Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
Departamento de Ingeniería Matemática

BANACH SPACES-BASED MIXED-PRIMAL FINITE ELEMENT METHODS FOR THE COUPLING OF THE NAVIER-STOKES-BRINKMAN AND NATURAL CONVECTION EQUATIONS

por

Nicolás Abraham Núñez Lira

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Profesores Guía: Gabriel N. Gatica, Ricardo Ruiz-Baier

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MÉTODOS DE ELEMENTOS FINITOS MIXTOS BASADOS EN ESPACIOS DE BANACH PARA LAS ECUACIONES ACOPLADAS DE NAVIER-STOKES-BRINKMAN Y CONVECCIÓN NATURAL

Comisión evaluadora

Dr. Gabriel N. Gatica [Profesor guía]

CI^{2}MA y Departamento de Ingeniería Matemática, Universidad de Concepción.

Dr. Ricardo Ruiz-Baier [Profesor co-guía]

School of Mathematics, Monash University, Melbourne, Australia.

Dra. Jessika Camaño

Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción.

Dr. Eligio Colmenares

Departamento de Ciencias Básicas, Universidad del Bío Bío.

Dr. Mauricio Sepulveda

CI^{2}MA y Departamento de Ingenieria Matematica, Universidad de Concepción

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Abstract

This work is divided in two main parts. In the first part we consider the Navier–Stokes–Brinkman equations, which constitute one of the most common nonlinear models utilized to simulate viscous fluids through porous media, and propose and analyze a Banach spaces-based approach yielding new mixed finite element methods for its numerical solution. In addition to the velocity and pressure, the strain rate tensor, the vorticity, and the stress tensor are introduced as auxiliary unknowns, and then the incompressibility condition is used to eliminate the pressure, which is computed afterwards by a postprocessing formula depending on the stress and the velocity. The resulting continuous formulation becomes a nonlinear perturbation of, in turn, a perturbed saddle point linear system, which is then rewritten as an equivalent fixed-point equation whose operator involved maps the velocity space into itself. The well-posedness of it is then analyzed by applying the classical Banach fixed point theorem, along with a smallness assumption on the data, the Babuška–Brezzi theory in Banach spaces, and a slight variant of a recently obtained solvability result for perturbed saddle point formulations in Banach spaces as well. The resulting Galerkin scheme is momentum-conservative. Its unique solvability is analyzed, under suitable hypotheses on the finite element subspaces, using a similar fixed-point strategy as in the continuous problem. A priori error estimates are rigorously derived, including also that for the pressure. We show that PEERS and AFW elements for the stress, the velocity and the rotation, together with piecewise polynomials of a proper degree for the strain rate tensor, yield stable discrete schemes. Then, the approximation properties of these subspaces and the Céa estimate imply the respective rates of convergence. Finally, we include two and three dimensional numerical experiments that serve to corroborate the theoretical findings, and these tests illustrate the performance of the proposed mixed finite element methods. This part yielded the following work, presently submitted:

> G.N. Gatica, N. Núñez and R. Ruiz-Baier, *New non-augmented mixed finite element methods for the Navier-Stokes-Brinkman equations using Banach spaces.* Preprint 2022-14, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile, (2022).

On the other hand, in the second part we consider a steady phase change problem for non-isothermal incompressible viscous flow in porous media with an enthalpy-porosity-viscosity coupling mechanism, and introduce and analyze a Banach spaces-based variational formulation yielding a new mixed-primal finite element method for its numerical solution. The momentum and mass conservation equations are formulated in terms of velocity and the tensors of strain rate, vorticity, and stress; and the incompressibility constraint is used to eliminate the pressure, which is computed afterwards by a postprocessing formula depending on the stress and the velocity. The resulting continuous formulation for the flow becomes a nonlinear perturbation of a perturbed saddle point linear system. The energy conservation equation is written as a nonlinear primal formulation that incorporates the additional unknown of boundary heat flux. The whole mixed-primal formulation is regarded as a fixed-point operator equation, so that its well-posedness hinges on Banach's theorem, along with smallness assumptions on the data. In turn, the solvability analysis of the uncoupled problem in the fluid employs the Babuška–Brezzi theory, a recently obtained result for perturbed saddle-point problems, and the Banach–Nečas–Babuška Theorem, all them in Banach spaces, whereas the one for the uncoupled energy equation applies a nonlinear version of the Babuška–Brezzi theory in Hilbert spaces. An analogue fixed-point strategy is employed for the analysis of the associated Galerkin scheme, using in this case Brouwer's theorem and assuming suitable conditions on the respective discrete subspaces. The error analysis is conducted under appropriate assumptions, and selecting specific finite element families that fit the theory. We finally report on the verification of theoretical convergence rates with the help of numerical examples. This part yielded the following work, presently submitted:

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New non-augmented mixed finite element methods for the Navier-Stokes-Brinkman equations using Banach spaces

CHAPTER 1

Introduction

The Navier–Stokes–Brinkman equations are nowadays present in a wide range of applications, among which we highlight the flow of a viscous fluid through porous media with adsorption, and the phase change models for natural convection in porous media as well. The former arises, for instance, in petroleum engineering [24], chromatography [63], and water decontamination [72], particularly in the design of water filtering devices [14], whereas the latter appears in melting and solidification processes [42, 73], design of energy storage devices [44], and ocean and atmosphere dynamics [43], to name a few. Motivated by the above, the devising of suitable numerical procedures to solve these problems, most of them within a Hilbertian framework, has gained increasing interest in recent years. The variational formulations utilized, which include the case of axisymmetric flow and time-dependent models, are based on velocity and pressure, stress, pseudostress, vorticity, or stream function, as main unknowns, whereas the techniques employed are basically finite element, mixed finite element, finite volume, stabilized finite element, spectral, mortar, and augmented finite element methods. For an overview of some contributions in these directions, we refer to [14, 24, 58] and [8, 9, 54], and the references therein, in the case of the aforementioned first and second model, respectively.

Aiming to provide further details on the state of the art, as well as to explain the main motivation of this part, we now refer specifically to [8], where rigorous mathematical and numerical analyses of mixed-primal and fully mixed methods for phase change models for natural convection, are provided, up to our knowledge, for the first time. Indeed, the problem under consideration there is the one originally proposed in [9], where a fully-primal formulation for the non-stationary case was analyzed. The governing equations are given by the Navier–Stokes–Brinkman equations coupled with a generalized energy equation, in addition to Dirichlet boundary conditions for the velocity and the temperature. The fluid part of the coupled model is handled similarly to [5] by introducing, besides the velocity, the strain rate tensor and the stress tensor relating the latter with the convective term, as auxiliary unknowns, so that the pressure is eliminated by using the incompressibility condition, and recovered later on via a postprocessing formula depending on the stress and the velocity. In turn, due to the convective term, and in order to stay within a Hilbertian framework, the velocity is sought in the Sobolev space of order 1, which requires the incorporation into the variational formulation of additional Galerkin-type terms arising from the constitutive and equilibrium equations. Furthermore, the symmetry of the stress is imposed in an ultra-weak sense (cf. [6]), which avoids to include the vorticity as a fourth unknown. Nevertheless, and while the augmentation procedure allows to circumvent the necessity of proving continuous and discrete inf-sup conditions, which yields, in particular, more flexibility for choosing the finite element subspaces, it is no less true that the complexity of both the resulting system and its associated computational implementation increases considerably, thus leading to much more expensive schemes. This last remark constitutes our main motivation to look now for non-augmented schemes.

A similar procedure to the one from [8] for the Navier–Stokes–Brinkman equations was introduced and analyzed in [50]. However, differently from [8], the authors do not include the strain rate tensor as an unknown, though it can also be computed via a postprocess, and instead of employing the stress and imposing the incompressibility condition, they use the pseudostress and consider a nonselenoidal condition, respectively. Besides these aspects and a minor difference related to the handling of the equilibrium equation, the rest of the variational formulation proceeds analogously by forcing as well a Hilbert spaces-based framework by means of the introduction of residual terms arising from the constitutive equation and the Dirichlet boundary condition. In addition to [14] and [50], just a few other contributions dealing with numerical methods for the Navier-Stokes-Brinkman equations seem to be available in the literature, among which we refer to [15, 51, 69]. More extensive is the list of references dealing with the numerical analysis of the related Stokes-Brinkman model (see, e.g. [22, 53, 74, 75]).

On the other hand, a significant amount of contributions showing the suitability of Banach spaces-based approaches to analyze the continuous and discrete formulations of diverse linear, nonlinear, and coupled problems in continuum mechanics, have appeared in recent years. A non-exhaustive list of them includes [16, 25, 33, 34, 37, 39, 49, 52], and among the different models addressed we can mention Poisson, Brinkman–Forchheimer, Darcy–Forchheimer, Navier-Stokes, Boussinesq, coupled flow-transport, and fluidized beds, most of which share a Banach saddle-point structure for the resulting variational formulations. The main advantage of employing this Banach framework is, precisely as sought, the fact that no augmentation is required, and hence the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy–Schwarz and Hölder inequalities to the tested and eventually integrated by parts equations. In this way, simpler and closer to the original physical model formulations are obtained. Moreover, it also allows to derive momentum conservative schemes, and to obtain direct approximations of further variables of physical interest, either by incorporating them into the formulation or by employing postprocessing formulae in terms of the discrete solution.

According to all the previous discussion, and bearing in mind that we finally aim at developing a non-augmented finite element method for the model from [8], the purpose of the present chapter is to advance toward that goal by introducing and analyzing first a Banach spaces-based mixed finite element method for the Navier–Stokes–Brinkman equations. The extension of it to the phase change model for natural convection in a porous medium will be reported in a separate work. The first part is organized as follows. The rest of this chapter collects some preliminary notations and results to be employed throughout this thesis. In Chapter 2 we set the model of interest, define the auxiliary unknowns to be considered, and eliminate the pressure. The variational formulation is introduced and analyzed in Chapter 3. In fact, in

Chapter 3.1 we describe the mixed approach and realize that the resulting continuous system, which is very close to the one from [8] before augmenting it, can be written as a nonlinear perturbation of a perturbed saddle point formulation in Banach spaces. Then, some abstract results that include a slight variant of the continuous and discrete well-posedness of the latter, as well as the Babuška–Brezzi theory in Banach spaces, are recalled in Chapter 3.2. The solvability analysis itself is developed in Chapter 3.3 by employing a fixed-point strategy along with the theorems from Chapter 3.2. Next, in Chapter 4 we introduce and analyze the associated Galerkin scheme under suitable assumptions on the finite element subspaces to be employed, adopting an analogous fixed-point strategy, and making use of the discrete versions of the theoretical results from Chapter 3.2. In addition, a priori error estimates are derived, specific finite element subspaces satisfying the aforementioned assumptions are described, and corresponding rates of convergence are established. Finally, several illustrative numerical results are reported in Chapter 5.

1.1 Preliminaty notations

Throughout this work, Ω is a given bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward unit normal at its boundary Γ is denoted $\boldsymbol{\nu}$. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^r(\Omega)$, with $r \in (1,\infty)$, and Sobolev spaces $\mathrm{W}^{s,r}(\Omega)$, with $s \geqslant 0$, endowed with the norms $\|\cdot\|_{0,r;\Omega}$ and $\|\cdot\|_{s,r;\Omega}$, respectively, whose vector and tensor versions are denoted in the same way. In particular, note that $\mathrm{W}^{0,r}(\Omega) = \mathrm{L}^r(\Omega)$, and that when $r = 2$ we simply write $\mathrm{H}^s(\Omega)$ in place of $\mathrm{W}^{s,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{s,\Omega}$, respectively. We also set $|\cdot|_{s,\Omega}$ for the seminorm of $\mathrm{H}^s(\Omega)$. In turn, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, $\mathrm{H}^{-1/2}(\Gamma)$ is its dual, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between them. On the other hand, by $\mathbf{S}$ and $\mathbb{S}$ we mean the corresponding vector and tensor counterparts, respectively, of a generic scalar functional space S. Furthermore, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, symmetric part of the gradient (also named strain rate tensor), divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \mathbf{e}(\mathbf{v}) \;:=\; \frac{1}{2}\big(\nabla \mathbf{v} + (\nabla \mathbf{v})^{\mathrm{t}}\big)\,,$$
$$\mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \qquad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}\,,$$

where the superscript $^{\mathrm{t}}$ stands for the matrix transpose. Next, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R} := \mathrm{R}^{n\times n}$. On the other hand, for each $r \in [1, +\infty]$ we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_r;\Omega) \;:=\; \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^r(\Omega)\Big\}\,, \tag{1.1}$$

which is endowed with the natural norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_r;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,\Omega} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_r;\Omega)\,, \tag{1.2}$$

and recall that, proceeding as in [47, eq. (1.43), Section 1.3.4] (see also [28, Section 4.1] and [37, Section 3.1]), one can prove that for each $r \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle \;=\; \int_\Omega \Big\{\boldsymbol{\tau} : \nabla \mathbf{v} \;+\; \mathbf{v}\cdot \mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_r;\Omega) \times \mathbf{H}^1(\Omega)\,, \tag{1.3}$$

where $\langle\cdot,\cdot\rangle$ stands as well for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Finally, bear in mind that when $r = 2$, the Hilbert space $\mathbb{H}(\mathbf{div}_2;\Omega)$ and its norm $\|\cdot\|_{\mathbf{div}_2;\Omega}$ are simply denoted $\mathbb{H}(\mathbf{div};\Omega)$ and $\|\cdot\|_{\mathbf{div};\Omega}$, respectively.

CHAPTER 2

The model problem

The modelling of a viscous fluid within a porous medium occupying the domain Ω, is described by the Navier–Stokes–Brinkman problem, which reduces to finding a velocity vector field $\mathbf{u} : \Omega \to \mathbf{R}$ and a pressure scalar field $p : \Omega \to \mathrm{R}$ satisfying the following system of partial differential equations:

$$\begin{aligned} \eta\,\mathbf{u} - \lambda\,\mathbf{div}\big(\mu\,\mathbf{e}(\mathbf{u})\big) + \big(\nabla\mathbf{u}\big)\mathbf{u} + \nabla p &= \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \mathrm{div}(\mathbf{u}) &= 0 \quad \text{in} \quad \Omega\,, \\ \mathbf{u} &= \mathbf{u}_D \quad \text{on} \quad \Gamma\,, \\ \int_\Omega p &= 0\,, \end{aligned} \tag{2.1}$$

where η is the scaled inverse permeability of the porous media, $\lambda := \mathrm{Re}^{-1}$, where Re is the Reynolds number, μ is the dynamic viscosity of the fluid, $\mathbf{f}$ is an external body force, and $\mathbf{u}_D$ is a Dirichlet datum for $\mathbf{u}$. The right spaces to which $\mathbf{f}$ and $\mathbf{u}_D$ belong will be precise later on. The functions η and μ are supposed to be bounded, which means that there exist positive constants η_0, η_1, μ_0, and μ_1, such that

$$0 < \eta_0 \leqslant \eta(\mathbf{x}) \leqslant \eta_1 \quad \text{and} \quad 0 < \mu_0 \leqslant \mu(\mathbf{x}) \leqslant \mu_1 \qquad \forall\, \mathbf{x} \in \Omega\,. \tag{2.2}$$

In turn, note that the incompressibility of the fluid (cf. second equation of (2.1)) imposes on $\mathbf{u}_D$ the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} \;=\; 0\,, \tag{2.3}$$

and that the last equation of (2.1) has been included for sake of uniqueness of p.

We now proceed as in [8] and [5] (see, also [25], [27], [29], [36], [38]) and transform (2.1) into an equivalent system of first order equations. To this end, we introduce the strain rate tensor $\mathbf{t}$, the vorticity $\boldsymbol{\gamma}$, and the stress tensor $\boldsymbol{\sigma}$ as auxiliary unknowns, namely

$$\mathbf{t} \;:=\; \mathbf{e}(\mathbf{u}) \;=\; \nabla\mathbf{u} \;-\; \boldsymbol{\gamma}\,, \qquad \text{where} \qquad \boldsymbol{\gamma} \;:=\; \frac{1}{2}\big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}}\big)\,, \tag{2.4}$$

and

$$\boldsymbol{\sigma} \;:=\; \lambda\,\mu\,\mathbf{t} \;-\; (\mathbf{u}\otimes\mathbf{u}) \;-\; p\,\mathbb{I}\,, \tag{2.5}$$

so that, thanks to the incompressibility of the fluid, the first equation of (2.1) is rewritten as

$$\eta\,\mathbf{u} \;-\; \mathbf{div}(\boldsymbol{\sigma}) \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

Moreover, it is easy to see that, precisely the second equation of (2.1), which becomes $\mathrm{tr}(\mathbf{t}) \;=\; 0$, together with (2.5), are equivalent to the pair of equations given by

$$\boldsymbol{\sigma}^{\mathrm{d}} \;=\; \lambda\,\mu\,\mathbf{t} \;-\; (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{and} \quad p \;=\; -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) \quad \text{in} \quad \Omega\,. \tag{2.7}$$

Consequently, the pressure unknown is eliminated from the formulation and computed afterwards, as suggested by the foregoing identity, in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$. In this way, (2.1) can be equivalently reformulated as

$$\begin{aligned}
\mathbf{t} + \boldsymbol{\gamma} &= \nabla \mathbf{u} && \text{in} \quad \Omega\,, \\
\lambda\,\mu\,\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} &= \boldsymbol{\sigma}^{\mathrm{d}} && \text{in} \quad \Omega\,, \\
\eta\,\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) &= \mathbf{f} && \text{in} \quad \Omega\,, \\
\mathbf{u} &= \mathbf{u}_D && \text{on} \quad \Gamma\,, \\
\int_\Omega \operatorname{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) &= 0\,.
\end{aligned} \tag{2.8}$$

CHAPTER 3

The continuous formulation

In this chapter we introduce and analyze the variational formulation of (2.8), which, differently from [8] and [50], does not include any augmentation procedure, and employs the natural spaces arising from the application of the Cauchy–Schwarz and Hölder inequalities to the terms, suitably tested and integrated by parts, if necessary, of the equations in (2.8).

3.1 The mixed approach

We begin by originally seeking $\mathbf{u}$ in $\mathbf{H}^1(\Omega)$, for which we assume from now on that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$. Then, given $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_r;\Omega)$, with $r \geqslant \frac{2n}{n+2}$, a straightforward application of (1.3) along with the fact that $\mathbf{u} = \mathbf{u}_D$ on Γ, yield

$$\int_\Omega \boldsymbol{\tau} : \nabla \mathbf{u} \,=\, -\int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,+\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle\,, \tag{3.1}$$

and hence the corresponding testing of the first equation of (2.8) becomes

$$\int_\Omega \mathbf{t} : \boldsymbol{\tau} \,+\, \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} \,+\, \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \,=\, \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_r;\Omega)\,. \tag{3.2}$$

We observe here, thanks to Cauchy-Schwarz's inequality and the fact that $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, that the first two terms of (3.2) make sense for both $\mathbf{t}$ and $\boldsymbol{\gamma}$ in $\mathbb{L}^2(\Omega)$. Thus, bearing in mind the free trace property of $\mathbf{t}$ and the skew symmetry of $\boldsymbol{\gamma}$ (cf. (2.4)), we look for these unknowns in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \,:=\, \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) = 0 \Big\}\,, \tag{3.3}$$

and

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) \,:=\, \Big\{ \boldsymbol{\delta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\delta}^{\mathrm{t}} = -\boldsymbol{\delta} \Big\}\,. \tag{3.4}$$

In turn, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^r(\Omega)$, and employing Hölder's inequality, we notice from the third term of (3.2) that, instead of $\mathbf{H}^1(\Omega)$, it would actually suffice to look for $\mathbf{u}$ in $\mathbf{L}^{r'}(\Omega)$, where r' is the conjugate of r, that is $r' \in [1, +\infty]$ is such that $\frac{1}{r} + \frac{1}{r'} = 1$. On the other hand, testing the second equation of (2.8) against $\mathbf{s} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, we formally obtain

$$\lambda \int_\Omega \mu\, \mathbf{t} : \mathbf{s} \,-\, \int_\Omega (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} : \mathbf{s} \,=\, \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s}\,,$$

which, using the fact that $\mathrm{tr}(\mathbf{s})$ also vanishes, becomes

$$\lambda \int_\Omega \mu\, \mathbf{t} : \mathbf{s} \,-\, \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} \,=\, \int_\Omega \boldsymbol{\sigma} : \mathbf{s}\,. \tag{3.5}$$

The boundedness of μ (cf. (2.2)) and the fact that both $\mathbf{t}$ and $\mathbf{s}$ lay in $\mathbb{L}^2(\Omega)$, guarantee that the first term of (3.5) is finite, whereas the last one is as well if $\boldsymbol{\sigma}$ (and hence $\boldsymbol{\sigma}^{\mathrm{d}}$) belongs to $\mathbb{L}^2(\Omega)$. Regarding the second one, straightforward applications of the Cauchy-Schwarz and Hölder inequalities imply that, for each ℓ, $j \in (1, +\infty)$ such that $\frac{1}{\ell} + \frac{1}{j}$, there holds

$$\Big| \int_\Omega (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} : \mathbf{s} \Big| \,=\, \Big| \int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s} \Big| \,\leqslant\, \|\mathbf{u}\|_{0,2\ell;\Omega}\, \|\mathbf{u}\|_{0,2j;\Omega}\, \|\mathbf{s}\|_{0,\Omega}\,, \tag{3.6}$$

which says that this term makes sense for $\mathbf{u} \in \mathbf{L}^{2\ell}(\Omega) \cap \mathbf{L}^{2j}(\Omega)$, that is, choosing in particular $l = j = 2$, for $\mathbf{u} \in \mathbf{L}^4(\Omega)$. In this way, our previous analysis on the first equation of (2.8) is restricted hereafter to $r' = 4$, and hence to $r = 4/3$. Moreover, aiming to keep the same space for the unknown $\boldsymbol{\sigma}$ and its associated test functions $\boldsymbol{\tau}$, we will seek $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Therefore, knowing now that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, and assuming that the datum $\mathbf{f}$ lays also in $\mathbf{L}^{4/3}(\Omega)$, we proceed to test the third equation of (2.8) against $\mathbf{v} \in \mathbf{L}^4(\Omega)$, which yields

$$\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \;-\; \int_\Omega \eta\, \mathbf{u} \cdot \mathbf{v} \;=\; -\int_\Omega \mathbf{f} \cdot \mathbf{v}\,. \tag{3.7}$$

Finally, the symmetry of $\boldsymbol{\sigma}$, which, according to (2.5), is equivalent to that of $\mathbf{t}$, is imposed weakly as

$$\int_\Omega \boldsymbol{\delta} : \boldsymbol{\sigma} \;=\; 0 \qquad \forall\, \boldsymbol{\delta} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,. \tag{3.8}$$

At this point, and before reordering the equations (3.2), (3.5), (3.7), and (3.8) in a suitable way, we consider, for sake of convenience of the subsequent analysis, the decomposition (see, e.g. [37, eqs. (3.12) - (3.13)], [49, eqs. (3.1) - (3.2)])

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \;:=\; \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{3.9}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \;:=\; \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \;=\; 0 \right\}. \tag{3.10}$$

In particular, the unknown $\boldsymbol{\sigma}$ can be uniquely decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\,\mathbb{I}$, where $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and, employing the last equation of (2.8),

$$c_0 \;:=\; \frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \;=\; -\frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\mathbf{u} \otimes \mathbf{u})\,. \tag{3.11}$$

In this way, knowing explicitly c_0 in terms of $\mathbf{u}$, it remains to find the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$-component $\boldsymbol{\sigma}_0$ of $\boldsymbol{\sigma}$ to fully determine it. In this regard, we readily observe that equations (3.5), (3.7), and (3.8) remain unchanged if $\boldsymbol{\sigma}$ is replaced there by $\boldsymbol{\sigma}_0$. Moreover, it is easy to see, thanks to the compatibility condition (2.3) satisfied by the Dirichlet datum $\mathbf{u}_D$, that both sides of

(3.2) vanish for $\boldsymbol{\tau} = \mathbb{I}$, and hence, testing this equation against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Consequently, redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, introducing the spaces

$$\mathbf{H} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \qquad \mathbf{Q} := \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{3.12}$$

setting the notations

$$\vec{\mathbf{t}} := (\mathbf{t},\boldsymbol{\sigma})\,,\ \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau})\,,\ \vec{\mathbf{r}} := (\mathbf{r},\boldsymbol{\zeta}) \in \mathbf{H}\,, \qquad \vec{\mathbf{u}} := (\mathbf{u},\boldsymbol{\gamma})\,,\ \vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta})\,,\ \vec{\mathbf{w}} := (\mathbf{w},\boldsymbol{\xi}) \in \mathbf{Q}\,, \tag{3.13}$$

endowing $\mathbf{H}$ and $\mathbf{Q}$ with the norms

$$\begin{aligned} \|\vec{\mathbf{s}}\|_{\mathbf{H}} &:= \|\mathbf{s}\|_{0,\Omega} + \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau}) \in \mathbf{H}\,,\\ \|\vec{\mathbf{v}}\|_{\mathbf{Q}} &:= \|\mathbf{v}\|_{0,4;\Omega} + \|\boldsymbol{\delta}\|_{0,\Omega} \qquad \forall\, \vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta}) \in \mathbf{Q}\,, \end{aligned} \tag{3.14}$$

and gathering (3.5), (3.2), and (3.7) + (3.8), we arrive at the following variational formulation of (2.8): Find $(\vec{\mathbf{t}},\vec{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} a(\mathbf{t},\mathbf{s}) + b_1(\mathbf{s},\boldsymbol{\sigma}) \qquad\qquad + b(\mathbf{u};\mathbf{u},\mathbf{s}) &= 0\,,\\ b_2(\mathbf{t},\boldsymbol{\tau}) \qquad + \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{u}}) &= \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D\rangle\,,\\ \mathbf{b}(\vec{\mathbf{t}},\vec{\mathbf{v}}) \qquad - \mathbf{c}(\vec{\mathbf{u}},\vec{\mathbf{v}}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v}\,, \end{aligned} \tag{3.15}$$

for all $(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}$, where the bilinear forms $a : \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \to \mathrm{R}$, $b_i : \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \to \mathrm{R}$, $i \in \{1,2\}$, $\mathbf{b} : \mathbf{H} \times \mathbf{Q} \to \mathrm{R}$, and $\mathbf{c} : \mathbf{Q} \times \mathbf{Q} \to \mathrm{R}$, are defined by

$$a(\mathbf{r},\mathbf{s}) := \lambda \int_\Omega \mu\,\mathbf{r} : \mathbf{s} \qquad \forall\,\mathbf{r},\,\mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.16a}$$

$$b_1(\mathbf{s},\boldsymbol{\tau}) := -\int_\Omega \mathbf{s} : \boldsymbol{\tau}\,, \quad b_2(\mathbf{s},\boldsymbol{\tau}) := \int_\Omega \mathbf{s} : \boldsymbol{\tau}\,, \qquad \forall\,(\mathbf{s},\boldsymbol{\tau}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{3.16b}$$

$$\mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{v}}) := \int_\Omega \boldsymbol{\delta} : \boldsymbol{\tau} + \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{3.16c}$$

$$\mathbf{c}(\vec{\mathbf{w}},\vec{\mathbf{v}}) := \int_\Omega \eta\,\mathbf{w}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}} \in \mathbf{Q}\,, \tag{3.16d}$$

whereas for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$, $b(\mathbf{w};\cdot,\cdot) : \mathbf{L}^4(\Omega) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega) \to \mathrm{R}$ is the bilinear form given by

$$b(\mathbf{w};\mathbf{v},\mathbf{s}) \,:=\, -\int_\Omega (\mathbf{w}\otimes\mathbf{v}) : \mathbf{s} \qquad \forall\,(\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.17}$$

Equivalently, letting $\mathbf{a} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ be the bilinear form that arises from the block $\begin{pmatrix} a & b_1 \\ b_2 & \end{pmatrix}$ by adding the first two equations of (3.15), that is

$$\mathbf{a}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \,:=\, a(\mathbf{r},\mathbf{s}) \,+\, b_1(\mathbf{s},\boldsymbol{\zeta}) \,+\, b_2(\mathbf{r},\boldsymbol{\tau}) \qquad \forall\,\vec{\mathbf{r}},\,\vec{\mathbf{s}} \in \mathbf{H}\,, \tag{3.18}$$

we find that (3.15) can be rewritten as: Find $(\vec{\mathbf{t}},\vec{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{t}},\vec{\mathbf{s}}) \,+\, \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{u}}) \,+\, b(\mathbf{u};\mathbf{u},\mathbf{s}) \;&=\; \langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{u}_D\rangle \qquad \forall\,\vec{\mathbf{s}}\in\mathbf{H}\,,\\ \mathbf{b}(\vec{\mathbf{t}},\vec{\mathbf{v}}) \,-\, \mathbf{c}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \;&=\; -\int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{Q}\,. \end{aligned} \tag{3.19}$$

Moreover, letting now $\mathbf{A} : \big(\mathbf{H}\times\mathbf{Q}\big)\times\big(\mathbf{H}\times\mathbf{Q}\big) \to \mathrm{R}$ be the bilinear from that arises from the block $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & -\mathbf{c} \end{pmatrix}$ by adding both equations of (3.19), that is

$$\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) \,:=\, \mathbf{a}(\vec{\mathbf{r}},\vec{\mathbf{s}}) \,+\, \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{w}}) \,+\, \mathbf{b}(\vec{\mathbf{r}},\vec{\mathbf{v}}) \,-\, \mathbf{c}(\vec{\mathbf{w}},\vec{\mathbf{v}}) \quad \forall\,(\vec{\mathbf{r}},\vec{\mathbf{w}}),\,(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}\,, \tag{3.20}$$

we deduce that (3.19) (and hence (3.15)) can be stated, equivalently as well, as: Find $(\vec{\mathbf{t}},\vec{\mathbf{u}}) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\mathbf{A}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) \,+\, b(\mathbf{u};\mathbf{u},\mathbf{s}) \,=\, \mathbf{F}(\vec{\mathbf{s}},\vec{\mathbf{v}}) \qquad \forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}\,, \tag{3.21}$$

where $\mathbf{F} \in \big(\mathbf{H}\times\mathbf{Q}\big)'$ is defined by

$$\mathbf{F}(\vec{\mathbf{s}},\vec{\mathbf{v}}) \,:=\, \langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{u}_D\rangle \,-\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}\,. \tag{3.22}$$

Our next goal is to analyze the solvability of (3.21) (equivalently, that of (3.19) or (3.15)), for which we will apply the abstract results collected in the following chapter. We stress that, except for the handling of the rotation, (3.15) coincides with the variational formulation for the fluid part of the phase change model for natural convection (cf. [8, first three rows of eq. (3.6)]), but before augmenting it, thus emphasizing that this procedure will not be employed here. In addition, we remark that (3.21) can be seen as a nonlinear perturbation of a perturbed saddle-point formulation in Banach spaces, for which continuous and discrete well-posedness results have been recently shown in [40].

3.2 Some abstract results

We begin by recalling the Babuška–Brezzi theory in Banach spaces.

Theorem 3.1. *Let H_1, H_2, Q_1, and Q_2 be real reflexive Banach spaces, and let $a : H_2 \times H_1 \to \mathrm{R}$ and $b_i : H_i \times Q_i \to \mathrm{R}$, $i \in \{1, 2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1, 2\}$, respectively. In addition, for each $i \in \{1, 2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i := \Big\{ v \in H_i : \quad b_i(v, q) = 0 \quad \forall\, q \in Q_i \Big\} . \tag{3.23}$$

Assume that

i) *there exists a constant $\alpha > 0$ such that*

$$\sup_{\substack{v \in \mathcal{K}_1 \\ v \neq 0}} \frac{a(w, v)}{\|v\|_{H_1}} \geqslant \alpha \, \|w\|_{H_2} \qquad \forall\, w \in \mathcal{K}_2 , \tag{3.24}$$

ii) *there holds*

$$\sup_{w \in \mathcal{K}_2} a(w, v) > 0 \qquad \forall\, v \in \mathcal{K}_1 , \ v \neq 0 , \tag{3.25}$$

iii) *for each $i \in \{1, 2\}$ there exists a constant $\beta_i > 0$ such that*

$$\sup_{\substack{v \in H_i \\ v \neq 0}} \frac{b_i(v, q)}{\|v\|_{H_i}} \geqslant \beta_i \, \|q\|_{Q_i} \qquad \forall\, q \in Q_i . \tag{3.26}$$

Then, for each $(F, G) \in H_1' \times Q_2'$ *there exists a unique* $(u, p) \in H_2 \times Q_1$ *such that*

$$\begin{aligned} a(u, v) + b_1(v, p) &= F(v) \qquad \forall\, v \in H_1\,, \\ b_2(u, q) &= G(q) \qquad \forall\, q \in Q_2\,, \end{aligned} \tag{3.27}$$

and the following a priori estimates hold:

$$\begin{aligned} \|u\|_{H_2} &\leqslant \frac{1}{\alpha}\,\|F\|_{H_1'} + \frac{1}{\beta_2}\left(1 + \frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,, \\ \|p\|_{Q_1} &\leqslant \frac{1}{\beta_1}\left(1 + \frac{\|a\|}{\alpha}\right)\|F\|_{H_1'} + \frac{\|a\|}{\beta_1\,\beta_2}\left(1 + \frac{\|a\|}{\alpha}\right)\|G\|_{Q_2'}\,. \end{aligned} \tag{3.28}$$

Moreover, i)*,* ii)*, and* iii) *are also necessary conditions for the well-posedness of* (3.27)*.*

Proof. See [18, Theorem 2.1, Corollary 2.1, Section 2.1] for the original version and its proof. For the particular case given by $H_1 = H_2$, $Q_1 = Q_2$, and $b_1 = b_2$, we also refer to [46, Theorem 2.34]. □

We remark here that the roles of $\mathcal{K}_1$ and $\mathcal{K}_2$ in the assumptions i) and ii) of Theorem 3.1 can be exchanged without altering the joint meaning of these hypotheses (cf. [18, eqs. (2.10) and (2.11)]). In addition, it is important to stress that (3.28) is equivalent to an inf-sup condition for the bilinear form arising after adding the left-hand sides of (3.27), which means that there exists a constant $C > 0$, depending only on α, β_1, β_2, and $\|a\|$, such that

$$\sup_{\substack{(v,q)\in H_1\times Q_2 \\ (v,q)\neq \mathbf{0}}} \frac{a(u, v) + b_1(v, p) + b_2(u, q)}{\|(v, q)\|_{H_1\times Q_2}} \geqslant C\,\|(u, p)\|_{H_2\times Q_1} \quad \forall\,(u, p) \in H_2 \times Q_1\,. \tag{3.29}$$

We continue with the following abstract result, which constitutes a slight variation of the recent result [40, Theorem 3.4] tailored for perturbed saddle-point problems in Banach spaces.

Theorem 3.2. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H}\times\mathrm{H} \to \mathrm{R}$*,* $b : \mathrm{H}\times\mathrm{Q} \to \mathrm{R}$*, and* $c : \mathrm{Q}\times\mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms. In addition, let* $\mathbf{B} : \mathrm{H} \to \mathrm{Q}'$ *be the bounded linear operator induced by* b*, and let* $\mathrm{V} := \mathrm{N}(\mathbf{B})$ *be the respective null space. Assume that:*

i) *a and c are positive semi-definite, that is*

$$a(\tau,\tau) \geqslant 0 \quad \forall\, \tau \in \mathrm{H} \quad \textit{and} \quad c(v,v) \geqslant 0 \quad \forall\, v \in \mathrm{Q}\,, \tag{3.30}$$

and that c is symmetric.

ii) *there exists a constant $\alpha > 0$ such that*

$$\sup_{\substack{\tau\in \mathrm{V}\\ \tau\neq 0}} \frac{a(\vartheta,\tau)}{\|\tau\|_{\mathrm{H}}} \geqslant \alpha\, \|\vartheta\|_{\mathrm{H}} \qquad \forall\, \vartheta \in \mathrm{V}\,, \tag{3.31}$$

and

$$\sup_{\substack{\vartheta\in \mathrm{V}\\ \vartheta\neq 0}} \frac{a(\vartheta,\tau)}{\|\vartheta\|_{\mathrm{H}}} \geqslant \alpha\, \|\tau\|_{\mathrm{H}} \qquad \forall\, \tau \in \mathrm{V}\,, \tag{3.32}$$

iii) *and there exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\tau\in \mathrm{H}\\ \tau\neq 0}} \frac{b(\tau,v)}{\|\tau\|_{\mathrm{H}}} \geqslant \beta\, \|v\|_{\mathrm{Q}} \qquad \forall\, v \in \mathrm{Q}\,, \tag{3.33}$$

Then, for each pair $(f,g) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(\sigma,u) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\begin{aligned} a(\sigma,\tau) + b(\tau,u) &= f(\tau) \qquad \forall\, \tau \in \mathrm{H}\,,\\ b(\sigma,v) - c(u,v) &= g(v) \qquad \forall\, v \in \mathrm{Q}\,. \end{aligned} \tag{3.34}$$

Moreover, there exists a constant $\widetilde{C} > 0$, depending only on $\|a\|$, $\|c\|$, α, and β, such that

$$\|(\sigma,u)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant \widetilde{C}\left\{\|f\|_{\mathrm{H}'} + \|g\|_{\mathrm{Q}'}\right\}. \tag{3.35}$$

The foregoing theorem is referred to as a slight variant of the original version given by [40, Theorem 3.4] because, on one hand, it does not assume symmetry of a, as the latter does, but on the other hand, it does require the second inf-sup condition (3.32) for this bilinear form, which the latter does not. Indeed, the proof of [40, Theorem 3.4] reduces basically to show that there exists a positive constant $\widehat{C}$, depending on $\|a\|$, $\|c\|$, α, and β, such that the bilinear form arising from adding the left hand sides of (3.34), say $A : \big(\mathrm{H} \times \mathrm{Q}\big) \times \big(\mathrm{H} \times \mathrm{Q}\big) \to \mathrm{R}$, satisfies the inf-sup condition

$$\sup_{\substack{(\tau,v)\in \mathrm{H}\times\mathrm{Q}\\ (\tau,v)\neq\mathbf{0}}} \frac{A\big((\zeta,w),(\tau,v)\big)}{\|(\tau,v)\|} \geqslant \widehat{C}\,\|(\zeta,w)\| \qquad \forall\,(\zeta,w)\in \mathrm{H}\times\mathrm{Q}\,. \tag{3.36}$$

In this way, thanks to the symmetry of a and c, A is obviously symmetric, and hence (3.36) suffices to conclude, via the Banach–Nečas–Babuška Theorem (cf. [46, Theorem 2.6]), also known as the generalized Lax–Milgram Lemma, the well-posedness of (3.34). However, if one drops the symmetry assumption on a (and therefore on A), as done in the present Theorem 3.2, the same conclusion is attained if additionally (3.36) is also satisfied by the bilinear form $\widetilde{A}$ that arises from A after exchanging its components. Thus, noting that the above reduces to fixing the second component of A and taking the supremum in (3.36) with respect to the first one, we realize that in order to prove this further inf-sup condition, the assumption (3.32) needs to be added, as we did in Theorem 3.2. Needless to say, and because of the same constant α in (3.31) and (3.32), the aforementioned further condition holds with the same constant $\widehat{C}$ from (3.36), that is

$$\sup_{\substack{(\zeta,w)\in \mathrm{H}\times\mathrm{Q}\\ (\zeta,w)\neq\mathbf{0}}} \frac{A\big((\zeta,w),(\tau,v)\big)}{\|(\zeta,w)\|} \geqslant \widehat{C}\,\|(\tau,v)\| \qquad \forall\,(\tau,v)\in \mathrm{H}\times\mathrm{Q}\,. \tag{3.37}$$

The Banach–Nečas–Babuška Theorem will also be employed in Chapter 3.3 below.

3.3 Solvability analysis

In this chapter we address the solvability of the variational formulation (3.21), for which we introduce the operator $\mathbf{T} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ defined by

$$\mathbf{T}(\mathbf{z}_0) \,:=\, \mathbf{u}_0 \qquad \forall\,\mathbf{z}_0 \in \mathbf{L}^4(\Omega)\,, \tag{3.38}$$

where $(\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0) = \big((\mathbf{t}_0, \boldsymbol{\sigma}_0), (\mathbf{u}_0, \boldsymbol{\gamma}_0)\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be derived below under what conditions it does exist) of the linear problem

$$\mathbf{A}\big((\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \,+\, b(\mathbf{z}_0; \mathbf{u}_0, \mathbf{s}) \,=\, \mathbf{F}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,. \tag{3.39}$$

It follows that (3.21) can be rewritten as the fixed-point equation: Find $\mathbf{u} \in \mathbf{L}^4(\Omega)$ such that

$$\mathbf{T}(\mathbf{u}) \,=\, \mathbf{u}\,, \tag{3.40}$$

so that, letting $(\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0)$ be the solution of (3.39) with $\mathbf{z}_0 := \mathbf{u}$, $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := (\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0) \in \mathbf{H} \times \mathbf{Q}$ is solution of (3.21), equivalently of (3.15) and (3.19).

We now aim at proving that the operator $\mathbf{T}$ is well-defined, which reduces to show that problem (3.39) is well-posed. To this end, we first state the boundedness of all the variational forms involved (cf. (3.16a), (3.16b), (3.16c), (3.16d), and (3.22)). Direct applications of the Cauchy–Schwarz and Hölder inequalities, along with the upper bounds of η and μ (cf. (2.2)), and the continuity of the normal trace operator in $\mathbb{H}(\mathbf{div}_{4/3}; \Omega)$, yield the existence of positive constants, denoted and given as:

$$\|a\| \,=\, \lambda\,\mu_1\,, \quad \|b_1\| \,=\, \|b_2\| \,=\, 1\,, \quad \|\mathbf{a}\| \,=\, \lambda\,\mu_1 \,+\, 2\,, \quad \|\mathbf{b}\| \,=\, 1\,, \quad \|\mathbf{c}\| \,=\, \eta_1\,|\Omega|^{1/2}\,, \tag{3.41a}$$

$$\|\mathbf{F}\| \,=\, \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\,, \tag{3.41b}$$

such that there hold

$$\begin{aligned}
|a(\mathbf{r}, \mathbf{s})| \,&\leqslant\, \|a\|\,\|\mathbf{r}\|_{0,\Omega}\,\|\mathbf{s}\|_{0,\Omega} && \forall\, \mathbf{r},\ \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\\
|b_i(\mathbf{s}, \boldsymbol{\tau})| \,&\leqslant\, \|b_i\|\,\|\mathbf{s}\|_{0,\Omega}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} && \forall\, (\mathbf{s}, \boldsymbol{\tau}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega) \times \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)\,,\\
|\mathbf{a}(\vec{\mathbf{r}}, \vec{\mathbf{s}})| \,&\leqslant\, \|\mathbf{a}\|\,\|\vec{\mathbf{r}}\|_{\mathbf{H}}\,\|\vec{\mathbf{s}}\|_{\mathbf{H}} && \forall\, (\vec{\mathbf{r}}, \vec{\mathbf{s}}) \in \mathbf{H} \times \mathbf{H}\,,\\
|\mathbf{b}(\vec{\mathbf{s}}, \vec{\mathbf{v}})| \,&\leqslant\, \|\mathbf{b}\|\,\|\vec{\mathbf{s}}\|_{\mathbf{H}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} && \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,,\\
|\mathbf{c}(\vec{\mathbf{w}}, \vec{\mathbf{v}})| \,&\leqslant\, \|\mathbf{c}\|\,\|\vec{\mathbf{w}}\|_{\mathbf{Q}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} && \forall\, \vec{\mathbf{w}},\ \vec{\mathbf{v}} \in \mathbf{Q}\,, \quad \text{and}\\
|\mathbf{F}(\vec{\mathbf{s}}, \vec{\mathbf{v}})| \,&\leqslant\, \|\mathbf{F}\|\,\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}} && \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,.
\end{aligned} \tag{3.42}$$

In turn, employing again Cauchy–Schwarz and Hölder inequalities, similarly as we did in (3.6), we find that for each $\mathbf{w} \in \mathbf{L}^4(\Omega)$ there holds (cf. (3.17))

$$|b(\mathbf{w};\mathbf{v},\mathbf{s})| \;\leqslant\; \|\mathbf{w}\|_{0,4;\Omega}\,\|\mathbf{v}\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega} \qquad \forall\,(\mathbf{v},\mathbf{s}) \in \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{3.43}$$

In what follows, and as suggested by the matrix representation $\begin{pmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{b} & -\mathbf{c} \end{pmatrix}$ of $\mathbf{A}$ (cf. (3.20)), we will apply Theorem 3.2 to derive global inf-sup conditions for this bilinear form. To this end, and due to the corresponding structure $\begin{pmatrix} a & b_1 \\ b_2 & \end{pmatrix}$ of $\mathbf{a}$, we will employ in turn Theorem 3.1 to establish the required assumptions on the latter. According to the above, we begin by deducing from the definition (3.16c) that the kernel $\mathbf{V}$ of $\mathbf{b}$ reduces to

$$\mathbf{V} \;:=\; \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times \mathrm{V}_0\,, \tag{3.44}$$

where

$$\mathrm{V}_0 \;:=\; \Big\{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega): \quad \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathrm{t}} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau})=\mathbf{0} \quad \text{in} \quad \Omega\Big\}\,. \tag{3.45}$$

Hereafter, we refer to the null space of the bounded linear operator induced by a bilinear form as the kernel of the latter. Then, for each $i\in\{1,2\}$ we let K_i be the kernel of $b_i|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathrm{V}_0}$, that is

$$\mathrm{K}_i \;:=\; \Big\{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega): \quad b_i(\mathbf{s},\boldsymbol{\tau})=0 \quad \forall\,\boldsymbol{\tau}\in\mathrm{V}_0\Big\}\,, \tag{3.46}$$

which, recalling from (3.16b) that $b_1=-b_2$, yields

$$\mathrm{K}_1 \;=\; \mathrm{K}_2 \;=\; \mathrm{K} \;:=\; \Big\{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega): \quad \int_\Omega \mathbf{s}:\boldsymbol{\tau}=0 \quad \forall\,\boldsymbol{\tau}\in\mathrm{V}_0\Big\}\,. \tag{3.47}$$

However, irrespective of the above, we readily observe, according to the definition of a (cf. (3.16a)) and the lower bound of μ (cf. (2.2)), that a is $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$-elliptic with the constant $\widetilde{\alpha} := \lambda\,\mu_0$, that is

$$a(\mathbf{s},\mathbf{s}) \;\geqslant\; \widetilde{\alpha}\,\|\mathbf{s}\|^2_{0,\Omega} \qquad \forall\,\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.48}$$

and hence, in particular, a is K-elliptic. Then it is fairly simple to see that a satisfies the assumptions i) (with constant $\alpha = \widetilde{\alpha}$) and ii) of Theorem 3.1. In turn, in order to prove that for each $i \in \{1,2\}$, $b_i|_{\mathbb{L}^2_{\mathrm{tr}}(\Omega)\times \mathrm{V}_0}$ satisfies hypothesis iii), we first need to recall a useful estimate for tensors in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Indeed, suitably modifying the proof of [47, Lemma 2.3] (or [21, Proposition 3.1, Chapter IV]), one can show (see also [25, Lemma 3.2]) that there exists a positive constant c_1, depending only on Ω, such that

$$c_1\,\|\boldsymbol{\tau}\|_{0,\Omega} \,\leqslant\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{3.49}$$

Then, we are in position to prove the following result.

Lemma 3.3. *There exists a positive constant* $\widetilde{\beta}$ *such that for each* $i \in \{1,2\}$ *there holds*

$$\sup_{\substack{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ \mathbf{s}\neq\mathbf{0}}} \frac{b_i(\mathbf{s},\boldsymbol{\tau})}{\|\mathbf{s}\|_{0,\Omega}} \,\geqslant\, \widetilde{\beta}\,\|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathrm{V}_0\,. \tag{3.50}$$

Proof. Since $b_1 = -b_2$, it suffices to show for one of these bilinear forms, so that we stay with b_2. Thus, given $\boldsymbol{\tau}\in\mathrm{V}_0$ (cf. (3.45)), such that $\boldsymbol{\tau}^{\mathrm{d}} \neq \mathbf{0}$, we have that $\boldsymbol{\tau}^{\mathrm{d}}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and hence, bounding from below the supremum in (3.50) with $\mathbf{s} = \boldsymbol{\tau}^{\mathrm{d}}$, and noting that $\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \boldsymbol{\tau} = \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega}$, we obtain

$$\sup_{\substack{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ \mathbf{s}\neq\mathbf{0}}} \frac{b_2(\mathbf{s},\boldsymbol{\tau})}{\|\mathbf{s}\|_{0,\Omega}} \,\geqslant\, \frac{b_2(\boldsymbol{\tau}^{\mathrm{d}},\boldsymbol{\tau})}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}} \,=\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}\,,$$

from which, using (3.49) and the fact that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$, it follows (3.50) with $\widetilde{\beta} := c_1$. Certainly, if $\boldsymbol{\tau}\in\mathrm{V}_0$ is such that $\boldsymbol{\tau}^{\mathrm{d}} = \mathbf{0}$, we deduce from (3.49) that $\boldsymbol{\tau} = \mathbf{0}$, whence (3.50) is trivially satisfied. □

As a consequence of Lemma 3.3 and the previous discussion on the bilinear form a, we conclude that a, b_1, and b_2 satisfy the hypotheses of Theorem 3.1, and hence, a straightforward application of this abstract result, though more specifically of the global inf-sup condition (3.29), yields the existence of a positive constant $\alpha_{\mathbf{a}}$, depending only on $\widetilde{\alpha} = \lambda\mu_0$, $\widetilde{\beta} = c_1$, and

$\|a\| = \lambda\,\mu_1$ (cf. (3.41a)), such that

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{V}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{a}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geqslant\, \alpha_{\mathbf{a}}\,\|\vec{\mathbf{r}}\|_{\mathbf{H}} \qquad \forall\,\vec{\mathbf{r}}\in\mathbf{V}\,. \tag{3.51}$$

Moreover, exchanging the roles of b_1 and b_2, so that, instead of the matrix structure $\begin{pmatrix} a & b_1 \\ b_2 & \end{pmatrix}$, we consider $\begin{pmatrix} a & b_2 \\ b_1 & \end{pmatrix}$, we can apply again Theorem 3.1 and (3.29) to conclude that, with the same constant $\alpha_{\mathbf{a}}$ from (3.51), there holds

$$\sup_{\substack{\vec{\mathbf{r}}\in\mathbf{V}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{\mathbf{a}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{r}}\|_{\mathbf{H}}} \,\geqslant\, \alpha_{\mathbf{a}}\,\|\vec{\mathbf{s}}\|_{\mathbf{H}} \qquad \forall\,\vec{\mathbf{s}}\in\mathbf{V}\,. \tag{3.52}$$

Furthermore, it is readily seen from (3.18) and the ellipticity of a in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$ (cf. (3.48)), that

$$\mathbf{a}(\vec{\mathbf{r}},\vec{\mathbf{r}}) \,=\, a(\mathbf{r},\mathbf{r}) \,\geqslant\, \widetilde{\alpha}\,\|\mathbf{r}\|^2_{0,\Omega} \qquad \forall\,\vec{\mathbf{r}} := (\mathbf{r},\boldsymbol{\zeta})\in\mathbf{H}\,, \tag{3.53}$$

which proves that $\mathbf{a}$ is positive semi-definite. In turn, it is clear from the definition of $\mathbf{c}$ (cf. (3.16d)) that this bilinear form is symmetric, and that, thanks to the lower bound of η (cf. (2.2)), there holds

$$\mathbf{c}(\vec{\mathbf{v}},\vec{\mathbf{v}}) \,\geqslant\, \eta_0\,\|\mathbf{v}\|^2_{0,\Omega} \qquad \forall\,\vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta})\in\mathbf{Q}\,, \tag{3.54}$$

which shows that $\mathbf{c}$ is positive semi-definite as well. In this way, we have proved that $\mathbf{a}$ and $\mathbf{c}$ verify the hypotheses **i)** and **ii)** of Theorem 3.2, and hence it only remains to show the corresponding assumption **iii)**, that is the continuous inf-sup condition for $\mathbf{b}$. This result has already been given in [49, Lemma 3.5], so that, in addition to its statement, and for sake of clearness, we provide next most details of the corresponding proof. For this purpose, we will make use of the Poincaré and the first Korn (cf. [60, Theorem 10.1] or [20, Corollaries 9.2.22 and 9.2.25]) inequalities, which establish that

$$\|\mathbf{v}\|^2_{1,\Omega} \,\leqslant\, c_{\mathrm{P}}\,|\mathbf{v}|^2_{1,\Omega} \quad\text{and}\quad |\mathbf{v}|^2_{1,\Omega} \,\leqslant\, 2\,\|\mathbf{e}(\mathbf{v})\|^2_{0,\Omega} \quad \forall\,\mathbf{v}\in\mathbf{H}^1_0(\Omega)\,, \tag{3.55}$$

respectively, with a positive constant $c_{\rm P}$ depending on Ω. In addition, we also let $\mathbf{i}_4$ be the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$. Then, the announced result is as follows.

Lemma 3.4. *There exists a positive constant* $\beta_{\mathbf{b}}$*, depending only on* $c_{\rm P}$ *and* $\|\mathbf{i}_4\|$*, such that*

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \,\geqslant\, \beta_{\mathbf{b}}\,\|\vec{\mathbf{v}}\|_{\mathbf{Q}} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{Q}\,. \tag{3.56}$$

Proof. Given $\vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta})\in\mathbf{Q}$, we set $\tilde{\mathbf{v}} := |\mathbf{v}|^2\,\mathbf{v}$ and notice that $\|\tilde{\mathbf{v}}\|_{0,4/3;\Omega}^{4/3} = \|\mathbf{v}\|_{0,4;\Omega}^4$, which says that $\tilde{\mathbf{v}}\in\mathbf{L}^{4/3}(\Omega)$, and additionally there holds

$$\int_\Omega \mathbf{v}\cdot\tilde{\mathbf{v}} \,=\, \|\mathbf{v}\|_{0,4;\Omega}^4 \,=\, \|\mathbf{v}\|_{0,4;\Omega}\,\|\tilde{\mathbf{v}}\|_{0,4/3;\Omega}\,. \tag{3.57}$$

Then, letting $\mathcal{A} : \mathbf{H}_0^1(\Omega)\times\mathbf{H}_0^1(\Omega)\to\mathrm{R}$ and $\mathcal{F} : \mathbf{H}_0^1(\Omega)\to\mathrm{R}$ be the bilinear form and linear functional, respectively, defined by

$$\mathcal{A}(\mathbf{w},\mathbf{z}) \,:=\, \int_\Omega \mathbf{e}(\mathbf{w}):\mathbf{e}(\mathbf{z}) \quad\text{and}\quad \mathcal{F}(\mathbf{z}) \,:=\, -\int_\Omega \tilde{\mathbf{v}}\cdot\mathbf{z} \qquad \forall\,\mathbf{w},\,\mathbf{z}\in\mathbf{H}_0^1(\Omega)\,,$$

we readily see that $\mathcal{A}$ is bounded, and that, using (3.55), it becomes $\mathbf{H}_0^1(\Omega)$-elliptic with constant $\alpha_{\mathcal{A}} := \frac{1}{2c_{\rm P}}$. In turn, thanks to Hölder's inequality and the continuous injection $\mathbf{i}_4$, it follows that $\mathcal{F}$ is well-defined and bounded with $\|\mathcal{F}\| \leqslant \|\mathbf{i}_4\|\,\|\tilde{\mathbf{v}}\|_{0,4/3;\Omega}$. Hence, a straightforward application of the classical Lax-Milgram Lemma implies the existence of a unique $\tilde{\mathbf{w}}\in\mathbf{H}_0^1(\Omega)$ such that $\mathcal{A}(\tilde{\mathbf{w}},\mathbf{z}) = \mathcal{F}(\mathbf{z})$ for all $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$, and $\|\tilde{\mathbf{w}}\|_{1,\Omega} \leqslant 2c_{\rm P}\|\mathbf{i}_4\|\|\tilde{\mathbf{v}}\|_{0,4/3;\Omega}$. Moreover, it is easy to see from the foregoing identity involving $\mathcal{A}$ and $\mathcal{F}$ that $\mathbf{div}\big(\mathbf{e}(\tilde{\mathbf{w}})\big) = \tilde{\mathbf{v}}$ in $\mathcal{D}'(\Omega)$, which together with the fact that $\mathbf{e}(\tilde{\mathbf{w}})\in\mathbb{L}^2(\Omega)$, proves that $\mathbf{e}(\tilde{\mathbf{w}})\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$. Then, letting $\tilde{\boldsymbol{\tau}}$ be the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ component of $\mathbf{e}(\tilde{\mathbf{w}})$, we readily find that $\mathbf{div}(\tilde{\boldsymbol{\tau}}) = \tilde{\mathbf{v}}$ and

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega} \,\leqslant\, \|\tilde{\mathbf{w}}\|_{1,\Omega} \,+\, \|\tilde{\mathbf{v}}\|_{0,4/3;\Omega} \,\leqslant\, \big(2c_{\rm P}\,\|\mathbf{i}_4\| + 1\big)\,\|\tilde{\mathbf{v}}\|_{0,4/3;\Omega}\,, \tag{3.58}$$

and hence, noting that $\tilde{\boldsymbol{\tau}}$ is symmetric, since $\mathbf{e}(\mathbf{w})$ and the identity matrix are, and employing

(3.57) and (3.58), we get

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \geqslant \frac{\mathbf{b}((\mathbf{0},\widetilde{\boldsymbol{\tau}}),\vec{\mathbf{v}})}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} = \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}})}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} = \frac{\displaystyle\int_\Omega \mathbf{v}\cdot\widetilde{\mathbf{v}}}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} \geqslant \widetilde{\beta}_{\mathbf{b}}\,\|\mathbf{v}\|_{0,4;\Omega}\,, \tag{3.59}$$

with $\widetilde{\beta}_{\mathbf{b}} := \big(2c_{\mathrm{P}}\,\|\mathbf{i}_4\|+1\big)^{-1}$. Similarly, introducing the bounded linear functional $\mathcal{G}:\mathbf{H}_0^1(\Omega)\to\mathrm{R}$ defined by

$$\mathcal{G}(\mathbf{z}) := -\int_\Omega \boldsymbol{\delta}:\mathbf{e}(\mathbf{z}) \qquad \forall\,\mathbf{z}\in\mathbf{H}_0^1(\Omega)\,,$$

we deduce that there exists a unique $\widehat{\mathbf{w}}\in\mathbf{H}_0^1(\Omega)$ such that $\mathcal{A}(\widehat{\mathbf{w}},\mathbf{z}) = \mathcal{G}(\mathbf{z})$ for all $\mathbf{z}\in\mathbf{H}_0^1(\Omega)$, and $\|\mathbf{e}(\widehat{\mathbf{w}})\|_{0,\Omega} \leqslant \|\boldsymbol{\delta}\|_{0,\Omega}$. It follows from the above that $\mathbf{div}\big(\mathbf{e}(\widehat{\mathbf{w}})+\boldsymbol{\delta}\big) = \mathbf{0}$ in $\mathcal{D}'(\Omega)$, so that $\mathbf{e}(\widehat{\mathbf{w}})+\boldsymbol{\delta}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and hence, letting now $\widehat{\boldsymbol{\tau}}$ be the $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ component of $\mathbf{e}(\widehat{\mathbf{w}})+\boldsymbol{\delta}$, we get $\mathbf{div}(\widehat{\boldsymbol{\tau}}) = \mathbf{0}$ and

$$\|\widehat{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega} = \|\widehat{\boldsymbol{\tau}}\|_{0,\Omega} \leqslant \|\mathbf{e}(\widehat{\mathbf{w}})\|_{0,\Omega} + \|\boldsymbol{\delta}\|_{0,\Omega} \leqslant 2\,\|\boldsymbol{\delta}\|_{0,\Omega}\,. \tag{3.60}$$

In this way, noting that $\widehat{\boldsymbol{\tau}}:\boldsymbol{\delta} = \boldsymbol{\delta}:\boldsymbol{\delta}$, and using (3.60), we obtain

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbf{H}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \geqslant \frac{\mathbf{b}((\mathbf{0},\widehat{\boldsymbol{\tau}}),\vec{\mathbf{v}})}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} = \frac{\displaystyle\int_\Omega \widehat{\boldsymbol{\tau}}:\boldsymbol{\delta}}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} = \frac{\|\boldsymbol{\delta}\|_{0,\Omega}^2}{\|\widehat{\boldsymbol{\tau}}\|_{\mathbf{div}_{4/3};\Omega}} \geqslant \widehat{\beta}_{\mathbf{b}}\,\|\boldsymbol{\delta}\|_{0,\Omega}\,, \tag{3.61}$$

with $\widehat{\beta}_{\mathbf{b}} := \frac{1}{2}$. Finally, the required inequality (3.56) is a direct consequence of (3.59) and (3.61), with $\beta_{\mathbf{b}} := \frac{1}{2}\min\big\{\widetilde{\beta}_{\mathbf{b}},\widehat{\beta}_{\mathbf{b}}\big\}$. □

Consequently, having the bilinear forms $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ satisfied the three hypotheses of Theorem 3.2, a straightforward application of this abstract result yields the existence of a positive constant $\alpha_{\mathbf{A}}$, depending on $\|\mathbf{a}\|$, $\|\mathbf{c}\|$, $\alpha_{\mathbf{a}}$, and $\beta_{\mathbf{b}}$, such that (cf. (3.36), (3.37))

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{A}}\,\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{H}\times\mathbf{Q}} \qquad \forall\,(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{H}\times\mathbf{Q}\,, \tag{3.62}$$

and

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{H}\times\mathbf{Q}\\(\vec{\mathbf{r}},\vec{\mathbf{w}})\neq\mathbf{0}}}\frac{\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)}{\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{H}\times\mathbf{Q}}}\geqslant\alpha_{\mathbf{A}}\,\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}\qquad\forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\,.\tag{3.63}$$

Moreover, employing (3.62) and the boundedness property from (3.43), it readily follows that, given $\mathbf{z}\in\mathbf{L}^4(\Omega)$, there holds

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\(\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}}\frac{\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)+b(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}\geqslant\big(\alpha_{\mathbf{A}}-\|\mathbf{z}\|_{0,4;\Omega}\big)\,\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{H}\times\mathbf{Q}}\quad\forall\,(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{H}\times\mathbf{Q}\,,$$

and hence, for each $\mathbf{z}\in\mathbf{L}^4(\Omega)$ such that, say $\|\mathbf{z}\|_{0,4;\Omega}\leqslant\dfrac{\alpha_{\mathbf{A}}}{2}$, we get

$$\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\(\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}}\frac{\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)+b(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}\geqslant\frac{\alpha_{\mathbf{A}}}{2}\,\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{H}\times\mathbf{Q}}\qquad\forall\,(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{H}\times\mathbf{Q}\,.\tag{3.64}$$

Similarly, but now using (3.63) and (3.43), and under the same assumption on $\mathbf{z}$, we arrive at

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\mathbf{w}})\in\mathbf{H}\times\mathbf{Q}\\(\vec{\mathbf{r}},\vec{\mathbf{w}})\neq\mathbf{0}}}\frac{\mathbf{A}\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)+b(\mathbf{z};\mathbf{w},\mathbf{s})}{\|(\vec{\mathbf{r}},\vec{\mathbf{w}})\|_{\mathbf{H}\times\mathbf{Q}}}\geqslant\frac{\alpha_{\mathbf{A}}}{2}\,\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}\qquad\forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\,.\tag{3.65}$$

We are now in a position to prove that the operator $\mathbf{T}$ (cf. (3.38)) is well-defined, equivalently that problem (3.39) is well-posed.

Lemma 3.5. *For each $\mathbf{z}_0\in\mathbf{L}^4(\Omega)$ such that $\|\mathbf{z}_0\|_{0,4;\Omega}\leqslant\dfrac{\alpha_{\mathbf{A}}}{2}$, problem (3.39) has a unique solution $(\vec{\mathbf{t}}_0,\vec{\mathbf{u}}_0)=\big((\mathbf{t}_0,\boldsymbol{\sigma}_0),(\mathbf{u}_0,\boldsymbol{\gamma}_0)\big)\in\mathbf{H}\times\mathbf{Q}$, and hence $\mathbf{T}(\mathbf{z}_0):=\mathbf{u}_0\in\mathbf{L}^4(\Omega)$ is well-defined. Moreover, there holds*

$$\|\mathbf{T}(\mathbf{z}_0)\|_{0,4;\Omega}=\|\mathbf{u}_0\|_{0,4;\Omega}\leqslant\|(\vec{\mathbf{t}}_0,\vec{\mathbf{u}}_0)\|_{\mathbf{H}\times\mathbf{Q}}\leqslant\frac{2}{\alpha_{\mathbf{A}}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma}+\|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,.\tag{3.66}$$

Proof. Given $\mathbf{z}_0$ as indicated, the existence of a unique solution of (3.39) follows from (3.64), (3.65), and a straightforward application of the Banach–Nečas–Babuška Theorem (cf. [46, Theorem 2.6]). In turn, the corresponding a priori estimate and the boundedness of $\mathbf{F}$ (cf. (3.41b), (3.42)) yield (3.66). □

Next, we introduce the ball

$$\mathbf{W} := \left\{ \mathbf{z} \in \mathbf{L}^4(\Omega) : \quad \|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\alpha_{\mathbf{A}}}{2} \right\}, \tag{3.67}$$

and prove that, under sufficiently small data, $\mathbf{T}$ maps $\mathbf{W}$ into itself.

Lemma 3.6. *Assume that*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \leqslant \frac{\alpha_{\mathbf{A}}^2}{4}. \tag{3.68}$$

Then, there holds $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$.

Proof. It is a direct consequence of the a priori estimate (3.66) and the assumption (3.68). □

The main result concerning the solvability of the fixed-point equation (3.40), and hence, equivalently, that of (3.21), (3.19), or (3.15), is stated as follows.

Theorem 3.7. *Assume that*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} < \frac{\alpha_{\mathbf{A}}^2}{4}. \tag{3.69}$$

Then, the operator $\mathbf{T}$ *has a unique fixed-point* $\mathbf{u} \in \mathbf{W}$. *Equivalently,* (3.21) *has a unique solution* $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := (\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0) \in \mathbf{H} \times \mathbf{Q}$ *with* $\mathbf{u} \in \mathbf{W}$, *where* $(\vec{\mathbf{t}}_0, \vec{\mathbf{u}}_0)$ *is the unique solution of* (3.39) *with* $\mathbf{z}_0 = \mathbf{u}$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}}, \vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A}}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \right\}. \tag{3.70}$$

Proof. It is clear, thanks to (3.69) and Lemma 3.6, that $\mathbf{T}$ maps $\mathbf{W}$ into itself, so that aiming to apply the classical Banach fixed-point theorem, it only remains to show that $\mathbf{T}$ is a contraction. To this end, given $\mathbf{z}_i \in \mathbf{W}$, $i \in \{1, 2\}$, we let $\mathbf{T}(\mathbf{z}_i) := \mathbf{u}_i$, where $(\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i) := \big((\mathbf{t}_i, \boldsymbol{\sigma}_i), (\mathbf{u}_i, \boldsymbol{\gamma}_i)\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (3.39) with $\mathbf{z}_0 := \mathbf{z}_i$, that is

$$\mathbf{A}\big((\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + b(\mathbf{z}_i; \mathbf{u}_i, \mathbf{s}) = \mathbf{F}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}. \tag{3.71}$$

Now, applying the inf-sup condition (3.64) with $\mathbf{z} = \mathbf{z}_1$ to $(\vec{\mathbf{r}}, \vec{\mathbf{w}}) := (\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)$, we obtain

$$\|(\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A}}} \sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q} \\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + b(\mathbf{z}_1; \mathbf{u}_1 - \mathbf{u}_2, \mathbf{s})}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}, \quad (3.72)$$

from which, adding and subtracting $b(\mathbf{z}_2; \mathbf{u}_2, \mathbf{s})$, and then employing (3.71), we arrive at

$$\|(\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A}}} \sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q} \\ (\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{b(\mathbf{z}_2 - \mathbf{z}_1; \mathbf{u}_2, \mathbf{s})}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}. \quad (3.73)$$

In turn, using the boundedness of b (cf. (3.43)) and the a priori estimate for $\|\mathbf{u}_2\|_{0,4;\Omega} = \|\mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega}$ provided by (3.66) (cf. Lemma 3.5), it follows from (3.73) that

$$\begin{aligned} \|\mathbf{T}(\mathbf{z}_1) - \mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega} &= \|\mathbf{u}_1 - \mathbf{u}_2\|_{0,4;\Omega} \leqslant \frac{2}{\alpha_{\mathbf{A}}} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega} \|\mathbf{u}_2\|_{0,4;\Omega} \\ &\leqslant \frac{4}{\alpha_{\mathbf{A}}^2} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}, \end{aligned} \quad (3.74)$$

which, according to (3.69), confirms the announced property on $\mathbf{T}$, thus ending the proof for the existence of a unique fixed-point $\mathbf{u}$ in $\mathbf{W}$ of this operator. Finally, the a priori estimate (3.70) is a straightforward consequence of (3.66) (cf. Lemma 3.5). □

CHAPTER 4

The discrete formulation

In this chapter we approximate the solution of (3.21) (equivalently, that of (3.19) or (3.15)) by introducing and analyzing the associated Galerkin scheme. To this end, similar tools to those employed in Chapter 3.3 will be utilized here.

4.1 The Galerkin scheme

We begin by considering arbitrary finite element subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively. Hereafter, h stands for both the sub-index of each foregoing subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (in R^2) or tetrahedra K (in R^3) of diameter h_K, that is $h := \max\big\{h_K : \quad K \in \mathcal{T}_h\big\}$. Specific finite element subspaces satisfying suitable hypotheses to be introduced in due course will be provided later on in Chapter 4.4. Then, letting

$$\mathbb{H}_h^{\boldsymbol{\sigma}} \ := \ \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \, \cap \, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{4.1}$$

defining the product spaces

$$\mathbf{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}, \qquad \mathbf{Q}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}, \tag{4.2}$$

and setting the notations

$$\begin{aligned} \vec{\mathbf{t}}_h &:= (\mathbf{t}_h, \boldsymbol{\sigma}_h),\ \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h),\ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h) \in \mathbf{H}_h, \\ \vec{\mathbf{u}}_h &:= (\mathbf{u}_h, \boldsymbol{\gamma}_h),\ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\delta}_h),\ \vec{\mathbf{w}}_h := (\mathbf{w}_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h, \end{aligned} \tag{4.3}$$

the Galerkin scheme associated with (3.15) reads as follows: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{lllll} a(\mathbf{t}_h, \mathbf{s}_h) + b_1(\mathbf{s}_h, \boldsymbol{\sigma}_h) & & + b(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) & = & 0, \\ b_2(\mathbf{t}_h, \boldsymbol{\tau}_h) & + \mathbf{b}(\vec{\mathbf{s}}_h, \vec{\mathbf{u}}_h) & & = & \langle \boldsymbol{\tau}_h \boldsymbol{\nu}, \mathbf{u}_D \rangle, \\ \quad \mathbf{b}(\vec{\mathbf{t}}_h, \vec{\mathbf{v}}_h) & - \mathbf{c}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) & & = & -\displaystyle\int_\Omega \mathbf{f} \cdot \mathbf{v}_h, \end{array} \tag{4.4}$$

for all $(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$. Similarly, the ones associated with (3.19) and (3.21), which are certainly equivalent to (4.4), become, respectively: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{b}(\vec{\mathbf{s}}_h, \vec{\mathbf{u}}_h) + b(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) &= \langle \boldsymbol{\tau}_h \boldsymbol{\nu}, \mathbf{u}_D \rangle \qquad \forall\, \vec{\mathbf{s}}_h \in \mathbf{H}_h, \\ \mathbf{b}(\vec{\mathbf{t}}_h, \vec{\mathbf{v}}_h) - \mathbf{c}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) &= -\int_\Omega \mathbf{f} \cdot \mathbf{v}_h \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Q}_h, \end{aligned} \tag{4.5}$$

and: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$\mathbf{A}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) + b(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) = \mathbf{F}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h. \tag{4.6}$$

In order to analyze the solvability of (4.6) (equivalently that of (4.5) or (4.4)) in Chapter 4.2 below, we will require the finite dimensional versions of the Babuška–Brezzi theory in Banach spaces (cf. Theorem 3.1) and the Banach–Nečas–Babuška theorem, which are available in [18, Sections 2.2 and 2.3] and [46, Theorem 2.22], respectively. In turn, we will also need the discrete analogue of Theorem 3.2, which is given by the slight improvement of [40, Theorem

3.5] that is stated next.

Theorem 4.1. *Let* H *and* Q *be reflexive Banach spaces, and let* $a : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$, $b : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$, *and* $c : \mathrm{Q} \times \mathrm{Q} \to \mathrm{R}$ *be given bounded bilinear forms. In addition, let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be families of finite dimensional subspaces of* H *and* Q*, respectively, and let* V_h *be the kernel of* $b|_{\mathrm{H}_h \times \mathrm{Q}_h}$*, that is*

$$\mathrm{V}_h \;:=\; \Big\{ \boldsymbol{\tau}_h \in \mathrm{H}_h : \quad b(\boldsymbol{\tau}_h, v_h) \;=\; 0 \quad \forall\, v_h \in \mathrm{Q}_h \Big\} \,. \tag{4.7}$$

Assume that:

i) *a and c are positive semi-definite, and that c is symmetric.*

ii) *there exists a constant $\widetilde{\alpha}_{\mathrm{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{V}_h \\ \tau_h \neq 0}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_{\mathrm{H}}} \;\geqslant\; \widetilde{\alpha}_{\mathrm{d}}\, \|\vartheta_h\|_{\mathrm{H}} \qquad \forall\, \vartheta_h \in \mathrm{V}_h \,, \tag{4.8}$$

iii) *and there exists a constant $\widetilde{\beta}_{\mathrm{d}} > 0$, independent of h, such that*

$$\sup_{\substack{\tau_h \in \mathrm{H}_h \\ \tau_h \neq 0}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_{\mathrm{H}}} \;\geqslant\; \widetilde{\beta}_{\mathrm{d}}\, \|v_h\|_{\mathrm{Q}} \qquad \forall\, v_h \in \mathrm{Q}_h \,. \tag{4.9}$$

Then, for each pair $(f, g) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(\sigma_h, u_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ such that

$$\begin{aligned} a(\sigma_h, \tau_h) + b(\tau_h, u_h) &= f(\tau_h) \qquad \forall\, \tau_h \in \mathrm{H}_h \,, \\ b(\sigma_h, v_h) - c(u_h, v_h) &= g(v_h) \qquad \forall\, v_h \in \mathrm{Q}_h \,. \end{aligned} \tag{4.10}$$

Moreover, there exists a constant $\widetilde{C}_{\mathrm{d}} > 0$, depending only on $\|a\|$, $\|c\|$, $\widetilde{\alpha}_{\mathrm{d}}$, and $\widetilde{\beta}_{\mathrm{d}}$, such that

$$\|\sigma_h\|_{\mathrm{H}} \;+\; \|u_h\|_{\mathrm{Q}} \;\leqslant\; \widetilde{C}_{\mathrm{d}} \Big\{ \|f\|_{\mathrm{H}'} \;+\; \|g\|_{\mathrm{Q}'} \Big\} \,. \tag{4.11}$$

We stress here that the aforementioned improvement refers to the fact that the symmetry of a, originally assumed in [40, Theorem 3.5], is actually not needed for Theorem 4.1. In addition to the above, note as well that the discrete analogue of (3.32) is not required either. The reason for these simplifications of the analysis is due to the fact that $\mathrm{H}_h \times \mathrm{Q}_h$ is the space to which both the unknowns and test functions of (4.10) belong, and hence, as stipulated by the finite dimensional version of the Banach–Nečas–Babuška theorem (cf. [46, Theorem 2.22]), in this case one only needs to prove the discrete analogue of (3.36). In this way, it is easy to see, as done in [40, Theorems 3.4 and 3.5], that in order to achieve the latter, it suffices to assume the already described hypotheses of Theorem 4.1.

4.2 Solvability analysis

In this chapter we adopt the discrete version of the fixed-point strategy employed in Chapter 3.3 to study the solvability of (4.6). For this purpose, we now let $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\mathbf{T}_h(\mathbf{z}_{0,h}) := \mathbf{u}_{0,h} \qquad \forall\, \mathbf{z}_{0,h} \in \mathbf{H}_h^{\mathbf{u}}\,, \tag{4.12}$$

where $(\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h}) = \big((\mathbf{t}_{0,h}, \boldsymbol{\sigma}_{0,h}), (\mathbf{u}_{0,h}, \boldsymbol{\gamma}_{0,h})\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be derived below under what conditions it does exist) of the linear problem

$$\mathbf{A}\big((\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h}), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) + b(\mathbf{z}_{0,h}; \mathbf{u}_{0,h}, \mathbf{s}_h) = \mathbf{F}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h\,. \tag{4.13}$$

Then, it is easily seen that (4.6) can be rewritten as the fixed-point equation: Find $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h\,, \tag{4.14}$$

so that, letting $(\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h})$ be the solution of (4.13) with $\mathbf{z}_{0,h} := \mathbf{u}_h$, $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := (\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h}) \in \mathbf{H}_h \times \mathbf{Q}_h$ is solution of (4.6), equivalently of (4.4) and (4.5).

In what follows we derive the preliminary results needed to address later on the solvabilities of (4.13) and (4.14), and hence of (4.6). Indeed, following a similar procedure to the one from Chapter 3.3, we first observe that the kernel $\mathbf{V}_h$ of $\mathbf{b}|_{\mathbf{H}_h \times \mathbf{Q}_h}$ reduces to

$$\mathbf{V}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathrm{V}_{0,h}\,, \tag{4.15}$$

where

$$\mathrm{V}_{0,h} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\delta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}_h^{\gamma} \text{ and } \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \right\}. \tag{4.16}$$

At this point, we introduce our first hypotheses on the finite element subspaces, namely

(**H.0**) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiplies of the identity tensor $\mathbb{I}$.

(**H.1**) $\mathbf{div}(\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{H}_h^{\mathbf{u}}$.

As a consequence of (**H.0**) and the decomposition (3.9), $\mathbb{H}_h^{\boldsymbol{\sigma}}$ (cf. (4.1)) can be redefined as

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \right\}. \tag{4.17}$$

We remark in advance, however, that for the computational implementation of the Galerkin scheme (4.6), which will be addressed later on in Chapter 5, we will utilize a real Lagrange multiplier to impose the mean value condition on the trace of the unknown tensor lying in $\mathbb{H}_h^{\boldsymbol{\sigma}}$.

In turn, thanks to (**H.1**), $\mathrm{V}_{0,h}$ becomes

$$\mathrm{V}_{0,h} := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\delta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}_h^{\gamma} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \right\}. \tag{4.18}$$

Next, for each $i \in \{1,2\}$ we let $\mathrm{K}_{i,h}$ be the kernel of $b_i|_{\mathbb{H}_h^{\mathbf{t}} \times \mathrm{V}_{0,h}}$, and notice, similarly as for the continuous case (cf. (3.47)), that

$$\mathrm{K}_{1,h} = \mathrm{K}_{2,h} = \mathrm{K}_h := \left\{ \mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} : \quad \int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h = 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathrm{V}_{0,h} \right\}. \tag{4.19}$$

While, as in the continuous case, the above does not allow us to derive an explicit characterization for the elements of K_h, this is actually unnecessary since, having already stated that the bilinear form a is $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$-elliptic (cf. (3.48)), this property is certainly valid for the subspace K_h. Consequently, the corresponding hypotheses on a, $\mathrm{K}_{1,h}$, and $\mathrm{K}_{2,h}$ specified in the discrete version of Theorem 3.1 (cf. [18, eqs. (2.19) and (2.20)]) are clearly satisfied with the same constant $\widetilde{\alpha}$ from (3.48). Nevertheless, we notice that [18, eq. (2.20)] is not required in the present case since obviously the dimensions of $\mathrm{K}_{1,h}$ and $\mathrm{K}_{2,h}$ coincide (cf. [18, eq. (2.21)] and the remark right before it).

Furthermore, in order to show that for each $i \in \{1, 2\}$, $b_i|_{\mathbb{H}_h^{\mathbf{t}} \times \mathrm{V}_{0,h}}$ satisfies the discrete version of the hypothesis iii) of Theorem 3.1, namely eq. $(2.22)_i$ in [18], we consider the following additional hypothesis:

(H.2) $\left(\mathrm{V}_{0,h}\right)^{\mathrm{d}} := \left\{\boldsymbol{\tau}_h^{\mathrm{d}} : \quad \boldsymbol{\tau}_h \in \mathrm{V}_{0,h}\right\} \subseteq \mathbb{H}_h^{\mathbf{t}}$.

In this way, proceeding analogously as for the proof of Lemma 3.3, that is, given $\boldsymbol{\tau}_h \in \mathrm{V}_{0,h}$, bounding from below with $\mathbf{s}_h = \boldsymbol{\tau}_h^{\mathrm{d}} \in \mathbb{H}_h^{\mathbf{t}}$, we find that

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{b_2(\mathbf{s}_h, \boldsymbol{\tau}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} \geqslant \frac{b_2(\boldsymbol{\tau}_h^{\mathrm{d}}, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega}} = \|\boldsymbol{\tau}_h^{\mathrm{d}}\|_{0,\Omega} ,$$

which, using (3.49) and the fact that $\mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0}$, yields

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{b_2(\mathbf{s}_h, \boldsymbol{\tau}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} \geqslant \widetilde{\beta} \, \|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in \mathrm{V}_{0,h} , \tag{4.20}$$

with $\widetilde{\beta} = c_1$. A similar reasoning provides the corresponding discrete inf-sup condition for b_1 with the same constant $\widetilde{\beta}$.

Therefore, having a, b_1, and b_2 satisfied the hypotheses of the discrete version of Theorem 3.1 (cf. [18, Corollary 2.2]), we conclude the discrete analogue of the global inf-sup condition (3.29), namely, with the same constant $\alpha_{\mathbf{a}}$ from (3.51), there holds

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{V}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathbf{a}(\vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}} \geqslant \alpha_{\mathbf{a}} \, \|\vec{\mathbf{r}}_h\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{r}}_h \in \mathbf{V}_h . \tag{4.21}$$

In addition, we know from the continuous analysis (cf. (3.53) and (3.54)) that $\mathbf{a}$ and $\mathbf{c}$ are positive semi-definite on $\mathbf{H}$ and $\mathbf{Q}$, respectively, so that they certainly keep this property on $\mathbf{H}_h$ and $\mathbf{Q}_h$. We have thus shown that the bilinear forms $\mathbf{a}$ and $\mathbf{c}$ satisfy the hypotheses **i)** and **ii)** of Theorem 4.1, and hence, in order to be able to apply this abstract result, we now add the remaining hypothesis **iii)** as an assumption:

(**H.3**) there exists a positive constant $\beta_{\mathbf{b},\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{H}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{H}}} \geqslant \beta_{\mathbf{b},\mathrm{d}} \, \|\vec{\mathbf{v}}_h\|_{\mathbf{Q}} \qquad \forall\, \vec{\mathbf{v}}_h \in \mathbf{Q}_h\,. \tag{4.22}$$

As already announced, specific finite element subspaces satisfying the four hypotheses (**H.0**) - (**H.3**) will be detailed later on in Chapter 4.4.

Now, having **a**, **b**, and **c** satisfied the hypotheses of Theorem 4.1, we conclude, similarly to the continuous case (cf. (3.62), (3.64)), the existence of a positive constant $\alpha_{\mathbf{A},\mathrm{d}}$, depending on $\|\mathbf{a}\|$, $\|\mathbf{c}\|$, $\alpha_{\mathbf{a}}$, and $\beta_{\mathbf{b},\mathrm{d}}$, and hence independent of h, such that

$$\sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{A},\mathrm{d}} \, \|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \qquad \forall\,(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h\,, \tag{4.23}$$

and thus, for each $\mathbf{z}_h \in \mathbf{H}^{\mathbf{u}}_h$ such that $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \dfrac{\alpha_{\mathbf{A},\mathrm{d}}}{2}$, there holds

$$\sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h) \in \mathbf{H}_h\times\mathbf{Q}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big) + b(\mathbf{z}_h;\mathbf{w}_h,\mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}} \geqslant \frac{\alpha_{\mathbf{A},\mathrm{d}}}{2} \, \|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \tag{4.24}$$

for all $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$.

According to the above, we are now in a position to present the discrete analogues of Lemmas 3.5 and 3.6, and Theorem 3.7, whose proofs follow almost verbatim to those for the continuous case, and hence only some remarks are provided. We begin with the well-posedness of (4.13), which is the same as establishing that $\mathbf{T}_h$ is well-defined.

Lemma 4.2. *For each $\mathbf{z}_{0,h} \in \mathbf{H}^{\mathbf{u}}_h$ such that $\|\mathbf{z}_{0,h}\|_{0,4;\Omega} \leqslant \dfrac{\alpha_{\mathbf{A},\mathrm{d}}}{2}$, problem* (4.13) *has a unique solution $(\vec{\mathbf{t}}_{0,h},\vec{\mathbf{u}}_{0,h}) = \big((\mathbf{t}_{0,h},\boldsymbol{\sigma}_{0,h}),(\mathbf{u}_{0,h},\gamma_{0,h})\big) \in \mathbf{H}_h \times \mathbf{Q}_h$, and hence $\mathbf{T}_h(\mathbf{z}_{0,h}) := \mathbf{u}_{0,h} \in \mathbf{H}^{\mathbf{u}}_h$ is well-defined. Moreover, there holds*

$$\|\mathbf{T}_h(\mathbf{z}_{0,h})\|_{0,4;\Omega} = \|\mathbf{u}_{0,h}\|_{0,4;\Omega} \leqslant \|(\vec{\mathbf{t}}_{0,h},\vec{\mathbf{u}}_{0.h})\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A},\mathrm{d}}} \Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,. \tag{4.25}$$

Proof. Given $\mathbf{z}_{0,h}$ as indicated, and bearing in mind (4.24), it suffices to apply the discrete version of the Banach–Nečas–Babuška Theorem (cf. [46, Theorem 2.22]) and its corresponding a priori error estimate. □

We continue with the result ensuring that $\mathbf{T}_h$ maps a ball of $\mathbf{H}_h^{\mathbf{u}}$ into itself.

Lemma 4.3. *Let* $\mathbf{W}_h$ *be the ball*

$$\mathbf{W}_h := \left\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\alpha_{\mathbf{A},\mathrm{d}}}{2} \right\}, \tag{4.26}$$

and assume that

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \leqslant \frac{\alpha_{\mathbf{A},\mathrm{d}}^2}{4}. \tag{4.27}$$

Then, there holds $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$.

Proof. It follows straightforwardly from (4.25) and (4.27). □

The unique solvability of (4.14), and hence, equivalently that of (4.6), is stated next.

Theorem 4.4. *Assume that*

$$\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} < \frac{\alpha_{\mathbf{A},\mathrm{d}}^2}{4}. \tag{4.28}$$

Then, the operator $\mathbf{T}_h$ *has a unique fixed-point* $\mathbf{u}_h \in \mathbf{W}_h$. *Equivalently,* (4.6) *has a unique solution* $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := (\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h}) \in \mathbf{H}_h \times \mathbf{Q}_h$ *with* $\mathbf{u}_h \in \mathbf{W}_h$*, where* $(\vec{\mathbf{t}}_{0,h}, \vec{\mathbf{u}}_{0,h})$ *is the unique solution of* (4.13) *with* $\mathbf{z}_{0,h} = \mathbf{u}_h$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A},\mathrm{d}}} \left\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \right\}. \tag{4.29}$$

Proof. Similarly to the proof of Theorem 3.7, it reduces to employ (4.24), (4.13), (4.25), and (3.43) to prove that $\mathbf{T}_h : \mathbf{W}_h \to \mathbf{W}_h$ is a contraction, and then apply the Banach fixed-point theorem. □

4.3 A priori error analysis

In this chapter we derive an a priori error estimate for the Galerkin scheme (4.6) with arbitrary finite element subspaces satisfying the hypotheses (**H.0**) up to (**H.3**) specified in Chapter 4.2. In other words, our main goal is to establish a Céa estimate for the error

$$\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,,$$

where $(\vec{\mathbf{t}},\vec{\mathbf{u}}) := \big((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\gamma)\big) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) := \big((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\gamma_h)\big) \in \mathbf{H}_h\times\mathbf{Q}_h$ are the unique solutions of (3.21) and (4.6), respectively, with $\mathbf{u}\in\mathbf{W}$ (cf. (3.67)) and $\mathbf{u}_h\in\mathbf{W}_h$ (cf. (4.26)). As a byproduct of this, we also derive an a priori estimate for $\|p-p_h\|_{0,\Omega}$, where p_h is the discrete pressure computed according to the postprocessing formula suggested by the second identity in (2.7), that is

$$p_h \;=\; -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma}_h + (\mathbf{u}_h\otimes\mathbf{u}_h)\big)\,. \tag{4.30}$$

We begin by observing from (3.21) that for each $(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$ there holds

$$\mathbf{A}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big) \,+\, b(\mathbf{u};\mathbf{u},\mathbf{s}_h) \;=\; \mathbf{F}(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\,,$$

which, combined with (4.6), yields for each $(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$

$$\mathbf{A}\big((\vec{\mathbf{t}},\vec{\mathbf{u}})-(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big) \;=\; b(\mathbf{u}_h;\mathbf{u}_h,\mathbf{s}_h) \,-\, b(\mathbf{u};\mathbf{u},\mathbf{s}_h)\,. \tag{4.31}$$

Now, the triangle inequality gives for each $(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h$

$$\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \|(\vec{\mathbf{t}},\vec{\mathbf{u}}) - (\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) - (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,, \tag{4.32}$$

and then, applying (4.23), subtracting and adding $(\vec{\mathbf{t}},\vec{\mathbf{u}})$ in the first component of $\mathbf{A}$, using the boundedness of $\mathbf{A}$ with constant $\|\mathbf{A}\|$, which depends on $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, and $\|\mathbf{c}\|$ (cf. (3.41a)), and employing the identity (4.31), we find that

$$\begin{aligned}
\alpha_{\mathbf{A},\mathrm{d}} \, \|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,&\leqslant\, \sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}} \\
&\leqslant\, \|\mathbf{A}\| \; \|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{\mathbf{A}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h),(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\big)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}} \\
&=\, \|\mathbf{A}\| \; \|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \sup_{\substack{(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\in\mathbf{H}_h\times\mathbf{Q}_h \\ (\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\neq\mathbf{0}}} \frac{b(\mathbf{u}_h;\mathbf{u}_h,\mathbf{s}_h) \,-\, b(\mathbf{u};\mathbf{u},\mathbf{s}_h)}{\|(\vec{\mathbf{s}}_h,\vec{\mathbf{v}}_h)\|_{\mathbf{H}\times\mathbf{Q}}}\,.
\end{aligned} \tag{4.33}$$

In turn, subtracting and adding $\mathbf{u}$ in the second component of the first term, and then invoking the boundedness property of b (3.43), and the a priori estimates (3.70) and (4.29) for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, respectively, we obtain

$$\begin{aligned}
b(\mathbf{u}_h;\mathbf{u}_h,\mathbf{s}_h) \,-\, b(\mathbf{u};\mathbf{u},\mathbf{s}_h) \,&=\, b(\mathbf{u}_h;\mathbf{u}_h-\mathbf{u},\mathbf{s}_h) \,+\, b(\mathbf{u}_h-\mathbf{u};\mathbf{u},\mathbf{s}_h) \\
&\leqslant\, \frac{4}{\widetilde{\alpha}_{\mathbf{A}}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,\|\mathbf{s}\|_{0,\Omega}\,,
\end{aligned} \tag{4.34}$$

where $\widetilde{\alpha}_{\mathbf{A}} \,:=\, \min\big\{\alpha_{\mathbf{A}},\alpha_{\mathbf{A},\mathrm{d}}\big\}$. In this way, using (4.34) in the last term of (4.33), we obtain

$$\begin{aligned}
&\|(\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \\
&\qquad\leqslant\, \frac{\|\mathbf{A}\|}{\alpha_{\mathbf{A},\mathrm{d}}}\,\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \frac{4}{\widetilde{\alpha}_{\mathbf{A}}^2}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,
\end{aligned} \tag{4.35}$$

which, replaced back into (4.32), leads to

$$\begin{aligned}
&\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \\
&\qquad\leqslant\, \Big(1+\frac{\|\mathbf{A}\|}{\alpha_{\mathbf{A},\mathrm{d}}}\Big)\,\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{r}}_h,\vec{\mathbf{w}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \frac{4}{\widetilde{\alpha}_{\mathbf{A}}^2}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,
\end{aligned}$$

for each $(\vec{\mathbf{r}}_h, \vec{\mathbf{w}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, and hence we conclude that

$$\begin{aligned} &\|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \\ &\qquad \leqslant \Big(1 + \frac{\|\mathbf{A}\|}{\alpha_{\mathbf{A},\mathrm{d}}}\Big) \operatorname{dist}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \mathbf{H}_h \times \mathbf{Q}_h\big) + \frac{4}{\widetilde{\alpha}_{\mathbf{A}}^2} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} . \end{aligned} \tag{4.36}$$

Hereafter, given a subspace X_h of a generic Banach space $\big(X, \|\cdot\|_X\big)$, we set for each $x \in X$

$$\operatorname{dist}(x, X_h) := \inf_{x_h \in X_h} \|x - x_h\|_X .$$

The Céa estimate for the error $\|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}}$ is stated then as follows.

Theorem 4.5. *Assume that for some $\delta \in (0,1)$ there holds*

$$\Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \leqslant \frac{\delta\, \widetilde{\alpha}_{\mathbf{A}}^2}{4} . \tag{4.37}$$

Then, there exists a positive constant C_d, depending only on $\|\mathbf{A}\|$, $\alpha_{\mathbf{A},\mathrm{d}}$, and δ, and hence independent of h, such that

$$\|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant C_\mathrm{d} \operatorname{dist}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \mathbf{H}_h \times \mathbf{Q}_h\big) . \tag{4.38}$$

Proof. It suffices to use (4.37) in (4.36), which yields (4.38) with $C_\mathrm{d} := (1-\delta)^{-1}\Big(1+\dfrac{\|\mathbf{A}\|}{\alpha_{\mathbf{A},\mathrm{d}}}\Big)$. □

Regarding the pressure error, we readily deduce from (2.7) and (4.30), applying Cauchy-Schwarz's inequality, performing some algebraic manipulations, and employing again the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$ (cf. (3.70) and (4.29)), that there exists a positive constant $\widetilde{C}$, depending only on n, $\widetilde{\alpha}_{\mathbf{A}}$, $\|\mathbf{u}_D\|_{1/2,\Gamma}$, and $\|\mathbf{f}\|_{0,4/3;\Omega}$, and hence, independent of h, such that

$$\|p - p_h\|_{0,\Omega} \leqslant \widetilde{C} \Big\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} \Big\} . \tag{4.39}$$

Thus, combining (4.38) and (4.39), we conclude the existence of a positive constant $\widetilde{C}_\mathrm{d}$, independent of h, such that

$$\|(\vec{\mathbf{t}},\vec{\mathbf{u}}) \,-\, (\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega} \,\leqslant\, \widetilde{C}_{\mathrm{d}}\,\mathrm{dist}\big((\vec{\mathbf{t}},\vec{\mathbf{u}}),\mathbf{H}_h \times \mathbf{Q}_h)\big)\,. \tag{4.40}$$

We end this chapter by stressing that (4.37) and the fact that $\widetilde{\alpha}_{\mathbf{A}} \,:=\, \min\big\{\alpha_{\mathbf{A}},\alpha_{\mathbf{A},\mathrm{d}}\big\}$ guarantee that the assumptions (3.69) and (4.28) of Theorems 3.7 and 4.4, respectively, are satisfied.

4.4 Specific finite element subspaces

In this chapter we resort to [49, Section 4.4] to specify two examples of finite element subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, satisfying the hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**) that were introduced in Chapter 4.2.

4.4.1 Preliminaries

Here we collect some definitions and results that are employed in what follows. Indeed, given an integer $\ell \geqslant 0$ and $K \in \mathcal{T}_h$, we first let $\mathrm{P}_\ell(K)$ be the space of polynomials of degree $\leqslant \ell$ defined on K, whose vector and tensor versions are denoted $\mathbf{P}_\ell(K) \,:=\, [\mathrm{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) \,=\, [\mathrm{P}_\ell(K)]^{n\times n}$, respectively. Also, we let $\mathbf{RT}_\ell(K) \,:=\, \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\,\mathbf{x}$ be the local Raviart–Thomas space of order ℓ defined on K, where $\mathbf{x}$ stands for a generic vector in $\mathbf{R} := \mathrm{R}^n$. Furthermore, we let b_K be the bubble function on K, which is defined as the product of its $n+1$ barycentric coordinates, and introduce the local bubble spaces of order ℓ as

$$\mathbf{B}_\ell(K) \,:=\, \mathrm{curl}\big(b_K\,\mathrm{P}_\ell(K)\big) \quad \text{if} \quad n=2\,, \quad \text{and} \quad \mathbf{B}_\ell(K) \,:=\, \mathrm{curl}\big(b_K\,\mathbf{P}_\ell(K)\big) \quad \text{if} \quad n=3\,,$$

where $\mathrm{curl}(v) \,:=\, \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ if $n = 2$ and $v : K \to \mathrm{R}$, and $\mathrm{curl}(\mathbf{v}) := \nabla\times\mathbf{v}$ if $n = 3$ and $\mathbf{v} : K \to \mathrm{R}^3$. In addition, we need to set the global spaces

$$\mathbf{P}_\ell(\Omega) \,:=\, \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \,\in\, \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,$$

$$\mathbb{P}_\ell(\Omega) \;:=\; \Big\{ \boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\delta}_h|_K \,\in\, \mathbb{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

$$\mathbb{RT}_\ell(\Omega) \;:=\; \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_K \,\in\, \mathbf{RT}_\ell(K) \quad \forall\, i \in \{1,...,n\}\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

and

$$\mathbb{B}_\ell(\Omega) \;:=\; \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_{h,i}|_K \,\in\, \mathbf{B}_\ell(K) \quad \forall\, i \in \{1,...,n\}\,, \quad \forall\, K \in \mathcal{T}_h \Big\}\,,$$

where $\boldsymbol{\tau}_{h,i}$ stands for the ith-row of $\boldsymbol{\tau}_h$. As noticed in [49], it is easily seen that $\mathbf{P}_\ell(\Omega)$ and $\mathbb{P}_\ell(\Omega)$ are also subspaces of $\mathbf{L}^4(\Omega)$ and $\mathbb{L}^4(\Omega)$, respectively, and that $\mathbb{RT}_\ell(\Omega)$ and $\mathbb{B}_\ell(\Omega)$ are both subspaces of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. Actually, since $\mathbb{H}(\mathbf{div};\Omega)$ is clearly contained in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, any subspace of the former is also subspace of the latter.

Next, defining $\mathbb{H}_0(\mathbf{div};\Omega) := \big\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0 \big\}$, we recall that a triplet of subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ of $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{L}^2(\Omega)$, and $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, respectively, is said to be stable for the classical Hilbertian mixed formulation of linear elasticity, if, denoting $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div};\Omega)$, there exists a positive constant $\beta_{\mathtt{e}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}_h + \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div};\Omega}} \;\geqslant\; \beta_{\mathtt{e}} \Big\{ \|\mathbf{v}_h\|_{0,\Omega} + \|\boldsymbol{\delta}_h\|_{0,\Omega} \Big\} \qquad \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\,. \tag{4.41}$$

In turn, since the definition of the bilinear form $\mathbf{b}$ (cf. (3.16c)) does not involve the $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$-variable, we notice that hypothesis (**H.3**) (cf. (4.22)) becomes

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \boldsymbol{\delta}_h : \boldsymbol{\tau}_h + \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega}} \;\geqslant\; \beta_{\mathbf{b},\mathtt{d}} \Big\{ \|\mathbf{v}_h\|_{0,4;\Omega} + \|\boldsymbol{\delta}_h\|_{0,\Omega} \Big\} \quad \forall\, (\mathbf{v}_h, \boldsymbol{\delta}_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\,. \tag{4.42}$$

Certainly, the inequalities (4.41) and (4.42) do not coincide since the spaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$ employ different norms in them. However, the following result, already proved in [49, Lemma 4.8], establishes a very suitable connection between these discrete inf-sup conditions.

Lemma 4.6. *Let* $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, *and* $\mathbb{H}_h^{\gamma}$ *be subspaces of* $\mathbb{H}(\mathbf{div};\Omega)$, $\mathbf{L}^2(\Omega)$, *and* $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, *respectively, such that they satisfy* (4.41). *In addition, assume that there exists an integer* $\ell \geqslant 0$ *such that* $\mathbb{RT}_\ell(\Omega) \subseteq \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ *and* $\mathbf{H}_h^{\mathbf{u}} \subseteq \mathbf{P}_\ell(\Omega)$. *Then* $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{H}_h^{\mathbf{u}}$, *and* $\mathbb{H}_h^{\gamma}$ *satisfy* (4.42) *with a positive constant* $\beta_{\mathbf{b},\mathrm{d}}$*, independente of* h*.*

According to the above, we now employ the stable triplets for elasticity proposed in [49, Section 4.4] to describe two examples of finite element subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ satisfying the hypotheses (**H.0**), (**H.1**), (**H.2**), and (**H.3**) from Chapter 4.2.

4.4.2 PEERS-based finite element subspaces

We first consider the plane elasticity element with reduced symmetry (PEERS) of order $\ell \geqslant 0$, whose stability was originally proved in [11] for $\ell = 0$ and $n = 2$, and later on in [57] for $\ell \geqslant 0$ and $n \in \{2,3\}$. In fact, denoting $\mathbb{C}(\bar{\Omega}) := [C(\bar{\Omega})]^{n\times n}$, the corresponding subspaces are given by

$$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and} \quad \mathbb{H}_h^{\gamma} := \mathbb{C}(\bar{\Omega}) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_{\ell+1}(\Omega)\,. \tag{4.43}$$

It is easily seen that $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$ satisfy (**H.0**) and (**H.1**), and, thanks to Lemma 4.6, whose hypotheses on $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$ are also guaranteed, it is clear that $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ satisfy (**H.3**) (cf. (4.42)). Next, in order to check (**H.2**), we recall from (4.18) that

$$\mathrm{V}_{0,h} := \left\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\delta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}_h^{\gamma} \quad \text{and} \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega\right\},$$

which, noting that $\mathbb{B}_\ell(\Omega)$ is divergence free, recalling that the divergence free tensors of $\mathbb{RT}_\ell(\Omega)$ are contained in $\mathbb{P}_\ell(\Omega)$ (cf. [47, proof of Theorem 3.3]), and observing that $\mathbb{B}_\ell(\Omega) \subseteq \mathbb{P}_{\ell+n}(\Omega)$, we deduce that

$$\mathrm{V}_{0,h} \subseteq \mathbb{P}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega) \subseteq \mathbb{P}_{\ell+n}(\Omega)\,,$$

so that, to accomplish (**H.2**), that is $\left(\mathrm{V}_{0,h}\right)^{\mathrm{d}} \subseteq \mathbb{H}_h^{\mathbf{t}}$, it suffices to choose

$$\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{\ell+n}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{4.44}$$

4.4.3 AFW-based finite element subspaces

Our second example is the Arnold–Falk–Winther (AFW) element of order $\ell \geqslant 0$, which is defined as

$$\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)\,, \quad \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and} \quad \mathbb{H}_h^{\gamma} := \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_\ell(\Omega)\,, \tag{4.45}$$

and whose stability for the Hilbertian mixed formulation of linear elasticity is proved in [12]. In this case, it is also straightforward to see that $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$ satisfy (**H.0**) and (**H.1**), as well as the hypotheses required by Lemma 4.6, and hence $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$ satisfy (**H.3**). In turn, for (**H.2**), and since $\mathrm{V}_{0,h}$ does not seem to be additionally simplifiable, it suffices to take

$$\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,. \tag{4.46}$$

4.4.4 The rates of convergence

The approximation properties of $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, for PEERS (cf. (4.43)) as well as for AFW (cf. (4.45)), whose derivations follow basically from the error estimates of the Raviart–Thomas and AFW interpolation operators, and of projectors onto piecewise vector and tensor polynomials (cf. [46, Proposition 1.135]), and which make use of the commuting diagram properties and of the interpolation estimates of Sobolev spaces, are given as follows (see also [19], [21], [37, eqs. (5.37) and (5.40)]):

$\left(\mathbf{AP}_h^{\boldsymbol{\sigma}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{r,4/3}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}\big) \leqslant C\, h^r \Big\{ \|\boldsymbol{\tau}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{r,4/3;\Omega} \Big\}\,, \tag{4.47}$$

$\left(\mathbf{AP}_h^{\mathbf{u}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell+1]$, and for each $\mathbf{v} \in \mathbf{W}^{r,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}\big) \,\leqslant\, C\, h^r \|\mathbf{v}\|_{r,4;\Omega}\,, \tag{4.48}$$

and

$\big(\mathbf{AP}_h^{\gamma}\big)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell + 1]$, and for each $\boldsymbol{\delta} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\delta}, \mathbb{H}_h^{\gamma}\big) \,\leqslant\, C\, h^r \|\boldsymbol{\delta}\|_{r,\Omega}\,. \tag{4.49}$$

In turn, denoting $\ell^* := \begin{cases} \ell + n & \text{for PEERS-based} \\ \ell + 1 & \text{for AFW-based} \end{cases}$, the approximation property for $\mathbb{H}_h^{\mathbf{t}}$ is similar to that of $\mathbf{H}_h^{\mathbf{u}}$, that is:

$\big(\mathbf{AP}_h^{\mathbf{t}}\big)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell^* + 1]$, and for each $\mathbf{s} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}\big) \,\leqslant\, C\, h^r \|\mathbf{s}\|_{r,\Omega}\,. \tag{4.50}$$

We are now in a position to provide the rates of convergence of the Galerkin scheme (4.6) with the finite element subspaces defined in Sections 4.4.2 and 4.4.3.

Theorem 4.7. *Assume that for some* $\delta \in (0,1)$ *the data satisfies* (4.37)*, and let* $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{H} \times \mathbf{Q}$ *and* $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (3.21) *and* (4.6)*, respectively, with* $\mathbf{u} \in \mathbf{W}$ *(cf.* (3.67)*) and* $\mathbf{u}_h \in \mathbf{W}_h$ *(cf.* (4.26)*), whose existences are guaranteed by Theorems* 3.7 *and* 4.4*, respectively. In turn, let* p *and* p_h *be the exact and approximate pressure defined by the second identity in* (2.7) *and* (4.30)*, respectively. Furthermore, given an integer* $\ell \geqslant 0$*, assume that there exists* $r \in [0, \ell+1]$ *such that* $\mathbf{t} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$*,* $\boldsymbol{\sigma} \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{r,4/3}(\Omega)$*,* $\mathbf{u} \in \mathbf{W}^{r,4}(\Omega)$*, and* $\boldsymbol{\gamma} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$*.*

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} &\|(\vec{\mathbf{t}}, \vec{\mathbf{u}}) - (\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \, + \, \|p - p_h\|_{0,\Omega} \\ &\qquad \leqslant \, C\, h^r \left\{ \|\mathbf{t}\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,4/3;\Omega} + \|\mathbf{u}\|_{r,4;\Omega} + \|\boldsymbol{\gamma}\|_{r,\Omega} \right\}. \end{aligned} \tag{4.51}$$

Proof. It follows straightforwardly from the final Céa estimate (4.40) and the approximation properties $\big(\mathbf{AP}_h^{\boldsymbol{\sigma}}\big)$, $\big(\mathbf{AP}_h^{\mathbf{u}}\big)$, $\big(\mathbf{AP}_h^{\boldsymbol{\gamma}}\big)$, and $\big(\mathbf{AP}_h^{\mathbf{t}}\big)$. □

CHAPTER 5

Numerical results

We report on the performance of the proposed numerical methods. The set of computational tests collected in this chapter have been implemented using the open source finite element library `FEniCS` [7]. A Newton-Raphson algorithm with null initial guess is used for the resolution of all nonlinear problems, and the solution of tangent systems resulting from the linearization is carried out with the multifrontal massively parallel sparse direct solver MUMPS [10].

5.1 Accuracy verification

The convergence of the methods is assessed in 2D and 3D. We consider the unit square $(0,1)^2$ and unit cube $(0,1)^3$ domains, discretized into meshes that are successively refined. We fix $\lambda = 0.2$ together with the heterogeneous viscosity and inverse permeabilities $\mu(x_1,x_2) = \exp(-x_1x_2), \eta(x_1,x_2) = 2+\sin(x_1x_2)$ (in 2D) and $\mu(x_1,x_2,x_3) = \exp(-x_1x_2x_3), \eta(x_1,x_2,x_3) = 2+\sin(x_1x_2x_3)$ (in 3D). And we choose a boundary velocity $\mathbf{u}_D$ and a forcing term $\mathbf{f}$ such that the exact solutions are

$$\mathbf{u}_{\mathrm{ex}}(x_1,x_2) = \begin{pmatrix} \cos(\pi x_1)\sin(\pi x_2) \\ -\sin(\pi x_1)\cos(\pi x_2) \end{pmatrix}, \quad p_{\mathrm{ex}}(x_1,x_2) = \sin(x_1 x_2),$$

and

$$\mathbf{u}_{\mathrm{ex}}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \quad p_{\mathrm{ex}}(x_1,x_2) = \sin(x_1 x_2 x_3),$$

for the 2D and 3D cases, respectively.

The condition of zero-average pressure (which, owing to (2.7), entails to fix the trace of the tensor quantity $\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}$) is imposed by means of a real Lagrange multiplier ξ. The modified system (cf. (3.19)) is then of the form

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{t}},\vec{\mathbf{s}}) \,+\, \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{u}}) \,+\, b(\mathbf{u};\mathbf{u},\mathbf{s}) \,+\, \widehat{b}_1(\boldsymbol{\tau},\xi) &= \langle \boldsymbol{\tau}\boldsymbol{\nu},\mathbf{u}_D\rangle && \forall\,\vec{\mathbf{s}}\in\mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{t}},\vec{\mathbf{v}}) \,-\, \mathbf{c}(\vec{\mathbf{u}},\vec{\mathbf{v}}) &= -\int_\Omega \mathbf{f}\cdot\mathbf{v} && \forall\,\vec{\mathbf{v}}\in\mathbf{Q}\,, \\ \widehat{b}_2(\boldsymbol{\sigma},\mathbf{u},\zeta) &= \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_{\mathrm{ex}} + \mathbf{u}_{\mathrm{ex}}\otimes\mathbf{u}_{\mathrm{ex}})\zeta && \forall\,\zeta\in \mathrm{R}, \end{aligned} \tag{5.1}$$

where $\widehat{b}_1(\boldsymbol{\tau},\xi) := \int_\Omega \mathrm{tr}(\boldsymbol{\tau})\xi$, and $\widehat{b}_2$ is associated with the term $\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u})\zeta$. Note that we do not have a zero-mean manufactured pressure, and in this case we require the additional right-hand side term in the third equation of (5.1).

Errors between exact and approximate solutions relevant to the norms used in the analysis of Chapter 4 are denoted as

$$\mathsf{e}(\mathbf{t}) := \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega} \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\mathbf{div}_{4/3};\Omega}\,, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\,,$$
$$\mathsf{e}(\boldsymbol{\gamma}) := \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,\Omega}\,, \quad \mathsf{e}(p) := \|p-p_h\|_{0,\Omega}\,.$$

The error decay according to the mesh refinement is reported in Figure 5.1. We plot, in log-log scale, errors for the individual variables in the norms above vs the number of degrees of

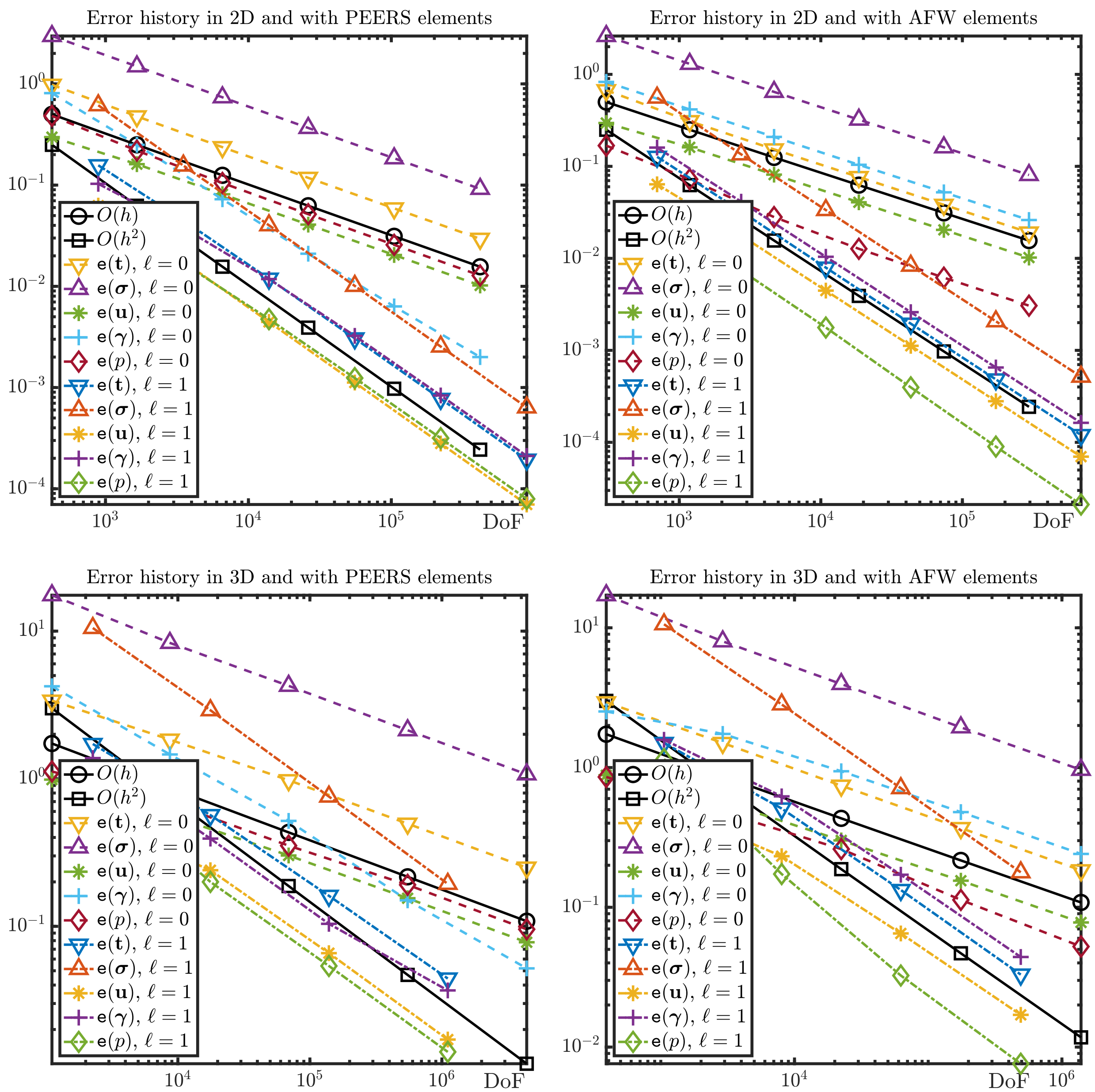

Figure 5.1: Error history for the mixed methods defined using the spaces in (4.43)-(4.44) (left panels) and in (4.45)-(4.46) (right panels), using manufactured solutions in 2D (top) and 3D (bottom).

freedom associated with each triangulation. Apart from the rotation tensor, which has a slightly better convergence than the optimal for the PEERS-based family and for the lowest-order case only, the convergences observed for all fields, even for coarser meshes, and for the two methods in 2D and 3D and using polynomial degrees $\ell = 0$ (dashed lines) and $\ell = 1$ (dot-dashed lines)

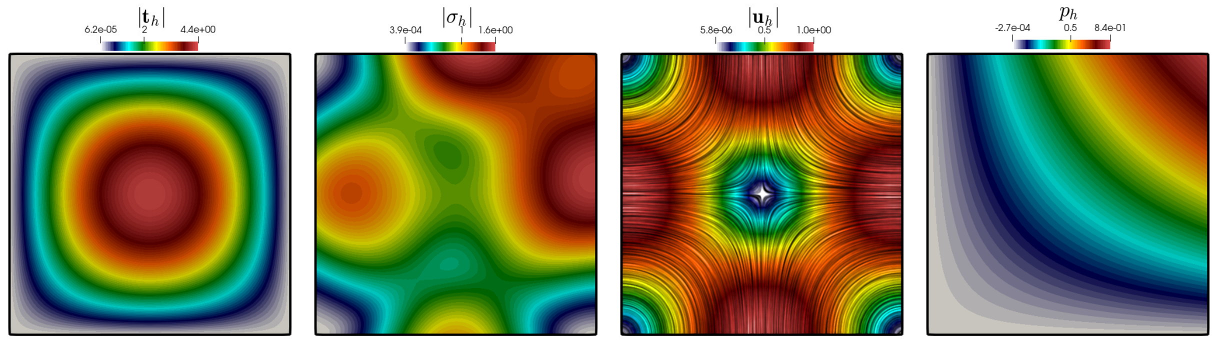

Figure 5.2: Sample of approximate solutions (velocity with line integral convolution) for the convergence test, obtained using the second-order AFW-based finite element family.

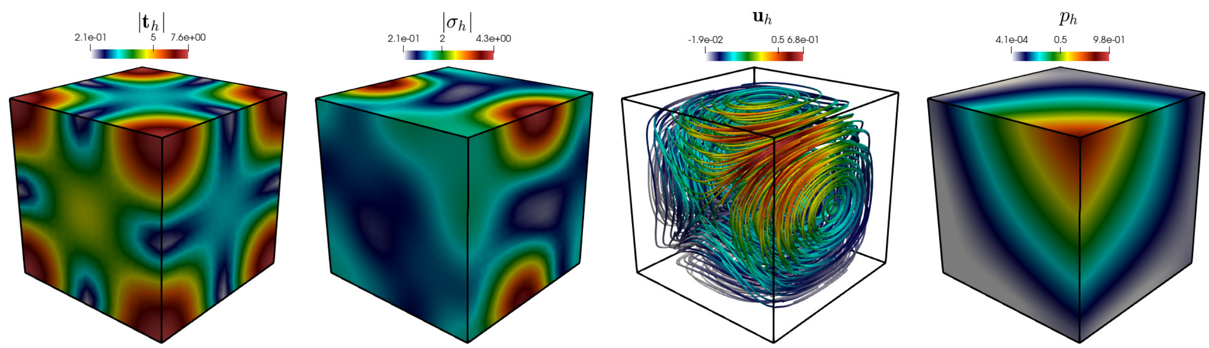

Figure 5.3: Sample of approximate solutions (velocity with streamlines) for the convergence test, obtained using the first-order PEERS-based finite element family.

are all optimal, $\mathcal{O}(h^{l+1})$, in accordance with Theorem 4.7. In addition, we show in Figures 5.2 and 5.3 approximate solutions after 4 steps of uniform mesh refinement. All field variables are well resolved.

5.2 Channel flow

Next we test the performance of the mixed finite element methods in reproducing flow patterns on a channel with three obstacles (using the domain and boundary configuration from the micro-macro models for incompressible flow introduced in [70]), and including mixed boundary conditions. An external forcing term is imposed $\mathbf{f} = (0,1)^{\mathrm{t}}$. On the inlet (the bottom horizontal section of the boundary defined by $(0,1) \times \{-2\}$) we prescribe a parabolic inflow velocity

$\mathbf{u}_{\text{in}} = (0, x_1(1-x_1))^{\text{t}}$. On the outlet (the vertical segment on the top left part of the boundary, defined by $\{-2\} \times (0,1)$) we impose a zero normal Cauchy stress, which means that we need to set

$$(\boldsymbol{\sigma} + \mathbf{u} \otimes \mathbf{u})\boldsymbol{\nu} = \mathbf{0} \qquad \text{on } \Gamma_{\text{out}},$$

and on the remainder of the boundary we set no-slip velocity $\mathbf{u} = \mathbf{0}$. The above Neumann boundary condition can be easily incorporated in the analysis developed in Chapters 3 and 4 by imposing it via either a Nitsche method or a Lagrange multiplier. We proceed with the former for the present numerical example. We use $\eta = 0.1 + x_1^2 + x_2^2$, $\lambda = 0.02$, and $\mu = \exp(-x_1 x_2)$. No closed-form solution is available for this problem. For this test we use $\ell = 1$ and the PEERS-based finite element family. The computed flow profiles are shown in Figure 5.4.

5.3 Flow on an intracranial aneurysm

We finalize this chapter by computing numerical solutions on a section of the middle cerebral artery with an aneurysm (abnormal bulge of a blood vessel). The surface mesh was obtained from the Gmsh repository[1], and it was then truncated and volume-meshed into 68'024 unstructured tetrahedral elements. For this test we use the AFW-based finite element family of second-order.

As a typical indicator of a risk factor for aneurysm rupture, we compute the wall shear stress (see, e.g., [64]). Its magnitude on the boundary (representing the tangential drag exerted by flowing blood on the aneurysmal sac and in general, on the vessel wall) is computed as the vector field $\mathbf{w}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\sum_{e \in \mathcal{E}_{h,\mathrm{w}}} \int_e \mathbf{w}_h \cdot \mathbf{v}_h = \sum_{e \in \mathcal{E}_{h,\mathrm{w}}} \frac{1}{h_e} \int_e (\boldsymbol{\sigma}_h + \mathbf{u}_h \otimes \mathbf{u}_h)_s \cdot \mathbf{v}_h \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}},$$

where $\mathcal{E}_{h,\mathrm{w}}$ stands for the set of faces e that are contained in the polyhedral approximation of the vessel wall that is inherited from the triangulation $\mathcal{T}_h$, and $\boldsymbol{\tau}_s := \boldsymbol{\tau}\boldsymbol{\nu} - (\boldsymbol{\tau}\boldsymbol{\nu} \cdot \boldsymbol{\nu})\boldsymbol{\nu}$ denotes

[1] https://gitlab.onelab.info/gmsh/gmsh/-/blob/master/examples/api/aneurysm2.stl

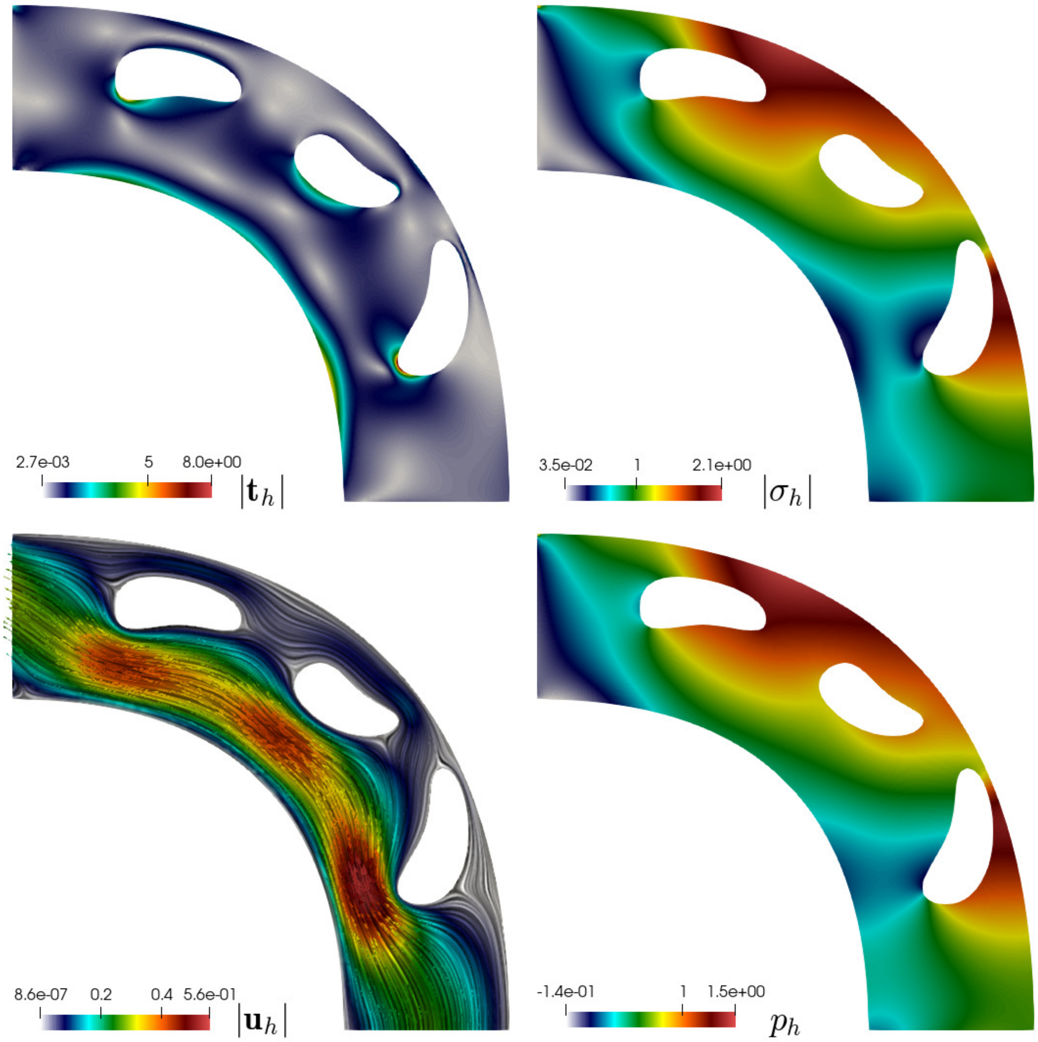

Figure 5.4: Approximate strain rate magnitude, total stress magnitude, velocity magnitude and line integral contour, and postprocessed pressure for the Navier–Stokes–Brinkman equations on a complex channel with obstacles. Solutions computed with a PEERS-based method using $\ell = 1$.

the tangential part of $\boldsymbol{\tau}$. We do not require differentiation of the velocity as in the usual postprocess-based computation of the wall shear stress.

The parameters for the incompressible fluid (in this case, blood) were defined by a constant density of 1g/cm^3 and a dynamic viscosity $\mu = 3.5 \cdot 10^{-3}\,\text{Kg} \cdot \text{m}^{-1}\text{s}^{-1}$ (and we take $\lambda = 1$ and $\eta = 10$). We impose a zero external force. At the vessel walls the no-slip condition $\mathbf{u} = \mathbf{0}$ is imposed. On the inlet (a disk-shaped surface on the parent artery branch near to

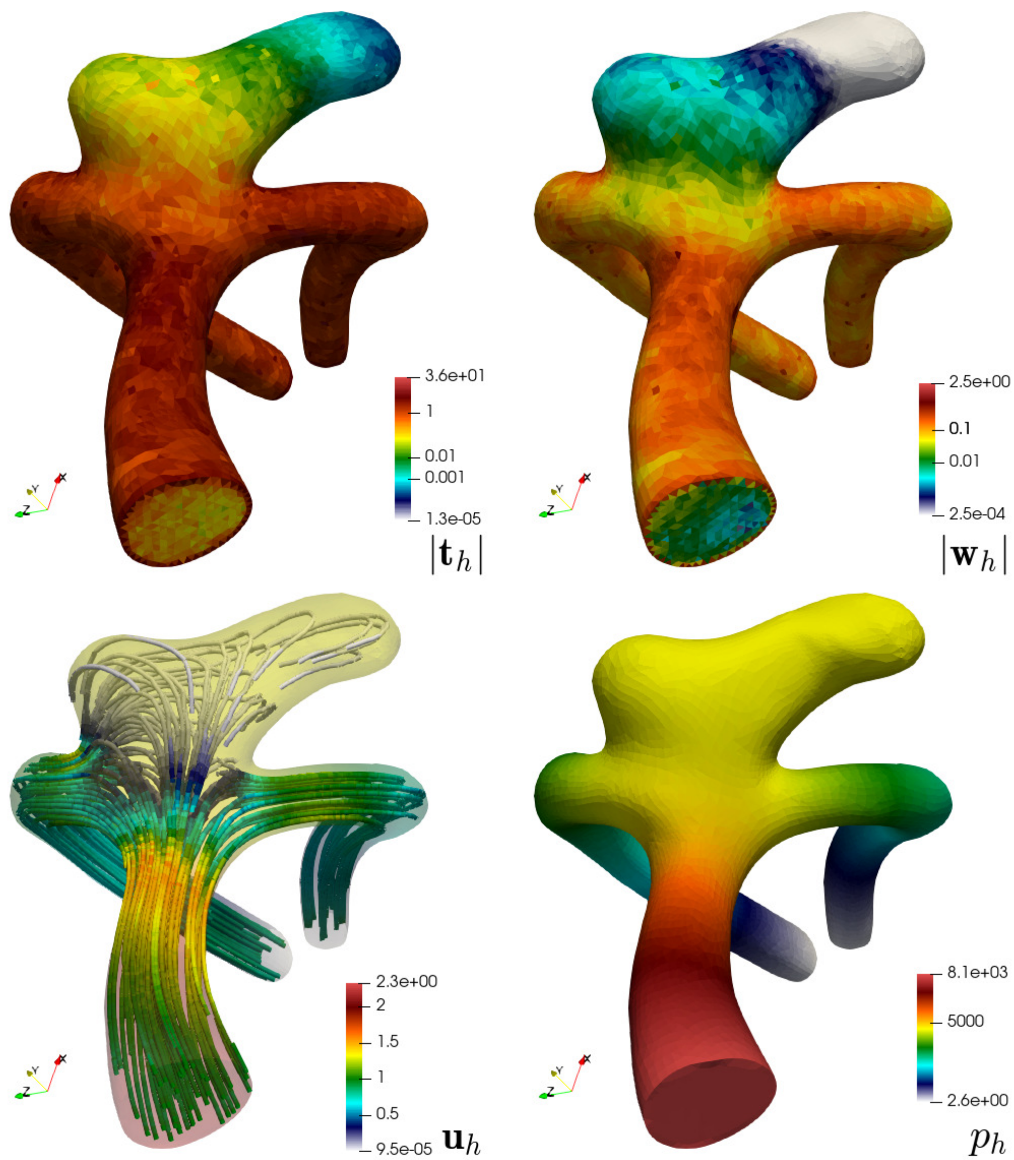

Figure 5.5: Approximate strain rate magnitude, wall shear stress magnitude, velocity streamlines, and postprocessed pressure for the Navier–Stokes–Brinkman equations on a cerebral aneurysm. Solutions computed with an AFW-based method using $\ell = 1$.

the visualization center) we impose a constant velocity profile $\mathbf{u} = -u_m \boldsymbol{\nu}$ (with $u_m = 1\,\mathrm{cm/s}$), while at the outlet (the caps at the two remaining distal ends), and differently than the previous example, we set $\boldsymbol{\sigma\nu} = \mathbf{0}$. This condition is simply included in the definitions of the spaces to which $\boldsymbol{\sigma}$ and $\boldsymbol{\sigma}_h$ belong, so that the continuous and discrete analyses remain basically unchanged. Under physiological circumstances the wall shear stress magnitude is of the order of 10 dyne/cm^2. The initiation of atherosclerosis is associated with a decrease in wall shear stress and a reduction in the function of several endothelial cell mechanisms. We plot in Figure 5.5 the obtained numerical solutions. It is observed that the wall shear stress is very low (magnitude less than 0.1 dyne/cm^2) in the aneurysm and we also see a large recirculation with a much lower velocity in that region. These findings are in qualitative agreement with, e.g., [59, 68].

Part II

Mixed-primal methods for natural convection driven phase change with Navier–Stokes–Brinkman equations

CHAPTER 6

Introduction

Heat driven flow is a class of physical phenomena that has been extensively studied and it has practical applications in many branches of science and engineering. Specific mechanisms such as natural convection lay the foundation for other– more involved– processes including heat and mass transfer, phase change such as melting and solidification [42,73], the design of energy storage devices [44], the description of ocean and atmosphere dynamics [43], and crystallization in magma chambers [71].

Throughout the literature, phase change is incorporated into the Boussinesq approximation by means of enthalpy-porosity methods [67] or enthalpy-viscosity models [42]. Numerical methods proposed for the former include a class of stabilised discontinuous Galerkin [67] and finite volume methods [73], whereas a primal finite element scheme [42] is employed for the latter. Other techniques used for either case include primal formulations with Taylor–Hood discretization, projection schemes, variational multiscale stabilization, and other variants [3,45,62,65,76]. Here we consider the general case where viscosity, enthalpy and porosity all depend on temperature. In turn, in the recent work [9] the authors introduced a phase change model for natural convection in porous media, where the problem is modeled as a viscous Newtonian fluid and the change of phase is encoded in the viscosity itself, and using a Brinkman–Boussinesq approximation where the solidification process influences the drag directly. A fully-primal formulation for the non-stationary case was analyzed in [9, Section 4.2], while rigorous mathematical and numerical analyses for mixed-primal and fully-mixed methods for the stationary case were provided in [8]. These numerical methods, as well as the related weak formulations, have been analyzed in Hilbert spaces-based frameworks.

Using the more general approach of working with Banach spaces framework permits us to avoid augmentation techniques, maintaining a structure much closer to the initial physical model in mixed form. This type of formalisms has other benefits such as enforcing strongly (momentum and mass and energy) conservative schemes. The numerical analysis of Banach spaces formulations for linear, nonlinear, and coupled problems in continuum mechanics has been carried out in the very recent contributions [16, 25, 30, 33, 37, 39, 41, 48, 49] (see also the references therein), which consider Poisson, Brinkman–Forchheimer, Darcy–Forchheimer, Navier–Stokes, chemotaxis/Navier–Stokes, Boussinesq, coupled flow–transport, and fluidized beds, among others models. The purpose of the present manuscript is to extend and adapt the analysis developed in [48] for the Navier–Stokes–Brinkman equations, to accommodate the analysis of the coupling with phase change models such as that of [8]. We recall that in [8] it is necessary to augment the formulation for sake of the analysis (since one cannot complete the norms and conveniently control the terms that appear naturally in the formulation due to the use of a functional structure based only on Hilbert spaces) and currently we are not aware of non-augmented formulations specifically aimed for such a system. We also stress that the fixed-point strategy used herein differs substantially from that used in [8].

We have laid out the remainder of this part in the following manner. Before the end of this chapter we collect some preliminary notational formalisms and recall some auxiliary results to be employed throughout this part. In Chapter 7 we introduce the model problem, define auxiliary variables to be employed in the setting of the mixed-primal formulation, and eliminate the pressure unknown. In Chapter 8 we derive the continuous formulation, and adopt a fixed-point strategy to analyze the corresponding solvability. Recent results on perturbed saddle-point problems, as well as the Babuška–Brezzi theory, both in Banach spaces, are employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution. The associated Galerkin scheme is introduced in Chapter 9, where, under suitable assumptions on finite element subspaces, the discrete analogue of the methodology from Chapter 8, along with the Brouwer theorem instead of the Banach one, are utilized to prove existence of solution. In addition, *ad-hoc* Strang-type lemmas in Banach spaces are applied to derive a priori error estimates, specific finite element subspaces satisfying the aforementioned assumptions are introduced, and corresponding rates of convergence are established. Finally, the performance of the method is illustrated in Chapter 10 with several numerical examples.

Background and preliminary notation. Throughout this part, Ω is a given bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward unit normal at its boundary Γ is denoted $\boldsymbol{\nu}$. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^r(\Omega)$, with $r \in (1,\infty)$, and Sobolev spaces $\mathrm{W}^{s,r}(\Omega)$, with $s \geqslant 0$, endowed with the norms $\|\cdot\|_{0,r;\Omega}$ and $\|\cdot\|_{s,r;\Omega}$, respectively, whose vector and tensor versions are denoted in the same way. In particular, note that $\mathrm{W}^{0,r}(\Omega) = \mathrm{L}^r(\Omega)$, and that when $r = 2$ we simply write $\mathrm{H}^s(\Omega)$ in place of $\mathrm{W}^{s,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0;\Omega}$ and $\|\cdot\|_{s;\Omega}$, respectively. We also set $|\cdot|_{s;\Omega}$ for the seminorm of $\mathrm{H}^s(\Omega)$. In turn, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, $\mathrm{H}^{-1/2}(\Gamma)$ is its dual, and $\langle\cdot,\cdot\rangle$ denotes the duality pairing between them. On the other hand, by $\mathbf{S}$ and $\mathbb{S}$ we mean the corresponding vector and tensor counterparts, respectively, of a generic scalar functional space S. Furthermore, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, symmetric part of the gradient (also named strain rate tensor), divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathbf{e}(\mathbf{v}) := \frac{1}{2}\big(\nabla\mathbf{v} + (\nabla\mathbf{v})^{\mathrm{t}}\big), \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}, \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n},$$

where the superscript $(\cdot)^{\mathrm{t}}$ stands for the matrix transposition. In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\mathrm{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\,,$$

where $\mathbb{I}$ is the identity matrix in $\mathbb{R} := \mathrm{R}^{n\times n}$. On the other hand, for each $r \in [1, +\infty]$ we introduce the Banach space

$$\mathbb{H}(\mathbf{div}_r;\Omega) \;:=\; \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^r(\Omega) \Big\}\,,$$

which is endowed with the natural norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_r;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0;\Omega} \;+\; \|\mathbf{div}(\boldsymbol{\tau})\|_{0,r;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_r;\Omega)\,,$$

and recall that, proceeding as in [47, eq. (1.43), Sect. 1.3.4] one can prove that for each $r \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{v}\rangle \;=\; \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla\mathbf{v} \;+\; \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_r;\Omega) \times \mathbf{H}^1(\Omega)\,,$$

where $\langle\cdot,\cdot\rangle$ stands as well for the duality pairing between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. Finally, bear in mind that when $r = 2$, the Hilbert space $\mathbb{H}(\mathbf{div}_2;\Omega)$ and its norm $\|\cdot\|_{\mathbf{div}_2;\Omega}$ are simply denoted $\mathbb{H}(\mathbf{div};\Omega)$ and $\|\cdot\|_{\mathbf{div};\Omega}$, respectively.

Finally, the symbol $[\,\cdot,\cdot\,]$ will denote a duality pairing induced by an appropriately defined operator.

CHAPTER 7

The model problem

Let us consider the following Navier–Stokes–Brinkman equations coupled with a generalized energy equation, describing phase change mechanisms involving viscous fluids within porous media:

$$
\begin{aligned}
\eta(\varphi)\,\mathbf{u} - \lambda\,\mathbf{div}\big(\mu(\varphi)\,\mathbf{e}(\mathbf{u})\big) + \big(\nabla\mathbf{u}\big)\mathbf{u} + \nabla p = f(\varphi)\mathbf{k} & \quad \text{in} \quad \Omega\,,\\
\operatorname{div}(\mathbf{u}) = 0 & \quad \text{in} \quad \Omega\,,\\
-\rho\operatorname{div}\left(\kappa\nabla\varphi\right) + \mathbf{u}\cdot\nabla\varphi + \mathbf{u}\cdot\nabla s(\varphi) = 0 & \quad \text{in} \quad \Omega\,,\\
\mathbf{u} = \mathbf{u}_D \quad \text{and} \quad \varphi = \varphi_D & \quad \text{on} \quad \Gamma\,,\\
\int_\Omega p = 0\,, &
\end{aligned}
\tag{7.1}
$$

with $\lambda := \mathrm{R}e^{-1}$, $\rho := (C\,\mathrm{Pr})^{-1}$, where Re and Pr are the Reynolds and Prandtl numbers, respectively, κ and C are the non-dimensional heat conductivity tensor (here assumed isotropic) and specific heat, respectively, $\mathbf{k}$ stands for the unit vector pointing oppositely to gravity, and $\mathbf{u} : \Omega \to \mathrm{R}^n$, $p : \Omega \to \mathrm{R}$ and $\varphi : \Omega \to \mathrm{R}$, correspond to the velocity, pressure, and the temperature of the fluid flow, respectively. Finally, μ, η, s and f are the nonlinear viscosity, porosity, enthalpy and buoyancy terms, respectively, which depend on the temperature. Here $s(\varphi)$ denotes an enthalpy function that accounts for the latent heat of fusion, i.e., the energy needed to change the phase of a material (cf. [9]).

Typical constitutive forms for the permeability-viscosity-enthalpy functions include, for example, the well-known Carman–Kozeny, exponential, and polynomial laws

$$\eta(\phi) = \epsilon_1 \frac{(1-\phi)^2}{\phi^3 + \epsilon_2}, \qquad \mu(\varphi) = \epsilon_3 \exp(-\varphi^{\epsilon_4}), \qquad s(\varphi) = \begin{cases} s_1 \varphi & \text{if } \varphi < \varphi_\epsilon, \\ s_2 + s_3(\varphi - \varphi_\epsilon) & \text{if } \varphi \geqslant \varphi_\epsilon, \end{cases}$$

respectively, where $\phi(\varphi) = \hat{\epsilon}_1 + \hat{\epsilon}_2(1 + \tanh[\varphi - \varphi_\epsilon])$ is a sharp liquid fraction field (porosity). For the subsequent analysis, however, we assume a regular porosity-enthalpy hypothesis. In particular, this implies that the functions μ, η, s are uniformly bounded and Lipschitz continuous, which means that there exist positive constants μ_0, μ_1, η_0, η_1, s_0, s_1, L_μ, L_η and L_s, such that

$$\begin{aligned} &\mu_0 \leqslant \mu(\psi) \leqslant \mu_1, \quad &&|\mu(\psi) - \mu(\phi)| \leqslant L_\mu |\psi - \phi| \quad &&\forall\, \psi, \phi \in \mathrm{R}, \\ &\eta_0 \leqslant \eta(\psi) \leqslant \eta_1, \quad &&|\eta(\psi) - \eta(\phi)| \leqslant L_\eta |\psi - \phi| \quad &&\forall\, \psi, \phi \in \mathrm{R}, \\ &s_0 \leqslant s(\psi) \leqslant s_1, \quad &&|s(\psi) - s(\phi)| \leqslant L_s |\psi - \phi| \quad &&\forall\, \psi, \phi \in \mathrm{R}. \end{aligned} \tag{7.2}$$

Similar assumptions are placed on the buoyancy f: there exist positive constants C_f and L_f such that

$$|f(\psi)| \leqslant C_f |\psi|, \quad |f(\psi) - f(\phi)| \leqslant L_f |\psi - \phi| \qquad \forall\, \psi, \phi \in \mathrm{R}. \tag{7.3}$$

On the other hand, we will suppose that for every $\psi \in \mathrm{H}^1(\Omega)$, we have $s(\psi) \in \mathrm{H}^1(\Omega)$, and that there exist positive constants s_3 and $L_{\hat{s}}$ such that

$$|\nabla s(\psi)| \leqslant s_3 |\nabla \psi|, \quad |\nabla s(\psi) - \nabla s(\phi)| \leqslant L_{\hat{s}} |\psi - \phi| \qquad \forall\, \psi, \phi \in \mathrm{R}. \tag{7.4}$$

Finally, we suppose that κ and κ^{-1} are uniformly bounded and uniformly positive definite tensors, meaning that there exist positive constants κ_0, κ_1, $\widetilde{\kappa}_0$ and $\widetilde{\kappa}_1$ such that

$$|\kappa| \leqslant \kappa_1, \quad \kappa \mathbf{v} \cdot \mathbf{v} \geqslant \kappa_0 |\mathbf{v}|^2, \quad |\kappa^{-1}| \leqslant \widetilde{\kappa}_1, \quad \kappa^{-1} \mathbf{v} \cdot \mathbf{v} \geqslant \widetilde{\kappa}_0 |\mathbf{v}|^2 \quad \forall\, \mathbf{v} \in \mathrm{R}^n. \tag{7.5}$$

In turn, note that the incompressibility constraint imposes on $\mathbf{u}_D$ the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} \,=\, 0\,,$$

and we also recall (see, e.g., [61]) that uniqueness of pressure is ensured in the space

$$\mathrm{L}^2_0(\Omega) \,=\, \left\{ q \in \mathrm{L}^2(\Omega) : \int_\Omega q \,=\, 0 \right\}.$$

We now proceed as in [8] (see also [5, 29, 38, 48]), and transform (7.1) into an equivalent first-order system without pressure. We introduce the strain rate $\mathbf{t}$, vorticity $\boldsymbol{\gamma}$, and stress $\boldsymbol{\sigma}$ as auxiliary tensor unknowns

$$\mathbf{t} \,:=\, \mathbf{e}(\mathbf{u}) \,=\, \nabla\mathbf{u} - \boldsymbol{\gamma}\,, \qquad \boldsymbol{\gamma} \,:=\, \frac{1}{2}\big(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{t}}\big)\,, \qquad \boldsymbol{\sigma} \,:=\, \lambda\,\mu(\varphi)\,\mathbf{t} - (\mathbf{u}\otimes\mathbf{u}) - p\,\mathbb{I}\,, \tag{7.6}$$

so that, thanks to the incompressibility of the fluid, the first equation of (7.1) is rewritten as

$$\eta(\varphi)\,\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) \,=\, f(\varphi)\,\mathbf{k} \quad \text{in} \quad \Omega\,.$$

Moreover, the second equation of (7.1) (written in the form $\mathrm{tr}(\mathbf{t}) \,=\, 0$) together with (7.6), are equivalent to the pair of equations given by

$$\boldsymbol{\sigma}^{\mathrm{d}} \,=\, \lambda\,\mu(\varphi)\,\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{and} \quad p \,=\, -\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) \quad \text{in} \quad \Omega\,. \tag{7.7}$$

In summary, (7.1) can be equivalently reformulated as

$$\begin{aligned}
\mathbf{t} + \boldsymbol{\gamma} &= \nabla\mathbf{u} && \text{in} \quad \Omega\,,\\
\lambda\,\mu(\varphi)\,\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} &= \boldsymbol{\sigma}^{\mathrm{d}} && \text{in} \quad \Omega\,,\\
\eta(\varphi)\,\mathbf{u} - \mathbf{div}(\boldsymbol{\sigma}) &= f(\varphi)\mathbf{k} && \text{in} \quad \Omega\,,\\
-\rho\,\mathrm{div}(\kappa\,\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi + \mathbf{u}\cdot\nabla s(\varphi) &= 0 && \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_D \quad \text{and} \quad \varphi &= \varphi_D && \text{on} \quad \Gamma\,,
\end{aligned} \tag{7.8}$$

$$\int_{\Omega} \operatorname{tr}\big(\boldsymbol{\sigma} + (\mathbf{u} \otimes \mathbf{u})\big) = 0\,.$$

CHAPTER 8

Continuous weak formulation

In this chapter we use a Banach framework for the continuous weak formulation of (7.8) and analyze its solvability by means of a fixed-point approach. More precisely, we follow [48] and introduce a mixed method for the Navier–Stokes–Brinkman equations, whereas for the energy equation we propose a primal method, which, differently from [8,9], is formulated in a nonlinear version.

8.1 Mixed-primal approach

Note that the uncoupled Navier–Stokes–Brinkman problem – described by the first three equations of (7.8) and the respective boundary condition for the velocity – has been analyzed in detail in [48] by using the abstract results for perturbed saddle-point problems derived in [40], along with the Banach–Nečas–Babuška theorem. Following [48], we recall the definitions

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) := \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) = 0 \Big\} \quad \text{and} \quad \mathbb{L}^2_{\mathtt{skew}}(\Omega) := \Big\{ \boldsymbol{\delta} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\delta}^{\mathrm{t}} = -\boldsymbol{\delta} \Big\},$$

and the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}\,,$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega):\, \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) \,=\, 0\Big\}\,.$$

In particular, the unknown $\boldsymbol{\sigma}$ can be uniquely decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\,\mathbb{I}$, where $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and, from the last equation of (7.8), we have

$$c_0 \,:=\, \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, -\frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,.$$

Consequently, re-denoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, introducing the spaces

$$\mathbf{H} \,:=\, \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \qquad \mathbf{Q} \,:=\, \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\,,$$

setting the notations

$$\vec{\mathbf{t}} := (\mathbf{t},\boldsymbol{\sigma})\,,\ \vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau})\,,\ \vec{\mathbf{r}} := (\mathbf{r},\boldsymbol{\zeta})\in\mathbf{H}\,, \qquad \vec{\mathbf{u}} := (\mathbf{u},\boldsymbol{\gamma})\,,\ \vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta})\,,\ \vec{\mathbf{w}} := (\mathbf{w},\boldsymbol{\xi})\in\mathbf{Q}\,,$$

equipping $\mathbf{H}$ and $\mathbf{Q}$ with the norms

$$\begin{aligned} \|\vec{\mathbf{s}}\|_{\mathbf{H}} &:= \|\mathbf{s}\|_{0,\Omega} + \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} && \forall\,\vec{\mathbf{s}} := (\mathbf{s},\boldsymbol{\tau})\in\mathbf{H}\,,\\ \|\vec{\mathbf{v}}\|_{\mathbf{Q}} &:= \|\mathbf{v}\|_{0,4;\Omega} + \|\boldsymbol{\delta}\|_{0,\Omega} && \forall\,\vec{\mathbf{v}} := (\mathbf{v},\boldsymbol{\delta})\in\mathbf{Q}\,, \end{aligned}$$

following [48], and assuming that the temperature dependency of μ,η,f does not affect the aforementioned analysis, we arrive at the following formulation: Find $(\vec{\mathbf{t}},\vec{\mathbf{u}})\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} a_\varphi(\mathbf{t},\mathbf{s}) + b_1(\mathbf{s},\boldsymbol{\sigma}) \qquad\qquad + b(\mathbf{u};\mathbf{u},\mathbf{s}) &= 0\,,\\ b_2(\mathbf{t},\boldsymbol{\tau}) \qquad + \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{u}}) \qquad\qquad &= \langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{u}_D\rangle\,,\\ \mathbf{b}(\vec{\mathbf{t}},\vec{\mathbf{v}}) \qquad - \mathbf{c}_\varphi(\vec{\mathbf{u}},\vec{\mathbf{v}}) \qquad\qquad &= -\int_\Omega f(\varphi)\mathbf{k}\cdot\mathbf{v}\,, \end{aligned} \tag{8.1}$$

for all $(\vec{\mathbf{s}},\vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q}$, where the bilinear forms $a_\phi : \mathbb{L}^2_{\rm tr}(\Omega)\times\mathbb{L}^2_{\rm tr}(\Omega) \to \mathrm{R}$, $b_i : \mathbb{L}^2_{\rm tr}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \to \mathrm{R}$, $i\in\{1,2\}$, $\mathbf{b} : \mathbf{H}\times\mathbf{Q}\to\mathrm{R}$, and $\mathbf{c}_\phi : \mathbf{Q}\times\mathbf{Q}\to\mathrm{R}$, with $\phi\in\mathrm{H}^1(\Omega)$, are defined, respectively, as

$$\begin{aligned}
a_\phi(\mathbf{r},\mathbf{s}) &:= \lambda\int_\Omega \mu(\phi)\,\mathbf{r}:\mathbf{s} \qquad \forall\,\mathbf{r},\,\mathbf{s}\in\mathbb{L}^2_{\rm tr}(\Omega)\,,\\
b_1(\mathbf{s},\boldsymbol{\tau}) &:= -\int_\Omega \mathbf{s}:\boldsymbol{\tau}\,,\quad b_2(\mathbf{s},\boldsymbol{\tau}) := \int_\Omega \mathbf{s}:\boldsymbol{\tau}\,, \qquad \forall\,(\mathbf{s},\boldsymbol{\tau})\in\mathbb{L}^2_{\rm tr}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{v}}) &:= \int_\Omega \boldsymbol{\delta}:\boldsymbol{\tau} + \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\,,\\
\mathbf{c}_\phi(\vec{\mathbf{w}},\vec{\mathbf{v}}) &:= \int_\Omega \eta(\phi)\,\mathbf{w}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}}\in\mathbf{Q}\,,
\end{aligned}$$

whereas for each $\mathbf{w}\in\mathbf{L}^4(\Omega)$, $b(\mathbf{w};\cdot,\cdot) : \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\rm tr}(\Omega)\to\mathrm{R}$ is the bilinear form given by

$$b(\mathbf{w};\mathbf{v},\mathbf{s}) := -\int_\Omega (\mathbf{w}\otimes\mathbf{v}):\mathbf{s} \qquad \forall\,(\mathbf{v},\mathbf{s})\in\mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\rm tr}(\Omega)\,.$$

Next, and letting, for each $\phi\in\mathrm{H}^1(\Omega)$, $\mathbf{a}_\phi : \mathbf{H}\times\mathbf{H}\to\mathrm{R}$ be the bilinear form that arises from the block $\begin{pmatrix} a_\phi & b_1 \\ b_2 & \end{pmatrix}$ by adding the first two equations of (8.1), that is

$$\mathbf{a}_\phi(\vec{\mathbf{r}},\vec{\mathbf{s}}) := a_\phi(\mathbf{r},\mathbf{s}) + b_1(\mathbf{s},\boldsymbol{\zeta}) + b_2(\mathbf{r},\boldsymbol{\tau}) \qquad \forall\,\vec{\mathbf{r}},\,\vec{\mathbf{s}}\in\mathbf{H}\,,$$

we find that (8.1) can be rewritten as: Find $(\vec{\mathbf{t}},\vec{\mathbf{u}})\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned}
\mathbf{a}_\varphi(\vec{\mathbf{t}},\vec{\mathbf{s}}) + \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{u}}) + b(\mathbf{u};\mathbf{u},\mathbf{s}) &= \langle\boldsymbol{\tau}\,\boldsymbol{\nu},\mathbf{u}_D\rangle \qquad \forall\,\vec{\mathbf{s}}\in\mathbf{H}\,,\\
\mathbf{b}(\vec{\mathbf{t}},\vec{\mathbf{v}}) - \mathbf{c}_\varphi(\vec{\mathbf{u}},\vec{\mathbf{v}}) &= -\int_\Omega f(\varphi)\mathbf{k}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{Q}\,.
\end{aligned} \tag{8.2}$$

Moreover, letting now $\mathbf{A}_\phi : (\mathbf{H}\times\mathbf{Q})\times(\mathbf{H}\times\mathbf{Q})\to\mathrm{R}$ be the bilinear form that arises from the block $\begin{pmatrix} \mathbf{a}_\phi & \mathbf{b} \\ \mathbf{b} & -\mathbf{c}_\phi \end{pmatrix}$, for each $\phi\in\mathrm{H}^1(\Omega)$, by adding both equations of (8.2), that is

$$\mathbf{A}_\phi\big((\vec{\mathbf{r}},\vec{\mathbf{w}}),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big) := \mathbf{a}_\phi(\vec{\mathbf{r}},\vec{\mathbf{s}}) + \mathbf{b}(\vec{\mathbf{s}},\vec{\mathbf{w}}) + \mathbf{b}(\vec{\mathbf{r}},\vec{\mathbf{v}}) - \mathbf{c}_\phi(\vec{\mathbf{w}},\vec{\mathbf{v}}) \quad \forall\,(\vec{\mathbf{r}},\vec{\mathbf{w}}),\,(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\,,$$

we deduce that (8.2) (and hence (8.1)) can be stated equivalently as: Find $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\mathbf{A}_\varphi\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \;+\; b(\mathbf{u}; \mathbf{u}, \mathbf{s}) \;=\; \mathbf{F}_\varphi(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{8.3}$$

where, for each $\phi \in \mathrm{H}^1(\Omega)$, the functional $\mathbf{F}_\phi \in \big(\mathbf{H} \times \mathbf{Q}\big)'$ is defined by

$$\mathbf{F}_\phi(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \;:=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{u}_D \rangle \;-\; \int_\Omega f(\phi)\mathbf{k} \cdot \mathbf{v} \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,.$$

On the other hand, in order to derive a weak form for the energy equation, we recall that the injection $\mathrm{i}_4 \,:\, \mathrm{H}^1(\Omega) \to \mathrm{L}^4(\Omega)$ is continuous (cf. [61, Theorem 1.3.4]), which is valid in R^n, $n \in \{2, 3\}$:

$$\|\psi\|_{0,4;\Omega} \leqslant \|\mathrm{i}_4\|\; \|\psi\|_{1;\Omega} \quad \forall\, \psi \in \mathrm{H}^1(\Omega)\,. \tag{8.4}$$

Proceeding as in [8, Section 3.1], we test the fourth equation of (7.8) against $\psi \in \mathrm{H}^1(\Omega)$, integrate by parts, introduce the normal heat flux $\chi \;:=\; -\rho\kappa\nabla\varphi \cdot \boldsymbol{\nu} \in \mathrm{H}^{-1/2}(\Gamma)$ as a new unknown, and impose the Dirichlet boundary condition for φ in a weak sense, so that we get

$$\begin{aligned} \rho \int_\Omega \kappa\nabla\varphi \cdot \nabla\psi \;+\; \int_\Omega \psi\mathbf{u} \cdot \nabla\big(\varphi + s(\varphi)\big) \;+\; \langle \chi, \psi \rangle_\Gamma \;&=\; 0 && \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ \langle \xi, \varphi \rangle_\Gamma \;&=\; \langle \xi, \varphi_D \rangle_\Gamma && \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{8.5}$$

Here we readily note that, in order for the second term in the first equation of (8.5) to be well-defined, and thanks to the continuous injection i_4 (cf. (8.4)) and the assumption on s (cf. Chapter 7), we require that $(\mathbf{u}, \varphi)$ lies in $\mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega)$. Then, given $\mathbf{u} \in \mathbf{L}^4(\Omega)$, we now consider the following primal formulation for the energy equation: Find $(\varphi, \chi) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} [\mathcal{A}_\mathbf{u}(\varphi), \psi] \;+\; [\mathcal{B}(\psi), \chi] \;&=\; 0 && \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ [\mathcal{B}(\varphi), \xi] \;&=\; \mathcal{G}(\xi) && \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,, \end{aligned} \tag{8.6}$$

where given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, the nonlinear operator $\mathcal{A}_\mathbf{z} : \mathrm{H}^1(\Omega) \to \mathrm{H}^1(\Omega)'$ and the linear operator $\mathcal{B} : \mathrm{H}^1(\Omega) \to \mathrm{H}^{-1/2}(\Gamma)'$ are defined by

$$[\mathcal{A}_\mathbf{z}(\phi), \psi] \;:=\; \rho \int_\Omega \kappa\nabla\phi \cdot \nabla\psi \;+\; \int_\Omega \psi\mathbf{z} \cdot \nabla\big(\phi + s(\phi)\big) \qquad \forall\, \phi, \psi \in \mathrm{H}^1(\Omega)\,, \tag{8.7}$$

and

$$[\mathcal{B}(\phi), \xi] := \langle \xi, \phi \rangle_\Gamma \qquad \forall\, \phi \in \mathrm{H}^1(\Omega),\ \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,,$$

whereas $\mathcal{G} \in \mathrm{H}^{-1/2}(\Gamma)'$ is the functional given by

$$\mathcal{G}(\xi) = \langle \xi, \varphi_D \rangle_\Gamma \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,.$$

Summarizing, the non-augmented mixed-primal formulation for (7.8) reduces to (8.3) and (8.6), that is: Find $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) \in \mathbf{H} \times \mathbf{Q}$ and $(\varphi, \chi) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\mathbf{A}_\varphi\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \;+\; b(\mathbf{u}; \mathbf{u}, \mathbf{s}) \;= \mathbf{F}_\varphi(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,, \tag{8.8a}$$

$$[\mathcal{A}_{\mathbf{u}}(\varphi), \psi] \;+\; [\mathcal{B}(\psi), \chi] \;= 0 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \tag{8.8b}$$

$$[\mathcal{B}(\varphi), \xi] \;= \mathcal{G}(\xi) \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \tag{8.8c}$$

8.2 Fixed-point strategy

Let $\mathbf{S} : \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$ be defined by

$$\mathbf{S}(\mathbf{z}, \phi) \;=\; \mathbf{u} \qquad \forall\, (\mathbf{z}, \phi) \in \mathbf{L}^4(\Omega) \times \mathrm{H}^1(\Omega)\,,$$

where $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of

$$\mathbf{A}_\phi\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \;+\; b(\mathbf{z}; \mathbf{u}, \mathbf{s}) \;=\; \mathbf{F}_\phi(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,. \tag{8.9}$$

In turn, we let $\widetilde{\mathbf{S}} : \mathbf{L}^4(\Omega) \to \mathrm{H}^1(\Omega)$ be the operator given by

$$\widetilde{\mathbf{S}}(\mathbf{z}) \;:=\; \varphi \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega)\,,$$

where $(\varphi, \chi) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ is the unique solution (to be confirmed below) of

$$\begin{aligned} [\mathcal{A}_{\mathbf{z}}(\varphi), \psi] + [\mathcal{B}(\psi), \chi] &= 0 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ [\mathcal{B}(\varphi), \xi] &= \mathcal{G}(\xi) \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Gamma)\,. \end{aligned} \tag{8.10}$$

Then, we define the operator $\mathbf{T} : \mathbf{L}^4(\Omega) \to \mathbf{L}^4(\Omega)$ by

$$\mathbf{T}(\mathbf{z}) := \mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z})\big) \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega)\,. \tag{8.11}$$

Solving (8.8) is equivalent to seeking a fixed point of $\mathbf{T}$, that is, finding $\mathbf{z} \in \mathbf{L}^4(\Omega)$ such that

$$\mathbf{T}(\mathbf{z}) = \mathbf{z}\,. \tag{8.12}$$

8.3 Well-posedness of the uncoupled problems

We now show that the uncoupled problems (8.3) and (8.6) are well-posed. We remark again that the only difference between (8.3) and the formulation in [48] is that μ, η, f are temperature-dependent, but in virtue of assumptions (7.2) and (7.3), we can simply state the following result (with an almost verbatim proof).

Lemma 8.1. *For any* $(\mathbf{z}, \phi) \in \mathrm{L}^4(\Omega) \times \mathrm{H}^1(\Omega)$ *such that* $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\alpha_{\mathbf{A}}}{2}$*, problem* (8.9) *has a unique solution* $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{H} \times \mathbf{Q}$*, and hence* $\mathbf{S}(\mathbf{z}, \phi) := \mathbf{u} \in \mathbf{L}^4(\Omega)$ *is well-defined. Moreover, there exists* $C_{\mathbf{S}} > 0$*, depending only on* $\alpha_{\mathbf{A}}$*,* C_f *(cf.* (7.3)*),* $|\Omega|$ *and* $\|\mathbf{k}\|_\infty$*, such that*

$$\|\mathbf{S}(\mathbf{z}, \phi)\|_{0,4;\Omega} = \|\mathbf{u}\|_{0,4;\Omega} \leqslant \|(\vec{\mathbf{t}}, \vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \leqslant C_{\mathbf{S}} \left\{ \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\phi\|_{1;\Omega} \right\}. \tag{8.13}$$

Proof. It follows directly from [48, Lemma 3.5], with the exception that now there holds

$$\|\mathbf{F}_\phi\| \leqslant C_{\mathbf{F}} \Big\{ \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\phi\|_{1;\Omega} \Big\}, \tag{8.14}$$

where $C_{\mathbf{F}} := \max\big\{1,\, C_f |\Omega|^{1/4} \|\mathbf{k}\|_\infty\big\}$. □

The previous lemma suggests to consider the ball (which will be employed below in Chapter 8.4)

$$\mathbf{W_S} := \left\{ \mathbf{z} \in \mathbf{L}^4(\Omega) \,:\, \|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\alpha_{\mathbf{A}}}{2} \right\}.$$

It remains to prove that $\widetilde{\mathbf{S}}$ is well-defined. To this end, and in order to proceed similarly to [17], we state next an abstract result that will be utilized to establish the well-posedness of problem (8.10), and which can be viewed as a nonlinear version of the Babuška–Brezzi theory. We notice in advance that, while the above is valid within a Banach spaces framework, its application below is just for a particular Hilbertian case.

Theorem 8.2. *Let* H *and* Q *be separable and reflexive Banach spaces, with* H *uniformly convex, and let* $a : \mathrm{H} \to \mathrm{H}'$ *be a nonlinear operator and* $b \in \mathcal{L}(\mathrm{H}, \mathrm{Q}')$*. Let* V *be the null space of* b*, and assume that*

i) *a is Lipschitz-continuous, that is there exists $L > 0$ such that*

$$\|a(u) - a(v)\|_{\mathrm{H}'} \leqslant L\|u - v\|_{\mathrm{H}} \quad \forall\, u, v \in \mathrm{H}\,.$$

ii) *The family of operators $a(\cdot + t) : \mathrm{V} \to \mathrm{V}'$, with $t \in \mathrm{H}$, is uniformly strongly monotone, that is there exists a positive constant α such that*

$$[a(u + t) - a(v + t), u - v] \geqslant \alpha\|u - v\|^2_{\mathrm{H}} \quad \forall\, t \in \mathrm{H},\, \forall\, u, v \in \mathrm{V}\,. \tag{8.15}$$

iii) *There exists a positive constant β such that*

$$\sup_{\substack{v \in \mathrm{H} \\ v \neq 0}} \frac{[b(v), \tau]}{\|v\|_{\mathrm{H}}} \geqslant \beta\|\tau\|_{\mathrm{Q}} \quad \forall\, \tau \in \mathrm{Q}\,.$$

Then, for each $(F, G) \in \mathrm{H}' \times \mathrm{Q}'$ there exists a unique $(u, \sigma) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} [a(u), v] \,+\, [b(v), \sigma] &= [F, v] \qquad \forall\, v \in \mathrm{H}\,, \\ [b(u), \tau] &= [G, \tau] \qquad \forall\, \tau \in Q\,. \end{aligned}$$

Furthermore, there hold

$$\begin{aligned}\|u\|_{\mathrm{H}} &\leqslant \frac{1}{\alpha}\|F\|_{\mathrm{H}'} + \frac{1}{\beta}\left(1+\frac{L}{\alpha}\right)\|G\|_{\mathrm{Q}'} + \frac{1}{\alpha}\|a(0)\|_{\mathrm{H}'}\,, \quad \textit{and} \\ \|\sigma\|_{\mathrm{Q}} &\leqslant \frac{1}{\beta}\left(1+\frac{L}{\alpha}\right)\|F\|_{\mathrm{H}'} + \frac{L}{\beta^2}\left(1+\frac{L}{\alpha}\right)\|G\|_{\mathrm{Q}'} + \frac{1}{\beta}\left(1+\frac{L}{\alpha}\right)\|a(0)\|_{\mathrm{H}'}\,.\end{aligned} \tag{8.16}$$

Proof. It follows from a slight adaptation of [66, Proposition 2.3] with $p = 2$ (see also [31, Theorem 3.1] with $p_1 = p_2 = 2$). □

Next, in order to apply Theorem 8.2 to problem (8.10), we first observe, thanks to the duality between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, that the linear operator $\mathcal{B}$ and the functional $\mathcal{G}$ are bounded, that is

$$|[\mathcal{B}(\phi),\xi]| \;\leqslant\; \|\phi\|_{1;\Omega}\|\xi\|_{-1/2;\Gamma} \qquad \forall\,\phi\in\mathrm{H}^1(\Omega),\,\forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma)\,, \qquad \text{and} \tag{8.17a}$$

$$\|\mathcal{G}\| \;:=\; \sup_{\substack{\xi\in\mathrm{H}^{-1/2}(\Gamma)\\ \xi\neq\mathbf{0}}} \frac{|\mathcal{G}(\xi)|}{\|\xi\|_{-1/2;\Gamma}} \leqslant \|\varphi_D\|_{1/2;\Gamma}\,. \tag{8.17b}$$

We continue our analysis by proving that for each $\mathbf{z}\in\mathbf{L}^4(\Omega)$, $\mathcal{A}_{\mathbf{z}}$ is Lipschitz continuous.

Lemma 8.3. *There exists a positive constant $L_{\mathcal{A}}$, depending only on ρ, κ_1, $L_{\widehat{s}}$ and $\|\mathrm{i}_4\|$, such that*

$$\|\mathcal{A}_{\mathbf{z}}(\phi_1)-\mathcal{A}_{\mathbf{z}}(\phi_2)\|_{\mathrm{H}^1(\Omega)'} \leqslant L_{\mathcal{A}}\left(1\;+\;\|\mathbf{z}\|_{0,4;\Omega}\right)\|\phi_1-\phi_2\|_{1;\Omega}, \tag{8.18}$$

for all $\mathbf{z}\in\mathbf{L}^4(\Omega)$, and for all $\phi_1,\phi_2\in\mathrm{H}^1(\Omega)$.

Proof. Given $\mathbf{z}\in\mathbf{L}^4(\Omega)$ and $\phi_1,\phi_2,\psi\in\mathrm{H}^1(\Omega)$, using (8.7), the upper bounds (7.4) and (7.5), the Cauchy–Schwarz and triangle inequalities, and the continuous injection i_4 (cf. (8.4)), we deduce that

$$\begin{aligned}&|[\mathcal{A}_{\mathbf{z}}(\phi_1)-\mathcal{A}_{\mathbf{z}}(\phi_2),\psi]| \\ &\quad\leqslant\; \rho\left|\int_\Omega \kappa\nabla(\phi_1-\phi_2)\cdot\nabla\psi\right| \;+\; \left|\int_\Omega \psi\mathbf{z}\cdot\nabla\big((\phi_1-\phi_2)+(s(\phi_1)-s(\phi_2))\big)\right| \\ &\quad\leqslant\; \rho\kappa_1|\phi_1-\phi_2|_{1;\Omega}|\psi|_{1;\Omega} \;+\; \big(|\phi_1-\phi_2|_{1;\Omega}+|s(\phi_1)-s(\phi_2)|_{1;\Omega}\big)\|\mathbf{z}\|_{0,4;\Omega}\|\psi\|_{0,4;\Omega} \\ &\quad\leqslant\; \big(\rho\kappa_1+(1+L_{\widehat{s}})\|\mathrm{i}_4\|\|\mathbf{z}\|_{0,4;\Omega}\big)\|\phi_1-\phi_2\|_{1;\Omega}\|\psi\|_{1;\Omega}\,,\end{aligned}$$

which confirms the mentioned property on $\mathcal{A}_{\mathbf{z}}$ with $L_{\mathcal{A}} := \max\left\{\rho\kappa_1,\, (1+L_{\widehat{s}})\|\mathrm{i}_4\|\right\}$. $\square$

Now, aiming to prove that $\mathcal{A}_{\mathbf{z}}$ satisfies (8.15), we require the Friedrichs–Poincaré inequality, which establishes the existence of a positive constant c_{P}, depending only on Ω, such that

$$|\phi|^2_{1;\Omega} \geqslant c_{\mathrm{P}}\|\phi\|^2_{1;\Omega} \qquad \forall\, \phi \in \mathrm{H}^1_0(\Omega)\,. \tag{8.19}$$

In addition, we note that the kernel $\widetilde{\mathrm{V}}$ of the operator $\mathcal{B}$ is given by

$$\widetilde{\mathrm{V}} \,:=\, \left\{\phi \in \mathrm{H}^1(\Omega)\,:\quad \langle \xi, \phi\rangle_\Gamma = 0 \quad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\right\} \,=\, \mathrm{H}^1_0(\Omega)\,, \tag{8.20}$$

and introduce the ball

$$\mathbf{W}_{\widetilde{\mathbf{S}}} \,:=\, \left\{\mathbf{z} \in \mathbf{L}^4(\Omega)\,:\quad \|\mathbf{z}\|_{0,4;\Omega} \leqslant \frac{\rho\kappa_0 c_{\mathrm{P}}}{2(1+L_{\widehat{s}})\|\mathrm{i}_4\|}\right\}\,.$$

Then, the following result states that $\mathcal{A}_{\mathbf{z}}$ satisfies hypothesis ii) of Theorem 8.2.

Lemma 8.4. *There exists a positive constant $\alpha_{\mathcal{A}}$, depending only on ρ, κ_0 and c_{P}, such that for each $\mathbf{z} \in \mathbf{W}_{\widetilde{\mathbf{S}}}$, the family of operators $\mathcal{A}_{\mathbf{z}}(\,\cdot + \phi)$ with $\phi \in \mathrm{H}^1(\Omega)$, is uniformly strongly monotone in $\widetilde{\mathrm{V}}$:*

$$[\mathcal{A}_{\mathbf{z}}(\theta_1+\phi) - \mathcal{A}_{\mathbf{z}}(\theta_2+\phi), \theta_1-\theta_2] \geqslant \alpha_{\mathcal{A}}\|\theta_1-\theta_2\|^2_{1;\Omega} \qquad \forall\, \phi \in \mathrm{H}^1(\Omega)\,, \forall\, \theta_1,\theta_2 \in \widetilde{\mathrm{V}}\,. \tag{8.21}$$

Proof. Given $\mathbf{z} \in \mathbf{L}^4(\Omega)$, $\phi \in \mathrm{H}^1(\Omega)$ and $\theta_1,\theta_2 \in \widetilde{\mathrm{V}}$, using (8.7), (7.5), (7.4), Friedrichs–Poincaré inequality (8.19), the continuous injection (8.4), and the Cauchy–Schwarz inequality, it follows that

$$\begin{aligned}
&[\mathcal{A}_{\mathbf{z}}(\theta_1+\phi) - \mathcal{A}_{\mathbf{z}}(\theta_2+\phi), \theta_1-\theta_2] \\
&\qquad = \rho\int_\Omega \kappa\nabla(\theta_1-\theta_2)\cdot\nabla(\theta_1-\theta_2) \,+\, \int_\Omega (\theta_1-\theta_2)\mathbf{z}\cdot\nabla\Big((\theta_1-\theta_2) + \big(s(\theta_1+\phi) - s(\theta_2+\phi)\big)\Big) \\
&\qquad \geqslant \rho\kappa_0|\theta_1-\theta_2|^2_{1;\Omega} - \|\theta_1-\theta_2\|_{0,4;\Omega}\|\mathbf{z}\|_{0,4;\Omega}\big(|\theta_1-\theta_2|_{1;\Omega} + |s(\theta_1+\phi) - s(\theta_2+\phi)|_{1;\Omega}\big) \\
&\qquad \geqslant \big(\rho\kappa_0 c_{\mathrm{P}} - (1+L_{\widehat{s}})\|\mathrm{i}_4\|\|\mathbf{z}\|_{0,4;\Omega}\big)\|\theta_1-\theta_2\|^2_{1;\Omega}\,.
\end{aligned}$$

In this way, defining $\alpha_{\mathcal{A}} := \dfrac{\rho\kappa_0 c_{\mathrm{P}}}{2}$, we obtain

$$[\mathcal{A}_{\mathbf{z}}(\theta_1+\phi)-\mathcal{A}_{\mathbf{z}}(\theta_2+\phi),\theta_1-\theta_2] \geqslant \big(2\alpha_{\mathcal{A}}-(1+L_{\hat{s}})\|\mathrm{i}_4\|\|\mathbf{z}\|_{0,4;\Omega}\big)\|\theta_1-\theta_2\|^2_{1;\Omega}\,,$$

from which, using that $\mathbf{z}\in\mathbf{W}_{\widetilde{\mathbf{S}}}$, we readily conclude the proof. □

We observe here that, instead of imposing $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \alpha_{\mathcal{A}}/\big((1+L_{\hat{s}})\|\mathrm{i}_4\|\big)$, we could have assumed that $\|\mathbf{z}\|_{0,4;\Omega} \leqslant 2\delta\alpha_{\mathcal{A}}/\big((1+L_{\hat{s}})\|\mathrm{i}_4\|\big)$, with $\delta\in(0,1)$. Then choosing δ closer to 1, the larger the resulting range of $\|\mathbf{z}\|_{0,4;\Omega}$, but then the strong monotonicity constant approaches 0. Conversely, the closer δ to 0, the smaller the range for $\|\mathbf{z}\|_{0,4;\Omega}$, but then the strong monotonicity constant approaches $2\alpha_{\mathcal{A}}$. Hence the choice $\delta=\frac{1}{2}$ aims to balance both aspects.

We complete the verification of the hypotheses of Theorem 8.2 with the inf-sup condition for $\mathcal{B}$, which can be found in [47, section 2.4.4].

Lemma 8.5. *The following inf-sup condition holds with inf-sup constant equal to 1*

$$\sup_{\substack{\psi\in\mathrm{H}^1(\Omega)\\ \psi\neq 0}}\frac{[\mathcal{B}(\psi),\xi]}{\|\psi\|_{1;\Omega}} \geqslant \|\xi\|_{-1/2;\Gamma} \qquad \forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma)\,.$$

Now, we are in position to establish the unique solvability of the nonlinear problem (8.10).

Lemma 8.6. *For each* $\mathbf{z}\in\mathbf{W}_{\widetilde{\mathbf{S}}}$*, the problem* (8.10) *has a unique solution* $(\varphi,\chi)\in\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$*, and hence* $\widetilde{\mathbf{S}}(\mathbf{z}) := \varphi\in\mathrm{H}^1(\Omega)$ *is well-defined. Moreover, there exist positive constants* $C_{\widetilde{\mathbf{S}}}$ *and* $\widetilde{C}_{\widetilde{\mathbf{S}}}$*, depending only on* $L_{\mathcal{A}}$ *(cf. proof of Lemma* (8.3)*) and* $\alpha_{\mathcal{A}}$ *(cf. proof of Lemma* 8.4)*, such that*

$$\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1;\Omega} := \|\varphi\|_{1;\Omega} \leqslant C_{\widetilde{\mathbf{S}}}\,\|\varphi_D\|_{1/2;\Gamma} \quad \textit{and} \quad \|\chi\|_{-1/2;\Gamma} \leqslant \widetilde{C}_{\widetilde{\mathbf{S}}}\,\|\varphi_D\|_{1/2;\Gamma}\,. \tag{8.22}$$

Proof. We first recall from (8.17a) and (8.17b) that $\mathcal{B}$ and $\mathcal{G}$ are linear and bounded. Thus, using Lemmas 8.3, 8.4 and 8.5, and applying Theorem 8.2 to problem (8.9) implies the well-definedness of the operator $\widetilde{\mathbf{S}}$ for each $\mathbf{z}\in\mathbf{W}_{\widetilde{\mathbf{S}}}$. Moreover, noting that $\mathcal{A}_{\mathbf{z}}(0)\in\mathrm{H}^1(\Omega)'$ is the null functional, recalling from Lemma 8.5 that the inf-sup constant is 1, and denoting

$\widetilde{L}_{\mathcal{A}} := L_{\mathcal{A}}(1+\alpha_{\mathcal{A}})$, the a priori estimate (8.16) yields

$$\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1;\Omega} \,=\, \|\varphi\|_{1;\Omega} \leqslant \Big(1+\frac{\widetilde{L}_{\mathcal{A}}}{\alpha_{\mathcal{A}}}\Big)\|\mathcal{G}\| \quad \text{and} \quad \|\chi\|_{-1/2;\Gamma} \leqslant \widetilde{L}_{\mathcal{A}}\Big(1+\frac{\widetilde{L}_{\mathcal{A}}}{\alpha_{\mathcal{A}}}\Big)\|\mathcal{G}\|\,,$$

which, along with the upper bound of $\|\mathcal{G}\|$ (cf. (8.17b)), implies (8.22). □

8.4 Solvability analysis

Consider now the ball

$$\mathbf{W} := \mathbf{W}_{\mathbf{S}} \,\cap\, \mathbf{W}_{\widetilde{\mathbf{S}}} \,=\, \Big\{\mathbf{z} \in \mathbf{L}^4(\Omega) \,:\, \|\mathbf{z}\|_{0,4;\Omega} \,\leqslant\, \varrho\Big\} \tag{8.23}$$

of radius

$$\varrho \,:=\, \min\left\{\frac{\alpha_{\mathbf{A}}}{2}\,,\, \frac{\alpha_{\mathcal{A}}}{(1+L_{\widehat{s}})\|\mathrm{i}_4\|}\right\}\,.$$

We proceed to prove that, under sufficiently small data, $\mathbf{T}$ maps $\mathbf{W}$ into itself.

Lemma 8.7. *Assume that the data satisfy*

$$C_{\mathbf{T}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/2;\Gamma}\Big\} \,\leqslant\, \varrho\,, \tag{8.24}$$

where $C_{\mathbf{T}} \,:=\, C_{\mathbf{S}}\max\big\{1, C_{\widetilde{\mathbf{S}}}\big\}$, *and* $C_{\mathbf{S}}$ *and* $C_{\widetilde{\mathbf{S}}}$ *are the constants specified in Lemmas* 8.1 *and* 8.6*. Then, there holds* $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$*.*

Proof. Given $\mathbf{z} \in \mathbf{W}$, we have that $\mathbf{z}$ satisfies the well-defined conditions for $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, and hence for $\mathbf{T}$. Moreover, the corresponding estimate (8.13) yields

$$\|\mathbf{T}(\mathbf{z})\|_{0,4;\Omega} \,=\, \|\mathbf{S}\big(\mathbf{z},\widetilde{\mathbf{S}}(\mathbf{z})\big)\|_{0,4;\Omega} \,\leqslant\, C_{\mathbf{S}}\left\{\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1;\Omega}\right\}\,.$$

Then, bounding $\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1;\Omega}$ in the foregoing inequality according to the estimate (8.22) and using the assumption (8.24), we get $\|\mathbf{T}(\mathbf{z})\|_{0,4;\Omega} \,\leqslant\, \varrho$, which completes the proof. □

We now prove that $\mathbf{T}$ is Lipschitz continuous (it suffices to show that $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ satisfy this property). For $\mathbf{S}$ we assume the further regularity $\mathbf{u}_D \in \mathbf{H}^{1/2+\epsilon}(\Gamma)$ for some $\epsilon \in [1/2, 1)$ (when $n = 2$) or $\epsilon \in [3/4, 1)$ (when $n = 3$), and that for each $(\mathbf{z}, \phi) \in \mathbf{W_S} \times \mathrm{H}^1(\Omega)$ there holds $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) = \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \Big(\big(\mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{H}^\epsilon(\Omega)\big) \times \big(\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \cap \mathbb{H}^\epsilon(\Omega)\big)\Big) \times \Big(\big(\mathbf{L}^4(\Omega) \cap \mathbf{W}^{\epsilon,4}(\Omega)\big) \times \big(\mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{H}^\epsilon(\Omega)\big)\Big)$ with $\mathbf{S}(\mathbf{z}, \phi) := \mathbf{u}$ and

$$\|\mathbf{t}\|_{\epsilon;\Omega} + \|\boldsymbol{\sigma}\|_{\epsilon;\Omega} + \|\mathbf{u}\|_{\epsilon,4;\Omega} + \|\boldsymbol{\gamma}\|_{\epsilon} \leqslant c_{\mathbf{S}} \Big\{ \|\mathbf{u}_D\|_{1/2+\epsilon;\Gamma} + \|\phi\|_{1;\Omega} \Big\}, \tag{8.25}$$

with a positive constant $c_{\mathbf{S}}$ independent of the given $(\mathbf{z}, \phi)$. The chosen range for ϵ will be clarified in the proof of the following lemma.

Lemma 8.8. *There exists a positive constant $L_{\mathbf{S}}$, depending on $|\Omega|$, $\|\mathbf{k}\|_\infty$, L_μ, L_η, $\|\mathrm{i}_4\|$, $\alpha_{\mathbf{A}}$ and ϵ, such that*

$$\begin{aligned}\|\mathbf{S}(\mathbf{z}_1, \phi_1) - \mathbf{S}(\mathbf{z}_2, \phi_2)\|_{0,4;\Omega} \\ \leqslant L_{\mathbf{S}} \Big\{ \|\mathbf{S}(\mathbf{z}_2, \phi_2)\|_{0,4;\Omega} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega} + \big(\|\mathbf{t}\|_{\epsilon;\Omega} + \|\mathbf{S}(\mathbf{z}_2, \phi_2)\|_{0,4;\Omega} + L_f \big) \|\phi_1 - \phi_2\|_{1;\Omega} \Big\},\end{aligned} \tag{8.26}$$

for all $(\mathbf{z}_1, \phi_1), (\mathbf{z}_2, \phi_2) \in \mathbf{W_S} \times \mathrm{H}^1(\Omega)$.

Proof. Given $(\mathbf{z}_i, \phi_i) \in \mathbf{W_S} \times \mathrm{H}^1(\Omega)$, for each $i \in \{1, 2\}$, we let $\mathbf{S}(\mathbf{z}_i, \phi_i) := \mathbf{u}_i$, where $(\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i) := \big((\mathbf{t}_i, \boldsymbol{\sigma}_i), (\mathbf{u}_i, \boldsymbol{\gamma}_i)\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (8.9) with $(\mathbf{z}, \phi) := (\mathbf{z}_i, \phi_i)$, that is

$$\mathbf{A}_{\phi_i}\big((\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + b(\mathbf{z}_i; \mathbf{u}_i, \mathbf{s}) = \mathbf{F}_{\phi_i}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,. \tag{8.27}$$

Now, applying the inf-sup condition for the bilinear form in the left hand side of the foregoing equation (cf. [48, eq. (3.64)]) with $(\mathbf{z}, \phi) = (\mathbf{z}_1, \phi_1)$ to $(\vec{\mathbf{r}}, \vec{\mathbf{w}}) := (\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)$, we obtain

$$\|(\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{A}}} \sup_{\substack{(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H}\times\mathbf{Q} \\ (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \neq \mathbf{0}}} \frac{\mathbf{A}_{\phi_1}\big((\vec{\mathbf{t}}_1, \vec{\mathbf{u}}_1) - (\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) + b(\mathbf{z}_1; \mathbf{u}_1 - \mathbf{u}_2, \mathbf{s})}{\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}},$$

from which, adding and subtracting $b(\mathbf{z}_2;\mathbf{u}_2,\mathbf{s})$, and then employing (8.27), we obtain

$$\begin{aligned}&\|(\vec{\mathbf{t}}_1,\vec{\mathbf{u}}_1)-(\vec{\mathbf{t}}_2,\vec{\mathbf{u}}_2)\|_{\mathbf{H}\times\mathbf{Q}}\\ &\quad\leqslant\frac{2}{\alpha_{\mathbf{A}}}\sup_{\substack{(\vec{\mathbf{s}},\vec{\mathbf{v}})\in\mathbf{H}\times\mathbf{Q}\\(\vec{\mathbf{s}},\vec{\mathbf{v}})\neq\mathbf{0}}}\frac{\big(\mathbf{A}_{\phi_2}-\mathbf{A}_{\phi_1}\big)\big((\vec{\mathbf{t}}_2,\vec{\mathbf{u}}_2),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)\;+\;b(\mathbf{z}_2-\mathbf{z}_1;\mathbf{u}_2,\mathbf{s})\;+\;\big(\mathbf{F}_{\phi_1}-\mathbf{F}_{\phi_2}\big)(\vec{\mathbf{s}},\vec{\mathbf{v}})}{\|(\vec{\mathbf{s}},\vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}}\,.\end{aligned}\tag{8.28}$$

We now estimate the right-hand side of (8.28) by separating its numerator into three suitable terms. Indeed, we first observe that

$$\begin{aligned}&\big(\mathbf{A}_{\phi_2}-\mathbf{A}_{\phi_1}\big)\big((\vec{\mathbf{t}}_2,\vec{\mathbf{u}}_2),(\vec{\mathbf{s}},\vec{\mathbf{v}})\big)\\ &\qquad=\big(a_{\phi_2}-a_{\phi_1}\big)(\mathbf{t}_2,\mathbf{s})\;+\;\big(\mathbf{c}_{\phi_1}-\mathbf{c}_{\phi_2}\big)(\vec{\mathbf{u}}_2,\vec{\mathbf{v}})\\ &\qquad=\lambda\int_\Omega\big(\mu(\phi_2)-\mu(\phi_1)\big)\,\mathbf{t}_2:\mathbf{s}\;+\;\int_\Omega\big(\eta(\phi_1)-\eta(\phi_2)\big)\,\mathbf{u}_2\cdot\mathbf{v}\\ &\qquad\leqslant\lambda\,L_\mu\,\|\phi_2-\phi_1\|_{2p;\Omega}\|\mathbf{t}_2\|_{2q;\Omega}\|\mathbf{s}\|_{0;\Omega}+\;L_\eta\|\phi_1-\phi_2\|_{0;\Omega}\|\mathbf{u}_2\|_{0,4;\Omega}\|\mathbf{v}\|_{0,4;\Omega}\,,\end{aligned}\tag{8.29}$$

where $p,q\in[1,\infty)$ are such that $\frac{1}{p}+\frac{1}{q}=1$. In this way, bearing in mind the further regularity (8.25), we recall that the Sobolev embedding Theorem [1, Theorem 4.12] establishes the continuous injection $\mathrm{i}_\epsilon:\mathbb{H}^\epsilon(\Omega)\to\mathbb{L}^{\epsilon^*}(\Omega)$, where $\epsilon^*=\begin{cases}\frac{2}{1-\epsilon} & \text{if } n=2\,,\\ \frac{6}{3-2\epsilon} & \text{if } n=3\end{cases}$. Thus, choosing q such that $2q=\epsilon^*$, there holds $\mathbf{t}_2\in\mathbb{L}^{2q}(\Omega)$ and

$$\|\mathbf{t}_2\|_{0,2q;\Omega}\leqslant\|\mathrm{i}_\epsilon\|\,\|\mathbf{t}_2\|_{\epsilon;\Omega}\,.\tag{8.30}$$

In turn, with that choice of $2q$, we obtain that $2p=n/\epsilon$ and hence, using now that for the specified ranges of ϵ the injection $\widetilde{\mathrm{i}}_\epsilon$ of $\mathrm{L}^4(\Omega)$ into $\mathrm{L}^{n/\epsilon}(\Omega)$ is continuous, and applying that $\mathrm{H}^1(\Omega)$ is continuously embedded into $\mathrm{L}^4(\Omega)$ (cf. (8.4)), there holds

$$\|\varphi_2-\varphi_1\|_{0,n/\epsilon;\Omega}\;\leqslant\;\|\widetilde{\mathrm{i}}_\epsilon\|\,\|\varphi_2-\varphi_1\|_{0,4;\Omega}\;\leqslant\;\|\widetilde{\mathrm{i}}_\epsilon\|\,\|\mathrm{i}_4\|\,\|\varphi_2-\varphi_1\|_{1;\Omega}\,.\tag{8.31}$$

Then, putting (8.30) and (8.31) back into (8.29), and denoting

$$L_{\mathbf{A}}\;:=\;\max\big\{\lambda\,L_\mu\,\|\widetilde{\mathrm{i}}_\epsilon\|\,\|\mathrm{i}_4\|\,\|i_\epsilon\|,\,L_\eta\big\}\,,$$

gives

$$\big(\mathbf{A}_{\phi_2} - \mathbf{A}_{\phi_1}\big)\big((\vec{\mathbf{t}}_2, \vec{\mathbf{u}}_2), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \leqslant L_{\mathbf{A}}\Big\{\|\mathbf{t}_2\|_{\epsilon;\Omega} \,+\, \|\mathbf{u}_2\|_{0,4;\Omega}\Big\}\|\phi_2 - \phi_1\|_{1;\Omega}\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}\,. \tag{8.32}$$

Next, it is easy to see that

$$b(\mathbf{z}_2 - \mathbf{z}_1; \mathbf{u}_2, \mathbf{s}) \,\leqslant\, \|\mathbf{u}_2\|_{0,4;\Omega}\|\mathbf{z}_2 - \mathbf{z}_1\|_{0,4;\Omega}\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}\,. \tag{8.33}$$

Now, thanks to the properties of f (cf. (7.3)) together with the Cauchy–Schwarz inequality, we have

$$\big(\mathbf{F}_{\phi_1} - \mathbf{F}_{\phi_2}\big)(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \,\leqslant\, L_f L_{\mathbf{F}}\|\phi_1 - \phi_2\|_{1;\Omega}\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}\,, \tag{8.34}$$

with $L_{\mathbf{F}} := |\Omega|^{1/4}\|\mathbf{k}\|_\infty$. Finally, replacing (8.32), (8.33) and (8.34) back into (8.28), and then simplifying by $\|(\vec{\mathbf{s}}, \vec{\mathbf{v}})\|_{\mathbf{H}\times\mathbf{Q}}$, we obtain (8.26) with

$$L_{\mathbf{S}} \,:=\, \frac{2}{\alpha_{\mathbf{A}}}\max\big\{L_{\mathbf{A}},\, 1,\, L_{\mathbf{F}}\big\}\,.$$

□

We now focus on proving the Lipschitz-continuity of $\widetilde{\mathbf{S}}$.

Lemma 8.9. *There exists a positive constant* $L_{\widetilde{\mathbf{S}}}$*, depending only on* s_3*,* $\|\mathrm{i}_4\|$ *and* $\alpha_{\mathcal{A}}$ *(cf. proof of Lemma* 8.4*), such that for all* $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}_{\widetilde{\mathbf{S}}}$*, there holds*

$$\|\widetilde{\mathbf{S}}(\mathbf{z}_1) - \widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega} \,\leqslant\, L_{\widetilde{\mathbf{S}}}\,\|\widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega}\|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}\,. \tag{8.35}$$

Proof. Given $\mathbf{z}_i \in \mathbf{W}_{\widetilde{\mathbf{S}}}$, $i \in \big\{1, 2\big\}$, we let $\widetilde{\mathbf{S}}(\mathbf{z}_i) \,=\, \varphi_i$, where $(\varphi_i, \chi_i) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ is the unique solution of (8.10) with $\mathbf{z} := \mathbf{z}_i$, that is

$$\begin{aligned}
[\mathcal{A}_{\mathbf{z}_i}(\varphi_i), \psi] \,+\, [\mathcal{B}(\psi), \chi_i] \;&=\; 0 \qquad\quad \forall\, \psi \in \mathrm{H}^1(\Omega)\,,\\
[\mathcal{B}(\varphi_i), \xi] \;&=\; \mathcal{G}(\xi) \qquad \forall\, \xi \in \mathrm{H}^{1/2}(\Gamma)\,.
\end{aligned}$$

Then, subtracting the two problems, we obtain

$$\begin{aligned} [\mathcal{A}_{\mathbf{z}_1}(\varphi_1) - \mathcal{A}_{\mathbf{z}_2}(\varphi_2), \psi] + [\mathcal{B}(\psi), \chi_1 - \chi_2] &= 0 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ [\mathcal{B}(\varphi_1 - \varphi_2), \xi] &= 0 \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \end{aligned} \tag{8.36}$$

It follows from the second equation of (8.36) that $\varphi_1 - \varphi_2 \in \widetilde{\mathrm{V}}$ (cf. (8.20)), and hence, using that $\mathcal{A}_{\mathbf{z}_1}$ is uniformly strongly monotone on $\widetilde{\mathrm{V}}$ (cf. (8.21)), with $\varphi_2 \in \mathrm{H}^1(\Omega)$ and $0, \varphi_1 - \varphi_2 \in \widetilde{\mathrm{V}}$, we get

$$\alpha_{\mathcal{A}} \|\varphi_1 - \varphi_2\|_{1;\Omega}^2 \leqslant [\mathcal{A}_{\mathbf{z}_1}(\varphi_1) - \mathcal{A}_{\mathbf{z}_1}(\varphi_2), \varphi_1 - \varphi_2]\,. \tag{8.37}$$

Now, using (8.7), adding and subtracting $\mathcal{A}_{\mathbf{z}_2}(\varphi_2)$ in the first term on the right-hand side of (8.37), using the first equation of (8.36) and Cauchy–Schwarz and Hölder inequalities, we have

$$\begin{aligned} \alpha_{\mathcal{A}} \|\varphi_1 - \varphi_2\|_{1;\Omega}^2 &\leqslant [\mathcal{A}_{\mathbf{z}_1}(\varphi_1) - \mathcal{A}_{\mathbf{z}_2}(\varphi_2), \varphi_1 - \varphi_2] - [\mathcal{A}_{\mathbf{z}_1}(\varphi_2) - \mathcal{A}_{\mathbf{z}_2}(\varphi_2), \varphi_1 - \varphi_2] \\ &\leqslant \left| \int_\Omega (\varphi_1 - \varphi_2)(\mathbf{z}_1 - \mathbf{z}_2) \cdot \nabla\big(\varphi_2 + s(\varphi_2)\big) \right| \\ &\leqslant \|\varphi_1 - \varphi_2\|_{0,4;\Omega} |\varphi_2 + s(\varphi_2)|_{1;\Omega} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}\,. \end{aligned}$$

Then, using the triangle inequality, the upper bound for the gradient of s (cf. (7.4)) and (8.4), we get

$$\|\varphi_1 - \varphi_2\|_{1;\Omega} \leqslant \frac{(1 + s_3)\|\mathrm{i}_4\|}{\alpha_{\mathcal{A}}} \|\widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}\,,$$

which yields (8.35) and ends the proof. □

As a consequence of the previous lemmas, we establish now the Lipschitz-continuity of $\mathbf{T}$.

Lemma 8.10. *There exists a positive constant $L_{\mathbf{T}}$, depending only on $C_{\widetilde{\mathbf{S}}}$, $C_{\mathbf{T}}$, $c_{\mathbf{S}}$, $L_{\mathbf{S}}$, and $L_{\widetilde{\mathbf{S}}}$, such that for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}$, there holds*

$$\|\mathbf{T}(\mathbf{z}_1) - \mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega} \leqslant L_{\mathbf{T}} \Big\{ \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma} + C(\mathbf{u}_D, \varphi_D)\, \|\varphi_D\|_{1/2;\Gamma} \Big\} \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}\,, \tag{8.38}$$

where

$$C(\mathbf{u}_D, \varphi_D) := \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\mathbf{u}_D\|_{1/2+\epsilon;\Gamma} + \|\varphi_D\|_{1/2;\Gamma} + L_f\,. \tag{8.39}$$

Proof. Given $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}$, and according to (8.11) and (8.26), we first obtain

$$\begin{aligned} \|\mathbf{T}(\mathbf{z}_1) - \mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega} \;&=\; \|\mathbf{S}\big(\mathbf{z}_1, \widetilde{\mathbf{S}}(\mathbf{z}_1)\big) - \mathbf{S}\big(\mathbf{z}_2, \widetilde{\mathbf{S}}(\mathbf{z}_2)\big)\|_{0,4;\Omega} \\ &\leqslant\; L_{\mathbf{S}}\Big\{\|\mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega}\|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega} \;+\; \Big(\|\mathbf{t}_2\|_{\epsilon;\Omega} + \|\mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega} + L_f\Big)\|\widetilde{\mathbf{S}}(\mathbf{z}_1) - \widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega}\Big\}\,, \end{aligned} \tag{8.40}$$

where for each $i \in \{1,2\}$, $(\vec{\mathbf{t}}_i, \vec{\mathbf{u}}_i) \;:=\; \big((\mathbf{t}_i, \boldsymbol{\sigma}_i), (\mathbf{u}_i, \boldsymbol{\gamma}_i)\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution of (8.9) with $\big(\mathbf{z}_i, \widetilde{\mathbf{S}}(\mathbf{z}_i)\big)$ instead of $(\mathbf{z}, \phi)$. In turn, the a priori estimate for $\widetilde{\mathbf{S}}$ (cf. (8.22)) holds

$$\|\widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega} \;\leqslant\; C_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2;\Gamma}\,, \tag{8.41}$$

whereas the Lipschitz-continuity of $\widetilde{\mathbf{S}}$ (cf. (8.35)) with (8.41), gives

$$\|\widetilde{\mathbf{S}}(\mathbf{z}_1) - \widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1;\Omega} \;\leqslant\; L_{\widetilde{\mathbf{S}}}C_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2;\Gamma}\|\mathbf{z}_1 - \mathbf{z}_2\|_{0,4;\Omega}\,, \tag{8.42}$$

and the a priori estimates for $\mathbf{T}$ (cf. Lemma 8.7) yields

$$\|\mathbf{T}(\mathbf{z}_2)\|_{0,4;\Omega} \;\leqslant\; C_{\mathbf{T}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\}\,, \tag{8.43}$$

and finally, replacing (8.41) on the regularity assumption (8.25) for $\mathbf{t}_2$, we find that

$$\|\mathbf{t}_2\|_{\epsilon;\Omega} \;\leqslant\; c_{\mathbf{S}}\Big\{\|\mathbf{u}_D\|_{1/2+\epsilon;\Gamma} + C_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2;\Gamma}\Big\}\,. \tag{8.44}$$

In this way, replacing (8.42), (8.43) and (8.44) in (8.40), and performing several algebraic manipulations aiming to simplify the whole writing, we are lead to (8.38) with

$$L_{\mathbf{T}} \;:=\; 2L_{\mathbf{S}}\max\Big\{\,c_{\mathbf{S}},\, c_{\mathbf{S}}C_{\widetilde{\mathbf{S}}},\, C_{\mathbf{T}},\, 1\Big\}\max\Big\{1,\, L_{\widetilde{\mathbf{S}}}C_{\widetilde{\mathbf{S}}}\Big\}\,.$$

□

The main result of this chapter is given as follows.

Theorem 8.11. *Assume the data satisfies* (8.24), *that is*

$$C_{\mathbf{T}}\Big\{\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\} \leqslant \varrho\,,$$

and

$$L_{\mathbf{T}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma} + C(\mathbf{u}_D,\varphi_D)\|\varphi_D\|_{1/2;\Gamma}\Big\} < 1\,. \tag{8.45}$$

Then $\mathbf{T}$ *has a unique fixed point* $\mathbf{u} \in \mathbf{W}$. *Equivalently, the coupled problem* (8.8) *has a unique solution* $(\vec{\mathbf{t}},\vec{\mathbf{u}}) := \big((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma})\big) \in \mathbf{H}\times\mathbf{Q}$ *and* $(\varphi,\chi) \in \mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$, *with* $\mathbf{u} \in \mathbf{W}$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}},\vec{\mathbf{u}})\|_{\mathbf{H}\times\mathbf{Q}} \leqslant C_{\mathbf{T}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\}\,, \tag{8.46a}$$

$$\|\varphi\|_{1;\Omega} \leqslant C_{\widetilde{\mathbf{S}}}\,\|\varphi_D\|_{1/2;\Gamma} \quad \textit{and} \quad \|\chi\|_{-1/2;\Gamma} \leqslant \widetilde{C}_{\widetilde{\mathbf{S}}}\,\|\varphi_D\|_{1/2;\Gamma}\,. \tag{8.46b}$$

Proof. It is clear, thanks to assumption (8.45) and Lemma 8.10, that $\mathbf{T}$ is a contraction, which together with Lemma 8.7, proves that the fixed point operator $\mathbf{T}$ satisfies the hypotheses of Banach's fixed-point theorem, which implies the solvability of the problem (8.12), equivalently, the solvability of (8.8). Consequently, the a priori estimates (8.46a) and (8.46b) follow from (8.13) and (8.22), respectively. □

CHAPTER 9

The Galerkin scheme

In this chapter, we introduce and analyze the Galerkin scheme associated with (8.8). The solvability of this scheme is addressed following basically the same techniques employed throughout Chapter 8. To this end, we let $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, H_h^{φ} and H_h^{χ} be arbitrary finite element subspaces of $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$, respectively. Hereafter, $h := \max\left\{h_K : \quad K \in \mathcal{T}_h\right\}$ stands for the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$. Specific finite element subspaces satisfying suitable hypotheses to be introduced along the analysis will be provided later on in Chapter 9.5. Then, letting

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \tag{9.1}$$

defining the product spaces

$$\mathbf{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}}\,, \quad \text{and} \quad \mathbf{Q}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\gamma}\,, \tag{9.2}$$

and setting the notations

$$\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h)\,,\ \vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h)\,,\ \vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\zeta}_h) \in \mathbf{H}_h\,,$$

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \boldsymbol{\gamma}_h)\,,\ \vec{\mathbf{v}}_h := (\mathbf{v}_h, \boldsymbol{\delta}_h)\,,\ \vec{\mathbf{w}}_h := (\mathbf{w}_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h\,,$$

the Galerkin scheme associated with (8.8) reads as follows: Find $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $(\varphi_h, \chi_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\chi}$ such that

$$\mathbf{A}_{\varphi_h}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,+\, b(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) \,= \mathbf{F}_{\varphi_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h\,, \tag{9.3a}$$

$$[\mathcal{A}_{\mathbf{u}_h}(\varphi_h), \psi_h] \,+\, [\mathcal{B}(\psi_h), \chi_h] \,= 0 \qquad \forall\, \psi_h \in \mathrm{H}_h^{\varphi}\,, \tag{9.3b}$$

$$[\mathcal{B}(\varphi_h), \xi_h] \,= \mathcal{G}(\xi_h) \qquad \forall\, \xi_h \in \mathrm{H}_h^{\chi}\,. \tag{9.3c}$$

9.1 The discrete fixed point strategy

We adopt the discrete analogue of Chapter 8.2 to analyze (9.3). Let $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator given by

$$\mathbf{S}_h(\mathbf{z}_h, \phi_h) \,=\, \mathbf{u}_h \qquad \forall\, (\mathbf{z}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}\,,$$

where $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the linear problem given by

$$\mathbf{A}_{\phi_h}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \,+\, b(\mathbf{z}_h; \mathbf{u}_h, \mathbf{s}_h) \,=\, \mathbf{F}_{\phi_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h\,. \tag{9.4}$$

In turn, we let $\widetilde{\mathbf{S}}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\varphi}$ be the operator defined by

$$\widetilde{\mathbf{S}}_h(\mathbf{z}_h) \,:=\, \varphi_h \qquad \forall\, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}\,,$$

where $(\varphi_h, \chi_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\chi}$ is the unique solution (to be confirmed below) of

$$
\begin{aligned}
[\mathcal{A}_{\mathbf{z}_h}(\varphi_h), \psi_h] + [\mathcal{B}(\psi_h), \chi_h] &= 0 \qquad \forall\, \psi_h \in \mathrm{H}_h^{\varphi}\,, \\
[\mathcal{B}(\varphi_h), \xi_h] &= \mathcal{G}(\xi_h) \qquad \forall\, \xi_h \in \mathrm{H}_h^{\chi}\,.
\end{aligned} \tag{9.5}
$$

Then, we define the operator $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ by

$$
\mathbf{T}_h(\mathbf{z}_h) := \mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathbf{S}}_h(\mathbf{z}_h)\big) \qquad \forall \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}\,, \tag{9.6}
$$

and realize that solving (9.3) is equivalent to seeking a fixed point of $\mathbf{T}_h$: Find $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$
\mathbf{T}_h(\mathbf{z}_h) = \mathbf{z}_h\,. \tag{9.7}
$$

9.2 Well-definedness of the discrete problems

In this chapter we apply the discrete versions of the solvability result for perturbed saddle-point problems and the nonlinear version of the Babuška–Brezzi theory employed in Chapter 8.3, to prove that the operators $\mathbf{S}_h$, $\widetilde{\mathbf{S}}_h$, and hence $\mathbf{T}_h$, are well-defined. As observed in the previous chapter, these goals reduce, equivalently, to establishing that the uncoupled problems (9.4) and (9.5) are well-posed. To this end, we begin by remarking, as in the continuous counterpart, that the solvability of the discrete problem (9.4) is addressed in [48, Section 4.2], and for this reason we just state the following result.

Lemma 9.1. *For each $(\mathbf{z}_h, \phi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\varphi}$ such that $\|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \dfrac{\alpha_{\mathbf{A},\mathrm{d}}}{2}$ (cf. [48, eq. 4.23]), problem (9.4) has a unique solution $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$, and hence $\mathbf{S}_h(\mathbf{z}_h, \phi_h) := \mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, depending only on $\alpha_{\mathbf{A},\mathrm{d}}$, C_f, $|\Omega|$ and $\|\mathbf{k}\|_\infty$, and hence independent of h, such that*

$$
\|\mathbf{S}_h(\mathbf{z}_h, \phi_h)\|_{0,4;\Omega} = \|\mathbf{u}_h\|_{0,4;\Omega} \leqslant \|(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}_h \times \mathbf{Q}_h} \leqslant C_{\mathbf{S},\mathrm{d}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\phi_h\|_{1;\Omega}\Big\}\,. \tag{9.8}
$$

Proof. It follows directly from [48, Lemma 4.2] and (8.14). □

The following assumptions, specified in [48, Section 4.2], are necessary to apply Lemma 9.1.

(**H.0**) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiplies of the identity tensor $\mathbb{I}$.

(**H.1**) $\mathbf{div}(\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}) \subseteq \mathbf{H}_h^{\mathbf{u}}$.

(**H.2**) $\big(\mathrm{V}_{0,h}\big)^{\mathrm{d}} \subseteq \mathbb{H}_h^{\mathbf{t}}$, where $\mathbf{V}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathrm{V}_{0,h}$ is the kernel of $\mathbf{b}|_{\mathbf{H}_h \times \mathbf{Q}_h}$, with

$$\mathrm{V}_{0,h} := \Big\{\boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \int_\Omega \boldsymbol{\tau}_h : \boldsymbol{\delta}_h = 0 \quad \forall\, \boldsymbol{\delta}_h \in \mathbb{H}_h^{\boldsymbol{\gamma}} \quad \text{and} \quad \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\Big\}.$$

(**H.3**) There exists a positive constant $\beta_{\mathbf{b},\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{s}} \in \mathbf{H}_h \\ \vec{\mathbf{s}} \neq 0}} \frac{\mathbf{b}(\vec{\mathbf{s}}, \vec{\mathbf{v}})}{\|\vec{\mathbf{s}}\|_{\mathbf{H}}} \geqslant \beta_{\mathbf{b},\mathrm{d}}\, \|\vec{\mathbf{v}}\|_{\mathbf{Q}} \quad \forall\, \vec{\mathbf{v}} \in \mathbf{Q}_h\,.$$

In addition, the previous lemma suggests to consider the ball

$$\mathbf{W}_{\mathbf{S},h} := \Big\{\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\alpha_{\mathbf{A},\mathrm{d}}}{2}\Big\},$$

which will be employed below in Chapter 9.3.

Next, aiming to prove the solvability of (9.5), we require a consequence of the generalized Poincaré inequality, which establishes the existence of a positive constant $\widehat{c}_{\mathrm{P}}$ such that

$$|\phi|_{1;\Omega}^2 \geqslant \widehat{c}_{\mathrm{P}}\, \|\phi\|_{1;\Omega}^2 \qquad \forall\, \phi \in \widehat{\mathrm{V}}\,, \tag{9.9}$$

where $\widehat{\mathrm{V}} := \big\{\phi \in \mathrm{H}^1(\Omega) : \int_\Gamma \phi = 0\big\}$. Then, in order to apply Theorem 8.2, we introduce appropriate hypotheses on the discrete spaces H_h^{φ} and H_h^{χ}:

(**H.4**) $\mathrm{P}_0(\Gamma) \subseteq \mathrm{H}_h^{\chi}$.

(**H.5**) There exists a positive constant $\beta_{\mathcal{B},\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\psi_h \in \mathrm{H}_h^{\varphi} \\ \psi_h \neq 0}} \frac{[\mathcal{B}(\psi_h), \xi_h]}{\|\psi_h\|_{1;\Omega}} \geqslant \beta_{\mathcal{B},\mathrm{d}}\, \|\xi_h\|_{-1/2;\Gamma} \quad \forall\, \xi_h \in \mathrm{H}_h^{\chi}\,. \tag{9.10}$$

Taking the above assumptions into account, and defining

$$\mathbf{W}_{\widetilde{\mathbf{S}},h} := \left\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \frac{\rho\,\kappa_0\,\widehat{c}_{\mathrm{P}}}{2(1 + L_{\widehat{s}})\|\mathrm{i}_4\|} \right\},$$

we can prove that the operator $\widetilde{\mathbf{S}}_h$ is well-posed, which is abridged in the following lemma.

Lemma 9.2. *For each $\mathbf{z}_h \in \mathbf{W}_{\widetilde{\mathbf{S}},h}$, problem* (9.5) *has a unique solution $(\varphi_h, \chi_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\chi}$, and hence $\widetilde{\mathbf{S}}_h(\mathbf{z}_h) := \varphi_h \in \mathrm{H}_h^{\varphi}$ is well-defined. Moreover, there exist positive constants $C_{\widetilde{\mathbf{S}},\mathrm{d}}$ and $\widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}$, depending on ρ, κ_0, $\widehat{c}_{\mathrm{P}}$ (cf.* (9.9)*) and κ_1, $\beta_{\mathcal{B},\mathrm{d}}$ (cf.* (9.10)*), such that*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_h)\|_{1;\Omega} := \|\varphi_h\|_{1;\Omega} \leqslant C_{\widetilde{\mathbf{S}},\mathrm{d}}\,\|\varphi_D\|_{1/2;\Gamma} \quad \textit{and} \quad \|\chi_h\|_{-1/2;\Gamma} \leqslant \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}\,\|\varphi_D\|_{1/2;\Gamma}\,. \tag{9.11}$$

Proof. We begin by introducing the discrete kernel of $\mathcal{B}$, namely

$$\widetilde{\mathrm{V}}_h := \left\{ \psi_h \in \mathrm{H}_h^{\varphi} : \quad \langle \xi_h, \psi_h \rangle_\Gamma = 0 \quad \forall\, \xi_h \in \mathrm{H}_h^{\chi} \right\},$$

which, as a consequence of (**H.4**), is clearly contained in $\widehat{\mathrm{V}}$, and thus, (9.9) is certainly valid in $\widetilde{\mathrm{V}}_h$. On the other hand, given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, $\phi_h \in \mathrm{H}_h^{\varphi}$ and $\theta_{1,h}, \theta_{2,h} \in \widetilde{\mathrm{V}}_h$, and proceeding as in Lemma 8.4, using in this case (9.9) instead of (8.19), we obtain

$$\begin{aligned}&[\mathcal{A}_{\mathbf{z}_h}(\theta_{1,h} + \phi_h) - \mathcal{A}_{\mathbf{z}_h}(\theta_{2,h} + \phi_h),\, \theta_{1,h} - \theta_{2,h}] \\ &\qquad \geqslant \big(\rho\kappa_0\widehat{c}_{\mathrm{P}} - (1 + L_{\widehat{s}})\|\mathrm{i}_4\|\,\|\mathbf{z}_h\|_{0,4;\Omega}\big)\|\theta_{1,h} - \theta_{2,h}\|_{1;\Omega}^2\,,\end{aligned}$$

from which, defining $\alpha_{\mathcal{A},\mathrm{d}} := \rho\,\kappa_0\,\widehat{c}_{\mathrm{P}}/2$ and using that $\mathbf{z}_h \in \mathbf{W}_{\widetilde{\mathbf{S}},h}$, we readily conclude that the family of operators $\mathcal{A}_{\mathbf{z}_h}(\cdot + \phi_h)$, with $\phi_h \in \mathrm{H}_h^{\varphi}$, is uniformly strongly monotone in $\widetilde{\mathrm{V}}_h$ with constant $\alpha_{\mathcal{A},\mathrm{d}}$. In addition, (8.18) and the specified bound on $\|\mathbf{z}_h\|_{0,4;\Omega}$ imply the Lipschitz-continuity of $\mathcal{A}_{\mathbf{z}_h}$ with constant $L_{\mathcal{A},\mathrm{d}} = \rho\kappa_1 + \alpha_{\mathcal{A},\mathrm{d}}$. Moreover, thanks to assumption (**H.5**) (cf (9.10)), a straightforward application of Theorem 8.2 and the upper bound for $\mathcal{G}$ (cf. (8.17b)), we obtain (9.11) with

$$C_{\widetilde{\mathbf{S}},\mathrm{d}} := \frac{1}{\beta_{\mathcal{B},\mathrm{d}}}\left(1 + \frac{L_{\mathcal{A},\mathrm{d}}}{\alpha_{\mathcal{A},\mathrm{d}}}\right) \quad \text{and} \quad \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}} := \frac{L_{\mathcal{A},\mathrm{d}}}{\beta_{\mathcal{B},\mathrm{d}}^2}\left(1 + \frac{L_{\mathcal{A},\mathrm{d}}}{\alpha_{\mathcal{A},\mathrm{d}}}\right).$$

□

9.3 Solvability analysis of the discrete fixed point

Having proved that $\mathbf{T}_h$ is well-defined, we now apply the following version of Brouwer's theorem (cf. [35, Theorem 9.9-2]) needed to show the solvability of (9.7).

Theorem 9.3. *Let* W *be a compact and convex subset of a finite dimensional Banach space* X *and* $T : \mathrm{W} \to \mathrm{W}$ *be a continuous mapping. Then* T *has at least one fixed-point.*

Similarly to Chapter 8.4, we introduce the ball

$$\mathbf{W}_h \;:=\; \mathbf{W}_{\mathbf{S},h} \cap \mathbf{W}_{\widetilde{\mathbf{S}},h} \;:=\; \Big\{ \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \;\; \|\mathbf{z}_h\|_{0,4;\Omega} \leqslant \varrho_{\mathrm{d}} \Big\} \tag{9.12}$$

with radius

$$\varrho_{\mathrm{d}} \;:=\; \min\left\{ \frac{\alpha_{\mathbf{A},\mathrm{d}}}{2}, \frac{\alpha_{\mathcal{A},\mathrm{d}}}{(1 + L_{\widehat{s}})\|\mathrm{i}_4\|} \right\} ,$$

which is a compact and convex subset of the finite dimensional space $\mathbf{H}_h^{\mathbf{u}}$. Then, the discrete analogue of Lemma 8.7 is stated as follows.

Lemma 9.4. *Assume that*

$$C_{\mathbf{T},\mathrm{d}} \Big\{ \|\mathbf{u}_D\|_{1/2,\Gamma} \;+\; \|\varphi_D\|_{1/2;\Gamma} \Big\} \;\leqslant\; \varrho_{\mathrm{d}} \,, \tag{9.13}$$

where $C_{\mathbf{T},\mathrm{d}} \;:=\; C_{\mathbf{S},\mathrm{d}} \max\{1, C_{\widetilde{\mathbf{S}},\mathrm{d}}\}$*, and* $C_{\mathbf{S},\mathrm{d}}$ *and* $C_{\widetilde{\mathbf{S}},\mathrm{d}}$ *are the constants specified in Lemmas* 9.1 *and* 9.2*, respectively. Then, there holds* $\mathbf{T}_h(\mathbf{W}_h) \subseteq \mathbf{W}_h$*.*

Proof. Similarly to the proof of Lemma 8.7, it is a direct consequence of the assumption (9.13) and Lemmas 9.1 and 9.2, particularly of the respective a priori bounds (9.8) and (9.11). □

We now aim to prove that $\mathbf{T}_h$ is continuous, for which we previously address the same property for $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$. Indeed, in what follows we state the discrete analogues of Lemmas 8.8 and 8.9.

Lemma 9.5. *There exists a positive constant* $L_{\mathbf{S},\mathrm{d}}$*, independent of* h*, depending only on* $\alpha_{\mathbf{A},\mathrm{d}}$*,* L_μ*,* L_η*,* $\|\mathrm{i}_4\|$*,* $|\Omega|$ *and* $\|\mathbf{k}\|_\infty$*, such that for all* $(\mathbf{z}_{1,h},\phi_{1,h}),(\mathbf{z}_{2,h},\phi_{2.h}) \in \mathbf{W}_{\mathbf{S},h}\times \mathrm{H}_h^\varphi$*, there holds*

$$\begin{aligned}\|\mathbf{S}_h(\mathbf{z}_{1,h},\phi_{1,h}) - \mathbf{S}_h(\mathbf{z}_{2,h},\phi_{2,h})\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, L_{\mathbf{S},\mathrm{d}}\Big\{\|\mathbf{S}_h(\mathbf{z}_{2,h},\phi_{2,h})\|_{0,4;\Omega}\|\mathbf{z}_{1,h}-\mathbf{z}_{2,h}\|_{0,4;\Omega} \\ + \big(\|\mathbf{t}_2\|_{0,4;\Omega} + \|\widetilde{\mathbf{S}}_h(\mathbf{z}_{2,h},\phi_{2,h})\|_{0,4;\Omega} + L_f\big)\|\phi_{1,h}-\phi_{2,h}\|_{1;\Omega}\Big\}\,.\end{aligned} \tag{9.14}$$

Proof. Given $(\mathbf{z}_{1,h},\phi_{1,h}),(\mathbf{z}_{2,h},\phi_{2.h}) \in \mathbf{W}_{\mathbf{S},h}\times \mathrm{H}_h^\varphi$, we let $\mathbf{S}_h(\mathbf{z}_{i,h},\phi_{i,h}) \,:=\, \mathbf{u}_{i,h}$, for each $i\in\{1,2\}$, where $(\vec{\mathbf{t}}_{i,h},\vec{\mathbf{u}}_{i,h}) \,=\, \big((\mathbf{t}_{i,h},\boldsymbol{\sigma}_{i,h}),(\mathbf{u}_{i,h},\boldsymbol{\gamma}_{i,h})\big)$ is the unique solution of (9.4) with $(\mathbf{z}_{i,h},\phi_{i,h})$ instead of $(\mathbf{z}_h,\phi_h)$. Then the proof of (9.14), starting now from the discrete global inf-sup condition [48, eq. (4.24)], is very similar to the one for Lemma 8.8. However, since a regularity assumption such as (8.25) is not available in the present discrete settings, we estimate $\mathbf{a}_{\phi_{2,h}} - \mathbf{a}_{\phi_{1,h}}$ by using an $\mathrm{L}^4(\Omega) - \mathbb{L}^4(\Omega) - \mathbb{L}^2(\Omega)$ argument along with (8.4). In this way, we obtain

$$(\mathbf{a}_{\phi_{2,h}} - \mathbf{a}_{\phi_{1,h}})(\mathbf{t}_{2,h},\mathbf{s}_h) \,\leqslant\, \lambda\, L_\mu\,\|\mathrm{i}_4\|\,\|\phi_{2,h}-\phi_{1,h}\|_{1;\Omega}\|\mathbf{t}_{2,h}\|_{0,4;\Omega}\|\mathbf{s}_h\|_{0;\Omega}\,.$$

The rest of the estimates are similar to those in the proof of Lemma 8.8, and are therefore omitted. □

Lemma 9.6. *There exists a positive constant* $L_{\widetilde{\mathbf{S}},\mathrm{d}}$*, independent of* h*, depending only on* s_3*,* $\|\mathrm{i}_4\|$ *and* $\alpha_{\mathcal{A},\mathrm{d}}$ *(cf. proof of Lemma 9.2), such that for all* $\mathbf{z}_{1,h},\mathbf{z}_{2,h}\in \mathbf{W}_{\widetilde{\mathbf{S}},h}$*, there holds*

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_{1,h}) - \widetilde{\mathbf{S}}_h(\mathbf{z}_{2,h})\|_{1;\Omega} \,\leqslant\, L_{\widetilde{\mathbf{S}},\mathrm{d}}\|\widetilde{\mathbf{S}}_h(\mathbf{z}_{2,h})\|_{1;\Omega}\|\mathbf{z}_{1,h}-\mathbf{z}_{2,h}\|_{0,4;\Omega}\,. \tag{9.15}$$

Proof. It follows very closely the arguments from the proof of Lemma 8.9. □

As a consequence of the previous two lemmas, we have the continuity of the operator $\mathbf{T}_h$.

Lemma 9.7. *There exists a positive constant* $L_{\mathbf{T},\mathrm{d}}$*, independent of* h*, depending only on* $C_{\widetilde{\mathbf{S}},\mathrm{d}}$*,* $C_{\mathbf{T},\mathrm{d}}$*,* $L_{\mathbf{S},\mathrm{d}}$ *and* $L_{\widetilde{\mathbf{S}},\mathrm{d}}$*, such that for all* $\mathbf{z}_{1,h},\mathbf{z}_{2,h}\in\mathbf{W}_h$*, there holds*

$$\begin{aligned}&\|\mathbf{T}_h(\mathbf{z}_{1,h}) - \mathbf{T}_h(\mathbf{z}_{2,h})\|_{0,4;\Omega} \\ &\quad\leqslant L_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} \,+\, \|\varphi_D\|_{1/2;\Gamma} \,+\, C_{\mathrm{d}}(\mathbf{u}_D,\varphi_D,\mathbf{t}_{h,2})\|\varphi_D\|_{1/2;\Gamma}\Big\}\|\mathbf{z}_{1,h}-\mathbf{z}_{2,h}\|_{0,4;\Omega}\,,\end{aligned} \tag{9.16}$$

where

$$C_{\mathrm{d}}(\mathbf{u}_D, \varphi_D, \mathbf{t}_{h,2}) \;:=\; \|\mathbf{u}_D\|_{1/2;\Gamma} \;+\; \|\varphi_D\|_{1/2;\Gamma} \;+\; \|\mathbf{t}_{2,h}\|_{0,4;\Omega} \;+\; L_f\,.$$

Proof. Given $\mathbf{z}_{1,h}, \mathbf{z}_{2,h} \in \mathbf{W}_h$, and proceeding as in the proof of Lemma 8.10, but now using the definition of $\mathbf{T}_h$ (cf. (9.6)) and the continuity of $\mathbf{S}_h$ (cf (9.5)), we readily find that

$$\begin{aligned} \|\mathbf{T}_h(\mathbf{z}_{1,h}) - \mathbf{T}_h(\mathbf{z}_{2,h})\|_{0,4;\Omega} \leqslant L_{\mathbf{S},\mathrm{d}}\Big\{ \|\mathbf{T}_h(\mathbf{z}_{2,h})\|_{0,4;\Omega}\|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{0,4;\Omega} \\ + \big(\|\mathbf{t}_{2,h}\|_{0,4;\Omega} + \|\mathbf{T}_h(\mathbf{z}_{2,h})\|_{0,4;\Omega} + L_f\big)\|\mathbf{S}_h(\mathbf{z}_{1,h}) - \mathbf{S}_h(\mathbf{z}_{2,h})\|_{1;\Omega}\Big\}\,. \end{aligned} \tag{9.17}$$

Then, thanks to the a priori estimate (9.8), the Lipschitz-continuity of $\widetilde{\mathbf{S}}_h$ (cf (9.15)) yields

$$\|\widetilde{\mathbf{S}}_h(\mathbf{z}_{1,h}) - \widetilde{\mathbf{S}}_h(\mathbf{z}_{2,h})\|_{1;\Omega} \leqslant L_{\widetilde{\mathbf{S}},\mathrm{d}} C_{\widetilde{\mathbf{S}},\mathrm{d}} \|\varphi_D\|_{1/2;\Gamma}\|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{0,4;\Omega}\,. \tag{9.18}$$

In addition, using the a priori estimates for $\mathbf{S}_h$ and $\widetilde{\mathbf{S}}_h$ (cf. (9.8) and (9.11)), we have

$$\|\mathbf{T}_h(\mathbf{z}_{2,h})\|_{0,4;\Omega} \;\leqslant\; C_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\}\,. \tag{9.19}$$

Finally, replacing (9.18) and (9.19) in (9.17), and performing some minor algebraic manipulations, we obtain (9.16) with the constant

$$L_{\mathbf{T},\mathrm{d}} \;:=\; L_{\mathbf{S},\mathrm{d}} \max\big\{C_{\mathbf{T},\mathrm{d}},\, 1\big\} \max\big\{1, L_{\widetilde{\mathbf{S}},\mathrm{d}} C_{\widetilde{\mathbf{S}},\mathrm{d}}\big\}\,.$$

□

We remark that, while the inequality (9.16) establishes the continuity of $\mathbf{T}_h$, the lack of control of the term $\|\mathbf{t}_{2,h}\|_{0,4;\Omega}$ prevents us from deducing Lipschitz-continuity and hence contractivity of $\mathbf{T}_h$. Consequently, we are only able to establish existence of a fixed point.

Theorem 9.8. *Assume that the data satisfy* (9.13). *Then, the Galerkin scheme* (9.3) *has at least a solution* $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) \;:=\; \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ *and* $(\varphi_h, \chi_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\chi}$, *with*

$\mathbf{u}_h \in \mathbf{W}_h$. *Moreover,*

$$\|(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \leqslant C_{\mathbf{T},\mathrm{d}}\Big\{\|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\},$$

$$\|\varphi_h\|_{0;\Omega} \leqslant C_{\widetilde{\mathbf{S}},\mathrm{d}}\,\|\varphi_D\|_{1/2;\Gamma} \quad \textit{and} \quad \|\chi_h\|_{-1/2;\Gamma} \leqslant \widetilde{C}_{\widetilde{\mathbf{S}},\mathrm{d}}\,\|\varphi_D\|_{1/2;\Gamma}\,.$$

Proof. Since $\mathbf{W}_h$ is compact and convex, and $\mathbf{T}_h$ maps $\mathbf{W}_h$ into itself (cf. Lemma 9.4), then Brouwer's theorem yields the existence of solution for (9.3). In turn, since $\mathbf{u}_h = \mathbf{T}_h(\mathbf{u}_h) = \mathbf{S}_h\big(\mathbf{u}_h, \widetilde{\mathbf{S}}_h(\mathbf{u}_h)\big)$ and $\varphi_h = \widetilde{\mathbf{S}}_h(\mathbf{u}_h)$, then (9.8) and (9.11) imply the continuous dependence on data of the solutions. □

9.4 A priori error analysis

In this chapter we derive a priori error estimates for the Galerkin scheme (9.3) with arbitrary finite element spaces satisfying the hypotheses (**H.0**)–(**H.5**) from Chapter 9.2. We focus on the global error

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} + \|\varphi - \varphi_h\|_{1;\Omega} + \|\chi - \chi_h\|_{-1/2;\Gamma}\,,$$

where $(\vec{\mathbf{t}}, \vec{\mathbf{u}}) := \big((\mathbf{t}, \boldsymbol{\sigma}), (\mathbf{u}, \boldsymbol{\gamma})\big) \in \mathbf{H} \times \mathbf{Q}$ and $(\varphi, \chi) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, with $\mathbf{u} \in \mathbf{W}$ (cf. (8.23)), is the unique solution of (8.8), and $(\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h) := \big((\mathbf{t}_h, \boldsymbol{\sigma}_h), (\mathbf{u}_h, \boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ and $(\varphi_h, \chi_h) \in \mathrm{H}_h^{\varphi} \times \mathrm{H}_h^{\chi}$, with $\mathbf{u}_h \in \mathbf{W}_h$ (cf. (9.12)), is a solution of the discrete coupled problem (9.3). To this end, we establish next two *ad-hoc* Strang-type estimates. Hereafter, given a subspace X_h of a generic Banach space $(\mathrm{X}, \|\cdot\|_{\mathrm{X}})$, we set as usual $\mathrm{dist}(x, \mathrm{X}_h) := \inf_{x_h \in \mathrm{X}_h} \|x - x_h\|_{\mathrm{X}}$ for all $x \in \mathrm{X}$.

Lemma 9.9. *Let* H *be a reflexive Banach space, and let* $a : \mathrm{H} \times \mathrm{H}$ *be a bounded bilinear form inducing the operator* $\mathcal{A} \in \mathcal{L}(\mathrm{H}, \mathrm{H}')$*, such that* a *satisfies the hypothesis of the Banach–Nečas–Babuška theorem* (cf. [46, Theorem 2.6]). *Furthermore, let* $\{\mathrm{H}_h\}_{h>0}$ *be a sequence of finite dimensional subspaces of* H*, and for each* $h > 0$*, consider a bounded bilinear form* $a_h : \mathrm{H}_h \times \mathrm{H}_h \to \mathrm{R}$ *inducing* $\mathcal{A}_h \in \mathcal{L}(\mathrm{H}_h, \mathrm{H}_h')$*, such that* $a_h|_{\mathrm{H}_h\times\mathrm{H}_h}$ *satisfies the hypotheses of Banach–*

Nečas–Babuška theorem as well, with constant $\widetilde{\alpha}$ *independent of* h*. In turn, given* $F \in \mathrm{H}'$*, and a sequence of functionals* $\{F_h\}_{h>0}$*, with* $F_h \in \mathrm{H}_h'$ *for each* $h > 0$*, we let* $u \in \mathrm{H}$ *and* $u_h \in \mathrm{H}_h$ *be the unique solutions to problems*

$$a(u,v) = F(v) \quad \forall v \in \mathrm{H}, \tag{9.20}$$

and

$$a_h(u_h,v_h) = F_h(v_h) \quad \forall v_h \in \mathrm{H}_h, \tag{9.21}$$

respectively. Then, there holds

$$\|u-u_h\|_{\mathrm{H}} \leqslant C_{S,1}\,\mathrm{dist}\,(u,\mathrm{H}_h) + C_{S,2}\left\{\|F-F_h\|_{\mathrm{H}_h'} + \|a(u,\cdot)-a_h(u,\cdot)\|_{\mathrm{H}_h'}\right\}, \tag{9.22}$$

where $C_{S,1}$ *and* $C_{S,2}$ *are the positive constants given by*

$$C_{S,1} := \left(1 + \frac{2\|\mathcal{A}\|}{\widetilde{\alpha}} + \frac{\|\mathcal{A}_h\|}{\widetilde{\alpha}}\right) \quad \textit{and} \quad C_{S,2} := \frac{1}{\widetilde{\alpha}}. \tag{9.23}$$

Proof. See [32, Lemma 5.1]. □

Lemma 9.10. *Let* H *and* Q *be separable and reflexive Banach spaces, with* H *uniformly convex, and let* $a : \mathrm{H} \to \mathrm{H}'$ *be a nonlinear operator and* $b \in \mathcal{L}(\mathrm{H},\mathrm{Q}')$ *satisfying the hypotheses of Theorem* 8.2 *with constants* L*,* α *and* β*. Furthermore, let* $\{\mathrm{H}_h\}_{h>0}$ *and* $\{\mathrm{Q}_h\}_{h>0}$ *be sequences of finite dimensional subspaces of* H *and* Q*, respectively, and for each* $h > 0$ *consider a nonlinear operator* $a_h : \mathrm{H} \to \mathrm{H}'$*, such that* $a|_{\mathrm{H}_h} : \mathrm{H}_h \to \mathrm{H}_h'$ *and* $b|_{\mathrm{H}_h} : \mathrm{H}_h \to \mathrm{Q}_h'$ *satisfy the hypothesis of Theorem* 8.2 *with constants* L_{d}*,* α_{d}*, and* β_{d}*, all independent of* h*. In turn, given* $F \in \mathrm{H}'$*,* $G \in \mathrm{Q}'$*, and sequences of functionals* $\{F_h\}_{h>0}$ *and* $\{G_h\}_{h>0}$*, with* $F_h \in \mathrm{H}_h'$ *and* $G_h \in \mathrm{Q}_h'$ *for each* $h > 0$*, we let* $(\sigma,u) \in \mathrm{H}\times\mathrm{Q}$ *and* $(\sigma_h,u_h) \in \mathrm{H}_h\times\mathrm{Q}_h$ *be the unique solutions to problems*

$$\begin{aligned} [a(\sigma),\tau] + [b(\tau),u] &= [F,\tau] \qquad \forall \tau \in \mathrm{H}, \\ [b(\sigma),v] &= [G,v] \qquad \forall v \in \mathrm{Q}, \end{aligned} \tag{9.24}$$

and

$$\begin{aligned} [a_h(\sigma_h), \tau_h] + [b_h(\tau_h), u_h] &= [F_h, \tau_h] \qquad \forall \tau_h \in \mathrm{H}_h, \\ [b_h(\sigma_h), v_h] &= [G_h, v_h] \qquad \forall v_h \in \mathrm{Q}_h, \end{aligned} \tag{9.25}$$

respectively. Then, there exists a positive constants $C_{S,i}$, depending only on L, α_{d}, β_{d}, and $\|b\|$, such that

$$\begin{aligned} \|\sigma - \sigma_h\|_{\mathrm{H}} + \|u - u_h\|_{\mathrm{Q}} &\leqslant C_{S,1}\,\mathrm{dist}\,(\sigma, \mathrm{H}_h) + C_{S,2}\,\mathrm{dist}\,(u, \mathrm{Q}_h) \\ &+ C_{S,3}\left\{\|F - F_h\|_{\mathrm{H}_h'} + \|G - G_h\|_{\mathrm{Q}_h'} + \|a(\sigma) - a_h(\sigma)\|_{\mathrm{H}_h'}\right\}. \end{aligned} \tag{9.26}$$

Proof. See [17, Lemma 5.1]. □

In order to apply Lemmas 9.9 and 9.10, we now observe that (8.8) and (9.3) can be rewritten as two pairs of continuous and discrete formulations as (9.20)-(9.21) and (9.24)-(9.25), respectively, namely

$$\begin{aligned} \mathbf{A}_{\varphi}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), (\vec{\mathbf{s}}, \vec{\mathbf{v}})\big) \;+\; b(\mathbf{u}; \mathbf{u}, \mathbf{s}) &= \mathbf{F}_{\varphi}(\vec{\mathbf{s}}, \vec{\mathbf{v}}) \qquad \forall\, (\vec{\mathbf{s}}, \vec{\mathbf{v}}) \in \mathbf{H} \times \mathbf{Q}\,, \\ \mathbf{A}_{\varphi_h}\big((\vec{\mathbf{t}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h)\big) \;+\; b(\mathbf{u}_h; \mathbf{u}_h, \mathbf{s}_h) &= \mathbf{F}_{\varphi_h}(\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \qquad \forall\, (\vec{\mathbf{s}}_h, \vec{\mathbf{v}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h\,, \end{aligned} \tag{9.27}$$

and

$$\begin{aligned} [\mathcal{A}_{\mathbf{u}}(\varphi), \psi] \;+\; [\mathcal{B}(\psi), \chi] &= 0 \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\ [\mathcal{B}(\varphi), \xi] &= [\mathcal{G}, \xi] \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,, \\ [\mathcal{A}_{\mathbf{u}_h}(\varphi_h), \psi_h] \;+\; [\mathcal{B}(\psi_h), \chi_h] &= 0 \qquad \forall\, \psi_h \in \mathrm{H}_h^{\varphi}\,, \\ [\mathcal{B}(\varphi_h), \xi_h] &= [\mathcal{G}, \xi_h] \qquad \forall\, \xi_h \in \mathrm{H}_h^{\chi}\,. \end{aligned} \tag{9.28}$$

The following lemma provides a preliminary estimate for the error $\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} \;+\; \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}}$.

Lemma 9.11. *There exists a positive constant C_{ST}, independent of h, such that*

$$\begin{aligned} \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} \;+\; \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} &\leqslant C_{ST}\Big\{\mathrm{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_h\big) \;+\; \mathrm{dist}\big(\vec{\mathbf{u}}, \mathbf{Q}_h\big) \\ &+\; C(\mathbf{u}_D, \varphi_D)\,\|\varphi - \varphi_h\|_{1;\Omega} + \big(\|\mathbf{u}_D\|_{1/2;\Gamma} \;+\; \|\varphi_D\|_{1/2;\Gamma}\big)\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,, \end{aligned} \tag{9.29}$$

where $C(\mathbf{u}_D, \varphi_D)$ is given by (8.39)*.*

Proof. We recall from Chapters 8.4 and 9.2 that $\mathbf{A}_{\varphi} + b(\mathbf{u}; \cdot, \cdot)$ and $\mathbf{A}_{\varphi_h} + b(\mathbf{u}_h; \cdot, \cdot)$, with $\mathbf{u} \in \mathbf{W}$ and $\mathbf{u}_h \in \mathbf{W}_h$, satisfy the hypotheses of Banach–Nečas–Babuška theorem on $\mathbf{H} \times \mathbf{Q}$ and $\mathbf{H}_h \times \mathbf{Q}_h$, respectively, the latter with constant $\alpha_{\mathbf{A},\mathrm{d}}/2$ (cf. [48, eq. (4.23)]). Then, applying Lemma 9.9 to (9.27), and according to (9.23), the estimates [48, eqs. (3.41a) and (3.43)], and the bounds (8.23) and (9.12), we conclude the existence of $C_{S,1} > 0$, independent of h, depending only on λ, μ_1, η_1, $|\Omega|$, $\alpha_{\mathbf{A},\mathrm{d}}$, ϱ and ϱ_{d}, such that

$$\begin{aligned}\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} \leqslant C_{S,1}\mathrm{dist}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \mathbf{H}_h \times \mathbf{Q}_h\big) + \frac{2}{\alpha_{\mathbf{A},\mathrm{d}}}\Big\{\|\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h}\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'} \\ +\|\mathbf{A}_\varphi\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \cdot\big) - \mathbf{A}_{\varphi_h}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \cdot\big)\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'} + \|b(\mathbf{u}; \mathbf{u}, \cdot) - b(\mathbf{u}_h; \mathbf{u}, \cdot)\|_{\mathbb{H}_h^{\mathbf{t}\,\prime}}\Big\}\,.\end{aligned} \tag{9.30}$$

Then, proceeding exactly as in Lemma 8.8, particularly from equations (8.32), (8.33) and (8.34), yields

$$\|\mathbf{F}_\varphi - \mathbf{F}_{\varphi_h}\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'} \leqslant L_f L_{\mathbf{F}}\|\varphi - \varphi_h\|_{1;\Omega}\,, \tag{9.31a}$$

$$\|\mathbf{A}_\varphi\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \cdot\big) - \mathbf{A}_{\varphi_h}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \cdot\big)\|_{(\mathbf{H}_h\times\mathbf{Q}_h)'} \leqslant L_{\mathbf{A}}\Big\{\|\mathbf{t}\|_{\epsilon;\Omega} + \|\mathbf{u}\|_{0,4;\Omega}\Big\}\|\varphi - \varphi_h\|_{1;\Omega} \quad \text{and} \tag{9.31b}$$

$$\|b(\mathbf{u}; \mathbf{u}, \cdot) - b(\mathbf{u}_h; \mathbf{u}, \cdot)\|_{\mathbb{H}_h^{\mathbf{t}\,\prime}} \leqslant \|\mathbf{u}\|_{0,4;\Omega}\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,. \tag{9.31c}$$

In this way, replacing (9.31) back into (9.30), using (8.25) and the bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\varphi\|_{1;\Omega}$ from Theorem 8.11, and performing algebraic manipulations, we obtain (9.29). □

Next, we have the following result concerning $\|\varphi - \varphi_h\|_{1;\Omega} + \|\chi - \chi_h\|_{-1/2;\Omega}$.

Lemma 9.12. *There exists a positive constant $\widetilde{C}_{ST}$, independent of h, depending only on s_3, $\|\mathrm{i}_4\|$, L, $\alpha_{\mathcal{A},\mathrm{d}}$, $\beta_{\mathcal{B},\mathrm{d}}$ and $C_{\widetilde{\mathbf{S}}}$, such that*

$$\|\varphi-\varphi_h\|_{1;\Omega} + \|\chi-\chi_h\|_{-1/2;\Gamma} \leqslant \widetilde{C}_{ST}\Big\{\mathrm{dist}\big(\varphi, \mathrm{H}_h^\varphi\big) + \mathrm{dist}\big(\chi, \mathrm{H}_h^\chi\big) + \|\varphi_D\|_{1/2;\Gamma}\|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\Big\}\,. \tag{9.32}$$

Proof. With $\mathbf{u} \in \mathbf{W}$ and $\mathbf{u}_h \in \mathbf{W}_h$ given, the continuous and discrete systems associated with (9.28) satisfy the hypothesis of Theorem 8.2, with constants $L_{\mathcal{A}}$, $\alpha_{\mathcal{A}}$, $\beta_{\mathcal{B}} = 1$, $L_{\mathcal{A},\mathrm{d}}$, $\alpha_{\mathcal{A},\mathrm{d}}$ and $\beta_{\mathcal{B},\mathrm{d}}$ (cf. Lemmas 8.3, 8.4, 8.5 and 9.2). Therefore, applying Lemma 9.10 to (9.28), we deduce the existence of a constant $\widehat{C}_{ST} > 0$, depending on $L_{\mathcal{A}}$, $\alpha_{\mathcal{A},\mathrm{d}}$ and $\beta_{\mathcal{B},\mathrm{d}}$, and hence independent of h, such that

$$\|\varphi-\varphi_h\|_{1;\Omega} + \|\chi-\chi_h\|_{-1/2;\Gamma} \leqslant \widehat{C}_{ST}\Big\{\mathrm{dist}\big(\varphi, \mathrm{H}_h^\varphi\big) + \mathrm{dist}\big(\chi, \mathrm{H}_h^\chi\big) + \|\mathcal{A}_{\mathbf{u}}(\varphi)-\mathcal{A}_{\mathbf{u}_h}(\varphi)\|_{\mathrm{H}_h^{\varphi\prime}}\Big\}\,. \tag{9.33}$$

Then, employing (7.4), (8.4) and Hölder inequality, we find that for each $\psi_h \in \mathrm{H}_h^{\varphi}$ there holds

$$|[\mathcal{A}_{\mathbf{u}}(\varphi) - \mathcal{A}_{\mathbf{u}_h}(\varphi), \psi_h]| \leqslant (1 + s_3)\|\mathrm{i}_4\|\|\varphi\|_{1;\Omega}\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\|\psi_h\|_{1;\Omega}\,,$$

which yields

$$\|\mathcal{A}_{\mathbf{u}}(\varphi) - \mathcal{A}_{\mathbf{u}_h}(\varphi)\|_{\mathrm{H}_h^{\varphi\prime}} \leqslant (1 + s_3)\|\mathrm{i}_4\|\|\varphi\|_{1;\Omega}\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,. \tag{9.34}$$

Then from (9.34), (9.33) and (8.22), we obtain (9.32) with $\widetilde{C}_{ST} := \widehat{C}_{ST}\max\big\{1, (1+s_3)\|\mathrm{i}_4\|C_{\widetilde{\mathbf{S}}}\big\}$.

□

The required Céa estimate will follow from Lemmas 9.11 and 9.12. Incorporating (9.32) into (9.29), and performing some algebraic manipulations, we find that there exist $\widetilde{C}_1, C_1 > 0$, independent of h, such that

$$\begin{aligned}\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} &\leqslant \widetilde{C}_1\Big\{\mathrm{dist}\big((\vec{\mathbf{t}}, \vec{\mathbf{u}}), \mathbf{H}_h \times \mathbf{Q}_h\big) + \mathrm{dist}(\varphi, \mathrm{H}_h^{\varphi}) + \mathrm{dist}(\chi, \mathrm{H}_h^{\chi})\Big\} \\ &+ C_1\Big\{C(\mathbf{u}_D, \varphi_D)\|\varphi_D\|_{1/2;\Gamma} + \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\}\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,.\end{aligned} \tag{9.35}$$

Thus, imposing the constant multiplying $\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}$ in (9.35) to be sufficient small, say less than or equal to 1/2, provides the a priori error estimate for $\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}}$, which, employed then to bound the third term on the right-hand side of (9.32), yields an upper bound for $\|\varphi - \varphi_h\|_{1;\Omega} + \|\chi - \chi_h\|_{-1/2;\Gamma}$. More precisely, we have proved the following result.

Theorem 9.13. *Assume that the data $\mathbf{u}_D$ and φ_D satisfy*

$$C_1\Big\{C(\mathbf{u}_D, \varphi_D)\|\varphi_D\|_{1/2;\Gamma} + \|\mathbf{u}_D\|_{1/2;\Gamma} + \|\varphi_D\|_{1/2;\Gamma}\Big\} \leqslant \frac{1}{2}\,. \tag{9.36}$$

Then, there exists a positive constant C_{d}, independent of h, such that

$$\begin{aligned}&\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} + \|\varphi - \varphi_h\|_{1;\Omega} + \|\chi - \chi_h\|_{-1/2;\Gamma} \\ &\leqslant C_{\mathrm{d}}\Big\{\mathrm{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_h\big) + \mathrm{dist}\big(\vec{\mathbf{u}}, \mathbf{Q}_h\big) + \mathrm{dist}\big(\varphi, \mathrm{H}_h^{\varphi}\big) + \mathrm{dist}\big(\chi, \mathrm{H}_h^{\chi}\big)\Big\}\,.\end{aligned} \tag{9.37}$$

Finally, regarding the pressure error $\|p - p_h\|_{0;\Omega}$, where p_h is the discrete pressure computed by the postprocessing formula suggested by the second identity in (7.7), that is

$$p_h \;=\; -\frac{1}{n}\operatorname{tr}\big(\boldsymbol{\sigma}_h + (\mathbf{u}_h \otimes \mathbf{u}_h)\big)\,, \tag{9.38}$$

we readily deduce from (9.37), similarly as in [26, Section 4] (see also [48, eq. (4.39)]), the existence of a positive constant $\widehat{C}$, independent of h, such that

$$\|p - p_h\|_{0;\Omega} \leqslant \widehat{C}\Big\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0;\Omega} \,+\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,. \tag{9.39}$$

Thus, combining (9.37) and (9.39), we conclude the existence of $\widehat{C}_{\mathrm{d}} > 0$, independent of h, such that

$$\begin{aligned} &\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{H}} \,+\, \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Q}} \,+\, \|p - p_h\|_{0;\Omega} \,+\, \|\varphi - \varphi_h\|_{1;\Omega} \,+\, \|\chi - \chi_h\|_{-1/2;\Gamma} \\ &\quad \leqslant \widehat{C}_{\mathrm{d}}\Big\{\operatorname{dist}\big(\vec{\mathbf{t}}, \mathbf{H}_h\big) \,+\, \operatorname{dist}\big(\vec{\mathbf{u}}, \mathbf{Q}_h\big) \,+\, \operatorname{dist}\big(\varphi, \mathrm{H}_h^{\varphi}\big) \,+\, \operatorname{dist}\big(\chi, \mathrm{H}_h^{\chi}\big)\Big\}\,. \end{aligned} \tag{9.40}$$

9.5 Specific finite element spaces

We refer to [48, Section 4.4] and [8, Section 3.5] to specify two examples of finite element subspaces $\mathbb{H}_h^{\mathbf{t}}$, $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\gamma}$, H_h^{φ} and H_h^{χ} satisfying the hypotheses (**H.0**), (**H.1**), (**H.2**), (**H.3**), (**H.4**) and (**H.5**) from Chapter 9.2, and establish the associated rates of convergence for the Galerkin scheme (9.3).

9.5.1 Preliminaries

Given an integer $\ell \geqslant 0$ and $K \in \mathcal{T}_h$, let $\mathrm{P}_\ell(K)$ denote the space of polynomials of degree $\leqslant \ell$ defined on K with vector and tensorial versions denoted by $\mathbf{P}_\ell(K) := [\mathrm{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) := [\mathrm{P}_\ell(K)]^{n\times n}$, respectively. By $\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) + \mathrm{P}_\ell(K)\mathbf{x}$ we denote the local Raviart–Thomas space of order ℓ defined on K, where $\mathbf{x}$ stands for a generic vector in R^n. Furthermore, denoting by b_K the bubble function on K (the product of its $n+1$ barycentric coordinates), we set the local bubble space of order ℓ as

$$\mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\,\mathrm{P}_\ell(K)\big) \quad \text{if} \quad n=2\,, \quad \text{and} \quad \mathbf{B}_\ell(K) := \operatorname{curl}\big(b_K\,\mathbf{P}_\ell(K)\big) \quad \text{if} \quad n=3\,,$$

where $\operatorname{curl}(v) := \big(\frac{\partial v}{\partial x_2}, -\frac{\partial v}{\partial x_1}\big)$ if $n=2$ and $v: K \to \mathrm{R}$, and $\operatorname{curl}(\mathbf{v}) := \nabla\times\mathbf{v}$ if $n=3$ and $\mathbf{v}: K \to \mathrm{R}^3$. In addition, we need to set the global spaces

$$\begin{aligned}
\mathbf{P}_\ell(\Omega) &:= \Big\{\mathbf{v}_h \in \mathbf{L}^2(\Omega): \quad \mathbf{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{P}_\ell(\Omega) &:= \Big\{\boldsymbol{\delta}_h \in \mathbb{L}^2(\Omega): \quad \boldsymbol{\delta}_h|_K \in \mathbb{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{RT}_\ell(\Omega) &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{RT}_\ell(K) \quad \forall\, i \in \big\{1,...,n\big\}\,, \quad \forall\, K \in \mathcal{T}_h\Big\}\,,\\
\mathbb{B}_\ell(\Omega) &:= \Big\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega): \quad \boldsymbol{\tau}_{h,i}|_K \in \mathbf{B}_\ell(K) \quad \forall\, i \in \big\{1,...,n\big\}\,, \quad \forall\, K \in \mathcal{T}_h\Big\}\,,
\end{aligned}$$

where $\boldsymbol{\tau}_{h,i}$ stands for the ith-row of $\boldsymbol{\tau}_h$. As noticed in [49], it is easily seen that $\mathbf{P}_\ell(\Omega)$ and $\mathbb{P}_\ell(\Omega)$ are also subspaces of $\mathbf{L}^4(\Omega)$ and $\mathbb{L}^4(\Omega)$, respectively, and that $\mathbb{RT}_\ell(\Omega)$ and $\mathbb{B}_\ell(\Omega)$ are both subspaces of $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. Actually, since $\mathbb{H}(\mathbf{div};\Omega)$ is clearly contained in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, any subspace of the former is also subspace of the latter.

9.5.2 Two specific examples

Similarly to [48, Section 4.4], we employ the stable triplets for linear elasticity proposed in [49, Section 4.4] to describe two examples of finite element subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$ and $\mathbb{H}_h^{\boldsymbol{\gamma}}$ and $\mathbb{H}_h^{\mathbf{t}}$ satisfying (**H.0**)-(**H.3**).

First, we consider PEERS$_\ell$ (plane elasticity element with reduced symmetry of order $\ell \geqslant 0$, [11,57]), and the subspace $\mathbb{H}_h^{\mathbf{t}}$ introduced in [48, Section 4.4.2]. Letting $\mathbb{C}(\bar\Omega) := [C(\bar\Omega)]^{n\times n}$, we have

$$\begin{gathered}
\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{\ell+n}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \mathbb{H}_h^{\boldsymbol{\sigma}} := \big(\mathbb{RT}_\ell(\Omega) \oplus \mathbb{B}_\ell(\Omega)\big) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,\\
\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \text{and} \quad \mathbb{H}_h^{\boldsymbol{\gamma}} := \mathbb{C}(\bar\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_{\ell+1}(\Omega)\,.
\end{gathered} \tag{9.41}$$

Secondly, AFW$_\ell$ (Arnold–Falk–Winther elements of order $\ell \geqslant 0$, [12]), and $\mathbb{H}_h^{\mathbf{t}}$ as in [48, Section

4.4.3]:

$$\begin{gathered}\mathbb{H}_h^{\mathbf{t}} := \mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \mathbb{H}_h^{\boldsymbol{\sigma}} := \big(\mathbb{P}_{\ell+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)\big) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,, \\ \mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_\ell(\Omega)\,, \quad \mathbb{H}_h^{\gamma} := \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{P}_\ell(\Omega)\,. \end{gathered} \tag{9.42}$$

In addition, and similarly to [8, Section 3.5] (see also [4, Section 4.3]), the approximation space for temperature will consist of continuous piecewise polynomials of degree $\leqslant \ell + 1$

$$\mathrm{H}_h^{\varphi} := \left\{\psi_h \in \mathrm{C}(\bar{\Omega}) : \quad \psi_h|_K \in \mathrm{P}_{\ell+1}(K) \quad \forall K \in \mathcal{T}_h\right\}, \tag{9.43}$$

and for the normal heat flux, we let $\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, ..., \widetilde{\Gamma}_m\}$ be an independent triangulation of Γ (made of straight segments in R^2, or triangles in R^3), and hence $\widetilde{h} := \max\limits_{j\in\{1,...,m\}} |\widetilde{\Gamma}_j|$. Then, we approximate χ by piecewise polynomials of degree $\leqslant \ell$ over this new mesh, that is

$$\mathrm{H}_{\tilde{h}}^{\chi} := \left\{\xi_{\tilde{h}} \in \mathrm{L}^2(\Gamma) : \quad \xi_{\tilde{h}}|_{\widetilde{\Gamma}_j} \in \mathrm{P}_\ell(\widetilde{\Gamma}_j) \quad \forall\, j \in \{1, ..., m\}\right\}. \tag{9.44}$$

Assumption (**H.4**) is trivially satisfied, whereas it can be proved (cf. [13, Section III], [38, Lemma 4.10], [47, Lemma 4.7]) that there exists a positive constant $\widetilde{c}_0 \in (0, 1]$ such that, provided that $h \leqslant \widetilde{c}_0\, \widetilde{h}$, $\mathrm{H}_{\tilde{h}}^{\chi}$ satisfies (**H.5**) as well.

9.5.3 The rates of convergence

According to [48, 49], and denoting $\ell^* := \begin{cases} \ell + n & \text{for PEERS-based} \\ \ell + 1 & \text{for AFW-based} \end{cases}$, the approximation properties of $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, and $\mathbb{H}_h^{\gamma}$, for PEERS (cf. (9.41)) as well as for AFW (cf. (9.42)), are given as follows:

$\left(\mathbf{AP}_h^{\mathbf{t}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell^* + 1]$, and for each $\mathbf{s} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}\big) \leqslant C\, h^r \|\mathbf{s}\|_{r,\Omega}\,, \tag{9.45}$$

$\left(\mathbf{AP}_h^{\boldsymbol{\sigma}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell + 1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^r(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{r,4/3}(\Omega)$, there holds

$$\operatorname{dist}\left(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}\right) \leqslant C\, h^r \left\{ \|\boldsymbol{\tau}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{r,4/3;\Omega} \right\}, \tag{9.46}$$

$\left(\mathbf{AP}_h^{\mathbf{u}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell + 1]$, and for each $\mathbf{v} \in \mathbf{W}^{r,4}(\Omega)$, there holds

$$\operatorname{dist}\left(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}\right) \leqslant C\, h^r \|\mathbf{v}\|_{r,4;\Omega}, \tag{9.47}$$

and

$\left(\mathbf{AP}_h^{\boldsymbol{\gamma}}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell + 1]$, and for each $\boldsymbol{\delta} \in \mathbb{H}^r(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\operatorname{dist}\left(\boldsymbol{\delta}, \mathbb{H}_h^{\boldsymbol{\gamma}}\right) \leqslant C\, h^r \|\boldsymbol{\delta}\|_{r,\Omega}. \tag{9.48}$$

Aditionally, the approximation properties for the subspaces H_h^{φ} and $\mathrm{H}_{\tilde{h}}^{\chi}$ (cf. [21] and [47]), are the following:

$\left(\mathbf{AP}_h^{\varphi}\right)$ there exists a positive constant C, independent of h, such that for each $r \in [0, \ell + 1]$, and for each $\psi \in \mathrm{H}^{1+r}(\Omega)$, there holds

$$\operatorname{dist}\left(\psi, \mathrm{H}_h^{\varphi}\right) \leqslant C\, h^r \|\psi\|_{1+r,\Omega}, \tag{9.49}$$

$\left(\mathbf{AP}_h^{\chi}\right)$ there exists a positive constant C, independent of $\tilde{h}$, such that for each $r \in [0, \ell + 1]$, and for each $\xi \in \mathrm{H}^{-1/2+r}(\Gamma)$, there holds

$$\operatorname{dist}\left(\xi, \mathrm{H}_{\tilde{h}}^{\chi}\right) \leqslant C\, \tilde{h}^r \|\xi\|_{-1/2+r,\Gamma}. \tag{9.50}$$

We are now in position to specify the rates of convergence of (9.3) with the spaces from

Chapter 9.5.2.

Theorem 9.14. *Assume that the data satisfy* (9.36), *and let* $(\vec{\mathbf{t}},\vec{\mathbf{u}}) := \big((\mathbf{t},\boldsymbol{\sigma}),(\mathbf{u},\boldsymbol{\gamma})\big) \in \mathbf{H}\times\mathbf{Q}$ *and* $(\varphi,\chi) \in \mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$, *and* $(\vec{\mathbf{t}}_h,\vec{\mathbf{u}}_h) := \big((\mathbf{t}_h,\boldsymbol{\sigma}_h),(\mathbf{u}_h,\boldsymbol{\gamma}_h)\big) \in \mathbf{H}_h\times\mathbf{Q}_h$ *and* $(\varphi_h,\chi_h) \in \mathrm{H}_h^{\varphi}\times\mathrm{H}_h^{\chi}$, *be solutions of* (8.8) *and* (9.3), *respectively, with* $\mathbf{u}\in\mathbf{W}$ (*cf.* (8.23)) *and* $\mathbf{u}_h\in\mathbf{W}_h$ (*cf.* (9.12)), *whose existences are guaranteed by Theorems* 8.11 *and* 9.8, *respectively. In turn, let* p *and* p_h *be the exact and approximate pressure defined by the second identity in* (7.7) *and* (9.38), *respectively. Furthermore, given an integer* $\ell \geqslant 0$, *assume that there exists* $r\in[0,\ell+1]$ *such that* $\mathbf{t}\in\mathbb{H}^r(\Omega)\cap\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\boldsymbol{\sigma}\in\mathbb{H}^r(\Omega)\cap\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma})\in\mathbf{W}^{r,4/3}(\Omega)$, $\mathbf{u}\in\mathbf{W}^{r,4}(\Omega)$, $\boldsymbol{\gamma}\in\mathbb{H}^r(\Omega)\cap\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $\varphi\in\mathrm{H}^{1+r}(\Omega)$, *and* $\chi\in\mathrm{H}^{-1/2+r}(\Gamma)$. *Then, there exist constants* $\widetilde{c}_0\in(0,1]$ *and* $C>0$, *independent of* h *and* $\widetilde{h}$, *such that for all* $h \leqslant \widetilde{c}_0\,\widetilde{h}$, *there holds*

$$\begin{aligned}
&\|\vec{\mathbf{t}}-\vec{\mathbf{t}}_h\|_{\mathbf{H}} + \|\vec{\mathbf{u}}-\vec{\mathbf{u}}_h\|_{\mathbf{Q}} + \|p-p_h\|_{0,\Omega} + \|\varphi-\varphi_h\|_{1;\Omega} + \|\chi-\chi_h\|_{-1/2;\Gamma}\\
&\leqslant C\,h^r\Big\{\|\mathbf{t}\|_{r,\Omega} + \|\boldsymbol{\sigma}\|_{r,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{r,4/3;\Omega} + \|\mathbf{u}\|_{r,4;\Omega} + \|\boldsymbol{\gamma}\|_{r,\Omega} + \|\varphi\|_{1+r;\Omega}\Big\} \qquad (9.51)\\
&\quad + C\,\widetilde{h}^r\|\chi\|_{-1/2+r,\Gamma}\,.
\end{aligned}$$

Proof. It follows straightforwardly from Céa's estimate (9.40) and the approximation properties $(\mathbf{AP}_h^{\mathbf{t}})$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, $(\mathbf{AP}_h^{\gamma})$, $(\mathbf{AP}_h^{\varphi})$ and $(\mathbf{AP}_h^{\chi})$. □

CHAPTER 10

Illustrative numerical examples

In this chapter we demonstrate properties of the proposed family of methods. Mesh generation, discretization, and solvers were implemented using the automated finite element library FEniCS [7] and, in particular, the specialized module FEniCS$_{ii}$ [55] required for the treatment of mixed-dimensional meshes of non-conforming type (and also instrumental to numerically realize the $H^{-1/2}(\Gamma)$ norm). The nonlinear algebraic equations were solved using a Newton–Raphson method with exact Jacobian, and the iterations were terminated once the ℓ^2-norm of either the relative or absolute residual drops below the prescribed tolerance 10^{-7}. The numerical tests are divided into three parts: a verification of convergence, the simulation of stationary phase change in 2D, and the extension to the 3D case.

Example 1. Let the square domain $\Omega = (0,1)^2$ meshed by successively refined regular triangles. We use this simple test case to assess the convergence of the finite element discretization, and consider the following smooth closed-form primary variables for an adaptation of the Burggraf flow [23] (a regularization of the well-known lid-driven cavity flow but here there is no velocity singularity at the top corners) to the case of thermally driven problems (see, e.g., [62])

$$\mathbf{u} = C_0 \begin{pmatrix} C_1'(x)C_2'(y) \\ -C_1''(x)C_2(y) \end{pmatrix},$$
$$p = \frac{C_0}{\mathrm{Re}}[C_2^{(3)}(y)C_1(x) + C_1''(x)C_2'(y)] + \frac{C_0^2}{2}C_1'(x)^2[C_2(y)C_2''(y) - C_2'(y)^2]\,,$$
$$\varphi = \varphi_0 + (\varphi_1 - \varphi_0)y + C_3(x)C_4(y),$$

with $C_0 > 0$ a scaling parameter and

$$C_1(x) = \frac{x^5}{5} - \frac{x^4}{2} + \frac{x^3}{3}, \quad C_2(x) = y^4 - y^2, \quad C_3(x) = \cos(\pi x), \quad C_4(y) = y(1-y).$$

These solutions are used to set boundary velocity and temperature to be imposed on the boundary. Also, as typically done when using manufactured solutions, after inserting these closed-form functions into the governing momentum and energy equations, additional source terms appear that constitute an augmented problem [77] (the mass conservation is satisfied as the manufactured velocity is divergence-free).

We consider the strong form (7.8) with the following constitutive equations and adimensional model parameters

$$\mu(\varphi) = \frac{1}{4}\exp(-\varphi), \quad f(\varphi) = \varphi(1-\varphi), \quad \eta(\varphi) = \frac{1}{4} + \frac{1}{2}\Big(1 + \tanh\Big(2\big(\frac{1}{4} - \varphi\big)\Big)\Big),$$
$$s(\varphi) = \frac{1}{2}\Big(1 + \tanh\Big(2\big(\frac{1}{4} - \varphi\big)\Big)\Big), \quad C_0 = \lambda = \rho = 1, \quad \kappa = 1.4, \quad \boldsymbol{k} = (0,1)^{\mathrm{t}}.$$

This choice of parameter regime is simply exemplary and similar in magnitude to the experiments considered in [62]. The null mean value for the trace of the stress is enforced through a real Lagrange multiplier method. Note that, as requested by the constraint $h \leqslant \widetilde{c}_0 \widetilde{h}$ (cf. remark on the verification of (**H.5**) at the end of Chapter 9.5.2), the mesh for the heat flux approximation is simply taken as two levels lower than a conforming mesh to the boundary of the bulk mesh (the former is constructed with $2^{j+2} + 4$ segments per side and the latter with $2^j + 1$ segments per side, giving $\widetilde{h} \approx 4h$).

Absolute errors are measured in the norms suggested by the analysis (where the exact solutions are evaluated at the quadrature points), which we denote – together with the experimental rates of convergence – as usual

$$\mathtt{e}(\mathbf{t}) = \|\mathbf{t}-\mathbf{t}_h\|_{0,\Omega}, \quad \mathtt{e}(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbb{H}(\mathbf{div}_{4/3};\Omega)}, \quad \mathtt{e}(\mathbf{u}) = \|\mathbf{u}-\mathbf{u}_h\|_{0,4,\Omega}, \quad \mathtt{e}(\boldsymbol{\gamma}) = \|\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\|_{0,\Omega},$$
$$\mathtt{e}(p) = \|p-p_h\|_{0,\Omega}, \quad \mathtt{e}(\varphi) = \|\varphi-\varphi_h\|_{1,\Omega}, \quad \mathtt{e}(\chi) = \|\chi-\chi_{\widetilde{h}}\|_{-1/2,\Gamma},$$
$$\mathtt{r}(\chi) = \frac{\log(\mathtt{e}(\chi)/\mathtt{e}'(\chi))}{\log(\widetilde{h}/\widetilde{h}')}, \quad \mathtt{r}(\%) = \frac{\log(\mathtt{e}(\%)/\mathtt{e}'(\%))}{\log(h/h')},$$

with $\% \in \{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p, \boldsymbol{\gamma}, \varphi\}$, and where $\mathtt{e}, \mathtt{e}'$ stand for errors generated on two consecutive meshes of sizes h, h' ($\widetilde{h}$ and $\widetilde{h}'$ for χ), respectively.

To compute $\|\chi-\chi_{\widetilde{h}}\|_{-1/2,\Gamma}$ we use the characterization of $\mathrm{H}^{-1/2}(\Gamma)$ in terms of the spectral decomposition of the Laplacian operator (see, e.g., [56, Sect. 2]). More precisely, let $S : \mathrm{H}_0^1(\Gamma) \longrightarrow \mathrm{H}_0^1(\Gamma)$ be the bounded linear operator defined by

$$(Su, v)_{1,\Gamma} \;=\; (u, v)_{0,\Gamma} \quad \text{for all } u,\, v \in \mathrm{H}_0^1(\Gamma)\,,$$

where $(\cdot,\cdot)_{1,\Gamma}$ and $(\cdot,\cdot)_{0,\Gamma}$ denote the inner products of $\mathrm{H}_0^1(\Gamma)$ and $\mathrm{L}^2(\Gamma)$, respectively. Then, one can find a basis $\{z_i\}_{i=1}^{\infty}$ of eigenfunctions of S with a non-increasing sequence of positive eigenvalues λ_i, and for any $u = \sum_{i=1}^{\infty} c_i z_i$ there holds

$$\|u\|_{-1/2,\Gamma}^2 = \sum_{i=1}^{\infty} c_i^2 \lambda_i^{-1/2}\,,$$

so that $\mathrm{H}^{-1/2}(\Gamma)$ becomes the closure of the span of the basis $\{z_i\}_{i=1}^{\infty}$ in this norm. Certainly, for the practical computation of $\|u\|_{-1/2,\Gamma}^2$ one utilizes a discrete approximation of the aforementioned spectral decomposition.

We take $\ell = 0, 1$ in the PEERS_ℓ- and AFW_ℓ-based families of finite elements (9.41) and (9.42), respectively; with (9.43),(9.44). We show the results of the convergence verification analysis in Table 10.1. There we depict the errors and decay rate and observe, for all field variables, the optimal convergence order $h^{\ell+1}$ predicted by (9.51). Sample approximate solutions

DoF	h	$\widetilde{h}$	e($\mathbf{t}$)	r	e($\boldsymbol{\sigma}$)	r	e($\mathbf{u}$)	r	e($\boldsymbol{\gamma}$)	r	e(φ)	r	e(χ)	r	e(p)	r	it
PEERS$_\ell$-based discretization with $\ell = 0$																	
2631	0.177	0.707	5.05e-02	*	2.63e-01	*	1.15e-02	*	2.33e-02	*	1.61e-01	*	7.72e-03	*	3.99e-02	*	4
5865	0.118	0.471	3.37e-02	0.99	1.76e-01	1.00	7.78e-03	0.97	1.54e-02	1.02	9.62e-02	1.28	3.80e-03	1.75	2.60e-02	1.06	4
16173	0.071	0.283	2.01e-02	1.01	1.05e-01	1.00	4.68e-03	1.00	8.23e-03	1.23	5.33e-02	1.15	1.41e-03	1.94	1.53e-02	1.03	4
52149	0.039	0.157	1.11e-02	1.02	5.83e-02	1.00	2.60e-03	1.00	3.67e-03	1.38	2.84e-02	1.07	7.30e-04	1.12	8.46e-03	1.01	4
185541	0.021	0.083	5.83e-03	1.01	3.08e-02	1.00	1.38e-03	1.00	1.46e-03	1.44	1.48e-02	1.03	3.85e-04	0.94	4.47e-03	1.00	5
698085	0.011	0.043	2.99e-03	1.01	1.58e-02	1.00	7.09e-04	1.00	5.51e-04	1.47	7.58e-03	1.01	1.93e-04	0.96	2.30e-03	1.00	4
2706213	0.005	0.022	1.51e-03	1.00	8.04e-03	1.00	3.60e-04	1.00	2.01e-04	1.48	3.84e-03	1.00	1.12e-04	0.87	1.17e-03	1.00	4
PEERS$_\ell$-based discretization with $\ell = 1$																	
6027	0.177	0.707	1.08e-02	*	4.25e-02	*	1.44e-03	*	8.32e-03	*	7.59e-02	*	9.00e-03	*	8.01e-03	*	5
13455	0.118	0.471	5.20e-03	1.79	2.01e-02	1.85	6.52e-04	1.95	4.27e-03	1.65	3.52e-02	1.90	4.33e-03	1.81	4.17e-03	1.71	5
37143	0.071	0.283	2.09e-03	1.89	7.66e-03	1.88	2.38e-04	1.97	1.82e-03	1.67	1.31e-02	1.93	1.67e-03	1.87	1.78e-03	1.77	6
119847	0.039	0.157	5.12e-04	1.91	2.50e-03	1.90	7.43e-05	1.98	6.59e-04	1.73	4.16e-03	1.96	5.40e-04	1.92	6.44e-04	1.83	5
426567	0.021	0.083	1.28e-04	1.94	7.39e-04	1.92	2.09e-05	1.99	2.08e-04	1.81	1.19e-03	1.97	1.56e-04	1.95	2.07e-04	1.88	6
1605255	0.011	0.043	3.02e-05	1.97	1.37e-04	1.98	5.12e-06	1.98	5.11e-05	1.90	2.63e-04	1.98	3.60e-05	1.96	5.03e-05	2.00	5
6223623	0.005	0.022	7.58e-06	1.96	3.49e-05	1.97	1.32e-06	1.99	1.27e-05	1.95	6.71e-05	1.98	6.49e-06	2.00	1.19e-05	1.99	5
AFW$_\ell$-based discretization with $\ell = 0$																	
2070	0.177	0.707	4.26e-02	*	2.79e-01	*	1.15e-02	*	3.70e-02	*	1.61e-01	*	7.67e-03	*	5.18e-02	*	4
4592	0.118	0.471	2.60e-02	1.22	1.75e-01	1.15	7.73e-03	0.97	2.40e-02	1.07	9.62e-02	1.28	3.78e-03	1.75	2.91e-02	1.42	4
12612	0.071	0.283	1.46e-02	1.13	9.96e-02	1.10	4.66e-03	0.99	1.41e-02	1.05	5.33e-02	1.15	1.41e-03	1.93	1.48e-02	1.33	4
40556	0.039	0.157	7.81e-03	1.06	5.37e-02	1.05	2.60e-03	1.00	7.72e-03	1.02	2.84e-02	1.07	7.31e-04	1.11	7.36e-03	1.18	4
144060	0.021	0.083	4.08e-03	1.02	2.81e-02	1.02	1.38e-03	1.00	4.06e-03	1.01	1.48e-02	1.03	4.85e-04	0.64	3.71e-03	1.08	5
541532	0.011	0.043	2.09e-03	1.01	1.44e-02	1.01	7.09e-04	1.00	2.09e-03	1.00	7.58e-03	1.01	2.93e-04	0.76	1.88e-03	1.03	5
2098332	0.005	0.022	1.06e-03	1.00	7.31e-03	1.00	3.60e-04	1.00	1.06e-03	1.00	3.84e-03	1.00	1.62e-04	0.88	9.50e-04	1.01	5
AFW$_\ell$-based discretization with $\ell = 1$																	
5002	0.177	0.707	1.36e-02	*	5.23e-02	*	1.43e-03	*	1.14e-02	*	7.59e-02	*	9.00e-03	*	1.44e-02	*	4
11150	0.118	0.471	7.25e-03	1.75	2.60e-02	1.93	6.50e-04	1.95	5.77e-03	1.98	3.52e-02	1.90	4.33e-03	1.81	7.55e-03	1.79	4
30742	0.071	0.283	3.20e-03	1.80	1.05e-02	1.96	2.38e-04	1.97	2.37e-03	1.94	1.31e-02	1.93	1.67e-03	1.87	3.20e-03	1.88	5
99110	0.039	0.157	1.24e-03	1.92	3.71e-03	1.97	7.42e-05	1.98	8.41e-04	1.96	4.16e-03	1.96	5.40e-04	1.92	1.16e-03	1.93	5
352582	0.021	0.083	4.46e-04	1.90	1.21e-03	1.96	2.09e-05	1.99	2.45e-04	1.96	1.19e-03	1.97	1.56e-04	1.95	3.84e-04	1.94	4
1326470	0.011	0.043	1.56e-04	1.98	3.88e-04	1.92	5.56e-06	2.00	8.72e-05	1.93	3.18e-04	1.99	4.21e-05	1.97	1.23e-04	1.91	4
5142022	0.005	0.022	3.46e-05	1.95	1.15e-04	1.97	1.44e-06	2.00	2.28e-05	1.97	8.22e-05	1.99	1.09e-05	1.99	3.78e-05	1.97	4

Table 10.1: Example 1. Accuracy test for four variants of the proposed numerical method in 2D, using the Burggraf solutions. Errors for each field variable on a sequence of successively refined grids versus the number of degrees of freedom, experimental rates of convergence, and Newton–Raphson iteration count.

are provided in Figure 10.1, which are confirmed to follow the flow patterns obtained in [62].

Example 2. Next we consider the steady regime of the phase change of a material adopting a 2D slice of a shell-and-tube geometry configuration, which is commonly used in thermal energy storage systems [2,65]. We construct a unit disk-shaped geometry with four circular inclusions of radius $\frac{1}{8}$. The inner tubes are kept hot with $\varphi_{\mathrm{hot}} = 1$ and the outer shell is kept cold $\varphi_{\mathrm{hot}} = -0.01$ (which differs from the mixed Dirichlet–Neumann conditions used in [65]). For the flow equations, all boundaries are equipped with no-slip velocity conditions. The meshes are unstructured, and the mesh sizes selected for the bulk and for the boundary are $h \approx 0.022$

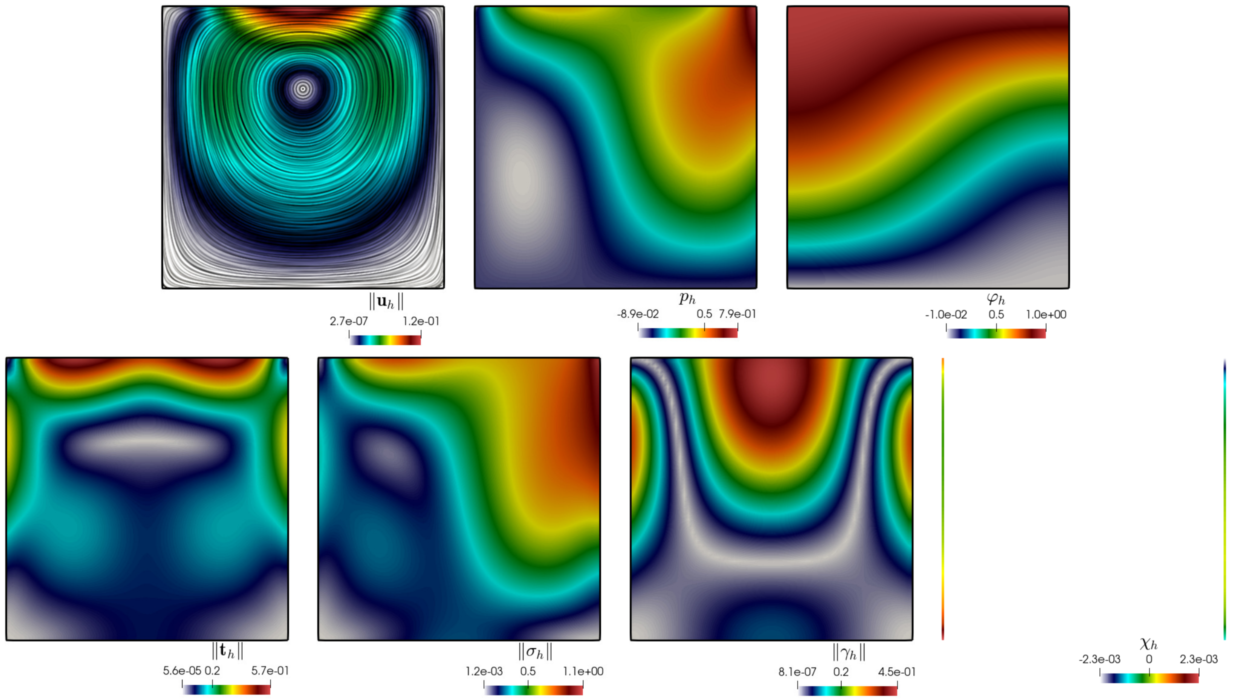

Figure 10.1: Example 1. Primal variables (velocity line integral convolution colored according to velocity magnitude, postprocessed pressure profile, and temperature distribution) and mixed unknowns (velocity gradient magnitude, stress magnitude, vorticity magnitude, and heat flux on the coarser boundary mesh) for the Burggraf stationary flow with thermal effects obtained after 4 steps of uniform refinement.

and $\widetilde{h} \approx 0.051$, respectively.

Similarly as in [9, 62], we use a porosity-enthalpy model, which means that the viscosity is taken constant. The temperature-dependent buoyancy, porosity and enthalpy functions are chosen as follows

$$f(\varphi) = \frac{\mathrm{Ra}}{\mathrm{Pr}}\varphi\mathbf{k}\,, \quad \eta(\varphi) = 10^5\Big(1 + \tanh\Big(\frac{0.01-\varphi}{0.2}\Big)\Big)\,,$$
$$s(\varphi) = \frac{1}{\mathrm{Ste}} - \frac{1}{2\cdot\mathrm{Ste}}\Big(1 + \tanh\Big(\frac{0.01-\varphi}{0.2}\Big)\Big)\,,$$

respectively, where the denominator in the argument of the hyperbolic tangent regularization indicates the size of the mushy zone (the region that approximates a sharp phase fraction jump).

The remaining coefficients assume the following values

$$\mu = \lambda = \rho = 1, \quad \mathrm{Pr} = 56.2, \quad \mathrm{Ste} = 0.02, \quad \mathrm{Ra} = 3.27 \times 10^5, \quad \kappa = \frac{1}{\mathrm{Pr}}, \quad \boldsymbol{k} = (0,0,1)^{\mathrm{t}},$$

where Ste denotes the Stefan number.

In Figure 10.2 we have portrayed the approximate solutions, generated with the second-order PEERS$_\ell$-based finite element family (9.41). In particular, the bottom-right panel of the figure shows the approximate heat flux on the (coarser) boundary mesh, and the top-right panel shows the typical counter rotating flow patterns expected in differentially heated enclosures. No closed-form solution is available for this problem but all fields exhibit a well resolved behavior, even on relative coarse meshes.

Example 3. Our last test, adapted from [8], simulates the phase change occurring in the melting of N–octadecane. The domain consists of the cuboid $\Omega = (0, 1.5, 0.3, 1.5)\,\mathrm{cm}^3$. For the thermal energy conservation, the boundary is split into two regions: $\Gamma_{\mathrm{hot}} \cup \Gamma_{\mathrm{cold}}$ (left and right ends) and Γ_{flux} (remainder of the boundary) where temperature and heat flux are prescribed, respectively. The molten material is on the "left" of the domain (towards the wall at $x = 0$ where we prescribe $\varphi_{\mathrm{hot}} = 1$). The low temperature imposed on the right wall $x = 1.5\,\mathrm{cm}$, $\varphi_{\mathrm{cold}} = -0.01$ is lower than the phase change temperature $\varphi = 0$, in order to allow the phase change to occur. The remaining boundaries are insulated (zero temperature flux), and on the whole boundary we impose no-slip conditions ($\mathbf{u} = \mathbf{0}$ everywhere on Γ). For this test we use a space resolution of $h \approx 0.07\,\mathrm{cm}$ and for the boundary sub-mesh we use a triangulation with $\widetilde{h} \approx 0.12\,\mathrm{cm}$.

As in example 2, here we use a porosity-enthalpy model together with the following constitutive relations and parameter scalings

$$\mu = \mathrm{Re} = 1, \quad f(\varphi) = \frac{\mathrm{Ra}}{\mathrm{Pr} \cdot \mathrm{Re}^2}\varphi\mathbf{k}, \quad \eta(\varphi) = 10^5\Big(1 + \tanh\Big(\frac{0.01 - \varphi}{0.1}\Big)\Big),$$
$$\mathrm{Pr} = 56.2, \quad \mathrm{Ra} = 3.27 \times 10^5, \quad s(\varphi) = \frac{1}{\mathrm{Ste}} - \frac{1}{2 \cdot \mathrm{Ste}}\Big(1 + \tanh\Big(\frac{0.01 - \varphi}{0.1}\Big)\Big),$$
$$\mathrm{Ste} = 0.045, \quad \lambda = \rho = 1, \quad \kappa = \frac{10}{\mathrm{Pr} \cdot \mathrm{Re}}, \quad \boldsymbol{k} = (0,0,1)^{\mathrm{t}}.$$

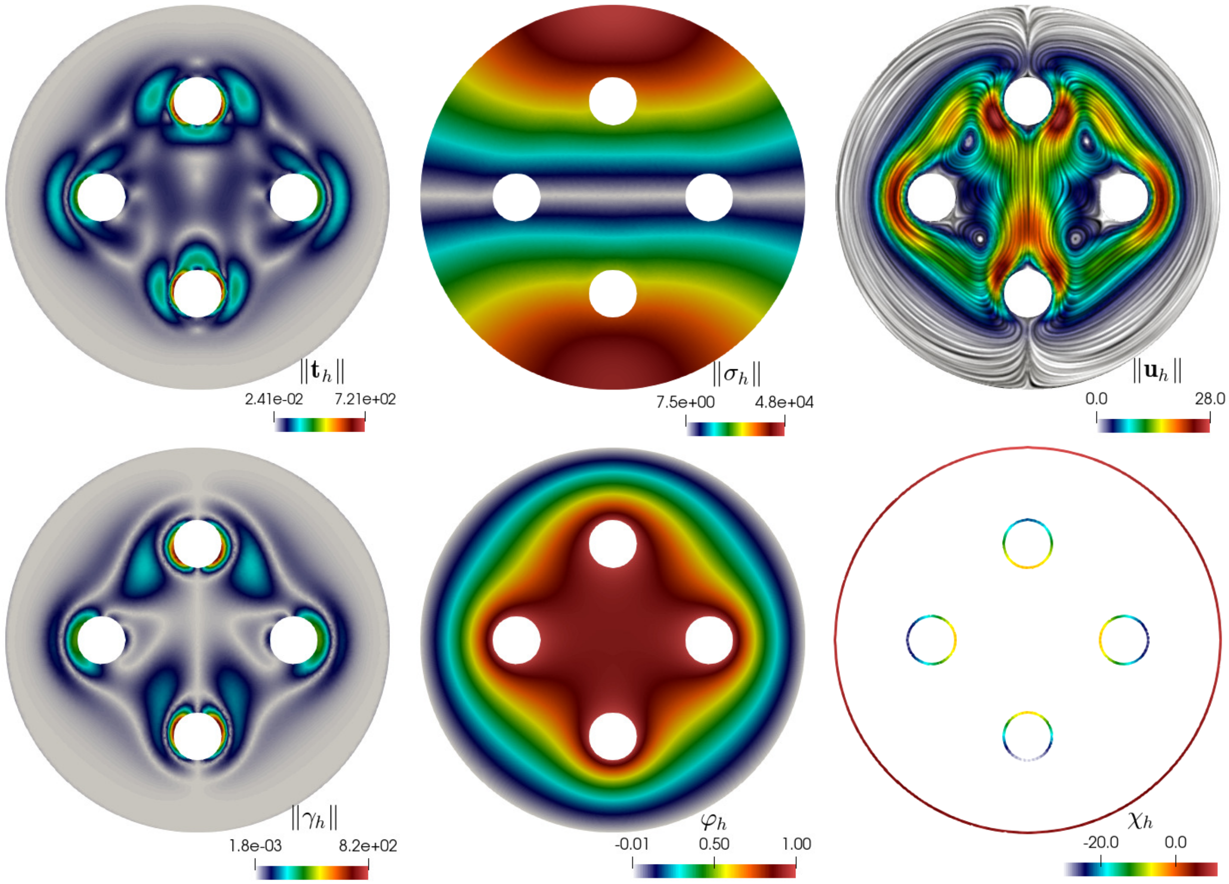

Figure 10.2: Example 2. Phase change on a differentially heated shell-tube system. Approximate solutions (velocity gradient magnitude, total stress magnitude, velocity streamlines, vorticity magnitude, dimensionless temperature, and heat flux) computed with the second-order PEERS$_\ell$-based mixed-primal method.

Given the strong nonlinearity of the non-isothermal coupling, it was necessary to use a continuation approach (the initial guess at each Newton–Raphson iteration is improved by solving intermediate problems with an increased value of a given parameter) and as continuation parameter we use the Rayleigh number starting from Ra$= 10^3$. Nine iterations are required in this case to reach the prescribed tolerance.

The thermal and fluid flow characteristics of the system are shown in Figure 10.3 where we plot temperature iso-surfaces, velocity streamlines, and all other computed numerical solutions using the lowest-order method based on the AFW$_\ell$ family of finite elements. The obtained flow structures are qualitatively similar to the expected behaviour for a stationary coupling (that

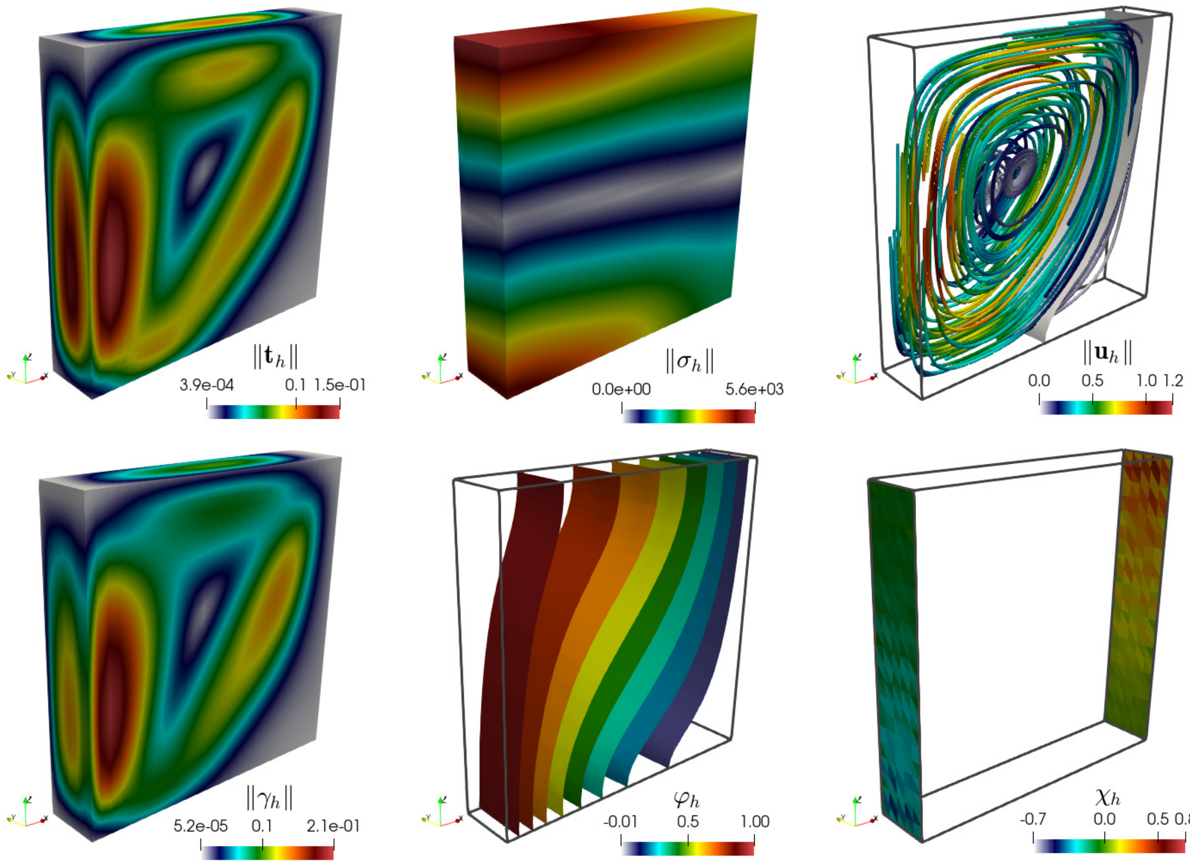

Figure 10.3: Example 3. Phase change of an octadecane specimen. Approximate solutions (velocity gradient magnitude, total stress magnitude, velocity streamlines, vorticity magnitude, temperature, and heat flux) computed with the lowest-order AFW_ℓ-based method.

is, a buoyancy-diven recirculation with a relatively large solid-liquid interface and the typical temperature distribution on the xz-plane).

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CHAPTER 11

Conclusions and Future Works

In this chapter we summarize the main contributions of this work and give a brief description of eventual future works.

11.1 Conclusions

Upon the results presented in the first part of this thesis, we can arrive to the following conclusions:

- We developed a new mixed formulation for Navier–Stokes–Brinkman equations, whose analysis made use of diverse tools and abstract results in Banach spaces.
- We showed that the pressure field can be obtained by using a post-processing formula based on the computed variables.
- We proved that is not necessary to use an augmented formulation to provide well posedness of the continuous and discrete formulations.

- We proved that the finite element method proposed here yields optimal convergence, which is confirmed through numerical examples.

According to the results presented in the second part of this work, we can state the following conclusions:

- We extended the analysis developed in the first part to introduce a mixed-primal formulation for the coupling of Navier–Stokes–Brinkman and natural convection equations.
- The mixed-primal finite element method proposed here was shown to be optimally convergent, which has been confirmed by several numerical examples.

11.2 Future Works

The methods developed and the results obtained here have motivated some posibilities of future work, which are described below:

- To extend the analysis and results to the unsteady state case.
- To develop the corresponding a posteriori error analysis.
- To extend the analysis to the case in which a mixed formulation is employed as well for the natural convection equations.

Bibliography

[1] R.A. Adams and J.J.F. Fournier, *Sobolev Spaces.* Academic Press, Elsevier Ltd, 2003.

[2] F. Agyenim, N. Hewitt, P. Eames and M. Smyth, *A review of materials, heat transfer and phase change problem formulation for latent heat thermal energy storage systems (LHTESS).* Renew. Sustain. Energy Rev. 14 (2010), no. 2, 615–628.

[3] R. Aldbaissy, F. Hecht, G. Mansour and T. Sayah, *A full discretisation of the time-dependent Boussinesq (buoyancy) model with nonlinear viscosity.* Calcolo 55 (2018), no. 4 49, Paper No 44.

[4] J.A. Almonacid, G.N. Gatica and R. Oyarzúa, *A mixed–primal finite element method for the Boussinesq problem with temperature-dependent viscosity.* Calcolo 55 (2018), no. 3, Paper No. 36.

[5] J.A. Almonacid, G.N. Gatica, R. Oyarzúa and R. Ruiz-Baier, *A new mixed finite element method for the n-dimensional Boussinesq problem with temperature-dependent viscosity.* Netw. Heterog. Media 15 (2020), no. 2, 215–245.

[6] J.A. Almonacid, G.N. Gatica and R. Ruiz-Baier, *Ultra-weak symmetry of stress for augmented mixed finite element formulations in continuum mechanics.* Calcolo 57 (2020), no. 1, Paper No. 2.

[7] M.S Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M.E. Rognes and G.N. Wells, *The FEniCS project version 1.5.* Arch. Numer. Softw., 3 (2015), no. 100, 9–23.

[8] M. Alvarez, G.N. Gatica, B. Gómez-Vargas and R. Ruiz-Baier, *New mixed finite element methods for natural convection with phase-change in porous media.* J. Sci. Comput. 80 (2019), no. 1, 141–174.

[9] M. Alvarez, B. Gómez-Vargas, R. Ruiz-Baier and J. Woodfield, *Stability and finite element approximation of phase change models for natural convection in porous media.* J. Comput. Appl. Math. 360 (2019), 117–137.

[10] P.R. Amestoy, I.S. Duff, and J.-Y. L'Excellent, *Multifrontal parallel distributed symmetric and unsymmetric solvers.* Comput. Methods Appl. Mech. Engrg. 184 (2000), 501–520.

[11] D.N. Arnold, F. Brezzi and J. Douglas, PEERS: *A new mixed finite element method for plane elasticity.* Jpn. J. Appl. Math. 1 (1984), 347–367.

[12] D.N. Arnold, R.S. Falk and R. Winther, *Mixed finite element methods for linear elasticity with weakly imposed symmetry¿* Math. Comp. 76 (2007) , no. 260, 1699–1723.

[13] I. Babuška and G.N. Gatica, *On the mixed finite element method with Lagrange multipliers.* Numer. Methods Partial Differential Equations 19 (2003), no. 2, 192–210.

[14] G. Baird, R. Bürger, P.E. Méndez and R. Ruiz-Baier, *Second-order schemes for axisymmetric Navier-Stokes-Brinkman and transport equations modelling water filters.* Numer. Math. 147 (2021), no. 2, 431–479.

[15] L. Balazi Atchy Nillama, J. Yang and L. Yang, *An explicit stabilised finite element method for Navier-Stokes-Brinkman equations.* J. Comput. Phys. 457 (2022), Paper No. 111033.

[16] G.A. Benavides, S. Caucao, G.N. Gatica and A.A. Hopper, *A Banach spaces–based analysis of a new mixed–primal finite element method for a coupled flow– transport problem*. Comput. Methods Appl. Mech. Engrg. 371 (2020), 113285.

[17] G.A. Benavides, S. Caucao, G.N. Gatica and A.A. Hopper, *A new non-augmented and momentum-conserving fully-mixed finite element method for a coupled flow-transport problem*. Calcolo 59 (2022), no. 1, Paper No. 6.

[18] C. Bernardi, C. Canuto and Y. Maday, *Generalized inf-sup conditions for Chebyshev spectral approximation of the Stokes problem.* SIAM J. Numer. Anal. 25 (1988), no. 6, 1237–1271.

[19] D. Boffi, F. Brezzi and M. Fortin, Mixed Finite Element Methods and Applications. Springer Series in Computational Mathematics, 44. Springer, Heidelberg, 2013.

[20] S.C. Brenner and L.R. Scott, The Mathematical Theory of Finite Element Methods, third ed., in: Texts in Applied Mathematics, vol. 15, Springer, New York, 2008.

[21] F. Brezzi and M. Fortin, Mixed and Hybrid Finite Element Methods. Springer-Verlag, 1991.

[22] P. Burda and M. Hasal, *An a posteriori error estimate for the Stokes-Brinkman problem in a polygonal domain*. Programs and algorithms of numerical mathematics 17, 32–40, Acad. Sci. Czech Repub. Inst. Math., Prague, 2015.

[23] O.R. Burggraf, *Analytical and numerical studies of the structure of steady separated flows*. J. Fluid Mech. 24 (1966), 113–151.

[24] R. Bürger, S.K. Kenettinkara, R. Ruiz-Baier and H. Torres, *Coupling of discontinuous Galerkin schemes for viscous flow in porous media with adsorption*. SIAM J. Sci. Comput. 40 (2018), no. 2, B637–B662.

[25] J. Camaño, C. García and R. Oyarzúa, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem.* Numer. Methods Partial Differential Equations 37 (2021), no. 5, 2895–2923.

[26] J. Camaño, G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier, *An augmented stress-based mixed finite element method for the steady state Navier-Stokes equations with nonlinear viscosity.* Numer. Methods Partial Differential Equations 33 (2017), no. 5, 1692–1725.

[27] J. Camaño, G.N. Gatica, R. Oyarzúa and G. Tierra, *An augmented mixed finite element method for the Navier-Stokes equations with variable viscosity.* SIAM J. Numer. Anal. 54 (2016), no. 2, 1069–1092.

[28] J. Camaño, C. Muñoz and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[29] J. Camaño, R. Oyarzúa, R. Ruiz-Baier and G. Tierra, *Error analysis of an augmented mixed method for the Navier-Stokes problem with mixed boundary conditions.* IMA J. Numer. Anal. 38 (2018), no. 3, 1452–1484.

[30] S. Caucao, E. Colmenares, G.N. Gatica and C. Inzunza, *A Banach spaces-based fully mixed finite element method for the stationary chemotaxis-Navier-Stokes problem.* Preprint 2022-16, Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción, (2022).

[31] S. Caucao, M. Discacciati, G.N. Gatica and R. Oyarzúa, *A conforming mixed finite element method for the Navier–Stokes/Darcy–Forchheimer coupled problem.* ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1689–1723.

[32] S. Caucao, G.N. Gatica, R. Oyarzúa and N. Sánchez, *A fully-mixed formulation for the steady double-diffusive convection system based upon Brinkman-Forchheimer equations.* J. Sci. Comput. 85 (2020), no. 2, Paper No. 44.

[33] S. Caucao, R. Oyarzúa and S. Villa-Fuentes, *A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy.* Calcolo 57 (2020), no. 4, Paper No. 36.

[34] S. Caucao and I. Yotov, *A Banach space mixed formulation for the unsteady Brinkman-Forchheimer equations.* IMA J. Numer. Anal. 41 (2021), no. 4, 2708–2743.

[35] P. Ciarlet, *Linear and Nonlinear Functional Analysis with Applications.* Society for Industrial and Applied Mathematics, Philadelphia, PA, 2013.

[36] E. Colmenares, G.N. Gatica and W. Miranda, *Analysis of an augmented fully-mixed finite element method for a bioconvective flows model.* J. Comput. Appl. Math. 393 (2021), Paper No. 113504.

[37] E. Colmenares, G.N. Gatica and S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem.* ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.

[38] E. Colmenares, G.N. Gatica and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem.* Numer. Methods Partial Differential Equations 32 (2016), no. 2, 445–478.

[39] E. Colmenares and M. Neilan, *Dual-mixed finite element methods for the stationary Boussinesq problem.* Comp. Math. Appl. 72 (2016), no. 7, 1828–1850.

[40] C.I. Correa and G.N. Gatica, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces.* Comput. Math. Appl. 117 (2022), 14–23.

[41] C.I. Correa, G.N. Gatica and R. Ruiz-Baier, *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces.* Preprint 2022-26, Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción, (2022).

[42] I. Danaila, R. Moglan, F. Hecht and S. Le Masson, *A Newton method with adaptive finite elements for solving phase-change problems with natural convection.* J. Comput. Phys. 274 (2014), 826–840.

[43] M.S. Dinniman, X.S. Asay-Davis, B.K. Galton-Fenzi, P.R. Holland, A. Jenkins and R. Timmermann, *Modeling ice shelf/ocean interaction in antarctica: a review.* Oceanography 29 (2016), no. 4, 144–153.

[44] Y. Dutil, D.R. Rousse, N.B. Salah, S. Lassue and L. Zalewski, *A review on phase-change materials: mathematical modeling and simulations.* Renew. Sustain. Energy Rev. 15 (2011), no. 1, 112–130.

[45] M. El-Hadda, Y. Belhamadia, J. Deteix and D. Yakoubi, *A projection scheme for phase change problems with convection.* Comput. Math. Appl. (2022), 109–122.

[46] A. Ern and J.-L Guermond, Theory and Practice of Finite Elements. Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[47] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* SpringerBriefs in Mathematics. Springer, Cham, 2014.

[48] G.N. Gatica, N. Núñez and R. Ruiz-Baier, *New non-augmented mixed finite element methods for the Navier-Stokes-Brinkman equations using Banach spaces.* Preprint 2022-14, Centro de Investigación en Ingeniería Matemática (CI^2MA), Universidad de Concepción, Chile, (2022).

[49] G.N. Gatica, R. Oyarzúa, R. Ruiz-Baier and Y.D. Sobral, *Banach spaces-based analysis of a fully-mixed finite element method for the steady-state model of fluidized beds.* Comput. Math. Appl. 84 (2021), 244–276.

[50] L.F. Gatica, R. Oyarzúa and N. Sánchez, *A priori and a posteriori error analysis of an augmented mixed-FEM for the Navier-Stokes-Brinkman problem.* Comput. Math. Appl. 75 (2018), no. 7, 2420–2444.

[51] L. Guta and S. Sundar, *Navier-Stokes-Brinkman system for interaction of viscous waves with a submerged porous structure*. Tamkang J. Math. 41 (2010), no. 3, 217–243.

[52] J. Howell and N. Walkington, *Dual-mixed finite element methods for the Navier-Stokes equations*. ESAIM Math. Model. Numer. Anal. 47 (2013), no. 3, 789–805.

[53] P. Huang and Z. Li, *A uniformly stable nonconforming FEM based on weighted interior penalties for Darcy-Stokes-Brinkman equations*. Numer. Math. Theory Methods Appl. 10 (2017), no. 1, 22–43.

[54] A.R. Khoei, D. Amini and S.M.S. Mortazavi, *Modeling non-isothermal two-phase fluid flow with phase change in deformable fractured porous media using extended finite element method*. Internat. J. Numer. Methods Engrg. 122 (2021), no. 16, 4378–4426.

[55] M. Kuchta, *Assembly of multiscale linear PDE operators*. In: F. J. Vermolen, C. Vuik (eds.) Numerical Mathematics and Advanced Applications ENUMATH 2019, (2021) pp. 641–650.

[56] M. Kuchta, M. Nordaas, J.C. Verschaeve, M. Mortensen and K.-A. Mardal, *Preconditioners for saddle point systems with trace constraints coupling 2d and 1d domains*. SIAM J. Sci. Comput. 38 (2016), B962–B987.

[57] M. Lonsing and R. Verfürth, *On the stability of* BDMS *and* PEERS *elements*. Numer. Math. 99 (2004), no. 1, 131–140.

[58] M.S. Mahmood, M. Hokr and M. Lukač, *Combined higher order finite volume and finite element scheme for double porosity and non-linear adsorption of transport problem in porous media*. J. Comput. Appl. Math. 235 (2011), no. 14, 5221–4236.

[59] E. Marchandise, P. Crosetto, C. Geuzaine, J.-F. Remacle, and E. Sauvage, *Quality open source mesh generation for cardiovascular flow simulation*. In: D. Ambrosi, A. Quarteroni, and G. Rozza, editors. Modeling of Physiological Flows. Milano: Springer (2011) 395–414.

[60] W. McLean, Strongly Elliptic Systems and Boundary Integral Equations, Cambridge University Press, 2000.

[61] A. Quarteroni and A. Valli, *Numerical Approximation of Partial Differential Equations*. Springer Series in Computational Mathematics, 23. Springer-Verlag, Berlin, 1994.

[62] A. Rakotondrandisa, G. Sadaka and I. Danaila, *A finite-element toolbox for the simulation of solid–liquid phase-change systems with natural convection*. Comput. Phys. Commun. 253 (2020), Paper 107188.

[63] C. Rana, M. Mishra and A. De Wit, *Effect of anti-Langmuir adsorption on spreading in porous media*. Europhysics Lett. 124 (2019), 64003.

[64] A.M. Robertson, A. Sequeira and R.G. Owense, *Cardiovascular Mathematics. Modeling and simulation of the circulatory system*, Volume 6. Springer Verlag Italia, 2009.

[65] G. Sadaka, A. Rakotondrandisa, P.H. Tournier, F. Luddens, C. Lothodé and I. Danaila, *Parallel finite-element codes for the simulation of two-dimensional and three-dimensional solid–liquid phase-change systems with natural convection*. Comput. Phys. Commun. 257 (2020), Paper 107492.

[66] B. Scheurer, *Existence et approximation de points selles pour certains problémes non linéaires*. RAIRO Anal. Numér. 11 (1977), no. 4, 369–400.

[67] P.W. Schroeder and G. Lube, *Stabilised dG-FEM for incompressible natural convection flows with boundary and moving interior layers on non-adapted meshes*. J. Comput. Phys. 335 (2017), 760–779.

[68] D.M. Sforza, C.M. Putman and J.R. Cebral, *Computational fluid dynamics in brain aneurysms*. Int. J. Numer. Methods Biomed. Engrg., 28 (2012), no. 6–7, 801–808.

[69] S. Sundar and L. Guta, *Navier-Stokes-Brinkman model for numerical simulation of free surface flows*. Math. Student 78 (2009), no. 1-4, 127–143 (2010).

[70] M. Torrilhon and N. Sarna, *Hierarchical Boltzmann simulations and model error estimation*. J. Comput. Phys. 342 (2017), 66–84.

[71] M. Ulvrová, S. Labrosse, N. Coltice, P. Røaback and P.J. Tackley, *Numerical modelling of convection interacting with a melting and solidification front: application to the thermal evolution of the basal magma ocean*. Phys. Earth Planet. Inter. 206–207 (2012), 51–66.

[72] J.-M. Vanson, A. Boutin, M. Klotz and F.-X. Coudert, *Transport and adsorption under liquid flow: the role of pore geometry*. Soft Matter 13 (2017), 875–885.

[73] S. Wang, A. Faghri and T.L. Bergman, *A comprehensive numerical model for melting with natural convection*. Int. J. Heat Mass Transfer. 53 (2010), no. 9–10, 1986–2000.

[74] K.A. Williamson, Accurate and Efficient Solution of the Stokes-Brinkman Problem. Thesis (Ph.D.)–University of Maryland, Baltimore County. 2020.

[75] K.A. Williamson, P. Burda and B. Sousedík, *A posteriori error estimates and adaptive mesh refinement for the Stokes-Brinkman problem*. Math. Comput. Simulation 166 (2019), 266–282. *Modeling ice shelf/ocean interaction in antarctica: a review*. Oceanography 29 (2016), no. 4, 144–153.

[76] Y. Zhang, Y. Hou and J. Zhao, *Error analysis of a fully discrete finite element variational multiscale method for the natural convection problem*. Comput. Math. Appl. 68 (2014), no. 4, 543–567.

[77] A.G. Zimmerman and J. Kowalski, *Mixed finite elements for convection-coupled phase-change in enthalpy form: open software verified and applied to 2D benchmarks*. Comput. Math. Appl. 84 (2021), 77–96.