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NEW MIXED FINITE ELEMENT METHODS FOR THE NAVIER-STOKES PROBLEM WITH VARIABLE VISCOSITY AND ITS COUPLING WITH DARCY EQUATIONS

por

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NEW MIXED FINITE ELEMENT METHODS FOR THE NAVIER-STOKES PROBLEM WITH VARIABLE VISCOSITY AND ITS COUPLING WITH DARCY EQUATIONS

NUEVOS MÉTODOS DE ELEMENTOS FINITOS MIXTOS PARA EL PROBLEMA DE NAVIER-STOKES CON VISCOSIDAD VARIABLE Y SU ACOPLAMIENTO CON LAS ECUACIONES DE DARCY

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Abstract

This work consists of two main parts. In the first part we propose and analyze a mixed variational formulation for the Navier-Stokes equations with variable viscosity that depends nonlinearly on the velocity gradient. Differently from previous works in which augmented terms are added to the formulation, here we employ a technique that had been previously applied to the stationary Boussinesq problem and the Navier-Stokes equations with constant viscosity. Firstly, a modified pseudostress tensor is introduced involving the diffusive and convective terms, and the pressure. Secondly, by using the incompressibility condition, the pressure is eliminated, and the gradient of velocity is incorporated as an auxiliary unknown to handle the aforementioned nonlinearity. As a consequence, a Banach spaces-based formulation is obtained, which can be written as a perturbed twofold saddle point operator equation. We address the continuous and discrete solvability of this problem by linearizing the perturbation and employing a fixed-point approach along with a particular case of a known abstract theory. Given an integer $\ell \geqslant 0$, feasible choices of finite element subspaces include discontinuous piecewise polynomials of degree $\leqslant \ell$ for each entry of the velocity gradient, Raviart-Thomas spaces of order ℓ for the pseudostress, and discontinuous piecewise polynomials of degree $\leqslant \ell$ for the velocity as well. Finally, optimal *a priori* error estimates are derived, and several numerical results confirming in general the theoretical rates of convergence, and illustrating the good performance of the scheme, are reported. This part yielded the following work already published:

I. Bermúdez, C.I. Correa, G.N. Gatica and J.P. Silva, *A perturbed twofold saddle point-based mixed finite element method for the Navier-Stokes equations with variable viscosity.* Appl. Numer. Math. 201 (2024), 465–487.

On the other hand, in the second part we propose and analyze a new fully-mixed finite element method for the coupled model arising from the Navier-Stokes equations, with variable viscosity, in an incompressible fluid, and the Darcy equations in an adjacent porous medium, so that suitable transmission conditions are considered on the corresponding interface. The approach is based on the introduction of the further unknowns in the fluid given by the veloc-

ity gradient and the pseudostress tensor, where the latter includes the respective diffusive and convective terms. The above allows the elimination from the system of the fluid pressure, which can be calculated later on via a postprocessing formula. In addition, the traces of the fluid velocity and the Darcy pressure become the Lagrange multipliers enforcing weakly the interface conditions. In this way, the resulting variational formulation is given by a nonlinear perturbation of a threefold saddle-point operator equation, where the saddle-point in the middle of them is, in turn, perturbed. A fixed-point strategy along with the generalized Babuška-Brezzi theory, a related abstract result for perturbed saddle-point problems, the Banach-Nečas-Babuška theorem, and the Banach fixed-point theorem, are employed to prove the well-posedness of the continuous and Galerkin schemes. In particular, Raviart-Thomas and piecewise polynomial subspaces of the lowest degree for the domain unknowns, as well as continuous piecewise linear polynomials for the Lagrange multipliers on the interface, constitute a feasible choice of the finite element subspaces. Optimal error estimates and associated rates of convergence are then established. Finally, several numerical results illustrating the good performance of the method in 2D and confirming the theoretical findings are reported. This part yielded the following work, presently submitted:

I. Bermúdez, G.N. Gatica and J.P. Silva, *A new Banach spaces-based mixed finite element method for the coupled Navier-Stokes and Darcy equations.* Preprint 2025-08, Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción, Chile, (2025).

Contents

List of Figures

Part I

A perturbed twofold saddle point-based mixed finite element method for the Navier-Stokes equations with variable viscosity

CHAPTER 1

Introduction

The development of Banach spaces-based mixed finite element methods for Newtonian and non-Newtonian incompressible fluids has received special attention by the community of numerical analysts of partial differential equations during the last decade. Indeed, in this paper we are interested in the Navier-Stokes problem with nonlinear viscosity, which refers to the mathematical description of the motion of a fluid whose viscosity coefficient is not constant but rather varies with respect to position and/or time. This problem is certainly more complex than the conventional Navier-Stokes problem for Newtonian fluids with constant viscosity since, in addition to the non-linearity arising from the convective term, one has to deal now with the one coming from the viscosity as well. In this sense, and according to what we have experienced in some of our own related contributions, the use of nonlinear saddle point formulations in Banach spaces has shown to be much more suitable for the corresponding continuous and discrete analyses than, for instance, classical Hilbertian approaches. In the context of augmentation techniques, mixed finite element methods for solving the Navier-Stokes equations with a viscosity that depends non-linearly on the magnitude of the velocity gradient have been recently introduced and analyzed in [11, 9]. In the first approach, the modified pseudostress tensor used in [10] is employed, which, like the one from [42], involves diffusive and convective terms as well as the pressure. The second approach takes into account the dependence of the viscosity on the strain rate tensor, resulting in a more physically relevant model that incorporates both deformation and vorticity as auxiliary unknowns. Additionally, in both works, the pressure unknown is eliminated through an equivalent statement implied by the incompressibility condition. In turn, due to the convective term, and in order to stay within a Hilbertian framework, the velocity is sought in the Sobolev space of order 1, which requires to augment the variational formulation with additional Galerkin-type terms arising from the constitutive and equilibrium equations. While the augmented methods avoid the need of proving continuous and discrete inf-sup conditions, thus allowing much more flexibility for choosing the finite element subspaces, it is no less true that the resulting Galerkin schemes and their corresponding computational implementations increase considerably in complexity, which leads to much more expensive discrete systems. This is the main reason for discouraging the use of augmented procedures. Regarding nonlinear twofold saddle point operator equations, also known as dual-dual variational formulations, there has been a diverse range of theories developed over the past two decades. These theories arose from the need of applying dual-mixed methods to a class of nonlinear boundary value problems in continuum mechanics. In [31], the Babuška-Brezzi theory in Hilbert spaces is generalized to a class of nonlinear variational problems, and in [32], a natural extension of the abstract framework for continuous and discrete nonlinear twofold saddle point formulations is derived. More recently, a fully-mixed finite element method has been developed and analyzed for the coupling of the Stokes and Darcy-Forchheimer problems in [1]. This method was later extended to the coupling of the Navier-Stokes and Darcy-Forchheimer problems with constant density and viscosity in [18]. The main novelty of these works is the use of a new approach that leads to Banach spaces and a twofold saddle point structure for the equation of the corresponding operator. The continuous and discrete solvabilities of this structure are analyzed in both papers using a suitable abstract theory developed for this purpose in the context of separable reflexive Banach spaces.

According to the previous discussion, the goal of the present paper is to extend the applicability of the Banach spaces framework discussed above by introducing a fully-mixed formulation for the Navier-Stokes equations with constant density and variable viscosity, without any augmentation procedure. The analysis and results from [18] are used to achieve this goal. The paper proves the well-posedness and uniqueness of both the continuous and discrete formulations using a fixed point argument and an abstract theory for twofold saddle point problems. An *a priori* analysis is also performed, and optimal rates of convergence are derived. Given an integer $\ell \geqslant 0$, discontinuous piecewise polynomials of degree $\leqslant \ell$ for each entry of the velocity gradient, Raviart-Thomas spaces of order ℓ for the pseudostress, and discontinuous piecewise polynomials of degree $\leqslant \ell$ for the velocity are feasible choices. The paper is structured as follows. In the rest of this chapter, we provide an overview of the standard notation and functional spaces that will be utilized throughout the paper. In Chapter 2 we introduce the model problem of interest and define the unknown to be considered in the variational formulation. Subsequently, in Chapter 3 we identify the twofold saddle-point structure of the corresponding variational system. We then proceed to analyze the continuous solvability and the equivalent fixed point setting in Chapter 4, and present the corresponding well-posedness result, assuming sufficiently small data. In Chapter 5, we investigate the associated Galerkin scheme by utilizing a discrete version of the fixed point strategy developed in Chapter 4 for the continuous case. Additionally, we derive the associated *a priori* error estimate in the same chapter. Furthermore, in Chapter 6 we specify particular choices of discrete subspaces that satisfy the hypotheses from Chapter 4 and provide the rates of convergence of the Galerkin schemes. Finally, we present several numerical examples in Chapter 7, which illustrate the good performance of the fully mixed finite element method and confirm the theoretical rates of convergence.

1.1 Preliminary notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n}\frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n}\boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n}\tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

On the other hand, given $t \in (1,+\infty)$, we also introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \Big\{\boldsymbol{\tau}\in\mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\tau})\in\mathrm{L}^t(\Omega)\Big\},$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \Big\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau})\in\mathbf{L}^t(\Omega)\Big\},$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}_t;\Omega),$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega).$$

Then, proceeding as in [30, eq. (1.43), Section 1.3.4] (see also [12, Section 4.1] and [20, Section 3.1]), it is easy to show that for each $t \in \begin{cases} (1,+\infty] & \text{if} \quad n = 2 \\ [6/5,+\infty] & \text{if} \quad n = 3 \end{cases}$, there holds

$$\langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, v \rangle \,=\, \int_\Omega \Big\{ \boldsymbol{\tau} \cdot \nabla v \,+\, v \, \mathrm{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t;\Omega) \times \mathrm{H}^1(\Omega)\,, \tag{1.1}$$

and analogously

$$\langle \boldsymbol{\tau}\, \boldsymbol{\nu}, \mathbf{v} \rangle \,=\, \int_\Omega \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} \,+\, \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \Big\} \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t;\Omega) \times \mathbf{H}^1(\Omega)\,, \tag{1.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. We find it important to stress here, as explained in the aforementioned references, that the second term on the right-hand side of (1.1) (resp. (1.2)) is well-defined because of the continuous embedding of $\mathrm{H}^1(\Omega)$ (resp. $\mathbf{H}^1(\Omega)$) into $\mathrm{L}^{t'}(\Omega)$ (resp. $\mathbf{L}^{t'}(\Omega)$), where t' is the conjugate of t, that is $t' \in [1,+\infty)$ such that $\frac{1}{t} + \frac{1}{t'} = 1$, which reduces to
$t' \in \begin{cases} [1,+\infty) & \text{if} \quad n = 2 \\ [1,6] & \text{if} \quad n = 3 \end{cases}$,

CHAPTER 2

The model problem

In what follows we consider the Navier-Stokes problem with variable viscosity consists of finding the velocity $\mathbf{u}$ and the pressure p of a fluid occupying the region Ω, such that

$$\begin{gathered} -\mathbf{div}(\mu(|\nabla\mathbf{u}|)\nabla\mathbf{u}) + (\nabla\mathbf{u})\mathbf{u} + \nabla p \;=\; \mathbf{f} \quad \text{in} \quad \Omega\,, \\ \operatorname{div}(\mathbf{u}) \;=\; 0 \quad \text{in} \quad \Omega\,, \quad \mathbf{u} \;=\; \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega p \;=\; 0\,, \end{gathered} \tag{2.1}$$

where the given data are a function $\mu : \mathrm{R}^+ \to \mathrm{R}^+$ describing the nonlinear viscosity, a volume force $\mathbf{f}$, and the boundary velocity $\mathbf{g}$. The right spaces to which $\mathbf{f}$ and $\mathbf{g}$ need to belong are specified later on. Note that $\mathbf{g}$ must formally satisfy the compatibility condition

$$\int_\Gamma \mathbf{g}\cdot\boldsymbol{\nu} \;=\; 0, \tag{2.2}$$

which arises from the incompressibility condition of the fluid. In addition, for the uniqueness of the pressure p in (2.1) we seek this unknown in the space

$$\mathrm{L}_0^2(\Omega) \;=\; \left\{ q \in \mathrm{L}^2(\Omega) : \quad \int_\Omega q \;=\; 0 \right\} . \tag{2.3}$$

Furthermore, we assume that μ is of class C^1, and that there exist constants μ_1, $\mu_2 > 0$, such that

$$\mu_1 \;\leqslant\; \mu(s) \;\leqslant\; \mu_2 \quad \text{and} \quad \mu_1 \;\leqslant\; \mu(s) + s\mu'(s) \;\leqslant\; \mu_2 \qquad s \;\geqslant\; 0\,, \tag{2.4}$$

which, according to the result provided by [40, Theorem 3.8], imply Lipschitz continuity and strong monotonicity of the nonlinear operator induced by μ, which is defined later on (cf. (3.8)). We will go back to this fact in Chapter 4. Also, it is important to remark here that the assumptions in (2.4) constitute the most commonly used sufficient conditions guaranteeing the aforementioned nonlinear operator to be Lipschitz-continuous and strongly monotone, which are, actually, the properties to be employed in our analysis. Some examples of nonlinear μ satisfying (2.4) are the following:

$$\mu(s) := \; 2 + \frac{1}{1+s} \quad \text{and} \quad \mu(s) \;:=\; \alpha_0 + \alpha_1(1+s^2)^{(\beta-2)/2}, \tag{2.5}$$

where α_0 , $\alpha_1 > 0$ and $\beta \in [1,2]$. The first example is basically academic but the second one corresponds to a particular case of the well-known Carreau law in fluid mechanics. It is easy to see that they both satisfy (2.4) with $(\mu_1, \mu_2) \;=\; (2,3)$ and $(\mu_1, \mu_2) \;=\; (\alpha_0, \alpha_0 + \alpha_1)$, respectively. The forthcoming analysis also applies to the slightly more general case of a viscosity function acting on $\Omega \times \mathrm{R}^+$. Next, proceeding similarly as in [11], we introduce the pseudostress tensor unknown, which is defined by

$$\boldsymbol{\sigma} := \; \mu(|\nabla \mathbf{u}|)\nabla \mathbf{u} - (\mathbf{u} \otimes \mathbf{u}) - p\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{2.6}$$

In this way, noting that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u}$, which makes uses of the fact that $\mathrm{div}(\mathbf{u}) = 0$, we find that the first equation of (2.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in} \quad \Omega\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of (2.6), which can be understood as the constitutive equation expressing $\boldsymbol{\sigma}$ in terms of $\mathbf{u}$, that the latter and the incompressibility condition are equivalent to the pair

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{d}} &= \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \quad \text{and} \\ p &= -\frac{1}{n}\mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) \quad \text{in} \quad \Omega\,. \end{aligned} \tag{2.7}$$

Thus, eliminating the pressure unknown which, anyway, can be approximated later on by the postprocessed formula suggested in (2.7), we arrive, at first instance, at the following system of equations with unknowns $\mathbf{u}$ and $\boldsymbol{\sigma}$:

$$\begin{gathered} \boldsymbol{\sigma}^{\mathrm{d}} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) = 0\,. \end{gathered} \tag{2.8}$$

Finally, since we are interested in a mixed variational formulation of our nonlinear problem, and in order to employ the integration by parts formula typically required by this approach, we introduce the auxiliary unknown $\mathbf{t} := \nabla\mathbf{u}$ in Ω. Consequently, instead of (2.8), we consider from now the set of equations with unknowns $\mathbf{t}$, $\mathbf{u}$, and $\boldsymbol{\sigma}$, given by

$$\begin{gathered} \mathbf{t} = \nabla\mathbf{u} \quad \text{in} \quad \Omega\,, \quad \boldsymbol{\sigma}^{\mathrm{d}} = \mu(|\mathbf{t}|)\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega\,, \\ -\mathbf{div}(\boldsymbol{\sigma}) = \mathbf{f} \quad \text{in} \quad \Omega\,, \quad \mathbf{u} = \mathbf{g} \quad \text{on} \quad \Gamma\,, \quad \int_\Omega \mathrm{tr}\big(\boldsymbol{\sigma} + (\mathbf{u}\otimes\mathbf{u})\big) = 0\,. \end{gathered} \tag{2.9}$$

Note that the incompressibility condition $0 = \mathrm{div}(\mathbf{u}) = \mathrm{tr}(\mathbf{t})$ is implicitly contained in the second equation of the first row of (2.9) since the matrix trace of each deviatoric tensor is 0.

We end this chapter by noticing that the formulation described by (2.9) is restricted to Dirichet boundary conditions only. In the case of Neumann boundary conditions, for instance, one would need either to consider a symmetric stress tensor instead of the present pseudostress, or use the latter to rewrite the condition in terms of it. However, it is not clear in advance a physical justification for having both tensors an advective component when imposing that condition.

CHAPTER 3

The fully mixed formulation

In this chapter we derive a Banach spaces-based fully-mixed formulation of (2.9). The integration by parts formula provided by (1.2), along with the Cauchy-Schwarz and Hölder inequalities, play a key role in this derivation. We begin by looking originally for $\mathbf{u} \in \mathbf{H}^1(\Omega)$. Then, multiplying the first equation of (2.9) by $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, with $t \in \begin{cases} (1,+\infty) & \text{if} \quad n = 2 \\ [6/5,+\infty) & \text{if} \quad n = 3 \end{cases}$, applying the integration by parts formula (1.2), and using the Dirichlet boundary conditions for $\mathbf{u}$, which implicity assumes that $\mathbf{g} \in \mathbf{H}^{1/2}(\Gamma)$, we find

$$\int_\Omega \boldsymbol{\tau} : \mathbf{t} + \int_\Omega \mathbf{u} \cdot \mathbf{div}(\boldsymbol{\tau}) \;=\; \langle \boldsymbol{\tau}\,\boldsymbol{\nu}, \mathbf{g}\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{3.1}$$

It is clear from (3.1) that its first term is well defined for $\mathbf{t} \in \mathbb{L}^2(\Omega)$, which, along with the free trace property of $\mathbf{t}$, suggests to look for $\mathbf{t} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, where

$$\mathbb{L}^2_{\mathrm{tr}}(\Omega) \;:=\; \Big\{ \mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathrm{tr}(\mathbf{s}) \;=\; 0 \Big\}\,. \tag{3.2}$$

In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, we realize from the second term and Hölder's inequality that it suffices to look for $\mathbf{u} \in \mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. Next, it follows from the second equation of (2.9), that formally

$$\int_\Omega \mu(|\mathbf{t}|)\mathbf{t} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}} : \mathbf{s} - \int_\Omega (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} : \mathbf{s} \;=\; 0 \qquad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \tag{3.3}$$

from which we notice that the first term is well-defined, whereas the second one makes sense if $\boldsymbol{\sigma}$ is sought in $\mathbb{L}^2(\Omega)$. In turn, for the third one there holds

$$\left|\int_\Omega (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} : \mathbf{s}\right| \;=\; \left|\int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s}\right| \;\leqslant\; \|\mathbf{u}\|_{0,4;\Omega}\, \|\mathbf{u}\|_{0,4;\Omega}\, \|\mathbf{s}\|_{0,\Omega}\,, \tag{3.4}$$

which, necessarily yields $t' \;=\; 4$, and thus $t \;=\; 4/3$.

Certainly, one could also consider arbitrary indexes ℓ, $j \in (1, +\infty)$ conjugate to each other, and then take $\mathbf{s} \in \mathbb{L}^j_0(\Omega)$ (defined analogously to (3.2)) instead of $\mathbf{s} \in \mathbb{L}^2_0(\Omega)$ in (3.4), thus obtaining

$$\left|\int_\Omega (\mathbf{u} \otimes \mathbf{u}) : \mathbf{s}\right| \;\leqslant\; \|\mathbf{u}\|_{0,2\ell;\Omega}\, \|\mathbf{u}\|_{0,2\ell;\Omega}\, \|\mathbf{s}\|_{0,j;\Omega}\,.$$

However, this would force all the remaining spaces involved to be modified, and particularly, because of the first term in (3.3), one would have to look for $\mathbf{t}$ in $\mathbb{L}^\ell_0(\Omega)$. As a consequence, the associated non-linear operator would not act from a Banach space onto its dual, which stops us of applying monotone operators theory, as we know it, to perform the corresponding analysis. This is the main reason for adopting here the simplest choice $\ell = j = 2$.

Finally, looking for $\boldsymbol{\sigma}$ in the same space of its corresponding test function $\boldsymbol{\tau}$, that is $\boldsymbol{\sigma} \in \mathbb{H}(\mathrm{div}_{4/3}; \Omega)$, the equilibrium equation in (2.9) is tested as

$$-\int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\sigma}) \;=\; \int_\Omega \mathbf{f} \cdot \mathbf{v} \quad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{3.5}$$

which forces $\mathbf{f}$ to belong to $\mathbf{L}^{4/3}(\Omega)$. Now we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega) \;=\; \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega) \,\oplus\, \mathrm{R}\,\mathbb{I}\,,$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega) : \quad \int_\Omega \operatorname{tr}(\boldsymbol{\tau}) = 0\right\}.$$

It follows that $\boldsymbol{\sigma}$ can be uniquely decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + c_0\mathbb{I}$, where, according to the third equation of the second row of (2.9),

$$\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \quad \text{and} \quad c_0 := \frac{1}{n|\Omega|}\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}) = -\frac{1}{n\,|\Omega|}\int_\Omega \operatorname{tr}(\mathbf{u}\otimes\mathbf{u}). \tag{3.6}$$

In this way, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that (3.3) and (3.5) remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. In addition, thanks to the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using the compatibility condition (2.2), we realize that testing (3.1) against $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Thus, redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, and suitably gathering (3.1), (3.3) and (3.5), we arrive at the following mixed formulation: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{L}^2_{\mathrm{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\times\mathbf{L}^4(\Omega)$ such that

$$\begin{aligned}
\int_\Omega \mu(|\mathbf{t}|)\mathbf{t}:\mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathrm{d}}:\mathbf{s} - \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}}:\mathbf{s} &= 0 && \forall\,\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega),\\
-\int_\Omega \boldsymbol{\tau}^{\mathrm{d}}:\mathbf{t} - \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) &= -\langle\boldsymbol{\tau}\boldsymbol{\nu},\mathbf{g}\rangle_\Gamma && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega),\\
-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) &= \int_\Omega \mathbf{f}\cdot\mathbf{v} && \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega).
\end{aligned} \tag{3.7}$$

Next, we observe that (3.7) has a perturbed twofold saddle point structure. Indeed, we first define the Banach spaces

$$\mathbb{H}_1 := \mathbb{L}^2_{\mathrm{tr}}(\Omega), \quad \mathbb{H}_2 := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \quad \text{and} \quad \mathbf{Q} := \mathbf{L}^4(\Omega),$$

which are endowed with the norms $\|\cdot\|_{0,\Omega}$, $\|\cdot\|_{\mathbf{div}_{4/3};\Omega}$, and $\|\cdot\|_{0,4;\Omega}$, respectively. Next, we introduce the nonlinear operator $\mathcal{A}:\mathbb{H}_1\to\mathbb{H}_1'$, and the bounded linear operators $\mathcal{B}_1:\mathbb{H}_1\to\mathbb{H}_2'$

and $\mathcal{B} : \mathbb{H}_2 \to \mathbf{Q}'$, given by

$$
\begin{aligned}
[\mathcal{A}(\mathbf{r}), \mathbf{s}] &:= \int_\Omega \mu(|\mathbf{r}|)\mathbf{r} : \mathbf{s} && \forall\, \mathbf{r}, \mathbf{s} \in \mathbb{H}_1, \\
[\mathcal{B}_1(\mathbf{s}), \boldsymbol{\tau}] &:= -\int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \mathbf{s} && \forall\, (\mathbf{s}, \boldsymbol{\tau}) \in \mathbb{H}_1 \times \mathbb{H}_2, \\
[\mathcal{B}(\zeta), \mathbf{v}] &:= -\int_\Omega \mathbf{v} \cdot \mathbf{div}(\zeta) && \forall\, (\zeta, \mathbf{v}) \in \mathbb{H}_2 \times \mathbf{Q}.
\end{aligned}
\tag{3.8}
$$

Hereafter, $[\cdot\,,\cdot]$ stands for the duality pairing between the corresponding Banach space involved and its dual. In turn, $\mathrm{G} \in \mathbb{H}_2'$, and $\mathrm{F} \in \mathbf{Q}'$ are the bounded linear functionals defined by

$$
[\mathrm{G}, \boldsymbol{\tau}] \;:=\; -\langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{g} \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2\,,
$$

and

$$
[\mathrm{F}, \mathbf{v}] \;:=\; \int_\Omega \mathbf{f} \cdot \mathbf{v} \qquad \forall\, \mathbf{v} \in \mathbf{Q}\,.
$$

Regarding the boundedness of G and F, we first observe, using the identity (1.2) and the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, that

$$
\big|[\mathrm{G}, \boldsymbol{\tau}]\big| \;\leqslant\; \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma}\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2\,. \tag{3.9}
$$

with $\widetilde{\mathbf{g}} := \max\big\{1, \|\mathbf{i}_4\|\big\}\,\mathbf{g}$. In addition, it follows by Hölder's inequality that

$$
\big|[\mathrm{F}, \mathbf{v}]\big| \;\leqslant\; \|\mathbf{f}\|_{0,4/3;\Omega}\, \|\mathbf{v}\|_{0,4;\Omega} \quad \forall\, \mathbf{v} \in \mathbf{Q}\,. \tag{3.10}
$$

According to the above, the fully mixed formulation (3.7) can be rewritten as: Find $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in$

$\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ such that

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}), \mathbf{s}] \quad + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma}] \quad - \int_\Omega (\mathbf{u} \otimes \mathbf{u})^{\mathrm{d}} : \mathbf{s} &&= 0 && \forall\, \mathbf{s} \in \mathbb{H}_1, \\
&[\mathcal{B}_1(\mathbf{t}), \boldsymbol{\tau}] \quad + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u}] &&= [\mathrm{G}, \boldsymbol{\tau}] && \forall\, \boldsymbol{\tau} \in \mathbb{H}_2, \\
&[\mathcal{B}(\boldsymbol{\sigma}), \mathbf{v}] &&= [\mathrm{F}, \mathbf{v}] && \forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned} \tag{3.11}$$

CHAPTER 4

The continuous solvability analysis

In this chapter, we analyze the solvability of (3.11) by applying a particular case of the more general result provided by [18, Theorem 3.4].

4.1 The fixed-point strategy

We begin by rewriting (3.11) as an equivalent fixed point equation. To this end, we proceed to linearize the perturbation (third term of the first equation of (3.11)) defining for each $\mathbf{w} \in \mathbf{Q}$ the functional $\mathrm{H}_{\mathbf{w}} : \mathbb{H}_1 \to \mathrm{R}$ by

$$[\mathrm{H}_{\mathbf{w}}, \mathbf{s}] \;:=\; \int_{\Omega} (\mathbf{w} \otimes \mathbf{w})^{\mathrm{d}} : \mathbf{s} \qquad \forall\, \mathbf{s} \in \mathbb{H}_1, \tag{4.1}$$

and let $\mathbf{T} : \mathbf{Q} \to \mathbf{Q}$ be the operator given by

$$\mathbf{T}(\mathbf{w}) \;=\; \mathbf{u} \qquad \forall\, \mathbf{w} \in \mathbf{Q}, \tag{4.2}$$

where $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ is the unique solution (to be proved later on) of the following system of equations:

$$
\begin{aligned}
&[\mathcal{A}(\mathbf{t}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma}] &&= [\mathrm{H}_{\mathbf{w}}, \mathbf{s}] \quad &&\forall\, \mathbf{s} \in \mathbb{H}_1,\\
&[\mathcal{B}_1(\mathbf{t}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u}] &&= [\mathrm{G}, \boldsymbol{\tau}] \quad &&\forall\, \boldsymbol{\tau} \in \mathbb{H}_2,\\
&[\mathcal{B}(\boldsymbol{\sigma}), \mathbf{v}] &&= [\mathrm{F}, \mathbf{v}] \quad &&\forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned}
\tag{4.3}
$$

Thus, we realize that solving (3.11) is equivalent to finding a fixed point of $\mathbf{T}$, that is $\mathbf{u} \in \mathbf{Q}$ such that

$$\mathbf{T}(\mathbf{u}) = \mathbf{u}.$$

4.2 Well-definedness of the operator T

We continue by establishing the well-definedness of the operator $\mathbf{T}$, equivalently, that problem (4.3) is well-posed. To this end, and as already announced, we make use of the following theorem.

Theorem 4.1. *Let $\mathbb{H}_1$, $\mathbb{H}_2$, and $\mathbf{Q}$ be separable and reflexive Banach spaces, and let $\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$ be a nonlinear operator, and $\mathcal{B}_1 : \mathbb{H}_1 \to \mathbb{H}_2'$ and $\mathcal{B} : \mathbb{H}_2 \to \mathbf{Q}'$ be bounded linear operators. In addition, let $\mathcal{K} := N(\mathcal{B})$ and assume that*

i) $\mathcal{A} : \mathbb{H}_1 \to \mathbb{H}_1'$ *is Lipschitz continuous, that is there exists a constant $\gamma > 0$ such that*

$$\|\mathcal{A}(\mathbf{r}) - \mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \leqslant \gamma \, \|\mathbf{r} - \mathbf{s}\|_{\mathbb{H}_1} \quad \forall\, \mathbf{r}, \mathbf{s} \in \mathbb{H}_1 ,$$

ii) *for each $\mathbf{s} \in \mathbb{H}_1$, the family of operators $\mathcal{A}(\cdot + \mathbf{s}) : \mathbb{H}_1 \to \mathbb{H}_1'$ is strictly monotone with a monotonicity constant $\alpha > 0$, independent of $\mathbf{s}$, that is*

$$[\mathcal{A}(\mathbf{t} + \mathbf{s}) - \mathcal{A}(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \geqslant \alpha \, \|\mathbf{t} - \mathbf{r}\|_{\mathbb{H}_1}^2 \quad \forall\, \mathbf{t}, \mathbf{r} \in \mathbb{H}_1 ,$$

iii) *there exists a positive constant* β *such that*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}_2\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}),\mathbf{v}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \,\geqslant\, \beta\,\|\mathbf{v}\|_{\mathbf{Q}} \quad \forall\,\mathbf{v}\in\mathbf{Q}\,, \quad \textit{and}$$

iv) *there exists a positive constant* β_1 *such that*

$$\sup_{\substack{\mathbf{s}\in\mathbb{H}_1\\ \mathbf{s}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} \,\geqslant\, \beta_1\,\|\boldsymbol{\tau}\|_{\mathbb{H}_2} \quad \forall\,\boldsymbol{\tau}\in\mathcal{K}\,.$$

Then, for each $(\mathrm{H},\mathrm{G},\mathrm{F})\in\mathbb{H}_1'\times\mathbb{H}_2'\times\mathbf{Q}'$ *there exists a unique* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}$ *solution of*

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}),\mathbf{s}] \quad + [\mathcal{B}_1(\mathbf{s}),\boldsymbol{\sigma}] && = [\mathrm{H},\mathbf{s}] \quad && \forall\,\mathbf{s}\in\mathbb{H}_1,\\
&[\mathcal{B}_1(\mathbf{t}),\boldsymbol{\tau}] \quad + [\mathcal{B}(\boldsymbol{\tau}),\mathbf{u}] && = [\mathrm{G},\boldsymbol{\tau}] \quad && \forall\,\boldsymbol{\tau}\in\mathbb{H}_2,\\
&[\mathcal{B}(\boldsymbol{\sigma}),\mathbf{v}] && = [\mathrm{F},\mathbf{v}] \quad && \forall\,\mathbf{v}\in\mathbf{Q}.
\end{aligned} \tag{4.4}$$

Moreover, there exists a constant $C>0$*, depending only on* γ*,* α*,* β*,* β_1*,* $\|\mathcal{B}_1\|$*, and* $\|\mathcal{B}_1'\|$*, such that*

$$\|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \,\leqslant\, C\left\{\|\mathrm{H}\|_{\mathbb{H}_1'} + \|\mathrm{G}\|_{\mathbb{H}_2'} + \|\mathrm{F}\|_{\mathbf{Q}'} + \|\mathcal{A}(\mathbf{0})\|_{\mathbb{H}_1'}\right\}. \tag{4.5}$$

Proof. It follows from a straightforward application of [18, Theorem 3.4] to the particular case $p_1=p_2=2$ of exponents p_1, $p_2\geqslant 2$ that appear there when specifying more general continuity and monotonicity properties. □

Now, if $\mathcal{A}$ becomes linear, the above theorem is simplified by keeping iii) and iv) as such, but assuming, instead of i) and ii), that $\mathcal{A}:\mathbb{H}_1\to\mathbb{H}_1'$ is bounded and $\mathbb{H}_1$-elliptic, which means that there exist constants γ, $\alpha>0$ such that

$$\|\mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \,\leqslant\, \gamma\,\|\mathbf{s}\|_{\mathbb{H}_1} \qquad \text{and} \qquad [\mathcal{A}(\mathbf{s}),\mathbf{s}] \,\geqslant\, \alpha\,\|\mathbf{s}\|_{\mathbb{H}_1}^2 \quad \forall\,\mathbf{s}\in\mathbb{H}_1\,.$$

Then, noting that the above certainly implies that $\mathcal{A}$ is Lipschitz continuous and strongly monotone, and that $\mathcal{A}(\mathbf{0}) = \mathbf{0}$, we conclude from Theorem 4.1 that for each $(\mathrm{H}, \mathrm{G}, \mathrm{F}) \in \mathbb{H}_1' \times \mathbb{H}_2' \times \mathbf{Q}'$ there exists a unique $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ solution of (4.4). Moreover, there exists a constant $C > 0$, depending only on γ, α, β, β_1, $\|\mathcal{B}_1\|$, and $\|\mathcal{B}_1'\|$, such that

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \leqslant C \Big\{ \|\mathrm{H}\|_{\mathbb{H}_1'} + \|\mathrm{G}\|_{\mathbb{H}_2'} + \|\mathrm{F}\|_{\mathbf{Q}'} \Big\} . \tag{4.6}$$

We remark here that (4.6) can be proved to be equivalent to an inf-sup condition involving the left-hand sides of (4.4) in the linear case of $\mathcal{A}$. Indeed, setting the notations $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\rho}, \mathbf{w})$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$, introducing the bounded bilinear form

$$\mathcal{S}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) := [\mathcal{A}(\mathbf{r}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}), \mathbf{v}] \quad \forall\, \vec{\mathbf{r}},\, \vec{\mathbf{s}} \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q} , \tag{4.7}$$

and defining the functionals $\mathrm{H}_{\vec{\mathbf{r}}} \in \mathbb{H}_1'$, $\mathrm{G}_{\vec{\mathbf{r}}} \in \mathbb{H}_2'$, and $\mathrm{F}_{\vec{\mathbf{r}}} \in \mathbf{Q}'$ given by

$$\mathrm{H}_{\vec{\mathbf{r}}}(\mathbf{s}) := \mathcal{S}\big(\vec{\mathbf{r}}, (\mathbf{s}, \mathbf{0}, \mathbf{0})\big) \quad \forall\, \mathbf{s} \in \mathbb{H}_1 , \quad \mathrm{G}_{\vec{\mathbf{r}}}(\boldsymbol{\tau}) := \mathcal{S}\big(\vec{\mathbf{r}}, (\mathbf{0}, \boldsymbol{\tau}, \mathbf{0})\big) \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2 ,$$

$$\text{and} \qquad \mathrm{F}_{\vec{\mathbf{r}}}(\mathbf{v}) := \mathcal{S}\big(\vec{\mathbf{r}}, (\mathbf{0}, \mathbf{0}, \mathbf{v})\big) \quad \forall\, \mathbf{v} \in \mathbf{Q} ,$$

we readily observe that $\mathcal{S}$ can be decomposed as

$$\mathcal{S}(\vec{\mathbf{r}}, \vec{\mathbf{s}}) = \mathrm{H}_{\vec{\mathbf{r}}}(\mathbf{s}) + \mathrm{G}_{\vec{\mathbf{r}}}(\boldsymbol{\tau}) + \mathrm{F}_{\vec{\mathbf{r}}}(\mathbf{v}) \qquad \forall\, \vec{\mathbf{r}},\, \vec{\mathbf{s}} \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q} .$$

Thus, it is not difficult to realize (see [21, Section 3.1, eqs. (3.6) - (3.8)] for a similar estimate) that there holds the equivalence

$$\begin{aligned} \frac{1}{3} \Big\{ \|\mathrm{H}_{\vec{\mathbf{r}}}\|_{\mathbb{H}_1'} + \|\mathrm{G}_{\vec{\mathbf{r}}}\|_{\mathbb{H}_2'} + \|\mathrm{F}_{\vec{\mathbf{r}}}\|_{\mathbf{Q}'} \Big\} &\leqslant \sup_{\substack{\vec{\mathbf{s}} \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q} \\ \vec{\mathbf{s}} \neq \mathbf{0}}} \frac{\mathcal{S}(\vec{\mathbf{r}}, \vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}}} \\ \leqslant \Big\{ \|\mathrm{H}_{\vec{\mathbf{r}}}\|_{\mathbb{H}_1'} + \|\mathrm{G}_{\vec{\mathbf{r}}}\|_{\mathbb{H}_2'} + \|\mathrm{F}_{\vec{\mathbf{r}}}\|_{\mathbf{Q}'} \Big\} &\qquad \forall\, \vec{\mathbf{r}} \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q} . \end{aligned} \tag{4.8}$$

Consequently, noting that $\vec{\mathbf{r}} \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ is certainly the solution of the linear version of (4.4) with the right-hand side given by the functionals $\mathrm{H}_{\vec{\mathbf{r}}} \in \mathbb{H}_1'$, $\mathrm{G}_{\vec{\mathbf{r}}} \in \mathbb{H}_2'$, and $\mathrm{F}_{\vec{\mathbf{r}}} \in \mathbf{Q}'$, we deduce from (4.6) and the lower bound of (4.8) that there exists a constant $\widetilde{C} > 0$, depending only on

γ, α, β, β_1, $\|\mathcal{B}_1\|$, and $\|\mathcal{B}_1'\|$, such that

$$\sup_{\substack{\vec{\mathbf{s}}\in\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}\\ \vec{\mathbf{s}}\neq\mathbf{0}}} \frac{\mathcal{S}(\vec{\mathbf{r}},\vec{\mathbf{s}})}{\|\vec{\mathbf{s}}\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}}} \geqslant \widetilde{C}\,\|\vec{\mathbf{r}}\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \qquad \forall\,\vec{\mathbf{r}}\in\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}\,. \tag{4.9}$$

Conversely, it is easy to see that (4.6) follows from (4.9) and the upper bound of (4.8).

We now verify that problem (4.3) satisfies the hypotheses of Theorem 4.1. To this end, in what follows we establish the Lipschitz continuity and strong monotonicity of $\mathcal{A}$, as well as the continuous inf-sup conditions for $\mathcal{B}$ and $\mathcal{B}_1$.

Lemma 4.2. *Let $\gamma_\mu := \max\{\mu_2, 2\mu_2-\mu_1\}$, where μ_1 and μ_2 are the bounds of μ given in (2.4). Then, for each $\mathbf{r},\mathbf{s}\in\mathbb{L}^2(\Omega)$, there hold the following inequalities:*

$$\|\mathcal{A}(\mathbf{r})-\mathcal{A}(\mathbf{s})\|_{\mathbb{H}_1'} \leqslant \gamma_\mu\,\|\mathbf{r}-\mathbf{s}\|_{\mathbb{H}_1}\,,$$

and

$$[\mathcal{A}(\mathbf{r})-\mathcal{A}(\mathbf{s}),\mathbf{r}-\mathbf{s}] \geqslant \mu_1\,\|\mathbf{r}-\mathbf{s}\|_{\mathbb{H}_1}^2\,. \tag{4.10}$$

Proof. See [40, Theorem 3.8] for details. □

Lemma 4.3. *There exists a constant $\beta>0$, such that*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}_2\\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}),\mathbf{v}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \geqslant \beta\,\|\mathbf{v}\|_{\mathbf{Q}} \quad \forall\,\mathbf{v}\in\mathbf{Q}.$$

Proof. See [35, Lemma 2.9] or [8, Lemma 3.5] for details. □

In turn, in order to prove that $\mathcal{B}_1$ satisfies hypothesis iv), we need to employ a useful estimate for tensors in $\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Indeed, suitably modifying the proof of [30, Lemma 2.3], one can show that there exists a positive constant $c_{4/3}$, depending only on Ω, such that

$$c_{4/3}\,\|\boldsymbol{\tau}\|_{0,\Omega} \leqslant \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,4/3;\Omega} \qquad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,. \tag{4.11}$$

Moreover, while (4.11) was first established in [8, Lemma 3.1], for sake of completeness we prove next a result that includes this inequality as a particular case.

Lemma 4.4. *For each* $t \in \begin{cases} (1,+\infty) & \text{if} \quad n = 2 \\ [6/5,+\infty) & \text{if} \quad n = 3 \end{cases}$, *there exists a constant* $c_t > 0$*, depending only on* Ω*, such that*

$$c_t \, \|\boldsymbol{\tau}\|_{0,\Omega} \,\leqslant\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)\,. \tag{4.12}$$

Proof. We begin by stressing that, exactly as for [30, Lemma 2.3] and [8, Lemma 3.1], the proof of (4.12) is based on the fact that the divergence operator div is an isomorphism from the closed subspace of $\mathbf{H}_0^1(\Omega)$ given by $\mathbf{W}^\perp$, where $\mathbf{W} := \Big\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \quad \mathrm{div}(\mathbf{v}) = 0 \Big\}$, onto $\mathrm{L}_0^2(\Omega)$ (cf. (2.3)). In this way, given $\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)$, that is $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$ and $\mathrm{tr}(\boldsymbol{\tau}) \in \mathrm{L}_0^2(\Omega)$, we let $\mathbf{v}$ be the unique element in $\mathbf{W}^\perp$ such that $\mathrm{div}(\mathbf{v}) = \mathrm{tr}(\boldsymbol{\tau})$ and $\|\mathbf{v}\|_{1,\Omega} \,\leqslant\, C_0 \, \|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}$, with a positive constant C_0 independent of $\mathbf{v}$ and $\boldsymbol{\tau}$. Then, as stated a few lines after [30, eq. (2.53)], it readily follows that

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}^2 \;=\; -n \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) - n \int_\Omega \boldsymbol{\tau}^{\mathrm{d}} : \nabla \mathbf{v}\,. \tag{4.13}$$

On the other hand, we let t' be the conjugate of t, and denote by $\mathbf{i}_{t'}$ the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^{t'}(\Omega)$, which holds for $t' \in [1,+\infty)$ in 2D and $t' \in [1,6]$ in 3D (as stated at the end of Chapter 1 as well). Hence, applying the Hölder and Cauchy-Schwarz inequalities in (4.13), and employing the boundedness of $\mathbf{i}_{t'}$, as well as the estimate bounding $\|\mathbf{v}\|_{1,\Omega}$ in terms of $\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}$, we find that

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}^2 \,\leqslant\, n\, C_0 \Big\{ \|\mathbf{i}_{t'}\| \, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \,+\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \Big\} \, \|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}\,,$$

that is

$$\|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega} \,\leqslant\, n\, C_0 \Big\{ \|\mathbf{i}_{t'}\| \, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \,+\, \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} \Big\}\,. \tag{4.14}$$

Finally, the fact that $\|\boldsymbol{\tau}\|_{0,\Omega}^2 = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 + \frac{1}{n}\, \|\mathrm{tr}(\boldsymbol{\tau})\|_{0,\Omega}^2$ along with (4.14) yield (4.12). □

Lemma 4.5. *There exists a constant* $\beta_1 > 0$*, such that*

$$\sup_{\substack{\mathbf{s}\in\mathbb{H}_1\\ \mathbf{s}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} \geqslant \beta_1\,\|\boldsymbol{\tau}\|_{\mathbb{H}_2} \quad \forall\,\boldsymbol{\tau}\in\mathcal{K}\,. \tag{4.15}$$

Proof. In order to satisfy the continuous inf-sup condition for $\mathcal{B}_1$, it is necessary to first realize that $\mathcal{K} := N(\mathcal{B})$ (cf. (3.8)), is given by

$$\mathcal{K} = \Big\{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) = \mathbf{0} \quad \text{in } \Omega\Big\}\,.$$

Then, given $\boldsymbol{\tau}\in\mathcal{K}$ such that $\boldsymbol{\tau}^{\mathrm{d}}\neq\mathbf{0}$, we have that $\boldsymbol{\tau}^{\mathrm{d}}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, so that bounding the supremum in (4.15) by below with $\mathbf{s} = -\boldsymbol{\tau}^{\mathrm{d}}$, it follows that

$$\sup_{\substack{\mathbf{s}\in\mathbb{H}_1\\ \mathbf{s}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}),\boldsymbol{\tau}]}{\|\mathbf{s}\|_{\mathbb{H}_1}} = \sup_{\substack{\mathbf{s}\in\mathbb{L}^2_{\mathrm{tr}}(\Omega)\\ \mathbf{s}\neq\mathbf{0}}} \frac{-\displaystyle\int_\Omega \boldsymbol{\tau}^{\mathrm{d}}:\mathbf{s}}{\|\mathbf{s}\|_{0,\Omega}} \geqslant \frac{\displaystyle\int_\Omega \boldsymbol{\tau}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}} = \|\boldsymbol{\tau}^{\mathrm{d}}\|_{0;\Omega}\,,$$

which, using (4.11) and the fact that $\mathbf{div}(\boldsymbol{\tau}) = \mathbf{0}$, implies that $\mathcal{B}_1$ satisfies the inf-sup condition with a constant $\beta_1 = c_{4/3}$. On the other hand, if $\boldsymbol{\tau}^{\mathrm{d}} = \mathbf{0}$, it is clear from (4.11) that $\boldsymbol{\tau} = \mathbf{0}$, and so (4.15) is trivially satisfied. □

Consequently, the well-definedness of the operator $\mathbf{T}$ can be stated as follows.

Theorem 4.6. *For each* $\mathbf{w}\in\mathbf{Q}$ *there exists a unique* $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}$ *solution to* (4.3)*, and hence we can define* $\mathbf{T}(\mathbf{w}) := \mathbf{u}\in\mathbf{Q}$*. Moreover, there exists a positive constant* $C_{\mathbf{T}}$*, depending only on* $\gamma_\mu, \mu_1, \beta, \beta_1, \|\mathcal{B}_1\|$*,* $\|\mathcal{B}_1'\|$*, and* $\|\mathbf{i}_4\|$*, and hence independent of* $\mathbf{w}$*, such that*

$$\|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} = \|\mathbf{u}\|_{\mathbf{Q}} \leqslant \|(\mathbf{t},\boldsymbol{\sigma},\mathbf{u})\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \leqslant C_{\mathbf{T}}\Big\{\|\mathbf{w}\|^2_{0,4;\Omega} + \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega}\Big\}\,. \tag{4.16}$$

Proof. It follows from Lemmas 4.2-4.5 and a straightforward application of Theorem 4.1. In turn, estimate (4.16) is a direct consequence of (4.5) (cf. Theorem 4.1) and the boundedness of G (cf. (3.9)) and F (cf. (3.10)). □

4.3 Solvability analysis of the fixed-point scheme

Knowing that the operator $\mathbf{T}$ is well-defined, in this section we address the solvability of the fixed-point equation (4.2). To this end, in what follows we first derive sufficient conditions on $\mathbf{T}$ to map a closed ball of $\mathbf{Q}$ into itself, and then we apply the Banach Theorem to conclude the unique solvability of (4.2). Indeed, given $\delta > 0$, from now on we let

$$\mathrm{W}(\delta) := \Big\{ \mathbf{w} \in \mathbf{Q} : \quad \|\mathbf{w}\|_{0,4;\Omega} \leqslant \delta \Big\} .$$

Lemma 4.7. *Assume that there holds*

$$C_{\mathbf{T}} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \leqslant \frac{\delta}{2} \quad \textit{and} \quad \delta \leqslant \frac{1}{2\,C_{\mathbf{T}}} . \tag{4.17}$$

Then $\mathbf{T}\big(\mathrm{W}(\delta)\big) \subseteq \mathrm{W}(\delta)$.

Proof. Given $\mathbf{w} \in \mathrm{W}(\delta)$, we know from Theorem 4.6 that $\mathbf{T}(\mathbf{w})$ is well defined and that there holds

$$\|\mathbf{T}(\mathbf{w})\|_{0,4;\Omega} \leqslant C_{\mathbf{T}} \Big\{ \|\mathbf{w}\|^2_{0,4;\Omega} + \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\} \leqslant C_{\mathbf{T}}\delta^2 + \frac{\delta}{2} \leqslant \delta,$$

which confirms that $\mathbf{T}(\mathbf{w}) \in W(\delta)$. □

We continue with the continuity property of the operator $\mathbf{T}$.

Lemma 4.8. *There exists a positive constant* $\mathrm{L}_{\mathbf{T}}$*, depending only on* β*,* $\|\mathcal{B}_1\|$*, and* μ_1*, such that*

$$\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widehat{\mathbf{w}})\|_{0,4;\Omega} \leqslant \mathrm{L}_{\mathbf{T}} \Big\{ \|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega} \Big\} \|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega} \tag{4.18}$$

for all $\mathbf{w}, \widehat{\mathbf{w}} \in \mathbf{Q}$.

Proof. Given $\mathbf{w}, \widehat{\mathbf{w}} \in \mathbf{Q}$, we let $\mathbf{T}(\mathbf{w}) := \mathbf{u}$ and $\mathbf{T}(\widehat{\mathbf{w}}) := \widehat{\mathbf{u}}$, where $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ and $(\widehat{\mathbf{t}}, \widehat{\boldsymbol{\sigma}}, \widehat{\mathbf{u}}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ are the corresponding unique solutions of (4.3). Then, subtracting both systems, we obtain

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}) - \mathcal{A}(\widehat{\mathbf{t}}), \mathbf{s}] \quad + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}] &&= [\mathrm{H}_{\mathbf{w}} - \mathrm{H}_{\widehat{\mathbf{w}}}, \mathbf{s}] \quad \forall\, \mathbf{s} \in \mathbb{H}_1, \\
&[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\tau}] \qquad\qquad + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{u} - \widehat{\mathbf{u}}] &&= 0 \qquad\qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_2, \\
&\qquad\qquad [\mathcal{B}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}), \mathbf{v}] &&= 0 \qquad\qquad \forall\, \mathbf{v} \in \mathbf{Q}.
\end{aligned} \tag{4.19}$$

In particular, taking $\mathbf{s} = \mathbf{t} - \widehat{\mathbf{t}}$ and $\boldsymbol{\tau} = \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}$, we realize from the second and third equations of (4.19) that

$$[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}] \,=\, -[\mathcal{B}(\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}), \mathbf{u} - \widehat{\mathbf{u}}] \,=\, 0\,,$$

which, along with the first equation of (4.19), yields

$$[\mathcal{A}(\mathbf{t}) - \mathcal{A}(\widehat{\mathbf{t}}), \mathbf{t} - \widehat{\mathbf{t}}] \,= [\mathrm{H}_{\mathbf{w}} - \mathrm{H}_{\widehat{\mathbf{w}}}, \mathbf{t} - \widehat{\mathbf{t}}]\,,$$

whence, using the stric monotonicity of $\mathcal{A}$ (cf. (4.10)) and the definition of $\mathrm{H}_{\mathbf{w}}$ (cf. (4.1)), we find that

$$\|\mathbf{t} - \widehat{\mathbf{t}}\|_{0,\Omega} \,\leqslant\, \frac{1}{\mu_1} \Big\{ \|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega} \Big\} \|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega}\,. \tag{4.20}$$

In turn, from Lemma 4.3 and the second equation of (4.19), we bound $\|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega}$ as follows:

$$\|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega} \,\leqslant\, \frac{1}{\beta} \sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}_2 \\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}), \mathbf{u} - \widehat{\mathbf{u}}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \,=\, \frac{1}{\beta} \sup_{\substack{\boldsymbol{\tau}\in\mathbb{H}_2 \\ \boldsymbol{\tau}\neq\mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{t} - \widehat{\mathbf{t}}), \boldsymbol{\tau}]}{\|\boldsymbol{\tau}\|_{\mathbb{H}_2}} \,\leqslant\, \frac{\|\mathcal{B}_1\|}{\beta} \|\mathbf{t} - \widehat{\mathbf{t}}\|_{0,\Omega}\,. \tag{4.21}$$

Finally, by combining (4.20) and (4.21), we have that

$$\|\mathbf{T}(\mathbf{w}) - \mathbf{T}(\widehat{\mathbf{w}})\|_{0,4;\Omega} \,=\, \|\mathbf{u} - \widehat{\mathbf{u}}\|_{0,4;\Omega} \,\leqslant\, \frac{\|\mathcal{B}_1\|}{\beta\mu_1} \Big\{ \|\mathbf{w}\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}\|_{0,4;\Omega} \Big\} \|\mathbf{w} - \widehat{\mathbf{w}}\|_{0,4;\Omega}\,,$$

which confirms the announced property on $\mathbf{T}$ (cf. (4.18)) with $\mathrm{L}_{\mathbf{T}} := \dfrac{\|\mathcal{B}_1\|}{\beta\mu_1}$. □

Owing to the above analysis, we now establish the main result of this section.

Theorem 4.9. *Assume that* $\delta < \dfrac{1}{2} \min\left\{ \dfrac{1}{C_{\mathbf{T}}}, \dfrac{\beta\mu_1}{\|\mathcal{B}_1\|} \right\}$ *and the data are sufficiently small so that* *the hypothesis of Lemma 4.7 holds, that is*

$$C_{\mathbf{T}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \right\} \,\leqslant\, \frac{\delta}{2}\,. \tag{4.22}$$

Then, the operator $\mathbf{T}$ *has a unique fixed point* $\mathbf{u} \in \mathrm{W}(\delta)$. *Equivalently, the problem* (3.11) *has a unique solution* $(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$. *Moreover, there holds*

$$\|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \,\leqslant\, 2 C_{\mathbf{T}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \right\}. \tag{4.23}$$

Proof. We first recall that the choice of δ and assumption (4.22) guarantee, thanks to Lemma 4.7, that $\mathbf{T}$ maps $\mathrm{W}(\delta)$ into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W}(\delta) \to \mathrm{W}(\delta)$ (cf. (4.18)), a straightforward application of the classical Banach theorem yields the existence of a unique fixed point $\mathbf{u} \in \mathrm{W}(\delta)$ of this operator, and hence a unique solution of (3.11). Finally, regarding the *a priori* estimate, we first observe from (4.16) that

$$\|\mathbf{T}(\mathbf{u})\|_{0,4;\Omega} \,=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \,\leqslant\, C_{\mathbf{T}} \left\{ \|\mathbf{u}\|_{0,4;\Omega}^2 + \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \right\},$$

from which, using that

$$\|\mathbf{u}\|_{0,4;\Omega}^2 \,\leqslant\, \delta\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \frac{1}{2 C_{\mathbf{T}}}\, \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}}\,,$$

we arrive at

$$\begin{aligned} \|\mathbf{T}(\mathbf{u})\|_{0,4;\Omega} \,&=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \\ &\leqslant\, \frac{1}{2}\, \|(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u})\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} + C_{\mathbf{T}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \right\}, \end{aligned}$$

which yields (4.23) and concludes the proof. □

CHAPTER 5

The Galerkin scheme

In order to approximate the solution of our fully-mixed variational formulation (3.11), we now introduce the associated Galerkin scheme, analyze its solvability by applying a discrete version of the fixed-point approach adopted in the previous chapter, and derive the corresponding *a priori* error estimate.

5.1 Preliminaries

We begin by considering arbitrary finite element subspaces $\mathbb{H}_{1,h}$, $\widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $\mathbf{L}^4(\Omega)$, respectively. Specific subspaces satisfying the assumptions and stability conditions to be indicated along the discussion, will be introduced later on in Chapter 6. Hereafter, $h := \max\{h_K : K \in \mathcal{T}_h\}$ denotes the size of a regular triangulation $\mathcal{T}_h$ of $\overline{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K.

Then, letting

$$\mathbb{H}_{2,h} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \cap \widetilde{\mathbb{H}}_{2,h}\,, \tag{5.1}$$

the Galerkin scheme associated with (3.11) reads: Find $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ such that

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}_h), \mathbf{s}_h] + [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] - \int_\Omega (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} : \mathbf{s}_h &&= 0 &&\forall\, \mathbf{s}_h \in \mathbb{H}_{1,h},\\
&[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] &&= [\mathrm{G}, \boldsymbol{\tau}_h] &&\forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{2,h},\\
&[\mathcal{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= [\mathrm{F}, \mathbf{v}_h] &&\forall\, \mathbf{v}_h \in \mathbf{Q}_h.
\end{aligned} \tag{5.2}$$

Then, we adopt the discrete version of the strategy employed in Section 4.2 to analyse the solvability of (5.2). To this end, we let $\mathbf{T}_h : \mathbf{Q}_h \to \mathbf{Q}_h$ be the discrete operator defined by

$$\mathbf{T}_h(\mathbf{w}_h) = \mathbf{u}_h \qquad \forall\, \mathbf{w}_h \in \mathbf{Q}_h,$$

where $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the following system of equations:

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}_h), \mathbf{s}_h] + [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] &&= [\mathrm{H}_{\mathbf{w}_h}, \mathbf{s}_h] &&\forall\, \mathbf{s}_h \in \mathbb{H}_{1,h},\\
&[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] &&= [\mathrm{G}, \boldsymbol{\tau}_h] &&\forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{2,h},\\
&[\mathcal{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= [\mathrm{F}, \mathbf{v}_h] &&\forall\, \mathbf{v}_h \in \mathbf{Q}_h.
\end{aligned} \tag{5.3}$$

Then, similarly as in the continuous case, we realize that solving (5.2) is equivalent to finding a fixed point of $\mathbf{T}_h$, that is $\mathbf{u}_h \in \mathbf{Q}_h$ such that

$$\mathbf{T}_h(\mathbf{u}_h) = \mathbf{u}_h. \tag{5.4}$$

5.2 Discrete solvability analysis

In this section we proceed analogously to Sections 4.2 and 4.3 and establish the well-posedness of the discrete system (5.2), equivalently of (5.4). To this end, we need to introduce certain hypotheses concerning the arbitrary spaces $\mathbb{H}_{1,h}$, $\widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$, and the discrete kernel associated with the linear operator $\mathcal{B}$, that is

$$\mathcal{K}_h \,:=\, \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_{2,h} : \quad [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h] \,=\, 0 \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h \Big\}\,. \tag{5.5}$$

More precisely, from now on we assume that:

(**H.1**) $\widetilde{\mathbb{H}}_{2,h}$ contains the multiplies of the identity tensor $\mathbb{I}$,

(**H.2**) $\mathbf{div}\big(\widetilde{\mathbb{H}}_{2,h}\big) \,\subseteq\, \mathbf{Q}_h$,

(**H.3**) $\mathcal{K}_h^{\mathrm{d}} \,:=\, \Big\{ \boldsymbol{\tau}_h^{\mathrm{d}} : \quad \boldsymbol{\tau}_h \in \mathcal{K}_h \Big\} \subseteq \mathbb{H}_{1,h}$, and

(**H.4**) there exists a positive constant $\boldsymbol{\beta}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbb{H}_{2,h} \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{[\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{v}_h]}{\|\boldsymbol{\tau}_h\|_{\mathbb{H}_2}} \,\geqslant\, \boldsymbol{\beta}_{\mathrm{d}}\, \|\mathbf{v}_h\|_{\mathbf{Q}} \quad \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,.$$

We highlight here that as a consequence of (**H.1**) we can employ the discrete version of the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}$, namely $\widetilde{\mathbb{H}}_{2,h} \,=\, \mathbb{H}_{2,h} \oplus \mathrm{R}\,\mathbb{I}$, thanks to which $\mathbb{H}_{2,h}$ (cf. (5.1)) can be used as the subspace where the unknown $\boldsymbol{\sigma}_h$ is sought. However, for the computational implementation of (5.2), which is addressed later on in Chapter 7, we actually look for $\boldsymbol{\sigma}_h$ in $\widetilde{\mathbb{H}}_{2,h}$, impose the null mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ through an additional equation tested against arbitrary $\eta_h \in \mathrm{R}$, and keep the symmetry of the resulting system by introducing an artificial unknown $\xi_h \in \mathrm{R}$, also known as Lagrange multiplier, which is shown in advance to be 0. In other words, we replace (5.2) by the modified Galerkin scheme: Find

$(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \xi_h) \in \mathbb{H}_{1,h} \times \widetilde{\mathbb{H}}_{2,h} \times \mathbf{Q}_h \times \mathrm{R}$ such that

$$\begin{aligned}
&[\mathcal{A}(\mathbf{t}_h), \mathbf{s}_h] + [\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\sigma}_h] - \int_\Omega (\mathbf{u}_h \otimes \mathbf{u}_h)^{\mathrm{d}} : \mathbf{s}_h &&= 0 && \forall\, \mathbf{s}_h \in \mathbb{H}_{1,h}\,,\\
&[\mathcal{B}_1(\mathbf{t}_h), \boldsymbol{\tau}_h] + [\mathcal{B}(\boldsymbol{\tau}_h), \mathbf{u}_h] + \xi_h \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) &&= [\mathrm{G}, \boldsymbol{\tau}_h] && \forall\, \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_{2,h}\,,\\
&[\mathcal{B}(\boldsymbol{\sigma}_h), \mathbf{v}_h] &&= [\mathrm{F}, \mathbf{v}_h] && \forall\, \mathbf{v}_h \in \mathbf{Q}_h\,,\\
&\eta_h \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}_h) &&= 0 && \forall\, \eta_h \in \mathrm{R}\,.
\end{aligned} \tag{5.6}$$

Clearly, the same modifications apply to (5.3). Note that when taking $\boldsymbol{\tau}_h = \mathbb{I}$ in the second row of (5.6), and using in particular (2.2), all the terms, except the third one on the left-hand side, vanish, so that this row becomes $n\,|\Omega|\,\xi_h = 0$, from which we clearly deduce, as previously announced, that $\xi_h = 0$. In addition, it is clear that the second row of (5.2) is recovered from the second row of (5.6) by simply taking $\boldsymbol{\tau}_h \in \mathbb{H}_{2,h}$. The above allows to prove that (5.2) and (5.6) are equivalent. Indeed, if $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ is solution of (5.2), then $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, 0) \in \mathbb{H}_{1,h} \times \widetilde{\mathbb{H}}_{2,h} \times \mathbf{Q}_h \times \mathrm{R}$ solves (5.6). Conversely, if $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \xi_h) \in \mathbb{H}_{1,h} \times \widetilde{\mathbb{H}}_{2,h} \times \mathbf{Q}_h \times \mathrm{R}$ is solution of (5.6), then $\boldsymbol{\sigma}_h \in \mathbb{H}_{2,h}$, $\xi_h = 0$, and $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ solves (5.2).

In turn, according to the definition of $\mathcal{B}$ (cf. (3.8)), it follows from (5.5) and (**H.2**) that

$$\mathcal{K}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_{2,h} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \Big\}, \tag{5.7}$$

which yields the discrete analogue of (4.15), that is, given $\boldsymbol{\tau}_h \in \mathcal{K}_h$ such that $\boldsymbol{\tau}_h^{\mathrm{d}} \neq \mathbf{0}$, we realize that $\mathbf{s}_h = -\boldsymbol{\tau}_h^{\mathrm{d}} \in \mathbb{H}_{1,h}$ (which follows from (**H.3**)), and thus

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_{1,h}\\ \mathbf{s}_h \neq \mathbf{0}}} \frac{[\mathcal{B}_1(\mathbf{s}_h), \boldsymbol{\tau}_h]}{\|\mathbf{s}_h\|_{\mathbb{H}_1}} \geqslant \beta_{1,\mathrm{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbb{H}_2}\,, \tag{5.8}$$

with constant $\beta_{1,\mathrm{d}} = c_{4/3}$ (cf. (4.11)). On the other hand, if $\boldsymbol{\tau}_h^{\mathrm{d}} = \mathbf{0}$, it is clear from (4.11) that $\boldsymbol{\tau}_h = \mathbf{0}$, and so the discrete inf-sup condition for $\mathcal{B}_1$ (cf. (5.8)) is trivially satisfied.

In addition, we recall that the Lipschitz-continuity and strict monotonicity of $\mathcal{A}$ (cf. Lemma

4.2), are also valid at the discrete level, that is, with the same constants γ_μ and μ_1, there hold

$$\|\mathcal{A}(\mathbf{r}_h) - \mathcal{A}(\mathbf{s}_h)\|_{\mathbb{H}'_{1,h}} := \sup_{\substack{\widetilde{\mathbf{r}}_h \in \mathbb{H}_{1,h} \\ \widetilde{\mathbf{r}}_h \neq \mathbf{0}}} \frac{[\mathcal{A}(\mathbf{r}_h) - \mathcal{A}(\mathbf{s}_h), \widetilde{\mathbf{r}}_h]}{\|\widetilde{\mathbf{r}}_h\|_{\mathbb{H}_1}} \leqslant \gamma_\mu \|\mathbf{r}_h - \mathbf{s}_h\|_{\mathbb{H}_1} \quad \forall\, \mathbf{r}_h, \mathbf{s}_h \in \mathbb{H}_{1,h}\,, \tag{5.9}$$

and

$$[\mathcal{A}(\mathbf{r}_h) - \mathcal{A}(\mathbf{s}_h), \mathbf{r}_h - \mathbf{s}_h] \geqslant \mu_1 \|\mathbf{r}_h - \mathbf{s}_h\|^2_{\mathbb{H}_1} \quad \forall\, \mathbf{r}_h, \mathbf{s}_h \in \mathbb{H}_{1,h}\,. \tag{5.10}$$

In this way, bearing the above discussion in mind, we are now in a position to establish the discrete analogue of Theorem 4.6.

Theorem 5.1. *For each $\mathbf{w}_h \in \mathbf{Q}_h$ there exists a unique $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ solution to (5.3), and hence we can define $\mathbf{T}_h(\mathbf{w}_h) := \mathbf{u}_h \in \mathbf{Q}_h$. Moreover, there exists a positive constant $C_{\mathbf{T},\mathrm{d}}$, depending only on $\gamma_\mu, \mu_1, \boldsymbol{\beta}_\mathrm{d}, \beta_{1,\mathrm{d}}, \|\mathcal{B}_1\|$, $\|\mathcal{B}'_1\|$, and $\|\mathbf{i}_4\|$, and hence independent of $\mathbf{w}_h$, such that*

$$\begin{aligned}\|\mathbf{T}_h(\mathbf{w}_h)\|_{0,4;\Omega} &= \|\mathbf{u}_h\|_{\mathbf{Q}} \leqslant \|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} \\ &\leqslant C_{\mathbf{T},\mathrm{d}} \left\{ \|\mathbf{w}_h\|^2_{0,4;\Omega} + \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \right\}.\end{aligned}$$

Proof. Thanks to the discrete inf-sup conditions for $\mathcal{B}$ (cf. (**H.4**)) and $\mathcal{B}_1$ (cf. (5.8)), and the inequalities (5.9) and (5.10), the proof follows from a direct application of Theorem 4.1. We omit further details. □

Having established that the discrete operator $\mathbf{T}_h$ is well defined, we now address the solvability of the corresponding fixed point equation (5.4). Then, letting δ_d be an arbitrary radius, we now set

$$\mathrm{W}(\delta_\mathrm{d}) := \left\{ \mathbf{w}_h \in \mathbf{Q}_h : \quad \|\mathbf{w}_h\|_{0,4;\Omega} \leqslant \delta_\mathrm{d} \right\}.$$

Then, reasoning analogously to the derivation of Lemma 4.7, we deduce that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_\mathrm{d})$ into itself under the analogue discrete assumptions, namely

$$C_{\mathbf{T},\mathrm{d}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \right\} \leqslant \frac{\delta_\mathrm{d}}{2} \quad \text{and} \quad \delta_\mathrm{d} \leqslant \frac{1}{2\,C_{\mathbf{T},\mathrm{d}}}\,. \tag{5.11}$$

We emphasize that the above is exactly the same as for the continuous case (cf. Lemma 4.7), except that the constant $C_\mathbf{T}$ and the radius δ are replaced by $C_{\mathbf{T},\mathrm{d}}$ and δ_d, respectively.

Moreover, employing similar arguments to those from the proof of Lemma 4.8, we are able to prove the discrete version of (4.18) with constant $\mathrm{L}_{\mathbf{T},\mathrm{d}} := \dfrac{\beta_{\mathrm{d}}\mu_1}{\|\mathcal{B}_1\|}$, that is

$$\|\mathbf{T}_h(\mathbf{w}_h) - \mathbf{T}_h(\widehat{\mathbf{w}}_h)\|_{0,4;\Omega} \leqslant \mathrm{L}_{\mathbf{T},\mathrm{d}} \Big\{ \|\mathbf{w}_h\|_{0,4;\Omega} + \|\widehat{\mathbf{w}}_h\|_{0,4;\Omega} \Big\} \|\mathbf{w}_h - \widehat{\mathbf{w}}_h\|_{0,4;\Omega} \tag{5.12}$$

for all $\mathbf{w}_h, \widehat{\mathbf{w}}_h \in \mathbf{Q}_h$, which proves the continuity of $\mathbf{T}_h$.

According to the above, the main result of this section is established as follows.

Theorem 5.2. *Assume that δ_{d} and the data are sufficiently small so that they satisfy (5.11). Then, the operator $\mathbf{T}_h$ has at least one fixed point $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathrm{d}})$. Equivalently, the problem (5.2) has at least one solution $(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$. Moreover, under the further assumption*

$$\delta_{\mathrm{d}} < \frac{1}{2} \min\left\{ \frac{1}{C_{\mathbf{T},\mathrm{d}}}, \frac{\beta_{\mathrm{d}}\mu_1}{\|\mathcal{B}_1\|} \right\}, \tag{5.13}$$

this solution is unique. In addition, there holds

$$\|(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}} \leqslant 2\, C_{\mathbf{T},\mathrm{d}} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} + \|\mathbf{f}\|_{0,4/3;\Omega} \Big\}.$$

Proof. The fact that $\mathbf{T}_h$ maps $\mathrm{W}(\delta_{\mathrm{d}})$ into itself, together with the continuity of $\mathbf{T}_h$ (cf. (5.12)), allow to apply the Brouwer Theorem to conclude the existence of a solution to (5.4), and hence to (5.2). Next, the assumption (5.13) and the Banach fixed-point Theorem imply the uniqueness. Finally, the *a priori* estimate is a consequence of Theorem 4.1 and analogue algebraic manipulations to those utilized in the proof of Theorem 4.9. □

5.3 A priori error analysis

In this section we consider finite element subspaces satisfying the assumptions specified in Section 5.2, and derive the Céa estimate for the Galerkin error

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} = \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,4/3,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega},$$

where $\vec{\mathbf{t}} := (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X} := \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ and $\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_h := \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ are the unique solutions of (3.11) and (5.2) respectively, with $\mathbf{u} \in \mathrm{W}(\delta)$ and $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathrm{d}})$. In what follows, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$\operatorname{dist}\big(z, Z_h\big) := \inf_{z_h \in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z\,.$$

In turn, in order to simplify our analysis, we recall a previous result concerning the operator $\mathcal{A}$. More precisely, we employ the following lemma.

Lemma 5.3. *The operator $\mathcal{A}$ defined in (3.8) has a first-order Gâteaux derivative $D\mathcal{A}$. Moreover, for any $\mathbf{s}_1 \in \mathbb{H}_1$, $D\mathcal{A}(\mathbf{s}_1)$ is a bounded and $\mathbb{H}_1$-elliptic bilinear form, with boundedness and ellipticity constants given by γ_μ and μ_1, respectively.*

Proof. See [32, Lemma 3.1]. □

We begin by introducing the global operator $\mathbf{P} : \mathbf{X} \to \mathbf{X}'$, and for each $\mathbf{w} \in \mathbf{Q}$ the linear functional $\mathrm{F}_{\mathbf{w}} : \mathbf{X} \to \mathrm{R}$ associated with the variational formulation (3.11), that is

$$[\mathbf{P}(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}), (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})] := [\mathcal{A}(\mathbf{r}), \mathbf{s}] + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}), \mathbf{v}]\,, \qquad (5.14)$$

$$[\mathrm{F}_{\mathbf{w}}, (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v})] := \int_\Omega (\mathbf{w} \otimes \mathbf{w})^{\mathrm{d}} : \mathbf{s} + [\mathrm{G}, \boldsymbol{\tau}] + [\mathrm{F}, \mathbf{v}],$$

for all $(\mathbf{r}, \boldsymbol{\rho}, \mathbf{w}), (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$. In this way, we realize from (3.11) and (5.2) that there holds

$$[\mathbf{P}(\vec{\mathbf{t}}), \vec{\mathbf{s}}_h] = [\mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] + [\mathrm{F}_{\mathbf{u}} - \mathrm{F}_{\mathbf{u}_h}, \vec{\mathbf{s}}_h] \quad \forall\, \vec{\mathbf{s}}_h \in \mathbf{X}_h\,, \qquad (5.15)$$

whereas the triangle inequality gives for each $\vec{\mathbf{r}}_h := (\mathbf{r}_h, \boldsymbol{\rho}_h, \mathbf{w}_h) \in \mathbf{X}_h$

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \leqslant \|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}} + \|\vec{\mathbf{r}}_h - \vec{\mathbf{t}}_h\|_{\mathbf{X}}\,. \qquad (5.16)$$

In order to establish a connection between the second term on the right-hand side of the above inequality and the operator $\mathbf{P}$, we proceed almost verbatim as in [32, Theorem 3.3]. In fact, given $\vec{\mathbf{s}}_h$, $\vec{\mathbf{r}}_h \in \mathbf{X}_h$, we can write

$$\begin{aligned}[\mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] \;&=\; \int_0^1 \frac{d}{d\mu}\{[\mathbf{P}(\mu\vec{\mathbf{t}}_h + (1-\mu)\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h]\}\,d\mu \\ &= \int_0^1 D\mathbf{P}(\mu\vec{\mathbf{t}}_h + (1-\mu)\vec{\mathbf{r}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)\,d\mu\,,\end{aligned} \tag{5.17}$$

where $D\mathbf{P} : \mathbf{X} \to \mathcal{L}(\mathbf{X}, \mathbf{X}')$ is the first-order Gâteaux derivative of the operator $\mathbf{P} : \mathbf{X} \to \mathbf{X}'$. More precisely, for any $\vec{\mathbf{s}}_1 := (\mathbf{s}_1, \boldsymbol{\tau}_1, \mathbf{v}_1)$, $\vec{\mathbf{r}} := (\mathbf{r}, \boldsymbol{\rho}, \mathbf{w})$, $\vec{\mathbf{s}} := (\mathbf{s}, \boldsymbol{\tau}, \mathbf{v}) \in \mathbf{X}$, $D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{r}}, \vec{\mathbf{s}})$ is obtained from (5.14) by replacing $[\mathcal{A}(\mathbf{r}), \mathbf{s}]$ by $D\mathcal{A}(\mathbf{s}_1)(\mathbf{r}, \mathbf{s})$, that is

$$D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{r}}, \vec{\mathbf{s}}) := D\mathcal{A}(\mathbf{s}_1)(\mathbf{r}, \mathbf{s}) + [\mathcal{B}_1(\mathbf{s}), \boldsymbol{\rho}] + [\mathcal{B}_1(\mathbf{r}), \boldsymbol{\tau}] + [\mathcal{B}(\boldsymbol{\tau}), \mathbf{w}] + [\mathcal{B}(\boldsymbol{\rho}), \mathbf{v}]\,. \tag{5.18}$$

Thus, for any $\vec{\mathbf{s}}_1 \in \mathbf{X}$, (5.18) induces the definition of an operator in $\mathcal{L}(\mathbf{X}, \mathbf{X}')$ (equivalently, a bilinear form as the one in (4.7)), which, according to Lemma 5.3, satisfies the hypotheses of the discrete version of the estimate (4.6) with constants independent of h and of $\vec{\mathbf{s}}_1$. Consequently, bearing in mind that the aforementioned estimate is equivalent to the discrete version of (4.9), we conclude that there exists $\widehat{C} > 0$, depending only on γ_μ, μ_1, β_1, $\|\mathcal{B}_1\|$, and β, such that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \;\leqslant\; \widehat{C} \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{X}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{D\mathbf{P}(\vec{\mathbf{s}}_1)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}} \qquad \forall\, \vec{\mathbf{s}}_1 \in \mathbf{X}. \tag{5.19}$$

On the other hand, the continuity of $D\mathcal{A}$ implies the same property for $D\mathbf{P}$, and hence there exists $\mu_0 \in (0,1)$ such that (5.17) becomes

$$[\mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] \;=\; D\mathbf{P}(\mu_0\vec{\mathbf{t}}_h + (1-\mu_0)\vec{\mathbf{r}}_h)(\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h, \vec{\mathbf{s}}_h)\,. \tag{5.20}$$

It follows from (5.19) (with $\vec{\mathbf{s}}_1 := \mu_0\vec{\mathbf{t}}_h + (1-\mu_0)\vec{\mathbf{r}}_h$) and (5.20) that

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \;\leqslant\; \widehat{C} \sup_{\substack{\vec{\mathbf{s}}_h \in \mathbf{X}_h \\ \vec{\mathbf{s}}_h \neq \mathbf{0}}} \frac{[\mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h]}{\|\vec{\mathbf{s}}_h\|_{\mathbf{X}}}\,. \tag{5.21}$$

Next, since $\mathbf{P}$ is Lipschitz continuous, with a constant $\widehat{\gamma}$, depending only on γ_μ, $\|\mathcal{B}_1\|$ and $\|\mathcal{B}_1'\|$, we subtract and add $\mathbf{P}(\vec{\mathbf{t}})$, and use (5.15), to find that

$$[\mathbf{P}(\vec{\mathbf{t}}_h), \vec{\mathbf{s}}_h] - [\mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h] \;=\; [\mathbf{P}(\vec{\mathbf{t}}_h) - \mathbf{P}(\vec{\mathbf{t}}), \vec{\mathbf{s}}_h] + [\mathbf{P}(\vec{\mathbf{t}}) - \mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h]$$

$$= [\mathrm{F}_{\mathbf{u}_h} - \mathrm{F}_{\mathbf{u}}, \vec{\mathbf{s}}_h] + [\mathbf{P}(\vec{\mathbf{t}}) - \mathbf{P}(\vec{\mathbf{r}}_h), \vec{\mathbf{s}}_h]$$

$$\leqslant \Big\{ \Big(\|\mathbf{u}\|_{\mathbf{Q}} + \|\mathbf{u}_h\|_{\mathbf{Q}} \Big) \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Q}} + \widehat{\gamma}\, \|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \Big\} \|\vec{\mathbf{s}}_h\|_{\mathbf{X}}\,,$$

which, replaced back into (5.21), gives

$$\|\vec{\mathbf{t}}_h - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \;\leqslant\; \widehat{C} \Big\{ \Big(\|\mathbf{u}\|_{\mathbf{Q}} + \|\mathbf{u}_h\|_{\mathbf{Q}} \Big) \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Q}} + \widehat{\gamma}\, \|\vec{\mathbf{t}} - \vec{\mathbf{r}}_h\|_{\mathbf{X}} \Big\}\,. \tag{5.22}$$

Finally, the triangle inequality (cf. (5.16)) along with (5.22) and the fact that $\|\mathbf{u}\|_{\mathbf{Q}}$ and $\|\mathbf{u}_h\|_{\mathbf{Q}}$ are bounded by δ and δ_{d}, respectively, yield

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \;\leqslant\; \big(1 + \widehat{\gamma}\, \widehat{C}\big) \inf_{\vec{\mathbf{s}}_h \in \mathbf{X}} \|\vec{\mathbf{t}} - \vec{\mathbf{s}}_h\|_{\mathbf{X}} + \widehat{C}\big(\delta + \delta_{\mathrm{d}}\big)\, \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Q}}\,. \tag{5.23}$$

In this way, our main result for the error $\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}}$ is stated as follows.

Theorem 5.4. *Assume that the hypotheses of Theorems 4.9 and 5.2 hold, and let* $\vec{\mathbf{t}} = (\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}) \in \mathbf{X}$ *and* $\vec{\mathbf{t}}_h = (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbf{X}_h$ *be the unique solutions of* (3.11) *and* (5.2)*, respectively. Assume further that*

$$(\delta + \delta_{\mathrm{d}}) \;\leqslant\; \frac{1}{2\,\widehat{C}}\,, \tag{5.24}$$

where $\widehat{C}$ *is the global inf-sup constant of* $D\mathbf{P}$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} \;\leqslant\; C\, \mathrm{dist}(\vec{\mathbf{t}}, \mathbf{X}_h)\,. \tag{5.25}$$

Proof. It suffices to use (5.24) in (5.23), which yields (5.25) with $C := 2\big(1 + \widehat{\gamma}\, \widehat{C}\big)$. □

Regarding the feasibility of (5.24), as compared with the previous assumptions on δ and δ_{d} given by (4.17) (cf. Lemma 4.7) and (5.11), we first notice that the latter can be rephrased, equivalently, as

$$2\,C_{\mathbf{T}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \delta \,\leqslant\, \frac{1}{2\,C_{\mathbf{T}}} \quad \text{and} \tag{5.26}$$

$$2\,C_{\mathbf{T},\mathrm{d}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \delta_{\mathrm{d}} \,\leqslant\, \frac{1}{2\,C_{\mathbf{T},\mathrm{d}}}\,, \tag{5.27}$$

respectively. In turn, it is clear that (5.24) holds, in particular, if both δ and δ_{d} are bounded above by $\dfrac{1}{4\,\widehat{C}}$, which, along with (5.26) and (5.27), yield the unified restrictions

$$2\,C_{\mathbf{T}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \delta \,\leqslant\, \min\left\{\frac{1}{2\,C_{\mathbf{T}}}, \frac{1}{4\,\widehat{C}}\right\} \quad \text{and} \tag{5.28}$$

$$2\,C_{\mathbf{T},\mathrm{d}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \delta_{\mathrm{d}} \,\leqslant\, \min\left\{\frac{1}{2\,C_{\mathbf{T},\mathrm{d}}}, \frac{1}{4\,\widehat{C}}\right\}\,. \tag{5.29}$$

Hence, in order to be able to ensure that it is possible to have δ and δ_{d} satisfying (5.28) and (5.29), it suffices to impose that

$$2\,C_{\mathbf{T}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \min\left\{\frac{1}{2\,C_{\mathbf{T}}}, \frac{1}{4\,\widehat{C}}\right\} \quad \text{and} \tag{5.30}$$

$$2\,C_{\mathbf{T},\mathrm{d}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,\Gamma} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega}\right\} \,\leqslant\, \min\left\{\frac{1}{2\,C_{\mathbf{T},\mathrm{d}}}, \frac{1}{4\,\widehat{C}}\right\}\,. \tag{5.31}$$

In other words, sufficiently small data $\mathbf{g}$ and $\mathbf{f}$ (according to (5.30) and (5.31)) guarantee that the restrictions on δ and δ_{d} are achievable.

We end this section by remarking that (2.7) and (3.6) suggest the following postprocessed approximation for the pressure p

$$p_h \,:=\, -\,\frac{1}{n}\mathrm{tr}\Big(\boldsymbol{\sigma}_h \,+\, (\mathbf{u}_h\otimes\mathbf{u}_h)\Big) \,-\, c_{0,h} \quad \text{in} \quad \Omega\,, \tag{5.32}$$

where

$$c_{0,h} \,:=\, -\frac{1}{n\,|\Omega|}\int_{\Omega}\mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\,. \tag{5.33}$$

Then, applying the Cauchy-Schwarz inequality, performing some algebraic manipulations, and employing the *a priori* bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p - p_h\|_{0,\Omega} \leqslant C\left\{\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,\Omega} + \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\right\}. \tag{5.34}$$

Thus, combining (5.25) and (5.34), we conclude the existence of a positive constant $\widetilde{C}$, independent of h, such that

$$\begin{aligned} &\|\mathbf{t} - \mathbf{t}_h\|_{\mathbb{H}_1} + \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbb{H}_2} + \|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{Q}} + \|p - p_h\|_{0,\Omega} \\ &\leqslant \widetilde{C}\left\{\operatorname{dist}(\mathbf{t}, \mathbb{H}_{1,h}) + \operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_{2,h}) + \operatorname{dist}(\mathbf{u}, \mathbf{Q}_h)\right\}. \end{aligned} \tag{5.35}$$

CHAPTER 6

Specific finite element subspaces

In this chapter, we introduce specific finite element subspaces $\mathbb{H}_{1,h}$, $\widetilde{\mathbb{H}}_{2,h}$, and $\mathbf{Q}_h$ of the spaces $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, and $\mathbf{L}^4(\Omega)$, respectively. These subspaces satisfy the hypotheses (**H.1**), (**H.2**), (**H.3**), and (**H.4**), which were introduced in Section 5.2 to ensure the well-posedness of our Galerkin scheme.

Preliminaries

In what follows, given an integer $\ell \geqslant 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_\ell(K)$ be the space of polynomials of degree $\leqslant \ell$ defined on K, whose vector and tensor versions are denoted by $\mathbf{P}_\ell(K) := [\mathrm{P}_\ell(K)]^n$ and $\mathbb{P}_\ell(K) := [\mathrm{P}_\ell(K)]^{n\times n}$, respectively. Next, we define the corresponding local Raviart-Thomas spaces of order ℓ as

$$\mathbf{RT}_\ell(K) := \mathbf{P}_\ell(K) \oplus \mathrm{P}_\ell(K)\mathbf{x} \quad \forall\, K \in \mathcal{T}_h\,,$$

and its associated tensor counterpart $\mathbb{RT}_\ell(K)$, which is defined row-wise by $\mathrm{RT}_\ell(K)$, where $\mathbf{x}$ is a generic vector in $\mathbf{R} := \mathrm{R}^n$. In turn, we let $\mathbf{P}_\ell(\mathcal{T}_h)$, $\mathbb{P}_\ell(\mathcal{T}_h)$ and $\mathbb{RT}_\ell(\mathcal{T}_h)$ be the global versions of $\mathbf{P}_\ell(K)$, $\mathbb{P}_\ell(K)$ and $\mathbb{RT}_\ell(K)$, respectively, that is

$$\begin{aligned}
\mathbf{P}_\ell(\mathcal{T}_h) &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{P}_\ell(\mathcal{T}_h) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{P}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{RT}_\ell(\mathcal{T}_h) &:= \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_\ell(K) \quad \forall\, K \in \mathcal{T}_h \Big\}.
\end{aligned}$$

We stress here that there hold $\mathbf{P}_\ell(\mathcal{T}_h) \subseteq \mathbf{L}^4(\Omega)$ and $\mathbb{RT}_\ell(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_{4/3};\Omega)$, inclusions that are implicitly utilized below to introduce the announced specific finite element subspaces. Indeed, we now define

$$\begin{gathered}
\mathbb{H}_{1,h} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_\ell(\mathcal{T}_h), \quad \widetilde{\mathbb{H}}_{2,h} := \mathbb{RT}_\ell(\mathcal{T}_h), \\
\mathbb{H}_{2,h} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \cap \widetilde{\mathbb{H}}_{2,h}, \quad \text{and} \quad \mathbf{Q}_h := \mathbf{L}^4(\Omega) \cap \mathbf{P}_\ell(\mathcal{T}_h).
\end{gathered} \tag{6.1}$$

Verification of the hypotheses (H.1) - (H.4)

We now confirm that the subspaces defined by (6.1) satisfy the hypotheses (**H.1**) - (**H.4**). Indeed, it is easily seen that $\widetilde{\mathbb{H}}_{2,h}$ satisfy (**H.1**) and (**H.2**). Next, in order to check (**H.3**), we recall from (5.7) that

$$\mathcal{K}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_{2,h} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \Big\},$$

from which, using that the divergence free tensors of $\mathbb{RT}_\ell(\mathcal{T}_h)$ are contained in $\mathbb{P}_\ell(\mathcal{T}_h)$ (cf. [30, Lemma 3.6]), it follows that $\mathcal{K}_h \subseteq \mathbb{P}_\ell(\mathcal{T}_h)$. Hence, noting that certainly $\mathrm{tr}(\boldsymbol{\tau}_h^{\mathrm{d}}) = 0$, for all $\boldsymbol{\tau}_h \in \mathcal{K}_h$, we deduce that $(\mathcal{K}_h)^{\mathrm{d}} \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_\ell(\mathcal{T}_h) = \mathbb{H}_{1,h}$, which proves (**H.3**). Finally, (**H.4**) is proved precisely in [16, Lemma 5.1] (see also [22, Lemma 6.1]).

The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (5.2) with the specific finite element subspaces introduced in Chapter 6, for which we previously collect the respective approximation properties. In fact, thanks to [27, Proposition 1.135] and its corresponding vector version, along with interpolation estimates of Sobolev spaces, those of $\mathbb{H}_{1,h}$, $\mathbb{H}_{2,h}$, and $\mathbf{Q}_h$, are given as follows:

$\big(\mathbf{AP}_h^{\mathbf{t}}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{s}, \mathbb{H}_{1,h}\big) := \inf_{\mathbf{s}_h \in \mathbb{H}_{1,h}} \|\mathbf{s} - \mathbf{s}_h\|_{0,\Omega} \leqslant C\, h^l\, \|\mathbf{s}\|_{l,\Omega}\,,$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\mathrm{dist}\big(\boldsymbol{\tau}, \mathbb{H}_{2,h}\big) := \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{2,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \leqslant C\, h^l \Big\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \Big\}\,,$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, \ell+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\mathrm{dist}\big(\mathbf{v}, \mathbf{Q}_h\big) := \inf_{\mathbf{v}_h \in \mathbf{Q}_h} \|\mathbf{v} - \mathbf{v}_h\|_{0,4;\Omega} \leqslant C\, h^l\, \|\mathbf{v}\|_{l,4;\Omega}\,.$$

The rates of convergence of (5.2) are now established by the following theorem.

Theorem 6.1. *Let* $\big(\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}\big) \in \mathbb{H}_1 \times \mathbb{H}_2 \times \mathbf{Q}$ *and* $\big(\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h\big) \in \mathbb{H}_{1,h} \times \mathbb{H}_{2,h} \times \mathbf{Q}_h$ *be the unique solutions of* (3.11) *and* (5.2) *with* $\mathbf{u} \in \mathrm{W}(\delta)$ *and* $\mathbf{u}_h \in \mathrm{W}(\delta_{\mathrm{d}})$*, whose existences are guaranteed by Theorems* 4.9 *and* 5.2*, respectively. In turn, let* p *and* p_h *given by* (2.7) *and* (5.32)*, respectively. Assume the hypotheses of Theorem* 5.4*, and that there exists* $l \in [1, \ell+1]$ *such that* $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega)$*,* $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$*, and* $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$ *. Then,* *there exists a positive constant C, independent of h, such that*

$$\|((\mathbf{t},\boldsymbol{\sigma},\mathbf{u}) - (\mathbf{t}_h,\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathbb{H}_1\times\mathbb{H}_2\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega}$$
$$\leqslant\, C\, h^l \left\{ \|\mathbf{u}\|_{l,4;\Omega} \,+\, \|\mathbf{t}\|_{l,\Omega} \,+\, \|\boldsymbol{\sigma}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} \right\}.$$

Proof. It follows straightforwardly from the Céa estimate (5.35), and the approximation properties $\big(\mathbf{AP}_h^{\mathbf{t}}\big)$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, and $(\mathbf{AP}_h^{\mathbf{u}})$. □

CHAPTER 7

Computational results

We now turn to the computational results, which mainly refer to the numerical verification of the rates of convergence anticipated by Theorem 6.1. The examples in 2D and 3D to be reported below have been developed with the finite element library FEniCS [2]. In all of them, the linear systems emanating from the Newton-Raphson linearisation, with the zero vector as initial guess and iterations stopped once the absolute or relative residual drops below 10^{-8}, have been solved with the multifrontal massively parallel sparse direct method MUMPS [4]. In turn, the null mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ is imposed via a real Lagrange multiplier as already described by (5.6). Then, $\boldsymbol{\sigma}_h$ is complemented by adding to it the expression $c_{0,h}\,\mathbb{I}$, where $c_{0,h}$ is the constant defined by (5.33). Subsequently, errors are defined as follows:

$$e(\mathbf{t}) = \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}\,, \qquad e(\boldsymbol{\sigma}) = \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}\,,$$

$$e(\mathbf{u}) = \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\,, \qquad e(p) = \|p - p_h\|_{0,\Omega}\,,$$

whereas convergence rates are set as

$$r(\star) = \frac{\log(e(\star)/\widehat{e}(\star))}{\log(h/\widehat{h})} \qquad \forall \star \in \left\{\mathbf{t}, \boldsymbol{\sigma}, \mathbf{u}, p\right\},$$

where e and $\widehat{e}$ denote errors computed on two consecutive meshes of sizes h and $\widehat{h}$. In addition, we refer to the number of degrees of freedom and the number of Newton iterations as `dof` and `iter`, respectively.

7.1 Example 1: 2D smooth solution

In our first numerical test, we consider the computational domain $\Omega = (0,1)^2$, and set the nonlinear viscosity to

$$\mu(s) := 2 + \frac{1}{1+s} \quad \forall\, s \geqslant 0\,.$$

In addition, we define the manufactured exact solution:

$$p = x^2 - y^2, \quad \mathbf{u} = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u},$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,,$$

so that the load function $\mathbf{f}$ and the Dirichlet datum $\mathbf{g}$ are computed accordingly. Table 7.1 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations for the approximations. The experiments confirm the theoretical rate of convergence $\mathcal{O}(h^{\ell+1})$ for $\ell \in \left\{0,1\right\}$, provided by Theorem 6.1. In addition, the number of Newton-Raphson iterations required to reach the convergence criterion based on the residuals with a tolerance of $1e-8$, was less than or equal to 4 in all runs. Sample of approximate solutions with $\ell = 1$ and $\texttt{dof} = 279041$ are shown in Figure 7.1.

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
121	0.7071	$1.26e+00$	$*$	$1.71e+01$	$*$	$4.11e-01$	$*$	$7.55e-01$	$*$	3
465	0.3536	$6.20e-01$	1.02	$8.99e+00$	0.93	$2.26e-01$	0.86	$3.69e-01$	1.03	3
1825	0.1768	$3.10e-01$	1.00	$4.59e+00$	0.97	$1.16e-01$	0.96	$1.82e-01$	1.02	4
7233	0.0884	$1.55e-01$	1.00	$2.31e+00$	0.99	$5.84e-02$	0.99	$8.86e-02$	1.04	4
28801	0.0442	$7.77e-02$	1.00	$1.16e+00$	1.00	$2.92e-02$	1.00	$4.33e-02$	1.03	4
114945	0.0221	$3.89e-02$	1.00	$5.79e-01$	1.00	$1.46e-02$	1.00	$2.15e-02$	1.01	4

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
289	0.7071	$2.75e-01$	$*$	$4.46e+00$	$*$	$1.55e-01$	$*$	$2.60e-01$	$*$	4
1121	0.3536	$7.35e-02$	1.90	$1.22e+00$	1.87	$4.11e-02$	1.91	$5.62e-02$	2.21	4
4417	0.1768	$1.93e-02$	1.93	$3.58e-01$	1.77	$1.05e-02$	1.97	$1.30e-02$	2.11	4
17537	0.0884	$4.93e-03$	1.97	$1.02e-01$	1.82	$2.64e-03$	1.99	$3.17e-03$	2.04	4
69889	0.0442	$1.24e-03$	1.99	$2.76e-02$	1.88	$6.62e-04$	2.00	$7.84e-04$	2.01	4
279041	0.0221	$3.12e-04$	1.99	$7.31e-03$	1.92	$1.66e-04$	2.00	$1.95e-04$	2.01	4

Table 7.1: Example 1, convergence history and Newton iteration count for the $\mathbb{P}_\ell - \mathbb{RT}_\ell - \mathbf{P}_\ell$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the $\mathrm{P}_\ell-$approximation of the postprocessed pressure field, with $\ell \in \{0,1\}$.

7.2 Example 2: 2D smooth solution in a non-convex domain

Now we illustrate the accuracy of our method in the non-convex domain $\Omega := (-1,1)^2 \setminus [0,1]^2$. The data $\mathbf{f}$ and $\mathbf{g}$ are computed so that the manufactured exact solution is defined as:

$$p = \sin(\pi x)\exp(y), \quad \mathbf{u} = \begin{pmatrix} -\cos(2\pi y)\sin(2\pi x) \\ \sin(2\pi y)\cos(2\pi x) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,.$$

The variable viscosity is defined in the same way as in Example 1. The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 1$ is shown in Table 7.2. We observe there that all variables, except $\boldsymbol{\sigma}$, converge optimally with $\mathcal{O}(h^2)$. Indeed, the non-convexity of the domain and the consequent lack of regularity of this unknown is the most probable reason, in our opinion, for its lower rate of convergence. Selected components of the numerical solution,

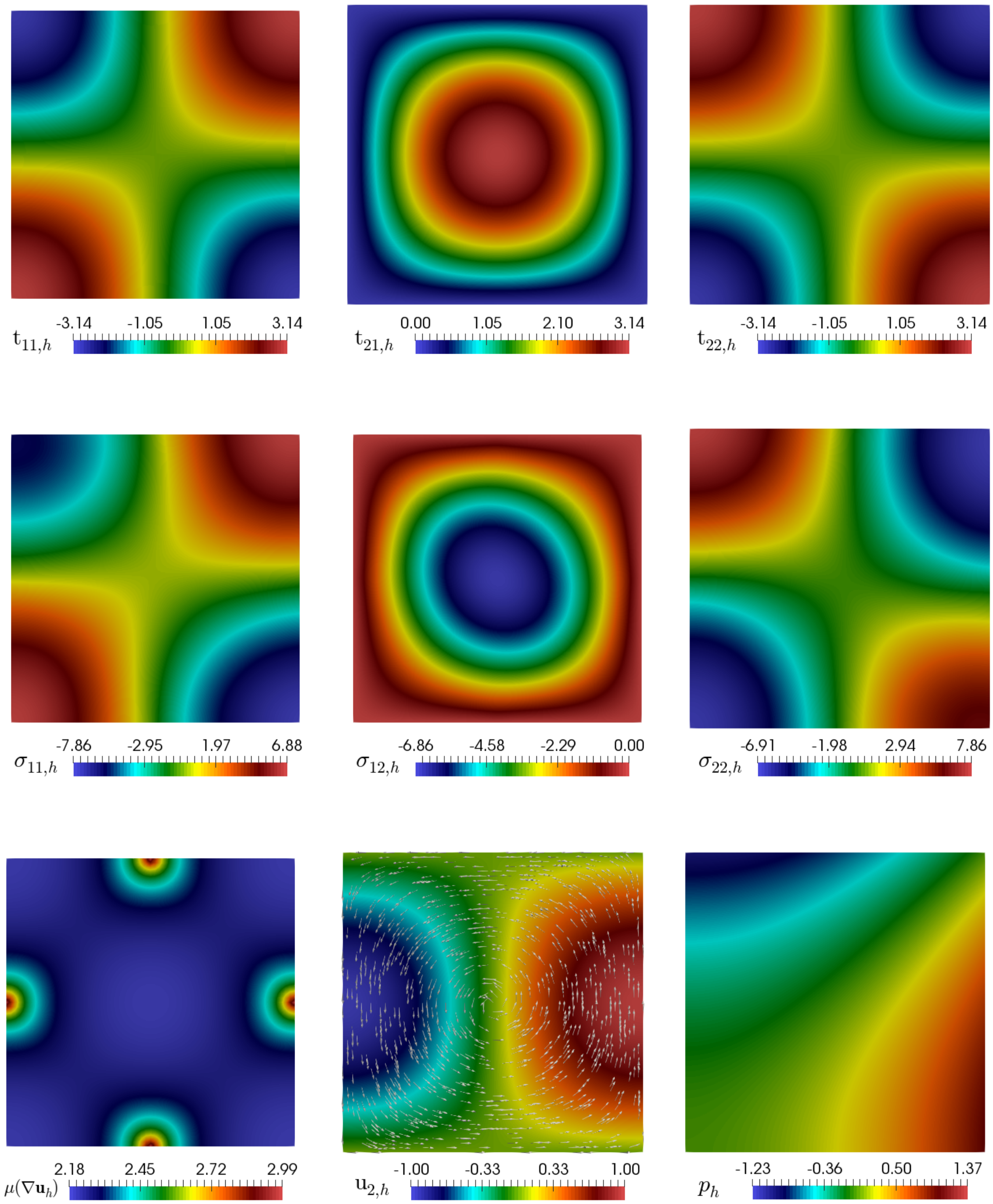

Figure 7.1: Example 1, $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation with `dof` $= 279041$ of velocity gradient components (top panels), pseudostress components (center panels), and viscosity, velocity component with vector directions, and postprocessed pressure field (bottom row).

which were obtained using the $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation with `dof` = 238603, are displayed in Figure 7.2.

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
`dof`	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	`iter`
383	1.1180	$8.59e+00$	$*$	$2.07e+02$	$*$	$8.07e-01$	$*$	$4.63e+00$	$*$	4
941	0.6212	$3.93e+00$	1.33	$9.23e+01$	1.38	$4.15e-01$	1.13	$2.46e+00$	1.08	4
3646	0.3171	$1.13e+00$	1.85	$2.40e+01$	2.00	$1.29e-01$	1.74	$5.48e-01$	2.23	4
15233	0.1582	$2.85e-01$	1.98	$6.25e+00$	1.94	$3.29e-02$	1.96	$1.29e-01$	2.08	4
59869	0.0795	$7.40e-02$	1.96	$1.94e+00$	1.70	$8.50e-03$	1.97	$3.38e-02$	1.95	4
238603	0.0398	$1.85e-02$	2.00	$6.03e-01$	1.69	$2.16e-03$	1.98	$8.50e-03$	1.99	4

Table 7.2: Example 2, convergence history and Newton iteration count for the fully-mixed $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the P_1-approximation of the postprocessed pressure field.

7.3 Example 3: 2D non-smooth solution in a non-convex domain

The third example is devoted to show that the rates of convergence are affected when the exact solution does not have enough regularity, in particular if it has a singularity near the vertex with major angle of a non-convex domain. In fact, here we consider again the L-shaped domain $\Omega := (-1,1)^2 \setminus [0,1]^2$, define the manufactured exact solution:

$$p = \frac{1-x}{2(x-0.02)^2 + 2(y-0.02)^2}\,, \quad \mathbf{u} = \begin{pmatrix} -\cos(\pi x)\sin(\pi y) \\ \sin(\pi x)\cos(\pi y) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,,$$

and compute the data $\mathbf{f}$ and $\mathbf{g}$ accordingly. The variable viscosity is defined in the same way as in Example 1. The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 1$ is shown in Table 7.3. As announced, suboptimal rates arise in this case, which is explained by the fact that the pressure exhibits high gradients near the corner region of the L-shaped domain. This is observed in Figure 7.3 below where selected components of the

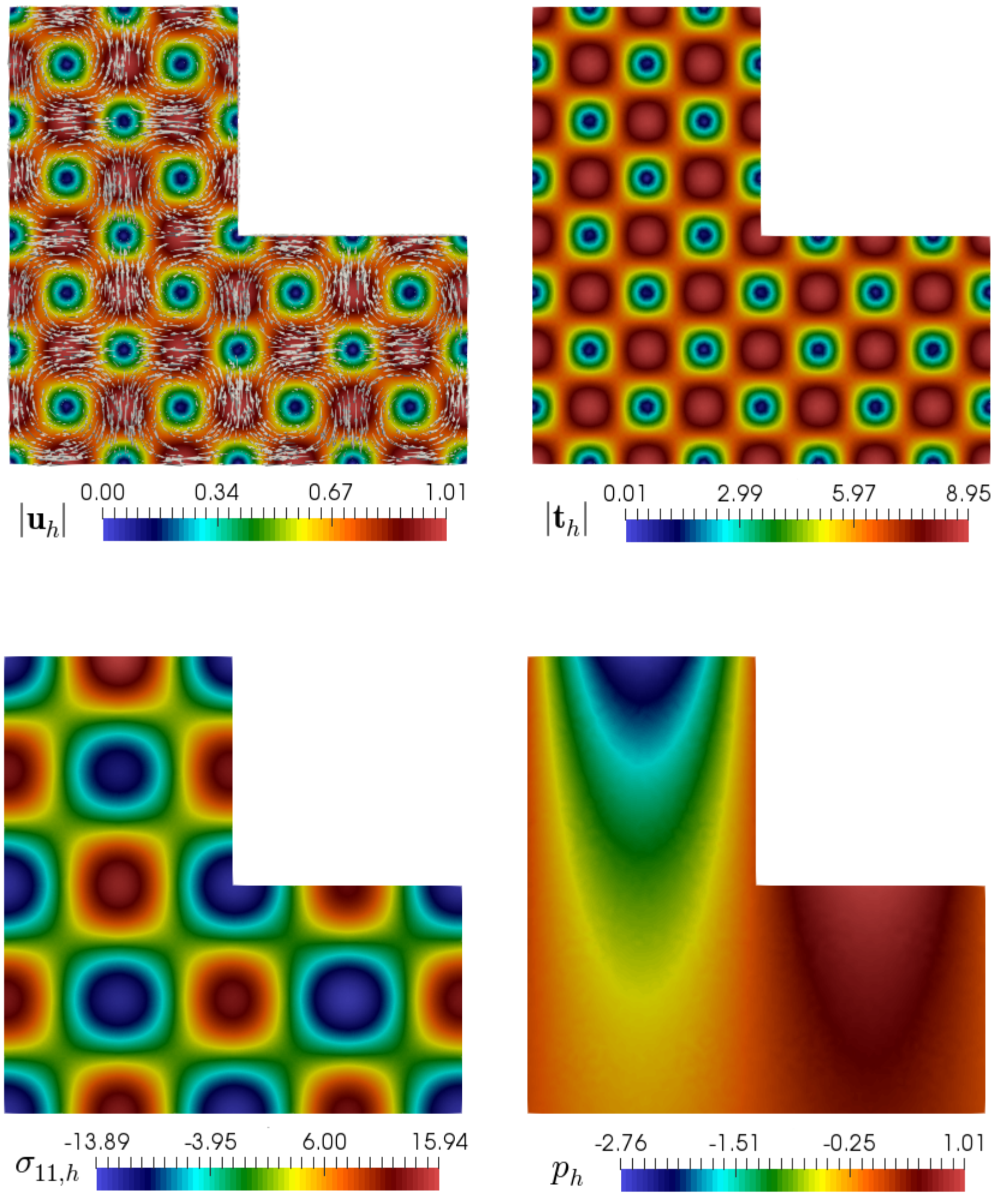

Figure 7.2: Example 2, $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation with dof $= 238603$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

numerical solution, obtained with the $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation and dof $= 238603$, are displayed.

$\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
383	1.1180	$1.10e+01$	$*$	$5.58e+02$	$*$	$1.20e+00$	$*$	$4.26e+01$	$*$	3
941	0.6212	$6.98e+00$	0.77	$3.72e+02$	0.69	$5.67e-01$	1.27	$2.35e+01$	1.01	3
3646	0.3171	$7.23e+00$	-0.05	$4.57e+02$	-0.31	$3.66e-01$	0.65	$1.88e+01$	0.33	3
15233	0.1582	$4.50e+00$	0.68	$3.74e+02$	0.29	$1.43e-01$	1.35	$1.17e+01$	0.68	3
59869	0.0795	$2.57e+00$	0.82	$1.83e+02$	1.04	$6.60e-02$	1.12	$6.21e+00$	0.92	3
238603	0.0398	$1.22e+00$	1.07	$7.49e+01$	1.29	$2.05e-02$	1.68	$2.64e+00$	1.23	5

Table 7.3: Example 3, convergence history and Newton iteration count for the $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximations of the Navier-Stokes model with variable viscosity, and convergence of the P_1-approximation of the postprocessed pressure field.

7.4 Example 4: 3D smooth solution

Next we illustrate a three-dimensional problem. In this case, we consider the cube domain $\Omega = (0,1)^3$, and define the nonlinear viscosity as

$$\mu(s) \;:=\; \alpha_0 + \alpha_1 (1+s^2)^{(\beta-2)/2}\,,$$

with $\alpha_0 = 2/5$, $\alpha_1 = 1/2$, and $\beta = 1$. The data are suitably adjusted according to the exact solution defined by the functions:

$$p = \sin(xyz)\,, \quad \mathbf{u} = \begin{pmatrix} \sin(\pi\,x)\cos(\pi\,y)\cos(\pi\,z) \\ -2\,\cos(\pi\,x)\,\sin(\pi\,y)\,\cos(\pi\,z) \\ \cos(\pi\,x)\,\cos(\pi y)\,\sin(\pi\,z) \end{pmatrix}, \quad \mathbf{t} = \nabla\mathbf{u}\,,$$
$$\text{and} \quad \boldsymbol{\sigma} = \mu(|\nabla\mathbf{u}|)\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u}) - p\mathbb{I}\,.$$

The convergence history for a sequence of quasi-uniform mesh refinements with $\ell = 0$ is shown in Table 7.4, while some components of the approximate solutions with `dof` $= 3360769$ are displayed in Figure 7.4. We observe that the Newton method exhibits a behavior independent of the meshsize, achieving the tolerance of $1e-8$ in four iterations in all cases. Again, the mixed finite element method converges optimally with $\mathcal{O}(h)$, as it was proved by Theorem 6.1.

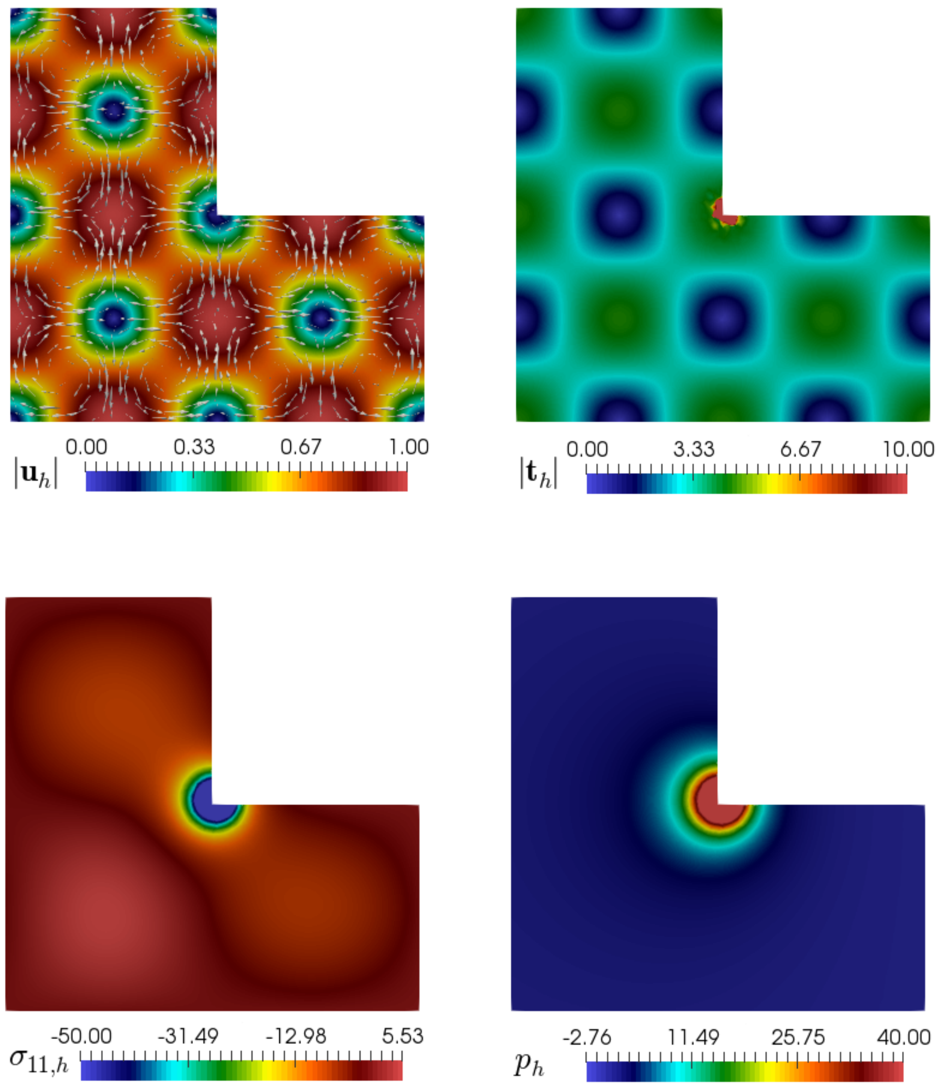

Figure 7.3: Example 3, $\mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{P}_1$ approximation with $\texttt{dof} = 238603$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

7.5 Example 5: 3D cavity problem

To conclude the set of numerical examples, we apply our mixed method with $\ell = 0$ to the driven cavity flow problem in the cube domain $\Omega = (0,1)^3$ by using the same sequence of quasi-uniform mesh refinements from Example 4. Again, the viscosity is taken as the Carreau law (2.5) with

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$										
dof	h	$e(\mathbf{t})$	$r(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$r(\mathbf{u})$	$e(p)$	$r(p)$	iter
889	0.8660	$2.61e+00$	$*$	$8.08e+00$	$*$	$5.65e-01$	$*$	$2.61e-01$	$*$	4
6817	0.4330	$1.41e+00$	0.89	$4.21e+00$	0.94	$3.01e-01$	0.91	$2.02e-01$	0.37	4
53377	0.2165	$7.31e-01$	0.95	$2.14e+00$	0.97	$1.55e-01$	0.96	$1.15e-01$	0.82	4
422401	0.1083	$3.71e-01$	0.98	$1.07e+00$	1.00	$7.79e-02$	0.99	$5.34e-02$	1.10	4
3360769	0.0541	$1.87e-01$	0.99	$5.36e-01$	1.00	$3.90e-02$	1.00	$2.40e-02$	1.15	4

Table 7.4: Example 4, convergence history and Newton iteration count for the $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$ approximation of the Navier-Stokes model with variable viscosity, and convergence of the P_0-approximation of the postprocessed pressure field.

$\alpha_0 = 1$, $\alpha_1 = 0.1$, and $\beta = 1$. The external body force is zero, and the three-dimensional flow patterns are determined by the boundary conditions only: a unidirectional Dirichlet velocity is set on the top lid $\mathbf{g} := (1,0,0)^{\mathrm{t}}$, and no-slip velocity $\mathbf{u} = \mathbf{0}$ are imposed elsewhere on Γ. Some approximate solutions obtained with dof $= 3360769$ are depicted in Figure 7.5. As expected, abrupt changes are observed near the top corners of the domain, where the Dirichlet datum is discontinuous, and where the pseudostress is concentrated. The maximum number of iterations required over the course of the Newton-Raphson loop was 3. On the other hand, if we wondered about the eventual influence of a Reynolds number Re in the present example, we would expect at least two different scenarios. Firstly, and similarly to what happens with a constant viscosity, we might have Re to be inversely proportional to either the lower bound μ_1 or the upper bound μ_2 of μ (cf. (2.4)), in which case a large Re will certainly affect our stability estimates. Secondly, if Re is involved in the definition of the nonlinear viscosity μ in such a way that both μ_1 and μ_2 are independent of this number, then we might expect a robust method not being affected by the range of Re.

We end this section by remarking that the mesh independence of the Newton iterations, observed in all the examples, except possibly in the non-smooth one given by Example 3, was actually to be expected. In fact, this property has been proved theoretically and is known to hold for a large class of problems (see, e.g. [45] and the references therein).

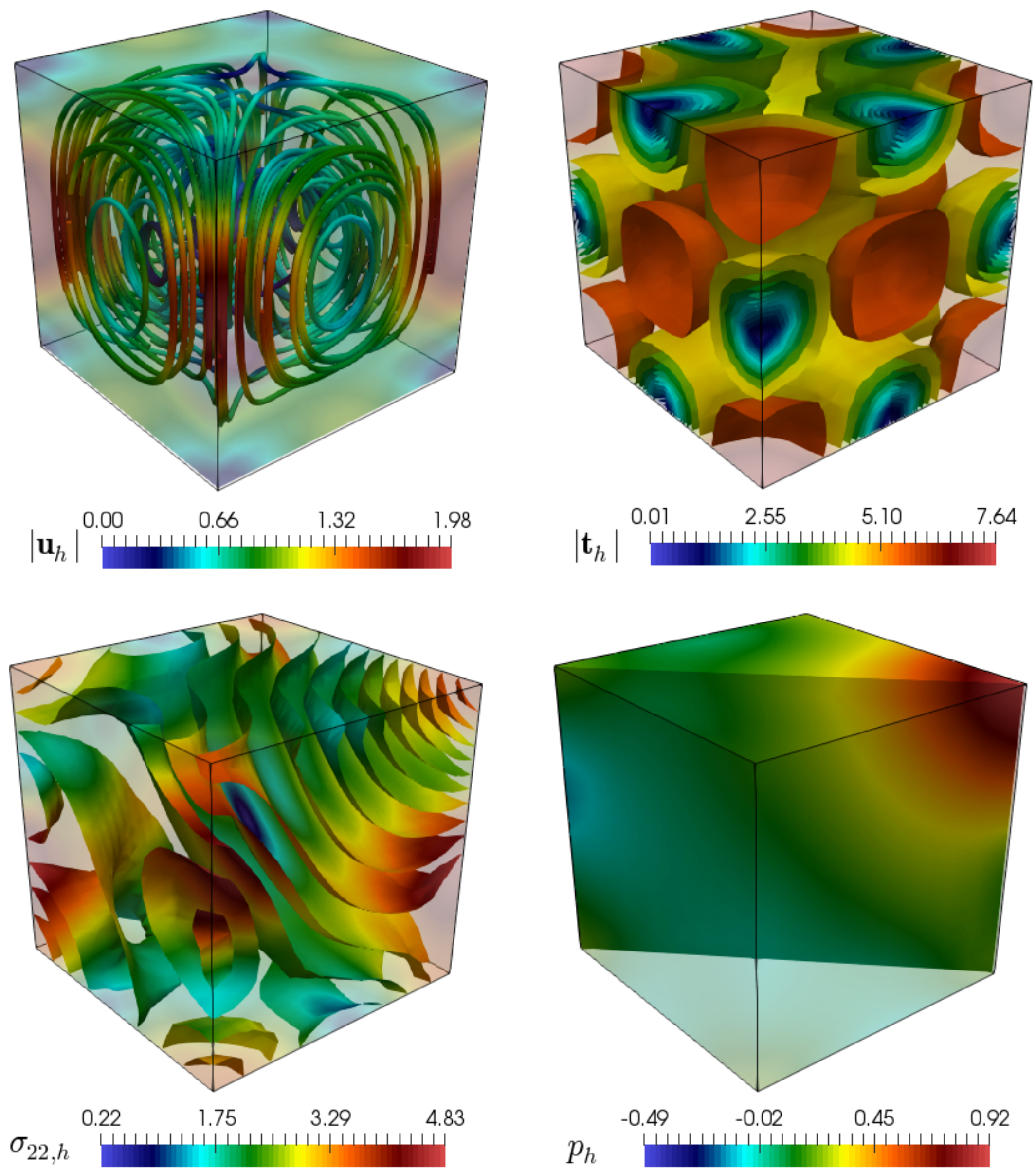

Figure 7.4: Example 4, numerical solutions using $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$ approximations with $\texttt{dof} = 3360769$ of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

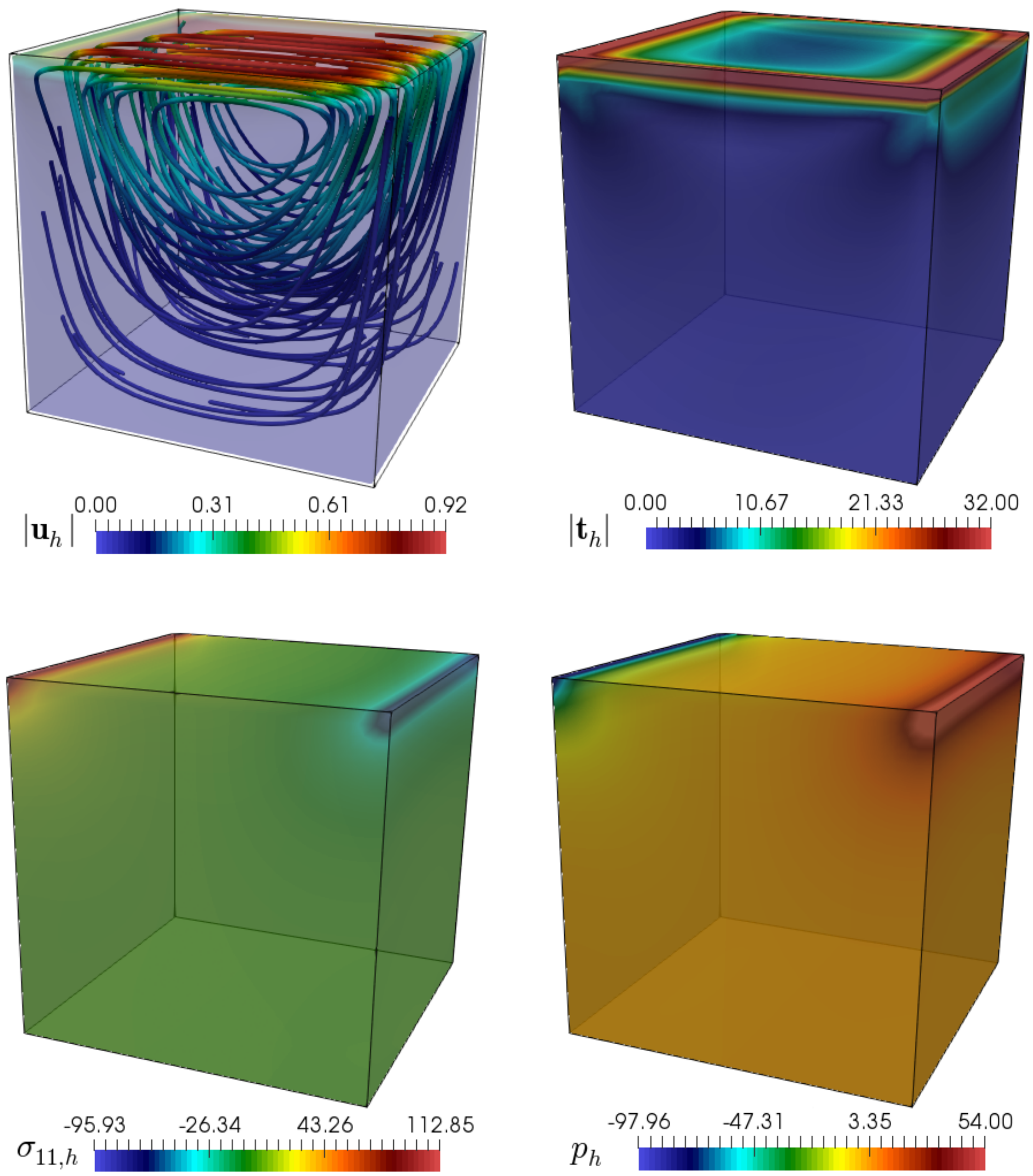

Figure 7.5: Example 5, numerical solutions using $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0$ approximations with `dof` = 3360769 of the fluid velocity magnitude, velocity gradient magnitude, pseudostress component, and postprocessed pressure field.

Part II

A new Banach spaces-based mixed finite element method for the coupled Navier-Stokes and Darcy equations

CHAPTER 8

Introduction

The study of coupled fluid systems, particularly those involving free and porous media flows, governed by the Navier–Stokes and Darcy equations, respectively, and connected through a set of suitable interface conditions, has received significant attention because of their wide range of applications. In particular, the latter includes environmental, biological, and industrial processes, such as the interaction of surface and subsurface flows, modeling of blood flow, and others. Over the years, several papers have been devoted to numerical modeling and analysis of the Navier–Stokes/Darcy and related coupled problems (see, e.g.,[6, 17, 24, 25, 34, 37, 38, 39, 43]). In the context of the Stokes–Darcy coupled problem, the first theoretical results go back to [43] and [24]. In [24] the authors introduce an iterative subdomain method that employs the standard velocity-pressure formulation for the Stokes equation and the primal one in the Darcy domain, whereas in [43] they apply the primal method in the fluid and the dual-mixed one in the porous medium, which means that only the original velocity and pressure unknowns are considered in the Stokes domain, whereas a further unknown (velocity) is added in the Darcy region. In turn, a conforming mixed finite element discretization of the variational formulation from [43] was introduced and analyzed in [34]. In this work, the porous medium is assumed to be entirely enclosed within a fluid region, and, as in [43], the corresponding interface conditions refer to mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman (BJS) law. As a consequence, the trace of the porous medium pressure needs to be introduced as a suitable Lagrange multiplier. In addition, Bernardi–Raugel and Raviart–Thomas elements for the velocities, piecewise constants for the pressures, and continuous piecewise-linear elements for the aforementioned multiplier, yield a stable Galerkin scheme. The results from [34] are then improved in [39] where a classical result on projection methods for Fredholm operators of index zero is employed to show that the use, not only of the one in [34], but of any pair of stable Stokes and Darcy elements, implies the stability of the corresponding Stokes-Darcy Galerkin scheme. Later one, a fully-mixed finite element method was proposed and analyzed in [37] for the Stokes–Darcy coupled problem, where the Babuška-Brezzi theories were used to derive sufficient conditions for the unique solvability of the resulting continuous and discrete formulations. Subsequently, in [38] the authors extend the previous results in [37] to the case of a two-dimensional nonlinear Stokes–Darcy coupled problem. Both *a priori* and *a posteriori* error analyses were developed in this work. As part of augmentation approaches, a fully-mixed finite element method for the Navier–Stokes/Darcy coupled problem with nonlinear viscosity has been introduced and analyzed in [17]. We also refer to [25] for the analysis of a conforming mixed finite element method for the Navier-–Stokes/Darcy coupled problem. In both works, and in order to stay within a Hilbertian framework, the velocity is sought in the Sobolev space of order 1, which requires to augment the variational formulation with additional Galerkin-type terms arising from the constitutive and equilibrium equations.

Although augmented methods are effective in ensuring stability, they significantly increase complexity and computational cost. This issue motivates the exploration of alternative approaches, such as those based on Banach spaces, whose main advantage is that no augmentation is required, and hence the spaces to which the unknowns belong are the natural ones arising from the application of the Cauchy–Schwarz and Hölder inequalities to the tested and eventually integrated by parts equations. A significant number of works have demonstrated the advantage of using this approach to analyze the continuous and discrete formulation of diverse problems (see, e.g, [5, 6, 16, 20, 22]). In particular a non-augmented mixed finite element method for the Navier–Stokes equations with variable viscosity was studied in [6]. More recently, a mass conservative finite element method for the Navier–Stokes/Darcy coupled system, which revisits the original primal-mixed approach from [25], was proposed in [13], whereas a conforming finite element method for a nonisothermal fluid-membrane interaction problem, modeled by the Navier-Stokes/heat system in the free-fluid region, and a Darcy-heat coupled system in the membrane, was introduced and analyzed in [14].

According to the above bibliographic discussion, the goal of this work is to extend the applicability of the Banach spaces framework by introducing a fully–mixed formulation for the coupling of fluid flow with porous media flow, without any augmentation procedure. To this end, we consider a similar approach to the one presented in [6] for the Navier-Stokes domain and adapt it to the coupled Navier-Stokes/Darcy problem. The remainder of this paper is organized as follows. In Chapter 9 we introduce the governing equations and the mathematical model. Subsequently, in Chapter 10 we present the fully-mixed variational formulation within a Banach space framework and prove the well-posedness of the continuous problem. The corresponding Galerkin system is introduced and analyzed in Chapter 11, where a discrete version of the fixed-point strategy developed in Chapter 10 is used. In addition, we derive the associated *a priori* error estimate in the same chapter. In Chapter 12 we specify particular choices of discrete subspaces, in 2D and 3D, that satisfy the hypotheses from Chapter 11 and establish the rates of convergence. Finally, in Chapter 13 we report on 2D numerical examples that validate the method and showcase its practical applications.

Preliminary notations

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\mathbf{n}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar or vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$ will be employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong. Furthermore, as usual, $\mathbb{I}$ stands for the identity tensor in $\mathbb{R} := \mathrm{R}^{n\times n}$, and $|\cdot|$ denotes the Euclidean norm in $\mathbf{R} := \mathrm{R}^n$. Also, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product, respectively, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \quad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} = (\tau_{ji})_{i,j=1,n}, \quad \mathrm{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau}:\boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}, \quad \text{and} \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}.$$

On the other hand, given $t \in (1, +\infty)$, we also introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) := \left\{\boldsymbol{\tau}\in\mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\right\},$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) := \left\{\boldsymbol{\tau}\in\mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\right\},$$

which are endowed with the natural norms defined, respectively, by

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbf{H}(\mathrm{div}_t;\Omega),$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_t;\Omega).$$

Then, proceeding as in [30, eq. (1.43), Section 1.3.4] (see also [12, Section 4.1] and [20, Section 3.1]), it is easy to show that for each $t \in \begin{cases} (1,+\infty) & \text{if } \ n \,=\, 2 \\ [6/5,+\infty) & \text{if } \ n \,=\, 3 \end{cases}$, there holds

$$\langle \boldsymbol{\tau} \cdot \mathbf{n}, v \rangle \,=\, \int_\Omega \{ \boldsymbol{\tau} \cdot \nabla v \,+\, v \, \mathrm{div}(\boldsymbol{\tau}) \} \quad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{H}(\mathrm{div}_t; \Omega) \times \mathrm{H}^1(\Omega)\,, \tag{8.1}$$

and analogously

$$\langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{v} \rangle \,=\, \int_\Omega \{ \boldsymbol{\tau} : \nabla \mathbf{v} \,+\, \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \} \quad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega)\,, \tag{8.2}$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, as well as between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$. We find it important to stress here, as explained in the aforementioned references, that the second term on the right-hand side of (8.1) (resp. (8.2)) is well-defined because of the continuous embedding of $\mathrm{H}^1(\Omega)$ (resp. $\mathbf{H}^1(\Omega)$) into $\mathrm{L}^{t'}(\Omega)$ (resp. $\mathbf{L}^{t'}(\Omega)$), where t' is the conjugate of t, that is $t' \in [1,+\infty)$ such that $\frac{1}{t} + \frac{1}{t'} \,=\, 1$, which holds for $t' \in \begin{cases} [1,+\infty) & \text{if } \ n \,=\, 2 \\ [1,6] & \text{if } \ n \,=\, 3 \end{cases}$.

CHAPTER 9

The model problem

In this chapter we introduce the model of interest, namely the coupled Navier-Stokes and Darcy equations with variable viscosity. To this end, we first let Ω_S and Ω_D be bounded and simply connected open polyhedral domains in R^n, such that $\Omega_S \cap \Omega_D = \varnothing$ and $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \varnothing$. The parts of the boundaries are $\Gamma_S := \partial\Omega_S \setminus \overline{\Sigma}$, $\Gamma_D := \partial\Omega_D \setminus \overline{\Sigma}$, and $\mathbf{n}$ denotes the unit normal vector on them, which is chosen pointing outward from $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider unit tangent vectors, which are given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ and by $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$ (see Fig. 9.1 below for a 2D illustration of the geometry involved). The mathematical model is defined by two separate groups of equations and by a set of coupling terms. Here, Ω_S and Ω_D represent the domains in the free and porous media, respectively.

The governing equations in Ω_S are those of the Navier-Stokes problem with constant density ρ and variable viscosity μ, which are written in terms of the velocity $\mathbf{u}_S$ and the pressure p_S of

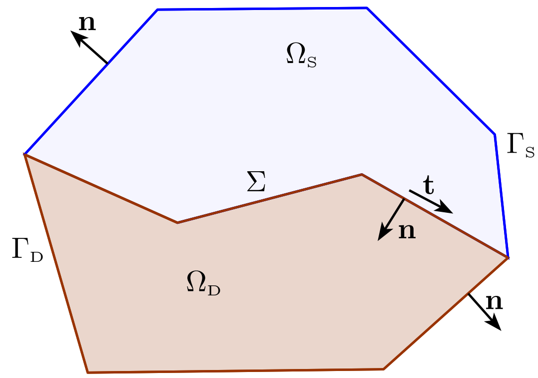

Figure 9.1: geometry of the coupled model

the fluid, that is

$$\begin{aligned} -\mathbf{div}(\mu\nabla\mathbf{u}_S) + \rho(\nabla\mathbf{u}_S)\mathbf{u}_S + \nabla p_S \;=\; \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \\ \operatorname{div}(\mathbf{u}_S) \;=\; 0 \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S \;=\; \mathbf{g} \quad \text{on} \quad \Gamma_S\,, \end{aligned} \tag{9.1}$$

where the given data are a function $\mu : \Omega_S \to \mathrm{R}^+$ describing the viscosity, a volume force $\mathbf{f}_S$, and the boundary velocity $\mathbf{g}$. The right spaces to which $\mathbf{f}_S$ and $\mathbf{g}$ need to belong are specified later on. Furthermore, the function μ is supposed to be bounded, which means that there exist constants μ_1, $\mu_2 > 0$, such that

$$\mu_1 \;\leqslant\; \mu(\mathbf{x}) \;\leqslant\; \mu_2 \quad \forall\, \mathbf{x} \in \Omega_S. \tag{9.2}$$

Next, we introduce the pseudostress tensor unknown

$$\boldsymbol{\sigma}_S \;:=\; \mu\,\nabla\mathbf{u}_S - \rho(\mathbf{u}_S \otimes \mathbf{u}_S) - p_S\mathbb{I} \quad \text{in} \quad \Omega_S\,, \tag{9.3}$$

so that, nothing that $\mathbf{div}(\mathbf{u}_S \otimes \mathbf{u}_S) \;=\; (\nabla\mathbf{u}_S)\mathbf{u}_S$, which makes use of the fact that $\operatorname{div}(\mathbf{u}_S) \;=\; 0$, we find that the first equation of (9.1) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}_S) \;=\; \mathbf{f}_S \quad \text{in} \quad \Omega_S\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of (9.3), that the latter along with the incompressibility condition are equivalent to the pair

$$\begin{aligned} \boldsymbol{\sigma}_S^{\mathrm{d}} &= \mu \nabla \mathbf{u}_S - \rho\,(\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} \quad \text{in} \quad \Omega_S\,, \qquad \text{and} \\ p_S &= -\frac{1}{n}\mathrm{tr}\big(\boldsymbol{\sigma}_S \,+\, \rho\,(\mathbf{u}_S \otimes \mathbf{u}_S)\big) \quad \text{in} \quad \Omega_S\,. \end{aligned} \tag{9.4}$$

Thus, eliminating the pressure unknown which, anyway, can be approximated later on by the postprocessed formula suggested in (9.4), the Navier–Stokes problem (9.1) can be rewritten as:

$$\begin{aligned} \boldsymbol{\sigma}_S^{\mathrm{d}} &= \mu \nabla \mathbf{u}_S - \rho\,(\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} \quad \text{in} \quad \Omega_S\,, \\ -\mathbf{div}(\boldsymbol{\sigma}_S) &= \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S \,=\, \mathbf{g} \quad \text{on} \quad \Gamma_S\,. \end{aligned} \tag{9.5}$$

Next, since we are interested in a mixed variational formulation of our problem, and in order to employ the integration by parts formula typically required by this approach, we introduce the auxiliary unknown $\mathbf{t}_S := \nabla \mathbf{u}_S$ in Ω_S. Consequently, instead of (9.5), we consider from now the set of equations with unknowns $\mathbf{t}_S$, $\mathbf{u}_S$, and $\boldsymbol{\sigma}_S$, given by

$$\begin{aligned} \mathbf{t}_S \,=\, \nabla \mathbf{u}_S \quad \text{in} \quad \Omega_S\,, \quad \boldsymbol{\sigma}_S^{\mathrm{d}} \,=\, \mu\,\mathbf{t}_S - \rho\,(\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} \quad \text{in} \quad \Omega_S\,, \\ -\mathbf{div}(\boldsymbol{\sigma}_S) \,=\, \mathbf{f}_S \quad \text{in} \quad \Omega_S\,, \quad \mathbf{u}_S \,=\, \mathbf{g} \quad \text{on} \quad \Gamma_S\,. \end{aligned} \tag{9.6}$$

On the other hand, in Ω_D we consider the linearized Darcy model:

$$\begin{aligned} \mathbf{u}_D = -\mathbf{K}\nabla p_D \quad \text{in} \quad \Omega_D, \quad \mathrm{div}(\mathbf{u}_D) = f_D \quad \text{in} \quad \Omega_D\,, \\ \mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\,, \end{aligned} \tag{9.7}$$

where $\mathbf{u}_D$ and p_D denote the velocity and pressure, respectively, in the porous medium, $f_D \in L^2(\Omega_D)$ is a source term and $\mathbf{K} \in [L^\infty(\Omega_D)]^{n\times n}$ is a positive definite symmetric tensor describing the permeability of Ω_D divided by a constant approximation of the viscosity, satisfying with $C_{\mathbf{K}} > 0$

$$\mathbf{w} \cdot \mathbf{K}^{-1}(\mathbf{x})\,\mathbf{w} \,\geqslant\, C_{\mathbf{K}}\,|\mathbf{w}|^2 \quad \forall\,(a.e.)\,\mathbf{x} \in \Omega_D, \qquad \forall\,\mathbf{w} \in \mathrm{R}^n\,.$$

Finally, following [43] and [34], the transmission conditions on Σ are given by

$$\begin{aligned} \mathbf{u}_S \cdot \mathbf{n} &= \mathbf{u}_D \cdot \mathbf{n} \quad \text{on} \quad \Sigma\,, \\ \boldsymbol{\sigma}_S \mathbf{n} + \sum_{l=1}^{n-1} \omega_l^{-1} (\mathbf{u}_S \cdot \mathbf{t}_l) \mathbf{t}_l &= -p_D \mathbf{n} \quad \text{on} \quad \Sigma\,, \end{aligned} \tag{9.8}$$

where $\{\omega_1, ..., \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (9.8) corresponds to mass conservation on Σ, whereas the second one establishes the balance of normal forces and Beavers–Joseph–Saffman law. In addition, $\mathbf{g}$ and f_D must formally satisfy the compatibility condition

$$\int_{\Gamma_S} \mathbf{g} \cdot \mathbf{n} + \int_{\Omega_D} f_D \;=\; 0\,. \tag{9.9}$$

CHAPTER 10

The continuous analysis

In this chapter we derive a Banach spaces-based fully-mixed variational formulation of the coupled problem described by (9.6), (9.7), and (9.8), and then perform its solvability analysis by means of a fixed-point strategy.

10.1 Preliminaries

Here we introduce further notations and definitions. We begin with the spaces

$$\begin{aligned}\mathbf{H}_0(\mathrm{div};\Omega_D) &:= \Big\{\mathbf{v}_D \in \mathbf{H}(\mathrm{div};\Omega_D): \quad \mathbf{v}_D\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma_D\Big\},\\ \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) &:= \Big\{\mathbf{r}_S \in \mathbb{L}^2(\Omega_S): \quad \mathrm{tr}(\mathbf{r}_S) = 0\Big\}.\end{aligned}$$

Furthermore, for each $* \in \{\mathrm{S,D}\}$, and given $\widetilde{\Gamma} \subset \partial\Omega_*$, we denote the space of traces

$$\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}) := \Big\{v|_{\widetilde{\Gamma}}: \quad v \in \mathrm{H}^1(\Omega_*), \quad v = 0 \quad \text{on} \quad \partial\Omega_*\backslash\widetilde{\Gamma}\Big\}.$$

and its vector version $\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma}) = \left[\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma})\right]^n$. Observe that, if $E_{\widetilde{\Gamma},*} : \mathrm{H}^{1/2}(\widetilde{\Gamma}) \to \mathrm{L}^2(\partial\Omega_*)$ is the extension operator defined by

$$E_{\widetilde{\Gamma},*}(\psi) := \begin{cases} \psi & \text{on} \quad \widetilde{\Gamma} \\ 0 & \text{on} \quad \partial\Omega_* \backslash \widetilde{\Gamma} \end{cases} \qquad \forall\, \psi \in \mathrm{H}^{1/2}(\widetilde{\Gamma}) \,,$$

we have, alternatively, that

$$\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma}) = \left\{ \psi \in \mathrm{H}^{1/2}(\widetilde{\Gamma}) : \quad E_{\widetilde{\Gamma},*}(\psi) \in \mathrm{H}^{1/2}(\partial\Omega_*) \right\} \,,$$

which is endowed with the norm $\|\psi\|_{1/2,00;\widetilde{\Gamma}} := \|E_{\widetilde{\Gamma},*}(\psi)\|_{1/2,\partial\Omega_*}$. The dual of $\mathrm{H}^{1/2}_{00}(\widetilde{\Gamma})$ (respectively $\mathbf{H}^{1/2}_{00}(\widetilde{\Gamma})$) is denoted by $\mathrm{H}^{-1/2}_{00}(\widetilde{\Gamma})$ (respectively $\mathbf{H}^{-1/2}_{00}(\widetilde{\Gamma})$), and $\|\cdot\|_{-1/2,00;\widetilde{\Gamma}}$ is set as the corresponding norms. Next, in order to deduce the variational formulation of the Navier–Stokes problem, we first look originally for $\mathbf{u}_S \in \mathbf{H}^1(\Omega_S)$, for which we assume from now on, for simplicity, that $\mathbf{g} \in \mathbf{H}^{1/2}_{00}(\Gamma_S)$. Equivalently, letting

$$\mathbf{g}_{s} := E_{\Gamma_S,S}(\mathbf{g}) = \begin{cases} \mathbf{g} & \text{on} \quad \Gamma_S \\ \mathbf{0} & \text{on} \quad \Sigma \end{cases} \,,$$

there holds $\mathbf{g}_{s} \in \mathbf{H}^{1/2}(\partial\Omega_S)$, and hence, using the trace operator $\gamma_0 : \mathbf{H}^1(\Omega_S) \to \mathbf{H}^{1/2}(\partial\Omega_S)$ (see [30, Section 1.3.1]), we can write $\gamma_0(\mathbf{u}_S) = \mathbf{g}_{s} + (\gamma_0(\mathbf{u}_S) - \mathbf{g}_{s})$, where

$$\gamma_0(\mathbf{u}_S) - \mathbf{g}_{s} = \begin{cases} \mathbf{0} & \text{on} \quad \Gamma_S \\ \gamma_0(\mathbf{u}_S) & \text{on} \quad \Sigma \end{cases} = E_{\Sigma,S}(\gamma_0(\mathbf{u}_S)|_\Sigma) \in \mathbf{H}^{1/2}(\partial\Omega_S) \,,$$

which proves that

$$\boldsymbol{\varphi} := -\gamma_0(\mathbf{u}_S)|_\Sigma \in \mathbf{H}^{1/2}_{00}(\Sigma) \,.$$

As a consequence, for each $\chi \in \mathbf{H}^{-1/2}(\partial\Omega_S)$ we get

$$\begin{aligned} \langle \chi, \gamma_0(\mathbf{u}_S) \rangle_{\partial\Omega_S} &= \langle \chi, \mathbf{g}_S \rangle_{\partial\Omega_S} + \langle \chi, \gamma_0(\mathbf{u}_S) - \mathbf{g}_S \rangle_{\partial\Omega_S} \\ &= \langle \chi, E_{\Gamma_S,S}(\mathbf{g}) \rangle_{\partial\Omega_S} - \langle \chi, E_{\Sigma,S}(\boldsymbol{\varphi}) \rangle_{\partial\Omega_S} \\ &= \langle \chi|_{\Gamma_S}, \mathbf{g} \rangle_{\Gamma_S} - \langle \chi|_{\Sigma}, \boldsymbol{\varphi} \rangle_{\Sigma} \,, \end{aligned} \tag{10.1}$$

where $\langle \cdot, \cdot \rangle_{\Gamma_S}$ (respectively $\langle \cdot, \cdot \rangle_{\Sigma}$) stands for the duality pairing between $\mathbf{H}_{00}^{-1/2}(\Gamma_S)$ (respectively $\mathbf{H}_{00}^{-1/2}(\Sigma)$) and $\mathbf{H}_{00}^{1/2}(\Gamma_S)$ (respectively $\mathbf{H}_{00}^{1/2}(\Sigma)$).

10.2 The fully-mixed formulation

Having established the above, we now multiply the first equation of (9.6) by $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_t; \Omega_S)$, with $t \in \begin{cases} (1, +\infty) & \text{if} \quad n = 2 \\ [6/5, +\infty) & \text{if} \quad n = 3 \end{cases}$, apply the integration by parts formula (8.2), and use (10.1) with $\chi = \boldsymbol{\tau}_S \, \mathbf{n}$, to find that

$$\int_{\Omega_S} \boldsymbol{\tau}_S : \mathbf{t}_S + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) \;=\; \langle \boldsymbol{\tau}_S \, \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} - \langle \boldsymbol{\tau}_S \, \mathbf{n}, \boldsymbol{\varphi} \rangle_{\Sigma} \quad \forall \, \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_t; \Omega_S) \,. \tag{10.2}$$

It is clear from (10.2) that its first term is well-defined for $\mathbf{t}_S \in \mathbb{L}^2(\Omega_S)$, which, along with the free trace property of $\mathbf{t}_S$, suggests to look for $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$. In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}_S) \in \mathbf{L}^t(\Omega_S)$, we realize from the second term and Hölder's inequality that it suffices to look for $\mathbf{u}_S \in \mathbf{L}^{t'}(\Omega_S)$, where t' is the conjugate of t. Next, it follows from the second equation of (9.6), that formally

$$\int_{\Omega_S} \mu \, \mathbf{t}_S : \mathbf{r}_S - \int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathrm{d}} : \mathbf{r}_S - \rho \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} : \mathbf{r}_S \;=\; 0 \quad \forall \, \mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \,, \tag{10.3}$$

from which we notice that the first term is well-defined, whereas the second one makes sense if $\boldsymbol{\sigma}_S$ is sought in $\mathbb{L}^2(\Omega_S)$. In turn, for the third one there holds

$$\left| \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} : \mathbf{r}_S \right| = \left| \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S) : \mathbf{r}_S \right| \;\leqslant\; \|\mathbf{u}_S\|_{0,4;\Omega_S} \, \|\mathbf{u}_S\|_{0,4;\Omega_S} \, \|\mathbf{r}_S\|_{0,\Omega_S} \,,$$

which, necessarily yields $t' = 4$, and thus $t = 4/3$. Finally, looking for $\boldsymbol{\sigma}_S$ in the same space of its corresponding test function $\boldsymbol{\tau}_S$, that is $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, it follows from the third equation of (9.6) that

$$-\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) \;=\; \int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \quad \forall\, \mathbf{v}_S \in \mathbf{L}^4(\Omega_S)\,, \tag{10.4}$$

which forces $\mathbf{f}_S$ to belong to $\mathbf{L}^{4/3}(\Omega_S)$. Now for the Darcy equations given in (9.7) and the transmission conditions specified in (9.8), we proceed similarly as in [17], so that introducing the auxiliary unknown

$$\lambda := p_D|_\Sigma \in \mathrm{H}^{1/2}(\Sigma)\,,$$

we obtain the variational problem: Find $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, $p_D \in \mathrm{L}^2(\Omega_D)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ and $\boldsymbol{\varphi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, such that

$$\begin{aligned}
\int_{\Omega_S} \mathbf{t}_S : \boldsymbol{\tau}_S^{\mathrm{d}} + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi}\rangle_\Sigma \;&= \langle \boldsymbol{\tau}_S\mathbf{n}, \mathbf{g}\rangle_{\Gamma_S}\,,\\
\int_{\Omega_D} \mathbf{K}^{-1}\mathbf{u}_D \cdot \mathbf{v}_D - \int_{\Omega_D} p_D \,\mathrm{div}(\mathbf{v}_D) - \langle \mathbf{v}_D\cdot\mathbf{n}, \lambda\rangle_\Sigma \;&= 0\,,\\
\int_{\Omega_S} \mu\, \mathbf{t}_S : \mathbf{r}_S - \int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathrm{d}} : \mathbf{r}_S - \rho \int_{\Omega_S} (\mathbf{u}_S\otimes\mathbf{u}_S)^{\mathrm{d}} : \mathbf{r}_S \;&= 0\,,\\
-\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) \;&= \int_{\Omega_S} \mathbf{f}_S\cdot\mathbf{v}_S\,,\\
\int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{u}_D) \;&= \int_{\Omega_D} f_D\, q_D\,,\\
-\langle \boldsymbol{\varphi}\cdot\mathbf{n}, \xi\rangle_\Sigma - \langle \mathbf{u}_D\cdot\mathbf{n}, \xi\rangle_\Sigma \;&= 0\,,\\
\langle \boldsymbol{\sigma}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi}\cdot\mathbf{n}, \lambda\rangle_\Sigma \;&= 0\,,
\end{aligned} \tag{10.5}$$

for all $\mathbf{r}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{v}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, $q_D \in \mathrm{L}^2(\Omega_D)$, $\xi \in \mathrm{H}^{1/2}(\Sigma)$ and $\boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma)$, where:

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi}\rangle_{\mathbf{t},\Sigma} = \sum_{l=1}^{n-1} w_l^{-1} \langle \boldsymbol{\varphi}\cdot\mathbf{t}_l, \boldsymbol{\psi}\cdot\mathbf{t}_l\rangle_\Sigma\,.$$

It is not difficult to see that the system (10.5) is not uniquely solvable since, given any solution $(\mathbf{t}_S, \mathbf{u}_S, \boldsymbol{\sigma}_S, \mathbf{u}_D, p_D, \lambda, \boldsymbol{\varphi})$ in the indicated spaces, and given any constant $c \in \mathrm{R}$, the vector defined by $(\mathbf{t}_S, \mathbf{u}_S, \boldsymbol{\sigma}_S - c\mathbb{I}, \mathbf{u}_D, p_D - c, \lambda + c, \boldsymbol{\varphi})$ also becomes a solution. In order to avoid this non-uniqueness, from now on we require the Darcy pressure p_D to be in $\mathrm{L}^2_0(\Omega_D)$, where

$$\mathrm{L}^2_0(\Omega_D) := \left\{ q_D \in \mathrm{L}^2(\Omega_D) : \quad \int_{\Omega_D} q_D = 0 \right\} .$$

On the other hand, for convenience of the subsequent analysis, we consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3}; \Omega_S) = \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \oplus \mathrm{R}\,\mathbb{I} \,, \tag{10.6}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{4/3}; \Omega_S) : \quad \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\tau}) = 0 \right\} .$$

It follows that $\boldsymbol{\sigma}_S$ can be uniquely decomposed as $\boldsymbol{\sigma}_S = \boldsymbol{\sigma}_{S,0} + l\mathbb{I}$, where

$$\boldsymbol{\sigma}_{S,0} \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \quad \text{and} \quad l := \frac{1}{n|\Omega_S|} \int_{\Omega_S} \mathrm{tr}(\boldsymbol{\sigma}_S) \,. \tag{10.7}$$

In this regard, we notice that (10.3) and (10.4) remain unchanged if $\boldsymbol{\sigma}_S$ is replaced by $\boldsymbol{\sigma}_{S,0}$. In this way, using the compatibility condition (9.9), the first and last equations of (10.5) are rewritten equivalently as

$$\begin{aligned}
\int_{\Omega_S} \mathbf{t}_S : \boldsymbol{\tau}_S^{\mathrm{d}} + \int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma &= \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} && \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S) \,, \\
j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= j \langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} && \forall\, j \in \mathrm{R} \,, \\
\langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma + l \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= 0 && \forall\, \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma) \,.
\end{aligned}$$

As a consequence of the above, we find that the resulting variational formulation reduces to: Find $\mathbf{t}_S \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}_0(\mathrm{div}; \Omega_D)$, $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}_{4/3}; \Omega_S)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$, $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$,

$\boldsymbol{\varphi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, $p_D \in \mathrm{L}_0^2(\Omega_D)$ and $l \in \mathrm{R}$, such that

$$
\begin{array}{llllllll}
\displaystyle\int_{\Omega_S} \mu \mathbf{t}_S : \mathbf{r}_S & & \displaystyle -\int_{\Omega_S} \boldsymbol{\sigma}_S^{\mathrm{d}} : \mathbf{r}_S & & & \displaystyle -\rho \int_{\Omega_S} (\mathbf{u}_S \otimes \mathbf{u}_S)^{\mathrm{d}} : \mathbf{r}_S & = 0 \\
 & \displaystyle\int_{\Omega_D} \mathbf{K}^{-1} \mathbf{u}_D \cdot \mathbf{v}_D & & -\langle \mathbf{v}_D \cdot \mathbf{n}, \lambda \rangle_\Sigma & & \displaystyle -\int_{\Omega_D} p_D \operatorname{div}(\mathbf{v}_D) & = 0 \\
\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{t}_S & & & & +\langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma & \displaystyle +\int_{\Omega_S} \mathbf{u}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) & = \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} \\
 & \langle \mathbf{u}_D \cdot \mathbf{n}, \xi \rangle_\Sigma & & & +\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma & & = 0 \\
 & & \langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma & +\langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma & -\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma} & +l \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma & = 0 \\
 & & \displaystyle\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\sigma}_S) & & & & \displaystyle = -\int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \\
 & & & & j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & = j \langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} \\
 & \displaystyle -\int_{\Omega_D} q_D \operatorname{div}(\mathbf{u}_D) & & & & & \displaystyle = -\int_{\Omega_D} f_D q_D
\end{array}
\tag{10.8}
$$

for all $\mathbf{r}_S \in \mathbb{L}_{\mathrm{tr}}^2(\Omega_S)$, $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$, $\xi \in \mathrm{H}^{1/2}(\Sigma)$, $\mathbf{v}_S \in \mathbf{L}^4(\Omega_S)$, $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, $q_D \in \mathrm{L}_0^2(\Omega_D)$ and $j \in \mathrm{R}$. Now, we group the spaces, unknowns, and test functions as follows:

$$
\begin{aligned}
&\mathbf{X} := \mathbb{L}_{\mathrm{tr}}^2(\Omega_S) \times \mathbf{H}_0(\mathrm{div};\Omega_D)\,, && \mathbf{Y} := \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \times \mathrm{H}^{1/2}(\Sigma) \\
&\mathbf{Z} := \mathbf{L}^4(\Omega_S) \times \mathbf{H}_{00}^{1/2}(\Sigma)\,, && \mathbb{H} := \mathbf{X} \times \mathbf{Y} \times \mathbf{Z}\,, \\
&\mathbf{Q} := \mathrm{L}_0^2(\Omega_D) \times \mathrm{R}\,,
\end{aligned}
$$

$$
\begin{aligned}
&\vec{\mathbf{t}} := (\mathbf{t}_S, \mathbf{u}_D) \in \mathbf{X}\,, && \vec{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \lambda) \in \mathbf{Y}\,, && \vec{\mathbf{u}} := (\mathbf{u}_S, \boldsymbol{\varphi}) \in \mathbf{Z}\,, && \vec{\boldsymbol{p}} := (p_D, l) \in \mathbf{Q}\,, \\
&\vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \mathbf{X}\,, && \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y}\,, && \vec{\mathbf{v}} := (\mathbf{v}_S, \boldsymbol{\psi}) \in \mathbf{Z}\,, && \vec{\boldsymbol{q}} := (q_D, j) \in \mathbf{Q}\,, \\
&\vec{\boldsymbol{\zeta}} := (\boldsymbol{\zeta}_S, \boldsymbol{z}_D) \in \mathbf{X}\,, && \vec{\boldsymbol{\eta}} := (\boldsymbol{\eta}_S, \vartheta) \in \mathbf{Y}\,, && \vec{\boldsymbol{z}} := (\boldsymbol{z}_S, \boldsymbol{\phi}) \in \mathbf{Z}\,, && \vec{\mathbf{s}} := (s_D, k) \in \mathbf{Q}\,,
\end{aligned}
$$

where $\mathbf{X}$, $\mathbf{Y}$, $\mathbf{Z}$, $\mathbb{H}$ and $\mathbf{Q}$ are respectively endowed with the norms

$$
\begin{aligned}
&\|\vec{\mathbf{r}}\|_{\mathbf{X}} := \|\mathbf{r}_S\|_{0,\Omega_S} + \|\mathbf{v}_D\|_{\mathrm{div},\Omega_D}\,, && \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} := \|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} + \|\xi\|_{1/2,\Sigma}\,, \\
&\|\vec{\mathbf{v}}\|_{\mathbf{Z}} := \|\mathbf{v}_S\|_{0,4;\Omega_S} + \|\boldsymbol{\psi}\|_{1/2,00;\Sigma}\,, && \|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} := \|\vec{\mathbf{r}}\|_{\mathbf{X}} + \|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} + \|\vec{\mathbf{v}}\|_{\mathbf{Z}}\,, \\
&\|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} := \|q_D\|_{0,\Omega_D} + |j|\,.
\end{aligned}
$$

Hence, using the same colors from (10.8), this formulation can be rewritten as: Find

$((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$, such that

$$\begin{array}{llllll}
[a(\vec{\mathbf{t}}), \vec{\mathbf{r}}] & +[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\sigma}}] & & -\int_{\Omega_D} p_D \operatorname{div}(\mathbf{v}_D) & +b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_S) & = 0 \\
[b_2(\vec{\mathbf{t}}), \vec{\boldsymbol{\tau}}] & & +[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{u}}] & & & = \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} \\
 & [\mathbf{B}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}), \vec{\mathbf{v}}] & -[\mathbf{C}(\vec{\mathbf{u}}), \vec{\mathbf{v}}] & +l\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & = -\int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \\
 & & +j\langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma & & & = j\langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S} \\
-\int_{\Omega_D} q_D \operatorname{div}(\mathbf{u}_D) & & & & & = -\int_{\Omega_D} f_D\, q_D
\end{array} \tag{10.9}$$

for all $((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}$, where $a : \mathbf{X} \times \mathbf{X} \to \mathrm{R}$, $b_1 : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$, $b_2 : \mathbf{X} \times \mathbf{Y} \to \mathrm{R}$, $\mathbf{B} : \mathbb{H} \to \mathrm{R}$, and $\mathbf{C} : \mathbf{Z} \times \mathbf{Z} \to \mathrm{R}$, are the bilinear forms defined by

$$\begin{aligned}
[a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}] &:= \int_{\Omega_S} \mu\, \boldsymbol{\zeta}_S : \mathbf{r}_S + \int_{\Omega_D} \mathbf{K}^{-1} \boldsymbol{z}_D \cdot \mathbf{v}_D && \forall\, \vec{\boldsymbol{\zeta}}, \vec{\mathbf{r}} \in \mathbf{X}\,, \\
[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] &:= -\langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma - \int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{r}_S && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}\,, \\
[b_2(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] &:= -[b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}\,, \\
[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}] &:= \langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi \rangle_\Sigma + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma + \int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) && \forall\, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}\,, \\
[\mathbf{C}(\vec{\boldsymbol{z}}), \vec{\mathbf{v}}] &:= \langle \boldsymbol{\phi}, \boldsymbol{\psi} \rangle_{\mathbf{t},\Sigma}, && \forall\, \vec{\boldsymbol{z}}, \vec{\mathbf{v}} \in \mathbf{Z}\,,
\end{aligned} \tag{10.10}$$

whereas for each $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$, $b(\mathbf{w}_S; \cdot, \cdot) : \mathbf{L}^4(\Omega_S) \times \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \to \mathrm{R}$ is the bilinear form given by

$$b(\mathbf{w}_S; \mathbf{v}_S, \mathbf{r}_S) := -\rho \int_{\Omega_S} (\mathbf{w}_S \otimes \mathbf{v}_S)^{\mathrm{d}} : \mathbf{r}_S\,. \tag{10.11}$$

As announced in the abstract, we notice here that (10.9) can be seen as a nonlinear perturbation, given by the term $b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S)$, of a threefold saddle point operator equation, whose main operator $\widetilde{\mathbf{A}}$, to be introduced below, shows a perturbed saddle-point structure (cf. [21]). Indeed, letting $\mathbf{A} : (\mathbf{X} \times \mathbf{Y}) \times (\mathbf{X} \times \mathbf{Y}) \to \mathrm{R}$ be the bilinear form that arises from the block $\begin{pmatrix} a & b_1 \\ b_2 & \end{pmatrix}$ by adding the first two equations of (10.9), that is

$$[\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})] := [a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}] + [b_1(\vec{\mathbf{r}}), \vec{\boldsymbol{\eta}}] + [b_2(\vec{\boldsymbol{\zeta}}), \vec{\boldsymbol{\tau}}] \quad \forall\, (\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X} \times \mathbf{Y}\,, \tag{10.12}$$

and letting $\widetilde{\mathbf{A}} : \mathbb{H} \times \mathbb{H} \to \mathrm{R}$ be the bilinear form that is derived from the block $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}$ by adding the first three equations from (10.9), that is

$$[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] := [\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})] + [\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\boldsymbol{z}}] + [\mathbf{B}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), \vec{\mathbf{v}}] - [\mathbf{C}(\vec{\boldsymbol{z}}), \vec{\mathbf{v}}] \tag{10.13}$$

for all $(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}$, we find that (10.9) becomes: Find $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ such that

$$\begin{aligned} [\widetilde{\mathbf{A}}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] \; + \; [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{p}}] \; + \; b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S) &= [\mathrm{G}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})]\,, \\ [\widetilde{\mathbf{B}}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{q}}] &= [\mathrm{F}, \vec{\boldsymbol{q}}]\,, \end{aligned} \tag{10.14}$$

for all $(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}$, for all $\vec{\boldsymbol{q}} \in \mathbf{Q}$, where

$$\begin{gathered} [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}] := -\int_{\Omega_D} q_D \operatorname{div}(\mathbf{v}_D) \; + \; j\,\langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma\,, \\ [\mathrm{G}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] := \langle \boldsymbol{\tau}_S \mathbf{n}, \mathbf{g} \rangle_{\Gamma_S} - \int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_S \quad \text{and} \quad [\mathrm{F}, \vec{\boldsymbol{q}}] := -\int_{\Omega_D} f_D q_D + j\langle \mathbf{g} \cdot \mathbf{n}, 1 \rangle_{\Gamma_S}\,. \end{gathered} \tag{10.15}$$

Moreover, letting now $\mathbf{P} : (\mathbb{H} \times \mathbf{Q}) \times (\mathbb{H} \times \mathbf{Q}) \to \mathrm{R}$ be the bilinear that arises from the block $\begin{pmatrix} \widetilde{\mathbf{A}} & \widetilde{\mathbf{B}} \\ \widetilde{\mathbf{B}} & \end{pmatrix}$ by adding both equations of (10.14), that is

$$[\mathbf{P}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}, \vec{\mathbf{s}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] := [\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] + [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\mathbf{s}}] + [\widetilde{\mathbf{B}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), \vec{\boldsymbol{q}}] \tag{10.16}$$

for all $((\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), \vec{\mathbf{s}}), ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}$, we deduce that (10.14) (and hence (10.9)) can be stated, equivalently as well, as: Find $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ such that

$$[\mathbf{P}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S) = [\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] \quad \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}\,, \tag{10.17}$$

where $\mathrm{H} \in (\mathbb{H} \times \mathbf{Q})'$ is defined by $[\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] = [\mathrm{G}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})] + [\mathrm{F}, \vec{\boldsymbol{q}}]$. Furthermore, let us introduce the operator $\mathbf{T} : \mathbf{L}^4(\Omega_S) \to \mathbf{L}^4(\Omega_S)$ defined as

$$\mathbf{T}(\mathbf{w}_S) \; := \; \mathbf{u}_S \quad \forall\, \mathbf{w}_S \in \mathbf{L}^4(\Omega_S)\,, \tag{10.18}$$

where $\mathbf{u}_S$ is the first component of $\vec{\mathbf{u}} \in \mathbf{Z}$, which, in turn, is the third component of the unique solution $\big((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}\big) \in \mathbb{H} \times \mathbf{Q}$ (to be proved later on) of the linearized problem arising from (10.17) after replacing $b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_S)$ by $b(\mathbf{w}_S; \mathbf{u}_S, \mathbf{r}_S)$, namely:

$$[\mathbf{P}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\mathbf{w}_S; \mathbf{u}_S, \mathbf{r}_S) = [\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] \quad \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}\,. \tag{10.19}$$

Thus, we realize that solving (10.14) (or (10.17)) is equivalent to finding a fixed-point of $\mathbf{T}$, that is $\mathbf{u}_S \in \mathbf{L}^4(\Omega_S)$ such that

$$\mathbf{T}(\mathbf{u}_S) \;=\; \mathbf{u}_S. \tag{10.20}$$

10.3 Solvability analysis

In this section we analyze the solvability of (10.17) (which is equivalent to (10.9) or (10.14)), by means of the fixed-point strategy that was depicted at the end of the previous section. To this end, we first recall next some theoretical results to be applied later on.

10.3.1 Some useful abstract results

We begin with the generalized Babuška-Brezzi theory.

Theorem 10.1. *Let H_1, H_2, Q_1 and Q_2 be reflexive Banach spaces, and let $b_i : H_i \times Q_i \to R$, $i \in \{1,2\}$, be bounded bilinear forms with boundedness constants given by $\|a\|$ and $\|b_i\|$, $i \in \{1,2\}$, respectively. In addition, for each $i \in \{1,2\}$, let $\mathcal{K}_i$ be the kernel of the operator induced by b_i, that is*

$$\mathcal{K}_i \;:=\; \Big\{ v \in H_i : \quad b_i(v,q) = 0 \quad \forall\, q \in Q_i \Big\}\,,$$

and assume that

i) *there exists a positive constant α such that*

$$\sup_{\substack{v \in \mathcal{K}_1 \\ v \neq 0}} \frac{a(w,v)}{\|v\|_{H_1}} \;\geqslant\; \alpha\, \|w\|_{H_2} \quad \forall\, w \in \mathcal{K}_2\,,$$

ii) *there holds*

$$\sup_{w\in\mathcal{K}_2} a(w,v) \,>\, 0 \quad \forall\, v\in\mathcal{K}_1,\, v\neq 0\,, \quad \textit{and}$$

iii) *for each* $i\in\{1,2\}$ *there exists a positive constant* β_i *such that*

$$\sup_{\substack{v\in H_i\\ v\neq 0}} \frac{b_i(v,q)}{\|v\|_{H_i}} \,\geqslant\, \beta_i\,\|q\|_{Q_i} \quad \forall\, q\in Q_i\,.$$

Then, for each $(\mathrm{F},\mathrm{G})\in H_1'\times Q_2'$ *there exists a unique* $(u,p)\in H_2\times Q_1$ *such that*

$$\begin{aligned} a(u,v) + \quad b_1(v,p) &= \mathrm{F}(v) \qquad \forall\, v\in H_1\,,\\ b_2(u,q) \qquad\qquad &= \mathrm{G}(q) \qquad \forall\, q\in Q_2\,, \end{aligned} \tag{10.21}$$

and the following a priori estimates hold

$$\begin{aligned} \|u\|_{H_2} &\leqslant \frac{1}{\alpha}\|\mathrm{F}\|_{H_1'} + \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{G}\|_{Q_2'} \\ \|p\|_{Q_1} &\leqslant \frac{1}{\beta}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{F}\|_{H_1'} + \frac{\|a\|}{\beta_1\beta_1}\left(1+\frac{\|a\|}{\alpha}\right)\|\mathrm{G}\|_{Q_2'}\,. \end{aligned} \tag{10.22}$$

Moreover, i), ii) *and* iii) *are also necessary conditions for the well-posedness of* (10.21).

Proof. See [7, Theorem 2.1, Corollary 2.1, Section 2.1] for the original version and its proof. For the particular case given by $H_1 = H_2$, $Q_1 = Q_2$, and $b_1 = b_2$, we also refer to [30, Theorem 2.34]. □

We remark here that the roles of $\mathcal{K}_1$ and $\mathcal{K}_2$ in the assumptions i) and ii) of Theorem 10.1 can be exchanged without altering the joint meaning of these hypotheses. In addition, it is important to stress that (10.22) is equivalent to an inf-sup condition for the bilinear form arising after adding the left-hand sides of (10.21), which means that there exists a constant $C > 0$, depending only on α, β_1, β_2 and $\|a\|$, such that

$$\sup_{\substack{(v,q)\in H_1\times Q_2\\ (v,q)\neq\mathbf{0}}} \frac{a(w,v)+b_1(v,r)+b_2(w,q)}{\|(v,q)\|_{H_1\times Q_2}} \,\geqslant\, C\,\|(w,r)\|_{H_2\times Q_1} \quad \forall\,(w,r)\in H_2\times Q_1\,. \tag{10.23}$$

Next, we recall from [36, Theorem 3.2] (see also [21, Theorem 3.4] for the original version of it) a result providing sufficient conditions for the well-posedness of a perturbed saddle-point problem.

Theorem 10.2. *Let H and Q be reflexive Banach spaces, and let $a : H \times H \to R$, $b : H \times Q \to R$ and $c : Q \times Q \to R$ be given bounded bilinear forms. In addition, let $\mathbf{B} : H \to Q'$ be the bounded linear operator induced by b, and let $V := N(\mathbf{B})$ be the respective null space. Assume that:*

i) *a and c are positive semi-definite, that is*

$$a(\tau, \tau) \geqslant 0 \quad \forall\, \tau \in H \quad \textit{and} \quad c(v, v) \geqslant 0 \quad \forall\, v \in Q, \tag{10.24}$$

and that c is symmetric,

ii) *there exists a constant $\alpha > 0$ such that*

$$\sup_{\substack{\tau \in V \\ \tau \neq 0}} \frac{a(\vartheta, \tau)}{\|\tau\|_H} \geqslant \alpha\, \|\vartheta\|_H \quad \forall\, \vartheta \in V\,, \quad \textit{and} \tag{10.25}$$

$$\sup_{\substack{\vartheta \in V \\ \vartheta \neq 0}} \frac{a(\vartheta, \tau)}{\|\vartheta\|_H} \geqslant \alpha\, \|\tau\|_H \quad \forall\, \tau \in V, \tag{10.26}$$

iii) *and there exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\tau \in H \\ \tau \neq 0}} \frac{b(\tau, v)}{\|\tau\|_H} \geqslant \beta\, \|v\|_Q \quad \forall\, v \in Q\,.$$

Then, for each pair $(f, g) \in H' \times Q'$ there exists a unique $(\sigma, u) \in H \times Q$ such that

$$\begin{aligned} a(\sigma, \tau) \;+\; b(\tau, u) \;&=\; f(\tau) \qquad \forall\, \tau \in H\,, \\ b(\sigma, v) \;-\; c(u, v) \;&=\; g(v) \qquad \forall\, v \in Q\,. \end{aligned} \tag{10.27}$$

Moreover, there exists a constant $\widetilde{C} > 0$, depending only on $\|a\|, \|c\|, \alpha$, and β, such that

$$\|(\sigma, u)\|_{H \times Q} \leqslant \widetilde{C} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\}. \tag{10.28}$$

As announced before, we stress here that the foregoing theorem is referred to as a slight variant of the original version given by [21, Theorem 3.4], which requires a to be symmetric as well. Indeed, the proof reduces basically to show that there exists a positive constant $\widehat{C}$, depending on $\|a\|$, $\|c\|$, α, and β, such that the bilinear form arising from adding the left hand sides of (10.27), say $A : (H \times Q) \times (H \times Q) \to \mathrm{R}$, satisfies the inf-sup condition

$$\sup_{\substack{(\tau,v)\in H\times Q\\ (\tau,v)\neq \mathbf{0}}} \frac{A((\zeta,w),(\tau,v))}{\|(\tau,v)\|_{H\times Q}} \geqslant \widehat{C}\,\|(\zeta,w)\|_{H\times Q} \quad \forall\,(\zeta,w)\in H\times Q\,. \tag{10.29}$$

In this way, thanks to the symmetry of a and c, A is obviously symmetric, and thus (10.29) is sufficient to conclude, using the Banach–Nečas–Babuška Theorem (cf.[27, Theorem 2.6], also known as the generalized Lax–Milgram Lemma, the well-posedness of (10.27). However, if the symmetry assumption on a (and consequently on A) is dropped, as done in the present Theorem 10.2, the same conclusion is attained if additionally (10.29) is also satisfied by the bilinear form $\widetilde{A}$ that arises from A after exchanging its components. Thus, noting that the above reduces to fixing the second component of A and taking the supremum in (10.29) with respect to the first one, we realize that in order to prove this further inf-sup condition, the assumption (10.25) needs to be added, as we did in Theorem 10.2. Needless to say, and because of the same constant α in (10.24) and (10.25), the aforementioned further condition holds with the same constant $\widehat{C}$ from (10.29).

10.3.2 Well-definedness of the operator T

We continue by establishing the well-definedness of the operator $\mathbf{T}$, equivalently, that problem (10.19) is well-posed. To this end, we first state the boundedness of all the variational forms involved by employing the Cauchy–Schwarz and Hölder inequalities, the upper bounds of μ, the continuity of the normal trace operator in $\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$ (which follows from (8.2)), the boundedness of the injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega_S) \to \mathbf{L}^4(\Omega_S)$, the boundedness of a suitable extension operator $E_D : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\partial\Omega_D)$ to be defined later on in (10.37) - (10.38), and the existence of a positive constant c_s, depending only on $\partial\Omega_S$, such that $\|\boldsymbol{\psi}\|_{0,\Sigma} \leqslant c_s\|\boldsymbol{\psi}\|_{1/2,\Sigma} \quad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Sigma)$, which yields, in particular, $\|\boldsymbol{\psi}\|_{0,\Sigma} \leqslant c_s\|\boldsymbol{\psi}\|_{1/2,00;\Sigma} \quad \forall \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$ (see [5, Appendix A.1]). In this way, we deduce the existence of positive constants, denoted and given as:

$$\begin{gathered}\|a\| := \max\{\mu_2, \|\mathbf{K}^{-1}\|_\infty\}\,, \quad \|b_1\| = \|b_2\| := \max\left\{1, \|E_D\|\right\},\\ \|\mathbf{A}\| = \|a\| + 2\|b_1\|\,, \quad \|\mathbf{B}\| = \max\{1, \|\mathbf{i}_4\|, c_s^2\}\,,\\ \|\mathbf{C}\| := c_s^2(n-1)\max\{\omega_1^{-1}, ..., \omega_{n-1}^{-1}\}\,, \quad \|\widetilde{\mathbf{A}}\| := \|\mathbf{A}\| + 2\|\mathbf{B}\| + \|\mathbf{C}\|\,,\\ \|\widetilde{\mathbf{B}}\| := \max\{1, c_s|\Sigma|^{1/2}\}\,, \quad \text{and} \quad \|\mathrm{H}\| := \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D}\,,\end{gathered} \tag{10.30}$$

with $\widetilde{\mathbf{g}} := \max\{1, \|\mathbf{i}_4\|, c_s|\Sigma|^{1/2}\}\mathbf{g}$, such that

$$\begin{array}{rcll}|[a(\vec{\boldsymbol{\zeta}}), \vec{\mathbf{r}}]| & \leqslant & \|a\|\,\|\vec{\boldsymbol{\zeta}}\|_{\mathbf{X}}\|\vec{\mathbf{r}}\|_{\mathbf{X}} & \forall\,\vec{\boldsymbol{\zeta}}, \vec{\mathbf{r}} \in \mathbf{X}\,,\\ |[b_i(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}]| & \leqslant & \|b_i\|\,\|\vec{\mathbf{r}}\|_{\mathbf{X}}\|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} & \forall\,(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X}\times\mathbf{Y}\,,\\ |[\mathbf{A}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})]| & \leqslant & \|\mathbf{A}\|\,\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}}\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}} & \forall\,(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}) \in \mathbf{X}\times\mathbf{Y},\\ |[\mathbf{B}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}), \vec{\mathbf{v}}]| & \leqslant & \|\mathbf{B}\|\,\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}}\|\vec{\mathbf{v}}\|_{\mathbf{Z}} & \forall\,(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}\,,\\ |[\mathbf{C}(\vec{\mathbf{v}}), \vec{\boldsymbol{z}}]| & \leqslant & \|\mathbf{C}\|\,\|\vec{\mathbf{v}}\|_{\mathbf{X}}\|\vec{\boldsymbol{z}}\|_{\mathbf{X}} & \forall\,\boldsymbol{\psi}, \boldsymbol{\phi} \in \mathbf{H}^{1/2}_{00}(\Sigma),\\ |[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})]| & \leqslant & \|\widetilde{\mathbf{A}}\|\,\|(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}})\|_{\mathbb{H}}\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}} & \forall(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H}\,,\\ |[\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}]| & \leqslant & \|\widetilde{\mathbf{B}}\|\,\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}})\|_{\mathbb{H}}\|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} & \forall\,((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q}\,,\\ |[\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})]| & \leqslant & \|\mathrm{H}\|\,\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} & \forall\,((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H}\times\mathbf{Q}\,.\end{array} \tag{10.31}$$

In turn, employing the Cauchy–Schwarz inequality twice, we find that

$$\begin{gathered}|b(\mathbf{w}_S; \mathbf{v}_S, \mathbf{r}_S)| \leqslant \rho\,\|\mathbf{w}_S\|_{0,4;\Omega_S}\,\|\mathbf{v}_S\|_{0,4;\Omega_S}\,\|\mathbf{r}_S\|_{0,\Omega_S}\\ \forall\,(\mathbf{w}_S, \mathbf{v}_S, \mathbf{r}_S) \in \mathbf{L}^4(\Omega_S)\times\mathbf{L}^4(\Omega_S)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\,.\end{gathered} \tag{10.32}$$

In what follows, and as suggested by the matrix representation $\begin{pmatrix}\widetilde{\mathbf{A}} & \widetilde{\mathbf{B}}\\ \widetilde{\mathbf{B}} & \mathbf{0}\end{pmatrix}$, we apply the symmetric case of Theorem 10.1. In particular, in order to derive the inf-sup conditions of the bilinear form $\widetilde{\mathbf{A}}$, and according to its structure given by $\begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B} & -\mathbf{C} \end{pmatrix}$ (cf. (10.13)), we employ Theorem 10.2. In turn, and due to the corresponding structure $\begin{pmatrix} a & b_1 \\ b_2 & 0 \end{pmatrix}$ of $\mathbf{A}$ (cf. (10.12)), we employ Theorem 10.1 to establish the required assumptions on $\mathbf{A}$. For the above purposes, we begin by deducing from the definition (10.15) that the kernel $\widetilde{\mathrm{V}}$ of $\widetilde{\mathbf{B}}$ reduces to

$$\widetilde{\mathrm{V}} := \Big\{ (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}) \in \mathbb{H} : \quad [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}] = 0 \quad \forall\, \vec{\boldsymbol{q}} \in \mathbf{Q} \Big\} = \widetilde{\mathbf{X}} \times \mathbf{Y} \times \widetilde{\mathbf{Z}}, \tag{10.33}$$

where

$$\widetilde{\mathbf{X}} = \mathbb{L}^2_{\mathrm{tr}}(\Omega_S) \times \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D), \qquad \widetilde{\mathbf{Z}} = \mathbf{L}^4(\Omega_S) \times \widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma), \tag{10.34}$$

with

$$\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D) := \Big\{ \mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D) : \quad \mathrm{div}(\mathbf{v}_D) \in \mathrm{P}_0(\Omega_D) \Big\},$$

$$\widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma) := \Big\{ \boldsymbol{\psi} \in \mathbf{H}^{1/2}_{00}(\Sigma) : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma = 0 \Big\}.$$

Hereafter, we refer to the null space of the bounded linear operator induced by a bilinear form as the kernel of the latter. Then we let V be the kernel of $\mathbf{B}|_{\widetilde{\mathrm{V}}}$, that is

$$\mathrm{V} = \widetilde{\mathbf{X}} \times \overline{\mathbf{Y}},$$

where

$$\begin{aligned} \overline{\mathbf{Y}} &:= \Big\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y} : \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma + \langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S) = 0 \quad \forall\, (\mathbf{v}_S, \boldsymbol{\psi}) \in \widetilde{\mathbf{Z}} \Big\}, \\ &= \Big\{ \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \mathbf{Y} : \quad \mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0}, \quad \langle \boldsymbol{\psi}\cdot\mathbf{n}, \xi\rangle_\Sigma = -\langle \boldsymbol{\tau}_S\mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma, \quad \forall\, \boldsymbol{\psi} \in \widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma) \Big\}. \end{aligned}$$

Then for each $i \in \{1,2\}$ we let $\mathcal{K}_i$ be the kernel of $b_i|_{\mathrm{V}}$, that is

$$\mathcal{K}_i := \left\{ \vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \widetilde{\mathbf{X}} : \quad [b_i(\vec{\mathbf{r}}), \vec{\boldsymbol{\tau}}] = 0 \quad \forall\, \vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \xi) \in \overline{\mathbf{Y}} \right\},$$

which, recalling from (10.10) that $b_1 = -b_2$, yields

$$\mathcal{K}_1 = \mathcal{K}_2 = \mathcal{K} \subseteq \widetilde{\mathbf{X}}.$$

At this point we recall, for later use, that there exist positive constants $c_{4/3}(\Omega_S)$ and C_{div}, such that (see, [6, Lemma 4.4] and [37, Lemma 3.2], respectively, for details)

$$c_{4/3}(\Omega)\, \|\boldsymbol{\tau}_S\|_{0,\Omega_S} \leqslant \|\boldsymbol{\tau}_S^{\mathrm{d}}\|_{0,\Omega_S} + \|\mathbf{div}(\boldsymbol{\tau}_S)\|_{0,4/3;\Omega_S} \quad \forall\, \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \tag{10.35}$$

and

$$\|\mathbf{v}_D\|^2_{0,\Omega_D} \geqslant C_{\mathrm{div}} \|\mathbf{v}_D\|^2_{\mathrm{div},\Omega_D} \quad \forall\, \mathbf{v}_D \in \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D). \tag{10.36}$$

We now follow [38] to recall some preliminary results concerning boundary conditions and extension operators. Given $\mathbf{v}_D \in \mathbf{H}_0(\mathrm{div};\Omega_D)$, the boundary condition $\mathbf{v}_D \cdot \mathbf{n} = 0$ on Γ_D means

$$\langle \mathbf{v}_D \cdot \mathbf{n}, E_{\Gamma_D,D}(\zeta) \rangle_{\partial\Omega_D} = 0 \quad \forall\, \zeta \in \mathrm{H}^{1/2}_{00}(\Gamma_D).$$

As a consequence, it is not difficult to show (see [29, Section 2]) that the restriction of $\mathbf{v}_D \cdot \mathbf{n}$ to Σ can be identified with an element of $\mathrm{H}^{-1/2}(\Sigma)$, namely

$$\langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma := \langle \mathbf{v}_D \cdot \mathbf{n}, E_D(\xi) \rangle_{\partial\Omega_D} \quad \forall\, \xi \in \mathrm{H}^{1/2}(\Sigma),$$

where $E_D : \mathrm{H}^{1/2}(\Sigma) \to \mathrm{H}^{1/2}(\partial\Omega_D)$ is any bounded extension operator. In particular, given $\xi \in \mathrm{H}^{1/2}(\Sigma)$, one could define $E_D(\xi) := z|_{\partial\Omega_D}$, where $z \in \mathrm{H}^1(\Omega_D)$ is the unique solution of the boundary value problem:

$$\Delta z = 0 \quad \text{in} \quad \Omega_D, \quad z = \xi \quad \text{on} \quad \Sigma, \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_D, \tag{10.37}$$

whose continuous dependence estimate yields $E_D \in \mathcal{L}\big(\mathrm{H}^{1/2}(\Sigma), \mathrm{H}^{1/2}(\partial\Omega_D)\big)$, and hence

$$\|E_D(\xi)\|_{1/2,\partial\Omega_D} \,\leqslant\, \|E_D\| \, \|\xi\|_{1/2,\Sigma} \,. \tag{10.38}$$

In addition, one can show (see [29, Lemma 2.2]) that for all $\zeta \in \mathrm{H}^{1/2}(\partial\Omega_D)$ there exist unique elements $\zeta_\Sigma \in \mathrm{H}^{1/2}(\Sigma)$ and $\zeta_{\Gamma_D} \in \mathrm{H}^{1/2}_{00}(\Gamma_D)$ such that

$$\zeta \,=\, E_D(\zeta_\Sigma) + E_{\Gamma_D,D}(\zeta_{\Gamma_D})\,, \tag{10.39}$$

and

$$C_1\left\{\|\zeta_\Sigma\|_{1/2,\Sigma} + \|\zeta_{\Gamma_D}\|_{1/2,00;\Gamma_D}\right\} \,\leqslant\, \|\zeta\|_{1/2,\partial\Omega_D} \,\leqslant\, C_2\left\{\|\zeta_\Sigma\|_{1/2,\Sigma} + \|\zeta_{\Gamma_D}\|_{1/2,00;\Gamma_D}\right\},$$

with positive constants C_1 and C_2, independent of Σ.

Then, we are in position to prove the results stated by the following lemmas.

Lemma 10.3. *For each $i \in \{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{\vec{\mathbf{r}}\in\widetilde{\mathbf{X}}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{[b_i(\vec{\mathbf{r}}),\vec{\boldsymbol{\tau}}]}{\|\vec{\mathbf{r}}\|_{\mathbf{X}}} \,\geqslant\, \beta_i\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} \quad \forall\,\vec{\boldsymbol{\tau}}\in\overline{\mathbf{Y}}\,. \tag{10.40}$$

Proof. Since $b_1 \,=\, -\,b_2$, it suffices to show for one of these bilinear forms, so that we stay with b_1. Moreover, considering that $\overline{\mathbf{Y}} \subseteq \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) \times \mathrm{H}^{1/2}(\Sigma)$, with

$$\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S) := \left\{\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0}\right\}\,,$$

we need to prove that there exists a positive constant β_1 such that

$$\sup_{\substack{\vec{\mathbf{r}}\in\widetilde{\mathbf{X}}\\ \vec{\mathbf{r}}\neq\mathbf{0}}} \frac{[b_1(\vec{\mathbf{r}}),\vec{\boldsymbol{\tau}}]}{\|\vec{\mathbf{r}}\|_{\mathbf{X}}} \,\geqslant\, \beta_1\,\|\vec{\boldsymbol{\tau}}\|_{\mathbf{Y}} \quad \forall\,\vec{\boldsymbol{\tau}}\in\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S)\times\mathrm{H}^{1/2}(\Sigma)\,. \tag{10.41}$$

In addition, due to the diagonal character of b_1 (cf. (10.10)), the proof of (10.41) reduces to establishing the following two independent inf-sup conditions

$$\sup_{\substack{\mathbf{v}_D\in\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\langle \mathbf{v}_D\cdot\mathbf{n},\xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}} \geqslant \beta_{1,\Sigma}\,\|\xi\|_{1/2,\Sigma} \quad \forall\,\xi\in \mathrm{H}^{1/2}(\Sigma)\,, \quad \text{and} \tag{10.42}$$

$$\sup_{\substack{\mathbf{r}_S\in\mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\\ \mathbf{r}_S\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_S}\boldsymbol{\tau}_S^{\mathrm{d}}:\mathbf{r}_S}{\|\mathbf{r}_S\|_{0,\Omega_S}} \geqslant \beta_{1,S}\,\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} \quad \forall\,\boldsymbol{\tau}_S\in\widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S)\,, \tag{10.43}$$

with $\beta_{1,\Sigma},\ \beta_{1,S} > 0$. Indeed, for (10.42) we refer to [38, Lemma 3.3]. However, for sake of completeness, most details are given in what follows. Given $\phi\in\mathrm{H}^{-1/2}(\Sigma)$, we define $\eta\in\mathrm{H}^{-1/2}(\partial\Omega_D)$ as

$$\langle\eta,\zeta\rangle_{\partial\Omega_D} := \langle\phi,\zeta_\Sigma\rangle_\Sigma \quad \forall\,\zeta\in\mathrm{H}^{1/2}(\partial\Omega_D)\,, \tag{10.44}$$

where ζ_Σ is given by the decomposition (10.39). It is not difficult to see that

$$\langle\eta, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = 0 \quad \forall\,\rho\in\mathrm{H}^{1/2}_{00}(\Gamma_D)\,, \tag{10.45}$$

$$\langle\eta, E_D(\xi)\rangle_{\partial\Omega_D} = \langle\phi,\xi\rangle_\Sigma \quad \forall\xi\in\mathrm{H}^{1/2}(\Sigma) \tag{10.46}$$

and

$$\|\eta\|_{-1/2,\partial\Omega_D} \leqslant C\|\phi\|_{-1/2,\Sigma}\,. \tag{10.47}$$

Hence, we now define $\mathbf{w}_D := \nabla z\in\Omega_D$, where $z\in H^1(\Omega_D)$ is the unique solution of the boundary value problem

$$\Delta z = \frac{1}{|\Omega_D|}\langle\eta,1\rangle_{\partial\Omega_D} \quad \text{in} \quad \Omega_D, \qquad \nabla z\cdot\mathbf{n} = \eta \quad \text{on} \quad \partial\Omega_D, \qquad \int_{\partial\Omega_D} z = 0\,.$$

It follows that $\mathrm{div}(\mathbf{w}_D) = \frac{1}{|\Omega_D|}\langle\eta,1\rangle_{\partial\Omega_D}\in\mathbb{P}_0(\Omega_D)$, $\mathbf{w}_D\cdot\mathbf{n}=\eta$ on $\partial\Omega_D$, and, using the estimate (10.47), $\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D} \leqslant C\|\eta\|_{-1/2,\partial\Omega_D} \leqslant C\|\phi\|_{-1/2,\Sigma}$. In addition, according to (10.44), (10.45) and (10.46), we find, respectively, that

$$\langle\mathbf{w}_D\cdot\mathbf{n},\xi\rangle_\Sigma = \langle\mathbf{w}_D\cdot\mathbf{n},E_D(\xi)\rangle_{\partial\Omega_D} = \langle\eta,E_D(\xi)\rangle_{\partial\Omega_D} = \langle\phi,\xi\rangle_\Sigma$$

and

$$\langle \mathbf{w}_D \cdot \mathbf{n}, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = \langle \eta, E_{\Gamma_D,D}(\rho)\rangle_{\partial\Omega_D} = 0 \quad \forall\, \rho \in \mathrm{H}^{1/2}_{00}(\Gamma_D)\,,$$

which implies that $\mathbf{w}_D \in \widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D)$. In this way, we conclude that

$$\sup_{\substack{\mathbf{v}_D\in\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}} \geqslant \frac{|\langle \mathbf{w}_D \cdot \mathbf{n}, \xi\rangle_\Sigma|}{\|\mathbf{w}_D\|_{\mathrm{div};\Omega_D}} \geqslant C\,\frac{|\langle \phi, \xi\rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} \quad \forall\, \phi \in \mathrm{H}^{-1/2}(\Sigma)\,,$$

and hence

$$\sup_{\substack{\mathbf{v}_D\in\widetilde{\mathbf{H}}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \xi\rangle_\Sigma}{\|\mathbf{v}_D\|_{\mathrm{div};\Omega_D}} \geqslant C \sup_{\substack{\phi\in\mathrm{H}^{-1/2}(\Sigma)\\ \phi\neq\mathbf{0}}} \frac{|\langle \phi, \xi\rangle_\Sigma|}{\|\phi\|_{-1/2,\Sigma}} = C\,\|\xi\|_{1/2,\Sigma}\,,$$

which confirms (10.42). On the other hand, given $\boldsymbol{\tau}_S \in \widetilde{\mathbb{H}}_0(\mathbf{div}_{4/3};\Omega_S)$ such that $\boldsymbol{\tau}_S^{\mathrm{d}} \neq \mathbf{0}$, we have that $\boldsymbol{\tau}_S^{\mathrm{d}} \in \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, so that bounding the supremum in (10.43) by below with $\mathbf{r}_S = -\boldsymbol{\tau}_S^{\mathrm{d}}$, it follows that

$$\sup_{\substack{\mathbf{r}_S\in\mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\\ \mathbf{r}_S\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \mathbf{r}_S}{\|\mathbf{r}_S\|_{0,\Omega_S}} \geqslant \frac{\displaystyle\int_{\Omega_S} \boldsymbol{\tau}_S^{\mathrm{d}} : \boldsymbol{\tau}_S^{\mathrm{d}}}{\|\boldsymbol{\tau}_S^{\mathrm{d}}\|_{0,\Omega_S}} = \|\boldsymbol{\tau}_S^{\mathrm{d}}\|_{0;\Omega_S}\,,$$

which, using (10.35) and the fact that $\mathbf{div}(\boldsymbol{\tau}_S) = \mathbf{0}$, implies that (10.43) is satisfied with constant $\beta_{1,S} = c_{4/3}(\Omega_S)$. On the other hand, if $\boldsymbol{\tau}_S^{\mathrm{d}} = \mathbf{0}$, it is clear from (10.35) that $\boldsymbol{\tau}_S = \mathbf{0}$, and so (10.43) is trivially satisfied. □

Lemma 10.4. *There exists a positive constant α such that*

$$[a(\vec{\mathbf{r}}), \vec{\mathbf{r}}] \geqslant \alpha_a\,\|\vec{\mathbf{r}}\|^2_{\mathbf{X}} \quad \forall\, \vec{\mathbf{r}} \in \widetilde{\mathbf{X}}\,.$$

Proof. Given $\vec{\mathbf{r}} := (\mathbf{r}_S, \mathbf{v}_D) \in \widetilde{\mathbf{X}}$, we use the definition of a (cf. (10.10)), (9.2), and (10.36), to obtain

$$[a(\vec{\mathbf{r}}), \vec{\mathbf{r}}] = \int_{\Omega_S} \mu\,\mathbf{r}_S : \mathbf{r}_S + \int_{\Omega_D} \mathbf{K}^{-1}\mathbf{v}_D \cdot \mathbf{v}_D \geqslant \mu_1\,\|\mathbf{r}_S\|^2_{0,\Omega_S} + C_{\mathbf{K}}\,\|\mathbf{v}_D\|^2_{0,\Omega_D} \geqslant \alpha_a\,\|\vec{\mathbf{r}}\|_{\mathbf{X}}\,,$$

with $\alpha_a := \dfrac{1}{2}\min\{\mu_1, C_{\mathrm{div}}C_{\mathbf{K}}\}$, thus confirming the required property on a. In particular, since

$\mathcal{K} \subset \widetilde{\mathbf{X}}$, it is clear that a is $\mathcal{K}$-elliptic. □

As a consequence of Lemma 10.3 and Lemma 10.4, we conclude that a, b_1 and b_2 satisfy the hypotheses of Theorem 10.1, and hence, a straightforward application of this abstract result yields the existence of a positive constant $\alpha_{\mathbf{A}}$, depending on $\|a\|$, α_a and β_1, such that

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in \mathrm{V} \\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\neq \mathbf{0}}} \frac{[\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}}} \geqslant \alpha_{\mathbf{A}}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}} \quad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}}) \in \mathrm{V}\,. \tag{10.48}$$

Moreover, if we swap the roles of b_1 and b_2, changing the matrix from $\begin{pmatrix} a & b_1 \\ b_2 & 0 \end{pmatrix}$ to $\begin{pmatrix} a & b_2 \\ b_1 & 0 \end{pmatrix}$, we can reapply Theorem 10.1 and (10.23) to conclude that, with the same constant $\alpha_{\mathbf{A}}$ from (10.48), there holds

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\in \mathrm{V} \\ (\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\neq \mathbf{0}}} \frac{[\mathbf{A}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})]}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}})\|_{\mathbf{X}\times\mathbf{Y}}} \geqslant \alpha_{\mathbf{A}}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}} \quad \forall\,(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}) \in \mathrm{V}\,.$$

Furthermore, it is evident from (10.12) and the ellipticity of a in $\widetilde{\mathbf{X}}$, that

$$[\mathbf{A}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})] \;=\; [a(\vec{\mathbf{r}}),\vec{\mathbf{r}}] \;\geqslant\; \alpha_a\,\|\vec{\mathbf{r}}\|_{\mathbf{X}} \quad \forall\,(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}) \in \widetilde{\mathbf{X}}\times\mathbf{Y}\,,$$

which proves that $\mathbf{A}$ is positive semi-definite.

Lemma 10.5. *There holds*

$$[\mathbf{C}(\vec{\mathbf{v}}),\vec{\mathbf{v}}] \geqslant 0 \quad \forall\,\vec{\mathbf{v}} \in \mathbf{Z}\,.$$

Proof. From the definition of the operator $\mathbf{C}$ (cf. (10.10)), it readily follows that

$$[\mathbf{C}(\vec{\mathbf{v}}),\vec{\mathbf{v}}] = \sum_{l=1}^{n-1} w_l^{-1}\|\boldsymbol{\psi}\cdot\mathbf{t}_l\|_{0,\Sigma}^2 \geqslant 0 \quad \vec{\mathbf{v}} \in \mathbf{Z}\,,$$

which confirms that $\mathbf{C}$ is positive semi-definite. □

In this way, we have demonstrated that $\mathbf{A}$ and $\mathbf{C}$ satisfy hypotheses i) and ii) of Theorem

10.2, and hence it only remains to show the corresponding assumption iii), which is the continuous inf-sup condition for $\mathbf{B}$ with respect to the third component $\widetilde{\mathbf{Z}}$ of the kernel $\widetilde{\mathrm{V}}$ of $\widetilde{\mathbf{B}}$ (cf. (10.33), (10.34)).

Lemma 10.6. *There exists a positive constant $\boldsymbol{\beta}_S$ such that*

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in\widetilde{\mathbf{X}}\times\mathbf{Y}\\(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbf{X}\times\mathbf{Y}}} \geqslant \boldsymbol{\beta}_S\,\|\vec{\mathbf{v}}\|_{\mathbf{Z}} \qquad \forall\,\vec{\mathbf{v}}\in\widetilde{\mathbf{Z}}\,. \tag{10.49}$$

Proof. Given $\vec{\mathbf{v}} := (\mathbf{v}_S,\boldsymbol{\psi}) \in \widetilde{\mathbf{Z}} := \mathbf{L}^4(\Omega_S)\times\widetilde{\mathbf{H}}^{1/2}_{00}(\Sigma)$, we first realize, taking $\vec{\mathbf{r}} := (\mathbf{r}_S,\mathbf{v}_D) = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S,\xi) = (\boldsymbol{\tau}_S,0)$, that

$$\begin{aligned}\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\in\mathbb{H}\\(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}}),\vec{\mathbf{v}}]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}})\|_{\mathbb{H}}} &\geqslant \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\\\boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{[\mathbf{B}(\vec{\mathbf{0}},(\boldsymbol{\tau}_S,0)),\vec{\mathbf{v}}]}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\\ &= \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\\\boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\,.\end{aligned} \tag{10.50}$$

Next, setting $\boldsymbol{\tau}_S := \boldsymbol{\tau}_{S,0} + c\mathbb{I} \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)$ with the respective components $c\in\mathrm{R}$ and $\boldsymbol{\tau}_{S,0}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$, we observe that

$$\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S) = \int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_{S,0})\,, \qquad \langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma = \langle\boldsymbol{\tau}_{S,0}\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma\,, \quad \text{and}$$

$$\|\boldsymbol{\tau}_S\|^2_{\mathbf{div}_{4/3};\Omega_S} \;=\; \|\boldsymbol{\tau}_{S,0}\|^2_{\mathbf{div}_{4/3};\Omega_S} \;+\; 2\,c^2\,|\Omega_S|\,.$$

Hence, noting that $\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S} \geqslant \|\boldsymbol{\tau}_{S,0}\|_{\mathbf{div}_{4/3};\Omega_S}$, we find that

$$\sup_{\substack{\boldsymbol{\tau}_{S,0}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\\\boldsymbol{\tau}_{S,0}\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_{S,0}\,\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_{S,0})}{\|\boldsymbol{\tau}_{S,0}\|_{\mathbf{div}_{4/3};\Omega_S}} \geqslant \sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\\boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\,,$$

which, along with (10.50), implies that in order to conclude (10.49), it suffices to show that there exists a positive constant $\boldsymbol{\beta}_S$, independent of the given $\vec{\mathbf{v}} := (\mathbf{v}_S, \boldsymbol{\psi}) \in \widetilde{\mathbf{Z}}$, such that

$$\sup_{\substack{\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S} \mathbf{v}_S \cdot \mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \;\geqslant\; \boldsymbol{\beta}_S \left\{ \|\boldsymbol{\psi}\|_{1/2,00;\Sigma} + \|\mathbf{v}_S\|_{0,4;\Omega_S} \right\}. \tag{10.51}$$

To this end, we now set $\widehat{\mathbf{v}}_S := |\mathbf{v}_S|^2\, \mathbf{v}_S$ and notice that $\|\widehat{\mathbf{v}}_S\|^{4/3}_{0,4/3;\Omega_S} = \|\mathbf{v}_S\|^4_{0,4;\Omega_S}$, which says that $\widehat{\mathbf{v}}_S \in \mathbf{L}^{4/3}(\Omega_S)$, and

$$\int_{\Omega_S} \mathbf{v}_S \cdot \widehat{\mathbf{v}}_S \;=\; \|\mathbf{v}_S\|_{0,4;\Omega_S}\, \|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,. \tag{10.52}$$

Then, we let $\boldsymbol{z} \in \mathbf{H}^1(\Omega_S)$ be the unique solution of

$$-\Delta \boldsymbol{z} \;=\; \widehat{\mathbf{v}}_S \quad \text{in} \quad \Omega_S, \qquad \boldsymbol{z} = 0 \quad \text{on} \quad \Gamma_S\,, \qquad \text{and} \qquad \nabla \boldsymbol{z}\, \mathbf{n} = 0 \quad \text{on} \quad \Sigma\,,$$

whose variational formulation reads: Find $\boldsymbol{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ such that

$$\int_{\Omega_S} \nabla \boldsymbol{z} \cdot \nabla \mathbf{w} = \int_{\Omega_S} \widehat{\mathbf{v}}_S \cdot \mathbf{w} \quad \forall\, \mathbf{w} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)\,, \tag{10.53}$$

where

$$\mathbf{H}^1_{\Gamma_S}(\Omega_S) \;:=\; \Big\{ \mathbf{w} \in \mathbf{H}^1(\Omega_S) : \quad \mathbf{w} = \mathbf{0} \quad \text{on} \quad \Gamma_S \Big\}\,.$$

In fact, we first notice that the left-hand side of (10.53) defines an $\mathbf{H}^1_{\Gamma_S}(\Omega_S)$-elliptic bilinear form. In addition, Hölder's inequality and the continuous injection $\mathbf{i}_4$ from $\mathbf{H}^1(\Omega_S)$ into $\mathbf{L}^4(\Omega_S)$ guarantee that the right-hand side of (10.53) constitutes a functional in $\mathbf{H}^1_{\Gamma_S}(\Omega_S)'$. Consequently, a straightforward application of the classical Lax–Milgram Lemma implies the existence of a unique $\boldsymbol{z} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$ solution to (10.53). Moreover, it follows from (10.53) that

$$|\boldsymbol{z}|_{1,\Omega_S} \;\leqslant\; c_s\, \|\mathbf{i}_4\|\, \|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,, \tag{10.54}$$

where c_s is the positive constant, depending only on Ω_S, provided by the Poincaré inequality, that is such that $\|\mathbf{v}\|_{1,\Omega_S} \leqslant c_s\, |\mathbf{v}|_{1,\Omega_S}$ for all $\mathbf{v} \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$. Then, defining $\widetilde{\boldsymbol{\tau}}_S := -\nabla \boldsymbol{z} \in \mathbf{L}^2(\Omega_S)$, we see that $\mathbf{div}(\widetilde{\boldsymbol{\tau}}_S) = \widehat{\mathbf{v}}_S$ in Ω_S, which says that actually $\widetilde{\boldsymbol{\tau}}_S \in \mathbb{H}(\mathbf{div}_{4/3}, \Omega_S)$, and that $\widetilde{\boldsymbol{\tau}}_S \mathbf{n} = \mathbf{0}$

on Σ, so that using (10.54), we get

$$\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathrm{div}_{4/3};\Omega_S} \,=\, |\boldsymbol{z}|_{1,\Omega_S} + \|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S} \,\leqslant\, \big(1 \,+\, c_s\,\|\mathbf{i}_4\|\big)\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}\,. \tag{10.55}$$

In this way, bounding by below with $\widetilde{\boldsymbol{\tau}}_S$, and employing (10.52) and (10.55), we deduce that

$$\begin{aligned}\sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} &\geqslant \frac{\displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\widetilde{\boldsymbol{\tau}}_S)}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}}\\ = \frac{\displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\widehat{\mathbf{v}}_S}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \,=\, \frac{\|\mathbf{v}_S\|_{0,4;\Omega_S}\,\|\widehat{\mathbf{v}}_S\|_{0,4/3;\Omega_S}}{\|\widetilde{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S}} &\geqslant \boldsymbol{\beta}_{S,1}\,\|\mathbf{v}_S\|_{0,4;\Omega_S}\,,\end{aligned} \tag{10.56}$$

with $\boldsymbol{\beta}_{S,1} \,:=\, \big(1 \,+\, c_s\,\|\mathbf{i}_4\|\big)^{-1}$. On the other hand, given $\boldsymbol{\eta}\in\mathbf{H}_{00}^{-1/2}(\Sigma)$, we let $\widehat{\boldsymbol{z}}\in\mathbf{H}^1_{\Gamma_S}(\Omega_S)$ be the unique solution of

$$-\,\Delta\widehat{\boldsymbol{z}} \,=\, \mathbf{0} \quad\text{in}\quad \Omega_S\,,\qquad \widehat{\boldsymbol{z}} \,=\, \mathbf{0} \quad\text{on}\quad \Gamma_S\,,\qquad \nabla\widehat{\boldsymbol{z}}\,\mathbf{n} \,=\, \boldsymbol{\eta} \quad\text{on}\quad \Sigma\,,$$

and define $\widehat{\boldsymbol{\tau}}_S := \nabla\widehat{\boldsymbol{z}}$ in Ω_S. It follows that $\mathbf{div}(\widehat{\boldsymbol{\tau}}_S) = \mathbf{0}$ in Ω_S, $\widehat{\boldsymbol{\tau}}_S\,\mathbf{n} = \boldsymbol{\eta}$ on Γ_S, and $\|\widehat{\boldsymbol{\tau}}_S\|_{\mathbf{div}_{4/3};\Omega_S} \,=\, \|\widehat{\boldsymbol{\tau}}_S\|_{0,\Omega_S} \,\leqslant\, \widehat{C}\,\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}$, which yields

$$\sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \,\geqslant\, \frac{\langle\widehat{\boldsymbol{\tau}}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma}{\|\widehat{\boldsymbol{\tau}}_S\|_{\mathrm{div}_{4/3};\Omega_S}} \,\geqslant\, \boldsymbol{\beta}_{S,2}\,\frac{\big|\langle\boldsymbol{\eta},\boldsymbol{\psi}\rangle_\Sigma\big|}{\|\boldsymbol{\eta}\|_{-1/2,00;\Sigma}}\,,$$

with $\boldsymbol{\beta}_{S,2} \,:=\, \widehat{C}^{-1}$. Since $\boldsymbol{\eta}\in\mathbf{H}_{00}^{-1/2}(\Sigma)$ is arbitrary, the foregoing inequality leads to

$$\sup_{\substack{\boldsymbol{\tau}_S\in\mathbb{H}(\mathbf{div}_{4/3};\Omega_S)\\ \boldsymbol{\tau}_S\neq\mathbf{0}}} \frac{\langle\boldsymbol{\tau}_S\mathbf{n},\boldsymbol{\psi}\rangle_\Sigma + \displaystyle\int_{\Omega_S}\mathbf{v}_S\cdot\mathbf{div}(\boldsymbol{\tau}_S)}{\|\boldsymbol{\tau}_S\|_{\mathbf{div}_{4/3};\Omega_S}} \,\geqslant\, \boldsymbol{\beta}_{S,2}\,\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}\,,$$

which, along with (10.56), shows (10.51), and hence (10.49), with $\boldsymbol{\beta}_S \,:=\, \frac{1}{2}\,\min\{\boldsymbol{\beta}_{S,1},\boldsymbol{\beta}_{S,2}\}$. $\square$

Consequently, having the bilinear forms $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$ satisfied the three hypotheses of Theorem

10.2, a straightforward application of this abstract result yields the existence of a positive constant $\widetilde{\alpha}$, depending on $\|\mathbf{A}\|$, $\|\mathbf{C}\|$, $\alpha_{\mathbf{A}}$, and $\boldsymbol{\beta}_S$ such that

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\widetilde{\mathrm{V}}\\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}}} \geqslant \widetilde{\alpha}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}})\|_{\mathbb{H}} \qquad \forall\,(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}})\in\widetilde{\mathrm{V}}\,,$$

and

$$\sup_{\substack{(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}})\in\widetilde{\mathrm{V}}\\ (\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}})\neq\mathbf{0}}} \frac{[\widetilde{\mathbf{A}}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})]}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}})\|_{\mathbb{H}}} \geqslant \widetilde{\alpha}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}} \qquad \forall\,(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\widetilde{\mathrm{V}}\,,$$

which means that $\widetilde{\mathbf{A}}$ satisfies the assumptions i) and ii) of Theorem 10.1. Thus, it only remains to demonstrate the corresponding assumption iii), which is the continuous inf-sup condition for $\widetilde{\mathbf{B}}$.

Lemma 10.7. *There exists a positive constant $\widetilde{\beta}$ such that*

$$\sup_{\substack{(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\in\mathbb{H}\\ (\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\neq\mathbf{0}}} \frac{[\widetilde{\mathbf{B}}(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}}]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}})\|_{\mathbb{H}}} \geqslant \widetilde{\beta}\,\|\vec{\boldsymbol{q}}\|_{\mathbf{Q}} \qquad \forall\,\vec{\boldsymbol{q}}\in\mathbf{Q}\,. \tag{10.57}$$

Proof. We first observe that the diagonal character of $\widetilde{\mathbf{B}}$ (cf. (10.15)) says that proving (10.57) is equivalent to establishing the following two independent inf–sup conditions

$$\sup_{\substack{\mathbf{v}_D\in\mathbf{H}_0(\mathrm{div};\Omega_D)\\ \mathbf{v}_D\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{v}_D)}{\|\mathbf{v}_D\|_{\mathrm{div},\Omega_D}} \geqslant \widetilde{\beta}_D\,\|q_D\|_{0,\Omega_D} \quad \forall\,q_D\in\mathrm{L}^2_0(\Omega_D)\,, \tag{10.58}$$

$$\sup_{\substack{\boldsymbol{\psi}\in\mathbf{H}^{1/2}_{00}(\Sigma)\\ \boldsymbol{\psi}\neq\mathbf{0}}} \frac{j\,\langle\boldsymbol{\psi}\cdot\mathbf{n},1\rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}} \geqslant \widetilde{\beta}_S\,|j| \quad \forall\,j\in\mathrm{R}\,. \tag{10.59}$$

To this end, we proceed similarly to the proof of [37, Lemma 3.6]. We define $\mathbf{v}_D := \nabla z$, where $z\in\mathrm{H}^1_\Sigma(\Omega_D)$ is the unique solution of the boundary value problem:

$$\Delta z \,=\, q_D \quad\text{in}\quad \Omega_D\,, \quad z \,=\, 0 \quad\text{on}\quad \Sigma\,, \quad \nabla z\cdot\mathbf{n} \,=\, \mathbf{0} \quad\text{on}\quad \Gamma_D\,.$$

It follows that $\mathbf{v}_D\in\mathbf{H}_0(\mathrm{div};\Omega_D)$ and $\mathrm{div}(\mathbf{v}_D) = q_D$, which yields the surjectivity of the operator div : $\mathbf{H}_0(\mathrm{div};\Omega_D) \to \mathrm{L}_0^2(\Omega_D)$, which is (10.58). On the other hand, the inf-sup condition (10.59) reduces to the surjectivity of the operator $\boldsymbol{\psi} \to \langle \boldsymbol{\psi}\cdot\mathbf{n}, 1\rangle_\Sigma$ from $\mathrm{H}^{1/2}(\Sigma) \to \mathrm{R}$, which in turn is equivalent to showing the existence of $\boldsymbol{\psi}_0 \in \mathrm{H}^{1/2}(\Sigma)$ such that $\langle \boldsymbol{\psi}_0\cdot\mathbf{n}, 1\rangle_\Sigma \neq 0$. In fact, we pick one corner point of Σ and define a function v that is continuous, linear on each side of Σ, equal to one in the chosen vertex, and zero on all other ones. If $\mathbf{n}_1$ and $\mathbf{n}_2$ are the normal vectors on the two sides of Σ that meet at the corner point, then $\boldsymbol{\psi}_0 := \nu(\mathbf{n}_1 + \mathbf{n}_2)$ satisfies the required property. Finally, the required inequality (10.57) is obtained with $\widetilde{\beta} := \min\big\{\widetilde{\beta}_S, \widetilde{\beta}_D\big\}$. $\square$

Now, having the bilinear forms $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ satisfied the assumptions of Theorem 10.1, a direct application of this abstract result guarantees the global inf-sup condition for $\mathbf{P}$ (cf. (10.16)), that is the existence of a positive constant $\alpha_{\mathbf{P}}$, depending on $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\|\widetilde{\mathbf{A}}\|$, such that

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q} \\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{P}}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}} \quad \forall\,((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q}\,. \tag{10.60}$$

In turn, if we consider the transpose of $\mathbf{P}$, which simply reduces to exchange the bilinear forms b_1 and b_2 in (10.12), we conclude that inf-sup conditions are satisfied by $\mathbf{P}$ with respect to the other component, that is

$$\sup_{\substack{((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q} \\ ((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\mathbf{s})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{P}}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} \quad \forall\,((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\,. \tag{10.61}$$

Moreover, employing (10.60) and the boundedness property of b (cf. (10.32)), it readily follows that, given $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$, there holds

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q} \\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})] + b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \big(\alpha_{\mathbf{P}} - \rho\,\|\mathbf{w}_S\|_{0,4;\Omega_S}\big)\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}}$$

for all $((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}}) \in \mathbb{H}\times\mathbf{Q}$, and hence, for each $\mathbf{w}_S \in \mathbf{L}^4(\Omega_S)$ such that $\|\mathbf{w}_S\|_{0,4;\Omega_S} \leqslant \dfrac{\alpha_{\mathbf{P}}}{2\rho}$, we get

$$\sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]+b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \frac{\alpha_{\mathbf{P}}}{2}\,\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}} \tag{10.62}$$

for all $((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q}$. Similarly, but now using (10.61), and under the same assumption on $\mathbf{w}_S$, we arrive at

$$\sup_{\substack{((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\in\mathbb{H}\times\mathbf{Q}\\((\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}}),\vec{\mathbf{s}})\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}}),(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})]+b(\mathbf{w}_S;\mathbf{u}_S,\mathbf{r}_S)}{\|(\vec{\boldsymbol{\zeta}},\vec{\boldsymbol{\eta}},\vec{\boldsymbol{z}},\vec{\mathbf{s}})\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \frac{\alpha_{\mathbf{P}}}{2}\,\|(\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}},\vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}} \tag{10.63}$$

for all $((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}$.

Consequently, the well-definedness of the operator $\mathbf{T}$ can be stated as follows.

Theorem 10.8. *For each $\mathbf{w}_S\in\mathbf{L}^4(\Omega_S)$ such that $\|\mathbf{w}_S\|_{0,4;\Omega_S}\leqslant\dfrac{\alpha_{\mathbf{P}}}{2\rho}$, there exists a unique solution $\big((\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}}),\vec{\boldsymbol{p}}\big)\in\mathbb{H}\times\mathbf{Q}$ solution to (10.19), and hence we can define $\mathbf{T}(\mathbf{w}_S):=\mathbf{u}_S\in\mathbf{L}^4(\Omega_S)$. Moreover, there holds*

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}_S)\|_{0,4;\Omega_S}=\|\mathbf{u}_S\|_{0,4;\Omega_S}&\leqslant\|(\vec{\mathbf{t}},\vec{\boldsymbol{\sigma}},\vec{\mathbf{u}},\vec{\boldsymbol{p}})\|_{\mathbb{H}\times\mathbf{Q}}\\&\leqslant\frac{2}{\alpha_{\mathbf{P}}}\Big\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S}+\|\mathbf{f}_S\|_{0,4/3;\Omega_S}+\|f_D\|_{0,\Omega_D}\Big\}.\end{aligned} \tag{10.64}$$

Proof. Given $\mathbf{w}_S$ as indicated, the existence of a unique solution to (10.19) follows from (10.62), (10.63), and a direct application of the Banach–Nečas–Babuška Theorem (see [27, Theorem 2.6]). In turn, the corresponding *a priori* estimate and the boundedness of H (cf. (10.31)) yield (10.64). □

10.3.3 Solvability analysis of the fixed-point scheme

Knowing that the operator $\mathbf{T}$ (cf. (10.18)) is well-defined, in this section we proceed to establish the existence of a unique solution of the fixed-point equation (10.20). To this end, in what follows we will first derive sufficient conditions on $\mathbf{T}$ to map a closed ball of $\mathbf{L}^4(\Omega_S)$ into itself.

This will allow us to apply the Banach Theorem later on. Indeed, from now on we let

$$\mathrm{W} := \left\{ \mathbf{w}_S \in \mathbf{L}^4(\Omega_S) : \quad \|\mathbf{w}_S\|_{0,4;\Omega_S} \leqslant \frac{\alpha_{\mathbf{P}}}{2\rho} \right\}.$$

Lemma 10.9. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \leqslant \frac{\alpha_{\mathbf{P}}^2}{4\rho}. \tag{10.65}$$

Then, there holds $\mathbf{T}\big(\mathrm{W}\big) \subseteq \mathrm{W}$.

Proof. Given $\mathbf{w}_S \in \mathrm{W}$, we know from Theorem 10.8 that $\mathbf{T}(\mathbf{w}_S)$ is well-defined and that there holds

$$\|\mathbf{T}(\mathbf{w}_S)\|_{0,4;\Omega_S} \leqslant \frac{2}{\alpha_{\mathbf{P}}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega} + \|f_D\|_{0,\Omega} \right\} \leqslant \frac{\alpha_{\mathbf{P}}}{2\rho}, \tag{10.66}$$

which shows that $\mathbf{T}(\mathbf{w}_S) \in \mathrm{W}$. □

We continue with the following result providing the required continuity of $\mathbf{T}$.

Lemma 10.10. *There holds*

$$\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4:\Omega_S} \leqslant \frac{4\rho}{\alpha_{\mathbf{P}}^2} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \right\} \|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S} \tag{10.67}$$

for all $\mathbf{w}_S, \underline{\mathbf{w}}_S \in \mathrm{W}$.

Proof. Given $\mathbf{w}_S, \underline{\mathbf{w}}_S \in \mathbf{L}^4(\Omega_S)$, we let $\mathbf{T}(\mathbf{w}_S) := \mathbf{u}_S$ and $\mathbf{T}(\underline{\mathbf{w}}_S) := \underline{\mathbf{u}}_S$, where $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ and $((\vec{\underline{\mathbf{t}}}, \vec{\underline{\boldsymbol{\sigma}}}, \vec{\underline{\mathbf{u}}}), \vec{\underline{\boldsymbol{p}}}) \in \mathbb{H} \times \mathbf{Q}$ are the corresponding unique solutions of (10.19), that is

$$[\mathbf{P}(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \boldsymbol{p}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\mathbf{w}_S; \mathbf{u}_S, \mathbf{r}_S) = [\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] \quad \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q} \tag{10.68}$$

and

$$[\mathbf{P}(\vec{\underline{\mathbf{t}}}, \vec{\underline{\boldsymbol{\sigma}}}, \vec{\underline{\mathbf{u}}}, \vec{\underline{\boldsymbol{p}}}), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\underline{\mathbf{w}}_S; \underline{\mathbf{u}}_S, \mathbf{r}_S) = [\mathrm{H}, (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] \quad \forall\, ((\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}), \vec{\boldsymbol{q}}) \in \mathbb{H} \times \mathbf{Q}\,. \tag{10.69}$$

Then, applying the inf-sup condition (10.62) to $(\vec{\boldsymbol{\zeta}}, \vec{\boldsymbol{\eta}}, \vec{\boldsymbol{z}}, \mathbf{s}) = (\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})$, we obtain

$$\frac{\alpha_{\mathbf{P}}}{2} \|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})\|_{\mathbb{H}\times\mathbf{Q}}$$
$$\leqslant \sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{[\mathbf{P}((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})), (\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})] + b(\mathbf{w}_S; \mathbf{u}_S - \underline{\mathbf{u}}_S, \mathbf{r}_S)}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}},$$

from which, employing (10.68) and (10.69), we arrive at

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\underline{\vec{\mathbf{t}}}, \underline{\vec{\boldsymbol{\sigma}}}, \underline{\vec{\mathbf{u}}}, \underline{\vec{\boldsymbol{p}}})\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{P}}} \sup_{\substack{((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\in\mathbb{H}\times\mathbf{Q}\\ ((\vec{\mathbf{r}},\vec{\boldsymbol{\tau}},\vec{\mathbf{v}}),\vec{\boldsymbol{q}})\neq\mathbf{0}}} \frac{b(\underline{\mathbf{w}}_S - \mathbf{w}_S; \underline{\mathbf{u}}_S, \mathbf{r}_S)}{\|(\vec{\mathbf{r}}, \vec{\boldsymbol{\tau}}, \vec{\mathbf{v}}, \vec{\boldsymbol{q}})\|_{\mathbb{H}\times\mathbf{Q}}}. \tag{10.70}$$

In turn, using the boundedness of b (cf. (10.32)) and the *a priori* estimate for

$$\|\underline{\mathbf{u}}_S\|_{0,4;\Omega_S} = \|\mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4;\Omega_S}$$

given by (10.64) (cf. Theorem 10.8), it follows from (10.70) that

$$\begin{aligned}\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\underline{\mathbf{w}}_S)\|_{0,4;\Omega_S} &= \|\mathbf{u}_S - \underline{\mathbf{u}}_S\|_{0,4;\Omega_S} \leqslant \frac{2\rho}{\alpha_{\mathbf{P}}} \|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S} \|\underline{\mathbf{u}}_S\|_{0,4;\Omega_S} \\ &\leqslant \frac{4\rho}{\alpha_{\mathbf{P}}^2} \left\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D}\right\} \|\mathbf{w}_S - \underline{\mathbf{w}}_S\|_{0,4;\Omega_S},\end{aligned}$$

which confirms the announced property on $\mathbf{T}$ (cf. (10.67)). □

The main result concerning the solvability of the fixed-point equation (10.20) is stated as follows.

Theorem 10.11. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} < \frac{\alpha_{\mathbf{P}}^2}{4\rho}.$$

Then, the operator $\mathbf{T}$ *has a unique fixed-point* $\mathbf{u}_S \in \mathrm{W}$. *Equivalently, problem* (10.17) *has a* *unique solution* $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ *with* $\mathbf{u}_S \in \mathrm{W}$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}})\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{P}}} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \Big\} . \tag{10.71}$$

Proof. Thanks to Lemma 10.9, we have that $\mathbf{T}$ maps W into itself. Then, bearing in mind the Lipschitz-continuity of $\mathbf{T} : \mathrm{W} \to \mathrm{W}$ (cf. (10.67)) and the assumption (10.65), a straightforward application of the classical Banach theorem yields the existence of a unique fixed-point $\mathbf{u}_S \in \mathrm{W}$ of this operator, and hence a unique solution to (10.14). Finally, it is easy to see that the *a priori* estimate is provided by (10.28) (cf. Theorem 10.1), which finishes the proof. □

CHAPTER 11

The discrete analysis

In order to approximate the solution of (10.9), we now introduce its associated Galerkin scheme, analyze its solvability by applying a discrete version of the fixed-point approach introduced for the continuous analysis, and derive the corresponding *a priori* error estimates.

11.1 The Galerkin scheme

We first consider a set of arbitrary discrete subspaces, namely

$$\begin{gathered}\mathrm{L}^2_h(\Omega_*) \subset \mathrm{L}^2(\Omega_*) \quad * \in \{S, D\}\,, \quad \mathbf{H}_h(\Omega_D) \subset \mathbf{H}(\mathrm{div};\Omega_D)\,, \quad \mathbf{H}_h(\Omega_S) \subset \mathbf{H}(\mathrm{div}_{4/3};\Omega_S)\,, \\ \mathbf{L}^4_h(\Omega_S) \subset \mathbf{L}^4(\Omega_S)\,, \qquad \Lambda^S_h(\Sigma) \subset \mathrm{H}^{1/2}_{00}(\Sigma)\,, \quad \text{and} \quad \Lambda^D_h(\Sigma) \subset \mathrm{H}^{1/2}(\Sigma)\,,\end{gathered} \tag{11.1}$$

so that, denoting by $\boldsymbol{\tau}_{S,i}$ the i-th row of a tensor $\boldsymbol{\tau}_S$, we set

$$\begin{aligned} &\mathbb{L}^2_{\mathrm{tr},h}(\Omega_S) := [\mathrm{L}^2_h(\Omega_S)]^{n\times n} \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)\,, \qquad \mathbf{H}_{h,0}(\Omega_D) := \mathbf{H}_h(\Omega_D) \cap \mathbf{H}_0(\mathrm{div};\Omega_D)\,, \\ &\mathbb{H}_h(\Omega_S) := \Big\{ \boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}_{4/3};\Omega_S) : \quad \boldsymbol{\tau}_{S,i} \in \mathbf{H}_h(\Omega_S) \quad \forall\, i \Big\}\,, \quad \boldsymbol{\Lambda}^S_h(\Sigma) := [\Lambda^S_h(\Sigma)]^n\,, \\ &\mathbb{H}_{h,0}(\Omega_S) := \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)\,, \quad \text{and} \quad \mathrm{L}^2_{h,0}(\Omega_D) := \mathrm{L}^2_h(\Omega_D) \cap \mathrm{L}^2_0(\Omega_D)\,. \end{aligned} \tag{11.2}$$

Then, defining the global spaces, unknowns, and test functions as follows

$$\begin{aligned} &\mathbf{X}_h := \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S) \times \mathbf{H}_{h,0}(\Omega_D)\,, \quad \mathbf{Y}_h := \mathbb{H}_{h,0}(\Omega_S) \times \Lambda^D_h(\Sigma)\,, \quad \mathbf{Z}_h := \mathbf{L}^4_h(\Omega_S) \times \boldsymbol{\Lambda}^S_h(\Sigma)\,, \\ &\mathbb{H}_h := \mathbf{X}_h \times \mathbf{Y}_h \times \mathbf{Z}_h\,, \qquad \mathbf{Q}_h := \mathrm{L}^2_{h,0}(\Omega_D) \times \mathrm{R}\,, \\ &\quad \vec{\mathbf{t}}_h := (\mathbf{t}_{S,h}, \mathbf{u}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{S,h}, \lambda_h) \in \mathbf{Y}_h\,, \quad \vec{\mathbf{u}}_h := (\mathbf{u}_{S,h}, \boldsymbol{\varphi}_h) \in \mathbf{Z}_h\,, \\ &\quad \vec{\mathbf{r}}_h := (\mathbf{r}_{S,h}, \mathbf{v}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h}, \xi_h) \in \mathbf{Y}_h\,, \quad \vec{\mathbf{v}}_h := (\mathbf{v}_{S,h}, \boldsymbol{\psi}_h) \in \mathbf{Z}_h\,, \\ &\quad \vec{\boldsymbol{\zeta}}_h := (\boldsymbol{\zeta}_{S,h}, \boldsymbol{z}_{D,h}) \in \mathbf{X}_h\,, \quad \vec{\boldsymbol{\eta}}_h := (\boldsymbol{\eta}_{S,h}, \vartheta_h) \in \mathbf{Y}_h\,, \quad \vec{\boldsymbol{z}}_h := (\boldsymbol{z}_{S,h}, \boldsymbol{\phi}_h) \in \mathbf{Z}_h\,, \\ &\quad \vec{\boldsymbol{p}}_h := (p_{D,h}, l_h) \in \mathbf{Q}_h\,, \quad \vec{\boldsymbol{q}}_h := (q_{D,h}, j) \in \mathbf{Q}_h\,, \quad \vec{\mathbf{s}}_h := (s_{D,h}, k) \in \mathbf{Q}_h\,, \end{aligned} \tag{11.3}$$

the Galerkin scheme associated with (10.9) reads: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{llllll} [a(\vec{\mathbf{t}}_h), \vec{\mathbf{r}}_h] & +[b_1(\vec{\mathbf{r}}_h), \vec{\boldsymbol{\sigma}}_h] & & -\displaystyle\int_{\Omega_D} p_{D,h}\,\mathrm{div}(\mathbf{v}_{D,h}) & -b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) & = 0 \\ [b_2(\vec{\mathbf{t}}_h), \vec{\boldsymbol{\tau}}_h] & & +[\mathbf{B}(\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h), \vec{\mathbf{u}}_h] & & & = \langle \boldsymbol{\tau}_{S,h}\mathbf{n}, g\rangle_{\Gamma_S} \\ & [\mathbf{B}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h), \vec{\mathbf{v}}_h] & -[\mathbf{C}(\vec{\mathbf{v}}_h), \vec{\mathbf{u}}_h] & +l\langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1\rangle_\Sigma & & = -\displaystyle\int_{\Omega_S} \mathbf{f}_S \cdot \mathbf{v}_{S,h} \\ & & +j\langle \boldsymbol{\varphi}_h \cdot \mathbf{n}, 1\rangle_\Sigma & & & = j\langle \mathbf{g}\cdot\mathbf{n}, 1\rangle_{\Gamma_S} \\ -\displaystyle\int_{\Omega_D} q_{D,h}\,\mathrm{div}(\mathbf{u}_{D,h}) & & & & & = -\displaystyle\int_{\Omega_D} f_D\, q_{D,h} \end{array} \tag{11.4}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$. Similarly, the ones associated with (10.14) and (10.17), which are certainly equivalent to (11.4), become, respectively: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \boldsymbol{p}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$\begin{array}{lll} [\widetilde{\mathbf{A}}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)] \;+\; [\widetilde{\mathbf{B}}(\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{p}}_h] \;+\; b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) & = & [\mathrm{G}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h)] \\ \qquad\qquad [\widetilde{\mathbf{B}}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{q}}_h] & = & [\mathrm{F}, \vec{\boldsymbol{q}}_h] \end{array}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ and: Find $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ such that

$$[\mathbf{P}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)] + b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) = [\mathrm{H}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)]\,, \tag{11.5}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$.

In what follows, we adopt the discrete version of the fixed-point strategy employed in Chapter 10 (at the end of Section 10.2) to study the solvability of (11.5). For this purpose, we now let $\mathbf{T}_h : \mathbf{L}^4_h(\Omega_S) \to \mathbf{L}^4_h(\Omega_S)$ be the operator defined by

$$\mathbf{T}_h(\mathbf{w}_{S,h}) \;:=\; \mathbf{u}_{S,h} \quad \forall\, \mathbf{w}_{S,h} \in \mathbf{L}^4_h(\Omega_S)\,, \tag{11.6}$$

where $\mathbf{u}_{S,h}$ is the first component of $\vec{\mathbf{u}}_h \in \mathbf{Z}_h$, which in turn is the third component of the unique solution $(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h)$ (to be proved later on) of the linearized problem arising from (11.5) after replacing $b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h})$ by $b(\mathbf{w}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h})$, namely:

$$[\mathbf{P}(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h), (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)] + b(\mathbf{w}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) = [\mathrm{H}, (\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h, \vec{\boldsymbol{q}}_h)]\,, \tag{11.7}$$

for all $((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$. Thus, we realize that solving (11.5) is equivalent to finding a fixed-point of $\mathbf{T}_h$, that is $\mathbf{u}_{S,h} \in \mathbf{L}^4_h(\Omega_S)$ such that

$$\mathbf{T}_h(\mathbf{u}_{S,h}) \;=\; \mathbf{u}_{S,h}\,. \tag{11.8}$$

11.2 Solvability analysis

Similarly to Section 10.3, in what follows we address the solvability of (11.5) by means of the corresponding analysis of (11.8).

11.2.1 Preliminaries

In addition to the finite dimensional versions of the Babuška-Brezzi theory in Banach spaces (cf. Theorem 10.1) and the Banach-Nečas-Babuška theorem, here we will also need the discrete version of Theorem 10.2, which is stated next.

Theorem 11.1. *Let H and Q be reflexive Banach spaces, and let $a : H \times H \to \mathrm{R}$, $b : H \times Q \to \mathrm{R}$ and $c : Q \times Q \to \mathrm{R}$ be given bounded bilinear forms. In addition, let $\{H_h\}_{h>0}$ and $\{Q_h\}_{h>0}$ be families of finite dimensional subspaces of H and Q, respectively, and let V_h be the kernel of $b|_{H_h \times Q_h}$ that is*

$$\mathrm{V}_h \;:=\; \Big\{ \tau_h \in H_h : \quad b(\tau_h, v_h) = 0 \quad \forall\, v_h \in Q_h \Big\}\,.$$

Assume that

i) *a and c are positive semi-definite, and that c is symmetric,*

ii) *there exists a constant $\alpha_{\mathrm{d}} > 0$ such that*

$$\sup_{\substack{\tau_h \in V_h \\ \tau_h \neq \mathbf{0}}} \frac{a(\vartheta_h, \tau_h)}{\|\tau_h\|_H} \;\geqslant\; \alpha_{\mathrm{d}}\, \|\vartheta_h\|_H \quad \forall\, \vartheta_h \in V_h,$$

iii) *and there exists a constant $\beta_{\mathrm{d}} > 0$ such that*

$$\sup_{\substack{\tau_h \in H_h \\ \tau_h \neq \mathbf{0}}} \frac{b(\tau_h, v_h)}{\|\tau_h\|_H} \;\geqslant\; \beta_{\mathrm{d}}\, \|v_h\|_Q \quad \forall\, v_h \in Q_h\,.$$

Then, for each pair $(f, g) \in H' \times Q'$ there exists a unique $(\sigma_h, u_h) \in H_h \times Q_h$ such that

$$\begin{array}{rcrcll} a(\sigma_h, \tau_h) & + & b(\tau_h, u_h) & = & f(\tau_h) & \quad \forall\, \tau_h \in H_h\,, \\ b(\sigma_h, v_h) & - & c(u_h, v_h) & = & g(v_h) & \quad \forall\, v_h \in Q_h\,. \end{array} \tag{11.9}$$

Moreover, there exists a constant $\widetilde{C}_{\mathrm{d}} > 0$, depending only on $\|a\|, \|c\|, \alpha_{\mathrm{d}}$, and β_{d}, such that

$$\|(\sigma_h, u_h)\|_{H \times Q} \leqslant \widetilde{C}_{\mathrm{d}} \left\{ \|f\|_{H'} + \|g\|_{Q'} \right\}\,.$$

We stress here that the discrete analogue of (10.26) is not required for Theorem 11.1. Indeed, since $H_h \times Q_h$ is the space to which both the unknowns and test functions of (11.9) belong, the corresponding finite dimensional version of the Banach–Nečas–Babuška Theorem (cf. [27, Theorem 2.22]) only requires the discrete analogue of (10.29), for which the already described hypotheses of Theorem 11.1 suffice.

11.2.2 Well-definedness of the operator $\mathbf{T}_h$

We begin by providing the preliminary results that are necessary to show that (11.7) is uniquely solvable. Once this is established, we address later on the well-posedness of (11.8), and consequently of (11.5). Indeed, following a similar procedure to that of Section 10.3.2, we first note that the kernel $\widetilde{\mathrm{V}}_h$ of $\widetilde{\mathbf{B}}|_{\mathbb{H}_h\times \mathrm{Q}_h}$ reduces to

$$\widetilde{\mathrm{V}}_h := \widetilde{\mathbf{X}}_h \times \mathbf{Y}_h \times \widetilde{\mathbf{Z}}_h\,,$$

where

$$\widetilde{\mathbf{X}}_h \,:=\, \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)\times \widetilde{\mathbf{H}}_{h,0}(\Omega_D) \quad \text{and} \quad \widetilde{\mathbf{Z}}_h \,:=\, \mathbf{L}^4_h(\Omega_S)\times \widetilde{\mathbf{\Lambda}}^S_h(\Sigma)\,,$$

with

$$\begin{aligned}\widetilde{\mathbf{H}}_{h,0}(\Omega_D) \,&:=\, \left\{\mathbf{v}_D\in \mathbf{H}_{h,0}(\Omega_D): \quad \int_{\Omega_D} q_D\,\mathrm{div}(\mathbf{v}_{D,h}) = 0 \quad \forall\, q_D\in \mathrm{L}^2_{h,0}(\Omega_D)\right\}\,, \quad \text{and}\\ \widetilde{\mathbf{\Lambda}}^S_h(\Sigma) \,&:=\, \left\{\boldsymbol{\psi}_h\in \mathbf{\Lambda}^S_h(\Sigma): \quad \langle \boldsymbol{\psi}_h\cdot\mathbf{n},1\rangle_\Sigma = 0\right\}.\end{aligned} \tag{11.10}$$

Then, the kernel V_h of $\mathbf{B}|_{\widetilde{\mathrm{V}}_h}$ reduces to

$$\mathrm{V}_h \,=\, \widetilde{\mathbf{X}}_h \times \overline{\mathbf{Y}}_h\,,$$

where

$$\begin{aligned}\overline{\mathbf{Y}}_h \,:=\, \Bigg\{\vec{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_{S,h},\xi_h)\in \mathbf{Y}_h: \quad &\int_{\Omega_S} \mathbf{v}_{S,h}\cdot \mathbf{div}(\boldsymbol{\tau}_{S,h}) \,=\, 0 \quad \text{and}\\ \langle \boldsymbol{\psi}_h\cdot\mathbf{n},\xi_h\rangle_\Sigma \,=\, -\langle \boldsymbol{\tau}_{S,h}\mathbf{n},\boldsymbol{\psi}_h\rangle_\Sigma \quad &\forall\,\vec{\mathbf{v}}_{S,h} := (\mathbf{v}_{S,h},\boldsymbol{\psi}_h)\in \mathbf{Z}_h\Bigg\}.\end{aligned}$$

At this point, we notice that $\overline{\mathbf{Y}}_h \subseteq \widetilde{\mathbb{H}}_{h,0}(\Omega_S) \times \Lambda_h^D(\Sigma)$, where

$$\widetilde{\mathbb{H}}_{h,0}(\Omega_S) := \left\{ \boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) : \quad \int_{\Omega_S} \mathbf{v}_{S,h} \cdot \mathbf{div}(\boldsymbol{\tau}_{S,h}) = 0 \quad \forall\, \mathbf{v}_{S,h} \in \mathbf{L}_h^4(\Omega_S) \right\}. \tag{11.11}$$

We now proceed similarly to [17], and introduce suitable hypotheses on the spaces defined in (11.3) to ensure the well-posedness of (11.7). We begin by noticing that, in order to have meaningful spaces $\mathbb{H}_{h,0}(\Omega_S)$ and $\mathrm{L}^2_{h,0}(\Omega_D)$, we need to be able to eliminate multiples of the identity matrix and constant polynomials from $\mathbb{H}_{h,0}(\Omega_S)$ and $\mathrm{L}^2_{h,0}(\Omega_D)$, respectively. This is certainly satisfied if we assume:

(**H.0**) $\mathrm{P}_0(\Omega_D) \subseteq \mathrm{L}^2_h(\Omega_D)$ and $\mathbb{I} \in \mathbb{H}_h(\Omega_S)$.

In addition, we consider the following further hypotheses

(**H.1**) $\mathrm{div}(\mathbf{H}_h(\Omega_D)) \subseteq \mathrm{L}^2_h(\Omega_D)$,

(**H.2**) $\mathbf{div}(\mathbb{H}_h(\Omega_S) \subseteq \mathbf{L}^4_h(\Omega_S)$,

(**H.3**) $\widetilde{\mathbb{H}}^{\mathrm{d}}_{h,0} := \left\{ \boldsymbol{\tau}^{\mathrm{d}}_{S,h} : \quad \boldsymbol{\tau}_{S,h} \in \widetilde{\mathbb{H}}_{h,0} \right\} \subseteq \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$,

(**H.4**) there holds the discrete analogue of (10.42), that is there exists a positive constant $\beta^{\mathrm{d}}_{1,\Sigma}$, independent of h, such that

$$\sup_{\substack{\mathbf{v}_{D,h} \in \widetilde{\mathbf{H}}_{h,0}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{D,h} \cdot \mathbf{n}, \xi_h \rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \geqslant \beta^{\mathrm{d}}_{1,\Sigma} \, \|\xi_h\|_{1/2,\Sigma} \quad \forall\, \xi \in \Lambda^D_h(\Sigma)\,, \tag{11.12}$$

(**H.5**) there holds the discrete analogue of (10.51), that is there exists a positive constant $\boldsymbol{\beta}^{\mathrm{d}}_S$, independent of h, such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_h(\Omega_S) \\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{S,h}\mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma + \displaystyle\int_{\Omega_S} \mathbf{v}_{S,h} \cdot \mathbf{div}(\boldsymbol{\tau}_{S,h})}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \geqslant \boldsymbol{\beta}^{\mathrm{d}}_S \left\{ \|\mathbf{v}_{S,h}\|_{0,4;\Omega_S} + \|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \right\}, \tag{11.13}$$

for all $\vec{\mathbf{v}}_{S,h} := (\mathbf{v}_{S,h}, \boldsymbol{\psi}_h) \in \mathbf{L}^4_h(\Omega_S) \times \boldsymbol{\Lambda}^S_h(\Sigma)$, (**H.6**) there hold the discrete analogue of (10.58) and a sufficient condition for the discrete analogue of (10.59), that is there exist a positive constant $\widetilde{\beta}_D^{\mathrm{d}}$, independent of h, and $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$, such that

$$\sup_{\substack{\mathbf{v}_{D,h} \in \mathbf{H}_{h,0}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_{D,h} \operatorname{div}(\mathbf{v}_{D,h})}{\|\mathbf{v}_{D,h}\|_{\operatorname{div};\Omega_D}} \geqslant \widetilde{\beta}_D^{\mathrm{d}} \, \|q_{D,h}\|_{0,\Omega_D} \quad \forall \, q_{D,h} \in \mathrm{L}^2_{h,0}(\Omega_D) \,, \quad \text{and} \tag{11.14}$$

$$\boldsymbol{\psi}_0 \in \boldsymbol{\Lambda}_h^S(\Sigma) \quad \forall \, h, \qquad \langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1 \rangle_\Sigma \neq 0 \,. \tag{11.15}$$

We highlight here that as a consequence of (**H.0**) we can employ the discrete version of the decomposition $\mathbb{H}(\mathbf{div}_{4/3};\Omega_S) = \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S) \oplus \mathrm{R}\,\mathbb{I}$, namely $\mathbb{H}_h(\Omega_S) = \mathbb{H}_{h,0}(\Omega_S) \oplus \mathrm{R}\,\mathbb{I}$, thanks to which $\mathbb{H}_{h,0}(\Omega_S)$ can be used as the subspace where the unknown $\boldsymbol{\sigma}_{S,h}$ is sought. However, for the computational implementation of the Galerkin scheme (11.7), which will be addressed later on in Chapter 13, we will utilize a real Lagrange multiplier to impose the mean value condition on the trace of the unknown tensor lying in $\mathbb{H}_{0,h}(\Omega_S)$. In turn, it follows from (**H.1**) and (11.10) that $\widetilde{\mathbf{H}}_{h,0}(\Omega_D)$ reduces to

$$\widetilde{\mathbf{H}}_{h,0}(\Omega_D) \, := \, \Big\{ \mathbf{v}_{D,h} \in \mathbf{H}_{h,0}(\Omega_D) : \quad \operatorname{div}(\mathbf{v}_{D,h}) \in \mathrm{P}_0(\Omega_D) \Big\} \,.$$

Similarly, thanks to (**H.2**) and (11.11), $\widetilde{\mathbb{H}}_{h,0}(\Omega_S)$ becomes

$$\widetilde{\mathbb{H}}_{h,0}(\Omega_S) \, := \, \Big\{ \boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_{S,h}) \, = \, \mathbf{0} \Big\} \,, \tag{11.16}$$

which yields the discrete analogue of (10.43) with constant $\beta_{1,S}^{\mathrm{d}}$. In fact, given $\boldsymbol{\tau}_{S,h} \in \widetilde{\mathbb{H}}_{h,0}(\Omega_S)$ such that $\boldsymbol{\tau}_{S,h}^{\mathrm{d}} \neq \mathbf{0}$, we realize, thanks to (**H.3**), that $\mathbf{r}_{S,h} := -\boldsymbol{\tau}_{S,h}^{\mathrm{d}} \in \mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$, and hence, along with the inf-sup condition from (**H.4**), we deduce the discrete version of (10.40) holds, that is, the existence of positive constants β_i^{d}, $i \in \{1, 2\}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{r}}_h \in \widetilde{\mathbf{X}}_h \\ \vec{\mathbf{r}}_h \neq \mathbf{0}}} \frac{[b_i(\vec{\mathbf{r}}_h), \vec{\boldsymbol{\tau}}_h]}{\|\vec{\mathbf{r}}_h\|_{\mathbf{X}}} \geqslant \beta_i^{\mathrm{d}} \, \|\vec{\boldsymbol{\tau}}_h\|_{\mathbf{Y}} \quad \forall \, \vec{\boldsymbol{\tau}}_h \in \overline{\mathbf{Y}}_h \,.$$

Furthermore, we remark that, similarly to the analyses in the proofs of Lemmas 10.6 and 10.7, (11.13) (cf. (**H.5**)) is a sufficient condition for the discrete version of (10.49), whereas (11.14) and (11.15) (cf. (**H.6**)) are equivalent to the discrete version of (10.57). We denote the constants involved in these discrete inf-sup conditions by $\boldsymbol{\beta}_{\mathrm{d}}$ and $\widetilde{\beta}_{\mathrm{d}}$, respectively.

Thus, having $\widetilde{\mathbf{A}}$ and $\widetilde{\mathbf{B}}$ satisfied for the present discrete scheme the hypotheses of Theorem 10.1 with constants $\widetilde{\alpha}_{\mathrm{d}}$ and $\widetilde{\beta}_{\mathrm{d}}$, we conclude, similarly to the continuous case, the existence of a positive constant $\alpha_{\mathbf{P},\mathrm{d}}$, depending on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, and $\|\widetilde{\mathbf{A}}\|$, and hence independent of h, such that

$$\sup_{\substack{((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\ ((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h),(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)]}{\|((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\|_{\mathbb{H}\times\mathbf{Q}}} \geqslant \alpha_{\mathbf{P},\mathrm{d}}\,\|(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\,, \tag{11.17}$$

for all $((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h$, and thus, for each $\mathbf{w}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{0,4;\Omega_S}\leqslant \dfrac{\alpha_{\mathbf{P},\mathrm{d}}}{2\rho}$, there holds

$$\begin{gathered}\sup_{\substack{((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\\ ((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\neq\mathbf{0}}} \frac{[\mathbf{P}(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h),(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)] + b(\mathbf{w}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h})}{\|(\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h,\vec{\boldsymbol{q}}_h)\|_{\mathbb{H}\times\mathbf{Q}}}\\ \geqslant \frac{\alpha_{\mathbf{P},\mathrm{d}}}{2}\,\|(\vec{\boldsymbol{\zeta}}_h,\vec{\boldsymbol{\eta}}_h,\vec{\boldsymbol{z}}_h,\vec{\mathbf{s}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \qquad \forall\,((\vec{\mathbf{r}}_h,\vec{\boldsymbol{\tau}}_h,\vec{\mathbf{v}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}_h\times\mathbf{Q}_h\,.\end{gathered} \tag{11.18}$$

According to the above, we are now in a position to present the discrete analogues of Theorem 10.8, Lemma 10.9, and Theorem 10.11, whose proofs follow almost verbatim to those for the continuous case, and hence only some remarks are provided. We begin with the well-posedness of (11.7), which is the same as establishing that $\mathbf{T}_h$ is well-defined.

Lemma 11.2. *For each $\mathbf{w}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|\leqslant \dfrac{\alpha_{\mathbf{P},\mathrm{d}}}{2\rho}$, there exists a unique solution $((\vec{\mathbf{t}}_h,\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h),\vec{\boldsymbol{q}}_h)\in\mathbb{H}\times\mathbf{Q}$ to (11.7), and hence we can define $\mathbf{T}_h(\mathbf{w}_{S,h})=\mathbf{u}_{S,h}\in\mathbf{L}^4_h(\Omega_S)$. Moreover, there holds*

$$\begin{aligned}\|\mathbf{T}_h(\mathbf{w}_{S,h})\|_{0,4;\Omega_S} &= \|\mathbf{u}_{S,h}\|_{0,4;\Omega_S} \leqslant \|(\vec{\mathbf{t}}_h,\vec{\boldsymbol{\sigma}}_h,\vec{\mathbf{u}}_h,\vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}}\\ &\leqslant \frac{2}{\alpha_{\mathbf{P},\mathrm{d}}}\left\{\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D}\right\}.\end{aligned} \tag{11.19}$$

Proof. Given $\mathbf{w}_{S,h}$ as indicated, and bearing in mind (11.18), it suffices to apply the discrete version of the Banach–Nečas–Babuška Theorem (cf. [27, Theorem 2.22]) and its corresponding *a priori* error estimate. □

We continue with the discrete analogue of Lemma 10.9, that is the result ensuring that $\mathbf{T}_h$ maps a ball of $\mathbf{L}^4_h(\Omega_S)$ into itself.

Lemma 11.3. *Let* W_h *be the ball*

$$\mathrm{W}_h := \left\{ \mathbf{w}_{S,h} \in \mathbf{L}^4_h(\Omega_S) : \quad \|\mathbf{w}_{S,h}\|_{0,4;\Omega_S} \leqslant \frac{\alpha_{\mathbf{P},\mathrm{d}}}{2\rho} \right\},$$

and assume that

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \leqslant \frac{\alpha^2_{\mathbf{P},\mathrm{d}}}{4\rho}. \tag{11.20}$$

Then, there holds $\mathbf{T}_h(\mathrm{W}_h) \subseteq \mathrm{W}_h$.

Proof. It follows straightforwardly from (11.19) and (11.20). □

The discrete analogue of Theorem 10.11, that is the unique solvability of (11.8), and hence, equivalently that of (11.5), is stated next.

Theorem 11.4. *Assume that*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \leqslant \frac{\alpha^2_{\mathbf{P},\mathrm{d}}}{4\rho}.$$

Then, the operator $\mathbf{T}_h$ *has a unique fixed-point* $\mathbf{u}_{S,h} \in \mathrm{W}_h$. *Equivalently, problem* (11.5) *has a unique solution* $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h), \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}$ *with* $\mathbf{u}_{S,h} \in \mathrm{W}_h$. *Moreover, there holds*

$$\|(\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \frac{2}{\alpha_{\mathbf{P},\mathrm{d}}} \left\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} + \|\mathbf{f}_S\|_{0,4/3;\Omega_S} + \|f_D\|_{0,\Omega_D} \right\}. \tag{11.21}$$

Proof. Similarly to the proof of Theorem 10.11, it reduces to employ (10.32), (11.7), (11.18) and (11.19) to prove that $\mathbf{T}_h : \mathrm{W}_h \to \mathrm{W}_h$ is a contraction, and then apply the Banach fixed-point theorem. □

We end this section by providing sufficient conditions for (11.12) and the particular case arising from (11.13) when $\mathbf{v}_{S,h} = \mathbf{0}$, that is for the existence of positive constants $\beta^{\mathrm{d}}_{1,\Sigma}$ and $\boldsymbol{\beta}^{\mathrm{d}}_{S,2}$, such that

$$\sup_{\substack{\mathbf{v}_{D,h}\in\widetilde{\mathbf{H}}_{h,0}(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\langle \mathbf{v}_{D,h}\cdot\mathbf{n},\xi_h\rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathrm{div};\Omega_D}} \geqslant \beta^{\mathrm{d}}_{1,\Sigma}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\,\xi_h\in\Lambda^D_h(\Sigma)\,, \quad \text{and} \tag{11.22}$$

$$\sup_{\substack{\boldsymbol{\tau}_{S,h}\in\widetilde{\mathbb{H}}_h(\Omega_S)\\ \boldsymbol{\tau}_{S,h}\neq\mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{S,h}\mathbf{n},\boldsymbol{\psi}_h\rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \geqslant \boldsymbol{\beta}^{\mathrm{d}}_{S,2}\,\|\boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \qquad \forall\,\boldsymbol{\psi}_h\in\boldsymbol{\Lambda}^S_h(\Sigma)\,, \tag{11.23}$$

where $\widetilde{\mathbb{H}}_h(\Omega_S) := \Big\{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_h(\Omega_S) : \quad \mathbf{div}(\boldsymbol{\tau}_{S,h}) = \mathbf{0}\Big\}$. In this regard, we first notice that the above inequalities, which deal with how the normal components of elements of $\widetilde{\mathbf{H}}_{h,0}(\Omega_D)$ and $\widetilde{\mathbb{H}}_h(\Omega_S)$ are tested against $\Lambda^D_h(\Sigma)$ and $\boldsymbol{\Lambda}^S_h(\Sigma)$, respectively, are shown below to be related to the eventual existence of a stable discrete lifting of the normal traces on Σ. Indeed, in order to establish (11.22) and (11.23), it suffices to prove that for each $*\in\{D,S\}$ there exists a positive constant $\beta^{\mathrm{d}}_{*,\Sigma}$, such that

$$\sup_{\substack{\mathbf{v}_h\in\widetilde{\mathbf{H}}_h(\Omega_*)\\ \mathbf{v}_h\neq\mathbf{0}}} \frac{\langle \mathbf{v}_h\cdot\mathbf{n},\xi_h\rangle_\Sigma}{\|\mathbf{v}_h\|_{\mathrm{div};\Omega_*}} \geqslant \beta^{\mathrm{d}}_{*,\Sigma}\,\|\xi_h\|_{1/2,\Sigma} \qquad \forall\,\xi\in\Lambda^*_h(\Sigma)\,, \tag{11.24}$$

where

$$\widetilde{\mathbf{H}}_h(\Omega_D) := \Big\{\mathbf{v}_h\in\mathbf{H}_{h,0}(\Omega_D) : \quad \mathrm{div}(\mathbf{v}_h)\in\mathrm{P}_0(\Omega_D)\Big\}\,, \quad \text{and}$$

$$\widetilde{\mathbf{H}}_h(\Omega_S) := \Big\{\mathbf{v}_h\in\mathbf{H}_h(\Omega_S) : \quad \mathrm{div}(\mathbf{v}_h) = 0\Big\}\,.$$

Next, for each $*\in\{D,S\}$ we define

$$\Phi^*_h(\Sigma) := \Big\{\mathbf{v}_h\cdot\mathbf{n}|_\Sigma : \quad \mathbf{v}_h\in\widetilde{\mathbf{H}}_h(\Omega_*)\Big\}\,, \tag{11.25}$$

and assume that the linear operator $\mathbf{v}_h \to \mathbf{v}_h\cdot\mathbf{n}$ from $\widetilde{\mathbf{H}}_h(\Omega_*)$ to $\Phi^*_h(\Sigma)$ has a uniformly bounded right inverse, which means that there exists a linear operator $\mathcal{L}^*_h : \Phi^*_h(\Sigma)\to\widetilde{\mathbf{H}}_h(\Omega_*)$

and a constant $c_* > 0$, independent of h, such that

$$\begin{aligned} \|\mathcal{L}_h^*(\phi_h)\|_{\mathrm{div};\Omega_*} &\leqslant c_* \, \|\phi_h\|_{-1/2,\Sigma}\,, \quad \text{and} \\ \mathcal{L}_h^*(\phi_h)\cdot\mathbf{n} &= \phi_h \quad \text{on} \quad \Sigma \quad \forall\, \phi_h \in \Phi_h^*(\Sigma)\,. \end{aligned} \tag{11.26}$$

Such a uniformly bounded right inverse $\mathcal{L}_h^*$ of the normal trace will henceforth be referred to as a stable discrete lifting to Ω_*. Note that by [26], existence of $\mathcal{L}_h^*$ satisfying (11.26) is equivalent to the existence of a Scott–Zhang type linear and uniformly bounded operator $\pi_h^* : \mathbf{H}(\mathrm{div};\Omega_*) \to \widetilde{\mathbf{H}}_h(\Omega_*)$, such that

$$\pi_h^*(\mathbf{v}_h) = \mathbf{v}_h \quad \forall\, \mathbf{v}_h \in \widetilde{\mathbf{H}}_h(\Omega_*)\,, \quad \text{and} \quad \mathbf{v}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma \Longrightarrow (\pi_h^*(\mathbf{v}))\cdot\mathbf{n} = 0 \quad \text{on} \quad \Sigma\,.$$

The following lemma, taken from [37, Lemma 4.2], reduces (11.24) to the inherited interaction between the elements of $\Phi_h^*(\Sigma)$ and $\Lambda_h^*(\Sigma)$.

Lemma 11.5. *Assume that there exists a stable discrete lifting to Ω_*. Then (11.24) is equivalent to the existence of a positive constant β_*^{d}, independent of h, such that*

$$\sup_{\substack{\phi_h\in\Phi_h^*(\Sigma)\\ \phi_h\neq\mathbf{0}}} \frac{\langle \phi_h, \xi_h\rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geqslant \beta_*^{\mathrm{d}}\, \|\xi_h\|_{1/2,\Sigma} \qquad \forall\, \xi_h \in \Lambda_h^*(\Sigma)\,. \tag{11.27}$$

We have thus proved that the existence of stable discrete liftings to Ω_S and Ω_D together with the inf-sup condition (11.27) constitute sufficient conditions for (11.24) to hold. In this respect, we find it important to emphasize that (11.27) deals exclusively with spaces of functions defined on Σ.

11.3 *A priori* error analysis

In this section we consider finite element subspaces satisfying the assumptions specified in Section 11.2.2, and derive the Céa estimate for the Galerkin error

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} = \|\vec{\mathbf{t}} - \vec{\mathbf{t}}_h\|_{\mathbf{X}} + \|\vec{\boldsymbol{\sigma}} - \vec{\boldsymbol{\sigma}}_h\|_{\mathbf{Y}} + \|\vec{\mathbf{u}} - \vec{\mathbf{u}}_h\|_{\mathbf{Z}} + \|\vec{\boldsymbol{p}} - \vec{\boldsymbol{p}}_h\|_{\mathbf{Q}}\,,$$

where $\underline{\mathbf{t}} := (\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) \in \mathbb{H}\times\mathbf{Q}$ and $\underline{\mathbf{t}}_h := (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$ are the unique solutions of (10.17) and (11.5) respectively, with $\mathbf{u}_S \in \mathrm{W}$ and $\mathbf{u}_{S,h} \in \mathrm{W}_h$. In what follows, given a subspace Z_h of an arbitrary Banach space $\big(Z, \|\cdot\|_Z\big)$, we set

$$\operatorname{dist}\big(z, Z_h\big) := \inf_{z_h\in Z_h} \|z - z_h\|_Z \qquad \forall\, z \in Z\,.$$

We begin by observing from (10.16) that for each $\underline{\mathbf{r}}_h := ((\vec{\mathbf{r}}_h, \vec{\boldsymbol{\tau}}_h, \vec{\mathbf{v}}_h), \vec{\boldsymbol{q}}_h) \in \mathbb{H}_h\times\mathbf{Q}_h$ there holds

$$[\mathbf{P}(\underline{\mathbf{t}}), \underline{\mathbf{r}}_h] + b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_{S,h}) = [\mathrm{H}, \underline{\mathbf{r}}_h]\,,$$

which combined with (11.5), yields for each $\underline{\mathbf{r}}_h \in \mathbb{H}_h\times\mathbf{Q}_h$

$$[\mathbf{P}(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h), \underline{\mathbf{r}}_h] = b(\mathbf{u}_{S,h}; \mathbf{u}_{S,h}, \mathbf{r}_{S,h}) - b(\mathbf{u}_S; \mathbf{u}_S, \mathbf{r}_{S,h})\,. \tag{11.28}$$

Now, the triangle inequality gives for each $\underline{\boldsymbol{\zeta}}_h \in \mathbb{H}_h\times\mathbf{Q}_h$

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \leqslant \|\underline{\mathbf{t}} - \underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} + \|\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}}\,, \tag{11.29}$$

and then, applying (11.17) to $\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h$, subtracting and adding $\underline{\mathbf{t}}$ in the first component of $\mathbf{P}$, using the boundedness of $\mathbf{P}$ with constant $\|\mathbf{P}\|$, and employing the identity (11.28), we find that

$$\begin{aligned} \alpha_{\mathbf{P},\mathrm{d}} \, \|\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \, &\leqslant \, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h \\ \underline{\mathbf{r}}_h\neq\mathbf{0}}} \frac{[\mathbf{P}(\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h), \underline{\mathbf{r}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}} \\ &\leqslant \, \|\mathbf{P}\| \, \|\underline{\mathbf{t}} - \underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} \, + \, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h \\ \underline{\mathbf{r}}_h\neq\mathbf{0}}} \frac{[\mathbf{P}(\underline{\mathbf{t}} - \underline{\mathbf{t}}_h), \underline{\mathbf{r}}_h]}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}} \\ &\leqslant \, \|\mathbf{P}\| \, \|\underline{\mathbf{t}} - \underline{\boldsymbol{\zeta}}_h\|_{\mathbb{H}\times\mathbf{Q}} \, + \, \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h \\ \underline{\mathbf{r}}_h\neq\mathbf{0}}} \frac{b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h})}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}} \times \mathbf{Q}} \, . \end{aligned} \tag{11.30}$$

In this way, replacing the bound for $\|\underline{\boldsymbol{\zeta}}_h - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}}$ that arises from (11.30) back into (11.29), and taking infimum with respect to $\underline{\mathbf{r}}_h \in \mathbb{H}_h \times \mathbf{Q}_h$ we deduce that

$$\begin{aligned} \|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \, &\leqslant \, \left(1 + \frac{\|\mathbf{P}\|}{\alpha_{\mathbf{P},\mathrm{d}}}\right) \mathrm{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h) \\ &+ \frac{1}{\alpha_{\mathbf{P},\mathrm{d}}} \sup_{\substack{\underline{\mathbf{r}}_h\in\mathbb{H}_h\times\mathbf{Q}_h \\ \underline{\mathbf{r}}_h\neq\mathbf{0}}} \frac{b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h})}{\|\underline{\mathbf{r}}_h\|_{\mathbb{H}\times\mathbf{Q}}} \, , \end{aligned} \tag{11.31}$$

which basically constitutes the Strang-type estimate for the joint setting formed by (10.17) and (11.5). Next, in order to estimate the consistency term given by the supremum in (11.31), we subtract and add $\mathbf{u}_S$ in the second component of $b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h})$, and then invoke the boundedness property of b (10.32), and the *a priori* estimates (10.71) and (11.21) for $\|\mathbf{u}_S\|_{0,4;\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{0,4;\Omega_S}$, respectively, thanks to all of which we obtain

$$\begin{aligned} b(\mathbf{u}_{S,h};\mathbf{u}_{S,h},\mathbf{r}_{S,h}) - b(\mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h}) \, &= \, b(\mathbf{u}_{S,h};\mathbf{u}_{S,h} - \mathbf{u}_S,\mathbf{r}_{S,h}) \, + \, b(\mathbf{u}_{S,h} - \mathbf{u}_S;\mathbf{u}_S,\mathbf{r}_{S,h}) \\ \leqslant \, \frac{4\rho}{\overline{\alpha}_{\mathbf{P}}} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \, + \, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \, + \, \|f_D\|_{0,\Omega_D} \Big\} & \, \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S} \, \|\mathbf{r}_{S,h}\|_{0,\Omega_S} \, , \end{aligned} \tag{11.32}$$

where $\overline{\alpha}_{\mathbf{P}} \, := \, \min\big\{\alpha_{\mathbf{P}}, \alpha_{\mathbf{P},\mathrm{d}}\big\}$. Hence, replacing (11.31) in (11.32), we conclude that

$$\begin{aligned} \|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \, &\leqslant \, \left(1 + \frac{\|\mathbf{P}\|}{\alpha_{\mathbf{P},\mathrm{d}}}\right) \mathrm{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h) \\ &+ \frac{4\rho}{\overline{\alpha}_{\mathbf{P}}^2} \Big\{ \|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \, + \, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \, + \, \|f_D\|_{0,\Omega_D} \Big\} \, \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S} \, . \end{aligned} \tag{11.33}$$

We are then in position to state the following result.

Theorem 11.6. *Assume that for some $\delta \in (0,1)$ there holds*

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D} \,\leqslant\, \frac{\delta\,\alpha_{\mathbf{P},\mathrm{d}}^2}{4\rho}\,. \tag{11.34}$$

Then, there exists a positive constant C_{d}, depending only on $\|\mathbf{P}\|$, $\alpha_{\mathbf{P},\mathrm{d}}$, and δ, and hence independent of h, such that

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} \,\leqslant\, C_{\mathrm{d}}\,\mathrm{dist}\big(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h\big)\,. \tag{11.35}$$

Proof. It suffices to use (11.34) in (11.33), which yields (11.35) with $C_{\mathrm{d}} := (1-\delta)^{-1}\big(1 + \|\mathbf{P}\|/\alpha_{\mathbf{P},\mathrm{d}}\big)$. □

In particular, taking $\delta = 1/2$, we get $C_{\mathrm{d}} := 2\big(1 + \|\mathbf{P}\|/\alpha_{\mathbf{P},\mathrm{d}}\big)$ in the proof of Lemma 11.6, and (11.34) becomes

$$\|\widetilde{\mathbf{g}}\|_{1/2,00;\Gamma_S} \,+\, \|\mathbf{f}_S\|_{0,4/3;\Omega_S} \,+\, \|f_D\|_{0,\Omega_D} \,\leqslant\, \frac{\alpha_{\mathbf{P},\mathrm{d}}^2}{8\rho}\,. \tag{11.36}$$

We end this section by remarking that (9.4) and (10.7) suggest the following postprocessed approximation for the pressure p_S

$$p_{S,h} \,:=\, -\,\frac{1}{n}\,\mathrm{tr}\big(\boldsymbol{\sigma}_{S,h} \,+\, (\mathbf{u}_{S,h}\otimes\mathbf{u}_{S,h})\big) \,-\, l_h \quad \text{in} \quad \Omega_S\,, \tag{11.37}$$

where

$$l_h \,:=\, -\frac{1}{n\,|\Omega_S|}\int_{\Omega_S}\mathrm{tr}(\boldsymbol{\sigma}_{S,h})\,.$$

Then, applying the Cauchy–Schwarz inequality, performing some algebraic manipulations, and employing the *a priori* bounds for $\|\mathbf{u}_S\|_{0,4;\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{0,4;\Omega_S}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$\|p \,-\, p_h\|_{0,\Omega_S} \,\leqslant\, C\,\Big\{\|\boldsymbol{\sigma}_S \,-\, \boldsymbol{\sigma}_{S,h}\|_{0,\Omega} \,+\, \|\mathbf{u}_S \,-\, \mathbf{u}_{S,h}\|_{0,4;\Omega_S}\Big\}\,. \tag{11.38}$$

Thus, combining (11.35) and (11.38), we conclude the existence of a positive constant $\widetilde{C}_{\mathrm{d}}$, independent of h, such that

$$\|\underline{\mathbf{t}} - \underline{\mathbf{t}}_h\|_{\mathbb{H}\times\mathbf{Q}} + \|p - p_h\|_{0,\Omega_S} \leqslant \widetilde{C}_{\mathrm{d}}\,\mathrm{dist}(\underline{\mathbf{t}}, \mathbb{H}_h \times \mathbf{Q}_h)\,. \tag{11.39}$$

CHAPTER 12

Specific finite element subspaces

In what follows we proceed similarly to [37] (see also [15]) and specify discrete spaces satisfying the hypotheses (**H.0**) up to (**H.6**) in 2D and 3D, thus ensuring the well-posedness of the Galerkin scheme (11.5). Their approximation properties and associated rates of convergence are also established.

12.1 Preliminaries

We begin by letting $\mathcal{T}_h^S$ and $\mathcal{T}_h^D$ be respective triangulations of the domains Ω_S and Ω_D, which are formed by shape-regular triangles (in R^2) or tetrahedra (in R^3) of diameter h_T, and assume that they match in Σ so that $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. We also let Σ_h be the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$). Then, given $T \in \mathcal{T}_h^S \cup \mathcal{T}_h{}^D$, we let $\mathrm{P}_0(T)$ be the space of polynomials of degree $= 0$ defined on T, whose vector and tensor versions are denoted by $\mathbf{P}_0(T) := [\mathrm{P}_0(T)]^n$ and $\mathbb{P}_0(T) := [\mathrm{P}_0(T)]^{n\times n}$, respectively. Next, we define the corresponding local Raviart-Thomas spaces of order 0 as

$$\mathbf{RT}_0(T) \,:=\, \mathbf{P}_0(T) \oplus \mathrm{P}_0(T)\,\mathbf{x}$$

and its associated tensor counterpart $\mathbb{RT}_0(T)$, where $\mathbf{x}$ is a generic vector in $\mathbf{R} \,:=\, \mathrm{R}^n$. In turn, given $* \in \{S, D\}$, we let $\mathrm{P}_0(\mathcal{T}_h^*)$, $\mathbf{P}_0(\mathcal{T}_h^*)$ and $\mathbf{RT}_0(\mathcal{T}_h{}^*)$ be the global versions of $\mathrm{P}_0(T)$, $\mathbf{P}_0(T)$, $\mathbb{P}_0(T)$, $\mathbf{RT}_0(T)$ and $\mathbb{RT}_0(T)$, respectively, that is

$$\begin{aligned}
\mathrm{P}_0(\mathcal{T}_h{}^*) \,&:=\, \left\{\mathbf{v}_h \in \mathrm{L}^2(\Omega_*) : \quad \mathbf{v}_h|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\right\}\,,\\
\mathbf{P}_0(\mathcal{T}_h{}^*) \,&:=\, \left\{\boldsymbol{\tau}_h \in \mathbf{L}^2(\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbf{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\right\}\,,\\
\mathbb{P}_0(\mathcal{T}_h{}^*) \,&:=\, \left\{\boldsymbol{\tau}_h \in \mathbb{L}^2(\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbb{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\right\}\,,\\
\mathbf{RT}_0(\mathcal{T}_h{}^*) \,&:=\, \{\boldsymbol{\tau}_h \in \mathbf{H}(\mathbf{div};\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\}\,,\\
\mathbb{RT}_0(\mathcal{T}_h{}^*) \,&:=\, \{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega_*) : \quad \boldsymbol{\tau}_h|_T \in \mathbb{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^*\}\,.
\end{aligned}$$

Then, we introduce the corresponding discrete subspaces in (11.1) as

$$\mathrm{L}^2_h(\Omega_*) \,:=\, \mathrm{P}_0(\mathcal{T}_h{}^*)\,, \quad \mathbf{H}_h(\Omega_*) \,:=\, \mathbf{RT}_0(\mathcal{T}_h{}^*)\,, \quad \text{and} \quad \mathbf{L}^4_h(\Omega_S) \,:=\, \mathbf{L}^4(\Omega_S) \cap \mathbf{P}_0(\mathcal{T}_h^S)\,, \tag{12.1}$$

so that the associated global spaces $\mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$, $\mathbf{H}_{h,0}(\Omega_D)$, $\mathbb{H}_h(\Omega_S)$, $\mathbb{H}_{h,0}(\Omega_S)$, and $\mathrm{L}^2_{h,0}(\Omega_D)$, are defined according to (11.2). The interface spaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ will be specified later on by separating the 2D and 3D cases.

Next, for the verification of the hypotheses introduced in Section 11.2.2, we first realize that (**H.0**), (**H.1**), and (**H.2**) follow straightforwardly from the definitions in (12.1). In turn, regarding (**H.3**), we now recall that the divergence free tensors of $\mathbb{RT}_0(\mathcal{T}_h)$ are contained in $\mathbb{P}_0(\mathcal{T}_h)$ (cf. [30, Lemma 3.6]), so that, invoking (11.16), we deduce that $\widetilde{\mathbb{H}}_{h,0}(\Omega_S) \subseteq \mathbb{P}_0(\mathcal{T}_h)$. In this way, noting that certainly $\mathrm{tr}(\boldsymbol{\tau}_h^{\mathrm{d}}) = 0$ for all $\boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_{h,0}(\Omega_S)$, we find that $\widetilde{\mathbb{H}}^{\mathrm{d}}_{h,0}(\Omega_S) \subseteq \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_0(\mathcal{T}_h) = \mathbb{L}^2_{\mathrm{tr},h}(\Omega)$, thus confirming the occurrence of (**H.3**).

We now turn partially to (**H.5**) and (**H.6**) and establish first an inequality aiming to accomplish (11.13), and then the discrete inf-sup condition (11.14). More precisely, we have the following results taken from [22] and [30], respectively.

Lemma 12.1. *There exists a positive constant* $\boldsymbol{\beta}^{\mathrm{d}}_{S,1}$*, independent of* h*, such that*

$$\sup_{\substack{\boldsymbol{\tau}_{S,h}\in\mathbb{H}_{h,0}(\Omega_S)\\ \boldsymbol{\tau}_{S,h}\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_S} \mathbf{v}_{S,h}\cdot\mathbf{div}(\boldsymbol{\tau}_{S,h})}{\|\boldsymbol{\tau}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}} \geqslant \boldsymbol{\beta}^{\mathrm{d}}_{S,1}\,\|\mathbf{v}_h\|_{0,4;\Omega_S} \qquad \forall\,\mathbf{v}_{S,h}\in\mathbf{L}^4_h(\Omega_S)\,. \tag{12.2}$$

Proof. See [22, Lemma 6.1]. We just stress that it is mainly based on the introduction of a suitable auxiliary boundary value problem, and the utilization of the elliptic regularity result provided by [28, Corollary 1]. □

Lemma 12.2. *There exists a positive constant* $\widetilde{\beta}^{\mathrm{d}}_D$*, independent of* h*, such that*

$$\sup_{\substack{\mathbf{v}_{D,h}\in\mathbf{H}_{h,0}(\Omega_D)\\ \mathbf{v}_{D,h}\neq\mathbf{0}}} \frac{\displaystyle\int_{\Omega_D} q_{D,h}\,\mathrm{div}(\mathbf{v}_{D,h})}{\|\mathbf{v}_{D,h}\|_{\mathrm{div},\Omega_D}} \geqslant \widetilde{\beta}_D\,\|q_{D,h}\|_{0,\Omega_D} \qquad \forall\,q_{D,h}\in\mathrm{L}^2_{h,0}(\Omega_D)\,. \tag{12.3}$$

Proof. We refer to [30, Chapter IV, Section 4.2] for full details. It basically reduces to the verification of the hypotheses of Fortin's lemma (cf. [30, Lemma 2.6]), which makes use of an elliptic regularity result in convex domains, and the main properties of the Raviart-Thomas interpolation operator. □

We complete the accomplishment of the hypothesis (**H.6**) by remarking that the existence of $\boldsymbol{\psi}_{0,\mathrm{d}}\in\mathbf{H}^{1/2}_{00}(\Sigma)$ satisfying (11.15) is guaranteed at the beginning of [37, Section 5.3]. In particular, this holds if the sequence of subspaces $\{\boldsymbol{\Lambda}^S_h(\Sigma)\}_{h>0}$ is nested, which is confirmed below when defining $\Lambda^S_h(\Sigma)$. Thus, $\boldsymbol{\psi}_{0,\mathrm{d}}$ can be constructed as indicated in the proof of Lemma 10.7. A similar procedure applies to the 3D case.

12.2 The spaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ and the remaining hypotheses in 2D

We now introduce the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ in 2D by following the simplest approach suggested in [37]. Indeed, we first assume, without loss of generality, that the number of edges of Σ_h is even, and let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. Since Σ_h is inherited from the interior triangulations, it is automatically of bounded variation, which means that ratio of lengths of adjacent edges is bounded, and, therefore, so is Σ_{2h}. Now, if the number of edges of Σ_h were odd, we simply reduce it to the even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this reduced partition. In this way, denoting by x_0 and x_N the extreme points of Σ, we set

$$\begin{aligned} \Lambda_h^S(\Sigma) \,:=\, & \Big\{ \xi_h \in C(\Sigma) : \quad \xi|_e \in \mathrm{P}_1(e) \quad \forall \, \text{edge } e \in \Sigma_{2h}, \quad \xi_h(x_0) \,=\, \xi_h(x_N) \,=\, 0 \Big\}, \\ \Lambda_h^D(\Sigma) \,:=\, & \Big\{ \xi_h \in C(\Sigma) : \quad \xi_h|_e \in \mathrm{P}_1(e) \quad \forall \; \text{edge } e \in \Sigma_{2h} \Big\}. \end{aligned} \tag{12.4}$$

We now aim to establish the discrete inf-sup conditions (11.22) (or (11.12)) and (11.23) by applying Lemma 11.5. To this end, we suppose from now on that $\{\mathcal{T}_h^S\}_{h>0}$ and $\{\mathcal{T}_h^D\}_{h>0}$ are quasi-uniform in a neighborhood of Σ. More precisely, we assume that there is an open neighborhood of Σ, say Ω_Σ, with Lipschitz-continuous boundary $\partial\Omega_\Sigma$, such that the elements intersecting that region are roughly of the same size. In other words, defining

$$\mathcal{T}_{h,\Sigma} \,:=\, \Big\{ T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D : \quad T \cap \Omega_\Sigma \neq \varnothing \Big\}, \tag{12.5}$$

there exists a positive c, independent of h, such that

$$\max_{T \in \mathcal{T}_{h,\Sigma}} h_T \,\leqslant\, c \min_{T \in \mathcal{T}_{h,\Sigma}} h_T \,. \tag{12.6}$$

Under this quasi-uniformity condition, it was proved in [37, Lemma 5.1] that there exist stable discrete lifting operators $\mathcal{L}_h^*$ to Ω_*, $* \in \{S, D\}$, satisfying (11.26). Moreover, as a consequence of this result, it is easy to see that both $\Phi_h^S(\Sigma)$ and $\Phi_h^D(\Sigma)$ (cf. (11.25)) coincide with

$$\Phi_h(\Sigma) := \left\{ \phi_h \in \mathrm{L}^2(\Sigma) : \quad \phi_h|_e \in \mathrm{P}_0(e) \quad \forall \text{ edge } e \in \Sigma_h \right\}. \tag{12.7}$$

Hence, a straightforward application of Lemma 11.5 implies that, in order to conclude (11.24), which in turn yields (11.22) and (11.23), it suffices to show (11.27). In fact, this latter result, taken from [37], is stated as follows.

Lemma 12.3. *There exists a positive constant* $\beta_\Sigma^{\mathrm{d}} > 0$*, independent of* h*, such that*

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma) \\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_h \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geqslant \beta_\Sigma^{\mathrm{d}} \, \|\xi_h\|_{1/2,\Sigma} \quad \forall\, \xi_h \in \Lambda_h^S(\Sigma) \cup \Lambda_h^D(\Sigma)\,.$$

Proof. See [37, Lemma 5.2] for details. □

As previously remarked, Lemma 12.3 yields, in particular, the verification of (11.22), which is the same as (11.12), and thus (**H.4**) is accomplished. Similarly, having as well (11.23), a suitable combination of this inequality with the discrete inf-sup condition provided by Lemma 12.1 leads to (**H.5**), that is to (11.13), with a constant $\boldsymbol{\beta}_S^{\mathrm{d}}$ depending only on $\boldsymbol{\beta}_{S,1}^{\mathrm{d}}$ (cf. Lemma 12.1) and $\boldsymbol{\beta}_{S,2}^{\mathrm{d}}$ (cf. (11.23)).

12.3 The spaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ and the remaining hypotheses in 3D

In order to set the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$ in the 3D case, we need to introduce an independent triangulation $\Sigma_{\widehat{h}}$ of Σ, made up of triangles K of diameter $\widehat{h}_K$, so that we set the meshsize $\widehat{h} := \max\left\{ \widehat{h}_K : \quad K \in \Sigma_{\widehat{h}} \right\}$. Then, denoting by $\partial\Sigma$ the polygonal boundary of Σ, we define

$$\Lambda_{\widehat{h}}^S(\Sigma) := \left\{ \xi_{\widehat{h}} \in C(\Sigma) : \quad \xi_{\widehat{h}}|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}}, \quad \xi_{\widehat{h}} = 0 \quad \text{on} \quad \partial\Sigma \right\},$$

$$\Lambda_{\widehat{h}}^D(\Sigma) := \left\{ \xi_{\widehat{h}} \in C(\Sigma) : \quad \xi_{\widehat{h}}|_K \in \mathrm{P}_1(K) \quad \forall\, K \in \Sigma_{\widehat{h}} \right\}.$$

Next, as in Section 12.2, we assume here that the families $\{\mathcal{T}_h^S\}_{h>0}$ and $\{\mathcal{T}_h^D\}_{h>0}$ are quasi-uniform as well in a neighborhood of Σ. Hence, proceeding similarly to the proof of [37, Lemma 5.1], it was proved in [3, Lemma 4.4] that there exist stable discrete lifting operators $\mathcal{L}_h^*$ to Ω_*, $* \in \{S, D\}$, satisfying the 3D version of (11.26). Moreover, since Σ_h is the partition of Σ inherited from $\mathcal{T}_h^S$ (or $\mathcal{T}_h^D$), made up of triangles K of diameter h_K, we set the respective meshsize $h_\Sigma := \max\{h_K : \quad K \in \Sigma_h\}$, and observe, as for the 2D case, that both $\Phi_h^S(\Sigma)$ and $\Phi_h^D(\Sigma)$ (cf. (11.25)) coincide with the 3D version of (12.7), that is

$$\Phi_h(\Sigma) := \Big\{\phi_h \in \mathrm{L}^2(\Sigma) : \quad \phi_h|_K \in \mathrm{P}_0(K) \quad \forall \,\text{triangle } K \in \Sigma_h\Big\}. \tag{12.8}$$

Consequently, applying again Lemma 11.5 we conclude, by means of (11.24), that (11.22) and (11.23) follow from the 3D version of (11.27), which is stated below.

Lemma 12.4. *There exist positive constants $\beta_\Sigma^{\mathrm{d}}$ and C_0, independent of h, such that for all $h_\Sigma \leqslant C_0\,\widehat{h}$ there holds*

$$\sup_{\substack{\phi_h \in \Phi_h(\Sigma)\\ \phi_h \neq 0}} \frac{\langle \phi_h, \xi_{\widehat{h}} \rangle_\Sigma}{\|\phi_h\|_{-1/2,\Sigma}} \geqslant \beta_\Sigma^{\mathrm{d}}\, \|\xi_{\widehat{h}}\|_{1/2,\Sigma} \quad \forall\, \xi_{\widehat{h}} \in \Lambda_{\widehat{h}}^S(\Sigma) \cup \Lambda_{\widehat{h}}^D(\Sigma)\,.$$

Proof. We refer to [3, Lemma 4.5] for full details (see also part of the proof of [33, Lemma 7.5]). □

The discussion regarding the consequent accomplishment of (**H.4**) and (**H.5**) in the present 3D case is analogous to the one given at the end of Section 12.2, the only difference being now the incorporation of the restriction $h_\Sigma \leqslant C_0\,\widehat{h}$ in the respective statements.

12.4 The rates of convergence

Here we provide the rates of convergence of the Galerkin scheme (11.4) with the specific finite element subspaces introduced in Sections 12.1, 12.2, and 12.3. For this purpose, we collect next the respective approximation properties (cf. [27], [30]) under the notational convention that in 2D, $\widehat{h}$, $\Lambda_{\widehat{h}}^D(\Sigma)$, and $\Lambda_{\widehat{h}}^S(\Sigma)$ mean h, $\Lambda_h^D(\Sigma)$, and $\Lambda_h^S(\Sigma)$, respectively:

$\left(\mathbf{AP}_h^{\mathbf{t}_S}\right)$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $\mathbf{r}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{L}^2_{\rm tr}(\Omega_S)$, there holds

$$\operatorname{dist}\big(\mathbf{r}_S, \mathbb{L}^2_{{\rm tr},h}(\Omega_S)\big) \,\leqslant\, C\, h^\varrho \, \|\mathbf{r}_S\|_{\varrho,\Omega_S} \,,$$

$(\mathbf{AP}_h^{\mathbf{u}_D})$ there exists a positive constant C, independent of h, such that for each $\varrho \in (0,1]$, and for each $\mathbf{v}_D \in \mathbf{H}^\varrho(\Omega_D) \cap \mathbf{H}_0(\operatorname{div};\Omega_D)$ with $\operatorname{div}(\mathbf{v}_D) \in \mathrm{H}^\rho(\Omega_D)$, there holds

$$\operatorname{dist}\big(\mathbf{v}_D, \mathbf{H}_{h,0}(\Omega_D)\big) \,\leqslant\, C\, h^\varrho \left\{ \|\mathbf{v}_D\|_{\varrho,\Omega_D} \,+\, \|\operatorname{div}(\mathbf{v}_D)\|_{\varrho,\Omega_D} \right\},$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}_S})$ there exists a positive constant C, independent of h, such that for each $\varrho \in (0,1]$, and for each $\boldsymbol{\tau}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$ with $\mathbf{div}(\boldsymbol{\tau}_S) \in \mathbf{W}^{\varrho,4/3}(\Omega_S)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}_S, \mathbb{H}_{h,0}(\Omega_S)\big) \,\leqslant\, C\, h^\varrho \left\{ \|\boldsymbol{\tau}_S\|_{\varrho,\Omega_S} \,+\, \|\mathbf{div}(\boldsymbol{\tau}_S)\|_{\varrho,4/3;\Omega_S} \right\},$$

$\left(\mathbf{AP}_{\widehat{h}}^{\lambda}\right)$ there exists a positive constant C, independent of h and $\widehat{h}$, such that for each $\varrho \in [0,1]$, and for each $\xi \in \mathrm{H}^{1/2+\varrho}(\Sigma)$, there holds

$$\operatorname{dist}\big(\xi, \Lambda^D_{\widehat{h}}(\Sigma)\big) \,\leqslant\, C\, \widehat{h}^\varrho \, \|\xi\|_{1/2+\varrho,\Sigma} \,,$$

$(\mathbf{AP}_h^{\mathbf{u}_S})$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $\mathbf{v}_S \in \mathbf{W}^{\varrho,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}_S, \mathbf{L}^4_h(\Omega_S)\big) \,\leqslant\, C\, h^\varrho \, \|\mathbf{v}_S\|_{\varrho,4;\Omega_S} \,,$$

$\left(\mathbf{AP}_{\widehat{h}}^{\boldsymbol{\varphi}}\right)$ there exists a positive constant C, independent of h and $\widehat{h}$, such that for each $\varrho \in [0,1]$, and for each $\boldsymbol{\psi} \in \mathbf{H}^{1/2+\varrho}(\Sigma) \cap \mathbf{H}^{1/2}_{00}(\Sigma)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\psi}, \boldsymbol{\Lambda}^S_{\widehat{h}}(\Sigma)\big) \,\leqslant\, C\, \widehat{h}^\varrho \, \|\boldsymbol{\psi}\|_{1/2+\varrho,\Sigma} \,,$$

$(\mathbf{AP}_h^{p_D})$ there exists a positive constant C, independent of h, such that for each $\varrho \in [0,1]$, and for each $q_D \in \mathrm{H}^\varrho(\Omega_D) \cap \mathrm{L}^2_0(\Omega_D)$, there holds

$$\operatorname{dist}\big(q_D, \mathrm{L}^2_{h,0}(\Omega_D)\big) \leqslant C\, h^\varrho \, \|q_D\|_{\varrho,\Omega_D}\,.$$

The rates of convergence of (11.4) are now established by the following theorem.

Theorem 12.5. *Let* $((\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}), \vec{\boldsymbol{p}}) \in \mathbb{H} \times \mathbf{Q}$ *and* $((\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h,)\vec{\boldsymbol{p}}_h) \in \mathbb{H}_h \times \mathbf{Q}_h$ *be the unique solutions of* (10.9) (*or* (10.17)) *and* (11.4) (*or* (11.5)), *with* $\mathbf{u}_S \in \mathrm{W}$ *and* $\mathbf{u}_{S,h} \in \mathrm{W}_h$, *whose existences are guaranteed by Theorems* 10.11 *and* 11.4, *respectively. In turn, let* p *and* p_h *given by* (9.4) *and* (11.37), *respectively. Assume the hypotheses of Theorem* 11.6, *and that there exists* $\varrho \in (0,1]$ *such that* $\mathbf{t}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{L}^2_{\mathrm{tr}}(\Omega_S)$, $\mathbf{u}_D \in \mathbf{H}^\varrho(\Omega_D) \cap \mathbf{H}_0(\mathrm{div};\Omega_D)$, $\mathrm{div}(\mathbf{u}_D) \in \mathrm{H}^\varrho(\Omega_D)$, $\boldsymbol{\sigma}_S \in \mathbb{H}^\varrho(\Omega_S) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega_S)$, $\mathbf{div}(\boldsymbol{\sigma}_S) \in \mathbf{W}^{\varrho,4/3}(\Omega_S)$, $\lambda \in \mathrm{H}^{1/2+\varrho}(\Sigma)$, $\mathbf{u}_S \in \mathbf{W}^{\varrho,4}(\Omega_S)$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\varrho}(\Sigma) \cap \mathbf{H}^{1/2}_{00}(\Sigma)$, *and* $p_D \in \mathrm{H}^\varrho(\Omega_D) \cap \mathrm{L}^2_0(\Omega_D)$. *Then, there exists a positive constant* C, *independent of* h, *such that*

$$\begin{aligned}
&\|(\vec{\mathbf{t}}, \vec{\boldsymbol{\sigma}}, \vec{\mathbf{u}}, \vec{\boldsymbol{p}}) - (\vec{\mathbf{t}}_h, \vec{\boldsymbol{\sigma}}_h, \vec{\mathbf{u}}_h, \vec{\boldsymbol{p}}_h)\|_{\mathbb{H}\times\mathbf{Q}} + \|p_S - p_{S,h}\|_{0,\Omega_S} \\
&\quad \leqslant C \Big\{ h^\varrho \Big(\|\mathbf{t}_S\|_{\varrho,\Omega_S} + \|\mathbf{u}_D\|_{\varrho,\Omega_D} + \|\mathrm{div}(\mathbf{u}_D)\|_{\varrho,\Omega_D} + \|\boldsymbol{\sigma}_S\|_{\varrho,\Omega_S} + \|\mathbf{div}(\boldsymbol{\sigma}_S)\|_{\varrho,4/3;\Omega_S} \\
&\quad + \|\mathbf{u}_S\|_{\varrho,4;\Omega_S} + \|p_D\|_{\varrho,\Omega_D} \Big) + \widehat{h}^\varrho \Big(\|\lambda\|_{1/2+\varrho,\Sigma} + \|\boldsymbol{\varphi}\|_{1/2+\varrho,\Sigma} \Big) \Big\}\,.
\end{aligned}$$

Proof. It follows straightforwardly from the Céa estimate (11.39) and the approximation properties $(\mathbf{AP}_h^{\mathbf{t}_S})$, $(\mathbf{AP}_h^{\mathbf{u}_D})$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}_S})$, $(\mathbf{AP}_{\widehat{h}}^{\lambda})$, $(\mathbf{AP}_h^{\mathbf{u}_S})$, $\Big(\mathbf{AP}_{\widehat{h}}^{\boldsymbol{\varphi}}\Big)$ and $(\mathbf{AP}_h^{p_D})$. □

CHAPTER 13

Computational results

In this chapter we present numerical results that illustrate the behavior of the Galerkin scheme (11.4). The computational implementation is based on a `FreeFem++` code (cf. [41]) and the use of the direct linear solvers UMFPACK (cf. [23]). The iterative method comes straightforwardly from the discrete fixed-point strategy along with a Newton-type method. Then, as a stopping criteria, we finish the algorithm when the relative error between two consecutive iterations of the complete coefficient vector **coeff** is small enough, that is

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leqslant \mathtt{tol}\,,$$

where $\|\cdot\|_{l^2}$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{dof}}$ with `dof` denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{L}^2_{\mathrm{tr},h}(\Omega_S)$, $\mathbb{H}_{h,0}(\Omega_S)$, $\mathbf{L}^4_h(\Omega_S)$,

$\mathbf{H}_{h,0}(\mathrm{div};\Omega_D)$, $\mathbf{\Lambda}^S_{\widehat{h}}(\Sigma)$, $\Lambda^D_{\widehat{h}}(\Sigma)$, and $\mathrm{L}^2_{h,0}(\Omega_D)$. Subsequently, errors are defined as follows:

$$\mathsf{e}(\mathbf{t}_S) := \|\mathbf{t}_S - \mathbf{t}_{S,h}\|_{0,\Omega_S}, \quad \mathsf{e}(\boldsymbol{\sigma}_S) := \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\mathbf{div}_{4/3};\Omega_S}, \quad \mathsf{e}(\mathbf{u}_S) := \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{0,4;\Omega_S},$$

$$\mathsf{e}(\mathbf{u}_D) := \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathrm{div};\Omega_D}, \quad \mathsf{e}(\lambda) := \|\lambda - \lambda_{\widehat{h}}\|_{1/2,\Sigma}, \quad \mathsf{e}(\boldsymbol{\varphi}) := \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_{\widehat{h}}\|_{1/2,00;\Sigma},$$

$$\mathsf{e}(p_D) := \|p_D - p_{D,h}\|_{0,\Omega_D}.$$

Again, hereafter, $\widehat{h}$, $\Lambda^D_{\widehat{h}}(\Sigma)$, and $\Lambda^S_{\widehat{h}}(\Sigma)$ mean h, $\Lambda^D_h(\Sigma)$, and $\Lambda^S_h(\Sigma)$, respectively, in 2D. Notice that, for ease of computation, and owing to the fact that $\mathrm{H}^{1/2}(\Sigma)$ is the interpolation space with index 1/2 between $\mathrm{H}^1(\Sigma)$ and $\mathrm{L}^2(\Sigma)$, the interface norm $\|\lambda-\lambda_{\widehat{h}}\|_{1/2,\Sigma}$ is replaced by $\|\lambda-\lambda_{\widehat{h}}\|_{(0,1),\Sigma}$, where

$$\|\xi\|_{(0,1),\Sigma} := \|\xi\|^{1/2}_{0,\Sigma}\,\|\xi\|^{1/2}_{1,\Sigma} \quad \forall \xi \in \mathrm{H}^1(\Sigma).$$

Similarly, the interface norm $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_{\widehat{h}}\|_{1/2,00;\Sigma}$ is replaced by $\|\boldsymbol{\varphi}-\boldsymbol{\varphi}_h\|_{(0,1),\Sigma}$. In turn, convergence rates are set as

$$r(\star) := \frac{\log(\mathsf{e}(\star)/\mathsf{e}'(\star))}{\log(h/h')}, \quad \forall \star \in \{\mathbf{t}_S, \boldsymbol{\sigma}_S, \mathbf{u}_S, \mathbf{u}_D, \boldsymbol{\varphi}, \lambda, p_D\},$$

where e and e' denote errors computed on two consecutive meshes of sizes h and h', respectively. In addition, we refer to the number of degrees of freedom and the number of Newton iterations as `dof` and `iter`, respectively.

13.1 Example 1: Tombstone-shaped domain.

In our first example, a minor modification of [17, Example 1], we consider a porous unit square, coupled with a semi-disk-shaped fluid domain, that is,

$$\Omega_D := (-0.5, 0.5)^2 \quad \text{and} \quad \Omega_S := \Big\{(x_1,x_2): \quad x_1^2 + (x_2-0.5)^2 < 0.25, \quad x_2 > 0.5\Big\}.$$

We set the model parameters

$$\mathbf{K} := 10^{-3}\,\mathbb{I}\,, \quad \rho := 1\,, \quad \omega_1 := 1.0\,,$$

and choose the data $\mathbf{f}_S$, $\mathbf{g}_S$, and f_D such that the variable viscosity is defined as

$$\mu(\nabla\mathbf{u}_S) := 2 + \frac{1}{1+|\nabla\mathbf{u}_S|}\,,$$

where the exact solution in the domain $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$ is given by the smooth functions

$$p_S(\mathbf{x}) = \sin(\pi x_1)\sin(\pi x_2)\,, \quad \mathbf{u}_S(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix} \quad \forall\,\mathbf{x} := (x_1, x_2) \in \Omega_S\,,$$

$$p_D(\mathbf{x}) = \cos(\pi x_1)\,\exp(x_2 - 0.5)\,, \quad \text{and} \quad \mathbf{u}_D(\mathbf{x}) = -\mathbf{K}\,\nabla p_D(\mathbf{x}) \quad \forall\,\mathbf{x} := (x_1, x_2) \in \Omega_D\,.$$

Notice that $\mathbf{u}_S$, being the **curl** of a smooth function, satisfies the incompressibility condition, and also $\mathbf{u}_S \cdot \mathbf{n} = 0$ on Γ_D. Table 13.1 shows the convergence history for a sequence of quasi-uniform mesh refinements, including the resulting number of Newton iterations. According to the polynomial degree employed, the respective sets of finite element subspaces are denoted $\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_1$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$, for the fluid and the porous medium, respectively. This example confirms the theoretical rate of convergence $\mathcal{O}(h)$ provided by Theorem 12.5 with $\varrho = 1$. In addition, the aforementioned number of Newton iterations required to reach the convergence criterion based on the residuals with a tolerance of $1e-8$, was equal to 4 in all runs. Finally, samples of approximate solutions are shown in Figure 13.1.

13.2 Example 2: air flow through a filter.

This example is similar to the one presented in [44, Section 4] (see also [19]). More precisely, we apply our mixed method to simulate air flow through a filter. To this end, we consider a two-dimensional channel with lenght 0.75 m and width 0.25 m which is partially blocked by a rectangular porous medium of length 0.25 m and width 0.2 m as shown in Figure 13.2, with boundaries $\Gamma_S = \Gamma_S^{\mathrm{in}} \cup \Gamma_S^{\mathrm{top}} \cup \Gamma_S^{\mathrm{out}} \cup \Gamma_S^{\mathrm{bottom}}$ and $\Gamma_D^{\mathrm{bottom}} := \Gamma_D$. The permeability tensor in the

$\mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{P}_1$ and $\mathbf{RT}_0 - \mathrm{P}_0 - \mathrm{P}_1$									
$e(\mathbf{t}_S)$	$r(\mathbf{t}_S)$	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(p_S)$	$r(p_S)$
$3.18e-01$	$*$	$1.75e+00$	$*$	$1.27e-01$	$*$	$3.24e-01$	$*$	$2.65e-01$	$*$
$1.63e-01$	1.08	$8.83e-01$	1.11	$6.21e-02$	1.15	$1.64e-01$	1.10	$1.26e-01$	1.21
$8.32e-02$	0.96	$4.46e-01$	0.98	$3.12e-02$	0.98	$8.28e-02$	0.98	$6.31e-02$	0.98
$4.16e-02$	1.05	$2.23e-01$	1.05	$1.57e-02$	1.05	$4.16e-02$	1.05	$3.24e-02$	1.01
$2.06e-02$	1.01	$1.10e-01$	1.02	$7.78e-03$	1.01	$2.08e-02$	1.00	$1.58e-02$	1.03
$1.04e-02$	1.08	$5.54e-02$	1.09	$3.89e-03$	1.10	$1.05e-02$	1.09	$7.78e-03$	1.08

$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\lambda)$	$r(\lambda)$	dof	iter
$2.28e-04$	$*$	$5.23e-02$	$*$	$2.50e-01$	$*$	731	4
$1.06e-04$	1.23	$2.29e-02$	1.26	$1.26e-01$	1.02	2659	4
$4.25e-05$	1.36	$1.05e-02$	1.16	$4.99e-02$	1.38	10460	4
$2.00e-05$	1.08	$5.00e-03$	1.05	$2.33e-02$	1.09	41804	4
$9.94e-06$	1.58	$2.53e-03$	1.54	$1.19e-02$	1.52	167808	4
$4.95e-06$	0.93	$1.27e-03$	0.93	$5.79e-03$	0.97	660726	4

Table 13.1: Example 1, convergence history and Newton iteration count for the fully-mixed approximations of the Navier–Stokes/Darcy equations with variable viscosity, and convergence of the P_0-approximation of the postprocessed pressure field.

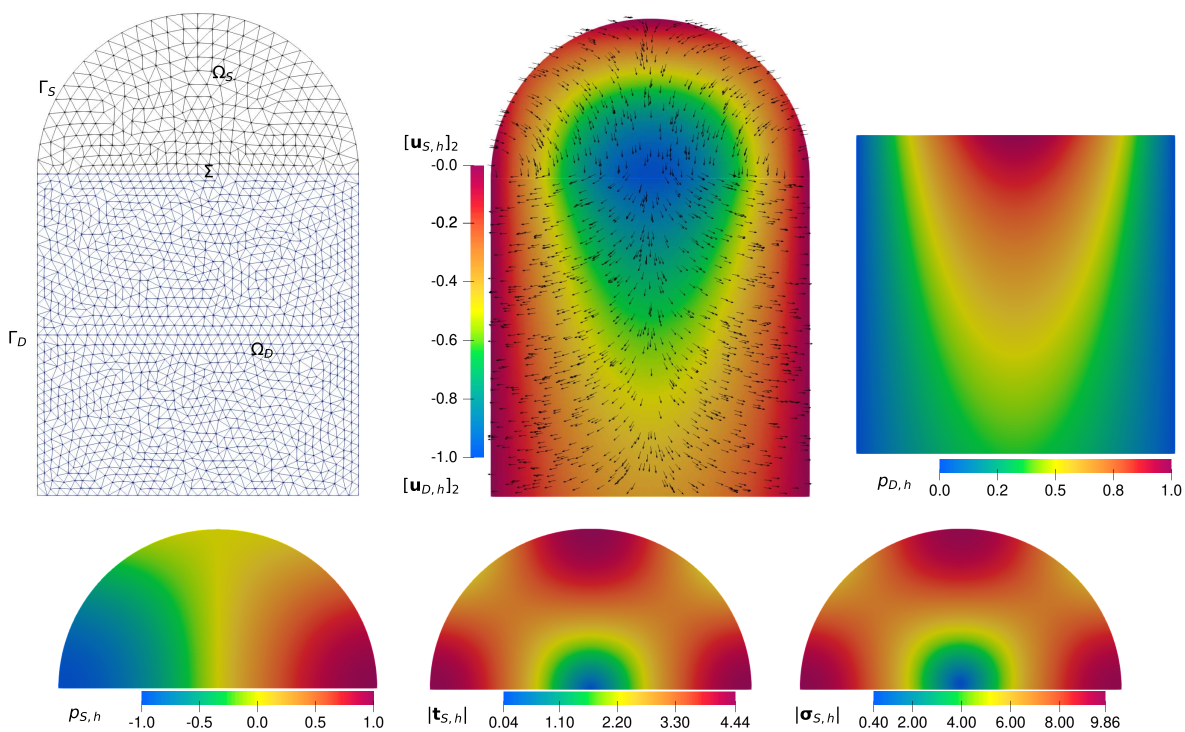

Figure 13.1: Example 1, domain configuration, approximated velocity component, Darcy pressure field, Navier–-Stokes pressure field, spectral norm of the Navier–-Stokes velocity gradient and pseudo-stress tensor.

porous medium is given as

$$\mathbf{K} \,=\, \mathrm{R}(\theta) \begin{pmatrix} \frac{1}{\delta}\kappa & 0 \\ 0 & \kappa \end{pmatrix} \mathrm{R}^{-1}(\theta)\,, \qquad \text{with } \mathrm{R}(\theta) \,=\, \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix},$$

where the angle $\theta = -45°$, the anisotropy ratio $\delta = 100$, and $\kappa = 10^{-6}\,m^2$. In turn, $\rho = 1.225 \times 10^{-5}\,\mathrm{Mg/m}^3$, $\omega_1 = 1.0$, and the top and bottom of the domain are impermeable walls. The flow is driven with an inlet mean velocity of 0.25 m/s. The force terms $\mathbf{f}_S$ and f_D are set to zero. As motivated again by [17], the viscosity follows the Carreau law given by

$$\mu \,=\, 1.81 + 1.81\,(1 + |\mathbf{t}_S|^2)^{-1/2} \times 10^{-5}\ \mathrm{Pa\,s}\,, \tag{13.1}$$

whereas the boundary conditions are

$$\mathbf{u}_S \,=\, \left[6\,\mathbf{u}_{\mathrm{in},S}\,\frac{x_2}{d}(1 - \frac{x_2}{d}), 0\right] \quad \text{on} \quad \Gamma_S^{\mathrm{in}}\,, \quad \mathbf{u}_S \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_S^{\mathrm{top}} \cup \Gamma_S^{\mathrm{bottom}}\,,$$

$$\boldsymbol{\sigma}_S\mathbf{n} \,=\, \mathbf{0} \quad \text{on} \quad \Gamma_S^{\mathrm{out}}, \quad \mathbf{u}_D \cdot \mathbf{n} \,=\, 0 \quad \text{on} \quad \Gamma_D^{\mathrm{bottom}}\,,$$

with $\mathbf{u}_{\mathrm{in},S} = 0.25$ m/s and $d = 0.2$ m. We stress here that, because of the fully nonlinear character of μ (cf. (13.1)), which depends on the unknown fluid velocity gradient $\mathbf{t}_S := \nabla\mathbf{u}_S$, the use of the Newton method to solve the corresponding Galerkin scheme (11.4) implies linearizing not only the convective term given by the form b (cf. (10.11)), but also the one arising from the form a (cf. (10.10)). In addition, we remark that the analysis developed in the previous chapters can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example. Now, using again a sequence of quasi-uniform mesh refinements, we find that the number of Newton iterations required to reach the convergence criterion, based on the residuals with a tolerance of $1e-8$, is 7. In Fig. 13.2 we display various components of the computed solution. As we expected, the top-left panel shows an increment in air flow in the surrounding region above the filter. This is caused by the flow resistance in the porous medium. The effect of anisotropy is also evident, as the air flow that passes through the porous block aligns with the angle $\theta = -45°$. In other words, the flow follows the inclined principal

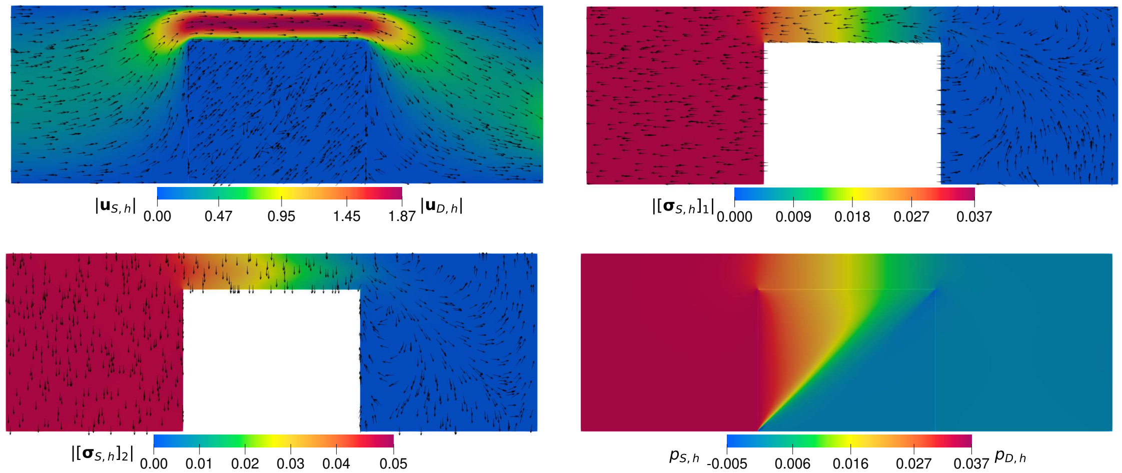

Figure 13.2: Example 2, approximated magnitude of the velocities (top-left), first rows (top-right) and second rows (bottom-left) of the pseudostress tensor with vector directions, and pressure fields (bottom-right).

direction of the permeability tensor. Furthermore, a continuous normal velocity is observed across all three interfaces, whereas the tangential velocity is discontinuous, especially at the interfaces with higher fluid velocity. This observation aligns with the continuity of flux and the BJS interface conditions. We also observe that the pressure drop is visible through the domain. Again, the effect of anisotropy is visible due to the inclined pressure drop in the porous domain. The pseudostress tensor $\boldsymbol{\sigma}_{S,h}$ is larger along the Γ_S^{in} boundary and zero at the Γ_S^{out} boundary, which is consistent with the boundary condition $\boldsymbol{\sigma}_S\,\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_S^{\text{out}}$.

CHAPTER 14

Conclusions and Future Works

In this chapter we summarize the main contributions of this work and give a brief description of eventual future works.

14.1 Conclusions

Upon the results presented in the first part of this thesis, we arrive at the following conclusions:

- We developed a new mixed formulation for the Navier-Stokes equations with variable viscosity that depends nonlinearly on the velocity gradient, whose analysis made use of diverse tools and abstract results in Banach spaces.
- We proved that is not necessary to use an augmented formulation to provide well posedness of the continuous and discrete formulations.
- The well-posedness of the continuous formulation was proved using a fixed point strategy in combination with the Banach theorem.

- An analogous approach is employed to conclude the existence and uniqueness of a solution for the associated Galerkin scheme. In addition, *a priori* error estimates are derived.
- We used Raviart-Thomas elements of order ℓ with their respective convergence rates, followed by several numerical experiments that confirmed the theoretical error bounds.

According to the results presented in the second part of this work, we can state the following conclusions:

- We develop a new mixed formulation in Banach spaces for the coupled problem given by Navier–Stokes and Darcy equations.
- We consider a similar approach to that presented in the first part for the Navier-Stokes domain and adapt it to the coupled Navier-Stokes/Darcy problem.
- Finally, several numerical results illustrating the good performance of the method in 2D and confirming the theoretical findings are reported.

14.2 Future Works

The methods developed and the results obtained here have motivated some possibilities of future work, which are described below:

- To extend the analysis of the coupled Navier–Stokes and Darcy equations with nonlinear viscosity.
- To develop the corresponding a posteriori error analyses for some of the above models.
- To extend the analysis and results to the time dependent case.

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