilde{\mathbf{v}})$ share the same $\widetilde{K}$-moments.

It remains to show that $\Pi_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbf{H}^\ell(\widetilde{K}), \mathbf{L}^2(\widetilde{K})\big)$, with $\|\Pi_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbf{H}^\ell(\widetilde{K}), \mathbf{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$. For this purpose, we first observe from (4.3.18) that

$$\|\Pi_k^{\widetilde{K}}(\mathbf{v})\|_{0,\widetilde{K}} \,\leq\, \sum_{j=1}^{n_k^{\widetilde{K}}} |m_{j,\widetilde{K}}(\mathbf{v})|\, \|\varphi_{j,\widetilde{K}}\|_{0,\widetilde{K}}\,,$$

where each $m_{j,\widetilde{K}}$ is defined according to (4.3.17) with $\mathcal{B}_{k-1}(\widetilde{K})$ and $B_k(\widetilde{e})$ $\forall$ edge $\widetilde{e} \subseteq \partial\widetilde{K}$, and $\left\{\varphi_{j,\widetilde{K}}\right\}_{j=1}^{n_k^{\widetilde{K}}}$ is the canonical basis of $H_h^{\widetilde{K}}$. Next, we proceed to bound the $\widetilde{K}$-moments in terms of $\|\mathbf{v}\|_{\ell,\widetilde{K}}$. In fact, given an edge $\widetilde{e} \subseteq \partial\widetilde{K}$, and $q \in \mathcal{B}_k(\widetilde{e})$, a simple computation shows that $\|q\|_{0,\widetilde{e}} \,\leq\, h_{\widetilde{e}}^{1/2} \,\leq\, h_{\widetilde{K}}^{1/2} \,=\, 1$, and then, applying the Cauchy-Schwarz and discrete trace inequalities, and using that $h_{\widetilde{e}} \geq C_{\mathcal{T}}$, we obtain

$$|m^{\mathbf{n}}_{q,\widetilde{e}}(\mathbf{v})| \,\leq\, \|\mathbf{v}\|_{0,\widetilde{e}}\,\|q\|_{0,\widetilde{e}} \,\leq\, c\left\{C_{\mathcal{T}}^{-1/2}\,\|\mathbf{v}\|_{0,\widetilde{K}} \,+\, |\mathbf{v}|_{1,\widetilde{K}}\right\} \,\leq\, C\,\|\mathbf{v}\|_{1,\widetilde{K}} \,\leq\, C\,\|\mathbf{v}\|_{\ell,\widetilde{K}}\,.$$

In turn, given now $q \in \mathcal{B}_{k-1}(\widetilde{K})$, it is easy to see that $\|q\|_{0,\widetilde{K}}$ and $|q|_{1,\widetilde{K}}$ are both bounded by constants independent of $\widetilde{K}$, and hence straightforward applications of the Cauchy-Schwarz inequality yield

$$|m^{\mathrm{div}}_{q,\widetilde{K}}(\mathbf{v})| \,+\, |m^{\mathrm{rot}}_{q,\widetilde{K}}(\mathbf{v})| \,\leq\, C\left\{\|\mathbf{v}\|_{0,\widetilde{K}} \,+\, |\mathbf{v}|_{1,\widetilde{K}}\right\} \,\leq\, C\,\|\mathbf{v}\|_{\ell,\widetilde{K}}\,,$$

Finally, we claim that, thanks to the assumptions a) and b) (cf. beginning of Section 4.3.2) and the choice of the normalized monomials given by $\mathcal{B}_\ell(e)$ and $\mathcal{B}_\ell(K)$ (cf. (4.3.4), (4.3.5)), there holds $\|\varphi_{j,\widetilde{K}}\|_{0,\widetilde{K}} \,=\, O(1)$, which would complete the boundedness of $\Pi^{\widetilde{K}}_k$. The specific technical details, however, will be given somewhere else. □

Lemma 4.3.7. Let k and r be integers such that $k \geq 1$ and $1 \leq r \leq k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\mathbf{v} \,-\, \Pi^K_k(\mathbf{v})\|_{0,K} \,\leq\, C\,h_K^r\,|\mathbf{v}|_{r,K} \qquad \forall\,\mathbf{v} \in \mathbf{H}^r(K)\,. \tag{4.3.23}$$

Proof. We proceed similarly as in the proof of Lemma 4.3.4. In fact, given integers k and r as stated, $K \in \mathcal{T}_h$, and $\mathbf{v} \in \mathbf{H}^r(K)$, we let $\widetilde{\mathbf{T}}^r(\widetilde{\mathbf{v}}) \,\in\, \mathbf{P}_{r-1}(\widetilde{K})$ be the vector version of the averaged Taylor polynomial of order r of $\widetilde{\mathbf{v}}$ (cf. Lemma 4.3.3), and observe, since $r-1 \leq k$, that $\Pi^{\widetilde{K}}_k\big(\widetilde{\mathbf{T}}^r(\widetilde{\mathbf{v}})\big) \,=\, \widetilde{\mathbf{T}}^r(\widetilde{\mathbf{v}})$. It follows, using Lemmas 4.3.6 and 4.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
\|\mathbf{v} - \Pi^K_k(\mathbf{v})\|_{0,K} \,&=\, h_K^{-1}\,\|\widetilde{\mathbf{v}} - \widetilde{\Pi^K_k(\mathbf{v})}\|_{0,\widetilde{K}} \,=\, h_K^{-1}\,\|\widetilde{\mathbf{v}} - \Pi^{\widetilde{K}}_k(\widetilde{\mathbf{v}})\|_{0,\widetilde{K}} \\
&=\, h_K^{-1}\,\|\big(\mathrm{I} - \Pi^{\widetilde{K}}_k\big)\big(\widetilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\widetilde{\mathbf{v}})\big)\|_{0,\widetilde{K}} \,\leq\, h_K^{-1}\,\|\mathrm{I} - \Pi^{\widetilde{K}}_k\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{L}^2(\widetilde{K}))}\,\|\widetilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\widetilde{\mathbf{v}})\|_{r,\widetilde{K}} \\
&\leq\, C\,h_K^{-1}\,|\widetilde{\mathbf{v}}|_{r,\widetilde{K}} \,=\, C\,h_K^r\,|\mathbf{v}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. □

As a corollary of Lemmas 4.3.5 and 4.3.7 we have the following result.

Lemma 4.3.8. Let k and r be integers such that $1 \leq r \leq k$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\mathbf{v} - \Pi_k^K(\mathbf{v})\|_{\mathrm{div};K} \,\leq\, C\, h_K^r \Big\{ |\mathbf{v}|_{r,K} + |\mathrm{div}\,\mathbf{v}|_{r,K} \Big\} \quad \forall\, \mathbf{v} \in \mathbf{H}^r(K) \quad \text{with} \quad \mathrm{div}\,\mathbf{v} \in H^r(K)\,.$$

Proof. It suffices to apply (4.3.16) with $\ell = 0$ and then combine it with the estimate provided by Lemma 4.3.7. □

4.3.4 The discrete bilinear forms

The ultimate purpose of this section is to define computable discrete versions $a_h : H_h \times H_h \longrightarrow \mathrm{R}$ and $b_h : H_h \times Q_h \longrightarrow \mathrm{R}$ of the bilinear forms a and b, respectively. To this end, we first observe that, given $(\mathbf{v}, q) \in H_h \times Q_h$, the expression

$$b\,(\mathbf{v}, q) \,:=\, \int_\Omega q\,\mathrm{div}\,\mathbf{v} \,=\, \sum_{K\in\mathcal{T}_h} \int_K q\,\mathrm{div}\,\mathbf{v}\,,$$

is explicitly calculable since, according to the definitions of H_h and Q_h (cf. (4.3.1), (4.3.2)), there holds $q|_K \in \mathrm{P}_{k-1}\,(K)$ and $\mathrm{div}\,\mathbf{v}|_K \in \mathrm{P}_{k-1}(K)$ on each element K, and hence we just set $b_h = b$. On the contrary, given $\mathbf{u}$, $\mathbf{v} \in H_h$, the expression

$$\mathbf{a}(\mathbf{u}, \mathbf{v}) \,:=\, \int_\Omega \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v} \,=\, \sum_{K\in\mathcal{T}_h} \int_K \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v}$$

is not explicitly calculable since in general $\mathbf{u}$ and $\mathbf{v}$ are not known on each $K \in \mathcal{T}_h$. In order to overcome this difficulty, we now proceed to introduce suitable spaces on which the elements of H_h will be projected later on, and for which the bilinear form a is computable. Indeed, let us first consider a particular choice of $\mathbf{u}$ given by $\mathbf{u} := \mathbb{K}\nabla q \in \mathbf{P}_k(K)$ with $q \in \mathrm{P}_{k+1}(K)$. It follows that for each $\mathbf{v} \in H_h$ there holds

$$\int_K \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v} \,=\, \int_K \nabla q\cdot\mathbf{v} \,=\, -\int_K q\,\mathrm{div}\,\mathbf{v} \,+\, \int_{\partial K} q\,\mathbf{v}\cdot\mathbf{n}\,, \tag{4.3.24}$$

which, bearing in mind from Lemma 4.3.1 that $\mathrm{div}\,\mathbf{v}|_K$ and $\mathbf{v}\cdot\mathbf{n}|_{\partial K}$ are explicitly known, shows that $a(\mathbf{u}, \mathbf{v})$ is in fact calculable in this case. The above suggests to define the subspace of $\mathbf{P}_k(K)$ given by

$$\widehat{H}_k^K \,:=\, \Big\{ \nabla q : \quad q \in \mathrm{P}_{k+1}(K) \Big\}.$$

The following lemma establishes a basic property of the above space.

Lemma 4.3.1. *There holds* $\dim \widehat{H}_k^K = \dfrac{(k+2)(k+3)}{2} - 1$.

Proof. We just need to prove that the set $\Big\{\nabla \mathbf{x}^\alpha : \quad 1 \le |\alpha| \le k+1\Big\}$ is a basis of $\widehat{H}_{k,\nabla}^K$. Indeed, the generation property is quite clear from the fact that $\Big\{\mathbf{x}^\alpha : \quad 0 \le |\alpha| \le k+1\Big\}$ is the canonical basis of $\mathrm{P}_{k+1}(K)$ and by observing that $\nabla q = \mathbf{0} \ \ \forall\, q \in \mathrm{P}_0(K)$. Next, we consider scalars $a_\alpha,\ 1 \le |\alpha| \le k+1$, and set $\nabla q = \mathbf{0}$ with $q := \displaystyle\sum_{1\le|\alpha|\le k+1} a_\alpha\, \mathbf{x}^\alpha$. It follows that q is a constant function from R into itself, that is

$$q \,=\, \sum_{1\le|\alpha|\le k+1} a_\alpha\, \mathbf{x}^\alpha \,=\, \text{constant} \quad \text{in} \quad K\,,$$

which yields $a_\alpha \,=\, 0$ for all $1 \le |\alpha| \le k+1$. Therefore, $q = 0$. Having thus identified a basis of $\widehat{H}_k^K$, whose cardinality is certainly given by $\dim \mathrm{P}_{k+1}(K) - \dim \mathrm{P}_0(K)$, we conclude that

$$\dim \widehat{H}_k^K \,=\, \frac{(k+2)(k+3)}{2} \,-\, 1\,,$$

which completes the proof. □

We now introduce a projection operator $\widehat{\Pi}_k^K : \mathbf{H}(\mathrm{div}\,; K) \longrightarrow \widehat{H}_k^K$. To this end, we set for each $K \in \mathcal{T}_h$ the local bilinear form

$$a^K(\mathbf{u},\mathbf{v}) \,:=\, \int_K \mathbb{K}^{-1}\mathbf{u}\cdot\mathbf{v} \qquad \forall\,\mathbf{u},\,\mathbf{v} \in \mathbf{L}^2(K)\,.$$

Then, we define $\hat{\mathbf{v}} \,:=\, \widehat{\Pi}_k^K(\mathbf{v}) \in \widehat{H}_k^K$ as the solution of the problem: Find $\hat{\mathbf{v}} \in \widehat{H}_k^K$ such that

$$a^K(\hat{\mathbf{v}},\mathbf{q}) = a^K(\mathbf{v},\mathbf{q}) \quad \forall\,\mathbf{q} \,\in\, \widehat{H}_k^K. \tag{4.3.25}$$

We remark that the unique solvability of (4.3.25) is guaranteed by the identity

$$a^K(\mathbf{v},\mathbf{v}) \,\ge\, C\,\|\mathbf{v}\|_{0,K}^2 > 0 \quad \forall \mathbf{v} \in \widehat{H}_k^K\,,$$

where the fact that $\mathbb{K}$ is a positive definite matrix was used. Furthermore, we have already noticed at the beginning of Section 4.3.4 that the right hand side of (4.3.25) is explicitly calculable when $\mathbf{v}$ belongs to $H_h^K \subseteq H(\mathrm{div}\,; K)$ (cf. (4.3.8)). Finally, it is straightforward to check from (4.3.25) that $\widehat{\Pi}_k^K(\mathbf{v}) = \mathbf{v} \quad \forall\, \mathbf{v} \in \widehat{H}_k^K$, which confirms that $\widehat{\Pi}_k^K$ is in fact a projector. Moreover, the following result establishes the uniform boundedness of the family $\left\{\widehat{\Pi}_k^K\right\}_{K\in\mathcal{T}_h} \subseteq \left\{\mathcal{L}(\mathbf{L}^2(K),\mathbf{L}^2(K))\right\}_{K\in\mathcal{T}_h}$.

Lemma 4.3.2. *There exists a constant $C>0$, depending only on $\mathbb{K}$, such that for each $K\in\mathcal{T}_h$ there holds*

$$\|\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K} \,\le\, C\,\|\mathbf{v}\|_{0,K} \qquad \forall\,\mathbf{v}\,\in\,\mathbf{L}^2(K)\,. \tag{4.3.26}$$

Proof. Given that $\mathbb{K}$ is a positive definite matrix, there exists $C>0$, depending only on $\mathbb{K}$, such that $\|\mathbf{v}\|_{0,K}\le C a^K(\mathbf{v},\mathbf{v}) \quad \forall\,\mathbf{v}\,\in\mathbf{L}^2(K)$. Then,

$$\|\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K}^2 \le C a^K\left(\widehat{\Pi}_k^K(\mathbf{v}),\widehat{\Pi}_k^K(\mathbf{v})\right) \le C a^K\left(\mathbf{v},\widehat{\Pi}_k^K(\mathbf{v})\right) \le C\|\mathbb{K}^{-1}\|\|\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K}\|\mathbf{v}\|_{0,K},$$

which leads directly to (4.3.26). □

The analogue of Lemma 4.3.6 is provided next.

Lemma 4.3.9. Given integers $k,\,\ell\ge 1$, and $K\in\mathcal{T}_h$, there holds $\widetilde{\widehat{\Pi}_k^K(\mathbf{v})} \,=\, \widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v}\in\mathbf{H}^k(K)$, and $\widehat{\Pi}_k^{\widetilde{K}}\in\mathcal{L}\big(\mathbf{H}^\ell(\widetilde{K}),\mathbf{L}^2(\widetilde{K})\big)$ with $\|\widehat{\Pi}_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbf{H}^\ell(\widetilde{K}),\mathbf{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$, namely depending only on k, $\widehat{\Delta}$, $\mathbb{K}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$.

Proof. Similarly as for Lemma 4.3.6, we first observe that $\mathbf{v}\in\widehat{H}_k^K$ if and only $\widetilde{\mathbf{v}} := \mathbf{v}\circ T_K^{-1}\,\in\,\widehat{H}_k^{\widetilde{K}}$. In particular, given $\mathbf{v}\in\mathbf{L}^2(K)$, there holds $\widetilde{\widehat{\Pi}_k^K(\mathbf{v})}\in\widehat{H}_k^{\widetilde{K}}$, and hence, in order to obtain the required identity, it suffices to show that $\widetilde{\widehat{\Pi}_k^K(\mathbf{v})}$ solves the same problem as $\widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\mathbf{v}})$, namely (4.3.25) with $K=\widetilde{K}$ and $\mathbf{v}=\widetilde{\mathbf{v}}$. In fact, we find, according to (4.3.25), that for each $\mathbf{q}\in\widehat{H}_k^K$ there holds

$$a^{\widetilde{K}}(\widetilde{\widehat{\Pi}_k^K(\mathbf{v})},\widetilde{\mathbf{q}}) \,=\, h_K^{-2}\,\mathbf{a}^K(\widehat{\Pi}_k^K(\mathbf{v}),\mathbf{q}) \,=\, h_K^{-2}\,\mathbf{a}^K(\mathbf{v},\mathbf{q}) \,=\, \mathbf{a}^{\widetilde{K}}(\widetilde{\mathbf{v}},\widetilde{\mathbf{q}})\,, \tag{4.3.27}$$

which confirms that $\widetilde{\widehat{\Pi}_k^K(\mathbf{v})}$ does solve the announced problem. Finally, since $h_{\widetilde{K}}=1$, a direct application of Lemma 4.3.2 implies the existence of a constant $C>0$, independent of $\widetilde{K}$, such that

$$\|\widehat{\Pi}_k^{\widetilde{K}}(\mathbf{v})\|_{0,\widetilde{K}} \,\le\, C\,\|\mathbf{v}\|_{0,\widetilde{K}} \,\le\, C\,\|\mathbf{v}\|_{\ell,\widetilde{K}} \qquad \forall\,\mathbf{v}\in\mathbf{H}^\ell(\widetilde{K})\,,$$

which completes the proof. $\square$

Before establishing the next result, we now recall that if $(\mathbf{u}, p) \in H \times Q$ is the solution of the continuous problem (4.2.3), then there holds $\mathbf{u} = \mathbb{K} \nabla p \in \mathbf{L}^2(\Omega)$, which motivates for each integer $r \geq 0$ the introduction of the space

$$\mathbf{H}^r_\nabla(K) := \Big\{ \mathbf{v} \in \mathbf{H}^r(K) : \quad \mathbf{v} = \mathbb{K} \nabla w \quad \text{for some} \quad w \in H^{r+1}(K) \Big\} .$$

Then, we have the following projection error for $\widehat{\Pi}^K_k$, which constitutes the analogue of Lemma 4.3.7.

Lemma 4.3.10. Let k and r be integers such that $k \geq 1$ and $1 \leq r \leq k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $\widehat{\Delta}$, $\mathbb{K}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\mathbf{v} - \widehat{\Pi}^K_k(\mathbf{v})\|_{0,K} \leq C\, h^r_K \, |\mathbf{v}|_{r,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^r_\nabla(K) .$$

Proof. We proceed similarly as in the proof of Lemma 4.3.7. In fact, given integers k and r as stated, $K \in \mathcal{T}_h$, and $\mathbf{v} \in \mathbf{H}^r_\nabla(K)$, we let $w \in \mathrm{H}^{r+1}(K)$ such that $\mathbf{v} = \mathbb{K} \nabla w$, set $\widetilde{w} \in \mathrm{H}^{r+1}(\widetilde{K})$ such that $\widetilde{\mathbf{v}} = \mathbb{K} \nabla \widetilde{w}$, denote by $\mathrm{T}^{r+1}(\widetilde{w}) \in \mathrm{P}_r(\widetilde{K})$ the averaged Taylor polynomial of order $r+1$ of $\widetilde{w}$ (cf. Lemma 4.3.3), and observe, since $r \leq k+1$, that

$$\widehat{\Pi}^{\widetilde{K}}_k(\mathbb{K} \nabla \mathrm{T}^{r+1}(\widetilde{w})) = \mathbb{K} \nabla \mathrm{T}^{r+1}(\widetilde{w}) .$$

It follows, using Lemmas 4.3.9 and 4.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
\|\mathbf{v} - \widehat{\Pi}^K_k(\mathbf{v})\|_{0,K} &= h^{-1}_K \, \|\widetilde{\mathbf{v}} - \widetilde{\widehat{\Pi}^K_k(\mathbf{v})}\|_{0,\widetilde{K}} = h^{-1}_K \, \|\widetilde{\mathbf{v}} - \widehat{\Pi}^{\widetilde{K}}_k(\widetilde{\mathbf{v}})\|_{0,\widetilde{K}} \\
&= h^{-1}_K \, \Big\| \big(\mathrm{I} - \widehat{\Pi}^{\widetilde{K}}_k\big) \big(\mathbf{v} - \mathbb{K} \nabla \mathrm{T}^{r+1}(\widetilde{w})\big) \Big\|_{0,\widetilde{K}} \\
&\leq h^{-1}_K \, \|\mathrm{I} - \widehat{\Pi}^{\widetilde{K}}_k\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}), \mathbf{L}^2(\widetilde{K}))} \, \|\mathbf{v} - \mathbb{K} \nabla \mathrm{T}^{r+1}(\widetilde{w})\|_{r,\widetilde{K}} \\
&\leq C\, h^{-1}_K \, \|\widetilde{w} - \mathrm{T}^{r+1}(\widetilde{w})\|_{r+1,\widetilde{K}} \leq C\, h^{-1}_K \, |\widetilde{w}|_{r+1,\widetilde{K}} \leq C\, h^{-1}_K \, |\widetilde{\mathbf{v}}|_{r,\widetilde{K}} = C\, h^r_K \, |\mathbf{v}|_{r,K} ,
\end{aligned}$$

which finishes the proof. $\square$

We now let $a^K_h : H^K_h \times H^K_h \longrightarrow \mathrm{R}$ be the local discrete bilinear form given by

$$a^K_h(\mathbf{u}, \mathbf{v}) := a^K\big(\widehat{\Pi}^K_k(\mathbf{u}), \widehat{\Pi}^K_k(\mathbf{v})\big) + \mathcal{S}^K\big(\mathbf{u} - \widehat{\Pi}^K_k(\mathbf{u}), \mathbf{v} - \widehat{\Pi}^K_k(\mathbf{v})\big) \qquad \forall\, \mathbf{u}, \mathbf{v} \in H^K_h , \tag{4.3.28}$$

where $\mathcal{S}^K : H_h^K \times H_h^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_k^K \times n_k^K}$ with respect to the basis $\{\varphi_{j,K}\}_{j=1}^{n_k^K}$ of H_k^K (cf. (4.3.17) - (4.3.18)), that is

$$\mathcal{S}^K(\mathbf{u},\mathbf{v}) := \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{u})\, m_{i,K}(\mathbf{v}) \qquad \forall\, \mathbf{u},\, \mathbf{v} \in H_h^K . \tag{4.3.29}$$

Next, as suggested by (4.3.28), we define the global discrete bilinear form $a_h : H_h \times H_h \longrightarrow \mathrm{R}$

$$a_h(\mathbf{u},\mathbf{v}) := \sum_{K\in\mathcal{T}_h} a_h^K(\mathbf{u},\mathbf{v}) \qquad \forall\, \mathbf{u},\, \mathbf{v} \in H_h . \tag{4.3.30}$$

Lemma 4.3.3. *There exist c_0, $c_1 > 0$, depending only on $C_{\mathcal{T}}$ and $\mathbb{K}$, such that*

$$c_0\, \|\mathbf{v}\|_{0,K}^2 \le \mathcal{S}^K(\mathbf{v},\mathbf{v}) \le c_1\, \|\mathbf{v}\|_{0,K}^2 \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \mathbf{v} \in H_h^K . \tag{4.3.31}$$

Proof. Since the corresponding proof is only sketched in [9, Remark 5.1], for the sake of completeness we provide in what follows further details on the derivation of the upper bound of (4.3.31). In fact, given $\mathbf{v} \in H_h^K$, we first notice from (4.3.29) that

$$\mathcal{S}^K(\mathbf{v},\mathbf{v}) = \sum_{j=1}^{n_k^K} m_{j,K}^2(\mathbf{v}) ,$$

and hence it suffices to estimate each one of the moments $m_{j,K}(\mathbf{v})$ (cf. (4.3.17)) in terms of $\|\mathbf{v}\|_{0,K}$. For this purpose, we employ the same partition and corresponding notations introduced in the proof of Lemma 4.3.2. We begin with $m_{q,e}^{\mathbf{n}}(\mathbf{v})$, where $e \subseteq \partial K$ is an edge of a triangle Δ_i and $q \in \mathcal{B}_k(e)$, by observing, thanks to the Cauchy-Schwarz and polynomial trace inequalities, that

$$|m_{q,e}^{\mathbf{n}}(\mathbf{v})| \le \|\mathbf{v}\|_{0,e}\, \|q\|_{0,e} \le C\, h_i^{-1/2}\, \|\mathbf{v}\|_{0,K}\, |e|^{1/2} \le C\, C_{\mathcal{T}}^{-1/2}\, h_K^{-1/2}\, \|\mathbf{v}\|_{0,K}\, h_e^{1/2} \le C_1\, \|\mathbf{v}\|_{0,K} .$$

In turn, given $q \in \mathcal{B}_{k-1}(K)$, we apply the inverse inequality on each triangle Δ_i, and then use that $|K| \le c\, h_K^2$, to find that

$$|m_{q,K}^{\mathrm{div}}(\mathbf{v})| \le \|\mathbf{v}\|_{0,K}\, |q|_{1,K} \le C\, C_{\mathcal{T}}^{-1}\, h_K^{-1}\, \|\mathbf{v}\|_{0,K}\, \|q\|_{0,K} \le C\, C_{\mathcal{T}}^{-1}\, h_K^{-1}\, \|\mathbf{v}\|_{0,K}\, |K|^{1/2} \le C_2\, \|\mathbf{v}\|_{0,K} .$$

Finally, given again $q \in \mathcal{B}_{k-1}(K)$, we integrate by parts in K to obtain

$$m^{\mathrm{rot}}_{q,K}(\mathbf{v}) \,=\, \int_K \mathbf{v}\cdot \underline{\mathrm{curl}}\, q \,-\, \int_{\partial K} \mathbf{v}\cdot \mathbf{t}\, q\,,$$

where $\mathbf{t}$ is the unit tangential vector along ∂K. In this way, employing basically the same arguments of the foregoing inequalities, we deduce that

$$|m^{\mathrm{rot}}_{q,K}(\mathbf{v})| \,\leq\, \|\mathbf{v}\|_{0,K}\,|q|_{1,K} \,+\, \sum_{e\subseteq\partial K} \|\mathbf{v}\|_{0,e}\,\|q\|_{0,e} \,\leq\, C_3\,\|\mathbf{v}\|_{0,K}\,,$$

where $C_3 \,=\, C_2 + N_{\mathcal{T}}\, C_1$, and C_1 and C_2 are positive constants depending only on $C_{\mathcal{T}}$. $\square$

The following result is a consequence of the properties of the projector $\widehat{\Pi}^K_k$ and the previous lemma.

Lemma 4.3.4. *For each $K \in \mathcal{T}_h$ there holds*

$$a^K_h(\mathbf{u},\mathbf{v}) \,=\, a^K(\mathbf{u},\mathbf{v}) \qquad \forall\,\mathbf{u}\in \widehat{H}^K_k, \quad \forall\,\mathbf{v}\in H^K_h\,, \tag{4.3.32}$$

and there exist positive constants α_1, α_2, independent of h and K, such that

$$|a^K_h(\mathbf{u},\mathbf{v})| \leq \alpha_1 \Big\{ \|\mathbf{u}\|_{0,K}\,\|\mathbf{v}\|_{0,K} + \|\mathbf{u}-\widehat{\Pi}^K_k(\mathbf{u})\|_{0,K}\,\|\mathbf{v}-\widehat{\Pi}^K_k(\mathbf{v})\|_{0,K} \Big\} \quad \forall\, K\in\mathcal{T}_h, \quad \forall\,\mathbf{u},\,\mathbf{v}\in H^K_h\,, \tag{4.3.33}$$

and

$$\alpha_2\,\|\mathbf{v}\|^2_{0,K} \,\leq\, a^K_h(\mathbf{v},\mathbf{v}) \,\leq\, \alpha_1 \Big\{ \|\mathbf{v}\|^2_{0,K} + \|\mathbf{v}-\widehat{\Pi}^K_k(\mathbf{v})\|^2_{0,K} \Big\} \quad \forall\, K\in\mathcal{T}_h, \quad \forall\,\mathbf{v}\in H^K_h\,. \tag{4.3.34}$$

Proof. Note first that a^K is a symmetric bilinear form since $\mathbb{K}$ is symmetric. Then, given $\mathbf{u}\in \widehat{H}^K_k$ and $\mathbf{v}\in H^K_k$, and bearing in mind problem 4.3.25, there holds

$$a^K_h(\mathbf{u},\mathbf{v}) \,=\, a^K(\widehat{\Pi}^K_k(\mathbf{u}),\widehat{\Pi}^K_k(\mathbf{v})) \,=\, a^K(\mathbf{u},\widehat{\Pi}^K_k(\mathbf{v})) \,=\, a^K(\widehat{\Pi}^K_k(\mathbf{v}),\mathbf{u}) = a^K(\mathbf{v},\mathbf{u}) \,=\, a^K(\mathbf{u},\mathbf{v})$$

which proves (4.3.32). Next, for the boundedness of a^K_h we apply the Cauchy-Schwarz inequality, the boundedness of $\widehat{\Pi}^K_k$, and the upper bound in (4.3.31) (cf. Lemma 4.3.3), to obtain

$$
\begin{aligned}
|a_h^K(\mathbf{u},\mathbf{v})| \;&\leq\; \|\widehat{\Pi}_k^K(\mathbf{u})\|_{0,K}\|\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K} \\
&+ \Big\{\mathcal{S}^K\big(\mathbf{u}-\widehat{\Pi}_k^K(\mathbf{u}),\mathbf{u}-\widehat{\Pi}_k^K(\mathbf{u})\big)\Big\}^{1/2}\,\Big\{S^K\big(\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v}),\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v})\big)\Big\}^{1/2} \\
&\leq\; \widehat{C}^2\|\mathbf{u}\|_{0,K}\,\|\mathbf{v}\|_{0,K} \,+\, c_1\,\|\mathbf{u}-\widehat{\Pi}_k^K(\mathbf{u})\|_{0,K}\;\|\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K} \qquad \forall\,\mathbf{u},\,\mathbf{v}\in H_h^K\,,
\end{aligned}
$$

which gives (4.3.33) with $\alpha_1 \;:=\; \max\{\widehat{C}^2, c_1\}$. Finally, concerning (4.3.34), it is clear that the corresponding upper bound follows from (4.3.33). In turn, applying the lower estimate in (4.3.31) (cf. Lemma 4.3.3) we find that

$$
\begin{aligned}
\|\mathbf{v}\|_{0,K}^2 \;&\leq\; 2\Big\{\|\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K}^2 \,+\, \|\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v})\|_{0,K}^2\Big\} \\
&\leq\; 2\,a^K\big(\widehat{\Pi}_k^K(\mathbf{v}),\widehat{\Pi}_k^K(\mathbf{v})\big) \,+\, 2\,\|\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}^2 \\
&\leq\; 2\,a^K\big(\widehat{\Pi}_k^K(\mathbf{v}),\widehat{\Pi}_k^K(\mathbf{v})\big) \,+\, \frac{2}{c_0}\,\mathcal{S}^K\big(\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v}),\mathbf{v}-\widehat{\Pi}_k^K(\mathbf{v})\big)\,,
\end{aligned}
$$

which yields the lower bound in (4.3.34) with $\alpha_2 \;:=\; 2\,\max\{1,\frac{1}{c_0}\}^{-1}$. □

4.3.5 The mixed virtual element scheme

According to the analysis from the foregoing section, we reformulate the Galerkin scheme associated with (4.2.4) as: Find $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ such that

$$
\begin{aligned}
a_h\,(\mathbf{u}_h,\mathbf{v}_h) \,+\, b\,(\mathbf{v}_h,p_h) \;&=\; 0 \qquad\qquad \forall\,\mathbf{v}_h\in H_h\,, \\
b\,(\mathbf{u}_h,q_h) \;&=\; -\int_\Omega f\,q_h \qquad \forall\,q_h\in Q_h\,.
\end{aligned}
\tag{4.3.35}
$$

In what follows we establish the well-posedness of (4.3.35). We begin with ellipticity of a_h in the discrete kernel of b.

Lemma 4.3.5. *Let* $V_h \;:=\; \Big\{\mathbf{v}_h\in H_h:\quad b(\mathbf{v}_h,q_h) \;=\; 0 \quad \forall\,q_h\in Q_h\Big\}$. *Then, there exists* $\alpha > 0$, *independent of* h, *such that*

$$
a_h(\mathbf{u}_h,\mathbf{u}_h) \;\geq\; \alpha\,\|\mathbf{u}_h\|_{\mathrm{div}\,;\Omega} \qquad \forall\,\mathbf{u}_h\in V_h\,. \tag{4.3.36}
$$

Proof. Recalling from (4.3.1) that for each $\mathbf{v}_h \in H_h$ there holds $\operatorname{div}\mathbf{v}_h|_K \in \mathrm{P}_{k-1}(K)$ $\forall\, K \in \mathcal{T}_h$, which actually says that $\operatorname{div}\mathbf{v}_h \in Q_h$, we find that

$$V_h \,:=\, \left\{ \mathbf{v}_h \in H_h : \quad \int_\Omega q_h \operatorname{div}(\mathbf{v}_h) \,=\, 0 \quad \forall\, q_h \in Q_h \right\} \,=\, \left\{ \mathbf{v}_h \in H_h : \quad \operatorname{div}\mathbf{v}_h = 0 \right\}.$$

Hence, according to the definition of a_h (cf. (4.3.30)), applying the lower bound in (4.3.34) and the fact that $\mathbb{K}$ is a positive definite tensor, we deduce that for each $\mathbf{v}_h \in V_h$ there holds

$$a_h(\mathbf{v}_h, \mathbf{v}_h) \,=\, \sum_{K\in\mathcal{T}_h} a_h^K(\mathbf{v}_h, \mathbf{v}_h) \,\geq\, C \sum_{K\in\mathcal{T}_h} \|\mathbf{v}_h\|_{0,K}^2 \,=\, C\,\|\mathbf{v}_h\|_{0,\Omega}^2 \,=\, \alpha\,\|\mathbf{v}_h\|_{\operatorname{div};\Omega}^2\,,$$

with $\alpha = C$, which ends the proof. $\square$

The following lemma provides the discrete inf-sup condition for b.

Lemma 4.3.11. Let H_h and Q_h be the virtual subspaces given by (4.3.1) and (4.3.2). Then, there exists $\beta > 0$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_h\in H_h\\ \mathbf{v}_h\neq \mathbf{0}}} \frac{\mathbf{b}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{\operatorname{div};\Omega}} \,\geq\, \beta\,\|q_h\|_{0,\Omega} \qquad \forall\, q_h \in Q_h\,. \tag{4.3.37}$$

Proof. Since b satisfies the continuous inf-sup condition, we proceed in the classical way (see, e.g. [23, Section 4.2]) by constructing a corresponding Fortin operator. In fact, given a convex and bounded domain G containing $\bar{\Omega}$, and given $\mathbf{v} \in H$ (cf. (4.2.5)), we let $z \in H_0^1(G) \cap \mathbf{H}^2(G)$ be the unique solution of the boundary value problem

$$\Delta\, z \,=\, \begin{cases} \operatorname{div}\mathbf{v} & \text{in} \quad \Omega\,, \\ \\ 0 & \text{in} \quad G\backslash\bar{\Omega}\,, \end{cases} \quad, \quad z = 0 \quad \text{on} \quad \partial G\,, \tag{4.3.38}$$

which, thanks to the corresponding elliptic regularity result, satisfies

$$\|z\|_{2,\Omega} \,\leq\, C\,\|\operatorname{div}\mathbf{v}\|_{0,\Omega}\,. \tag{4.3.39}$$

Then, recalling that Π_k^h denotes the interpolation operator mapping $\widetilde{H}$ onto our virtual subspace H_h (cf. (4.3.13), (4.3.14)), we now define the operator $\pi_k^h : H \to H_h$ as

$$\pi_k^h(\mathbf{v}) \,=\, \Pi_k^h(\nabla z).$$

It follows, using (4.3.15) and the fact that Π_k^K and $\mathcal{P}_{k-1}^K$ are the restrictions to $K \in \mathcal{T}_h$ of the operators Π_k^h and $\mathcal{P}_{k-1}^h$, respectively, that

$$\operatorname{div} \pi_k^h(\mathbf{v}) \,=\, \operatorname{div} \Pi_k^h(\nabla z) \,=\, \mathcal{P}_{k-1}^h(\operatorname{div} \nabla z) \,=\, \mathcal{P}_{k-1}^h\big(\operatorname{div} \mathbf{v}\big) \quad \text{in} \quad \Omega\,, \tag{4.3.40}$$

and hence for each $q_h \in Q_h$ we obtain

$$b(\pi_k^h(\mathbf{v}), q_h) \,=\, \int_\Omega q_h \operatorname{div} \pi_k^h(\mathbf{v}) \,=\, \int_\Omega q_h\, \mathcal{P}_{k-1}^h\big(\operatorname{div} \mathbf{v}\big) \,=\, \int_\Omega q_h \operatorname{div} \mathbf{v} \,=\, \mathbf{b}(\mathbf{v}, q_h)\,. \tag{4.3.41}$$

In turn, using (4.3.40), (4.3.23) (with $r = 1$), and (4.3.39), we find that

$$\|\pi_k^h(\mathbf{v})\|_{\operatorname{div};\Omega} \,\le\, \|\Pi_k^h(\nabla z)\|_{0,\Omega} \,+\, \|\operatorname{div} \mathbf{v}\|_{0,\Omega} \,\le\, \|\nabla z\|_{0,\Omega} \,+\, \|\nabla z - \Pi_k^h(\nabla z)\|_{0,\Omega} \,+\, \|\operatorname{div} \mathbf{v}\|_{0,\Omega}$$

$$\le\, \|\nabla z\|_{0,\Omega} \,+\, c\,h\,\|\nabla z\|_{1,\Omega} \,+\, \|\operatorname{div} \mathbf{v}\|_{0,\Omega} \,\le\, \bar{C}\,\|z\|_{2,\Omega} \,+\, \|\operatorname{div} \mathbf{v}\|_{0,\Omega} \,\le\, C\,\|\operatorname{div} \mathbf{v}\|_{0,\Omega}\,,$$

which proves the uniform boundedness of the operators $\{\pi_k^h\}_{h>0}$. This fact and the identity (4.3.41) confirm that $\{\pi_k^h\}_{h>0}$ constitutes a family of Fortin operators, which yields (4.3.37) and ends the proof. □

The unique solvability and stability of the actual Galerkin scheme (4.3.35) is established now.

Theorem 4.3.1. *There exists a unique $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ solution of (4.3.35), and there exists a positive constant C, independent of h, such that*

$$\|(\mathbf{u}_h, q_h)\|_{H\times Q} \,\le\, C\left\{\, \|f\|_{0,\Omega} \,+\, \|g\|_{-1/2,\Gamma} \,\right\}.$$

Proof. The boundedness of $a_h : H_h \times H_h \longrightarrow \mathrm{R}$ with respect to the norm $\|\cdot\|_{\operatorname{div};\Omega}$ of $H(\operatorname{div};\Omega)$ follows easily from (4.3.33) and (4.3.26) (cf. Lemma 4.3.2). In turn, it is quite clear that b is also bounded. Hence, thanks to Lemmas 4.3.5 and 4.3.11, a straightforward application of the Babuška–Brezzi theory completes the proof. □

We now aim to provide the corresponding a priori error estimates. To this end, and just for sake of clearness in what follows, we recall that $\mathcal{P}_{k-1}^h : L^2(\Omega) \longrightarrow Q_h$ and $\Pi_k^h : \widetilde{H} \longrightarrow H_h$ are the projector and interpolator, respectively, defined by (4.3.12) and (4.3.14), whose associated local operators are denoted by $\mathcal{P}_{k-1}^K$ and Π_k^K. In turn, given our local projector $\widehat{\Pi}_k^K$ defined by (4.3.25), we denote by $\widehat{\Pi}_k^h$ its global counterpart, that is, given $\mathbf{v} \in H(\mathrm{div}\,;\Omega)$, we let

$$\widehat{\Pi}_k^h(\mathbf{v})|_K \;:=\; \widehat{\Pi}_k^K(\mathbf{v}|_K) \qquad \forall\, K \in \mathcal{T}_h\,.$$

Then, we have the following main result.

Theorem 4.3.2. *Let* $(\mathbf{u}, p) \in H \times Q$ *and* $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ *be the unique solutions of the continuous and discrete schemes* (4.2.4) *and* (4.3.35)*, respectively. Then, there exist positive constants* C_1*,* C_2*, independent of* h*, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \;\leq\; C_1 \left\{ \|\mathbf{u} - \Pi_k^h(\mathbf{u})\|_{0,\Omega} \;+\; \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} \right\}, \tag{4.3.42}$$

and

$$\|p - p_h\|_{0,\Omega} \;\leq\; C_2 \left\{ \|\mathbf{u} - \Pi_k^h(\mathbf{u})\|_{0,\Omega} \;+\; \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} \;+\; \|p - \mathcal{P}_{k-1}^h(p)\|_{0,\Omega} \right\}. \tag{4.3.43}$$

Proof. We first have, thanks to the triangle inequality, that

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \;\leq\; \|\mathbf{u} - \Pi_k^h(\mathbf{u})\|_{0,\Omega} \;+\; \|\Pi_k^h(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h\|_{0,\Omega}\,, \tag{4.3.44}$$

whence it just remains to estimate $\boldsymbol{\delta}_h \;:=\; \Pi_k^h(\mathbf{u}) - \mathbf{u}_h$. We now observe from (4.3.15) and the second equation of (4.3.35) that $\mathrm{div}\,\big(\Pi_k^h(\mathbf{u})\big) \;=\; \mathcal{P}_{k-1}^h(\mathrm{div}\,\mathbf{u}) \;=\; \mathcal{P}_{k-1}^h(-f) \;=\; \mathrm{div}\,\mathbf{u}_h$, which says that $\boldsymbol{\delta}_h \in V_h$. It follows from (4.3.36) (cf. Lemma 4.3.5), adding and substracting $\widehat{\Pi}_k^h(\mathbf{u})$, using the first equations of (4.3.35) and (4.2.4), employing the identity (4.3.32), and applying the boundedness of a_h^K (cf. (4.3.33)), a^K and $\widehat{\Pi}_k^K$ (cf.

(4.3.26)), that

$$\begin{aligned}
\alpha\,\|\boldsymbol{\delta}_h\|^2_{\mathrm{div}\,;\Omega} \,&=\, \alpha\,\|\boldsymbol{\delta}_h\|^2_{0,\Omega} \,\le\, a_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \,=\, a_h(\Pi^h_k(\mathbf{u}),\boldsymbol{\delta}_h) \,-\, a_h(\mathbf{u}_h,\boldsymbol{\delta}_h)\\
&= a_h(\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k(\mathbf{u}),\boldsymbol{\delta}_h) \,+\, a_h(\widehat{\Pi}^h_k(\mathbf{u}),\boldsymbol{\delta}_h) \,-\, \langle \boldsymbol{\delta}_h\cdot\mathbf{n}, g\rangle\\
&= a_h(\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k(\mathbf{u}),\boldsymbol{\delta}_h) \,+\, a_h(\widehat{\Pi}^h_k(\mathbf{u}),\boldsymbol{\delta}_h) \,-\, a(\mathbf{u},\boldsymbol{\delta}_h)\\
&= \sum_{K\in\mathcal{T}_h}\left\{ a^K_h(\Pi^K_k(\mathbf{u}) - \widehat{\Pi}^K_k(\mathbf{u}),\boldsymbol{\delta}_h) \,-\, a^K(\mathbf{u} - \widehat{\Pi}^K_k(\mathbf{u}),\boldsymbol{\delta}_h)\right\}\\
&\le \alpha_1 \sum_{K\in\mathcal{T}_h}\left\{ \|\Pi^K_k(\mathbf{u}) - \widehat{\Pi}^K_k(\mathbf{u})\|_{0,K} \,+\, \|\Pi^K_k(\mathbf{u}) - \widehat{\Pi}^K_k\big\{\Pi^K_k(\mathbf{u})\big\}\|_{0,K}\right\}\|\boldsymbol{\delta}_h\|_{0,K}\\
&+\quad \|\mathbb{K}^{-1}\| \sum_{K\in\mathcal{T}_h} \|\mathbf{u} - \widehat{\Pi}^K_k(\mathbf{u})\|_{0,K}\,\|\boldsymbol{\delta}_h\|_{0,K}\,,
\end{aligned}$$

which yields, with $C \,:=\, \frac{1}{\alpha}\,\max\{\alpha_1, \|\mathbb{K}^{-1}\|\}$,

$$\|\boldsymbol{\delta}_h\|_{\mathrm{div}\,;\Omega} \,\le\, C\left\{\|\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega} \,+\, \|\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k\big\{\Pi^h_k(\mathbf{u})\big\}\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega}\right\}. \tag{4.3.45}$$

Next, adding and substracting $\mathbf{u}$, we deduce that

$$\|\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega} \,\le\, \|\mathbf{u} - \Pi^h_k(\mathbf{u})\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega}\,. \tag{4.3.46}$$

In turn, proceding in the same way and employing the boundedness of $\widehat{\Pi}^K_k$ (cf. (4.3.26)), we find that

$$\begin{aligned}
\|\Pi^h_k(\mathbf{u}) - \widehat{\Pi}^h_k\big\{\Pi^h_k(\mathbf{u})\big\}\|_{0,\Omega} \,&\le\, \|\mathbf{u} - \Pi^h_k(\mathbf{u})\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega} \,+\, \|\widehat{\Pi}^h_k\big\{\mathbf{u} - \Pi^h_k(\mathbf{u})\big\}\|_{0,\Omega}\\
&\le\, C\left\{\|\mathbf{u} - \Pi^h_k(\mathbf{u})\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}^h_k(\mathbf{u})\|_{0,\Omega}\right\}.
\end{aligned} \tag{4.3.47}$$

In this way, replacing (4.3.47) and (4.3.46) into (4.3.45), and then the resulting estimate back into (4.3.44), we conclude the upper bound for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ induced by (4.3.42). On the other hand, concerning the error $\|p - p_h\|_{0,\Omega}$, we begin with the triangle inequality again and obtain

$$\|p - p_h\|_{0,\Omega} \,\le\, \|p - \mathcal{P}^h_{k-1}(p)\|_{0,\Omega} \,+\, \|\mathcal{P}^h_{k-1}(p) - p_h\|_{0,\Omega}\,. \tag{4.3.48}$$

Next, proceeding as in the proof of Lemma 4.3.11, taking $\mathcal{P}_{k-1}^h(p) - p_h \in Q_h$ instead of $\operatorname{div}(\mathbf{v})$ in the definition of the auxiliary problem (4.3.38), we deduce the existence of $\mathbf{u}_h^* \in H_h$ such that

$$\operatorname{div}(\mathbf{u}_h^*) \,=\, \mathcal{P}_{k-1}^h(p) - p_h \quad \text{and} \quad \|\mathbf{u}_h^*\|_{\operatorname{div};\Omega} \,\leq\, c\,\|\mathcal{P}_{k-1}^h(p) - p_h\|_{0,\Omega}\,.$$

It follows, employing the first equations of (4.3.35) and (4.2.4), and the identity (4.3.32), that

$$\begin{aligned} \|\mathcal{P}_{k-1}^h(p) - p_h\|_{0,\Omega}^2 \,&=\, \int_\Omega \big(\mathcal{P}_{k-1}^h(p) - p_h\big) \operatorname{div}\mathbf{u}_h^* \,=\, \int_\Omega \big(p - p_h\big) \operatorname{div}\mathbf{u}_h^* \\ &=\, b(\mathbf{u}_h^*, p) \,-\, b(\mathbf{u}_h^*, p_h) \,=\, a_h(\mathbf{u}_h, \mathbf{u}_h^*) \,-\, a(\mathbf{u}, \mathbf{u}_h^*) \\ &=\, a_h(\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u}), \mathbf{u}_h^*) \,-\, a(\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u}), \mathbf{u}_h^*)\,, \end{aligned}$$

which, applying the boundedness of a_h^K (cf. (4.3.33)), a^K and $\widehat{\Pi}_k^K$ (cf. (4.3.26)), and observing in particular that $\|\mathbf{u}_h^* - \widehat{\Pi}_k^h(\mathbf{u}_h^*)\|_{0,\Omega} \,\leq\, c\,\|\mathbf{u}_h^*\|_{\operatorname{div};\Omega} \,\leq\, C\,\|\mathcal{P}_{k-1}^h(p) - p_h\|_{0,\Omega}$, gives

$$\|\mathcal{P}_{k-1}^h(p) - p_h\|_{0,\Omega} \,\leq\, C\left\{\|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} + \|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u}_h)\|_{0,\Omega} + \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega}\right\}. \tag{4.3.49}$$

Now, adding and substracting $\mathbf{u}$, we readily get

$$\|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} \,\leq\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega}\,. \tag{4.3.50}$$

Similarly, and utilizing once again the boundedness of $\widehat{\Pi}_k^K$ (cf. (4.3.26)), we can write

$$\begin{aligned} \|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u}_h)\|_{0,\Omega} \,&\leq\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} \,+\, \|\widehat{\Pi}_k^h(\mathbf{u} - \mathbf{u}_h)\|_{0,\Omega} \\ &\leq\, C\left\{\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,+\, \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega}\right\}. \end{aligned} \tag{4.3.51}$$

Consequently, replacing (4.3.51) and (4.3.50) into (4.3.49), and the already derived a priori error bound for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$, and then placing the resulting estimate back into (4.3.48), we arrive at (4.3.43) and conclude the proof. □

Having established the a priori error estimates for our unknowns, we now provide the corresponding rates of convergence.

Theorem 4.3.3. *Let* $(\mathbf{u}, p) \in H \times Q$ *and* $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ *be the unique solutions of the continuous and discrete schemes* (4.2.4) *and* (4.3.35)*, respectively. Assume that for some* $r \in [1, k+1]$ *and* $s \in [1, k]$ *there hold* $\mathbf{u}|_K \in \mathbf{H}^r_\nabla(K)$*,* $f|_K = -\mathrm{div}\,(\mathbf{u})|_K \in H^{r-1}(K)$*, and* $p|_K \in H^s(K)$ *for each* $K \in \mathcal{T}_h$*. Then, there exist positive constants* $\bar{C}_1$*,* $\bar{C}_2$*, independent of* h*, such that*

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,\leq\, \bar{C}_1\, h^r \left\{ \sum_{K\in\mathcal{T}_h} \left\{ \|\mathbf{u}\|^2_{r,K} \,+\, \|f\|^2_{r-1,K} \right\} \right\}^{1/2}, \tag{4.3.52}$$

and

$$\|p - p_h\|_{0,\Omega} \,\leq\, \bar{C}_2\, h^r \left\{ \sum_{K\in\mathcal{T}_h} \left\{ \|\mathbf{u}\|^2_{r,K} \,+\, \|f\|^2_{r-1,K} \right\} \right\}^{1/2} \,+\, \bar{C}_2\, h^s \left\{ \sum_{K\in\mathcal{T}_h} \|p\|^2_{s,K} \right\}^{1/2}. \tag{4.3.53}$$

Proof. The case of integers $r \in [1, k+1]$ and $s \in [1, k]$ follows from straightforward application of the approximation properties provided by Lemmas 4.3.7, 4.3.10, and 4.3.4, to the terms on the right hand sides of (4.3.42) and (4.3.43). In turn, the usual interpolation estimates of Sobolev spaces allow us to conclude for the remaining real values of r and s. We omit further details. □

We notice that if the assumed regularities in the foregoing theorem are global, then the estimates (4.3.52) and (4.3.53) become, respectively,

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,\leq\, \bar{C}_1\, h^r \left\{ \|\mathbf{u}\|_{r,\Omega} \,+\, \|f\|_{r-1,\Omega} \right\},$$

and

$$\|p - p_h\|_{0,\Omega} \,\leq\, \bar{C}_2\, h^r \left\{ \|\mathbf{u}\|_{r,\Omega} \,+\, \|f\|_{r-1,\Omega} \right\} \,+\, \bar{C}_2\, h^s\, |p|_{s,K}\,.$$

In turn, it is also clear from the range of variability of the integers r and s that the highest possible rate of convergence for $\mathbf{u}$ is h^{k+1}, whereas that of p is h^k.

We now introduce the fully computable approximation of $\mathbf{u}$ given by

$$\widehat{\mathbf{u}}_h \;:=\; \widehat{\Pi}^h_k(\mathbf{u}_h) \tag{4.3.54}$$

and establish next the corresponding a priori error estimates.

Theorem 4.3.4. *There exists a positive constant C_3, independent of h, such that*

$$\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega} \leq C_3 \left\{ \|\mathbf{u} - \Pi_k^h(\mathbf{u})\|_{0,\Omega} + \|\mathbf{u} - \widehat{\Pi}_k^h(\mathbf{u})\|_{0,\Omega} \right\}. \tag{4.3.55}$$

Proof. Similarly as at the end of the proof of Theorem 4.3.2 we have by adding and substracting $\mathbf{u}_h$

$$\|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega} \leq \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} + \|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u}_h)\|_{0,\Omega} .$$

In this way, utilizing the estimates for $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ and $\|\mathbf{u}_h - \widehat{\Pi}_k^h(\mathbf{u}_h)\|_{0,\Omega}$ given by (4.3.42) and (4.3.51), respectively, we arrive at (4.3.55) and complete the proof. □

We end this section by remarking, according to the upper bounds provided by (4.3.42) (cf. Theorem 4.3.2) and (4.3.55) (cf. Theorem 4.3.4), that $\mathbf{u}_h$ and $\mathbf{u}$ share exactly the same rates of convergence given by Theorem 4.3.3.

4.4 Computational implementation

4.4.1 Introduction

We consider the same notations as in previous sections, and given a descomposition $\mathcal{T}_h$ of Ω in polygons, we consider the discretized problem: Find $(\mathbf{u}_h, p_h) \in H_h \times Q_h$ such that

$$\begin{array}{rcl} a_h(\mathbf{u}_h, \mathbf{v}_h) + b(\mathbf{v}_h, p_h) & = & 0 \qquad \forall\, \mathbf{v}_h \in H_h \\ b(\mathbf{u}_h, q_h) & = & -\displaystyle\int_\Omega f\, q_h \quad \forall\, q_h \in Q_h, \end{array} \tag{4.4.1}$$

where H_h, Q_h, a_h and b were defined on (4.3.1), (4.3.2), (4.3.28) and (4.2.6). Now, we have by using Theorem 4.3.1, that (4.4.1) has a unique solution $(\mathbf{u}_h, p_h)$, for every given data f and g. Now, let $\{\boldsymbol{\varphi}_j\}_{j=1}^N$ be an edge-oriented basis of H_h and let $\{\psi_j\}_{j=1}^M$ be a basis of Q_h. If $\mathbf{u}_h := \displaystyle\sum_{j=1}^N u_j \boldsymbol{\varphi}_j$ and $p_h := \displaystyle\sum_{j=1}^M p_j \psi_j$, then (4.4.1) is turned into the linear system of equations

$$\begin{aligned} \sum_{j=1}^{N} u_j\, a_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \quad + \quad \sum_{l=1}^{M} p_l\, b(\boldsymbol{\varphi}_i, \psi_l) \quad &= \quad 0 \\ \sum_{j=1}^{N} u_j\, b(\boldsymbol{\varphi}_j, \psi_l) \quad &= \quad -\int_\Omega f\, \psi_l, \end{aligned} \tag{4.4.2}$$

for each $i \in \{1, \ldots, N\}$ and for each $l \in \{1, \ldots, M\}$. Equivalently,

$$Ax = b,$$

where $x = (u_1, \ldots, u_N, p_1, \ldots, p_M)^{\mathtt{t}}$, $A = \begin{pmatrix} \mathbf{A}_h & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix}$, $\mathbf{A}_h = (A_{ij})$, $\mathbf{B} = (B_{ij})$, $\mathbf{d} = (d_i)$ and $b = (\mathbf{c}, \mathbf{d})^{\mathtt{t}}$. Also,

$$\begin{aligned} A_{ij} &:= a_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \quad i, j \in \{1, \ldots, N\} \\ B_{ij} &:= b(\boldsymbol{\varphi}_i, \psi_j) \quad i \in \{1, \ldots, N\} \;\; j \in \{1, \ldots, M\} \\ d_i &:= -\left(f, \psi_i\right)_{0,\Omega} \quad i \in \{1, \ldots, M\}\,. \end{aligned} \tag{4.4.3}$$

Furthermore, recall that we can write $a_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = \sum_{K \in \mathcal{T}_h} a_h^K(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j)$ and do the same for the other functionals and bilinear forms defined above, and we can then localize the calculations for $\mathbf{A}_h$, $\mathbf{B}$, $\mathbf{c}$ and $\mathbf{d}$.

4.4.1.1 Notations

We start remembering that $\mathcal{T}_h$ is a decomposition of K in polygons and $k \geq 1$ is the polynomial degree of accuracy. Given $K \in \mathcal{T}_h$ we denote by h_K and $\mathbf{x}_K$ the diameter and the barycenter of K, respectively, and we define, for a multi-index α, the polynomial function $m_\alpha^K := \left(\dfrac{\mathbf{x} - \mathbf{x}_K}{h_K}\right)^\alpha$. From now on, we will use the identification $m_0 := 0$, $1 \leftrightarrow (0,0)$, $2 \leftrightarrow (1,0)$, $3 \leftrightarrow (0,1)$, $4 \leftrightarrow (2,0)$, $5 \leftrightarrow (1,1)$ and so on. On a similar way, we define the polynomial vector field $\mathbf{m}_{\alpha,\beta}^K := \left(m_\alpha^K, m_\beta^K\right)^{\mathtt{t}}$ and we will use the identification $\mathbf{m}_1^K \leftrightarrow \mathbf{m}_{1,0}^K$, $\mathbf{m}_2^K \leftrightarrow \mathbf{m}_{0,1}^K$, $\mathbf{m}_3^K \leftrightarrow \mathbf{m}_{2,0}^K$, $\mathbf{m}_4^K \leftrightarrow \mathbf{m}_{0,2}^K$ and so on. Then, as in (4.3.5), we have that

$$\mathcal{B}_k(K) := \left\{ \mathbf{m}_j^K : j \in \{1, \ldots, \dim \mathrm{P}_k\left(K\right)\} \right\}.$$

In turn, we choose a counterclockwise arrangement $\{V_1, \ldots, V_{d_K}\}$ of the vertices of K, where we have defined d_K as the number of vertices of K. If e_j is the edge that connects V_j with V_{j+1} (and using the identification $V_{d_K+1} \leftrightarrow V_1$), we denote h_{e_j} the length of e_j, x_{e_j} the middle point of e_j, and for $i \geq 1$, $j \in \{1, \ldots, d_K\}$ and $x \in e_j$, we define

$$m_i^{e_j}(x) := \left(\frac{x - x_{e_j}}{h_{e_j}} \right)^{i-1}.$$

Anagously, using the identification $m_0^{\partial K} \leftrightarrow 1$, $m_1^{\partial K} \leftrightarrow m_1^{e_1}$, $m_2^{\partial K} \leftrightarrow m_2^{e_1}$, $m_{(k+1)+1}^{\partial K} \leftrightarrow m_1^{e_2}$, we define

$$\mathcal{B}_k(\partial K) := \Big\{ m_j^{\partial K} : j \in \{1, \ldots,\, d_K\ (k+1)\, \} \Big\}.$$

Now, given $K \in \mathcal{T}_h$, we want to calculate the local matrix $\mathbf{A}_K$ that corresponds to a_h^K, i.e.

$$(\mathbf{A}_K)_{ij} := a_h^K\,(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \qquad i, j \in \left\{1, \ldots, n_k^K\right\},$$

where $\left\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \ldots, \boldsymbol{\varphi}_{n_k^K}\right\}$ is the canonical basis of H_h^K and n_k^K its dimension (cf. (4.3.7)). For this end, and recalling the definition of a_h^K (cf. (4.3.28)), we have to calculate the projector $\widehat{\Pi}_k^K(\boldsymbol{\varphi}_i)$ for each $i \in \{1, \ldots, n_k^K\}$.

4.4.2 Calculating local matrices

Given $K \in \mathcal{T}_h$ and according to (4.4.3), we need to calculate the local matrices, which will be denoted $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{c}_K$ and $\mathbf{d}_K$, respectively. Let us first calculate $\mathbf{A}_K$. Given $i, j \in \left\{1, \ldots, n_k^K\right\}$, recall that

$$a_h^K(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = a^K(\hat{\Pi}_k^K(\boldsymbol{\varphi}_i), \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)) + S^K(\boldsymbol{\varphi}_i - \widehat{\Pi}_k^K(\boldsymbol{\varphi}_i), \boldsymbol{\varphi}_j - \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)).$$

In turn, if $\widehat{n}_k := \dim \widehat{H}_k^K$ and $\widehat{\mathbf{m}}_j := h_K \mathbb{K} \nabla m_{j+1}$ for each $j \in \{1, \ldots, \widehat{n}_k\}$, we can write

$$\widehat{\Pi}_k^K(\boldsymbol{\varphi}_i) := \sum_{j=1}^{n} s_j^{(i)} \widehat{\mathbf{m}}_j.$$

Therefore, replacing the above expression into (4.3.25) and using the symmetry of $\mathbb{K}^{-1}$, we arrive at the system

$$\sum_{\alpha=1}^{\widehat{n}_k} s_j^{(i)} \int_K \mathbb{K}^{-1}\,\widehat{\mathbf{m}}_\alpha \cdot \widehat{\mathbf{m}}_\beta = \int_K \mathbb{K}^{-1}\,\widehat{\mathbf{m}}_\beta \cdot \boldsymbol{\varphi}_i \qquad \beta \in \{1,\ldots,\widehat{n}_k\}. \tag{4.4.4}$$

Moreover, the above system can be read in a more compact form:

$$\mathbf{G}\,\mathbf{s}^{(i)} = \mathbf{b}^{(i)},$$

where $\mathbf{s}^{(i)} := \left(s_1^{(i)}, s_2^{(i)}, \ldots, s_{\widehat{n}_k}^{(i)}\right)^{\mathtt{t}}$. In turn, if

$$\mathbf{T} := [\ \mathbf{b}^{(1)} \quad \mathbf{b}^{(2)} \quad \cdots \quad \mathbf{b}^{(n_k^K)}\ ] \in \mathrm{R}^{\widehat{n}_k \times n_k^K},$$

then the matrix representation $\widehat{\mathbf{\Pi}}_*$ of the operator $\widehat{\Pi}_k^K$, acting from H_k^K into $\widehat{H}_k^K$, is given by $(\widehat{\mathbf{\Pi}}_*)_{\alpha i} := s_\alpha^{(i)}$, that is

$$\widehat{\mathbf{\Pi}}_* \;=\; \mathbf{G}^{-1}\,\mathbf{T}.$$

4.4.2.1 Calculating T.

Recall first that K has d_K edges. In turn, in order to split the elements of the local basis H_k^K, we define the numbers

$$\begin{aligned} n_{k,0}^K &:= d_K(k+1) = \dim \mathrm{P}_k(\partial K) \\ n_{k,1}^K &:= n_{k,0}^K + \frac{k(k+1)}{2} - 1 = n_{k,0}^K + \dim \mathrm{P}_{k-1}(K) - 2. \end{aligned}$$

Also, extending the functions of $\mathcal{B}_k(\partial K)$ outside its edges by zero, we denote (cf. (4.3.17)),

$$\begin{aligned} m_{j,K}(\mathbf{v}) &:= \int_{\partial K} (\mathbf{v}\cdot\mathbf{n})\, m_j^{\partial K} \qquad j \in \left\{1,\ldots,n_{k,0}^K\right\}, \\ m_{j,K}(\mathbf{v}) &:= \int_K \mathbf{v} \,:\, \nabla\, m_{j+1-n_{k,0}^K}^K \qquad j \in \left\{n_{k,0}^K+1,\ldots,n_{k,1}^K\right\}, \quad k>1, \\ m_{j,K}(\mathbf{v}) &:= \int_K m_{j-n_{k,1}^K}^K \cdot \operatorname{rot}\mathbf{v} \qquad j \in \left\{n_{k,1}^K+1,\ldots,n_k^K\right\}, \end{aligned}$$

and we also remind that $m_{i,K}(\boldsymbol{\varphi}_j) = \delta_{ij} \quad \forall i, j \in \{1, \ldots, n_k^K\}$. Now, let $i \in \{1, \ldots, n_k^K\}$ and $\beta \in \{1, \ldots, \widehat{n}_k\}$. Then, we have that $\widehat{\mathbf{m}}_\beta = h_K \, \mathbb{K} \nabla m_{\beta+1}$. Thus,

$$\begin{aligned}
(\mathbf{b}^{(i)})_\beta : &= h_K \int_K \boldsymbol{\varphi}_i \cdot \nabla m_{\beta+1} = -\int_K m_{\beta+1} \operatorname{div} \boldsymbol{\varphi}_i + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \cdot \mathbf{n}) \, m_{\beta+1} \\
&= -\int_K \mathcal{P}_{k-1}^K(m_{\beta+1}) \operatorname{div} \boldsymbol{\varphi}_i + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \cdot \mathbf{n}) \, \mathcal{P}_k^{\partial K}(m_{\beta+1}) \\
&= h_K \int_K \boldsymbol{\varphi}_i \cdot \nabla \mathcal{P}_{k-1}^K(m_{\beta+1}) + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \cdot \mathbf{n}) \, \mathcal{P}_k^{\partial K}(m_{\beta+1} - \mathcal{P}_{k-1}^K(m_{\beta+1}))
\end{aligned}$$

Now, we can write

$$\mathcal{P}_{k-1}^K(m_{\beta+1}) = \sum_{\alpha=1}^{\frac{k(k+1)}{2}} p_\alpha m_\alpha^K ,$$

and if $\mathbf{P}_0$ is the Gram matrix asociated to $\mathcal{P}_{k-1}^K$, that is $(\mathbf{P}_0)_{ij} := (m_i^K, m_j^K)_{0,K}$, and $\mathbf{d}_0$ is the right-hand side of such a system, i.e. $(\mathbf{d}_0)_j := (m_{\beta+1}, m_j^K)_{0,K}$, then

$$\Big(p_1, \ldots, p_{k(k+1)}\Big)^{\mathtt{t}} =: \mathbf{p}_0 = \mathbf{P}_0^{-1} \mathbf{d}_\mathbf{0}.$$

Similarly, we can write

$$\mathcal{P}_k^{\partial K}(m_{\beta+1} - \mathcal{P}_{k-1}^K(m_{\beta+1})) = \sum_{\alpha=1}^{n_{k,0}^K} q_\alpha m_\alpha^{\partial K}$$

where, again, if $\mathbf{P}_1$ is the Gram matrix asociated to such a system and $\mathbf{d}_1$ its right hand side, i.e. $(\mathbf{d}_1)_j := (m_{\beta+1} - \mathcal{P}_{k-1}^K(m_{\beta+1}), \mathbf{m}_j^{\partial K})$, then $\Big(q_1, \ldots, q_{2d_K(k+1)}\Big)^{\mathtt{t}} =: \mathbf{p}_1 = \mathbf{P}_1^{-1} \mathbf{d}_1$. Now, using the degrees of freedom for $\boldsymbol{\varphi}_i$, we obtain

$$\mathbf{b}^{(i)} := h_K \begin{cases} \Big(\mathbf{p}_1^{\mathtt{t}}, 0\Big)^{\mathtt{t}} & , \quad k = 1, \\ \Big(\mathbf{p}_1^{\mathtt{t}}, p_3, p_4, \ldots, p_{k(k+1)}, \mathbf{0}\Big)^{\mathtt{t}} & , \quad k > 1, \end{cases} \tag{4.4.5}$$

which is the formula to calculate $\mathbf{b}^{(i)}$ and consequently $\mathbf{T}$.

Remark 4.4.1. Note that

$$\frac{1}{h_e} \int_e \left(\frac{x - x_e}{h_e}\right)^\alpha \left(\frac{x - x_e}{h_e}\right)^\beta = \frac{1}{2^{\alpha+\beta}(\alpha + \beta + 1)} ,$$

which implies that $\mathbf{P}_1$ is a matrix made of d_K identical blocks.

Now, we need to calculate the coordinates of $\widehat{\Pi}_k^K \boldsymbol{\varphi}_i$ into the space H_k^K. More specifically,

$$\widehat{\Pi}_k^K \boldsymbol{\varphi}_i = \sum_{j=1}^{n_k^K} \pi_j^{(i)} \boldsymbol{\varphi}_j,$$

whence $\pi_j^{(i)} = m_{j,K}(\widehat{\Pi}_k^K \boldsymbol{\varphi}_i)$. Thus,

$$\widehat{\Pi}_k^K \boldsymbol{\varphi}_i = \sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} \widehat{\mathbf{m}}_\alpha = \sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} \sum_{j=1}^{n_k^K} m_{j,K}(\widehat{\mathbf{m}}_\alpha) \boldsymbol{\varphi}_j,$$

and so

$$\pi_i^j = \sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} m_{j,K}(\widehat{\mathbf{m}}_\alpha).$$

Now, we define $\mathbf{D} := (\mathbf{D}_{i\alpha}) = m_{i,K}(\widehat{\mathbf{m}}_\alpha)$ for every $i \in \{1, \ldots, n_k^K\}$ and $\alpha \in \{1, \ldots, \widehat{n}_k\}$, and then

$$\pi_j^{(i)} = \sum_{\alpha=1}^{n_k^K} (\mathbf{G}^{-1}\mathbf{T})_{\alpha i} \mathbf{D}_{j\alpha} = (\mathbf{D}\mathbf{G}^{-1}\mathbf{T})_{ji}.$$

Consequently, the matrix representation $\widehat{\boldsymbol{\Pi}}$ of the operator $\widehat{\Pi}_k^K$, from H_k^K into itself, is given by $\widehat{\boldsymbol{\Pi}} := \mathbf{D}\mathbf{G}^{-1}\mathbf{T} = \mathbf{D}\widehat{\boldsymbol{\Pi}}_*$.

Proposition 4.4.1. For every element K, we have $\mathbf{G} = \mathbf{T}\mathbf{D}$.

Proof. Given $\beta \in \{1, \ldots, \widehat{n}_k\}$ and $\alpha \in \{1, \ldots, \widehat{n}_{k,\nabla}\}$, it follows that

$$\begin{aligned}
&\sum_{i=1}^{n_k^K} \mathbf{T}_{\alpha i} \mathbf{D}_{i\beta} \\
&\quad = \sum_{i=1}^{n_k^K} m_{i,K}(\widehat{\mathbf{m}}_\beta)(\widehat{\mathbf{m}}_\alpha, \boldsymbol{\varphi}_i)_{0,K} = (\widehat{\mathbf{m}}_\alpha, \sum_{i=1}^{n_k^K} m_{i,K}(\widehat{\mathbf{m}}_\beta)\boldsymbol{\varphi}_i)_{0,K} = (\widehat{\mathbf{m}}_\alpha, \widehat{\mathbf{m}}_\beta)_{0,K} = \mathbf{G}_{\alpha\beta}.
\end{aligned}$$

which finishes the proof. □

Remark 4.4.2. The above proposition implies an improvement of the code in terms of speed, that is, it is not necessary to calculate $\mathbf{G}$ by using its explicit expression given by the corresponding system that defines $\widetilde{\Pi}_k^K$; we just need to calculate $\mathbf{S}$, $\mathbf{D}$, and then we obtain $\mathbf{G}$ by doing $\mathbf{G} := \mathbf{S}\mathbf{D}$.

On the other hand, using that

$$\mathcal{S}^K\Big((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i),(I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_j)\Big)=\sum_{r=1}^{n_k^K} m_{r,K}((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i))m_{r,K}((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i)),$$

and

$$m_{r,K}((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i))=[(\mathbf{I}-\widehat{\mathbf{\Pi}})]_{ir},$$

we then have that $\mathcal{S}^K\Big((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i),(I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_j)\Big)=[(\mathbf{I}-\widehat{\mathbf{\Pi}})^{\mathrm{T}}(\mathbf{I}-\widehat{\mathbf{\Pi}})]_{ij}$. Therefore, the matrix expression for the local stiffness matrix $\mathbf{A}_h^K$ is given by

$$\mathbf{A}_h^K=(\widehat{\mathbf{\Pi}}_*)^{\mathrm{T}}\widetilde{\mathbf{G}}(\widehat{\mathbf{\Pi}}_*)+(\mathbf{I}-\widehat{\mathbf{\Pi}})^{\mathrm{T}}(\mathbf{I}-\widehat{\mathbf{\Pi}}). \tag{4.4.6}$$

Now, in order to calculate $\mathbf{B}_K$, we note that given indices $\alpha\in\{1,\ldots,\widehat{n}_k\}$ and $j\in\Big\{1,\ldots,\dfrac{k(k+1)}{2}\Big\}$,

$$\begin{aligned}b(\boldsymbol{\varphi}_\alpha,m_j^K)&=\int_K m_j^K\cdot\operatorname{div}\boldsymbol{\varphi}_\alpha\\&=-h_K^{-1}\int_K\boldsymbol{\varphi}_\alpha:\nabla m_j^K+\int_{\partial K}(\boldsymbol{\varphi}_\alpha\cdot\mathbf{n})\cdot m_j^K\\&=-h_K^{-1}\delta_{\alpha,j+n_{k,0}^K}+\delta_{\alpha j}q_j,\end{aligned}$$

where $(q_1,\ldots,q_{n_{k,0}^K})^{\mathtt{t}}:=\mathbf{P^{-1}d}$ and $\mathbf{d}:=\Big((m_j^K,m_1^{\partial K}),\ldots,(m_j^K,m_{n_{k,0}^K}^{\partial K})\Big)^{\mathtt{t}}$. Finally, $\mathbf{d}_K$ is simply calculated by considering an integration rule on K which is exact when integrating polynomials of degree up to $k-1$. In summary, we obtain the following procedure to calculate the local matrices on each element K:

- Calculate $\mathbf{S}$ and $\mathbf{D}$.
- Calculate $\mathbf{G}$ by doing $\mathbf{G}:=\mathbf{SD}$.
- Calculate $\mathbf{A}_h^K$ by using formula (5.4.5).
- Calculate $\mathbf{B}_K$, $\mathbf{c}_K$ and $\mathbf{d}_K$.

4.4.3 Assembling the global matrix and post-processing the solution

Using the fact that all the bilinear forms and functionals can be split into a sum element by element, we just need to assemble the local matrices and then sum them by its corresponding indices. We point out now that we are calculating the coefficients of $\mathbf{u}_h$ in H_h and the coefficients of p_h in Q_h. Note that $\mathbf{u}_h$ is not manipulable since it belongs to H_h. Consequently, the error $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$ is only theoretical. To create a fully computable solution, we use (4.3.54) and make

$$\widehat{\mathbf{u}}_h := \widehat{\Pi}_k^h(\mathbf{u}_h).$$

Now, the condition $\displaystyle\int_\Omega p_h = 0$ is not naturally imposed in the system and hence $\mathbf{u}_h \in \mathbb{H}_0$ does not always hold. In order to fix this issue, we introduce a Lagrange multiplier $\lambda \in \mathrm{R}$ and consider the augmented variational formulation: Find $((\mathbf{u}_h, p_h), \lambda) \in H \times Q \times \mathrm{R}$ such that

$$\begin{array}{lllcll} a(\mathbf{u}_h, \mathbf{v}_h) & + \; b(\mathbf{v}_h, p_h) & & = & 0 & \forall \mathbf{v}_h \in H_h, \\ b(\mathbf{u}_h, q_h) & & + \; \lambda \displaystyle\int_\Omega q_h & = & -\displaystyle\int_\Omega f\, q_h & \forall q_h \in Q_h, \\ & \mu \displaystyle\int_\Omega p_h & & = & 0 & \forall \mu \in \mathrm{R}. \end{array} \tag{4.4.7}$$

Since the bilinear form $b_0((\mathbf{v}_h, q_h), \mu) := \mu \displaystyle\int_\Omega q_h$ satisfies the inf-sup condition, as well as b, and a is elliptic, it follows that (4.4.7) has a unique solution $((\mathbf{u}_h, p_h), \lambda) \in H_h \times Q_h \times \mathrm{R}$ and consequently $p_h \in L_0^2(\Omega)$. For implementation purposes, the stiffness matrix associated to (4.4.7) has an extra column, which contains the values of $\displaystyle\int_\Omega p_h$ at the elements $K \in \mathcal{T}_h$ and the corresponding basis functions.

4.4.4 Numerical results

In this section we present three numerical examples illustrating the good performance of the virtual mixed finite element scheme (4.3.35), and confirming the rates of convergence predicted by Theorem 4.3.3. For all the computations we consider the virtual element subspaces H_h and Q_h given by (4.3.1) and (4.3.2), with $k = 1$. In turn, for each Example we assume first decompositions of Ω made of triangles. In addition, in Example 1 we also consider straight squares, whereas Examples 2 and 3 make use of general quadrilateral elements as well. We begin by introducing additional notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (5.3.48), that is, $N = \dim H_h + \dim Q_h$. More precisely, according to (4.3.6) and (4.3.2), and bearing in mind that $\dim P_k(e) = k+1 \quad \forall$ edge $e \in \mathcal{T}_h$, and $\dim P_{k-1}(K) = \dfrac{k(k+1)}{2} \quad \forall\, K \in \mathcal{T}_h$, we find that in general

$$N \;=\; (k+1) \;\times\; \text{number of edges } e \in \mathcal{T}_h \;+\; \left\{\frac{3k(k+1)}{2} - 1\right\} \;\times\; \text{number of } K \in \mathcal{T}_h\,,$$

which, in the case $k = 1$, becomes

$$N \;=\; 2 \;\times\; \Big\{\text{number of edges } e \in \mathcal{T}_h\Big\} + \Big\{\text{number of } K \in \mathcal{T}_h\Big\}.$$

Also, the individual errors are defined by

$$\mathtt{e}_0(\mathbf{u}) \;:=\; \|\mathbf{u} - \widehat{\mathbf{u}}_h\|_{0,\Omega}\,, \quad \text{and} \quad \mathtt{e}(p) \;:=\; \|p - \widehat{p}_h\|_{0,\Omega}\,,$$

where $\widehat{\mathbf{u}}_h$ is computed according to (4.3.54), and p_h is provided by (5.3.48). In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}_0(\mathbf{u}) \;:=\; \frac{\log\big(\mathtt{e}_0(\mathbf{u})/\mathtt{e}_0'(\mathbf{u})\big)}{\log(h/h')}\,, \quad \text{and} \quad \mathtt{r}(p) \;:=\; \frac{\log\big(\mathtt{e}(p)/\mathtt{e}'(p)\big)}{\log(h/h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the errors for two consecutive meshes with sizes h and h', respectively. The numerical results presented below were obtained using a Matlab code. The corresponding linear systems were solved using the Conjugate Gradient method as main solver, and applying a stopping criterion determined by a relative tolerance of 10^{-10}. The specific examples to be considered are described next.

In Example 1 we consider $\Omega =]0,1[^2$, and choose $\mathbb{K} = \mathbb{I}$ and the data f and g so that the exact solution of (4.2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$p(\mathbf{x}) \;:=\; \sin(\pi x_1)\,\cos(\pi x_2) - \frac{4}{\pi^2}\,.$$

In Example 2 we consider the L-shaped domain $\Omega := \,]-1,1[^2 - [0,1]^2$, and choose $\mathbb{K} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and the data f and g so that the exact solution of (4.2.1) is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$p(\mathbf{x}) \;:=\; (x_1+1)\,(x_2+1) \;-\; \frac{1}{x_1+x_2+3} \;-\; \frac{1}{3}\left\{\frac{7}{4} - 16\log 2 + 9\log 3\right\}.$$

Finally, in Example 3 we consider the same geometry of Example 1, that is $\Omega =]0,1[^2$, and choose the data f and g so that the exact solution is given for each $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \Omega$ by

$$p(\mathbf{x}) \;:=\; \left(x_1^2 + x_2^2\right)^{5/6} \;-\; \int_\Omega \left(x_1^2 + x_2^2\right)^{5/6}.$$

Note in this example that the partial derivatives of p, and hence, in particular $\operatorname{div}\mathbf{u}$, are singular at the origin. Moreover, because of the power $1/3$, there holds $\mathbf{u} \in \mathbf{H}^{5/3-\epsilon}(\Omega)$ and $\operatorname{div}\mathbf{u} \in H^{2/3-\epsilon}(\Omega)$ for each $\epsilon > 0$, which, applying Theorem 5.3.3 with $r = 5/3 - \epsilon$, should yield a rate of convergence very close to $O(h^{5/3})$ for $\mathbf{u}$.

In Tables 4.1 up to 4.4 we summarize the convergence history of the mixed virtual element scheme (4.3.35) as applied to Examples 1 and 2, for sequences of quasi-uniform refinements of each domain. We notice there that the rates of convergences $O(h^{k+1}) = O(h^2)$ and $O(h^k) = O(h)$ predicted by Theorem 4.3.3 (when $r = k+1$ and $s = k$) are attained by $\mathbf{u}$ and p, respectively, for triangular as well as for quadrilateral meshes. In turn, in Tables 4.5 and 4.6 we display the corresponding convergence history of Example 3. As predicted in advance, and due to the limited regularity of $\mathbf{u}$ in this case, we observe that the orders $O(h^{k+\frac{2}{3}}) = O(h^{5/3})$ and $O(h^k) = O(h)$ are attained by $\mathbf{u}$ and p, respectively, Finally, in order to illustrate the accurateness of the discrete scheme, in Figures 4.1 up to 4.18 we display several components of the approximate and exact solutions for each example.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
39	0.707	$5.678E-01$	$-$	$2.504E-01$	$-$
143	0.354	$1.383E-01$	2.038	$1.300E-01$	0.946
543	0.177	$3.436E-02$	2.009	$6.536E-02$	0.992
2111	0.088	$8.577E-03$	2.002	$3.271E-02$	0.998
8319	0.044	$2.143E-03$	2.000	$1.636E-02$	1.000
33023	0.022	$5.358E-04$	2.000	$8.181E-03$	1.000
131583	0.011	$1.340E-04$	2.000	$4.091E-03$	1.000

Table 4.1: Example 1, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
27	0.707	$8.784E-01$	$-$	$2.964E-01$	$-$
95	0.354	$2.123E-01$	2.049	$1.584E-01$	0.904
351	0.177	$5.265E-02$	2.012	$7.996E-02$	0.986
1343	0.088	$1.314E-02$	2.003	$4.006E-02$	0.997
5247	0.044	$3.283E-03$	2.001	$2.004E-02$	0.999
20735	0.022	$8.206E-04$	2.000	$1.002E-02$	1.000
82431	0.011	$2.051E-04$	2.000	$5.010E-03$	1.000
328703	0.006	$5.129E-05$	2.000	$2.505E-03$	1.000

Table 4.2: Example 1, quasi-uniform refinement with straight squares.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
111	0.707	$2.057E-01$	$-$	$3.761E-01$	$-$
415	0.354	$6.045E-02$	1.767	$1.881E-01$	1.000
1599	0.177	$1.587E-02$	1.929	$9.403E-02$	1.000
6271	0.088	$4.021E-03$	1.981	$4.701E-02$	1.000
24831	0.044	$1.008E-03$	1.995	$2.351E-02$	1.000
98815	0.022	$2.523E-04$	1.999	$1.175E-02$	1.000
394239	0.011	$6.309E-05$	2.000	$5.877E-03$	1.000

Table 4.3: Example 2, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
75	0.800	$2.258E-01$	$-$	$3.985E-01$	$-$
271	0.431	$5.886E-02$	2.173	$2.016E-01$	1.101
1023	0.215	$1.662E-02$	1.816	$1.011E-01$	0.991
3967	0.110	$4.277E-03$	2.021	$5.069E-02$	1.028
15615	0.055	$1.057E-03$	2.008	$2.532E-02$	0.998
61951	0.028	$2.618E-04$	2.082	$1.266E-02$	1.034
246783	0.014	$6.562E-05$	2.011	$6.330E-03$	1.007

Table 4.4: Example 2, quasi-uniform refinement with quadrilaterals.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
39	0.707	$7.263E-02$	$-$	$1.939E-01$	$-$
143	0.354	$2.723E-02$	1.416	$9.750E-02$	0.992
543	0.177	$9.820E-03$	1.471	$4.880E-02$	0.999
2111	0.088	$3.454E-03$	1.508	$2.440E-02$	1.000
8319	0.044	$1.194E-03$	1.533	$1.220E-02$	1.000
33023	0.022	$4.072E-04$	1.551	$6.099E-03$	1.000
131583	0.011	$1.376E-04$	1.566	$3.050E-03$	1.000
525311	0.006	$4.612E-05$	1.577	$1.525E-03$	1.000

Table 4.5: Example 3, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$
27	0.707	$6.759E-02$	$-$	$2.008E-01$	$-$
95	0.495	$3.203E-02$	2.094	$1.169E-01$	1.518
351	0.277	$1.186E-02$	1.709	$6.386E-02$	1.040
1343	0.143	$3.685E-03$	1.761	$3.279E-02$	1.004
5247	0.072	$1.130E-03$	1.724	$1.651E-02$	1.001
20735	0.036	$3.533E-04$	1.682	$8.270E-03$	1.000
82431	0.018	$1.125E-04$	1.653	$4.137E-03$	1.000
328703	0.009	$3.625E-05$	1.634	$2.069E-03$	1.000

Table 4.6: Example 3, quasi-uniform refinement with distorted squares.

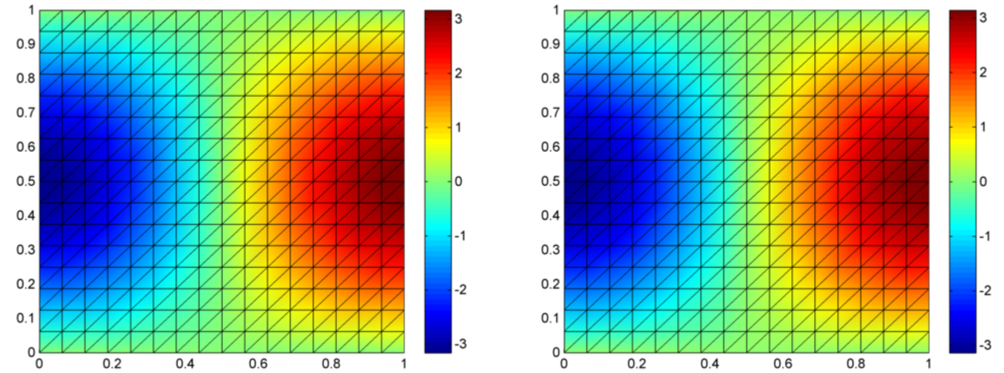

Figure 4.1: Example 1, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 2111$).

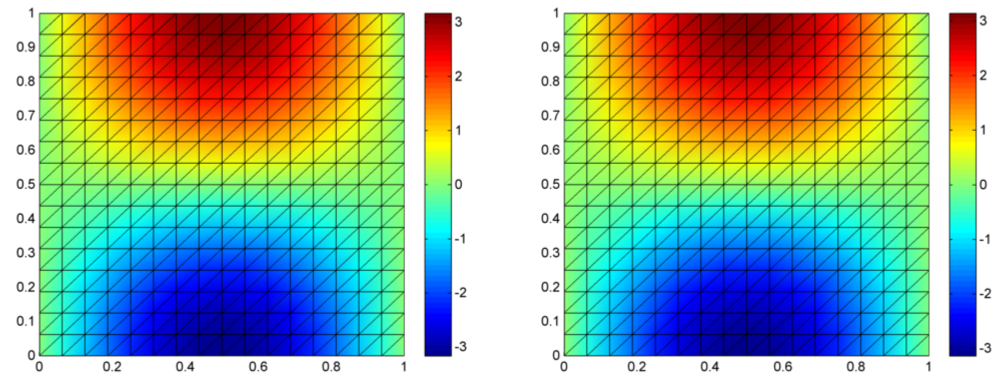

Figure 4.2: Example 1, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with triangles ($N = 2111$).

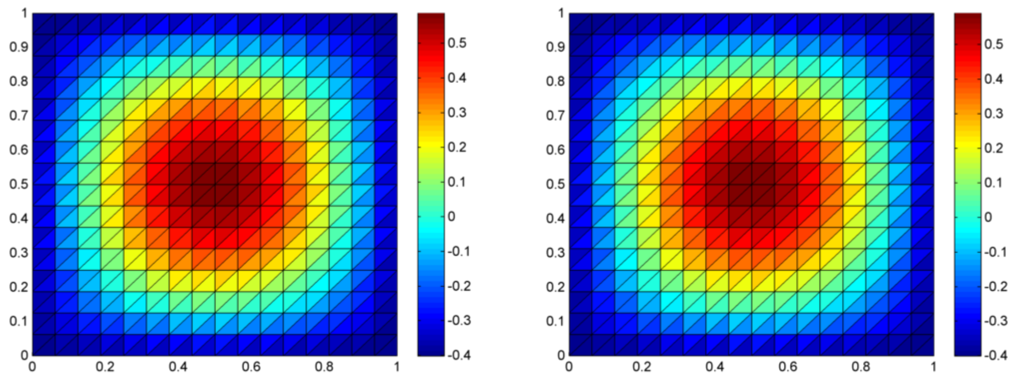

Figure 4.3: Example 1, p_h and p for a mesh with triangles ($N = 2111$).

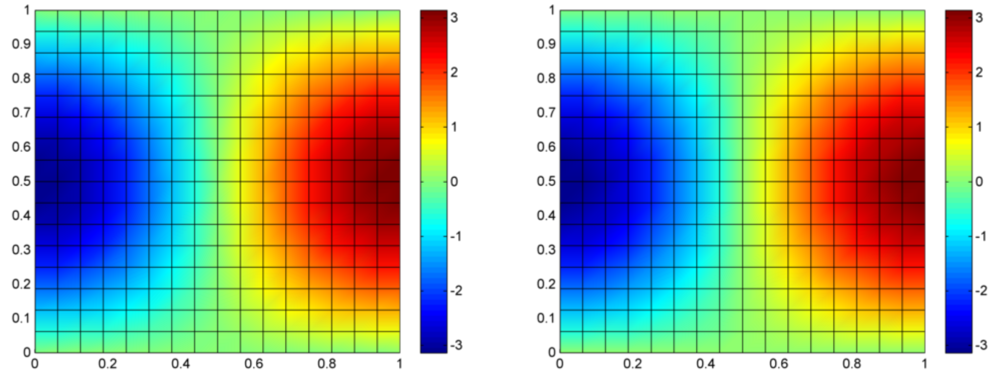

Figure 4.4: Example 1, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with straight squares ($N = 1343$).

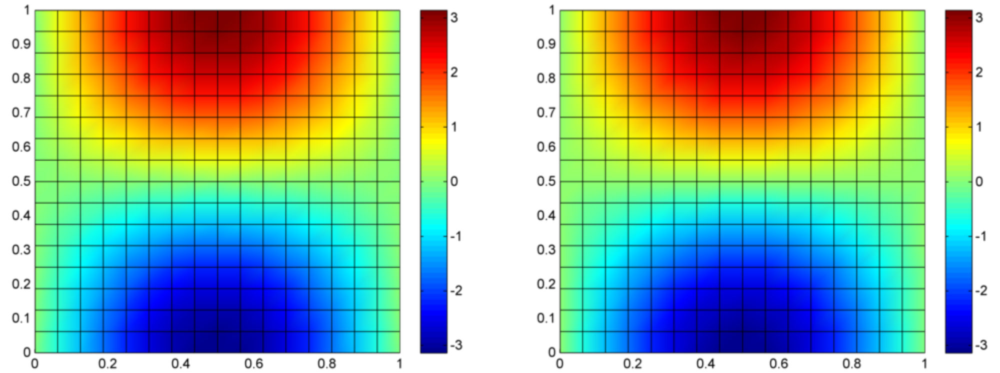

Figure 4.5: Example 1, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with straight squares ($N = 1343$).

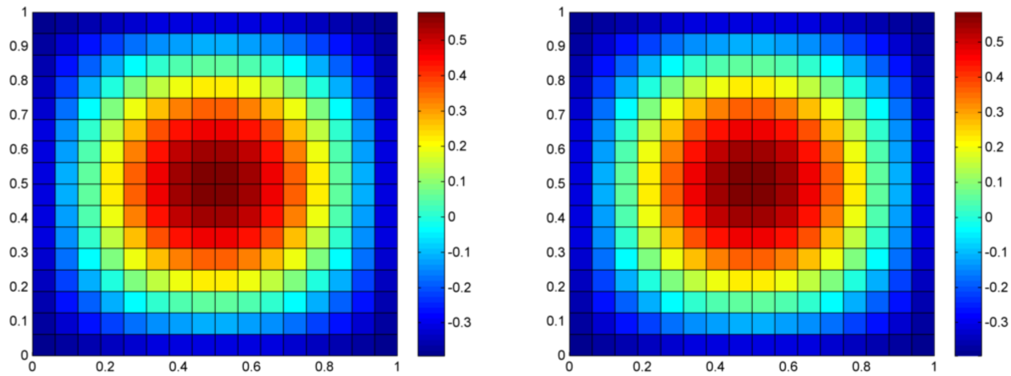

Figure 4.6: Example 1, p_h and p for a mesh with straight squares ($N = 1343$).

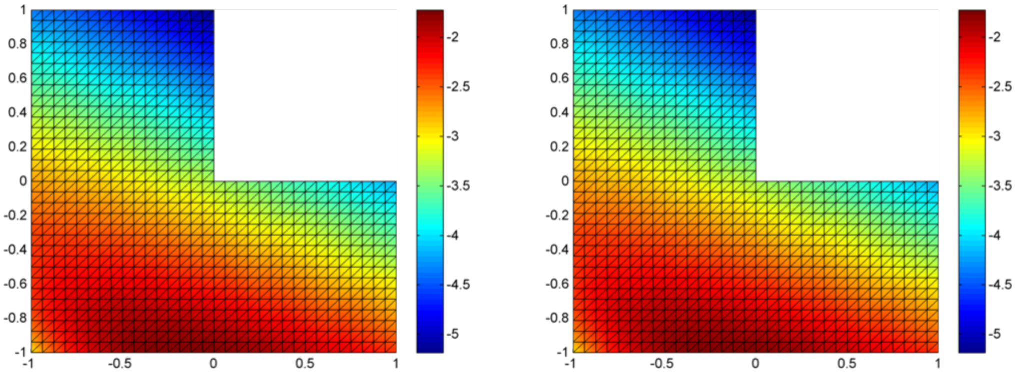

Figure 4.7: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 3967$).

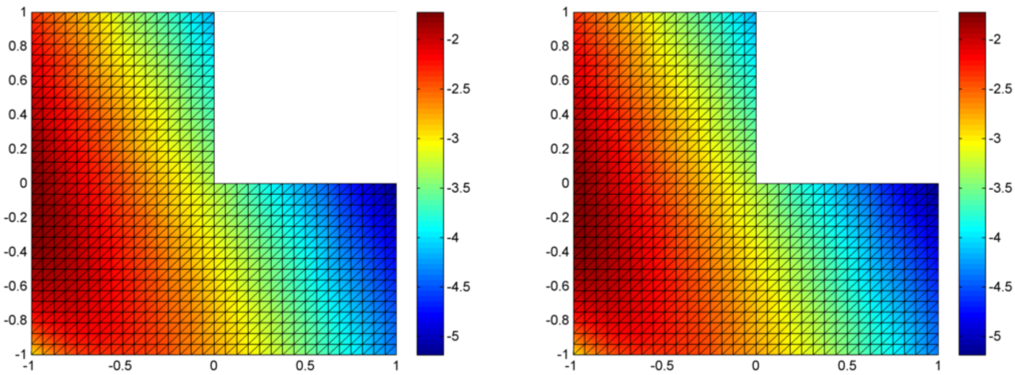

Figure 4.8: Example 2, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with triangles ($N = 3967$).

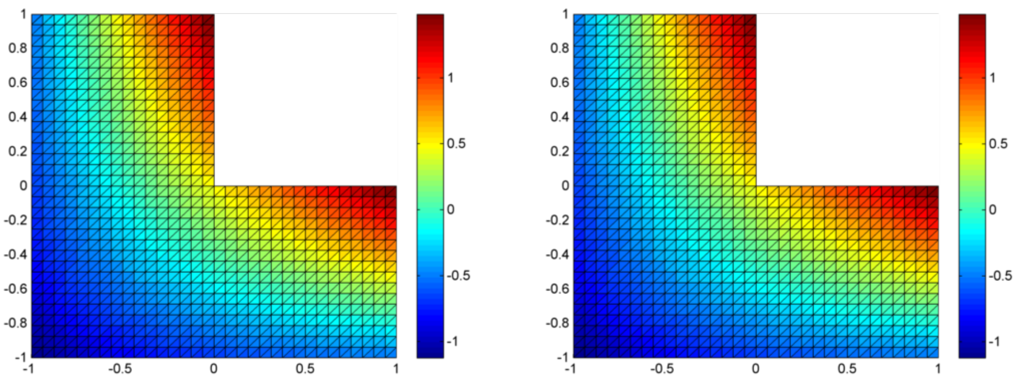

Figure 4.9: Example 2, p_h and p for a mesh with triangles ($N = 3967$).

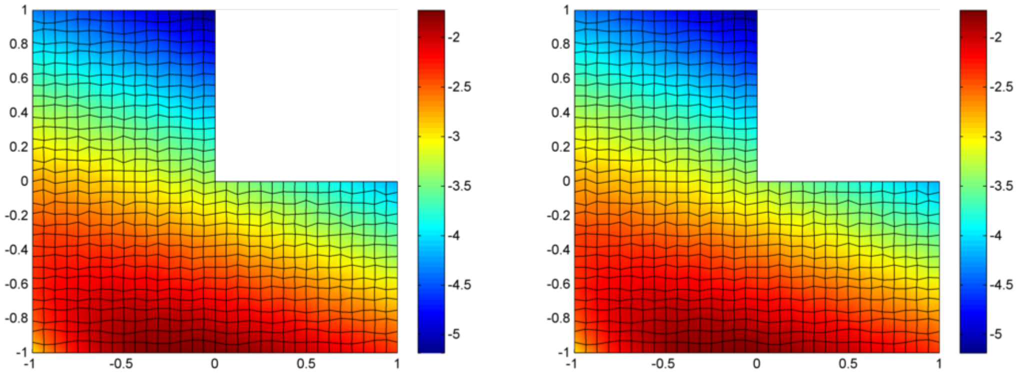

Figure 4.10: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with quadrilaterals ($N = 3967$).

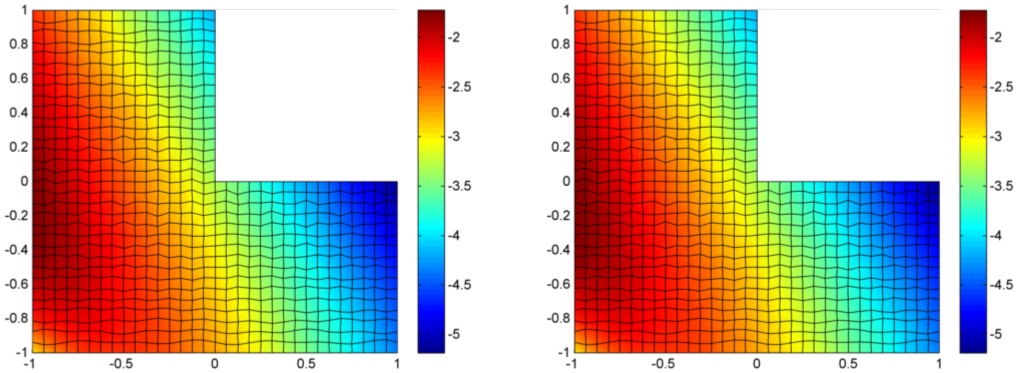

Figure 4.11: Example 2, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with quadrilaterals ($N = 3967$).

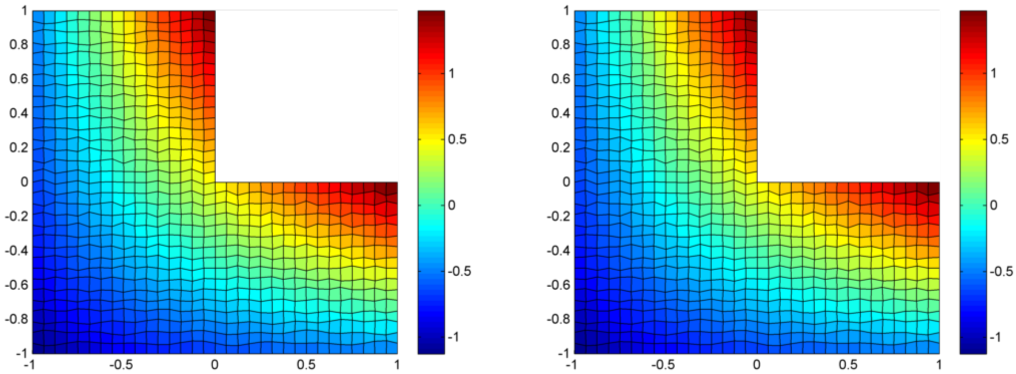

Figure 4.12: Example 2, p_h and p for a mesh with quadrilaterals ($N = 3967$).

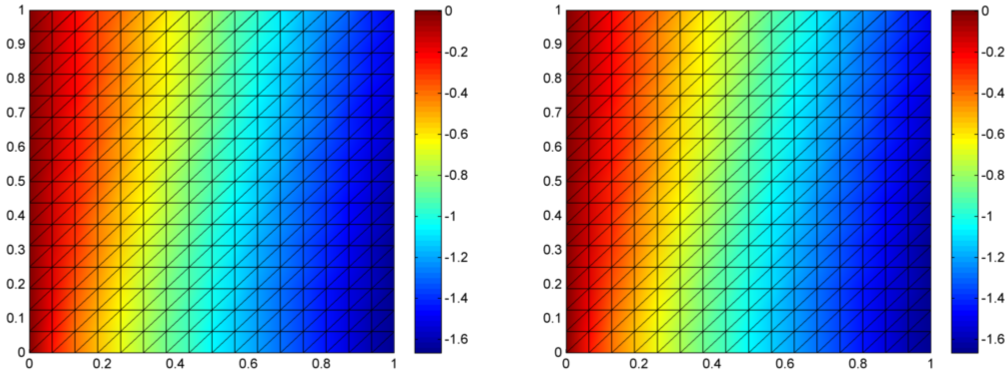

Figure 4.13: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 2111$).

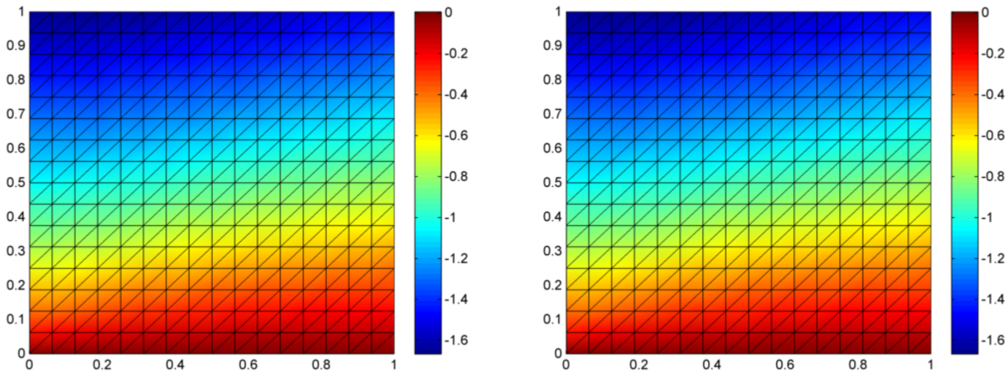

Figure 4.14: Example 3, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with triangles ($N = 2111$).

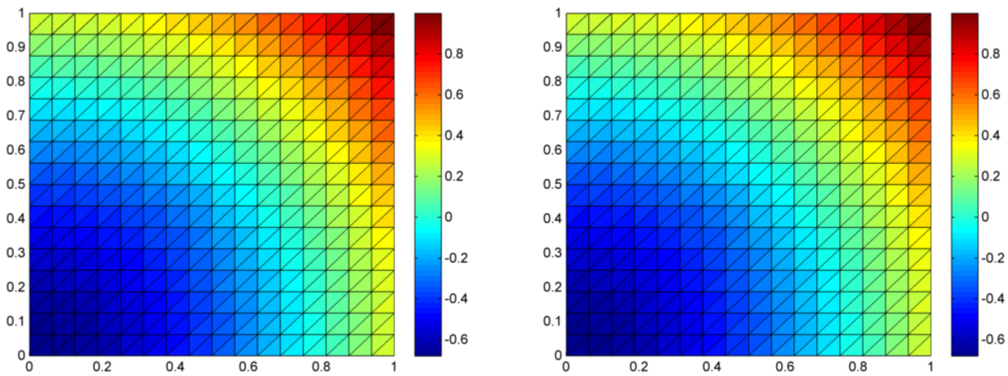

Figure 4.15: Example 3, p_h and p for a mesh with triangles ($N = 2111$).

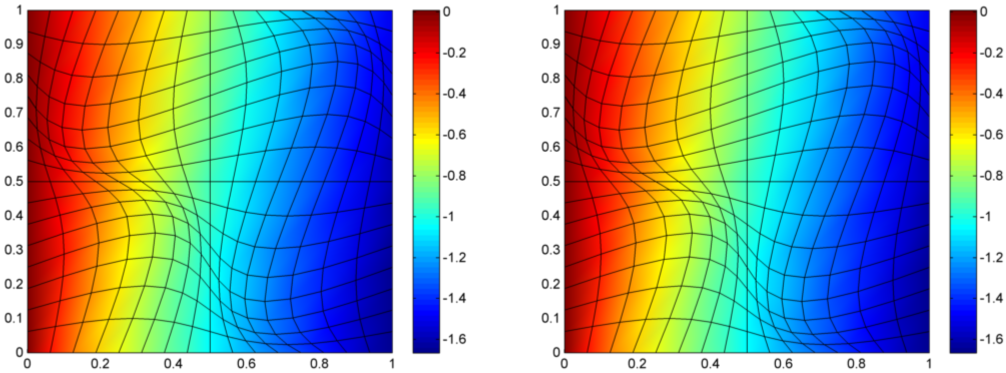

Figure 4.16: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with distorted squares ($N = 1343$).

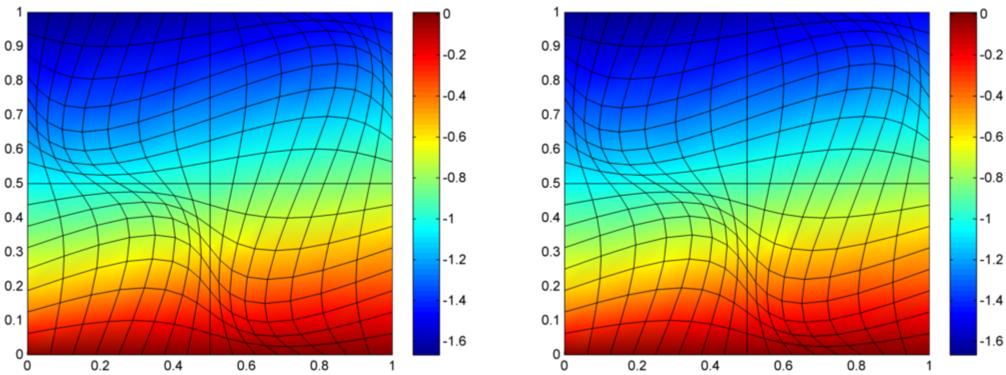

Figure 4.17: Example 3, $\widehat{\mathbf{u}}_{h,2}$ and $\mathbf{u}_2$ for a mesh with distorted squares ($N = 1343$).

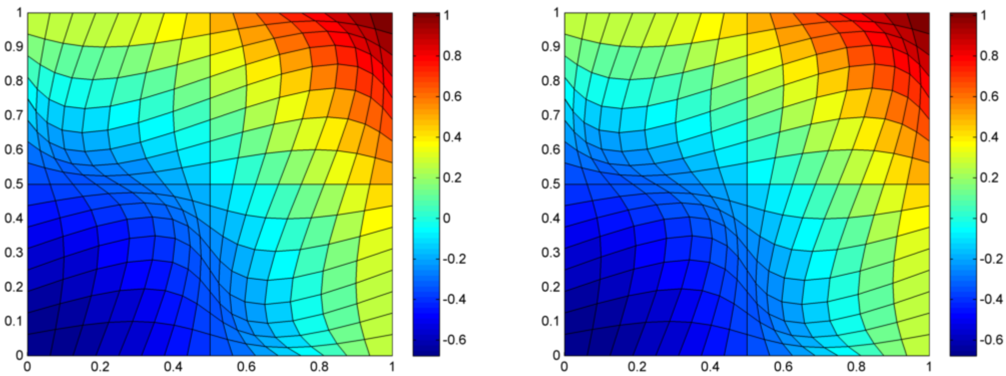

Figure 4.18: Example 3, p_h and p for a mesh with distorted squares ($N = 1343$).

Chapter 5

A mixed virtual element method for the Stokes problem

5.1 Introduction

Following previous related contributions on mixed finite element methods in fluid mechanics, we consider here the pseudostress–velocity formulation introduced first in [15], and furtherly developed, among others, in [27] and [28]. Indeed, the derivation of pseudostress–based mixed finite element methods for problems in continuum mechanics has become a very active research area lately, mainly due to the need of finding new ways of circumventing the symmetry requirement of the usual stress–based approach. While the weak imposition of this condition was suggested long before (see, e.g. [2]), the use of the pseudostress has become very popular in recent years, specially in the context of least-squares and augmented methods for incompressible flows, precisely because of the non–necessity of the symmetry condition. As a consequence, two new approaches appeared: the pseudostress–velocity–pressure and pseudostress–velocity formulations (see, e.g. [13], [14], [21], and the references therein). In particular, augmented mixed finite element methods for both pseudostress-based formulations of the stationary Stokes equations are studied in [21]. In addition, the pseudostress–velocity–pressure formulation has also been applied to nonlinear Stokes problems (see, e.g. [20], [26], [32]). Furthermore, the formulation from [15] is modified in [27] by incorporating the pressure into the discrete analysis, thus allowing further flexibility for approximating this unknown. More precisely, it is established there that the corresponding Galerkin scheme only makes sense for pressure finite element subspaces not containing the traces of the pseudostresses subspace. In particular, this is the case when Raviart–Thomas elements of index $k \geq 0$ for the pseudostress, and piecewise discontinuous polynomials of degree k for the velocity and the pressure, are utilized. On the other hand, for recent applications of the pseudostress–based approach in fluid mechanics we refer for instance to [24] and [25], where dual-mixed methods for the linear and nonlinear versions of the two–dimensional Brinkman problem are studied. Actually, the pseudostress is the main unknown of the resulting saddle point problems in [24] and [25], and the velocity and pressure are easily recovered through simple postprocessing formulae. In addition, as it is usual for dual–mixed methods, the Dirichlet boundary condition for the velocity becomes natural in this case, and the Neumann boundary condition, being essential, is imposed weakly through the introduction of the trace of the velocity on that boundary as the associated Lagrange multiplier. Additional contributions on this and related topics include [16], [17], [18], [29], and [34].

The rest of this chapter is organized as follows. In Section 5.2 we introduce the boundary value problem of interest, and recall from [27] its pseudostress–velocity mixed formulation and the associated well–posedness result. Then, in Section 5.3.1 we follow [9] to introduce the virtual element subspaces that will be employed, and then show the respective unisolvency, define the associated interpolation operators, and provide their approximation properties. Though some of the proofs of these results are sketched in [9], for sake of clearness and completeness, in the present paper we try to give as much detail as possible in some of them. In particular, a Bramble–Hilbert type theorem for averaged Taylor polynomials (cf. Chapter 2) plays a key role in our analysis. Next, fully calculable discrete bilinear forms are introduced in Section 5.3.4 and their boundedness and related properties are established. To this end, a new local projector onto a suitable space of polynomials is proposed here. This operator is somehow suggested by the main features of the continuous solution of the Stokes problem, and it also responds to the need of explicitly integrating the terms of the bilinear form that involves deviatoric tensors. The family of local projectors is shown to be uniformly bounded, and the aforementioned compactness theorem is applied to derive its approximation properties. The actual mixed virtual element method is then introduced and analyzed in Section 5.3.5. The classical discrete Babuška-Brezzi theory is applied to deduce the well–posedness of this scheme, and then suitable bounds and identities satisfied by the bilinear forms and the projectors and interpolators involved, allow to derive the a priori error estimates and corresponding rates of convergence for the virtual solution as well as for the projection of it. On the other hand, in Section 5.4 we use [5] to provide details on the computational implementation of MVEM, explaining how to assemble the global stiffness matrix and how to impose the associated extra condition (cf. Section 5.2) on the solution. Finally, several numerical examples showing the good performance of the method, confirming the rates of convergence for regular and singular solutions, and illustrating the accuracy obtained with the approximate solutions, are reported in Section 5.4.4 .

Finally, it is important to remark that all the contents of this chapter correspond exactly to what is provided in [12], with exception of the section corresponding to the computational implementation of the method (cf. Section 5.4).

5.2 The continuous problem and its mixed formulation

Let Ω be a bounded and simply connected polygonal domain in R^2 with boundary Γ. Our aim is to find the velocity $\mathbf{u}$, the pseudostress tensor $\boldsymbol{\sigma}$ and the pressure p of a steady flow occupying Ω, under the action of external forces. More precisely, given a volume force $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we seek a tensor field $\boldsymbol{\sigma}$, a vector field $\mathbf{u}$ and a scalar field p

such that

$$
\begin{aligned}
\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega\,,\\
\operatorname{div}\mathbf{u} = 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,,
\end{aligned}
\tag{5.2.1}
$$

where μ is the kinematic viscosity. As required by the incompressibility condition, we assume that $\mathbf{g}$ satisfies the compatibility condition

$$\int_\Gamma \mathbf{g}\cdot\mathbf{n} = 0\,.$$

It is easy to see, using that $\operatorname{tr}(\nabla\mathbf{u}) = \operatorname{div}\mathbf{u}$ in Ω, that the pair of equations given by

$$\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega \quad \text{and} \quad \operatorname{div}\,(\mathbf{u}) = 0 \quad \text{in} \quad \Omega\,,$$

is equivalent to

$$\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbb{I} \quad \text{in} \quad \Omega \quad \text{and} \quad p + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,,$$

and therefore, instead of (5.2.1), from now on we consider

$$
\begin{aligned}
\boldsymbol{\sigma} = 2\mu\nabla\mathbf{u} - p\mathbf{I} \quad \text{in} \quad \Omega\,, \qquad \mathbf{div}\,\boldsymbol{\sigma} = -\mathbf{f} \quad \text{in} \quad \Omega\,,\\
p + \frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}) = 0 \quad \text{in} \quad \Omega\,, \qquad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,.
\end{aligned}
\tag{5.2.2}
$$

Then, proceding as in [27], in particular eliminating the pressure from the third equation of (5.2.2), we arrive at the following mixed variational formulation: Find $(\boldsymbol{\sigma},\mathbf{u}) \in H \times Q$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{g}\rangle \qquad \forall\,\boldsymbol{\tau} \in H\,,\\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v} \in Q\,,
\end{aligned}
\tag{5.2.3}
$$

where

$$H := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,;\Omega) : \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\right\}, \qquad Q := \mathbf{L}^2(\Omega)\,, \tag{5.2.4}$$

and H is endowed with the usual norm $\|\cdot\|_{\mathbf{div};\Omega}$ of $\mathbb{H}(\mathbf{div};\Omega)$. In turn, $\mathbf{a} : H \times H \to \mathrm{R}$ and $\mathbf{b} : H \times Q \to \mathrm{R}$ are the bounded bilinear forms defined by

$$\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) := \frac{1}{2\mu}\int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} \qquad \forall\,(\boldsymbol{\sigma},\boldsymbol{\tau}) \in H \times H\,,$$

and

$$\mathbf{b}\,(\boldsymbol{\tau},\mathbf{v}) \,:=\, \int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\tau} \qquad \forall\,(\boldsymbol{\tau},\mathbf{v})\in H\times Q\,. \tag{5.2.5}$$

The unique solvability of (5.2.3) is established as follows.

Theorem 5.2.1. *There exists a unique* $(\boldsymbol{\sigma},\mathbf{u})\in H\times Q$ *solution of* (5.2.3). *Moreover, there exists a constant* $C>0$, *depending only on* Ω, *such that*

$$\|(\boldsymbol{\sigma},\mathbf{u})\|_{H\times Q} \,\leq\, C\left\{\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\,\right\}.$$

Proof. See [27, Theorem 2.1]. □

5.3 The discrete problem

5.3.1 The virtual element subspaces

5.3.1.1 Preliminaries

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω in polygonal elements. For each $K\in\mathcal{T}_h$ we denote its diameter by h_K, and define, as usual, $h\,:=\,\max\Big\{h_K:\quad K\in\mathcal{T}_h\Big\}$. Now, given an integer $k\geq 0$, we let $\mathrm{P}_k\,(K)$ be the space of polynomials on K of total degree up to k. Then, given an integer $k\geq 1$, we follow [9] and consider the following virtual element subspaces of H and Q, respectively:

$$\begin{aligned} H_h \,:=\, \Big\{\boldsymbol{\tau}\in H:\quad \boldsymbol{\tau}\,\mathbf{n}\Big|_e \in\mathbf{P}_k(e)\quad\forall\text{ edge } e\in\mathcal{T}_h,\quad \mathbf{div}\,\boldsymbol{\tau}\Big|_K\in\mathbf{P}_{k-1}(K)\,,\\ \mathbf{rot}\,\boldsymbol{\tau}\Big|_K\in\mathbf{P}_{k-1}(K)\quad\forall\,K\in\mathcal{T}_h\Big\}\,, \end{aligned} \tag{5.3.1}$$

and

$$Q_h \,:=\, \Big\{\mathbf{v}\in Q:\quad \mathbf{v}\Big|_K\in\mathbf{P}_{k-1}(K)\quad\forall\,K\in\mathcal{T}_h\Big\}\,, \tag{5.3.2}$$

where

$$\mathbf{rot}\,\boldsymbol{\tau} \,:=\, \begin{pmatrix} \dfrac{\partial\tau_{12}}{\partial x_1} - \dfrac{\partial\tau_{11}}{\partial x_2} \\[2ex] \dfrac{\partial\tau_{22}}{\partial x_1} - \dfrac{\partial\tau_{21}}{\partial x_2} \end{pmatrix} \qquad \forall\,\boldsymbol{\tau}\in H\,.$$

Then, the Galerkin scheme associated with (5.2.3) would read: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \langle \boldsymbol{\tau}_h \mathbf{n}, \mathbf{g} \rangle \qquad \forall\, \boldsymbol{\tau}_h \in H_h\,, \\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= -\int_\Omega \mathbf{f} \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in Q_h\,.
\end{aligned}
\tag{5.3.3}
$$

Nevertheless, we will observe later on that $\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h)$ can not be computed explicitly when $\boldsymbol{\sigma}_h$, $\boldsymbol{\tau}_h$ belongs to H_h, and hence a suitable approximation of this bilinear form, namely $\mathbf{a}_h$, will be introduced in Section 5.3.4 to redefine (5.3.3).

5.3.2 Unisolvency of the virtual element subspaces

In what follows we assume that there exists a constant $C_{\mathcal{T}} > 0$ such that for each decomposition $\mathcal{T}_h$ and for each $K \in \mathcal{T}_h$ there hold:

a) the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and

b) K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}}\, h_K$ and center $\mathbf{x}_B \in K$, that is, for each $x_0 \in B$, all the line segments joining x_0 with any $x \in K$ are contained in K, or, equivalently, for each $x \in K$, the closed convex hull of $\{x\} \cup B$ is contained in K.

As a consequence of the above hypotheses, one can show that each $K \in \mathcal{T}_h$ is simply connected, and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that the number of edges of each $K \in \mathcal{T}_h$ is bounded above by $N_{\mathcal{T}}$.

Next, in order to choose the degrees of freedom of H_h, given an edge $e \in \mathcal{T}_h$ with medium point x_e and length h_e, and given an integer $\ell \geq 0$, we first introduce the following set of $2(\ell+1)$ normalized monomials on e

$$
\mathcal{B}_\ell(e) := \left\{ \left(\left(\frac{x - x_e}{h_e} \right)^j, 0 \right)^{\mathtt{t}} \right\}_{0 \leq j \leq \ell} \bigcup \left\{ \left(0, \left(\frac{x - x_e}{h_e} \right)^j \right)^{\mathtt{t}} \right\}_{0 \leq j \leq \ell}, \tag{5.3.4}
$$

which certainly constitutes a basis of $\mathbf{P}_\ell(e)$. Similarly, given an element $K \in \mathcal{T}_h$ with barycenter $\mathbf{x}_K$, and given an integer $\ell \geq 0$, we define the following set of $(\ell+1)(\ell+2)$ normalized monomials

$$\mathcal{B}_\ell(K) := \left\{ \left(\left(\frac{\mathbf{x}-\mathbf{x}_K}{h_K} \right)^\alpha, 0 \right)^{\mathtt{t}} \right\}_{0\leq|\alpha|\leq\ell} \bigcup \left\{ \left(0, \left(\frac{\mathbf{x}-\mathbf{x}_K}{h_K} \right)^\alpha \right)^{\mathtt{t}} \right\}_{0\leq|\alpha|\leq\ell}, \tag{5.3.5}$$

which is a basis of $\mathbf{P}_\ell(K)$. Note that (5.3.5) makes use of the multi-index notation where, given $\mathbf{x} := (x_1, x_2)^{\mathtt{t}} \in \mathrm{R}^2$ and $\alpha := (\alpha_1, \alpha_2)^{\mathtt{t}}$, with nonnegative integers α_1, α_2, we set $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\alpha| := \alpha_1 + \alpha_2$. According to the above and the definition of H_h (cf. (5.3.1)), we propose the following degrees of freedom for a given $\boldsymbol{\tau} \in H_h$:

$$\begin{array}{llll}
\text{a)} & \displaystyle\int_e \boldsymbol{\tau}\mathbf{n}\cdot\mathbf{q} & \forall\,\mathbf{q} \in \mathcal{B}_k(e) & \forall \text{ edge } e \in \mathcal{T}_h\,, \\
\text{b)} & \displaystyle\int_K \boldsymbol{\tau} : \nabla\mathbf{q} & \forall\,\mathbf{q} \in \mathcal{B}_{k-1}(K) \setminus \left\{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\right\} & \forall\, K \in \mathcal{T}_h\,, \\
\text{c)} & \displaystyle\int_K \mathbf{q}\cdot\mathbf{rot}\,\boldsymbol{\tau} & \forall\,\mathbf{q} \in \mathcal{B}_{k-1}(K) & \forall\, K \in \mathcal{T}_h\,.
\end{array} \tag{5.3.6}$$

We now observe, according to the cardinalities of $\mathcal{B}_k(e)$ and $\mathcal{B}_{k-1}(K)$, that the amount of local degrees of freedom, that is those related to a given $K \in \mathcal{T}_h$, is given by

$$n_k^K := 2(k+1)\,d_K + \left\{k(k+1)-2\right\} + k(k+1) = 2\left\{(k+1)\,(d_K+k) - 1\right\}, \tag{5.3.7}$$

where d_K is the number of edges of K. Moreover, we have the following local unisolvence result.

Lemma 5.3.1. Given an integer $k \geq 1$, we define for each $K \in \mathcal{T}_h$ the local space

$$\begin{aligned} H_h^K := \Big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}\,; K) \cap \mathbb{H}(\mathbf{rot}\,; K) : \quad & \boldsymbol{\tau}\,\mathbf{n}\Big|_e \in \mathbf{P}_k(e) \quad \forall \text{ edge } e \subseteq \partial K\,, \\ & \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{P}_{k-1}(K)\,, \quad \mathbf{rot}\,\boldsymbol{\tau} \in \mathbf{P}_{k-1}(K) \Big\}\,. \end{aligned} \tag{5.3.8}$$

Then, the n_k^K local degrees of freedom arising from (5.3.6) are unisolvent in H_h^K.

Proof. Let $\boldsymbol{\tau} \in H_h^K$ such that

$$\begin{aligned}
\int_e \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,, \\
\int_K \boldsymbol{\tau} : \nabla \mathbf{q} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,, \\
\int_K \mathbf{q} \cdot \mathbf{rot}\, \boldsymbol{\tau} &= 0 \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,.
\end{aligned} \tag{5.3.9}$$

It follows easily from the definition (5.3.8) together with the first and third equations of (5.3.9) that

$$\boldsymbol{\tau}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial K\,, \quad \text{and} \quad \mathbf{rot}\, \boldsymbol{\tau} = \mathbf{0} \quad \text{in} \quad K\,. \tag{5.3.10}$$

In turn, integrating by parts the second equation in (5.3.9), we find that

$$0 = \int_K \boldsymbol{\tau} : \nabla \mathbf{q} = -\int_K \mathbf{q} \cdot \mathbf{div} \boldsymbol{\tau} + \int_{\partial K} \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} = -\int_K \mathbf{q} \cdot \mathbf{div} \boldsymbol{\tau} \qquad \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,,$$

which yields $\mathbf{div}\boldsymbol{\tau} = \mathbf{0}$ in K. Now, since K is simply connected, we know from the second identity in (5.3.10) and [30, Chapter I, Theorem 2.9] that there exists $\boldsymbol{\phi} \in \mathbf{H}^1(K)$ such that $\boldsymbol{\tau} = \nabla \boldsymbol{\phi}$ in K. In this way, the free divergence property of $\boldsymbol{\tau}$, and the fact that its normal component is the null vector on ∂K, can be rewritten as

$$\Delta \boldsymbol{\phi} = \mathbf{0} \quad \text{in} \quad K\,, \quad \nabla \boldsymbol{\phi}\, \mathbf{n} = \mathbf{0} \quad \text{on} \quad \partial K\,.$$

Thus, the classical solvability analysis of this Neumann problem implies that $\boldsymbol{\phi}$ is a constant vector, and hence $\boldsymbol{\tau}$ vanishes in K, which completes the proof. $\square$

5.3.3 Interpolation on H_h and Q_h

In this section we define suitable interpolation operators on our virtual element subspaces and establish their corresponding approximation properties. To this end, we need some preliminary notations and technical results. For each element $K \in \mathcal{T}_h$ we let $\widetilde{K} := T_K(K)$, where $T_K : \mathrm{R}^2 \longrightarrow \mathrm{R}^2$ is the bijective affine mapping defined by $T_K(\mathbf{x}) := \dfrac{\mathbf{x} - \mathbf{x}_B}{h_K}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$. Note that the diameter $h_{\widetilde{K}}$ of $\widetilde{K}$ is 1, and, according to the assumptions a) and b), it is easy to see that the shortest edge of $\widetilde{K}$ is bigger than $C_{\mathcal{T}}$, and that $\widetilde{K}$ is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. Recall here that $\mathbf{x}_B$ is the center of the ball B with respect to which K is star-shaped. Then, by connecting each vertex of $\widetilde{K}$ to the center of $\widetilde{B}$, that is to the origin, we generate a partition of $\widetilde{K}$ into $d_{\widetilde{K}}$ triangles $\widetilde{\Delta}_i$, $i \in \{1, 2, \dots, d_{\widetilde{K}}\}$, where $d_{\widetilde{K}} \leq N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\dfrac{\widetilde{h}_i}{\widetilde{\rho}_i} \leq c_{\mathcal{T}} \;\; \forall\, i \in \{1, 2, \dots, d_{\widetilde{K}}\}$, where $\widetilde{h}_i$ is the diameter of $\widetilde{\Delta}_i$ and $\widetilde{\rho}_i$ is the diameter of the largest ball contained in $\widetilde{\Delta}_i$. We also let $\widehat{\Delta}$ be the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$, and for each $i \in \{1, 2, \dots, d_{\widetilde{K}}\}$ we let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\,\mathbf{x}$ $\forall\, \mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \widetilde{\Delta}_i$. We remark that the fact that the origin is a vertex of each triangle $\widetilde{\Delta}_i$ allows to choose F_i as indicated.

In what follows, given $\mathbf{v} \in \mathbf{L}^2(K)$, we let $\widetilde{\mathbf{v}} := \mathbf{v} \circ T_K^{-1} \in \mathbf{L}^2(\widetilde{K})$. Then, we have the following result.

Lemma 5.3.2. Given an integer $\ell \geq 0$ and an element $K \in \mathcal{T}_h$, we let $\mathcal{P}_\ell^K : \mathbf{L}^2(K) \to \mathbf{P}_\ell(K)$ and $\mathcal{P}_\ell^{\widetilde{K}} : \mathbf{L}^2(\widetilde{K}) \to \mathbf{P}_\ell(\widetilde{K})$ be the corresponding orthogonal projectors. Then $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$ for all $\mathbf{v} \in \mathbf{L}^2(K)$, and for any pair of nonnegative integers r and s there holds $\mathcal{P}_\ell^{\widetilde{K}} \in \mathcal{L}(\mathbf{H}^r(\widetilde{K}), \mathbf{H}^s(\widetilde{K}))$, with $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))}$ independent of $\widetilde{K}$, namely depending only on ℓ, s, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$.

Proof. Denoting $N_\ell := (\ell+1)(\ell+2)$, we let $\left\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_{N_\ell}\right\}$ be a basis of $\mathbf{P}_\ell(K)$, in particular $\mathcal{B}_\ell(K)$ (cf. (5.3.5)), and observe that $\left\{\widetilde{\boldsymbol{\varphi}}_1, \widetilde{\boldsymbol{\varphi}}_2, \dots, \widetilde{\boldsymbol{\varphi}}_{N_\ell}\right\}$ becomes a basis of $\mathbf{P}_\ell(\widetilde{K})$. Hence, given $\mathbf{v} \in \mathbf{L}^2(K)$, and bearing in mind that the Jacobian of T_K is h_K^{-2}, we find that for each $j \in \{1, 2, \dots, N_\ell\}$ there holds

$$\int_{\widetilde{K}} \widetilde{\mathcal{P}_\ell^K(\mathbf{v})} \cdot \widetilde{\boldsymbol{\varphi}}_j \,=\, h_K^{-2} \int_K \mathcal{P}_\ell^K(\mathbf{v}) \cdot \boldsymbol{\varphi}_j \,=\, h_K^{-2} \int_K \mathbf{v} \cdot \boldsymbol{\varphi}_j \,=\, \int_{\widetilde{K}} \widetilde{\mathbf{v}} \cdot \widetilde{\boldsymbol{\varphi}}_j \,=\, \int_{\widetilde{K}} \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}}) \cdot \widetilde{\boldsymbol{\varphi}}_j\,,$$

which shows that $\widetilde{\mathcal{P}_\ell^K(\mathbf{v})} = \mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})$. Throughout the rest of the proof we assume for simplicity that $\left\{\widetilde{\boldsymbol{\varphi}}_1, \widetilde{\boldsymbol{\varphi}}_2, \dots, \widetilde{\boldsymbol{\varphi}}_{N_\ell}\right\}$ is orthonormal, which yields $\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}}) = \displaystyle\sum_{j=1}^{N_\ell} \langle \widetilde{\mathbf{v}}, \widetilde{\boldsymbol{\varphi}}_j \rangle_{0,\widetilde{K}}\, \widetilde{\boldsymbol{\varphi}}_j\,.$

Then, employing the Cauchy-Schwarz inequality, we obtain

$$\|\mathcal{P}_\ell^{\widetilde{K}}(\widetilde{\mathbf{v}})\|_{s,\widetilde{K}} \,\le\, \left\{ \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}} \right\} \|\widetilde{\mathbf{v}}\|_{0,\widetilde{K}} \,\le\, \left\{ \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}} \right\} \|\widetilde{\mathbf{v}}\|_{r,\widetilde{K}}\,,$$

which proves that $\mathcal{P}_\ell^{\widetilde{K}} \,\in\, \mathcal{L}(\mathbf{H}^r(\widetilde{K}), \mathbf{H}^s(\widetilde{K}))$, with

$$\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))} \,\le\, \sum_{j=1}^{N_\ell} \|\widetilde{\boldsymbol{\varphi}}_j\|_{s,\widetilde{K}} \,=\, \sum_{j=1}^{N_\ell} \left\{ \sum_{i=1}^{d_{\widetilde{K}}} \|\widetilde{\boldsymbol{\varphi}}_j\|^2_{s,\widetilde{\Delta}_i} \right\}^{1/2}\,, \tag{5.3.11}$$

where the last equality makes use of the aforementioned decomposition of $\widetilde{K}$. We now apply the usual scaling properties connecting the Sobolev integer seminorms in each $\widetilde{\Delta}_i$ with those in $\widehat{\Delta}$. In this way, denoting $\widehat{\boldsymbol{\varphi}}_{j,i} \,:=\, \widetilde{\boldsymbol{\varphi}}_j|_{\widetilde{\Delta}_i} \circ F_i \in \mathbf{P}_\ell(\widehat{\Delta})$, using the equivalence of norms in $\mathbf{P}_\ell(\widehat{\Delta})$, and noting that $\widetilde{\rho}_i^{-1} \,\le\, c_{\mathcal{T}}\,\widetilde{h}_i^{-1} \,\le\, c_{\mathcal{T}}\,C_{\mathcal{T}}^{-1}$, we deduce that for each integer $t \ge 0$ there holds

$$\begin{aligned} |\widetilde{\boldsymbol{\varphi}}_j|_{t,\widetilde{\Delta}_i} \,&\le\, C_t\,\widehat{h}^t\,\widetilde{\rho}_i^{-t}\,|\det B_i|^{1/2}\,\|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{t,\widehat{\Delta}} \,\le\, C_t\,\widehat{h}^t\,c_{\mathcal{T}}^t\,C_{\mathcal{T}}^{-t}\,|\det B_i|^{1/2}\,\widehat{c}\,\|\widehat{\boldsymbol{\varphi}}_{j,i}\|_{0,\widehat{\Delta}} \\ &=\, C_t\,\widehat{h}^t\,c_{\mathcal{T}}^t\,C_{\mathcal{T}}^{-t}\,\widehat{c}\,\|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{\Delta}_i} \,\le\, C_t\,\widehat{h}^t\,c_{\mathcal{T}}^t\,C_{\mathcal{T}}^{-t}\,\widehat{c}\,\|\widetilde{\boldsymbol{\varphi}}_j\|_{0,\widetilde{K}} \,=\, C\,, \end{aligned}$$

where C_t depends on t, whereas $\widehat{c}$ depends on $\mathbf{P}_\ell(\widehat{\Delta})$ and t, and $C \,=\, C_t\,\widehat{h}^t\,c_{\mathcal{T}}^t\,C_{\mathcal{T}}^{-t}\,\widehat{c}$. The foregoing inequality and (5.3.11) give the announced independence of $\|\mathcal{P}_\ell^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^s(\widetilde{K}))}$, which ends the proof.

□

The next result taken from [7, Lemma 4.3.8] (see also [19]) is required in what follows as well.

Lemma 5.3.3. Let $\mathcal{O}$ be a domain of R^2 with diameter 1, such that it is star-shaped with respect to a ball B of radius $> \frac{1}{2}\,\rho_{\mathtt{max}}$, where

$$\rho_{\mathtt{max}} \,:=\, \sup\Big\{\,\rho : \quad \mathcal{O} \text{ is star-shaped with respect to a ball of radius } \rho\Big\}.$$

In addition, given an integer $m \ge 1$ and $\mathbf{v} \in \mathbf{H}^m(\mathcal{O})$, we let $\mathbf{T}^m(\mathbf{v}) \,\in\, \mathbf{P}_{m-1}(\mathcal{O})$ be the Taylor polynomial of order m of $\mathbf{v}$ averaged over B. Then, there exists $C > 0$, depending only on m and $\rho_{\mathtt{max}}$, such that

$$|\mathbf{v} - \mathbf{T}^m(\mathbf{v})|_{\ell,\mathcal{O}} \,\le\, C\,|\mathbf{v}|_{m,\mathcal{O}} \qquad \forall\,\ell \in \{0, 1, \ldots, m\}\,.$$

We now proceed to define our interpolation operators. We begin by letting $\mathcal{P}_{k-1}^h : \mathbf{L}^2(\Omega) \longrightarrow Q_h$ be the orthogonal projector, that is, given $\mathbf{v} \in Q := \mathbf{L}^2(\Omega)$, $\mathcal{P}_{k-1}^h(\mathbf{v})$ is characterized by

$$\int_K \big(\mathbf{v} - \mathcal{P}_{k-1}^h(\mathbf{v})\big) \cdot \mathbf{q} = 0 \qquad \forall\, K \in \mathcal{T}_h\,, \quad \forall\, \mathbf{q} \in \mathbf{P}_{k-1}(K)\,, \tag{5.3.12}$$

which means, equivalently, that

$$\mathcal{P}_{k-1}^h(\mathbf{v})\big|_K = \mathcal{P}_{k-1}^K(\mathbf{v}|_K)\,,$$

where, as indicated in Lemma 5.3.2, $\mathcal{P}_{k-1}^K : \mathbf{L}^2(K) \to \mathbf{P}_{k-1}(K)$ is the local orthogonal projector. The following lemma establishes the approximation properties of this operator.

Lemma 5.3.4. Let k, ℓ and r be integers such that $1 \le r \le k$ and $0 \le \ell \le r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$|\mathbf{v} - \mathcal{P}_{k-1}^K(\mathbf{v})|_{\ell,K} \le C\, h_K^{r-\ell}\, |\mathbf{v}|_{r,K} \qquad \forall\, \mathbf{v} \in \mathbf{H}^r(K)\,.$$

Proof. Given integers k, ℓ and r as stated, $K \in \mathcal{T}_h$, and $\mathbf{v} \in \mathbf{H}^r(K)$, we first observe that there hold

$$|\widetilde{\mathbf{v}}|_{\ell,\widetilde{K}} = h_K^{\ell+1}\, |\mathbf{v}|_{\ell,K} \quad \text{and} \quad \mathcal{P}_{k-1}^{\tilde{K}}\big(\widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\big) = \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\,,$$

where $\widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}}) \in \mathbf{P}_{r-1}(\widetilde{K})$ is the Taylor polynomial of order r of $\widetilde{\mathbf{v}}$ averaged over a ball of radius $> \frac{1}{2}\,\widetilde{\rho}_{\mathtt{max}}$, where

$$\widetilde{\rho}_{\mathtt{max}} := \sup\Big\{\rho : \quad \widetilde{K} \text{ is star-shaped with respect to a ball of radius } \rho\Big\}.$$

Recall here that $\widetilde{K}$ has diameter 1 and is star-shaped with respect to a ball $\widetilde{B}$ of radius $C_{\mathcal{T}}$ and centered at the origin. It follows, using Lemmas 5.3.2 and 5.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
|\mathbf{v} - \mathcal{P}_{k-1}^K(\mathbf{v})|_{\ell,K} &= h_K^{-\ell-1}\, |\widetilde{\mathbf{v}} - \widetilde{\mathcal{P}_{k-1}^K(\mathbf{v})}|_{\ell,\widetilde{K}} = h_K^{-\ell-1}\, |\widetilde{\mathbf{v}} - \mathcal{P}_{k-1}^{\widetilde{K}}(\widetilde{\mathbf{v}})|_{\ell,\widetilde{K}} \\
&= h_K^{-\ell-1}\, |\big(\mathrm{I} - \mathcal{P}_{k-1}^{\widetilde{K}}\big)\big(\widetilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\big)|_{\ell,\widetilde{K}} \le h_K^{-\ell-1}\, \|\mathrm{I} - \mathcal{P}_{k-1}^{\widetilde{K}}\|_{\mathcal{L}(\mathbf{H}^r(\widetilde{K}),\mathbf{H}^\ell(\widetilde{K}))}\, \|\widetilde{\mathbf{v}} - \widetilde{\mathbf{T}}^r(\tilde{\mathbf{v}})\|_{r,\widetilde{K}} \\
&\le C\, h_K^{-\ell-1}\, |\widetilde{\mathbf{v}}|_{r,\widetilde{K}} = C\, h_K^{r-\ell}\, |\mathbf{v}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. $\square$

We now let

$$\widetilde{H} \,:=\, \Big\{\boldsymbol{\tau} \in H : \quad \boldsymbol{\tau}|_K \in \mathbb{L}^s(K) \text{ (for some } s > 2) \text{ and } \mathbf{rot}\,\boldsymbol{\tau}|_K \in \mathbf{L}^1(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,, \tag{5.3.13}$$

and introduce an interpolation operator $\Pi_k^h : \widetilde{H} \longrightarrow H_h$. Indeed, given $\boldsymbol{\tau} \in \widetilde{H}$, we let $\Pi_k^h(\boldsymbol{\tau})$ be the unique element in H_h such that

$$\begin{aligned}
0 \,&=\, \int_e \big(\boldsymbol{\tau} - \Pi_k^h(\boldsymbol{\tau})\big)\mathbf{n}\cdot\mathbf{q} && \forall\,\mathbf{q} \in \mathcal{B}_k(e) && \forall \text{ edge } e \in \mathcal{T}_h\,,\\
0 \,&=\, \int_K \big(\boldsymbol{\tau} - \Pi_k^h(\boldsymbol{\tau})\big) : \nabla\mathbf{q} && \forall\,\mathbf{q} \in \mathcal{B}_{k-1}(K)\setminus\big\{(1,0)^{\mathtt{t}},(0,1)^{\mathtt{t}}\big\} && \forall\, K \in \mathcal{T}_h\,,\\
0 \,&=\, \int_K \mathbf{q}\cdot\mathbf{rot}\,\big(\boldsymbol{\tau} - \Pi_k^h(\boldsymbol{\tau})\big) && \forall\,\mathbf{q} \in \mathcal{B}_{k-1}(K) && \forall\, K \in \mathcal{T}_h\,.
\end{aligned} \tag{5.3.14}$$

Note here that the extra local regularities on $\boldsymbol{\tau}$ and $\mathbf{rot}\,\boldsymbol{\tau}$ allow to define normal traces of $\boldsymbol{\tau}$ on the edges of $\mathcal{T}_h$ and the moments involving $\mathbf{rot}\,\boldsymbol{\tau}$ in each $K \in \mathcal{T}_h$, respectively. In addition, the uniqueness of $\Pi_k^h(\boldsymbol{\tau})$ is guaranteed by Lemma 5.3.1. Next, we define the local restriction of the interpolation operator as $\Pi_k^K(\boldsymbol{\tau}) \,:=\, \Pi_k^h(\boldsymbol{\tau})|_K \,\in\, H_h^K$. It follows that for each $\mathbf{q} \in \mathbf{P}_{k-1}(K)$ there holds

$$\int_K \mathbf{div}\,\big(\boldsymbol{\tau} \,-\, \Pi_k^K(\boldsymbol{\tau})\big)\cdot\mathbf{q} \,=\, -\int_K \big(\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\big) : \nabla\mathbf{q} \,+\, \int_{\partial K} \big(\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\big)\,\mathbf{n}\cdot\mathbf{q} \,=\, 0\,,$$

which, together with the fact that $\mathbf{div}\,\Pi_k^K(\boldsymbol{\tau}) \,\in\, \mathbf{P}_{k-1}(K)$, implies that

$$\mathbf{div}\,\Pi_k^K(\boldsymbol{\tau}) \,=\, \mathcal{P}_{k-1}^K(\mathbf{div}\,\boldsymbol{\tau})\,. \tag{5.3.15}$$

This identity implies the following result.

Lemma 5.3.1. *Let k, ℓ and r be integers satisfying $1 \le r \le k$ and $0 \le \ell \le r$. Then, there exists a constant $C > 0$, depending only on k, ℓ, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ and for any $\boldsymbol{\tau}$ verifying additionally that $\mathbf{div}\,\boldsymbol{\tau}|_K \in \mathbf{H}^r(K)$ there holds*

$$|\mathbf{div}\,\boldsymbol{\tau} - \mathbf{div}\,\Pi_k^K(\boldsymbol{\tau})|_{l,K} \,\le\, C\,h_K^{r-l}\,|\mathbf{div}\,\boldsymbol{\tau}|_{r,K}\,. \tag{5.3.16}$$

Proof. It follows from a straightforward application of Lemma 5.3.4. $\square$

We now consider $K \in \mathcal{T}_h$ and set the local moments defining H_h^K. Indeed, given $\boldsymbol{\tau}$ as required by (5.3.14), we define the K-moments:

$$\begin{aligned} m_{\mathbf{q},e}^{\mathbf{n}}(\boldsymbol{\tau}) &:= \int_e \boldsymbol{\tau}\mathbf{n} \cdot \mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_k(e)\,, \quad \forall \text{ edge } e \subseteq \partial K\,, \\ m_{\mathbf{q},K}^{\mathbf{div}}(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \nabla \mathbf{q} && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K) \setminus \big\{(1,0)^{\mathtt{t}}, (0,1)^{\mathtt{t}}\big\}\,, \\ m_{\mathbf{q},K}^{\mathbf{rot}}(\boldsymbol{\tau}) &:= \int_K \mathbf{q} \cdot \mathbf{rot}\,(\boldsymbol{\tau}) && \forall\, \mathbf{q} \in \mathcal{B}_{k-1}(K)\,, \end{aligned} \tag{5.3.17}$$

and gather all the above in the set $\Big\{m_{j,K}(\boldsymbol{\tau})\Big\}_{j=1}^{n_k^K}$. Then, we let $\{\boldsymbol{\varphi}_{j,K}\}_{j=1}^{n_k^K}$ be the canonical basis of H_h^K, that is, given $i \in \{1, 2, \ldots, n_k^K\}$, $\boldsymbol{\varphi}_{i,K}$ is the unique element in H_h^K such that

$$m_{j,K}(\boldsymbol{\varphi}_{i,K}) \,=\, \delta_{ij} \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,.$$

It follows easily that

$$\Pi_k^K(\boldsymbol{\tau}) \,:=\, \sum_{j=1}^{n_k^K} m_{j,K}(\boldsymbol{\tau})\, \boldsymbol{\varphi}_{j,K}\,, \tag{5.3.18}$$

or, equivalently, $\Pi_k^K(\boldsymbol{\tau})$ is the unique element in H_h^K such that

$$m_{j,K}(\Pi_k^K(\boldsymbol{\tau})) \,=\, m_{j,K}(\boldsymbol{\tau}) \qquad \forall\, j \in \{1, 2, \ldots, n_k^K\}\,. \tag{5.3.19}$$

We now provide the analogue of Lemmas 5.3.2 and 5.3.4 for the foregoing local operator Π_k^K.

Lemma 5.3.5. Given integers k, $\ell \geq 1$, and $K \in \mathcal{T}_h$, there holds $\widetilde{\Pi_k^K(\boldsymbol{\tau})} \,=\, \Pi_k^{\widetilde{K}}(\widetilde{\boldsymbol{\tau}})$ for all $\boldsymbol{\tau} \in \mathbb{H}^k(K)$, and $\Pi_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbb{H}^\ell(\widetilde{K}), \mathbb{L}^2(\widetilde{K})\big)$ with $\|\Pi_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbb{H}^\ell(\widetilde{K}),\mathbb{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$, namely depending only on k, ℓ, $N_{\mathcal{T}}$, and $C_{\mathcal{T}}$.

Proof. We first note that clearly $\boldsymbol{\varphi} \in H_h^K$ if and only if $\widetilde{\boldsymbol{\varphi}} \,:=\, \boldsymbol{\varphi} \circ T_K^{-1} \in H_h^{\widetilde{K}}$. In particular, given $\boldsymbol{\tau} \in \mathbb{H}^k(K)$, there holds $\widetilde{\Pi_k^K(\boldsymbol{\tau})} \in H_h^{\widetilde{K}}$, and hence the required identity holds if and only if the $\widetilde{K}$-moments of $\widetilde{\Pi_k^K(\boldsymbol{\tau})}$ and $\Pi_k^{\widetilde{K}}(\widetilde{\boldsymbol{\tau}})$ coincide. Indeed, let $\widetilde{e}$ be an edge of $\partial\widetilde{K}$, and let $e := T_K^{-1}(\widetilde{e})$ be the corresponding edge of ∂K. Then, for each $\mathbf{q} \in \mathcal{B}_k(e)$, and letting $\widetilde{\mathbf{q}} := \mathbf{q} \circ T_K^{-1} \in \mathbf{P}_k(\widetilde{e})$, we find, integrating by parts and using (5.3.19), that

$$\begin{aligned}\int_{\widetilde{e}} \widetilde{\Pi_k^K(\boldsymbol{\tau})}\widetilde{\mathbf{n}} \cdot \widetilde{\mathbf{q}} &= \int_{\widetilde{K}} \widetilde{\mathbf{q}} \cdot \mathbf{div}\, \widetilde{\Pi_k^K(\boldsymbol{\tau})} + \int_{\widetilde{K}} \widetilde{\Pi_k^K(\boldsymbol{\tau})} : \nabla \widetilde{\mathbf{q}} \\ &= h_K^{-3} \int_K \mathbf{q} \cdot \mathbf{div}\, \Pi_k^K(\boldsymbol{\tau}) + h_K^{-3} \int_K \Pi_k^K(\boldsymbol{\tau}) : \nabla \mathbf{q} = h_K^{-3} \int_e \Pi_k^K(\boldsymbol{\tau})\mathbf{n} \cdot \mathbf{q} \\ &= h_K^{-3} \int_e (\boldsymbol{\tau}\mathbf{n}) \cdot \mathbf{q} = h_K^{-3} \int_K \mathbf{q} \cdot \mathbf{div}\, \boldsymbol{\tau} + h_K^{-3} \int_K \boldsymbol{\tau} : \nabla \mathbf{q} \\ &= \int_K \widetilde{\mathbf{q}} \cdot \mathbf{div}\, \widetilde{\boldsymbol{\tau}} + \int_K \widetilde{\boldsymbol{\tau}} : \nabla \widetilde{\mathbf{q}} = \int_{\widetilde{e}} (\widetilde{\boldsymbol{\tau}}\widetilde{\mathbf{n}}) \cdot \widetilde{\mathbf{q}}\,.\end{aligned} \tag{5.3.20}$$

Next, if $k \geq 2$ and $\mathbf{q} \in \mathcal{B}_{k-1}(K)$, we easily obtain, using again (5.3.19), that

$$\int_{\widetilde{K}} \widetilde{\Pi_k^K(\boldsymbol{\tau})} : \nabla \widetilde{\mathbf{q}} = h_K^{-3} \int_K \Pi_k^K(\boldsymbol{\tau}) : \nabla \mathbf{q} = h_K^{-3} \int_K \boldsymbol{\tau} : \nabla \mathbf{q} = \int_{\widetilde{K}} \widetilde{\boldsymbol{\tau}} : \nabla \widetilde{\mathbf{q}}\,, \tag{5.3.21}$$

and

$$\int_{\widetilde{K}} \widetilde{\mathbf{q}} \cdot \mathbf{rot}\, \widetilde{\Pi_k^K(\boldsymbol{\tau})} = h_K^{-3} \int_K \mathbf{q} \cdot \mathbf{rot}\, \Pi_k^K(\boldsymbol{\tau}) = h_K^{-3} \int_K \mathbf{q} \cdot \mathbf{rot}\, \boldsymbol{\tau} = \int_{\widetilde{K}} \widetilde{\mathbf{q}} \cdot \mathbf{rot}\, \widetilde{\boldsymbol{\tau}}\,. \tag{5.3.22}$$

In this way, (5.3.17) together with (5.3.20), (5.3.21), and (5.3.22) confirm that $\widetilde{\Pi_k^K(\boldsymbol{\tau})}$ and $\Pi_k^{\widetilde{K}}(\widetilde{\boldsymbol{\tau}})$ share the same $\widetilde{K}$-moments.

It remains to show that $\Pi_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbb{H}^\ell(\widetilde{K}), \mathbb{L}^2(\widetilde{K})\big)$, with $\|\Pi_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbb{H}^\ell(\widetilde{K}), \mathbb{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$. For this purpose, we first observe from (5.3.18) that

$$\|\Pi_k^{\widetilde{K}}(\boldsymbol{\tau})\|_{0,\widetilde{K}} \leq \sum_{j=1}^{n_k^{\widetilde{K}}} |m_{j,\widetilde{K}}(\boldsymbol{\tau})|\, \|\boldsymbol{\varphi}_{j,\widetilde{K}}\|_{0,\widetilde{K}}\,,$$

where each $m_{j,\widetilde{K}}$ is defined according to (5.3.17) with $\mathcal{B}_{k-1}(\widetilde{K})$ and $B_k(\widetilde{e})$ $\forall$ edge $\widetilde{e} \subseteq \partial\widetilde{K}$, and $\left\{\boldsymbol{\varphi}_{j,\widetilde{K}}\right\}_{j=1}^{n_k^{\widetilde{K}}}$ is the canonical basis of $H_h^{\widetilde{K}}$. Next, we proceed to bound the $\widetilde{K}$-moments in terms of $\|\boldsymbol{\tau}\|_{\ell,\widetilde{K}}$. In fact, given an edge $\widetilde{e} \subseteq \partial\widetilde{K}$, and $\mathbf{q} \in \mathcal{B}_k(\widetilde{e})$, a simple computation shows that $\|\mathbf{q}\|_{0,\widetilde{e}} \leq h_{\widetilde{e}}^{1/2} \leq h_{\widetilde{K}}^{1/2} = 1$, and then, applying the Cauchy-Schwarz and discrete trace inequalities, and using that $h_{\widetilde{e}} \geq C_{\mathcal{T}}$, we obtain

$$|m_{\mathbf{q},\widetilde{e}}^{\mathbf{n}}(\boldsymbol{\tau})| \leq \|\boldsymbol{\tau}\|_{0,\widetilde{e}}\, \|\mathbf{q}\|_{0,\widetilde{e}} \leq c \left\{ C_{\mathcal{T}}^{-1/2}\, \|\boldsymbol{\tau}\|_{0,\widetilde{K}} + |\boldsymbol{\tau}|_{1,\widetilde{K}} \right\} \leq C\, \|\boldsymbol{\tau}\|_{1,\widetilde{K}} \leq C\, \|\boldsymbol{\tau}\|_{\ell,\widetilde{K}}\,.$$

In turn, given now $\mathbf{q} \in \mathcal{B}_{k-1}(\widetilde{K})$, it is easy to see that $\|\mathbf{q}\|_{0,\widetilde{K}}$ and $|\mathbf{q}|_{1,\widetilde{K}}$ are both bounded by constants independent of $\widetilde{K}$, and hence straightforward applications of the Cauchy-Schwarz inequality yield

$$|m^{\mathbf{div}}_{\mathbf{q},\widetilde{K}}(\boldsymbol{\tau})| \,+\, |m^{\mathbf{rot}}_{\mathbf{q},\widetilde{K}}(\boldsymbol{\tau})| \,\le\, C \left\{ \|\boldsymbol{\tau}\|_{0,\widetilde{K}} \,+\, |\boldsymbol{\tau}|_{1,\widetilde{K}} \right\} \,\le\, C\, \|\boldsymbol{\tau}\|_{\ell,\widetilde{K}} \,,$$

Finally, we claim that, thanks to the assumptions a) and b) (cf. beginning of Section 5.3.2) and the choice of the normalized monomials given by $\mathcal{B}_\ell(e)$ and $\mathcal{B}_\ell(K)$ (cf. (5.3.4), (5.3.5)), there holds $\|\boldsymbol{\varphi}_{j,\widetilde{K}}\|_{0,\widetilde{K}} \,=\, O(1)$, which would complete the boundedness of $\Pi_k^{\widetilde{K}}$. The specific technical details, however, will be given somewhere else. $\square$

Lemma 5.3.6. Let k and r be integers such that $k \ge 1$ and $1 \le r \le k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\boldsymbol{\tau} \,-\, \Pi_k^K(\boldsymbol{\tau})\|_{0,K} \,\le\, C\, h_K^r\, |\boldsymbol{\tau}|_{r,K} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K)\,. \tag{5.3.23}$$

Proof. We proceed similarly as in the proof of Lemma 5.3.4. In fact, given integers k and r as stated, $K \in \mathcal{T}_h$, and $\boldsymbol{\tau} \in \mathbb{H}^r(K)$, we let $\widetilde{\mathbb{T}}^r(\widetilde{\boldsymbol{\tau}}) \,\in\, \mathbb{P}_{r-1}(\widetilde{K})$ be the tensor version of the averaged Taylor polynomial of order r of $\widetilde{\boldsymbol{\tau}}$ (cf. Lemma 5.3.3), and observe, since $r-1 \le k$, that $\Pi_k^{\widetilde{K}}\big(\widetilde{\mathbb{T}}^r(\widetilde{\boldsymbol{\tau}})\big) \,=\, \widetilde{\mathbb{T}}^r(\widetilde{\boldsymbol{\tau}})$. It follows, using Lemmas 5.3.5 and 5.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
\|\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\|_{0,K} \,&=\, h_K^{-1}\, \|\widetilde{\boldsymbol{\tau}} - \widetilde{\Pi_k^K(\boldsymbol{\tau})}\|_{0,\widetilde{K}} \,=\, h_K^{-1}\, \|\widetilde{\boldsymbol{\tau}} - \Pi_k^{\widetilde{K}}(\widetilde{\boldsymbol{\tau}})\|_{0,\widetilde{K}} \\
&=\, h_K^{-1}\, \|\big(\mathrm{I} - \Pi_k^{\widetilde{K}}\big)\big(\widetilde{\boldsymbol{\tau}} - \widetilde{\mathbb{T}}^r(\widetilde{\boldsymbol{\tau}})\big)\|_{0,\widetilde{K}} \,\le\, h_K^{-1}\, \|\mathrm{I} - \Pi_k^{\widetilde{K}}\|_{\mathcal{L}(\mathbb{H}^r(\widetilde{K}),\mathbb{L}^2(\widetilde{K}))}\, \|\widetilde{\boldsymbol{\tau}} - \widetilde{\mathbb{T}}^r(\widetilde{\boldsymbol{\tau}})\|_{r,\widetilde{K}} \\
&\le\, C\, h_K^{-1}\, |\widetilde{\boldsymbol{\tau}}|_{r,\widetilde{K}} \,=\, C\, h_K^r\, |\boldsymbol{\tau}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. $\square$

As a corollary of Lemmas 5.3.1 and 5.3.6 we have the following result.

Lemma 5.3.7. Let k and r be integers such that $1 \le r \le k$. Then, there exists a constant $C > 0$, depending only on k, r, $c_{\mathcal{T}}$, $C_{\mathcal{T}}$, and $N_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\boldsymbol{\tau} - \Pi_k^K(\boldsymbol{\tau})\|_{\mathbf{div};K} \,\le\, C\, h_K^r \left\{ |\boldsymbol{\tau}|_{r,K} + |\mathbf{div}\,\boldsymbol{\tau}|_{r,K} \right\} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(K) \quad \text{with} \quad \mathbf{div}\,\boldsymbol{\tau} \in \mathbf{H}^r(K)\,.$$

Proof. It suffices to apply (5.3.16) with $\ell = 0$ and then combine it with the estimate provided by Lemma 5.3.6. $\square$

5.3.4 The discrete bilinear forms

The ultimate purpose of this section is to define computable discrete versions $\mathbf{a}_h : H_h \times H_h \longrightarrow \mathrm{R}$ and $\mathbf{b}_h : H_h \times Q_h \longrightarrow \mathrm{R}$ of the bilinear forms $\mathbf{a}$ and $\mathbf{b}$, respectively. To this end, we first observe that, given $(\boldsymbol{\tau}, \mathbf{v}) \in H_h \times Q_h$, the expression

$$\mathbf{b}\,(\boldsymbol{\tau}, \mathbf{v}) \,:=\, \int_\Omega \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau} \,=\, \sum_{K\in\mathcal{T}_h} \int_K \mathbf{v} \cdot \mathbf{div}\,\boldsymbol{\tau}\,,$$

is explicitly calculable since, according to the definitions of H_h and Q_h (cf. (5.3.1), (5.3.2)), there holds $\mathbf{v}|_K \in \mathbf{P}_{k-1}(K)$ and $\mathbf{div}\,\boldsymbol{\tau}|_K \in \mathbf{P}_{k-1}(K)$ on each element K, and hence we just set $\mathbf{b}_h = \mathbf{b}$. On the contrary, given $\boldsymbol{\zeta}$, $\boldsymbol{\tau} \in H_h$, the expression

$$\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) \,:=\, \frac{1}{2\mu} \int_\Omega \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \,=\, \frac{1}{2\mu} \sum_{K\in\mathcal{T}_h} \int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}}$$

is not explicitly calculable since in general $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}$ are not known on each $K \in \mathcal{T}_h$. In order to overcome this difficulty, we now proceed to introduce suitable spaces on which the elements of H_h will be projected later on, and for which the bilinear form $\mathbf{a}$ is computable. Indeed, let us first consider a particular choice of $\boldsymbol{\tau}$ given by $\boldsymbol{\tau} := \nabla\,\underline{\mathrm{curl}}\,q \in \mathbb{P}_k(K)$ with $q \in \mathrm{P}_{k+2}(K)$, where $\underline{\mathrm{curl}} := \left(\frac{\partial}{\partial x_2}, -\frac{\partial}{\partial x_1}\right)$, and observe that $\mathrm{tr}(\boldsymbol{\tau}) = \mathrm{div}\,(\underline{\mathrm{curl}}\,q) = 0$, whence $\boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau}$. It follows that for each $\boldsymbol{\zeta} \in H_h$ there holds

$$\int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \,=\, \int_K \boldsymbol{\zeta} : \boldsymbol{\tau} \,=\, \int_K \boldsymbol{\zeta} : \nabla\,\underline{\mathrm{curl}}\,q \,=\, -\int_K \underline{\mathrm{curl}}\,q \cdot \mathbf{div}\,\boldsymbol{\zeta} \,+\, \int_{\partial K} (\boldsymbol{\zeta}\mathbf{n}) \cdot \underline{\mathrm{curl}}\,q\,, \tag{5.3.24}$$

which, bearing in mind from Lemma 5.3.1 that $\mathbf{div}\,\boldsymbol{\zeta}|_K$ and $\boldsymbol{\zeta}\mathbf{n}|_{\partial K}$ are explicitly known, shows that $\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau})$ is in fact calculable in this case. In turn, it is also quite clear that, given $\boldsymbol{\tau} := q\,\mathbb{I} \in \mathbb{P}_k(K)$ with $q \in \mathrm{P}_k(K)$, there holds $\mathbf{a}(\boldsymbol{\zeta}, \boldsymbol{\tau}) = 0$ for all $\boldsymbol{\zeta} \in H_h$. The above suggests to define the subspace of $\mathbb{P}_k(K)$ given by

$$\widehat{H}_k^K \,:=\, \widehat{H}_{k,\nabla}^K \,+\, \widehat{H}_{k,\mathbb{I}}^K\,,$$

where

$$\widehat{H}^K_{k,\nabla} := \left\{ \nabla \underline{\mathrm{curl}}\, q : \quad q \in \mathrm{P}_{k+2}(K) \right\} \quad \text{and} \quad \widehat{H}^K_{k,\mathbb{I}} := \left\{ q\,\mathbb{I} : \quad q \in \mathrm{P}_k(K) \right\}.$$

The following lemma establishes the basic properties of the space $\widehat{H}^K_k$.

Lemma 5.3.2. *There holds* $\widehat{H}^K_k := \widehat{H}^K_{k,\nabla} \oplus \widehat{H}^K_{k,\mathbb{I}}$ *and* $\dim \widehat{H}^K_k = (k+1)\,(k+4)$.

Proof. Given $\boldsymbol{\tau} \in \widehat{H}^K_{k,\nabla} \cap \widehat{H}^K_{k,\mathbb{I}}$, we have on one hand $\boldsymbol{\tau} = \boldsymbol{\tau}^{\mathrm{d}}$, and on the other hand $\boldsymbol{\tau}^{\mathrm{d}} = \mathbf{0}$, so that necessarily $\boldsymbol{\tau} = \mathbf{0}$. This shows the required decomposition and hence

$$\dim \widehat{H}^K_k = \dim \widehat{H}^K_{k,\nabla} + \dim \widehat{H}^K_{k,\mathbb{I}} = \dim \widehat{H}^K_{k,\nabla} + \dim \mathrm{P}_k(K)\,. \tag{5.3.25}$$

Alternatively, it is easy to see that $\widehat{H}^K_{k,\nabla}$ and $\widehat{H}^K_{k,\mathbb{I}}$ are orthogonal with respect to the usual inner product of $\mathbb{H}(\mathbf{rot}\,;K)$. In order to determine the remaining dimension in (5.3.25) we prove now that the set $\left\{\nabla \underline{\mathrm{curl}}\, \mathbf{x}^\alpha : \quad 2 \le |\alpha| \le k+2\right\}$ is a basis of $\widehat{H}^K_{k,\nabla}$. Indeed, the generation property is quite clear from the fact that $\left\{\mathbf{x}^\alpha : \quad 0 \le |\alpha| \le k+2\right\}$ is the canonical basis of $\mathrm{P}_{k+2}(K)$ and by observing that $\nabla \underline{\mathrm{curl}}\, q = \mathbf{0} \quad \forall\, q \in \mathrm{P}_1(K)$. Next, we consider scalars a_α, $2 \le |\alpha| \le k+2$, and set $\nabla \underline{\mathrm{curl}}\, q = \mathbf{0}$ with $q := \sum_{2\le|\alpha|\le k+2} a_\alpha\, \mathbf{x}^\alpha$. It follows that $\underline{\mathrm{curl}}\, q$ is a constant vector of R^2, that is

$$\begin{aligned}
\frac{\partial q}{\partial x_1} &= \sum_{1\le|\beta|\le k+1} a_{\beta+(1,0)}\,(\beta_1+1)\,\mathbf{x}^\beta = \text{constant} \quad \text{in} \quad K\,,\\
\frac{\partial q}{\partial x_2} &= \sum_{1\le|\beta|\le k+1} a_{\beta+(0,1)}\,(\beta_2+1)\,\mathbf{x}^\beta = \text{constant} \quad \text{in} \quad K\,,
\end{aligned}$$

which yields $a_{\beta+(1,0)} = a_{\beta+(0,1)} = 0$ for all $1 \le |\beta| \le k+1$. In this way, since clearly

$$\left\{a_\alpha : \ 2 \le |\alpha| \le k+2\right\} = \left\{a_{\beta+(1,0)} : \ 1 \le |\beta| \le k+1\right\} \cup \left\{a_{\beta+(0,1)} : \ 1 \le |\beta| \le k+1\right\},$$

we deduce that $q = 0$. Having thus identified a basis of $\widehat{H}^K_{k,\nabla}$, whose cardinality is certainly given by $\dim \mathrm{P}_{k+2}(K) - \dim \mathrm{P}_1(K)$, we conclude from (5.3.25) and the foregoing expression that

$$\dim \widehat{H}^K_k = \frac{(k+3)(k+4)}{2} - 3 + \frac{(k+1)(k+2)}{2} = (k+1)(k+4)\,,$$

which completes the proof. $\square$

We now introduce a projection operator $\widehat{\Pi}_k^K : \mathbb{H}(\mathbf{div}; K) \longrightarrow \widehat{H}_k^K$. To this end, we set for each $K \in \mathcal{T}_h$ the local bilinear form

$$\mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \;:=\; \frac{1}{2\mu} \int_K \boldsymbol{\zeta}^{\mathtt{d}} : \boldsymbol{\tau}^{\mathtt{d}} \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in \mathbb{L}^2(K)\,.$$

Then, we define $\widehat{\boldsymbol{\zeta}} \,:=\, \widehat{\Pi}_k^K(\boldsymbol{\zeta}) \in \widehat{H}_k^K$ in terms of the decomposition:

$$\widehat{\boldsymbol{\zeta}} = \widehat{\boldsymbol{\zeta}}_\nabla \,+\, q_{\boldsymbol{\zeta}}\, \mathbb{I} \,+\, c_{\boldsymbol{\zeta}}\, \mathbb{I}\,, \tag{5.3.26}$$

where the components $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}_{k,\nabla}^K$, $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K) \,:=\, \operatorname{span}\Big\{ \mathbf{x}^\alpha : \quad 1 \leq |\alpha| \leq k \Big\}$, and $c_{\boldsymbol{\zeta}} \,\in\, \mathrm{R}$ are computed according to the following sequentially connected problems:

- Find $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}_{k,\nabla}^K$ such that

$$\mathbf{a}^K(\widehat{\boldsymbol{\zeta}}_\nabla, \boldsymbol{\tau}) \;=\; \mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \,\in\, \widehat{H}_{k,\nabla}^K\,, \tag{5.3.27}$$

- Find $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K)$ such that

$$(\mathbf{div}(q_{\boldsymbol{\zeta}}\, \mathbb{I}), \mathbf{div}(q\, \mathbb{I}))_{0,K} \;=\; (\mathbf{div}\big(\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_\nabla\big), \mathbf{div}(q\, \mathbb{I}))_{0,K} \qquad \forall\, q \,\in\, \widehat{\mathrm{P}}_k(K)\,, \tag{5.3.28}$$

- Find $c_{\boldsymbol{\zeta}} \,\in\, \mathrm{R}$ such that:

$$\int_K \operatorname{tr}(\widehat{\boldsymbol{\zeta}}) \;=\; \int_K \operatorname{tr}(\boldsymbol{\zeta})\,. \tag{5.3.29}$$

We remark that the unique solvability of (5.3.27) is guaranteed by the identity

$$\mathbf{a}^K(\boldsymbol{\tau}, \boldsymbol{\tau}) \;=\; \frac{1}{2\mu}\, \|\boldsymbol{\tau}^{\mathtt{d}}\|_{0,K}^2 \;=\; \frac{1}{2\mu}\, \|\boldsymbol{\tau}\|_{0,K}^2 \qquad \forall\, \boldsymbol{\tau} \in \widehat{H}_{k,\nabla}^K\,,$$

whereas that of (5.3.28) follows from the inequality

$$\|\mathbf{div}(q\, \mathbb{I})\|_{0,K}^2 \;=\; |q|_{1,K}^2 \;>\; 0 \qquad \forall\, q \,\in\, \widehat{\mathrm{P}}_k(K) \setminus \{0\}\,.$$

In this way, having computed $\widehat{\boldsymbol{\zeta}}_\nabla \,\in\, \widehat{H}_{k,\nabla}^K$ and then $q_{\boldsymbol{\zeta}} \,\in\, \widehat{\mathrm{P}}_k(K)$, we replace them into (5.3.29), which, using that $\operatorname{tr}(\widehat{\boldsymbol{\zeta}}_\nabla) = 0$, yields

$$c_{\boldsymbol{\zeta}} \;=\; \frac{1}{2|K|} \int_K \Big\{ \operatorname{tr}(\boldsymbol{\zeta}) \,-\, 2 q_{\boldsymbol{\zeta}} \Big\}\,. \tag{5.3.30}$$

Let us now check that the right hand sides of (5.3.27), (5.3.28), and (5.3.30) are indeed calculable when $\boldsymbol{\zeta}$ belongs to our local virtual space $H_h^K \subseteq \mathbb{H}(\mathbf{div}; K)$ (cf. (5.3.8)). Firstly, the fact that $\mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau})$ can be explicitly computed for $\boldsymbol{\zeta} \in H_h^K$ and $\boldsymbol{\tau} \in \widehat{H}_{k,\nabla}^K$, was already noticed at the beginning of Section 5.3.4 (cf. (5.3.24)). In turn, since $\mathbf{div}\boldsymbol{\zeta} \in \mathbf{P}_{k-1}(K)$ (cf. (5.3.8)) and $\widehat{\boldsymbol{\zeta}}_\nabla \in \mathbb{P}_k(K)$, it is quite clear that the expression $(\mathbf{div}\big(\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_\nabla\big), \mathbf{div}(q\,\mathbb{I}))_{0,K}$ is also calculable for each $q \in \widehat{\mathrm{P}}_k(K)$. Next, for the right hand side of (5.3.30) we simply observe that

$$\int_K \mathrm{tr}(\boldsymbol{\zeta}) \,=\, \int_K \boldsymbol{\zeta} : \mathbb{I} \,=\, \int_K \boldsymbol{\zeta} : \nabla \mathbf{x} \,=\, -\int_K \mathbf{x} \cdot \mathbf{div}\boldsymbol{\zeta} \,+\, \int_{\partial K} \boldsymbol{\zeta}\,\mathbf{n} \cdot \mathbf{x}\,,$$

which, according to (5.3.8), is calculable as well. Finally, it is straightforward to check from (5.3.27) - (5.3.29) that $\widehat{\Pi}_k^K(\boldsymbol{\zeta}) = \boldsymbol{\zeta} \quad \forall\, \boldsymbol{\zeta} \in \widehat{H}_k^K$, which confirms that $\widehat{\Pi}_k^K$ is in fact a projector. Moreover, the following result establishes the uniform boundedness of the family $\left\{\widehat{\Pi}_k^K\right\}_{K\in\mathcal{T}_h} \subseteq \left\{\mathcal{L}(\mathbb{H}(\mathbf{div}; K), \mathbb{L}^2(K))\right\}_{K\in\mathcal{T}_h}$.

Lemma 5.3.3. *There exists a constant $C > 0$, depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds*

$$\|\widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K} \,\leq\, C\Big\{\,\|\boldsymbol{\zeta}\|_{0,K} \,+\, h_K\,\|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K}\Big\} \qquad \forall\,\boldsymbol{\zeta} \,\in\, \mathbb{H}(\mathbf{div}; K)\,. \tag{5.3.31}$$

Proof. Given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}; K)$ we utilize again the decomposition (5.3.26) and set

$$\widehat{\boldsymbol{\zeta}} := \widehat{\Pi}_k^K(\boldsymbol{\zeta}) \,=\, \widehat{\boldsymbol{\zeta}}_\nabla \,+\, q_{\boldsymbol{\zeta}}\,\mathbb{I} \,+\, c_{\boldsymbol{\zeta}}\,\mathbb{I}\,, \tag{5.3.32}$$

with $\widehat{\boldsymbol{\zeta}}_\nabla \in \widehat{H}_{k,\nabla}^K$, $q_{\boldsymbol{\zeta}} \in \widehat{P}_k(K)$, and $c_{\boldsymbol{\zeta}} \in \mathrm{R}$. Then, it follows straightforwardly from (5.3.27), (5.3.28), and (5.3.30) that

$$\|\widehat{\boldsymbol{\zeta}}_\nabla\|_{0,K} \,\leq\, \|\boldsymbol{\zeta}\|_{0,K}\,, \quad |q_{\boldsymbol{\zeta}}|_{1,K} \,=\, \|\mathbf{div}(q_{\boldsymbol{\zeta}}\,\mathbb{I})\|_{0,K} \,\leq\, \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K} \,+\, \|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_\nabla)\|_{0,K}\,, \tag{5.3.33}$$

and

$$\|c_{\boldsymbol{\zeta}}\,\mathbb{I}\|_{0,K} \,\leq\, \|\boldsymbol{\zeta}\|_{0,K} \,+\, \sqrt{2}\,\|q_{\boldsymbol{\zeta}}\|_{0,K}\,. \tag{5.3.34}$$

In what follows we bound $\|q_{\boldsymbol{\zeta}}\|_{0,K}$ and $\|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_\nabla)\|_{0,K}$ in terms of $|q_{\boldsymbol{\zeta}}|_{1,K}$ and $\|\boldsymbol{\zeta}\|_{0,K}$, respectively. For the first estimate we assume, without loss of generality, that K is star-shaped with respect to a ball B centered at the origin. Otherwise, instead of K we consider the shifted region $\bar{K} := \bar{T}_K(K)$, where $\bar{T}_K(\mathbf{x}) := \mathbf{x} - \mathbf{x}_B \quad \forall\,\mathbf{x} \in K$, for which there holds $h_K = h_{\bar{K}}$. Then, analogously as described for $\widetilde{K}$ at the beginning of Section 5.3.3, we now let $\Big\{ \Delta_i : \quad i \in \{1, 2, ..., d_K\} \Big\}$ be the partition of K obtained by connecting each vertex of this element to the origin. In addition, for each $i \in \{1, 2, \ldots, d_K\}$ we let h_i and ρ_i be the geometric parameters of Δ_i, and let $F_i : \mathrm{R}^2 \to \mathrm{R}^2$ be the bijective linear mapping, say $F_i(\mathbf{x}) := B_i\,\mathbf{x} \ \ \forall\,\mathbf{x} \in \mathrm{R}^2$, with $B_i \in \mathrm{R}^{2\times 2}$ invertible, such that $F_i(\widehat{\Delta}) = \Delta_i$. Recall that $\widehat{\Delta}$ is the canonical triangle of R^2 with corresponding parameters $\widehat{h}$ and $\widehat{\rho}$. Hence, we can write

$$\|q_{\boldsymbol{\zeta}}\|^2_{0,K} \;=\; \sum_{i=1}^{d_K} \|q_{\boldsymbol{\zeta}}\|^2_{0,\Delta_i} \;=\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\,\|\widehat{q}_{\boldsymbol{\zeta},i}\|^2_{0,\widehat{\Delta}}\,, \tag{5.3.35}$$

where $\widehat{q}_{\boldsymbol{\zeta},i} := q_{\boldsymbol{\zeta}}|_{\Delta_i} \circ F_i \in \widehat{P}_k(\widehat{\Delta})$. We emphasize here that the fact that the origin is a vertex of each one of the triangles Δ_i has allowed to choose a linear (not affine) transformation F_i mapping $\widehat{\Delta}$ onto Δ_i, which, given that $q_{\boldsymbol{\zeta}}|_{\Delta_i} \in \widehat{\mathrm{P}}_k(\Delta_i)$, insures that $\widehat{q}_{\boldsymbol{\zeta},i}$ does belong to $\widehat{\mathrm{P}}_k(\widehat{\Delta})$. Moreover, the importance of it lies on the fact that $|\cdot|_{1,\widehat{\Delta}}$ is a norm on $\widehat{\mathrm{P}}_k(\widehat{\Delta})$, and therefore there exists $\widehat{c} > 0$, depending only on k and $\widehat{\Delta}$, such that, in particular, $\|\widehat{q}_{\boldsymbol{\zeta},i}\|^2_{0,\widehat{\Delta}} \le \widehat{c}\,|\widehat{q}_{\boldsymbol{\zeta},i}|^2_{1,\widehat{\Delta}}$. In this way, applying once more the scaling properties between Sobolev seminorms, we obtain from (5.3.35) that

$$\|q_{\boldsymbol{\zeta}}\|^2_{0,K} \;\le\; \sum_{i=1}^{d_K} |\mathrm{det} B_i|\,\widehat{c}\,|\widehat{q}_{\boldsymbol{\zeta},i}|^2_{1,\widehat{\Delta}} \;\le\; \widehat{c} \sum_{i=1}^{d_K} h_i^2\,\widehat{\rho}^{-2}\,|q_{\boldsymbol{\zeta},i}|^2_{1,\Delta_i} \;\le\; \widehat{C}\,h_K^2\,|q_{\boldsymbol{\zeta}}|^2_{1,K}\,, \tag{5.3.36}$$

which, together with the second inequality in (5.3.33), gives

$$\|q_{\boldsymbol{\zeta}}\|_{0,K} \;\le\; \widehat{C}\,h_K \Big\{ \|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K} \;+\; \|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_\nabla)\|_{0,K} \Big\}\,. \tag{5.3.37}$$

On the other hand, applying the inverse inequality in each triangle Δ_i, noting from the assumption a) at the beginning of Section 5.3.2 that $h_i^{-1} \le C_{\mathcal{T}}^{-1}\,h_K^{-1}$, and then using the first estimate in (5.3.33), we find that

$$\begin{aligned}\|\mathbf{div}(\widehat{\boldsymbol{\zeta}}_\nabla)\|^2_{0,K} \;&\le\; 2\,|\widehat{\boldsymbol{\zeta}}_\nabla|^2_{1,K} \;=\; 2 \sum_{i=1}^{d_K} |\widehat{\boldsymbol{\zeta}}_\nabla|^2_{1,\Delta_i} \;\le\; c \sum_{i=1}^{d_K} h_i^{-2}\,\|\widehat{\boldsymbol{\zeta}}_\nabla\|^2_{0,\Delta_i} \\ &\le\; c\,C_{\mathcal{T}}^{-2}\,h_K^{-2}\,\|\widehat{\boldsymbol{\zeta}}_\nabla\|^2_{0,K} \;\le\; c\,C_{\mathcal{T}}^{-2}\,h_K^{-2}\,\|\boldsymbol{\zeta}\|^2_{0,K}\,,\end{aligned}$$

which, replaced back into (5.3.37), yields

$$\|q_{\boldsymbol{\zeta}}\|_{0,K} \,\leq\, \widehat{C}\left\{\|\boldsymbol{\zeta}\|_{0,K} \,+\, h_K\,\|\mathbf{div}(\boldsymbol{\zeta})\|_{0,K}\right\}. \tag{5.3.38}$$

Finally, it is easy to see that (5.3.32), the first inequality in (5.3.33), (5.3.34), and (5.3.38) imply the required estimate (5.3.31), thus completing the proof. $\square$

We remark at this point that instead of using $\widehat{\mathrm{P}}_k(K)$ in the decomposition (5.3.26), one could also employ the space $\mathrm{P}_{k,0}(K) := \left\{q \in \mathrm{P}_k(K) : \quad \int_K q = 0\right\}$. In this case, the corresponding inequality (5.3.36) follows from the approximation property given by Lemma 5.3.4 with $k = 1$, $r = 1$, and $\ell = 0$, and noting that obviously $\mathcal{P}_0^K(q) = 0$ $\forall\, q \in \mathrm{P}_{k,0}(K)$. Nevertheless, we prefer to stay with $\widehat{\mathrm{P}}_k(K)$ because of the simplicity of its canonical basis for the implementation of (5.3.28).

The analogue of Lemma 5.3.5 is provided next.

Lemma 5.3.8. Given integers k, $\ell \geq 1$, and $K \in \mathcal{T}_h$, there holds $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} = \widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})$ for all $\boldsymbol{\zeta} \in \mathbb{H}^k(K)$, and $\widehat{\Pi}_k^{\widetilde{K}} \in \mathcal{L}\big(\mathbb{H}^\ell(\widetilde{K}), \mathbb{L}^2(\widetilde{K})\big)$ with $\|\widehat{\Pi}_k^{\widetilde{K}}\|_{\mathcal{L}\left(\mathbb{H}^\ell(\widetilde{K}),\mathbb{L}^2(\widetilde{K})\right)}$ independent of $\widetilde{K}$, namely depending only on k, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$.

Proof. Similarly as for Lemma 5.3.5, we first observe that $\boldsymbol{\tau} \in \widetilde{\widehat{H}_k^K}$ if and only $\widetilde{\boldsymbol{\tau}} := \boldsymbol{\tau} \circ T_K^{-1} \in \widehat{H}_k^{\widetilde{K}}$. In particular, given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div}; K)$, there holds $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} \in \widehat{H}_k^{\widetilde{K}}$, and hence, in order to obtain the required identity, it suffices to show that $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}$ solves the same problem as $\widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})$, namely (5.3.27) - (5.3.29) with $K = \widetilde{K}$ and $\boldsymbol{\zeta} = \widetilde{\boldsymbol{\zeta}}$. In fact, setting as before $\widehat{\Pi}_k^K(\boldsymbol{\zeta}) = \widehat{\boldsymbol{\zeta}}_\nabla \,+\, q_{\boldsymbol{\zeta}}\,\mathbb{I} \,+\, c_{\boldsymbol{\zeta}}\,\mathbb{I}$, where $\widehat{\boldsymbol{\zeta}}_\nabla \in \widehat{H}_{k,\nabla}^K$, $q_{\boldsymbol{\zeta}} \in \widehat{\mathrm{P}}_k(K)$, and $c_{\boldsymbol{\zeta}} \in \mathrm{R}$, we find, according to (5.3.27), that for each $\boldsymbol{\tau}_\nabla \in \widehat{H}_{k,\nabla}^K$ there holds

$$\mathbf{a}^{\widetilde{K}}(\widetilde{\widehat{\boldsymbol{\zeta}}_\nabla}, \widetilde{\boldsymbol{\tau}_\nabla}) \,=\, h_K^{-2}\,\mathbf{a}^K(\widehat{\boldsymbol{\zeta}}_\nabla, \boldsymbol{\tau}_\nabla) \,=\, h_K^{-2}\,\mathbf{a}^K(\boldsymbol{\zeta}, \boldsymbol{\tau}_\nabla) \,=\, \mathbf{a}^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}_\nabla})\,. \tag{5.3.39}$$

In turn, for each $q \in \widehat{\mathrm{P}}_k(K)$ we have, in virtue of (5.3.28), that

$$\begin{aligned}(\mathbf{div}(\widetilde{q}_{\boldsymbol{\zeta}}\,\mathbb{I}), \mathbf{div}(\widetilde{q}\,\mathbb{I}))_{0,\widetilde{K}} \,&=\, (\mathbf{div}(\widetilde{q_{\boldsymbol{\zeta}}\,\mathbb{I}}), \mathbf{div}(\widetilde{q\,\mathbb{I}}))_{0,\widetilde{K}} \,=\, h_K^{-4}\,(\mathbf{div}(q_{\boldsymbol{\zeta}}\,\mathbb{I}), \mathbf{div}(q\,\mathbb{I}))_{0,K} \\ &=\, h_K^{-4}\,(\mathbf{div}(\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_{\nabla}), \mathbf{div}(q\,\mathbb{I}))_{0,K} \,=\, (\mathbf{div}(\widetilde{\boldsymbol{\zeta} - \widehat{\boldsymbol{\zeta}}_{\nabla}}), \mathbf{div}(\widetilde{q\,\mathbb{I}}))_{0,\widetilde{K}} \\ &=\, (\mathbf{div}(\widetilde{\boldsymbol{\zeta}} - \widetilde{\widehat{\boldsymbol{\zeta}}_{\nabla}}), \mathbf{div}(\widetilde{q}\,\mathbb{I}))_{0,\widetilde{K}}\,.\end{aligned} \tag{5.3.40}$$

Next, it is easy to see, thanks to (5.3.29), that

$$\int_{\widetilde{K}} \mathrm{tr}\big(\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}\big) \,=\, \int_{\widetilde{K}} \widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})} : \mathbb{I} \,=\, h_K^{-2} \int_K \widehat{\Pi}_k^K(\boldsymbol{\zeta}) : \mathbb{I} = h_K^{-2} \int_K \boldsymbol{\zeta} : \mathbb{I} \,=\, \int_{\widetilde{K}} \widetilde{\boldsymbol{\zeta}} : \mathbb{I} \,=\, \int_{\widetilde{K}} \mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,,$$

which, together with (5.3.39) and (5.3.40), confirm that $\widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}$ does solve the announced problem. Finally, since $h_{\widetilde{K}} = 1$, a direct application of Lemma 5.3.3 implies the existence of a constant $C > 0$, independent of $\widetilde{K}$, such that

$$\|\widehat{\Pi}_k^{\widetilde{K}}(\boldsymbol{\zeta})\|_{0,\widetilde{K}} \;\le\; C\,\|\boldsymbol{\zeta}\|_{\mathbf{div};\widetilde{K}} \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\widetilde{K})\,,$$

and consequently, for each integer $\ell \ge 1$ there holds

$$\|\widehat{\Pi}_k^{\widetilde{K}}(\boldsymbol{\zeta})\|_{0,\widetilde{K}} \;\le\; C\,\|\boldsymbol{\zeta}\|_{\ell,\widetilde{K}} \qquad \forall\,\boldsymbol{\zeta} \in \mathbb{H}^{\ell}(\widetilde{K})\,,$$

which completes the proof. $\square$

Before establishing the next result, we now recall from [27] that if $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times Q$ is the solution of the continuous problem (5.2.3), then there holds $\boldsymbol{\sigma}^{\mathrm{d}} \,=\, 2\,\mu\,\nabla\mathbf{u} \in \mathbb{L}^2(\Omega)$ and $\mathrm{div}\ \mathbf{u} \,=\, 0$, which formally implies the existence of $w \in \mathrm{H}^2(\Omega)$ such that $\mathbf{u} \,=\, \underline{\mathrm{curl}}\,w$, and hence $\boldsymbol{\sigma}^{\mathrm{d}} \,=\, 2\,\mu\,\nabla\,\underline{\mathrm{curl}}\,w$. These remarks motivate for each integer $r \ge 0$ the introduction of the space

$$\mathbb{H}^r_{\nabla\,\underline{\mathrm{curl}}}(K) \;:=\; \Big\{\boldsymbol{\zeta} \in \mathbb{H}^r(K): \quad \boldsymbol{\zeta}^{\mathrm{d}} \;=\; \nabla\,\underline{\mathrm{curl}}\,w \quad \text{for some} \quad w \in \mathrm{H}^{r+2}(K)\Big\}\,.$$

Then, we have the following projection error for $\widehat{\Pi}_k^K$, which constitutes the analogue of Lemma 5.3.6.

Lemma 5.3.9. Let k and r be integers such that $k \geq 1$ and $1 \leq r \leq k+1$. Then, there exists a constant $C > 0$, depending only on k, r, $\widehat{\Delta}$, $c_{\mathcal{T}}$, and $C_{\mathcal{T}}$, such that for each $K \in \mathcal{T}_h$ there holds

$$\|\boldsymbol{\zeta} \,-\, \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K} \,\leq\, C\, h_K^r\, |\boldsymbol{\zeta}|_{r,K} \qquad \forall\, \boldsymbol{\zeta} \in \mathbb{H}^r_{\nabla\,\underline{\mathrm{curl}}}(K)\,.$$

Proof. We proceed similarly as in the proof of Lemma 5.3.6. In fact, given integers k and r as stated, $K \in \mathcal{T}_h$, and $\boldsymbol{\zeta} \in \mathbb{H}^r_{\nabla\,\underline{\mathrm{curl}}}(K)$, we let $w \in \mathrm{H}^{r+2}(K)$ such that $\boldsymbol{\zeta}^{\mathtt{d}} \,=\, \nabla\,\underline{\mathrm{curl}}\, w$, set $\widetilde{w} \in \mathrm{H}^{r+2}(\widetilde{K})$ such that $\widetilde{\boldsymbol{\zeta}}^{\mathtt{d}} \,=\, \nabla\,\underline{\mathrm{curl}}\, \widetilde{w}$, denote by $\mathrm{T}^{r+2}(\widetilde{w}) \,\in\, \mathrm{P}_{r+1}(\widetilde{K})$ and $\mathrm{T}^r\big(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\big) \,\in\, \mathrm{P}_{r-1}(\widetilde{K})$ the averaged Taylor polynomials of order $r+2$ and r of $\widetilde{w}$ and $\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})$, respectively (cf. Lemma 5.3.3), and observe, since $r+1 \leq k+2$ and $r-1 \leq k$, that

$$\widehat{\Pi}_k^{\widetilde{K}}(\nabla\,\underline{\mathrm{curl}}\,\mathrm{T}^{r+2}(\widetilde{w})) \,=\, \nabla\,\underline{\mathrm{curl}}\,\mathrm{T}^{r+2}(\widetilde{w}) \qquad \text{and} \qquad \widehat{\Pi}_k^{\widetilde{K}}\big(\mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I}\big) \,=\, \mathrm{T}^r\big(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\big)\,\mathbb{I}\,.$$

It follows, using Lemmas 5.3.8 and 5.3.3 (with $\mathcal{O} = \widetilde{K}$), that

$$\begin{aligned}
\|\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K} \,&=\, h_K^{-1}\,\|\widetilde{\boldsymbol{\zeta}} - \widetilde{\widehat{\Pi}_k^K(\boldsymbol{\zeta})}\|_{0,\widetilde{K}} \,=\, h_K^{-1}\,\|\widetilde{\boldsymbol{\zeta}} - \widehat{\Pi}_k^{\widetilde{K}}(\widetilde{\boldsymbol{\zeta}})\|_{0,\widetilde{K}} \\
&=\, h_K^{-1}\,\left\|\big(\mathrm{I} - \widehat{\Pi}_k^{\widetilde{K}}\big)\left(\widetilde{\boldsymbol{\zeta}}^{\mathtt{d}} \,-\, \frac{1}{2}\,\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,\mathbb{I} \,-\, \nabla\,\underline{\mathrm{curl}}\,\mathrm{T}^{r+2}(\widetilde{w}) \,+\, \frac{1}{2}\,\mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I}\right)\right\|_{0,\widetilde{K}} \\
&\leq\, h_K^{-1}\,\|\mathrm{I} - \widehat{\Pi}_k^{\widetilde{K}}\|_{\mathcal{L}(\mathbb{H}^r(\widetilde{K}),\mathbb{L}^2(\widetilde{K}))}\,\Big\{\|\widetilde{\boldsymbol{\zeta}}^{\mathtt{d}} \,-\, \nabla\,\underline{\mathrm{curl}}\,\mathrm{T}^{r+2}(\widetilde{w})\|_{r,\widetilde{K}} \,+\, \|\mathrm{tr}(\widetilde{\boldsymbol{\zeta}})\,\mathbb{I} \,-\, \mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\,\mathbb{I}\|_{r,\widetilde{K}}\Big\} \\
&\leq\, C\, h_K^{-1}\,\Big\{\|\widetilde{w} \,-\, \mathrm{T}^{r+2}(\widetilde{w})\|_{r+2,\widetilde{K}} \,+\, \|\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}) \,-\, \mathrm{T}^r(\mathrm{tr}(\widetilde{\boldsymbol{\zeta}}))\|_{r,\widetilde{K}}\Big\} \\
&\leq\, C\, h_K^{-1}\,\Big\{\,|\widetilde{w}|_{r+2,\widetilde{K}} \,+\, |\widetilde{\boldsymbol{\zeta}}|_{r,\widetilde{K}}\,\Big\} \,\leq\, C\, h_K^{-1}\,|\widetilde{\boldsymbol{\zeta}}|_{r,\widetilde{K}} \,=\, C\, h_K^r\,|\boldsymbol{\zeta}|_{r,K}\,,
\end{aligned}$$

which finishes the proof. $\square$

We now let $\mathbf{a}_h^K : H_h^K \times H_h^K \longrightarrow \mathrm{R}$ be the local discrete bilinear form given by

$$\mathbf{a}_h^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}),\widehat{\Pi}_k^K(\boldsymbol{\tau})\big) \,+\, \mathcal{S}^K\big(\boldsymbol{\zeta} - \widehat{\Pi}_k^K(\boldsymbol{\zeta}),\boldsymbol{\tau} - \widehat{\Pi}_k^K(\boldsymbol{\tau})\big) \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\,\in\,H_h^K\,, \tag{5.3.41}$$

where $\mathcal{S}^K : H_h^K \times H_h^K \to \mathrm{R}$ is the bilinear form associated to the identity matrix in $\mathrm{R}^{n_k^K \times n_k^K}$ with respect to the basis $\{\boldsymbol{\varphi}_{j,K}\}_{j=1}^{n_k^K}$ of H_k^K (cf. (5.3.17) - (5.3.18)), that is

$$\mathcal{S}^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \,:=\, \sum_{i=1}^{n_k^K} m_{i,K}(\boldsymbol{\zeta})\, m_{i,K}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\,\in\,H_h^K\,. \tag{5.3.42}$$

Next, as suggested by (5.3.41), we define the global discrete bilinear form $\mathbf{a}_h : H_h \times H_h \longrightarrow$ R

$$\mathbf{a}_h(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{K \in \mathcal{T}_h} \mathbf{a}_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\zeta},\, \boldsymbol{\tau} \in H_h\,. \tag{5.3.43}$$

The following result is the analogue of the inequality given in [9, eq. (5.8)] for the present case.

Lemma 5.3.4. *There exist c_0, $c_1 > 0$, depending only on $C_{\mathcal{T}}$, such that*

$$c_0\, \|\boldsymbol{\zeta}\|_{0,K}^2 \,\le\, \mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \,\le\, c_1\, \|\boldsymbol{\zeta}\|_{0,K}^2 \qquad \forall\, K \in \mathcal{T}_h, \quad \forall\, \boldsymbol{\zeta} \in H_h^K\,. \tag{5.3.44}$$

Proof. Actually, except for the presence of the norm of a matrix multiplying $\mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\zeta})$ in [9, eq. (5.8)], the inequalities are basically the same. However, since the corresponding proof is only sketched in [9, Remark 5.1], for sake of completeness we provide in what follows further details on the derivation of the upper bound of (5.3.44). In fact, given $\boldsymbol{\zeta} \in H_h^K$, we first notice from (5.3.42) that

$$\mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \,=\, \sum_{j=1}^{n_k^K} m_{j,K}^2(\boldsymbol{\zeta})\,,$$

and hence it suffices to estimate each one of the moments $m_{j,K}(\boldsymbol{\zeta})$ (cf. (5.3.17)) in terms of $\|\boldsymbol{\zeta}\|_{0,K}$. For this purpose, we employ the same partition and corresponding notations introduced in the proof of Lemma 5.3.3. We begin with $m_{\mathbf{q},e}^{\mathbf{n}}(\boldsymbol{\zeta})$, where $e \subseteq \partial K$ is an edge of a triangle Δ_i and $\mathbf{q} \in \mathcal{B}_k(e)$, by observing, thanks to the Cauchy-Schwarz and polynomial trace inequalities, that

$$|m_{\mathbf{q},e}^{\mathbf{n}}(\boldsymbol{\zeta})| \,\le\, \|\boldsymbol{\zeta}\|_{0,e}\, \|\mathbf{q}\|_{0,e} \,\le\, C\, h_i^{-1/2}\, \|\boldsymbol{\zeta}\|_{0,K}\, |e|^{1/2} \,\le\, C\, C_{\mathcal{T}}^{-1/2}\, h_K^{-1/2}\, \|\boldsymbol{\zeta}\|_{0,K}\, h_e^{1/2} \,\le\, C_1\, \|\boldsymbol{\zeta}\|_{0,K}\,.$$

In turn, given $\mathbf{q} \in \mathcal{B}_{k-1}(K)$, we apply the inverse inequality on each triangle Δ_i, and then use that $|K| \,\le\, c\, h_K^2$, to find that

$$|m_{\mathbf{q},K}^{\mathbf{div}}(\boldsymbol{\zeta})| \,\le\, \|\boldsymbol{\zeta}\|_{0,K}\, |\mathbf{q}|_{1,K} \,\le\, C\, C_{\mathcal{T}}^{-1}\, h_K^{-1}\, \|\boldsymbol{\zeta}\|_{0,K}\, \|\mathbf{q}\|_{0,K} \,\le\, C\, C_{\mathcal{T}}^{-1}\, h_K^{-1}\, \|\boldsymbol{\zeta}\|_{0,K}\, |K|^{1/2} \,\le\, C_2\, \|\boldsymbol{\zeta}\|_{0,K}\,.$$

Finally, given again $\mathbf{q} \in \mathcal{B}_{k-1}(K)$, we integrate by parts in K to obtain

$$m_{\mathbf{q},K}^{\mathbf{rot}}(\boldsymbol{\zeta}) \,=\, \int_K \boldsymbol{\zeta} : \underline{\mathbf{curl}}\, \mathbf{q} \,-\, \int_{\partial K} \boldsymbol{\zeta}\mathbf{t} \cdot \mathbf{q}\,,$$

where $\mathbf{t}$ is the unit tangential vector along ∂K and $\underline{\mathbf{curl}}$ is the operator $\underline{\mathrm{curl}}$ acting row-wise. In this way, employing basically the same arguments of the foregoing inequalities, we deduce that

$$|m^{\mathbf{rot}}_{\mathbf{q},K}(\boldsymbol{\zeta})| \,\le\, \|\boldsymbol{\zeta}\|_{0,K}\,|\mathbf{q}|_{1,K} \,+\, \sum_{e\subseteq\partial K} \|\boldsymbol{\zeta}\|_{0,e}\,\|\mathbf{q}\|_{0,e} \,\le\, C_3\,\|\boldsymbol{\zeta}\|_{0,K}\,,$$

where $C_3 \,=\, C_2 + N_{\mathcal{T}}\,C_1$, and C_1 and C_2 are positive constants depending only on $C_{\mathcal{T}}$. $\square$

The following result is consequence of the properties of the projector $\widehat{\Pi}^K_k$ and the previous lemma.

Lemma 5.3.5. *For each $K\in\mathcal{T}_h$ there holds*

$$\mathbf{a}^K_h(\boldsymbol{\zeta},\boldsymbol{\tau}) \,=\, \mathbf{a}^K(\boldsymbol{\zeta},\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\zeta}\in\widehat{H}^K_k, \quad \forall\,\boldsymbol{\tau}\in H^K_h\,, \tag{5.3.45}$$

and there exist positive constants α_1, α_2, independent of h and K, such that

$$|\mathbf{a}^K_h(\boldsymbol{\zeta},\boldsymbol{\tau})| \le \alpha_1 \left\{ \|\boldsymbol{\zeta}\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} + \|\boldsymbol{\zeta}-\widehat{\Pi}^K_k(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau}-\widehat{\Pi}^K_k(\boldsymbol{\tau})\|_{0,K} \right\} \quad \forall\,K\in\mathcal{T}_h, \quad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in H^K_h\,, \tag{5.3.46}$$

and

$$\alpha_2\,\|\boldsymbol{\zeta}^{\mathtt{d}}\|^2_{0,K} \,\le\, \mathbf{a}^K_h(\boldsymbol{\zeta},\boldsymbol{\zeta}) \,\le\, \alpha_1 \left\{ \|\boldsymbol{\zeta}\|^2_{0,K} + \|\boldsymbol{\zeta}-\widehat{\Pi}^K_k(\boldsymbol{\zeta})\|^2_{0,K} \right\} \quad \forall\,K\in\mathcal{T}_h, \quad \forall\,\boldsymbol{\zeta}\in H^K_h\,. \tag{5.3.47}$$

Proof. Given $\boldsymbol{\zeta}\in\widehat{H}^K_k$, we certainly have $\boldsymbol{\zeta} \,=\, \widehat{\Pi}^K_k(\boldsymbol{\zeta}) \,:=\, \widehat{\boldsymbol{\zeta}}_\nabla + q_{\boldsymbol{\zeta}}\,\mathbb{I} + c_{\boldsymbol{\zeta}}\,\mathbb{I}$, with $\widehat{\boldsymbol{\zeta}}_\nabla\in\widehat{H}^K_{k,\nabla}$, $q_{\boldsymbol{\zeta}}\in\widehat{\mathrm{P}}_k(K)$, and $c_{\boldsymbol{\zeta}}\in\mathrm{R}$. Hence, using the symmetry of $\mathbf{a}^K$, and bearing in mind problem (5.3.27), we deduce, starting from (5.3.41), that given $\boldsymbol{\tau}\in H^K_h$ and denoting the deviatoric tensor of $\widehat{\Pi}^K_k(\boldsymbol{\tau})$ by $\widehat{\boldsymbol{\tau}}_\nabla\in\widehat{H}^K_{k,\nabla}$, there holds

$$\begin{aligned}\mathbf{a}^K_h(\boldsymbol{\zeta},\boldsymbol{\tau}) \,&=\, \mathbf{a}^K(\widehat{\Pi}^K_k(\boldsymbol{\zeta}),\widehat{\Pi}^K_k(\boldsymbol{\tau})) \,=\, \mathbf{a}^K(\boldsymbol{\zeta},\widehat{\Pi}^K_k(\boldsymbol{\tau})) \,=\, \mathbf{a}^K(\widehat{\Pi}^K_k(\boldsymbol{\tau}),\boldsymbol{\zeta})\\ &=\, \mathbf{a}^K(\widehat{\boldsymbol{\tau}}_\nabla,\widehat{\boldsymbol{\zeta}}_\nabla) \,=\, \mathbf{a}^K(\boldsymbol{\tau},\widehat{\boldsymbol{\zeta}}_\nabla) \,=\, \mathbf{a}^K(\boldsymbol{\tau},\boldsymbol{\zeta}) \,=\, \mathbf{a}^K(\boldsymbol{\zeta},\boldsymbol{\tau})\,,\end{aligned}$$

which proves (5.3.45). Next, for the boundedness of $\mathbf{a}^K_h$ we apply the Cauchy-Schwarz inequality, the first estimate in (5.3.33), and the upper bound in (5.3.44) (cf. Lemma

5.3.4), to obtain

$$
\begin{aligned}
|\mathbf{a}_h^K(\boldsymbol{\zeta},\boldsymbol{\tau})| &\le \frac{1}{2\mu}\,\|\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathsf{d}}\|_{0,K}\|\big(\widehat{\Pi}_k^K(\boldsymbol{\tau})\big)^{\mathsf{d}}\|_{0,K}\\
&+\Big\{\mathcal{S}^K\big(\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta}),\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)\Big\}^{1/2}\Big\{S^K\big(\boldsymbol{\tau}-\widehat{\Pi}_k^K(\boldsymbol{\tau}),\boldsymbol{\tau}-\widehat{\Pi}_k^K(\boldsymbol{\tau})\big)\Big\}^{1/2}\\
&\le \frac{1}{2\mu}\,\|\boldsymbol{\zeta}\|_{0,K}\,\|\boldsymbol{\tau}\|_{0,K} + c_1\,\|\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}\,\|\boldsymbol{\tau}-\widehat{\Pi}_k^K(\boldsymbol{\tau})\|_{0,K} \qquad \forall\,\boldsymbol{\zeta},\,\boldsymbol{\tau}\in H_h^K\,,
\end{aligned}
$$

which gives (5.3.46) with $\alpha_1 := \max\{\frac{1}{2\mu}, c_1\}$. Finally, concerning (5.3.47), it is clear that the corresponding upper bound follows from (5.3.46). In turn, applying the lower estimate in (5.3.44) (cf. Lemma 5.3.4) we find that

$$
\begin{aligned}
\|\boldsymbol{\zeta}^{\mathsf{d}}\|_{0,K}^2 &\le 2\Big\{\|\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathsf{d}}\|_{0,K}^2 + \|\big(\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)^{\mathsf{d}}\|_{0,K}^2\Big\}\\
&\le 4\mu\,\mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}),\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big) + 2\,\|\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\|_{0,K}^2\\
&\le 4\mu\,\mathbf{a}^K\big(\widehat{\Pi}_k^K(\boldsymbol{\zeta}),\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big) + \frac{2}{c_0}\,\mathcal{S}^K\big(\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta}),\boldsymbol{\zeta}-\widehat{\Pi}_k^K(\boldsymbol{\zeta})\big)\,,
\end{aligned}
$$

which yields the lower bound in (5.3.47) with $\alpha_2 := \max\{4\mu, \frac{2}{c_0}\}^{-1}$. □

We end this section by observing that all the tensors in H_h^K vanishing $\mathbf{a}^K(\cdot,\cdot)$ also vanish $\mathbf{a}_h^K(\cdot,\cdot)$, which is important for the stability condition provided by (5.3.46) and (5.3.47). In fact, given $\boldsymbol{\tau}\in H_h^K$ such that $0=\mathbf{a}^K(\boldsymbol{\tau},\boldsymbol{\tau})=\frac{1}{2\mu}\,\|\boldsymbol{\tau}^{\mathsf{d}}\|_{0,K}^2$, we have $\boldsymbol{\tau}^{\mathsf{d}} = \mathbf{0}$, that is $\boldsymbol{\tau} = \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I}$, which implies, according to the definition of H_h^K (cf. (5.3.8)), that $\mathbf{div}\,\boldsymbol{\tau} = \frac{1}{2}\,\nabla\big(\mathrm{tr}(\boldsymbol{\tau})\big) \in \mathbf{P}_{k-1}(K)$. It follows that $\mathrm{tr}(\boldsymbol{\tau})\in \mathrm{P}_k(K)$, and hence $\boldsymbol{\tau}\in \widehat{H}_{k,\mathbb{I}}^K \subseteq \widehat{H}_k^K$. In this way, thanks to the consistency condition (5.3.45), we conclude that $\mathbf{a}_h^K(\boldsymbol{\tau},\boldsymbol{\tau})=\mathbf{a}^K(\boldsymbol{\tau},\boldsymbol{\tau})=0$.

5.3.5 The mixed virtual element scheme

According to the analysis from the foregoing section, we reformulate the Galerkin scheme associated with (5.2.3) as: Find $(\boldsymbol{\sigma}_h,\mathbf{u}_h)\in H_h\times Q_h$ such that

$$
\begin{aligned}
\mathbf{a}_h\,(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + \mathbf{b}\,(\boldsymbol{\tau}_h,\mathbf{u}_h) &= \langle\boldsymbol{\tau}_h\mathbf{n},\mathbf{g}\rangle \qquad \forall\,\boldsymbol{\tau}_h\in H_h\,,\\
\mathbf{b}\,(\boldsymbol{\sigma}_h,\mathbf{v}_h) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}_h \qquad \forall\,\mathbf{v}_h\in Q_h\,.
\end{aligned}
\tag{5.3.48}
$$

In addition, as suggested by the third equation of (5.2.2), the postprocessed virtual pressure is defined as follows:

$$p_h \,=\, -\,\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma}_h)\,. \tag{5.3.49}$$

In what follows we establish the well-posedness of (5.3.48). We begin the analysis with the following result from [10].

Lemma 5.3.6. *There exists $c_\Omega > 0$, depending only on Ω, such that*

$$c_\Omega\,\|\boldsymbol{\zeta}\|^2_{0,\Omega} \,\leq\, \|\boldsymbol{\zeta}^{\mathtt{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\zeta})\|^2_{0,\Omega} \quad \forall\,\boldsymbol{\zeta}\,\in\, H \text{ (cf. (5.2.4))}\,. \tag{5.3.50}$$

Proof. See [10, Chapter IV, Proposition 3.1]. □

The ellipticity of $\mathbf{a}_h$ in the discrete kernel of $\mathbf{b}$ is proved next.

Lemma 5.3.7. *Let* $V_h \,:=\, \Big\{\boldsymbol{\zeta}_h \in H_h : \quad \mathbf{b}(\boldsymbol{\zeta}_h, \mathbf{v}_h) \,=\, 0 \quad \forall\,\mathbf{v}_h \in Q_h\Big\}$. *Then, there exists $\alpha > 0$, independent of h, such that*

$$\mathbf{a}_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,\geq\, \alpha\,\|\boldsymbol{\zeta}_h\|_{\mathbf{div};\Omega} \qquad \forall\,\boldsymbol{\zeta}_h \in V_h\,. \tag{5.3.51}$$

Proof. Recalling from (5.3.1) that for each $\boldsymbol{\zeta}_h \in H_h$ there holds $\mathbf{div}\boldsymbol{\zeta}_h|_K \in \mathbf{P}_{k-1}(K)$ $\forall\,K \in \mathcal{T}_h$, which actually says that $\mathbf{div}(\boldsymbol{\zeta}_h) \in Q_h$, we find that

$$V_h \,:=\, \Big\{\boldsymbol{\zeta}_h \in H_h : \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\zeta}_h) \,=\, 0 \quad \forall\,\mathbf{v}_h \in Q_h\Big\} \,=\, \Big\{\boldsymbol{\zeta}_h \in H_h : \quad \mathbf{div}(\boldsymbol{\zeta}_h) = 0\Big\}\,.$$

Hence, according to the definition of $\mathbf{a}_h$ (cf. (5.3.43)), and applying the lower bound in (5.3.47) and the estimate (5.3.50) (cf. Lemma 5.3.6), we deduce that for each $\boldsymbol{\zeta}_h \in V_h$ there holds

$$\mathbf{a}_h(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,=\, \sum_{K\in\mathcal{T}_h} \mathbf{a}_h^K(\boldsymbol{\zeta}_h, \boldsymbol{\zeta}_h) \,\geq\, \alpha_2 \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\zeta}_h^{\mathtt{d}}\|^2_{0,K} \,=\, \alpha_2\,\|\boldsymbol{\zeta}_h^{\mathtt{d}}\|^2_{0,\Omega} \,\geq\, \alpha\,\|\boldsymbol{\zeta}_h\|^2_{\mathbf{div};\Omega}\,,$$

with $\alpha = c_\Omega\,\alpha_2$, which ends the proof. □

The following lemma provides the discrete inf-sup condition for $\mathbf{b}$.

Lemma 5.3.10. Let H_h and Q_h be the virtual subspaces given by (5.3.1) and (5.3.2). Then, there exists $\beta > 0$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in H_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}\,;\Omega}} \,\geq\, \beta\, \|\mathbf{v}_h\|_{0,\Omega} \qquad \forall\, \mathbf{v}_h \in Q_h\,. \tag{5.3.52}$$

Proof. Since $\mathbf{b}$ satisfies the continuous inf-sup condition, we proceed in the classical way (see, e.g. [23, Section 4.2]) by constructing a corresponding Fortin's operator. In fact, given a convex and bounded domain G containing $\bar{\Omega}$, and given $\boldsymbol{\tau} \in H$ (cf. (5.2.4)), we let $\mathbf{z} \in \mathbf{H}_0^1(G) \cap \mathbf{H}^2(G)$ be the unique solution of the boundary value problem

$$\Delta \mathbf{z} \,=\, \begin{cases} \mathbf{div}(\boldsymbol{\tau}) & \text{in} \quad \Omega\,, \\ \\ 0 & \text{in} \quad G\backslash\bar{\Omega}\,, \end{cases} \quad , \quad \mathbf{z} = 0 \quad \text{on} \quad \partial G\,, \tag{5.3.53}$$

which, thanks to the corresponding elliptic regularity result, satisfies

$$\|\mathbf{z}\|_{2,\Omega} \,\leq\, C\,\|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}\,. \tag{5.3.54}$$

Then, recalling that Π_k^h denotes the interpolation operator mapping $\widetilde{H}$ onto our virtual subspace H_h (cf. (5.3.13), (5.3.14)), we now define the operator $\pi_k^h : H \to H_h$ as

$$\pi_k^h(\boldsymbol{\tau}) \,=\, \Pi_k^h(\nabla\mathbf{z}) \,-\, \left\{\frac{1}{2\,|\Omega|}\int_\Omega \operatorname{tr}\left(\Pi_k^h(\nabla\mathbf{z})\right)\right\}\mathbb{I}\,.$$

It follows, using (5.3.15) and the fact that Π_k^K and $\mathcal{P}_{k-1}^K$ are the restrictions to $K \in \mathcal{T}_h$ of the operators Π_k^h and $\mathcal{P}_{k-1}^h$, respectively, that

$$\mathbf{div}\left(\pi_k^h(\boldsymbol{\tau})\right) \,=\, \mathbf{div}\left(\Pi_k^h(\nabla\mathbf{z})\right) \,=\, \mathcal{P}_{k-1}^h(\mathbf{div}\,\nabla\mathbf{z}) \,=\, \mathcal{P}_{k-1}^h\big(\mathbf{div}\,(\boldsymbol{\tau})\big) \quad \text{in} \quad \Omega\,, \tag{5.3.55}$$

and hence for each $\mathbf{v}_h \in Q_h$ we obtain

$$\mathbf{b}(\pi_k^h(\boldsymbol{\tau}), \mathbf{v}_h) \,=\, \int_\Omega \mathbf{v}_h\cdot\mathbf{div}\,(\pi_k^h(\boldsymbol{\tau}) \,=\, \int_\Omega \mathbf{v}_h\cdot\mathcal{P}_{k-1}^h\big(\mathbf{div}\,(\boldsymbol{\tau})\big) \,=\, \int_\Omega \mathbf{v}_h\cdot\mathbf{div}\,(\boldsymbol{\tau}) \,=\, \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}_h)\,. \tag{5.3.56}$$

In turn, using (5.3.55), (5.3.23) (with $r = 1$), and (5.3.54), we find that

$$\|\pi_k^h(\boldsymbol{\tau})\|_{\mathbf{div};\Omega} \,\leq\, \|\Pi_k^h(\nabla\mathbf{z})\|_{0,\Omega} \,+\, \|\mathbf{div}\,\boldsymbol{\tau}\|_{0,\Omega} \,\leq\, \|\nabla\mathbf{z}\|_{0,\Omega} \,+\, \|\nabla\mathbf{z} - \Pi_k^h(\nabla\mathbf{z})\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\Omega}$$

$$\leq\, \|\nabla\mathbf{z}\|_{0,\Omega} \,+\, c\,h\,\|\nabla\mathbf{z}\|_{1,\Omega} \,+\, \|\mathbf{div}\,(\boldsymbol{\tau})\|_{0,\Omega} \,\leq\, \bar{C}\,\|\mathbf{z}\|_{2,\Omega} \,+\, \|\mathbf{div}\,(\boldsymbol{\tau})\|_{0,\Omega} \,\leq\, C\,\|\mathbf{div}\,(\boldsymbol{\tau})\|_{0,\Omega}\,,$$

which proves the uniform boundedness of the operators $\{\pi_k^h\}_{h>0}$. This fact and the identity (5.3.56) confirm that $\{\pi_k^h\}_{h>0}$ constitutes a family of Fortin's operators, which yields (5.3.52) and ends the proof.

□

The unique solvability and stability of the actual Galerkin scheme (5.3.48) is established now.

Theorem 5.3.1. *There exists a unique* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ *solution of* (5.3.48)*, and there exists a positive constant* C*, independent of* h*, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{H\times Q} \,\leq\, C\Big\{\,\|\mathbf{f}\|_{0,\Omega} \,+\, \|\mathbf{g}\|_{1/2,\Gamma}\,\Big\}\,.$$

Proof. The boundedness of $\mathbf{a}_h : H_h \times H_h \longrightarrow \mathrm{R}$ with respect to the norm $\|\cdot\|_{\mathbf{div};\Omega}$ of $\mathbb{H}(\mathbf{div};\Omega)$ follows easily from (5.3.46) and (5.3.31) (cf. Lemma 5.3.3). In turn, it is quite clear that $\mathbf{b}$ is also bounded. Hence, thanks to Lemmas 5.3.7 and 5.3.10, a straightforward application of the Babuška-Brezzi theory completes the proof. □

We now aim to provide the corresponding a priori error estimates. To this end, and just for sake of clearness in what follows, we recall that $\mathcal{P}_{k-1}^h : \mathbf{L}^2(\Omega) \longrightarrow Q_h$ and $\Pi_k^h : \widetilde{H} \longrightarrow H_h$ are the projector and interpolator, respectively, defined by (5.3.12) and (5.3.14), whose associated local operators are denoted by $\mathcal{P}_{k-1}^K$ and Π_k^K. In turn, given our local projector $\widehat{\Pi}_k^K$ defined by (5.3.26) - (5.3.29), we denote by $\widehat{\Pi}_k^h$ its global counterpart, that is, given $\boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega)$, we let

$$\widehat{\Pi}_k^h(\boldsymbol{\zeta})|_K \;:=\; \widehat{\Pi}_k^K(\boldsymbol{\zeta}|_K) \qquad \forall\, K \in \mathcal{T}_h\,.$$

Then, we have the following main result.

Theorem 5.3.2. *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in H\times Q$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h\times Q_h$ *be the unique solutions of the continuous and discrete schemes* (5.2.3) *and* (5.3.48)*, respectively, and let* $p_h \in L^2(\Omega)$ *be the postprocessed virtual pressure defined in* (5.3.49)*. Then, there exist positive constants*

C_1, C_2, independent of h, such that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}+\|p-p_h\|_{0,\Omega} \,\leq\, C_1 \left\{ \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \right\}, \tag{5.3.57}$$

and

$$\|\mathbf{u}-\mathbf{u}_h\|_{0,\Omega} \,\leq\, C_2 \left\{ \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\mathbf{u}-\mathcal{P}_{k-1}^h(\mathbf{u})\|_{0,\Omega} + h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \right\}. \tag{5.3.58}$$

Proof. We proceed similarly as in [9, Theorem 6.1]. Indeed, we first have, thanks to the triangle inequality, that

$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} \,\leq\, \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\Pi_k^h(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h\|_{0,\Omega}\,, \tag{5.3.59}$$

whence it just remains to estimate $\boldsymbol{\delta}_h := \Pi_k^h(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h$. We now observe from (5.3.15) and the second equation of (5.3.48) that $\mathbf{div}\big(\Pi_k^h(\boldsymbol{\sigma})\big) = \mathcal{P}_{k-1}^h(\mathbf{div}\boldsymbol{\sigma}) = \mathcal{P}_{k-1}^h(-\mathbf{f}) = \mathbf{div}(\boldsymbol{\sigma}_h)$, which says that $\boldsymbol{\delta}_h \in V_h$. It follows from (5.3.51) (cf. Lemma 5.3.7), adding and substracting $\widehat{\Pi}_k^h(\boldsymbol{\sigma})$, using the first equations of (5.3.48) and (5.2.3), employing the identity (5.3.45), and applying the boundedness of $\mathbf{a}_h^K$ (cf. (5.3.46)), $\mathbf{a}^K$ and $\widehat{\Pi}_k^K$ (cf. (5.3.31)), that

$$\begin{aligned}
\alpha\,\|\boldsymbol{\delta}_h\|_{\mathbf{div};\Omega}^2 \,&=\, \alpha\,\|\boldsymbol{\delta}_h\|_{0,\Omega}^2 \,\leq\, \mathbf{a}_h(\boldsymbol{\delta}_h,\boldsymbol{\delta}_h) \,=\, \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}_h(\boldsymbol{\sigma}_h,\boldsymbol{\delta}_h)\\
&=\, \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,+\, \mathbf{a}_h(\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \langle \boldsymbol{\delta}_h\,\mathbf{n},\mathbf{g}\rangle\\
&=\, \mathbf{a}_h(\Pi_k^h(\boldsymbol{\sigma})-\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,+\, \mathbf{a}_h(\widehat{\Pi}_k^h(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\delta}_h)\\
&=\, \sum_{K\in\mathcal{T}_h} \left\{ \mathbf{a}_h^K(\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \,-\, \mathbf{a}^K(\boldsymbol{\sigma}-\widehat{\Pi}_k^K(\boldsymbol{\sigma}),\boldsymbol{\delta}_h) \right\}\\
&\leq\, \alpha_1 \sum_{K\in\mathcal{T}_h} \left\{ \|\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K(\boldsymbol{\sigma})\|_{0,K} \,+\, \|\Pi_k^K(\boldsymbol{\sigma})-\widehat{\Pi}_k^K\{\Pi_k^K(\boldsymbol{\sigma})\}\|_{0,K} \right\} \|\boldsymbol{\delta}_h\|_{0,K}\\
&+\, \frac{1}{2\mu} \sum_{K\in\mathcal{T}_h} \|\boldsymbol{\sigma}-\widehat{\Pi}_k^K(\boldsymbol{\sigma})\|_{0,K}\,\|\boldsymbol{\delta}_h\|_{0,K}\,,
\end{aligned}$$

which yields, with $C := \frac{1}{\alpha}\max\{\alpha_1, \frac{1}{2\mu}\}$,

$$\|\boldsymbol{\delta}_h\|_{\mathbf{div};\Omega} \,\le\, C\left\{\|\Pi_k^h(\boldsymbol{\sigma}) - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\Pi_k^h(\boldsymbol{\sigma}) - \widehat{\Pi}_k^h\{\Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\right\}. \tag{5.3.60}$$

Next, adding and substracting $\boldsymbol{\sigma}$, we deduce that

$$\|\Pi_k^h(\boldsymbol{\sigma}) - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,\le\, \|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,. \tag{5.3.61}$$

In turn, proceding in the same way and employing the boundedness of $\widehat{\Pi}_k^K$ (cf. (5.3.31)), we find that

$$\begin{aligned}
\|\Pi_k^h(\boldsymbol{\sigma}) - \widehat{\Pi}_k^h\{\Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \,&\le\, \|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\widehat{\Pi}_k^h\{\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\}\|_{0,\Omega} \\
&\le\, C\left\{\|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{div}\big(\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\big)\|_{0,\Omega}\right\} \\
&\le\, C\left\{\|\boldsymbol{\sigma} - \Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\right\}.
\end{aligned} \tag{5.3.62}$$

In this way, replacing (5.3.62) and (5.3.61) into (5.3.60), and then the resulting estimate back into (5.3.59), we conclude the upper bound for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$ induced by (5.3.57). In addition, using from the third equation of (5.2.2) that $p = -\frac{1}{2}\,\mathrm{tr}(\boldsymbol{\sigma})$, we obtain

$$\|p - p_h\|_{0,\Omega} \,=\, \frac{1}{2}\,\|\mathrm{tr}(\boldsymbol{\sigma}) - \mathrm{tr}(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,\le\, c\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}\,,$$

which completes the proof of (5.3.57).

On the other hand, concerning the error $\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}$, we begin with the triangle inequality again and obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \,\le\, \|\mathbf{u} - \mathcal{P}_{k-1}^h(\mathbf{u})\|_{0,\Omega} \,+\, \|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}\,. \tag{5.3.63}$$

Next, proceeding as in the proof of Lemma 5.3.10, taking $\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h \in Q_h$ instead of $\mathbf{div}(\boldsymbol{\tau})$ in the definition of the auxiliary problem (5.3.53), we deduce the existence of $\boldsymbol{\sigma}_h^* \in H_h$ such that

$$\mathbf{div}(\boldsymbol{\sigma}_h^*) \,=\, \mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h \quad \text{and} \quad \|\boldsymbol{\sigma}_h^*\|_{\mathbf{div};\Omega} \,\le\, c\,\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}\,.$$

It follows, employing the first equations of (5.3.48) and (5.2.3), and the identity (5.3.45), that

$$
\begin{aligned}
\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}^2 \,&=\, \int_\Omega \big(\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\big)\cdot \mathbf{div}(\boldsymbol{\sigma}_h^*) \,=\, \int_\Omega \big(\mathbf{u} - \mathbf{u}_h\big)\cdot \mathbf{div}(\boldsymbol{\sigma}_h^*)\\
&=\, \mathbf{b}(\boldsymbol{\sigma}_h^*, \mathbf{u}) \,-\, \mathbf{b}(\boldsymbol{\sigma}_h^*, \mathbf{u}_h) \,=\, \mathbf{a}_h(\boldsymbol{\sigma}_h, \boldsymbol{\sigma}_h^*) \,-\, \mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\sigma}_h^*)\\
&=\, \mathbf{a}_h(\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h^*) \,-\, \mathbf{a}(\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma}), \boldsymbol{\sigma}_h^*)\,,
\end{aligned}
$$

which, applying the boundedness of $\mathbf{a}_h^K$ (cf. (5.3.46)), $\mathbf{a}^K$ and $\widehat{\Pi}_k^K$ (cf. (5.3.31)), and observing in particular that $\|\boldsymbol{\sigma}_h^* - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h^*)\|_{0,\Omega} \,\le\, c\,\|\boldsymbol{\sigma}_h^*\|_{\mathbf{div};\Omega} \,\le\, C\,\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega}$, gives

$$
\|\mathcal{P}_{k-1}^h(\mathbf{u}) - \mathbf{u}_h\|_{0,\Omega} \,\le\, C\left\{\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\right\}. \tag{5.3.64}
$$

Now, adding and substracting $\boldsymbol{\sigma}$, we readily get

$$
\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,\le\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega}\,. \tag{5.3.65}
$$

Similarly, and utilizing once again the boundedness of $\widehat{\Pi}_k^K$ (cf. (5.3.31)), we can write

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h - \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega} \,&\le\, \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, \|\widehat{\Pi}_k^h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}\\
&\le\, C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma} - \widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} \,+\, h\,\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}\right\}.
\end{aligned} \tag{5.3.66}
$$

Furthermore, since $\mathbf{div}(\boldsymbol{\sigma}_h) \,=\, \mathcal{P}_{k-1}^h(-\mathbf{f})$ and $\mathbf{div}(\boldsymbol{\sigma}) \,=\, -\mathbf{f}$, we have

$$
\|\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega} \,=\, \|\mathbf{f} - \mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega}\,. \tag{5.3.67}
$$

Consequently, replacing (5.3.66) and (5.3.65) into (5.3.64), making use also of (5.3.67) and the already derived a priori error bound for $\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega}$, and then placing the resulting estimate back into (5.3.63), we arrive at (5.3.58) and conclude the proof. □

Having established the a priori error estimates for our unknowns, we now provide the corresponding rates of convergence.

Theorem 5.3.3. *Let* $(\boldsymbol{\sigma}, \mathbf{u}) \in H \times Q$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ *be the unique solutions of the continuous and discrete schemes* (5.2.3) *and* (5.3.48)*, respectively, and let* $p_h \in L^2(\Omega)$ *be the postprocessed virtual pressure defined in* (5.3.49)*. Assume that for some* $r \in [1, k+1]$ *and* $s \in [1, k]$ *there hold* $\boldsymbol{\sigma}|_K \in \mathbb{H}^r_{\nabla \underline{\mathrm{curl}}}(K)$*,* $\mathbf{f}|_K = -\mathbf{div}\boldsymbol{\sigma}|_K \in \mathbf{H}^{r-1}(K)$*, and* $\mathbf{u}|_K \in \mathbf{H}^s(K)$ *for each* $K \in \mathcal{T}_h$*. Then, there exist positive constants* $\bar{C}_1$*,* $\bar{C}_2$*, independent of* h*, such that*

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le \bar{C}_1 \, h^r \left\{ \sum_{K \in \mathcal{T}_h} \left\{ \|\boldsymbol{\sigma}\|^2_{r,K} + \|\mathbf{f}\|^2_{r-1,K} \right\} \right\}^{1/2}, \tag{5.3.68}$$

and

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \le \bar{C}_2 \, h^r \left\{ \sum_{K \in \mathcal{T}_h} \left\{ \|\boldsymbol{\sigma}\|^2_{r,K} + \|\mathbf{f}\|^2_{r-1,K} \right\} \right\}^{1/2} + \bar{C}_2 \, h^s \left\{ \sum_{K \in \mathcal{T}_h} \|\mathbf{u}\|^2_{s,K} \right\}^{1/2}. \tag{5.3.69}$$

Proof. The case of integers $r \in [1, k+1]$ and $s \in [1, k]$ follows from straightforward applications of the approximation properties provided by Lemmas 5.3.6, 5.3.9, and 5.3.4, to the terms on the right hand sides of (5.3.57) and (5.3.58). In turn, the usual interpolation estimates of Sobolev spaces allow to conclude for the remaining real values of r and s. We omit further details. □

We notice that if the assumed regularities in the foregoing theorem are global, then the estimates (5.3.68) and (5.3.69) become, respectively,

$$\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|p - p_h\|_{0,\Omega} \le \bar{C}_1 \, h^r \left\{ \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{f}\|_{r-1,\Omega} \right\},$$

and

$$\|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega} \le \bar{C}_2 \, h^r \left\{ \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{f}\|_{r-1,\Omega} \right\} + \bar{C}_2 \, h^s \, |\mathbf{u}|_{s,K} \, .$$

In turn, it is also clear from the range of variability of the integers r and s that the highest possible rate of convergence for $\boldsymbol{\sigma}$ and p is h^{k+1}, whereas that of $\mathbf{u}$ is h^k.

We now introduce the fully computable approximations of $\boldsymbol{\sigma}$ and p given by

$$\widehat{\boldsymbol{\sigma}}_h := \widehat{\Pi}^h_k(\boldsymbol{\sigma}_h) \qquad \text{and} \qquad \widehat{p}_h = -\frac{1}{2} \operatorname{tr}\big(\widehat{\boldsymbol{\sigma}}_h\big), \tag{5.3.70}$$

and establish next the corresponding a priori error estimates.

Theorem 5.3.4. *There exists a positive constant C_3, independent of h, such that*

$$\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}+\|p-\widehat{p}_h\|_{0,\Omega} \,\le\, C_3 \left\{ \|\boldsymbol{\sigma}-\Pi_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma}-\widehat{\Pi}_k^h(\boldsymbol{\sigma})\|_{0,\Omega} + h\,\|\mathbf{f}-\mathcal{P}_{k-1}^h(\mathbf{f})\|_{0,\Omega} \right\}. \tag{5.3.71}$$

Proof. Similarly as at the end of the proof of Theorem 5.3.2 we have

$$\|p-\widehat{p}_h\|_{0,\Omega} \,=\, \frac{1}{2}\,\|\mathrm{tr}(\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h)\|_{0,\Omega} \,\le\, c\,\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,,$$

and then, adding and substracting $\boldsymbol{\sigma}_h$, we get

$$\|\boldsymbol{\sigma}-\widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,\le\, \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\sigma}_h-\widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega}\,.$$

In this way, utilizing the estimates for $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$ and $\|\boldsymbol{\sigma}_h-\widehat{\Pi}_k^h(\boldsymbol{\sigma}_h)\|_{0,\Omega}$ given by (5.3.57) and (5.3.66), respectively, and employing (5.3.67) as well, we arrive at (5.3.71) and complete the proof. $\square$

We end this section by remarking, according to the upper bounds provided by (5.3.57) (cf. Theorem 5.3.2) and (5.3.71) (cf. Theorem 5.3.4), that the pairs $(\boldsymbol{\sigma}_h, p_h)$ and $(\widehat{\boldsymbol{\sigma}}_h, \widehat{p}_h)$ share exactly the same rates of convergence given by Theorem 5.3.3.

5.4 Computational implementation

5.4.1 Introduction and notations

We consider the same notations as in previous sections, and given a descomposition $\mathcal{T}_h$ of Ω in polygons, we consider the discretized problem: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in H_h \times Q_h$ such that

$$\begin{array}{lcl} \mathbf{a}_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \;+\; \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) & = & \langle \boldsymbol{\tau}_h \mathbf{n}, \mathbf{g}\rangle \quad \forall\, \boldsymbol{\tau}_h \in H_h \\ \mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) & = & -\displaystyle\int_\Omega \mathbf{f}\cdot\mathbf{v}_h \quad \forall\, \mathbf{v}_h \in Q_h, \end{array} \tag{5.4.1}$$

where H_h, Q_h, $\mathbf{a}_h$ and $\mathbf{b}$ were defined on (5.3.1), (5.3.2), (5.3.41) and (5.2.5). Now, we have by using Theorem 5.3.1, that (5.4.1) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$, for every given data $\mathbf{f}$ and $\mathbf{g}$. Now, let $\{\boldsymbol{\varphi}_j\}_{j=1}^N$ be an edge-oriented basis of H_h and let $\{\boldsymbol{\psi}_j\}_{j=1}^M$ be a basis of Q_h. If $\boldsymbol{\sigma}_h := \sum_{j=1}^N \sigma_j \boldsymbol{\varphi}_j$ and $\mathbf{u}_h := \sum_{j=1}^M u_j \boldsymbol{\psi}_j$, then (5.4.1) is turned into the linear system of equations

$$\begin{aligned} \sum_{j=1}^N \sigma_j \, \mathbf{a}_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \quad + \quad \sum_{l=1}^M u_l \, \mathbf{b}(\boldsymbol{\varphi}_i, \boldsymbol{\psi}_l) \quad &= \quad \langle \boldsymbol{\varphi}_i \mathbf{n}, \mathbf{g} \rangle_\Gamma \\ \sum_{j=1}^N \sigma_j \, \mathbf{b}(\boldsymbol{\varphi}_j, \boldsymbol{\psi}_l) \quad &= \quad -\int_\Omega \mathbf{f} \cdot \boldsymbol{\psi}_l, \end{aligned} \tag{5.4.2}$$

for each $i \in \{1, \ldots, N\}$ and for each $l \in \{1, \ldots, M\}$. Equivalently,

$$Ax = b,$$

where $x = (\sigma_1, \ldots, \sigma_N, u_1, \ldots, u_M)^{\mathtt{t}}$, $A = \begin{pmatrix} \mathbf{A}_h & \mathbf{B} \\ \mathbf{B}^t & 0 \end{pmatrix}$, $\mathbf{A}_h = (A_{ij})$, $\mathbf{B} = (B_{ij})$, $\mathbf{c} = (c_i)$, $\mathbf{d} = (d_i)$ and $b = (\mathbf{c}, \mathbf{d})^{\mathtt{t}}$. Also,

$$\begin{aligned} A_{ij} &:= \mathbf{a}_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \quad i, j \in \{1, \ldots, N\} \\ B_{ij} &:= \mathbf{b}(\boldsymbol{\varphi}_i, \boldsymbol{\psi}_j) \quad i \in \{1, \ldots, N\} \;\; j \in \{1, \ldots, M\} \\ c_i &:= \langle \boldsymbol{\varphi}_i \mathbf{n}, \mathbf{g} \rangle_\Gamma \quad i \in \{1, \ldots, N\} \\ d_i &:= -\left(\mathbf{f}, \boldsymbol{\psi}_i\right)_{0,\Omega} \quad i \in \{1, \ldots, M\}. \end{aligned} \tag{5.4.3}$$

Furthermore, recall that we can write $\mathbf{a}_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = \sum_{K \in \mathcal{T}_h} \mathbf{a}_h^K(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j)$ and do the same for the other functionals and bilinear forms defined above, and we can then localize the calculations for $\mathbf{A}_h$, $\mathbf{B}$, $\mathbf{c}$ and $\mathbf{d}$.

5.4.1.1 Notations

We start remembering that $\mathcal{T}_h$ is a decomposition of K in polygons and $k \geq 1$ is the polynomial degree of accuracy. Given $K \in \mathcal{T}_h$ we denote by h_K and $\mathbf{x}_K$ the diameter and the barycenter of K, respectively, and we define, for a multi-index α, the polynomial function $m_\alpha^K := \left(\dfrac{\mathbf{x} - \mathbf{x}_K}{h_K}\right)^\alpha$. From now on, we will use the identification $m_0 := 0$, $1 \leftrightarrow (0,0)$, $2 \leftrightarrow (1,0)$, $3 \leftrightarrow (0,1)$, $4 \leftrightarrow (2,0)$, $5 \leftrightarrow (1,1)$ and so on. In a similar way, we define the polynomial vector field $\mathbf{m}^K_{\alpha,\beta} := \left(m^K_\alpha, m^K_\beta\right)^{\mathtt{t}}$ and we will use the identification $\mathbf{m}^K_1 \leftrightarrow \mathbf{m}^K_{1,0}$, $\mathbf{m}^K_2 \leftrightarrow \mathbf{m}^K_{0,1}$, $\mathbf{m}^K_3 \leftrightarrow \mathbf{m}^K_{2,0}$, $\mathbf{m}^K_4 \leftrightarrow \mathbf{m}^K_{0,2}$ and so on. Then, as in (4.3.5), we have that

$$\mathcal{B}_k(K) := \Big\{\mathbf{m}^K_j : j \in \{1, \dots, 2\dim \mathrm{P}_k(K)\}\Big\}.$$

In turn, we choose a counterclockwise arrangement $\{V_1, \dots, V_{d_K}\}$ of the vertices of K, where we have defined d_K as the number of vertices of K. If e_j is the edge that connects V_j with V_{j+1} (and using the identification $V_{d_K+1} \leftrightarrow V_1$), we denote h_{e_j} the length of e_j, x_{e_j} the middle point of e_j, and for $i \geq 1$, $j \in \{1, \dots, d_K\}$ and $x \in e_j$, we define

$$m^{e_j}_i(x) := \left(\frac{x - x_{e_j}}{h_{e_j}}\right)^{i-1}.$$

Anagously, using the identification $m^{\partial K}_0 \leftrightarrow 1$, $m^{\partial K}_1 \leftrightarrow m^{e_1}_1$, $m^{\partial K}_2 \leftrightarrow m^{e_1}_2$, $m^{\partial K}_{(k+1)+1} \leftrightarrow m^{e_2}_1$ and so on, we define $\mathbf{m}^{\partial K}_{i,j} := \left(m^{\partial K}_i, m^{\partial K}_j\right)^t$, and we use the identification $\mathbf{m}^{\partial K}_1 \leftrightarrow \mathbf{m}^{\partial K}_{1,0}$, $\mathbf{m}^{\partial K}_2 \leftrightarrow \mathbf{m}^{\partial K}_{0,1}$, $\mathbf{m}^{\partial K}_3 \leftrightarrow \mathbf{m}^{\partial K}_{2,0}$ and so on. Then, we define

$$\mathcal{B}_k(\partial K) := \Big\{\mathbf{m}^{\partial K}_j : j \in \{1, \dots, 2\, d_K\ (k+1)\}\Big\}.$$

Now, given $K \in \mathcal{T}_h$, we want to calculate the local matrix $\mathbf{A}_K$ that corresponds to $\mathbf{a}^K_h$, i.e.

$$(\mathbf{A}_K)_{ij} := \mathbf{a}^K_h(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) \qquad i, j \in \left\{1, \dots, n^K_k\right\},$$

where $\left\{\boldsymbol{\varphi}_1, \boldsymbol{\varphi}_2, \dots, \boldsymbol{\varphi}_{n^K_k}\right\}$ is the canonical basis of H^K_h and n^K_k its dimension (cf. (5.3.7)). For this end, and recalling the definition of $\mathbf{a}^K_h$ (cf. (5.3.41)), we have to calculate the projector $\widehat{\Pi}^K_k(\boldsymbol{\varphi}_i)$ for each $i \in \{1, \dots, n^K_k\}$. In order to do this, we recall the decomposition introduced in (5.3.26), that is

$$\widehat{\Pi}^K_k(\boldsymbol{\varphi}_i) := \widehat{\boldsymbol{\zeta}} = \boldsymbol{\zeta}_\nabla + q_{\boldsymbol{\zeta}}\,\mathbb{I} + c_{\boldsymbol{\zeta}}\,\mathbb{I},$$

where $\boldsymbol{\zeta}_\nabla \in \widehat{H}^K_{k,\nabla}$, $q_{\boldsymbol{\zeta}} \in \widehat{P}_k(K)$ and $c_{\boldsymbol{\zeta}} \in \mathrm{R}$.

5.4.2 Calculating local matrices

Given $K \in \mathcal{T}_h$ and according to (5.4.3), we need to calculate the local matrices, which will be denoted $\mathbf{A}_K$, $\mathbf{B}_K$, $\mathbf{c}_K$ and $\mathbf{d}_K$, respectively. Let us first calculate $\mathbf{A}_K$. Given

$i, j \in \{1, \ldots, n_k^K\}$, recall that

$$\mathbf{a}_h^K(\boldsymbol{\varphi}_i, \boldsymbol{\varphi}_j) = \mathbf{a}^K(\hat{\Pi}_k^K(\boldsymbol{\varphi}_i), \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)) + S^K(\boldsymbol{\varphi}_i - \widehat{\Pi}_k^K(\boldsymbol{\varphi}_i), \boldsymbol{\varphi}_j - \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)).$$

In turn, if $\widehat{n}_{k,\nabla} := \dim \widehat{H}_{k,\nabla}^K$, $\widehat{n}_k := \dim \widehat{P}_k(K)$, $\widehat{N}_k := \dim \widehat{H}_k^K$, $\mathbf{M}_j := h_K^2 \nabla \underline{\mathrm{curl}}\, m_{j+3}$ for each $j \in \{1, \ldots, \widehat{n}_{k,\nabla}\}$ and $\widehat{m}_j := m_{j+1}$ for each $j \in \{1, \ldots, \widehat{n}_k\}$, we can write

$$\widehat{\Pi}_k^K(\boldsymbol{\varphi}_i) \;:=\; \sum_{j=1}^{n} s_{j,\nabla}^{(i)} \mathbf{M}_j \;+\; \sum_{j=1}^{\widehat{n}_k} s_j^{(i)} \widehat{m}_j \mathbb{I} \;+\; s_{\widehat{N}_k}^{(i)}\, \mathbb{I}.$$

Therefore, replacing the above expression into (5.3.27)-(5.3.29), we arrive at the system

$$\begin{array}{rcl}
\displaystyle\sum_{\alpha=1}^{\widehat{n}_{k,\nabla}} s_{\alpha,\nabla}^{(i)} (\mathbf{M}_\alpha, \mathbf{M}_\beta)_{0,K} & = & (\mathbf{M}_\beta, \boldsymbol{\varphi}_i)_{0,K} \\
\displaystyle\sum_{\alpha=1}^{\widehat{n}_{k,\nabla}} s_{\alpha,\nabla}^{(i)} (\mathbf{div}\, \mathbf{M}_\alpha, \mathbf{div}\, \mathbf{M}_\gamma)_{0,K} \;+\; \sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} (\nabla \widehat{m}_\alpha, \nabla \widehat{m}_\gamma)_{0,K} & = & (\mathbf{div}\, \boldsymbol{\varphi}_i, \nabla \widehat{m}_\gamma) \\
\displaystyle 2\sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} (\widehat{m}_\alpha, 1)_{0,K} \;+\; 2 s_{\widehat{N}_k}^{(i)} |K| & = & (\boldsymbol{\varphi}_i, 1)_{0,K}
\end{array}$$

for each $\beta \in \{1, \ldots, \widehat{n}_{k,\nabla}\}$ and $\gamma \in \{1, \ldots, \widehat{n}_k\}$. Moreover, the above system can be read in a more compact form:

$$\mathbf{G}\, \mathbf{s}^{(i)} = \mathbf{b}^{(i)},$$

where $\mathbf{s}^{(i)} := \left(s_{1,\nabla}^{(i)}, s_{2,\nabla}^{(i)}, \ldots, s_{\widehat{n}_{k,\nabla},\nabla}^{(i)}, s_1^{(i)}, s_2^{(i)}, \ldots, s_{\widehat{n}_k}^{(i)}, s_{\widehat{N}_k}^{(i)} \right)^{\mathtt{t}}$. In turn, if

$$\mathbf{T} := [\; \mathbf{b}^{(1)} \quad \mathbf{b}^{(2)} \quad \cdots \quad \mathbf{b}^{(n_k^K)} \;] \in \mathrm{R}^{\widehat{N}_k \times n_k^K},$$

then the matrix representation $\widehat{\boldsymbol{\Pi}}_*$ of the operator $\widehat{\Pi}_k^K$, acting from H_k^K into $\widehat{H}_k^K$, is given by $(\widehat{\boldsymbol{\Pi}}_*)_{\alpha i} := s_\alpha^{(i)}$, that is

$$\widehat{\boldsymbol{\Pi}}_* \;=\; \mathbf{G}^{-1}\, \mathbf{T}.$$

5.4.2.1 Calculating T.

Recall first that K has d_K edges. In turn, in order to split the elements of the local basis H_k^K, we define the numbers

$$n_{k,0}^K := 2 d_K (k+1) = \dim \mathbf{P}_k(\partial K)$$
$$n_{k,1}^K := n_{k,0}^K + k(k+1) - 2 = n_{k,0}^K + \dim \mathbf{P}_{k-1}(K) - 2.$$

Also, extending the functions of $\mathcal{B}_k(\partial K)$ outside its edges by zero, we denote (cf. (5.3.17)),

$$
\begin{aligned}
m_{j,K}(\boldsymbol{\tau}) &:= \int_{\partial K} (\boldsymbol{\tau}\,\mathbf{n}) \cdot \mathbf{m}_j^{\partial K} \qquad j \in \left\{1, \dots, n_{k,0}^K\right\}, \\
m_{j,K}(\boldsymbol{\tau}) &:= \int_K \boldsymbol{\tau} : \nabla\, \mathbf{m}_{j+2-n_{k,0}^K}^K \qquad j \in \left\{n_{k,0}^K + 1, \dots, n_{k,1}^K\right\}, \quad k > 1, \\
m_{j,K}(\boldsymbol{\tau}) &:= \int_K \mathbf{m}_{j-n_{k,1}^K}^K \cdot \mathbf{rot}\,\boldsymbol{\tau} \qquad j \in \left\{n_{k,1}^K + 1, \dots, n_k^K\right\},
\end{aligned}
$$

and we also remind that $m_{i,K}(\boldsymbol{\varphi}_j) = \delta_{ij} \quad \forall i, j \in \left\{1, \dots, n_k^K\right\}$. Now, let $i \in \{1, \dots, n_k^K\}$ and $\beta \in \{1, \dots, \widehat{n}_{k,\nabla}\}$. Then, we have that $\mathbf{M}_\beta = h_K^2 \nabla\, \underline{\mathrm{curl}}\, m_{\beta+3}^K$, hence, there exists $\gamma \in \{1, \dots, (k+2)(k+3)\}$ such that $\mathbf{M}_\beta = h_K \nabla \mathbf{m}_\gamma^K$. Thus,

$$
\begin{aligned}
(\mathbf{b}^{(i)})_\beta :&= -\int_K \mathbf{m}_\gamma^K \cdot \mathbf{div}\, \boldsymbol{\varphi}_i + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \mathbf{n}) \cdot \mathbf{m}_\gamma^K \\
&= -\int_K \mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K) \cdot \mathbf{div}\, \boldsymbol{\varphi}_i + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \mathbf{n}) \cdot \mathcal{P}_k^{\partial K}(\mathbf{m}_\gamma^K) \\
&= h_K \int_K \boldsymbol{\varphi}_i : \nabla \mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K) + h_K \int_{\partial K} (\boldsymbol{\varphi}_i \mathbf{n}) \cdot \mathcal{P}_k^{\partial K}(\mathbf{m}_\gamma^K - \mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K))
\end{aligned}
$$

Now, we can write

$$
\mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K) = \sum_{\alpha=1}^{k(k+1)} p_\alpha \mathbf{m}_\alpha^K,
$$

and if $\mathbf{P}_0$ is the Gram matrix asociated to $\mathcal{P}_{k-1}^K$, that is $(\mathbf{P}_0)_{ij} := (\mathbf{m}_i^K, \mathbf{m}_j^K)_{0,K}$, and $\mathbf{d}_0$ is the right-hand side of such a system, i.e. $(\mathbf{d}_0)_j := (\mathbf{m}_\gamma^K, \mathbf{m}_j^K)_{0,K}$, then $\left(p_1, \dots, p_{k(k+1)}\right)^{\mathtt{t}} =: \mathbf{p}_0 = \mathbf{P}_0^{-1}\mathbf{d_0}$. Furthermore, we can write

$$
\mathcal{P}_k^{\partial K}(\mathbf{m}_\gamma^K - \mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K)) = \sum_{\alpha=1}^{n_{k,0}^K} q_\alpha \mathbf{m}_\alpha^{\partial K}
$$

where, again, if $\mathbf{P}_1$ is the Gram matrix asociated to such a system and $\mathbf{d}_1$ its right hand side, i.e. $(\mathbf{d}_1)_j := (\mathbf{m}_\gamma^K - \mathcal{P}_{k-1}^K(\mathbf{m}_\gamma^K), \mathbf{m}_j^{\partial K})$, then $\left(q_1, \dots, q_{2d_K(k+1)}\right)^{\mathtt{t}} =: \mathbf{p}_1 = \mathbf{P}_1^{-1}\mathbf{d}_1$. Now, using the degrees of freedom for $\boldsymbol{\varphi}_i$, we obtain

$$
\mathbf{b}^{(i)} := h_K \begin{cases} \left(\mathbf{p}_1^{\mathtt{t}}, 0, 0\right)^{\mathtt{t}} & , \quad k = 1 \\ \left(\mathbf{p}_1^{\mathtt{t}}, p_3, p_4, \dots, p_{k(k+1)}, \mathbf{0}\right)^{\mathtt{t}} & , \quad k > 1 \end{cases} \tag{5.4.4}
$$

Remark 5.4.1. Note that

$$\frac{1}{h_e}\int_e \left(\frac{x-x_e}{h_e}\right)^{\alpha}\left(\frac{x-x_e}{h_e}\right)^{\beta} = \frac{1}{2^{\alpha+\beta}(\alpha+\beta+1)}\begin{cases}1 & , \ \alpha+\beta \text{ is even} \\ 0 & , \text{ otherwise}\end{cases},$$

which implies that $\mathbf{P}_1$ is a matrix made of d_K identical blocks.

It remains to consider the second set of equations defining $\widehat{\Pi}_k^K(\boldsymbol{\varphi}_i)$. On its last equation, we can take $\mathbf{m}_\gamma^K := \mathbf{m}_3^K + \mathbf{m}_6^K$ and repeat the analysis done before. Finally, if $\beta \in \{1,\ldots,\widehat{n}_k\}$, then $\widehat{m}_\beta$ is given directly by using the degrees of freedom of $\boldsymbol{\varphi}_i$. Thus,

$$\begin{aligned} &a^K(\widehat{\Pi}_k^K(\boldsymbol{\varphi}_i), \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)) \\ &\quad = \sum_{\alpha,\beta=1}^{\widehat{n}_k} s_\alpha^{(i)} s_\beta^{(j)} (\mathbf{M}_\alpha, \mathbf{M}_\beta)_{0,K} = \sum_{\alpha,\beta=1}^{\widehat{n}_k} (\widehat{\mathbf{\Pi}}_*)_{\alpha i}(\widehat{\mathbf{\Pi}}_*)_{\beta j}\widetilde{\mathbf{G}}_{\alpha\beta} = [(\widehat{\mathbf{\Pi}}_*)^{\mathrm{T}}\widetilde{\mathbf{G}}(\widehat{\mathbf{\Pi}}_*)]_{ij}, \end{aligned}$$

where $\widetilde{\mathbf{G}}$ is the matrix that is equal to $\mathbf{G}$, in exception of the rows $\widehat{n}_{k,\nabla}+1$ up to $\widehat{N}_k$ which are set to zero, since $(q\mathbb{I})^{\mathrm{d}} = \mathbf{0}$ if $q \in \mathrm{P}_k(K)$.

Now, we need to calculate the coordinates of $\widehat{\Pi}_k^K \boldsymbol{\varphi}_i$ into the space H_k^K. More specifically,

$$\widehat{\Pi}_k^K \boldsymbol{\varphi}_i = \sum_{j=1}^{n_k^K} \pi_j^{(i)} \boldsymbol{\varphi}_j,$$

whence $\pi_j^{(i)} = m_{j,K}(\widehat{\Pi}_k^K \boldsymbol{\varphi}_i)$. Then, defining $\mathbf{M}_{\alpha+\widehat{n}_{k,\nabla}} := \widehat{m}_\alpha$ if $\alpha \in \{1,\ldots,\widehat{n}_k\}$ and $\mathbf{M}_{\widehat{N}_k} := \mathbb{I}$, and using the expresion for the coordinates of $\widehat{\Pi}_k^K \boldsymbol{\varphi}_i$ into the space H_k^K, we get

$$\widehat{\Pi}_k^K \boldsymbol{\varphi}_i = \sum_{\alpha=1}^{\widehat{N}_k} s_\alpha^{(i)} \mathbf{M}_\alpha = \sum_{\alpha=1}^{\widehat{N}_k} s_\alpha^{(i)} \sum_{j=1}^{n_k^K} m_{j,K}(\mathbf{M}_\alpha)\boldsymbol{\varphi}_j,$$

and so

$$\pi_i^j = \sum_{\alpha=1}^{\widehat{n}_k} s_\alpha^{(i)} m_{j,K}(\mathbf{M}_\alpha).$$

Now, we define $\mathbf{D} := (\mathbf{D}_{i\alpha}) = m_{i,K}(\mathbf{M}_\alpha)$ for every $i \in \{1,\ldots,n_k^K\}$ and $\alpha \in \{1,\ldots,\widehat{N}_k\}$, and then

$$\pi_j^{(i)} = \sum_{\alpha=1}^{n_k^K} (\mathbf{G^{-1}T})_{\alpha i}\mathbf{D}_{j\alpha} = (\mathbf{DG^{-1}T})_{ji}.$$

Consequently, the matrix representation $\widehat{\mathbf{\Pi}}$ of the operator $\widehat{\Pi}_k^K$, from H_k^K into itself, is given by $\widehat{\mathbf{\Pi}} := \mathbf{D}\mathbf{G}^{-1}\mathbf{T} = \mathbf{D}\widehat{\mathbf{\Pi}}_*$.

Proposition 5.4.1. For every element K, we have $\mathbf{G} = \mathbf{T}\mathbf{D}$.

Proof. Given $\beta \in \{1, \ldots, \widehat{N}_k\}$ and for $\alpha \in \{1, \ldots, \widehat{n}_{k,\nabla}\}$, we have that

$$\begin{aligned}
&\sum_{i=1}^{n_k^K} \mathbf{T}_{\alpha i}\mathbf{D}_{i\beta} \\
&\quad = \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)(\mathbf{M}_\alpha, \boldsymbol{\varphi}_i)_{0,K} = (\mathbf{M}_\alpha, \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)\boldsymbol{\varphi}_i)_{0,K} = (\mathbf{M}_\alpha, \mathbf{M}_\beta)_{0,K} = \mathbf{G}_{\alpha\beta}.
\end{aligned}$$

Now, if $\alpha \in \{\widehat{n}_{k,\nabla} + 1, \ldots, \widehat{N}_k - 1\}$,

$$\begin{aligned}
\sum_{i=1}^{n_k^K} \mathbf{T}_{\alpha i}\mathbf{D}_{i\beta} &= \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)(\mathbf{div}\, \mathbf{M}_\beta, \mathbf{div}\, \boldsymbol{\varphi}_i)_{0,K} \\
&= \Big(\mathbf{div}\, \mathbf{M}_\beta, \mathbf{div} \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)\boldsymbol{\varphi}_i\Big)_{0,K} = (\mathbf{div}\, \mathbf{M}_\alpha, \mathbf{div}\, \mathbf{M}_\beta)_{0,K} = \mathbf{G}_{\alpha\beta}.
\end{aligned}$$

Finally, if $\alpha = \widehat{N}_k$,

$$\sum_{i=1}^{n_k^K} \mathbf{T}_{\alpha i}\mathbf{D}_{i\beta} = \sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)(\boldsymbol{\varphi}_i, \mathbf{I})_{0,K} = \Big(\sum_{i=1}^{n_k^K} m_{i,K}(\mathbf{M}_\beta)\boldsymbol{\varphi}_i, \mathbf{I}\Big)_{0,K} = (\mathbf{M}_\alpha, \mathbf{I})_{0,K} = \mathbf{G}_{\alpha\beta},$$

which finishes the proof. $\square$

Remark 5.4.2. The above proposition implies an improvement of the code in terms of speed, that is, it is not necessary to calculate $\mathbf{G}$ by using its explicit expression given by the corresponding system that defines $\widetilde{\Pi}_k^K$; we just need to calculate $\mathbf{T}$, $\mathbf{D}$, and then we obtain $\mathbf{G}$ by doing $\mathbf{G} := \mathbf{T}\mathbf{D}$.

On the other hand, using that

$$\mathcal{S}^K\Big((I - \widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i), (I - \widehat{\Pi}_k^K)(\boldsymbol{\varphi}_j)\Big) = \sum_{r=1}^{n_k^K} m_{r,K}((I - \widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i))m_{r,K}((I - \widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i)),$$

and

$$m_{r,K}((I - \widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i)) = [(\mathbf{I} - \widehat{\mathbf{\Pi}})]_{ir},$$

we then have that $\mathcal{S}^K\Big((I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_i),(I-\widehat{\Pi}_k^K)(\boldsymbol{\varphi}_j)\Big) = [(\mathbf{I}-\widehat{\mathbf{\Pi}})^{\mathrm{T}}(\mathbf{I}-\widehat{\mathbf{\Pi}})]_{ij}$. Therefore, the matrix expression for the local stiffness matrix $\mathbf{A}_h^K$ is given by

$$\mathbf{A}_h^K = (\widehat{\mathbf{\Pi}}_*)^{\mathrm{T}}\widetilde{\mathbf{G}}(\widehat{\mathbf{\Pi}}_*) + (\mathbf{I}-\widehat{\mathbf{\Pi}})^{\mathrm{T}}(\mathbf{I}-\widehat{\mathbf{\Pi}}). \tag{5.4.5}$$

Now, in order to calculate $\mathbf{B}_K$, we note that given indices $\alpha \in \{1,\dots,\widehat{N}_k\}$ and $j \in \{1,\dots,k(k+1)\}$,

$$\begin{aligned} b(\boldsymbol{\varphi}_\alpha,\mathbf{m}_j^K) &= \int_K \mathbf{m}_j^K \cdot \mathbf{div}\,\boldsymbol{\varphi}_\alpha \\ &= -h_K^{-1}\int_K \boldsymbol{\varphi}_\alpha : \nabla\mathbf{m}_j^K + \int_{\partial K} (\boldsymbol{\varphi}_\alpha\mathbf{n})\cdot\mathbf{m}_j^K \\ &= -h_K^{-1}\delta_{\alpha,j+n_{k,0}^K} + \delta_{\alpha j}q_j, \end{aligned}$$

where $(q_1,\dots,q_{n_{k,0}^K})^{\mathtt{t}} := \mathbf{P^{-1}d}$ and $\mathbf{d} := \Big((\mathbf{m}_j^K,\mathbf{m}_1^{\partial K}),\dots,(\mathbf{m}_j^K,\mathbf{m}_{n_{k,0}^K}^{\partial K})\Big)^{\mathtt{t}}$. To calculate $\mathbf{c}$, we need, again, the $[L^2(\partial K)]^2$ projection onto $[\mathrm{P}_k(\partial K)]^2$. So, we define $\mathbf{h} := \Big((\mathbf{g},\mathbf{m}_1^{\partial K})_{0,\partial K},\dots,(\mathbf{g},\mathbf{m}_{n_{k,0}^K}^{\partial K})_{0,\partial K}\Big)^{\mathtt{t}}$, and then, if $\mathbf{q} := \mathbf{P^{-1}h}$ and the corresponding edge lies on Γ (we set the corresponding elements by zero otherwise), then

$$\mathbf{c}_K = \mathbf{q}.$$

Finally, $\mathbf{d}_K$ is simply calculated by considering an integration rule on K which is exact when integrating polynomials of degree up to $k-1$. In summary, we obtain the following procedure to calculate the local matrices on each element K:

- Calculate $\mathbf{T}$ and $\mathbf{D}$.
- Calculate $\mathbf{G}$ by doing $\mathbf{G} := \mathbf{SD}$.
- Calculate $\mathbf{A}_h^K$ by using formula (5.4.5).
- Calculate $\mathbf{B}_K$, $\mathbf{c}_K$ and $\mathbf{d}_K$.

5.4.3 Assembling the global matrix and post-processing the solution

Using the fact that all the bilinear forms and functionals can be split into an element by element sum, we just need to assemble the local matrices and then sum them by its corresponding indices. We point out now that we are calculating the coefficients of $\boldsymbol{\sigma}_h$ in H_h, of $\mathbf{u}_h$ in Q_h, as well as the coefficients of p_h. Since $p_h = -\frac{1}{2}\operatorname{tr}(\boldsymbol{\sigma}_h)$ and $\boldsymbol{\sigma}_h$ is not manipulable, then p_h is not manipulable either. Consequently, the errors $\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega}$ and $\|p-p_h\|_{0,\Omega}$ are only theoretical. To create a fully computable solution, we use (5.3.70) and set

$$\widehat{\boldsymbol{\sigma}}_h := \widehat{\Pi}_k^h(\boldsymbol{\sigma}_h), \qquad \widehat{p}_h := -\frac{1}{2}\operatorname{tr}(\widehat{\boldsymbol{\sigma}}_h).$$

Now, the condition $\displaystyle\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) = 0$ was not explicitly introduced in the system and hence $\boldsymbol{\sigma}_h \in \mathbb{H}_0$ does not always hold. Note that if $K \in \mathcal{T}_h$ and $\tau_1, \ldots, \tau_{n_k^K}$ are the local coefficients of $\boldsymbol{\sigma}_h|_K$, then

$$\int_K \operatorname{tr}(\boldsymbol{\sigma}_h) = \sum_{j=1}^{n_k^K} \tau_j(\boldsymbol{\varphi}_j, \mathbf{I})_{0,K} = \sum_{j=1}^{n_k^K} \tau_j \mathbf{T}_{\widehat{N}_k, j},$$

that is, $(\boldsymbol{\sigma}_h, \mathbf{I})_{0,K}$ is the scalar multiplication between the $\widehat{N}_k$-th row of $\mathbf{T}$ and the foregoing local coefficients $\boldsymbol{\tau} := (\tau_1, \ldots, \tau_{n_k^K})^{\mathtt{t}}$. Then, the following procedure to find the solution $\boldsymbol{\sigma}_h \in \mathbb{H}_0$ can be used.

- Set $T = 0$.
- On each element $K \in \mathcal{T}_h$, re-define T as $T + x$, where x is the scalar multiplication between the $\widehat{N}_k$-th row of $\mathbf{T}$ and the local coefficients of $\boldsymbol{\sigma}_h$ in K.
- After finishing the step above, re-define $\boldsymbol{\sigma}_h$ as $\boldsymbol{\sigma}_h - \dfrac{T}{2\,|\Omega|}\mathbf{I}$.

Therefore, using that $\widehat{\Pi}_k^K \mathbb{I} = \mathbb{I}$, it follows that

$$\widehat{\Pi}_k^K(\boldsymbol{\sigma}_h|_K) = \widehat{\Pi}_k^K\left(\sum_{j=1}^{n_k^K} \tau_j \boldsymbol{\varphi}_j - \frac{T}{2\,|\Omega|}\mathbf{I}\right) = -\frac{T}{2\,|\Omega|}\mathbf{I} + \sum_{j=1}^{n_k^K} \tau_j \widehat{\Pi}_k^K(\boldsymbol{\varphi}_j)$$
$$= -\frac{T}{2\,|\Omega|}\mathbf{I} + \sum_{j=1}^{n_k^K} \tau_j \sum_{\alpha=1}^{\widehat{n}_k} (\widehat{\mathbf{\Pi}}_*)_{\alpha j}\mathbf{M}_\alpha = -\frac{T}{2\,|\Omega|}\mathbf{I} + \sum_{\alpha=1}^{\widehat{n}_k} (\widehat{\mathbf{\Pi}}_* \boldsymbol{\tau})_\alpha \mathbf{M}_\alpha,$$

that is, the coefficients of $\widehat{\boldsymbol{\sigma}}_h$ on $\widehat{H}_k^K$ are given by $\widehat{\mathbf{\Pi}}_* \boldsymbol{\tau} - \dfrac{T}{2\,|\Omega|} e_{\widehat{N}_k}$, where e_j is the j^{th} canonical vector on $\mathrm{R}^{\widehat{N}_k}$.

5.4.4 Numerical results

In this section we present three numerical examples illustrating the good performance of the virtual mixed finite element scheme (5.3.48), and confirming the rates of convergence predicted by Theorem 5.3.3. For all the computations we consider the virtual element subspaces H_h and Q_h given by (5.3.1) and (5.3.2), with $k = 1$. In turn, for each Example we take kinematic viscosity $\mu = 1$ and assume first decompositions of Ω made of triangles. In addition, in Example 1 we also consider straight squares, whereas Examples 2 and 3 make use of general quadrilateral elements as well. We begin by introducing additional notations. In what follows, N stands for the total number of degrees of freedom (unknowns) of (5.3.48), that is, $N = \dim H_h + \dim Q_h$. More precisely, according to (5.3.6) and (5.3.2), and bearing in mind that $\dim \mathbf{P}_k(e) = 2(k+1)$ $\forall$ edge $e \in \mathcal{T}_h$, and $\dim \mathbf{P}_{k-1}(K) = k(k+1)$ $\forall\, K \in \mathcal{T}_h$, we find that in general

$$N \;=\; 2(k+1) \;\times\; \text{number of edges } e \in \mathcal{T}_h \;+\; \Big\{3k(k+1) - 2\Big\} \;\times\; \text{number of } K \in \mathcal{T}_h\,,$$

which, in the case $k = 1$, becomes

$$N \;=\; 4 \;\times\; \Big\{\text{number of edges } e \in \mathcal{T}_h\Big\} \;+\; 2\Big\{\text{number of } K \in \mathcal{T}_h\Big\}.$$

Also, the individual errors are defined by

$$\mathsf{e}_0(\boldsymbol{\sigma}) \;:=\; \|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_{0,\Omega}\,, \quad \mathsf{e}(p) \;:=\; \|p - \widehat{p}_h\|_{0,\Omega}\,, \quad \text{and} \quad \mathsf{e}(\mathbf{u}) \;:=\; \|\mathbf{u} - \mathbf{u}_h\|_{0,\Omega}\,,$$

where $\widehat{\boldsymbol{\sigma}}_h$ and $\widehat{p}_h$ are computed according to (5.3.70), and $\mathbf{u}_h$ is provided by (5.3.48). In turn, the associated experimental rates of convergence are given by

$$\mathtt{r}_0(\boldsymbol{\sigma}) \,:=\, \frac{\log\big(\mathtt{e}_0(\boldsymbol{\sigma})/\mathtt{e}_0'(\boldsymbol{\sigma})\big)}{\log(h/h')}\,, \quad \mathtt{r}(p) \,:=\, \frac{\log\big(\mathtt{e}(p)/\mathtt{e}'(p)\big)}{\log(h/h')}\,, \quad \text{and} \quad \mathtt{r}(\mathbf{u}) \,:=\, \frac{\log\big(\mathtt{e}(\mathbf{u})/\mathtt{e}'(\mathbf{u})\big)}{\log(h/h')}\,,$$

where $\mathtt{e}$ and $\mathtt{e}'$ denote the errors for two consecutive meshes with sizes h and h', respectively. The numerical results presented below were obtained using a Matlab code. The corresponding linear systems were solved using the Conjugate Gradient method as main solver, and applying a stopping criterion determined by a relative tolerance of 10^{-10}. The specific examples to be considered are described next.

In Example 1 we consider $\Omega =]0,1[^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (5.2.1) is given for each $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, (\sin(\pi x_1)\,\cos(\pi x_2), -\cos(\pi x_1)\,\sin(\pi x_2))^{\mathtt{t}} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, \frac{1}{x_2^2+1} - \frac{\pi}{4}\,.$$

In Example 2 we consider the L-shaped domain $\Omega := \,]-1,1\,[^2 - [0,1]^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution of (5.2.1) is given for each $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, \frac{\mathbf{x}-(1,1)^{\mathtt{t}}}{(x_1-1)^2+(x_2-1)^2} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, x_1 + \frac{1}{6}\,.$$

Finally, in Example 3 we consider the same geometry of Example 1, that is $\Omega =]0,1[^2$, and choose the data $\mathbf{f}$ and $\mathbf{g}$ so that the exact solution is given for each $\mathbf{x} := (x_1,x_2)^{\mathtt{t}} \in \Omega$ by

$$\mathbf{u}(\mathbf{x}) \,:=\, (x_2^2, -x_1^2)^{\mathtt{t}} \qquad \text{and} \qquad p(\mathbf{x}) \,:=\, \big(x_1^2+x_2^2\big)^{1/3} - \int_\Omega \big(x_1^2+x_2^2\big)^{1/3}\,.$$

Note in this example that the partial derivatives of p, and hence, in particular $\mathbf{div}\,\boldsymbol{\sigma}$, are singular at the origin. Moreover, because of the power $1/3$, there holds $\boldsymbol{\sigma} \in \mathbb{H}^{5/3-\epsilon}(\Omega)$ and $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{H}^{2/3-\epsilon}(\Omega)$ for each $\epsilon > 0$, which, applying Theorem 5.3.3 with $r = 5/3-\epsilon$, should yield a rate of convergence very close to $O(h^{5/3})$ for $\boldsymbol{\sigma}$ and p.

In Tables 5.1 up to 5.4 we summarize the convergence history of the mixed virtual element scheme (5.3.48) as applied to Examples 1 and 2, for sequences of quasi-uniform refinements of each domain. We notice there that the rates of convergences $O(h^{k+1}) = O(h^2)$ and $O(h^k) = O(h)$ predicted by Theorem 5.3.3 (when $r = k+1$ and $s = k$) are attained by $(\boldsymbol{\sigma}, p)$ and $\mathbf{u}$, respectively, for triangular as well as for quadrilateral meshes. In turn, in Tables 5.5 and 5.6 we display the corresponding convergence history of Example 3. As predicted in advance, and due to the limited regularity of p and $\boldsymbol{\sigma}$ in this case, we observe that the orders $O(h^{k+\frac{2}{3}}) = O(h^{5/3})$ and $O(h^k) = O(h)$ are attained by $(\boldsymbol{\sigma}, p)$ and $\mathbf{u}$, respectively, Finally, in order to illustrate the accurateness of the discrete scheme, in Figures 5.1 up to 5.18 we display several components of the approximate and exact solutions for each example.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
352	0.354	$1.028E-01$	–	$1.212E-01$	–	$1.826E-01$	–
1344	0.177	$2.654E-02$	1.953	$3.036E-02$	1.998	$9.225E-02$	0.985
5248	0.088	$6.712E-03$	1.984	$7.660E-03$	1.986	$4.624E-02$	0.996
20736	0.044	$1.688E-03$	1.992	$1.931E-03$	1.988	$2.314E-02$	0.999
82432	0.022	$4.233E-04$	1.995	$4.852E-04$	1.992	$1.157E-02$	1.000
328704	0.011	$1.060E-04$	1.998	$1.217E-04$	1.996	$5.785E-03$	1.000
1312768	0.006	$2.653E-05$	1.999	$3.047E-05$	1.998	$2.892E-03$	1.000

Table 5.1: Example 1, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
224	0.354	$6.923E-02$	–	$4.429E-02$	–	$7.125E-02$	–
832	0.177	$1.756E-02$	1.979	$8.415E-03$	2.396	$3.582E-02$	0.992
3200	0.088	$4.347E-03$	2.014	$1.658E-03$	2.344	$1.792E-02$	0.999
12544	0.044	$1.081E-03$	2.007	$3.684E-04$	2.170	$8.962E-03$	1.000
49664	0.022	$2.698E-04$	2.003	$8.835E-05$	2.060	$4.481E-03$	1.000
197632	0.011	$6.740E-05$	2.001	$2.180E-05$	2.019	$2.241E-03$	1.000
788480	0.006	$1.685E-05$	2.000	$5.427E-06$	2.006	$1.120E-03$	1.000

Table 5.2: Example 1, quasi-uniform refinement with straight squares.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
272	0.707	$3.866E-02$	–	$3.770E-02$	–	$1.131E-01$	–
1024	0.354	$9.502E-03$	2.025	$7.461E-03$	2.337	$5.747E-02$	0.976
3968	0.177	$2.507E-03$	1.922	$1.851E-03$	2.011	$2.888E-02$	0.993
15616	0.088	$6.419E-04$	1.966	$4.549E-04$	2.025	$1.446E-02$	0.998
61952	0.044	$1.625E-04$	1.982	$1.141E-04$	1.996	$7.231E-03$	1.000
246784	0.022	$4.100E-05$	1.987	$2.867E-05$	1.992	$3.616E-03$	1.000
985088	0.011	$1.029E-05$	1.995	$7.179E-06$	1.998	$1.808E-03$	1.000

Table 5.3: Example 2, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
176	0.800	$6.059E-02$	–	$3.400E-02$	–	$1.352E-01$	–
640	0.431	$1.736E-02$	2.020	$8.099E-03$	2.319	$7.081E-02$	1.045
2432	0.215	$4.347E-03$	1.989	$1.352E-03$	2.571	$3.584E-02$	0.978
9472	0.110	$1.097E-03$	2.050	$3.182E-04$	2.154	$1.796E-02$	1.028
37376	0.055	$2.746E-04$	1.990	$6.277E-05$	2.332	$8.997E-03$	0.993
148480	0.028	$6.873E-05$	2.066	$1.328E-05$	2.317	$4.504E-03$	1.032
591872	0.014	$1.718E-05$	2.015	$2.987E-06$	2.168	$2.253E-03$	1.007
2363392	0.007	$4.295E-06$	2.002	$7.091E-07$	2.077	$1.127E-03$	1.001

Table 5.4: Example 2, quasi-uniform refinement with quadrilaterals.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
352	0.354	$3.626E-03$	–	$4.324E-03$	–	$9.552E-02$	–
1344	0.177	$1.215E-03$	1.578	$1.406E-03$	1.621	$4.802E-02$	0.992
5248	0.088	$3.963E-04$	1.616	$4.522E-04$	1.636	$2.405E-02$	0.998
20736	0.044	$1.275E-04$	1.636	$1.444E-04$	1.647	$1.203E-02$	1.000
82432	0.022	$4.070E-05$	1.648	$4.591E-05$	1.654	$6.014E-03$	1.000
328704	0.011	$1.293E-05$	1.655	$1.454E-05$	1.658	$3.007E-03$	1.000
1312768	0.006	$4.093E-06$	1.659	$4.598E-06$	1.661	$1.504E-03$	1.000

Table 5.5: Example 3, quasi-uniform refinement with triangles.

N	h	$\mathtt{e}_0(\boldsymbol{\sigma})$	$\mathtt{r}_0(\boldsymbol{\sigma})$	$\mathtt{e}(p)$	$\mathtt{r}(p)$	$\mathtt{e}(\mathbf{u})$	$\mathtt{r}(\mathbf{u})$
64	0.734	$3.147E-02$	–	$4.258E-02$	–	$2.254E-01$	–
224	0.394	$1.084E-02$	1.711	$1.453E-02$	1.726	$1.181E-01$	1.038
832	0.215	$3.374E-03$	1.931	$4.618E-03$	1.896	$5.987E-02$	1.123
3200	0.113	$1.180E-03$	1.636	$1.588E-03$	1.663	$3.016E-02$	1.068
12544	0.058	$3.760E-04$	1.703	$5.058E-04$	1.703	$1.508E-02$	1.033
49664	0.030	$1.175E-04$	1.776	$1.592E-04$	1.764	$7.535E-03$	1.058
197632	0.015	$3.691E-05$	1.683	$5.019E-05$	1.678	$3.768E-03$	1.008
788480	0.008	$1.162E-05$	1.681	$1.582E-05$	1.679	$1.885E-03$	1.008

Table 5.6: Example 3, quasi-uniform refinement with distorted squares.

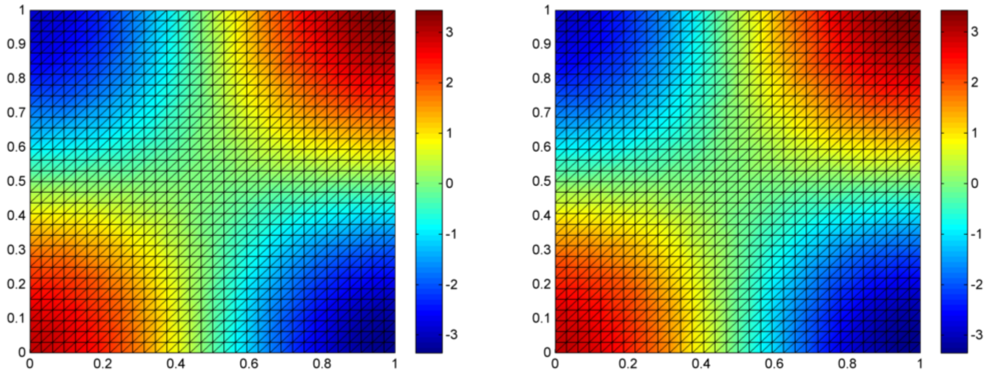

Figure 5.1: Example 1, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 20736$).

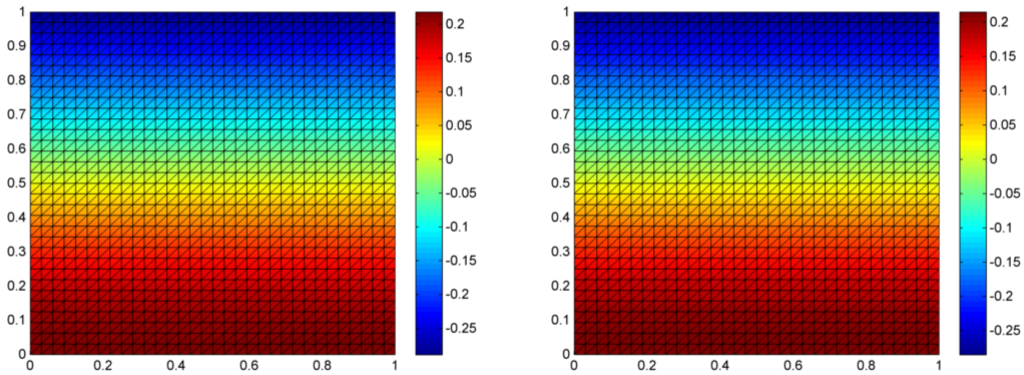

Figure 5.2: Example 1, $\widehat{p}_h$ and p for a mesh with triangles ($N = 20736$).

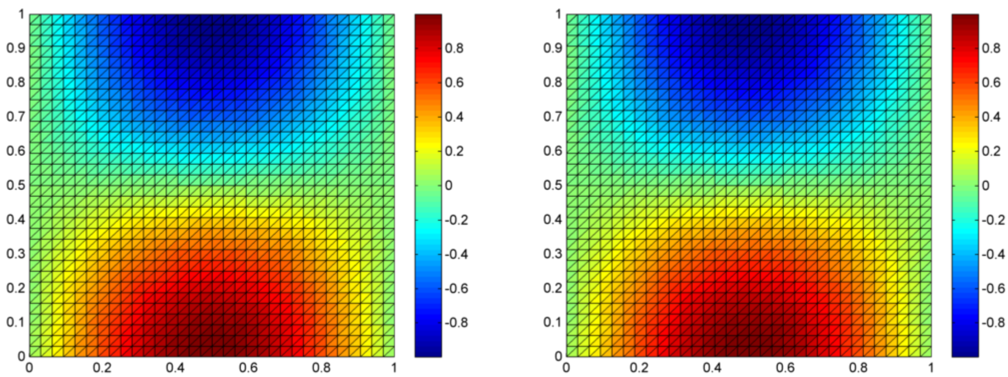

Figure 5.3: Example 1, $\mathbf{u}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 20736$).

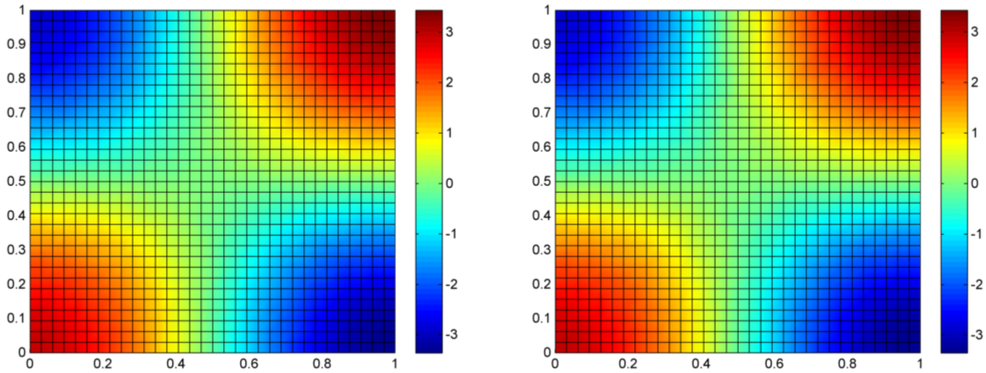

Figure 5.4: Example 1, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with straight squares ($N = 12544$).

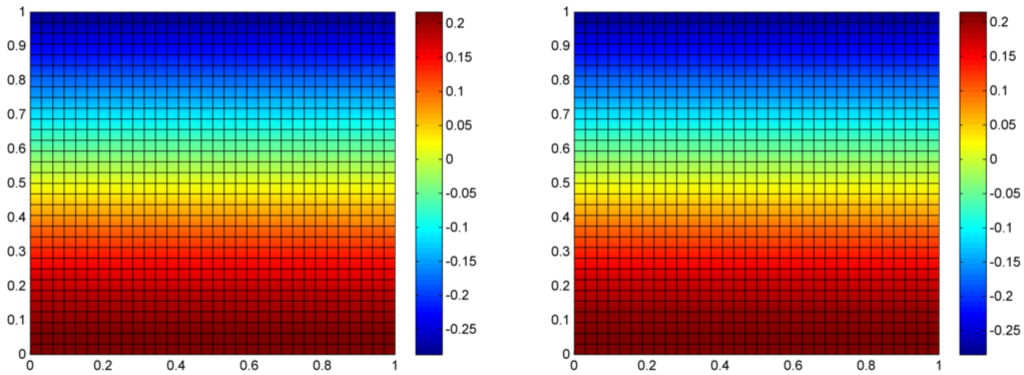

Figure 5.5: Example 1, $\widehat{p}_h$ and p for a mesh with straight squares ($N = 12544$).

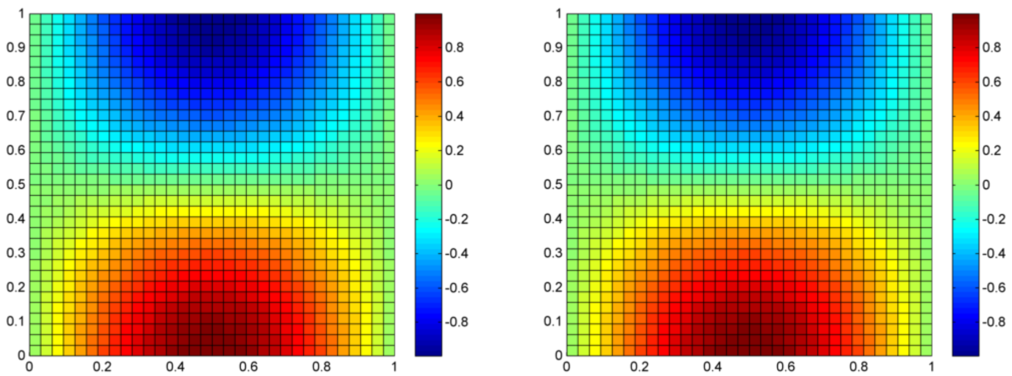

Figure 5.6: Example 1, $\mathbf{u}_{h,1}$ and $\mathbf{u}_1$ for a mesh with straight squares ($N = 12544$).

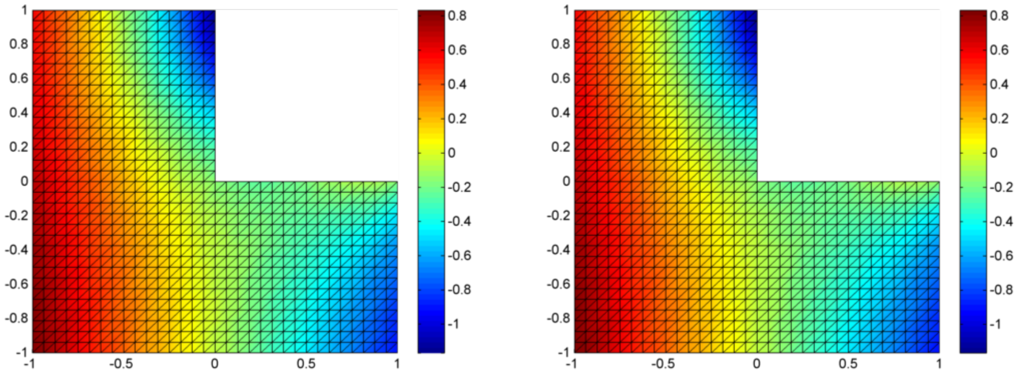

Figure 5.7: Example 2, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 15616$).

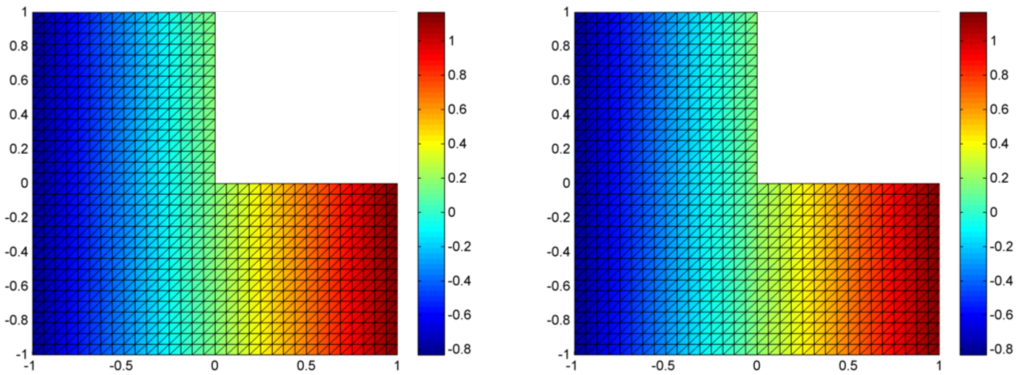

Figure 5.8: Example 2, $\widehat{p}_h$ and p for a mesh with triangles ($N = 15616$).

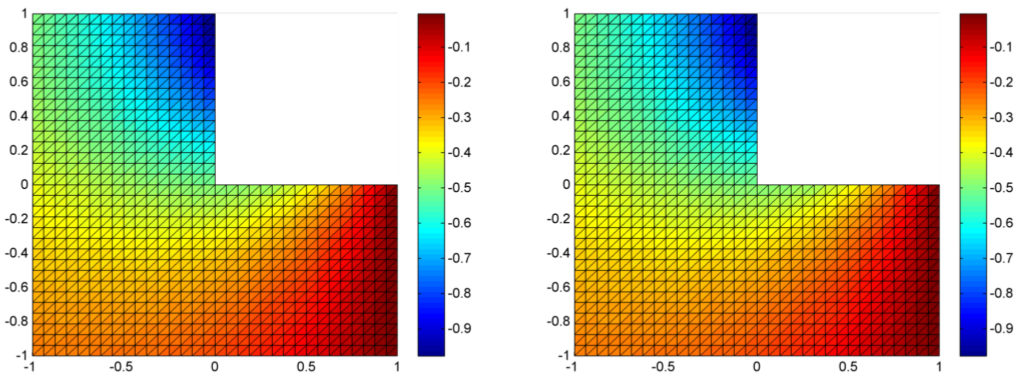

Figure 5.9: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 15616$).

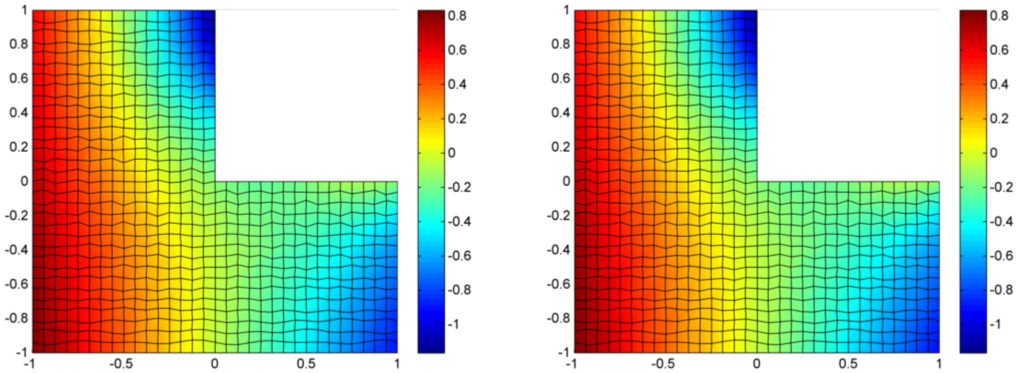

Figure 5.10: Example 2, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with quadrilaterals ($N = 9472$).

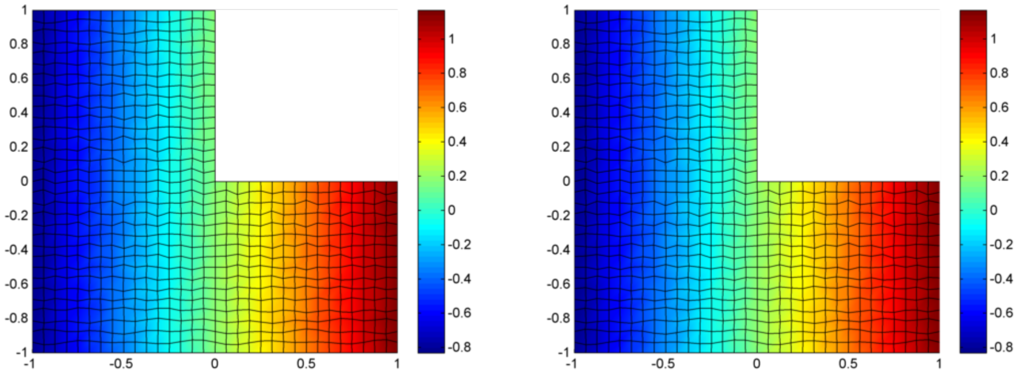

Figure 5.11: Example 2, $\widehat{p}_h$ and p for a mesh with quadrilaterals ($N = 9472$)

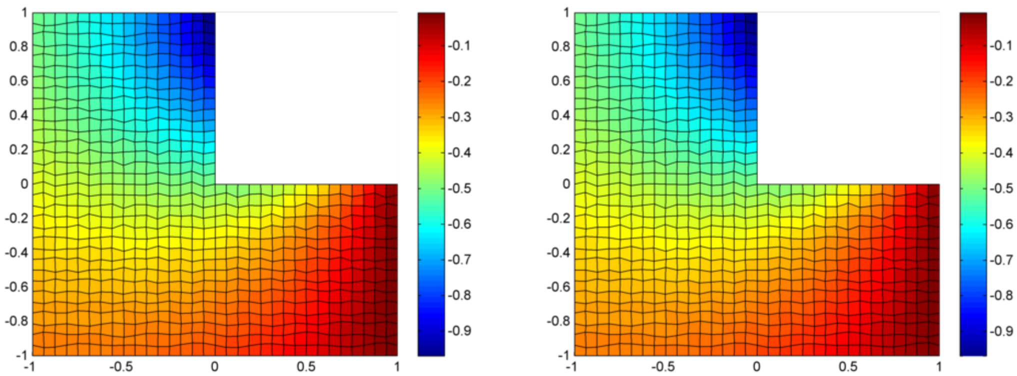

Figure 5.12: Example 2, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with quadrilaterals ($N = 9472$).

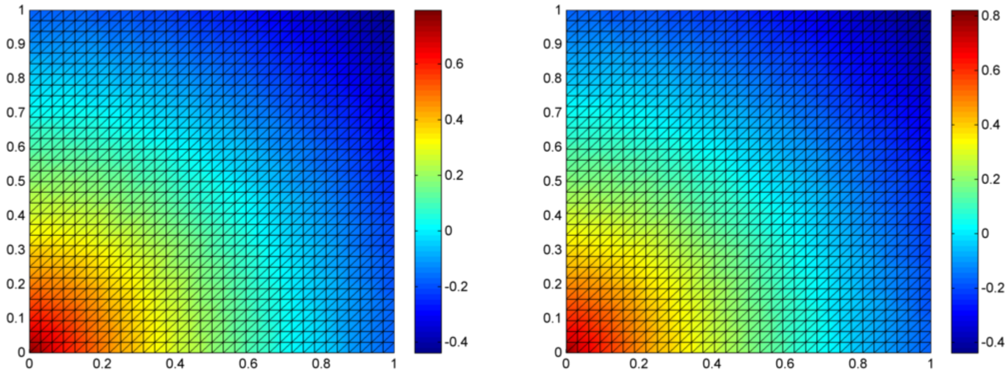

Figure 5.13: Example 3, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with triangles ($N = 20736$).

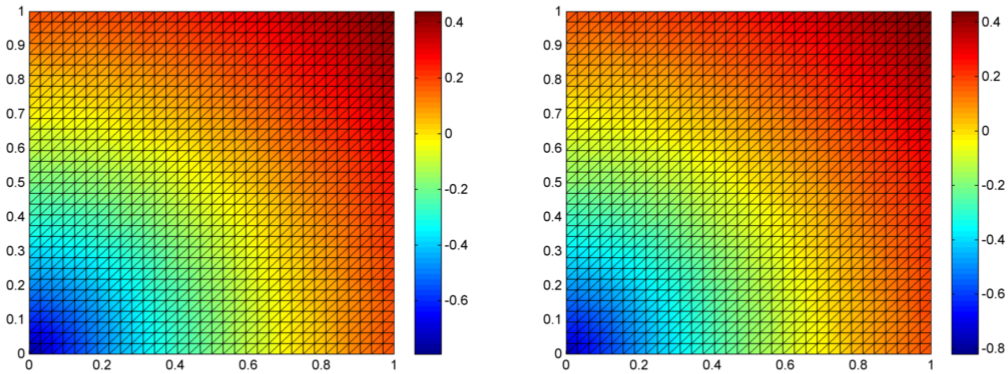

Figure 5.14: Example 3, $\widehat{p}_h$ and p for a mesh with triangles ($N = 20736$).

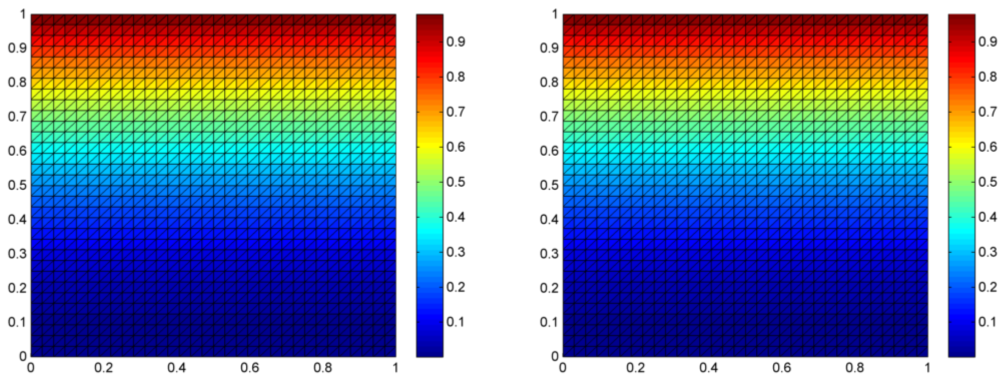

Figure 5.15: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with triangles ($N = 20736$).

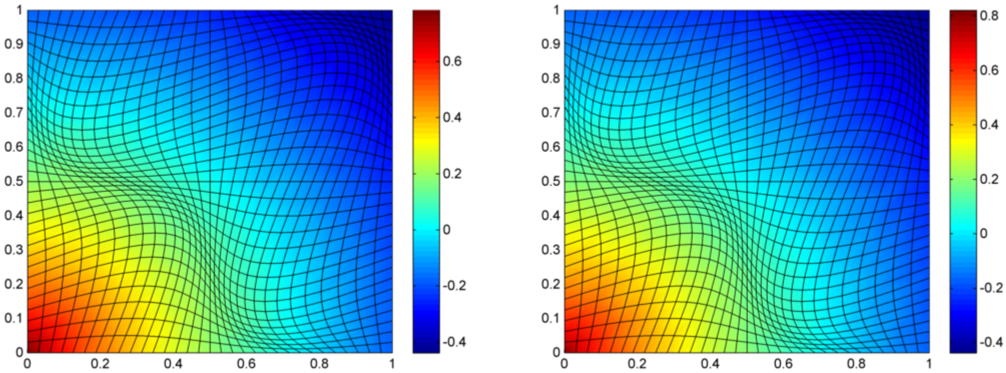

Figure 5.16: Example 3, $\widehat{\boldsymbol{\sigma}}_{h,11}$ and $\boldsymbol{\sigma}_{11}$ for a mesh with distorted squares ($N = 12544$).

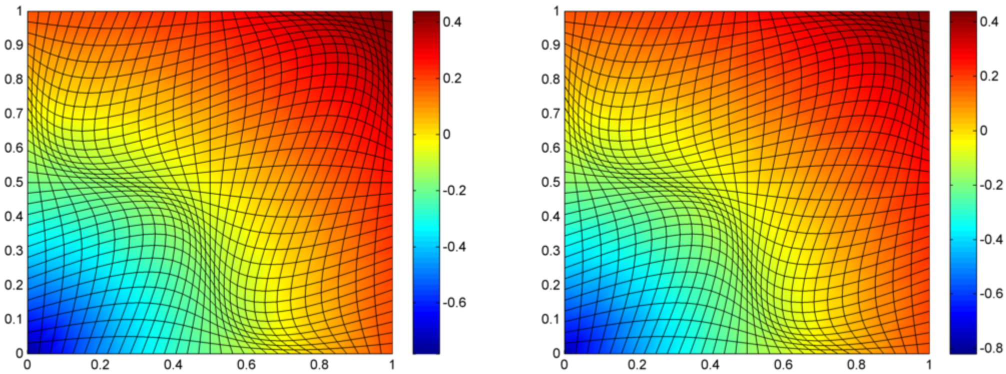

Figure 5.17: Example 3, $\widehat{p}_h$ and p for a mesh with distorted squares ($N = 12544$).

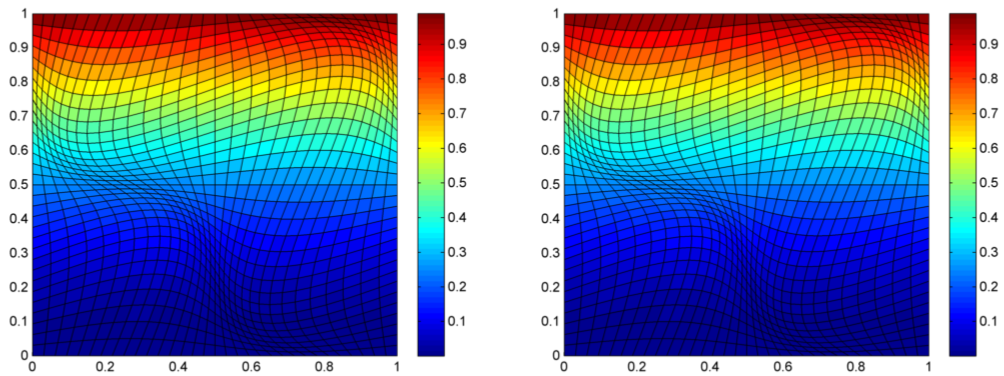

Figure 5.18: Example 3, $\widehat{\mathbf{u}}_{h,1}$ and $\mathbf{u}_1$ for a mesh with distorted squares ($N = 12544$).

Chapter 6

Conclusions and Future Works

In this chapter we summarize the main contributions of this work and give a brief description of further works.

6.1 Conclusions

The aim of this work was to analyze a convergent and stable virtual element scheme for the Poisson problem, and convergent and stables schemes for the Darcy and Stokes problem, respectively. To do so, and concerning the Poisson problem, we followed [3] and gave further details on the construction of the required interpolators, projectors and approximation properties of those. In turn, following [5], a computational implementation section was introduced. To finalize with, we presented several examples which confirm that the stablished error estimates are achieved. In what Darcy problem refers, we followed [9] and, anagously, gave further details on the construction of the virtual element subspaces and on the proofs of the associated results. We remark on this point that the error estimates given by [9, Corollary 5.2] are different than those given in here. Furthermore, the proofs of the error estimates given on this work are fully given and several examples showing the good performance of those are presented. In turn, as in the analysis of the Poisson problem, a computational implementation guide was presented.

The main contribution of this work concerns the analysis of a mixed virtual element method of the two–dimensional Stokes problem, in which the velocity and the pseudostress tensor are the only unknowns, whereas the pressure is computed via a post–processing formula. We tried to give as much details as possible on the construction of the method, that is, the discrete spaces and its corresponding interpolators and projectors, as well as the related proofs. Finally, we have provided several two–dimensional numerical examples which corfirm the good performance of the method, and as in the other problems, a computational implementation guide was also introduced.

6.2 Future works

Among the future works arising from this work, we can find:

1. The analysis of a posteriori error estimator for the primal–mixed formulation (5.2.3) and the corresponding discrete scheme (5.3.48).

2. The analysis of a Mixed Virtual Element Method for non–linear problems as, for instance, the Stokes problem.

3. The analysis of a Mixed Virtual Element Method for coupled problems as, for instance, the Stokes–Darcy problem.

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