Universidad de Concepción
Facultad de Ciencias Físicas y Matemáticas
Departamento de Ingeniería Matemática

ORTHOGONAL POLYNOMIALS WITH REFLECTION-INVARIANT WEIGHTS

por

Gonzalo Alejandro Benavides García

Tesis presentada a la Facultad de Ciencias Físicas y Matemáticas de la Universidad de Concepción para optar al título profesional de Ingeniero Civil Matemático

Profesor Guía: Dr. Leonardo E. Figueroa

Marzo de 2020
Concepción, Chile

ORTHOGONAL POLYNOMIALS WITH REFLECTION-INVARIANT WEIGHTS

POLINOMIOS ORTOGONALES CON PESOS INVARIANTES RESPECTO A REFLEXIONES

COMISIÓN EVALUADORA

Dr. Leonardo E. Figueroa [Profesor guía]

CI^{2}MA y Departamento de Ingeniería Matemática, Universidad de Concepción.

Dr. Norbert Heuer

Facultad de Matemáticas, Pontificia Universidad Católica de Chile.

Dr. Diego Paredes

CI^{2}MA y Departamento de Ingeniería Matemática, Universidad de Concepción.

FECHA DE DEFENSA: 27 de Marzo de 2020

A Emy, mi compañera perruna

Acknowledgements

Comienzo agradeciendo a los profesores del Departamento de Ingeniería Matemática, valoro enormemente su gran calidad académica y su altísima paciencia para explicar y responder dudas (y para lidiar con mi, a veces descontrolada, inquisitiva actitud), a ellos les debo lo que he aprendido durante estos seis años de carrera y en lo que soy como profesional. Gracias por confiar innumerablemente en mí como su ayudante, espero haber valido la pena.

Profesor Julio Aracena, a usted le debo el haber decidido ingresar a la carrera, siempre recordaré con cariño su disposición a recibirme como alumno de enseñanza media.

Profesor Freddy Paiva, infinitas gracias por la constante motivación, brindarme la confianza que muchas veces carecí, los libros y apuntes, preocuparse constantemente de mi alimentación y su humor sarcástico tan característico de usted. No sabe lo importante que fue para mí la franqueza de cada una de sus palabras.

Profesora Mónica Selva, conversar con usted siempre ha sido como conversar con un familiar, siempre recordaré con cariño todas las horas conversadas junto a una taza de café en donde discutíamos de lo que surgiera en el momento, la calidez de su trato y la alegría que irradia. Siempre he hallado en usted una palabra reconfortante que me ha dado las fuerzas para continuar en la vida académica. Gracias, además, por la prestancia como Jefa de Carrera a la hora de lidiar con la infinidad de trámites que le solicité.

Profesor Rommel Bustinza, a usted le debo el comienzo de mi interés en el Análisis Numérico, gracias por la completitud de su cátedra y la complejidad de sus evaluaciones, quizá sin ellas me habría formado una idea simplona del área. Más importante aún, agradezco la simpatía de su trato, el verme como un semejante, las excelentes recomendaciones culinarias, las interesantes conversaciones y los buenos memes que comparte.

Profesor Rodolfo Rodríguez, gracias por la excelentísima introducción al Análisis, pero más aún por su grandes capacidades pedagógicas, la humildad y respeto en el trato a los estudiantes, y el gran apoyo que me ha brindado en mi incursión a estudios de posgrado. A usted le debo que confiasen, contra todo pronóstico, en mí.

Profesor Gabriel Gatica, a usted no solo le agradezco las grandísimas y entretenidas clases de Análisis Funcional y Elementos Finitos, sino que también la filosa sinceridad de sus palabras, la preocupación constante por mi salud, y tenerme siempre presente para participar en actividades académicas. Gracias por la infinita paciencia y por impregnarme de su enfoque pragmático.

Profesor Leonardo Figueroa, mi profesor guía en este trabajo, ¡qué impresionante paciencia me ha tenido! A pesar de mi letargo y a veces carencia de creatividad, siempre confió en mí y me tendió una mano. Espero poder seguir aprendiendo de usted, no solo de Teoría de Aproximación, sino que también de su precisa y elegante forma de escribir.

Gracias a todos mis compañeros de carrera por soportar constantemente mis excentricidades y siempre haber sido amables conmigo. En particular, agradezco a Jaime Luckmann, Jaime Manríquez e Ivette Henríquez.

Gracias a mis compañeros de colegio, que desde siempre me han motivado y deseado lo mejor.

Mis mejores amigos, Fernando Gajardo y Carlos Retamal, con ustedes siempre he podido ser verdaderamente yo. Sin sus visitas sorpresivas a mi hogar, humor irreverente y alegría quizá me hubiera vuelto loco hace varios años atrás. Espero que nuestra amistad perdure por siempre.

Stefany Gajardo, gracias por haber sido parte de mi vida durante tantos años. Nuestros caminos se han separado, pero los recuerdos y aprendizajes perduran.

Por último, pero lo más importante, gracias a mi familia que desde que tengo memoria me han ayudado a superar mis demonios.

Mis padres Claudia García y Claudio Benavides, a ustedes les debo la vida, sin su amor y apoyo incondicional no habría logrado llegar hasta aquí.

Madre, gracias por toda la ayuda espiritual, los mimos, por contagiarme de tu optimismo y hacerme ver que muchas veces me ahogo en una gota de agua. Tus abrazos me dan fuerza para seguir viviendo.

Padre, todos los días me contagias la pasión desinteresada por tu trabajo y el deseo incansable de seguir perfeccionándose. Siempre has impulsado mi carrera académica y me recuerdas que

no es necesario hacer ruido para saber que uno avanza.

Antonia, mi hermana, tu mera presencia siempre me recuerda que hay muchas más cosas en la vida que las matemáticas, gracias por los excelentes platillos con los que siempre nos has deleitado.

Emy, mi hermana perruna, gracias por la alegría y amor incondicional que siempre me has brindado, tú me enseñas que en la vida se puede ser feliz con muy pocas cosas.

Un abrazo a mis primos, tíos y abuelas que siempre me apoyan a la distancia.

Esta tesis ha sido parcialmente financiada por CONICYT-Chile a través del proyecto Fondecyt Regular 1181957.

Contents

Listings

Abstract

In this thesis we study multivariate orthogonal polynomials in the d-dimensional unit ball associated with a class of reflection-invariant weights,

$$W_{\alpha,\gamma}(x) := (1 - \|x\|^2)^\alpha \prod_{i=1}^{d} |x_i|^{\gamma_i},$$

where $\alpha > -1$ and $\gamma = (\gamma_1, \ldots, \gamma_d) \in (-1, +\infty)^d$. We obtain approximation properties of the related $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomial projector measured in what we call Dunkl–Sobolev norms, characterize a certain family of Sobolev-type orthogonal polynomials associated to the weight above, obtain connection relations of a specific bivariate basis to then develop computational tools to easily operate with polynomials expanded in terms of this basis.

Resumen

En esta tesis estudiamos polinomios ortogonales en la bola unitaria d-dimensional asociados a una clase de pesos invariantes respecto a reflexiones,

$$W_{\alpha,\gamma}(x) := (1 - \|x\|^2)^\alpha \prod_{i=1}^{d} |x_i|^{\gamma_i},$$

donde $\alpha > -1$ y $\gamma = (\gamma_1, \ldots, \gamma_d) \in (-1, +\infty)^d$. Obtenemos propiedades de aproximación del respectivo proyector $\mathrm{L}^2_{\alpha,\gamma}$-ortogonal medidas en las que denominamos normas Dunkl–Sobolev, caracterizamos una cierta familia de polinomios ortogonales tipo Sobolev asociados al peso, obtenemos relaciones de conexión de una base bivariada específica, para luego desarrollar herramientas computacionales para operar fácilmente con polinomios expandidos en términos de esta base.

CHAPTER 1

Introduction

Orthogonal polynomials have been a subject of study in mathematics for at least two hundred of years, whose origins can be traced to Legendre's work [34] on planetary motions and Laplace studies [33] on probability theory. Arguably, Chebyshev was the first person who saw the possibility of a general theory and its applications. His work [43] was strongly motivated by the theory of least squares approximation and probability; he applied his results to interpolation, quadrature rules and mechanics (see [25] for a detailed review). Besides the aforementioned researchers, there were many other mathematicians in the 19th century who helped in establishing the foundations of orthogonal polynomials, such as Jacobi, Gauss and Gegenbauer, among others. Standard references on classical orthogonal polynomials are [42] and [11].

In the second half of the 20th century the topic received renewed interest as a result of the computer revolution and the increasing activity in approximation theory and numerical analysis. Moreover, in recent years the several applications of orthogonal polynomials in physics, chemistry, probability and statistics have made the interest in orthogonal polynomial increase even more. We refer the interested reader to [12] for a very personal view of orthogonal polynomials in the second half of the 20th century.

Within the vast study orthogonal polynomials there are two subareas that we would like to highlight, not only because of their intrinsic importance, but also because their intersection hosts the work presented in this thesis.

The first one corresponds to multivariate orthogonal polynomials. Multivariate orthogonal polynomials have been rigorously studied since the work of Jackson [28] on orthogonal polynomials of two variables. Despite having numerous similarities with the theory of univariate orthogonal polynomials, there are several and fundamental complications in studying multivariate orthogonal polynomials—that were realized even then—such as the non-uniqueness of orthogonal bases. With respect to this issue, it was pointed out in [19] that, as choosing a particular order of the orthogonal system usually destroys natural symmetries, it is often preferable to construct biorthogonal systems; in [31], the authors came with the brilliant observation that "if the results can be stated in terms of orthogonal polyomial spaces rather in terms of a particular basis, a degree of uniqueness is restored". As much as we can in this thesis we embrace this idea, not only because of its elegance, but also because it allows for obtaining results in general dimensions while avoiding long and painful algebraic manipulations. We are later forced to deal with specific bases, though, for the purposes of numerical computation. We refer to [30] and [27] for an accessible mid-seventies survey on bivariate orthogonal polynomials and an extensive historical and technical review of the topic, respectively. The standard reference is the monograph [17].

The second one is Sobolev orthogonal polynomials. Sobolev orthogonal polynomials are polynomials that are orthogonal with respect to an inner product involving derivatives. They originated in the 1960s by considering the problem of obtaining the polynomial that best approximates a function and, simultaneously, its derivatives; but the field started flourishing in the late 1980s when the concept of coherent measures was introduced. Unlike ordinary orthogonal polynomials, Sobolev orthogonal polynomials lack the three-term recurrence relation and therefore their study is usually far more challenging. Despite the difficulties, Sobolev orthogonal polynomials have attained great value as an important instrument in the numerical resolution of partial differential equations; from a heuristic point of view, Sobolev orthogonal polynomials should have a better approximation behavior than ordinary orthogonal polynomials, as the former require more information on the function being approximated. See [35] for a detailed survey on Sobolev orthogonal polynomials.

In this thesis we are interested in studying multivariate orthogonal polynomials in the d-dimensional unit ball associated with a class of reflection-invariant weights,

$$W_{\alpha,\gamma}(x) := (1 - \|x\|^2)^\alpha \prod_{i=1}^{d} |x_i|^{\gamma_i} ,$$

where $\alpha > -1$ and $\gamma = (\gamma_1, \ldots, \gamma_d) \in (-1, +\infty)^d$. Our aim is to obtain approximation properties of the related orthogonal polynomial projector measured in what we call Dunkl–Sobolev norms, characterize a certain family of Sobolev-type orthogonal polynomials associated to the weight above, obtain connection relations of a specific bivariate basis to then develop computational tools to easily operate with polynomials expanded in terms of this basis.

The study of orthogonal polynomials and approximation results involving weights such as $W_{\alpha,\gamma}$ is interesting, first, as an archetype of weights of interior singularities, as its highly symmetric form allows for sourcing useful results from the theory of reflection-invariant orthogonal polynomials [17, Ch. 6 and Ch. 7]. Secondly, there is an intimate connection between orthogonal polynomials in the ball with respect to $W_{\alpha,\gamma}$ and orthogonal polynomials in the unit simplex $T^d := \{x \in \mathbb{R}^d \mid x_1 \geq 0, \ldots, x_d \geq 0, 1 - \sum_{i=1}^{d} x_i \geq 0\}$ with respect to weights that are products of powers of distances to their faces [17, Subsec. 8.2], that is, weights of the form

$$x \mapsto \left(1 - \sum_{i=1}^{d} x_i\right)^{\kappa_{d+1}} \prod_{i=1}^{d} x_i^{\kappa_i}, \quad \kappa_1, \ldots, \kappa_{d+1} > -1;$$

as this reference attests, when mapping orthogonal polynomials in the ball to orthogonal polynomials in the simplex, only the fully reflection-symmetric of the former participate, and for these the Dunkl operators reduce to partial derivatives.

As it will be detailed in the following chapters, in the study of these orthogonal polynomials a certain class of differential-difference operators called Dunkl operators play an important role. First introduced in [15] by C.F. Dunkl, these operators are very important in pure mathematics and physics. To name a few of their applications, they provide an essential tool in the study of special functions with root systems (see [17, Ch. 6-7]), they are closely related to certain representations of degenerate affine Hecke algebras (see for instance [38] and its references therein), the commutative algebra generated by these operators has been used in the study of certain exactly solvable models of quantum mechanics, such as the Calogero–Sutherland–Moser models (see [24] and its references therein); moreover, a generalization of the Fourier transform associated with the Dunkl operators was first presented in [16] and studied in more detail in later publications and a vast theory generalizing harmonic analysis has been developed (see [14]).

1.1 Outline

The rest of this thesis is organized as follows. In Chapter 2 we start by introducing the Dunkl operators related to the weight $W_{\alpha,\gamma}$, study their properties on classical smooth functions and define appropriate ad-hoc weighted Dunkl–Sobolev spaces that will prove useful later. Then, we introduce the orthogonal polynomials spaces with respect to the inner product $(u, v) \mapsto \int_{B^d} u\, v\, W_{\alpha,\gamma}$, deduce some relations between orthogonal polynomials spaces, orthogonal projectors and Dunkl operators and then rewrite the associated second-order Sturm–Liouville problem satisfied by these orthogonal polynomials in a suitable weak form. The rest of the chapter focuses on cleverly handling this weak problem to obtain, at the end, a bound of the orthogonal projector error measured in Dunkl–Sobolev norms in terms of powers on the degree of approximation. We also show that, for a specific case, our bound is sharp (in the sense that the power on the degree of approximation cannot be lowered). The contents of this chapter originally appeared in the following preprint:

Gonzalo A. Benavides and Leonardo E. Figueroa, *Orthogonal polynomial projection error in Dunkl-Sobolev norms in the ball*, arXiv e-prints (2020), arXiv:2002.01638

In Chapter 3 we study a specific family of Dunkl–Sobolev spaces related to the weight $W_{\alpha,\gamma}$. We characterize these spaces in terms of the orthogonal polynomials spaces introduced in Chapter 2, deduce connection relations and finish by showing that these Dunkl–Sobolev orthogonal polynomials satisfy a second-order Sturm–Liouville problem strongly related with the one satisfied by the orthogonal polynomials of Chapter 2.

In Chapter 4 we study connection relations for a specific bivariate basis of orthogonal polynomials (Dunkl–Zernike polynomials). Namely, we deduce the explicit incarnations of the relations satisfied by these orthogonal polynomials previously obtained in Chapter 2. We finish this chapter by turning these connection relations between members of bases into corresponding relations between expansion coefficients of polynomials.

In Chapter 5 we describe a recently developed package for Julia 1.2.0, which implements the connection relations deduced in Chapter 4, allowing for easy and fast numerical computation with polynomials expressed in terms of the Dunkl–Zernike polynomial basis of Chapter 4.

CHAPTER 2

Orthogonal polynomial projection error in Dunkl–Sobolev norms in the ball

2.1 Introduction

Let B^d denote the unit ball of $\mathbb{R}^d$, $\alpha > -1$ and let the weight function $W_\alpha\colon B^d \to \mathbb{R}$ be defined by $W_\alpha(x) = (1 - \|x\|^2)^\alpha$ with $\|\cdot\|$ being the Euclidean norm. Let L^2_α denote the weighted Lebesgue space $\mathrm{L}^2(B^d, W_\alpha) := \{W_\alpha^{-1/2} f \mid f \in \mathrm{L}^2(B^d)\}$, whose natural squared norm is $\|u\|_\alpha^2 := \int_{B^d} |u|^2 W_\alpha$. In [22] it was proved that the orthogonal projector S_N^α mapping L^2_α onto Π_N^d (the space of d-variate polynomials of degree less than or equal to N) satisfies the bound

$$(\forall\, u \in \mathrm{H}^l_\alpha) \quad \|u - S_N^\alpha(u)\|_{\alpha;1} \leq C\, N^{3/2-l}\, \|u\|_{\alpha;l}\,, \tag{2.1.1}$$

where $C > 0$ depends on α and the integer $l \geq 1$ only, and, for every integer $m \geq 1$, H^α_m denotes the weighted Sobolev space whose natural squared norm is $\|u\|^2_{\alpha;m} := \sum_{k=0}^m \left\|\nabla^k u\right\|^2_\alpha$ (here ∇^k is the k-fold gradient).

The purpose of this chapter is proving an analogue of (2.1.1) for a class of reflection-invariant weights involving, fittingly, differential-difference Dunkl operators [17, Sec. 6.4] instead of partial derivatives. In order to state this analogue we introduce now the rest of the minimal necessary notation. Given $\alpha > -1$ and $\gamma = (\gamma_1, \dots, \gamma_d) \in (-1, \infty)^d$, let the weight function $W_{\alpha,\gamma} \colon B^d \to \mathbb{R}$ be defined by

$$W_{\alpha,\gamma}(x) := (1 - \|x\|^2)^\alpha \prod_{i=1}^{d} |x_i|^{\gamma_i} .$$

We denote by $\mathrm{L}^2_{\alpha,\gamma}$ the weighted Lebesgue space $\mathrm{L}^2(B^d, W_{\alpha,\gamma})$, whose natural inner product and squared norm are $\langle u, v\rangle_{\alpha,\gamma} := \int_{B^d} u\, v\, W_{\alpha,\gamma}$ and $\|u\|^2_{\alpha,\gamma} := \int_{B^d} |u|^2\, W_{\alpha,\gamma}$, respectively. Let $S_N^{(\alpha,\gamma)}$ be the orthogonal projector mapping $\mathrm{L}^2_{\alpha,\gamma}$ onto Π^d_N. For $j \in \{1, \dots, d\}$ the Dunkl operator $\mathcal{D}_j^{(\gamma)}$ is defined by

$$\mathcal{D}_j^{(\gamma)} u(x) := \partial_j u(x) + \frac{\gamma_j}{2}\left(u(x) - u(x_1, \dots, \overbrace{-x_j}^{j\text{-th entry}}, \dots, x_d)\right).$$

Given an integer $m \geq 0$, we define the *Dunkl–Sobolev space* $\mathrm{H}^m_{\alpha,\gamma}$ as the topological completion of $\mathrm{C}^m(\overline{B^d})$ with respect to the norm $\|u\|_{\alpha,\gamma;m} := \left(\sum_{k=0}^{m} \left\|(\mathcal{D}^{(\gamma)})^k u\right\|^2_{\alpha,\gamma}\right)^{1/2}$, where $(\mathcal{D}^{(\gamma)})^k$ is the k-fold Dunkl gradient constructed in terms of the Dunkl operators (we reintroduce the Dunkl operators and Dunkl–Sobolev spaces in their proper context in (2.2.12) and Definition 2.2.2, respectively). The main result of this chapter is

Theorem 2.1.1. *For all integers $1 \leq r \leq l$, $\alpha \in (-1, \infty)$ and $\gamma \in (-1, \infty)^d$, there exists $C = C(\alpha, \gamma, l, r) > 0$ such that*

$$(\forall\, u \in \mathrm{H}^l_{\alpha,\gamma}) \qquad \left\|u - S_N^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma;r} \leq C\, N^{-1/2+2r-l}\, \|u\|_{\alpha,\gamma;l} .$$

This chapter builds upon a lineage of works which proved results analogous to Theorem 2.1.1, all of which correspond, in our notation, to cases with $\gamma = 0$, so the involved weights lack interior singularities and the Dunkl operators reduce to partial derivatives. In [9, Th. 2.2 and Th. 2.4] our main result was proved in dimension $d = 1$ when the $\alpha = -1/2$ (Chebyshev case) and when $\alpha = 0$ (Legendre case); see also the streamlined proofs for these cases at [8, Ch. 5]. In [26, Th, 2.6], the one-dimensional case was proved for general α (Gegenbauer case). In [48, Th. 2.6], the one-dimensional case with general asymmetric $(1-x)^\alpha(1+x)^\beta$ weight (Jacobi case) was proved. In [23, Th. 3.11], Theorem 2.1.1 was extended to dimension $d = 2$ for general α. Finally, in [22, Th. 1.1], a new technique of proof, based on orthogonal polynomial *spaces* instead of orthogonal polynomial *bases* (thus circumventing the need for spectral differentiation formulas, which by [23] had made the necessary algebraic manipulation very long in comparison) allowed for extending the result to arbitrary dimension for general α.

In the $\gamma = 0$ cases cited above, the analogues of Theorem 2.1.1 are results of provably non-optimal polynomial approximation with respect to the power on N, caused by the mismatch between the orthogonality that defines the projection operator $S_N^{(\alpha,\gamma)}$ —which can be characterized as a generalized Fourier series truncation operator; cf. (2.3.4)— and the Hilbert norm in which the error is measured (see the references provided in [22, Sec. 1] for optimal polynomial approximation results). The same mismatch occurs in this chapter, so we expect Theorem 2.1.1 to be non-optimal too; however, we cannot be sure because we are not aware of best approximation results for the general γ case.

Our main result involves weighted Dunkl–Sobolev spaces instead of the better understood weighted Sobolev spaces because it is in terms of the former that the contours of the argument in [22] can be reproduced. This is readily apparent because the characterization of $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomials as eigenfunctions of Sturm–Liouville-type operators occurs in terms of Dunkl operators [17, Th. 8.1.3]; said characterization is essential for our way of inferring approximation rates out of the regularity of the function being approximated.

We remark that $r = 0$ case (i.e., approximation error measured in $\mathrm{L}^2_{\alpha,\gamma}$) lies outside of the scope of Theorem 2.1.1; indeed, in this case, the provably optimal power on N is $-l$ (cf. Corollary 2.4.5 below), outside of the pattern set by our main result.

The outline of this chapter is as follows. We finish this introductory Section 2.1 introducing some additional notation. In Section 2.2 we introduce the reflections, Dunkl operators and main Dunkl–Sobolev spaces that participate in this work. In Section 2.3 we introduce orthogonal polynomials spaces and their interaction with Dunkl operators and certain generalizations thereof. In Section 2.4 we put the differential-difference Sturm–Liouville operator the abovementioned orthogonal polynomial spaces are eigenspaces of in a suitable weak form, prove preliminary approximation results upon it and prove our main result. At last, in Section 2.5 we prove the sharpness of our main result for special values of its Dunkl–Sobolev regularity parameters.

Given, $j \in \{1, \dots, d\}$, let $e_j \in \mathbb{R}^d$ be Cartesian unit vector in the j-th direction; i.e., $(e_j)_i$ is 1 if $i = j$ and 0 otherwise. We will denote the Euclidean norm by $\|\cdot\|$. We will denote the space of d-variate polynomials by Π^d; we have already introduced its subspace Π^d_N consisting of polynomials of total degree less than or equal to N. We will adopt the convention that, for $N < 0$, $\Pi^d_N = \{0\}$.

Given an open $\Omega \subset \mathbb{R}^d$ we will denote the integral of functions $f\colon \Omega \to \mathbb{R}$ with respect to the Lebesgue measure simply by $\int_\Omega f(x)\,\mathrm{d}x$. We will denote by σ_{d-1} the surface measure of $\mathbb{S}^{d-1}$, the unit sphere of $\mathbb{R}^d$ [6, Ex. 3.10.82]. For all Lebesgue-integrable f,

$$\int_{\mathbb{R}^d} f(x)\,\mathrm{d}x = \int_0^\infty \int_{\mathbb{S}^{d-1}} f(ry)\, r^{d-1}\,\mathrm{d}\sigma_{d-1}(y)\,\mathrm{d}r. \tag{2.1.2}$$

We denote by $\mathbb{N}$ the set of strictly positive integers and $\mathbb{N}_0 := \{0\} \cup \mathbb{N}$. Members of $[\mathbb{N}_0]^d$ will be called multi-indices and for every multi-index $a \in [\mathbb{N}_0]^d$, point $x \in \mathbb{R}^d$ and regular enough real-valued function f defined on some open set of $\mathbb{R}^d$ we shall write $|a| = \sum_{i=1}^d a_i$, $x^a = \prod_{i=1}^d x_i^{a_i}$ and $\partial_a f = \partial^{|a|} f/(\partial x_1^{a_1} \cdots \partial x_d^{a_d})$.

Setting $a_i = 1$, $p_i = 2$, $t_1 = 0$, $t_2 = 1$, $\alpha_i = \gamma_i + 1$ and $f(u) = (1-u)^\alpha$ in [1, Th. 1.8.5] it readily follows that

$$\int_{B^d} W_{\alpha,\gamma}(x)\,\mathrm{d}x = \frac{\prod_{i=1}^d \Gamma\left(\frac{\gamma_i+1}{2}\right)}{\Gamma\left(\frac{d+\sum_{i=1}^d \gamma_i}{2}\right)}\, \mathrm{B}\left(\frac{d}{2} + \frac{1}{2}\sum_{i=1}^d \gamma_i, \alpha+1\right), \tag{2.1.3}$$

where Γ and B denote Gamma and Beta functions respectively; these functions being finite for positive arguments, it follows that the constraints $\alpha > -1$ and $\gamma \in (-1,\infty)^d$ are precisely those that ensure that the above integral is finite. As a consequence of (2.1.3), $\mathrm{L}^\infty(B^d) \subset \mathrm{L}^2_{\alpha,\gamma}$. In particular, every polynomial, being a bounded function in B^d, belongs to $\mathrm{L}^2_{\alpha,\gamma}$.

We finish this introductory section noting that we mostly omit the dimension d from the notation of e.g., function spaces, in order to avoid cluttering and because all of our arguments work independently of the dimension.

2.2 Dunkl operators and weighted Dunkl–Sobolev spaces

Given $j \in \{1, \dots, d\}$ let $\sigma_j \colon B^d \to B^d$ be the reflection defined by

$$(\forall\, x \in B^d) \quad \sigma_j x := x - 2\, x_j\, e_j; \tag{2.2.1}$$

that is, σ_j flips the sign of the j-th component of its argument. The group generated by $\{\sigma_j \mid 1 \le j \le d\}$ with function composition as the group operation is (isomorphic to) the Coxeter group $\mathbb{Z}_2^d$ [17, Sec. 7.5]. Given a scalar-, vector- or tensor-valued function f on B^d, we shall write $\sigma_j^* f := f \circ \sigma_j$. We will say that f is σ_j-even (resp. σ_j-odd) if $\sigma_j^* f = f$ (resp. $\sigma_j^* f = -f$) almost everywhere. On defining

$$\mathrm{Sym}_j(f) := \frac{f + \sigma_j^* f}{2} \qquad \text{and} \qquad \mathrm{Skew}_j(f) := \frac{f - \sigma_j^* f}{2}, \tag{2.2.2}$$

every such f admits

$$f = \mathrm{Sym}_j(f) + \mathrm{Sym}_j(f) \tag{2.2.3}$$

as its unique decomposition into a σ_j-even and a σ_j-odd part. For every $i, j \in \{1, \dots, d\}$, σ_i and σ_j commute. Therefore, so do the operator pairs (σ_i^*, σ_j^*), $(\mathrm{Sym}_i, \mathrm{Sym}_j)$ and $(\mathrm{Sym}_i, \mathrm{Skew}_j)$. It follows that

$$f = \mathrm{Sym}_i(\mathrm{Sym}_j(f)) + \mathrm{Sym}_i(\mathrm{Skew}_j(f)) + \mathrm{Skew}_i(\mathrm{Sym}_j(f)) + \mathrm{Skew}_i(\mathrm{Skew}_j(f)) \tag{2.2.4}$$

is the only decomposition of f into all four combinations of σ_i- and σ_j-parity. Following [17, Def. 6.4.4], we further introduce the operators ρ_j by

$$\rho_j(f)(x) := \frac{f(x) - f(\sigma_j x)}{x_j} = \frac{2\ \mathrm{Skew}_j(f)(x)}{x_j}, \tag{2.2.5}$$

where, whenever $x_j = 0$, the ratio must be interpreted as the corresponding limit; namely, $2\,\partial_j f(x)$. The following variant of Hadamard's lemma (cf. [39, Sec. 3.20]) encapsulates the properties of the ρ_j operators we shall need later.

Proposition 2.2.1. *Let $j \in \{1, \dots, d\}$ and $f \in \mathrm{C}^r(\overline{B^d})$, $r \geq 1$. Then, $\rho_j(f) \in \mathrm{C}^{r-1}(\overline{B^d})$ and, for all multi-indices a with $0 \leq |a| \leq r-1$,*

$$\|\partial_a \rho_j(f)\|_\infty \leq 2\,\|\partial_a \partial_j f\|_\infty . \tag{2.2.6}$$

If f happens to be a polynomial of degree n, $\rho_j(f)$ is also a polynomial of degree at most $n-1$.

Proof. Throughout this proof, for all $z \in \overline{B^d}$ we set $z' = (z_1, \dots, z_{d-1})$ and $z'' = (z_1, \dots, z_{d-2})$ so that $z = (z', z_d) = (z'', z_{d-1}, z_d)$. Also, given a function $h\colon \overline{B^d} \to \mathbb{R}$ we denote its modulus of continuity by $\omega(\,\cdot\,; h)$; that is, for all $t \in [0,\infty]$, $\omega(t;h) := \sup\left\{|h(x) - h(y)| \mid x, y \in \overline{B^d},\ |x-y| \leq t\right\}$. We also assume, without loss of generality, that $j = d$.

Given $k \in \mathbb{N}_0$ let the integral operator H_k be defined by

$$H_k(h)(x) := \int_{-1}^{1} s^k\, h(x', s\, x_d)\, \mathrm{d}s. \tag{2.2.7}$$

First, let us note that

$$(\forall\, h \in \mathrm{C}(\overline{B^d})) \quad H_k(h) \in \mathrm{C}(\overline{B^d}). \tag{2.2.8}$$

Indeed, let $h \in \mathrm{C}(\overline{B^d})$. Then, for all $x, y \in \overline{B^d}$,

$$|H_k(h)(x) - H_k(h)(y)| \leq \int_{-1}^{1} |h(x', s\, x_d) - h(y', s\, y_d)|\ \mathrm{d}s \leq 2\,\omega(|x-y|\,; h).$$

Thus, $0 \leq \omega(\,\cdot\,; H_k(h)) \leq 2\,\omega(\,\cdot\,; h)$ so $H_k(h)$ inherits the uniform continuity of h, which, in turn, stems from the fact that $\overline{B^d}$ is compact. Also, directly from the definition (2.2.7),

$$(\forall\, h \in \mathrm{C}(\overline{B^d})) \quad \|H_k(h)\|_\infty \leq 2\,\|h\|_\infty . \tag{2.2.9}$$

Next, we note that, as a consequence of the Fundamental Theorem of Calculus, for all $i \in$

$\{1,\dots,d\}$,

$$(\forall\, h\in \mathrm{C}^1(\overline{B^d}))\qquad \left|\frac{h(x+\eta\, e_i)-h(x)}{\eta}-\partial_i h(x)\right|\le \omega(|\eta|\,;\partial_i h). \tag{2.2.10}$$

Further, we affirm that

$$(\forall\, h\in \mathrm{C}^1(\overline{B^d}))\quad \partial_i H_k(h)=\begin{cases} H_k(\partial_i h) & \text{if } i\ne d,\\ H_{k+1}(\partial_d h) & \text{if } i=d.\end{cases} \tag{2.2.11}$$

Indeed, let $h\in \mathrm{C}^1(\overline{B^d})$. Let $i\in\{1,\dots,d-1\}$; without loss of generality we can assume that $i=d-1$. Then,

$$\begin{aligned}&\left|\frac{H_k(h)(x+\eta e_{d-1})-H_k(h)(x)}{\eta}-H_k(\partial_{d-1}h)(x)\right|\\ &\qquad\qquad\le \int_{-1}^{1}\left|\frac{h(x'',x_{d-1}+\eta,s\,x_d)-h(x'',x_{d-1},s\,x_d)}{\eta}-\partial_{d-1}h(x',s\,x_d)\right|\,\mathrm{d}s \xrightarrow{\eta\to 0} 0\end{aligned}$$

because, per (2.2.10), the last integrand tends to 0 as η tends to 0 uniformly with respect to s. If $i=d$,

$$\begin{aligned}&\left|\frac{H_k(h)(x+\eta e_d)-H_k(h)(x)}{\eta}-H_{k+1}(\partial_d h)(x)\right|\\ &\qquad\qquad\le \int_{-1}^{1}\left|\frac{h(x',s(x_d+\eta))-h(x',s\,x_d)}{\eta}-s\,\partial_d h(x',s\,x_d)\right|\,\mathrm{d}s\\ &\qquad\qquad\qquad\le \int_{-1}^{1}\left|\frac{h(x',s\,x_d+s\,\eta)-h(x',s\,x_d)}{s\,\eta}-\partial_d h(x',s\,x_d)\right|\,\mathrm{d}s \xrightarrow{\eta\to 0} 0,\end{aligned}$$

again by (2.2.10) and the fact that $|s\,\eta|\le|\eta|$. Thus we have justified (2.2.11).

Let $f\in \mathrm{C}^r(\overline{B^d})$. Then, $\rho_d(f)=H_0(\partial_d f)$. Indeed, if $x_d=0$, $\rho_d(f)(x)=2\,\partial_d f(x)$ and $H_0(\partial_d f)(x)$ obviously coincide. If $x_d\ne 0$, by the Fundamental Theorem of Calculus and the definition in (2.2.5),

$$\rho_d(f)(x)=\frac{1}{x_d}\int_{-x_d}^{x_d}\partial_d f(x',t)\,\mathrm{d}t=\int_{-1}^{1}\partial_d f(x',s\,x_d)\,\mathrm{d}s=H_0(\partial_d f)(x).$$

With $\rho_d f$ characterized in this way, its membership in $\mathrm{C}^{r-1}(\overline{B^d})$ and the bound (2.2.6) stem from (2.2.8), (2.2.9) and (2.2.11).

Let us note that if h happens to be the monomial $h(x) = \prod_{i=1}^{d} x_i^{\alpha_i}$, $\alpha_1, \ldots, \alpha_d \in \mathbb{N}_0$, a direct computation shows that $H_0(h) = \frac{1-(-1)^{\alpha_d+1}}{\alpha_d+1} h$. Thus, H_0 maps polynomials to polynomials of at most the same total degree. Hence, if f is a polynomial of total degree n, $\rho_d(f) = H_0(\partial_d f)$ is a polynomial of total degree at most $n-1$. $\square$

Given any $\gamma \in \mathbb{R}^d$, the map that to each e_j and $-e_j$, $j \in \{1, \ldots, d\}$ associates γ_j is $\mathbb{Z}_2^d$ invariant, so it is a multiplicity function in the sense of [17, Def. 6.4.1]. The Dunkl operators associated with (the multiplicity function induced by) γ [17, Def. 6.4.2] are

$$(\forall\, j \in \{1, \ldots, d\}) \quad \mathcal{D}_j^{(\gamma)} q(x) := \partial_j q(x) + \frac{\gamma_j}{2} \rho_j(q)(x) \overset{(2.2.5)}{=} \partial_j q(x) + \frac{\gamma_j}{2} \frac{q(x) - q(\sigma_j x)}{x_j}. \quad (2.2.12)$$

By simple computation, we deduce that given differentiable functions p and q,

$$\mathcal{D}_j^{(\gamma)}(p\, q)(x) = q(x) \mathcal{D}_j^{(\gamma)} p(x) + p(x) \partial_j q(x) + \frac{\gamma_j}{2} p(\sigma_j x) \frac{q(x) - q(\sigma_j x)}{x_j}. \quad (2.2.13)$$

Through Proposition 2.2.1 the Dunkl operators inherit from the standard partial derivatives the inclusions

$$\mathcal{D}_j^{(\gamma)} \left(\mathrm{C}^m(\overline{B^d}) \right) \subseteq \mathrm{C}^{m-1}(\overline{B^d}) \quad \text{and} \quad \mathcal{D}_j^{(\gamma)} \left(\Pi_m^d \right) \subseteq \Pi_{m-1}^d \quad (2.2.14)$$

for $m \in \mathbb{N}$ and $m \in \mathbb{N}_0$, respectively.

The following commutation relations are particularizations of Prop. 6.4.3, Th. 6.4.9 and Prop. 6.4.10 of [17], respectively:

$$\mathcal{D}_j^{(\gamma)} \sigma_i^* = \begin{cases} \sigma_i^* \mathcal{D}_j^{(\gamma)} & \text{if } i \neq j, \\ -\sigma_j^* \mathcal{D}_j^{(\gamma)} & \text{if } i = j, \end{cases} \quad (2.2.15)$$

$$\mathcal{D}_i^{(\gamma)} \mathcal{D}_j^{(\gamma)} = \mathcal{D}_j^{(\gamma)} \mathcal{D}_i^{(\gamma)}, \quad (2.2.16)$$

$$\mathcal{D}_j^{(\gamma)}(x_i q) = \begin{cases} x_i \mathcal{D}_j^{(\gamma)} q & \text{if } i \neq j, \\ x_j \mathcal{D}_j^{(\gamma)} q + q + \gamma_j \sigma_j^* q & \text{if } i = j. \end{cases} \quad (2.2.17)$$

Note that in (2.2.17) and in the sequel we commit the common abuse of notation of denoting maps of the form $x \mapsto x_i q(x)$ simply as $x_i q$. Some consequences of (2.2.15) are

$$\mathcal{D}_j^{(\gamma)} \operatorname{Sym}_i = \begin{cases} \operatorname{Sym}_i \mathcal{D}_j^{(\gamma)} & \text{if } i \neq j, \\ \operatorname{Skew}_i \mathcal{D}_j^{(\gamma)} & \text{if } i = j \end{cases} \quad \text{and} \quad \mathcal{D}_j^{(\gamma)} \operatorname{Skew}_i = \begin{cases} \operatorname{Skew}_i \mathcal{D}_j^{(\gamma)} & \text{if } i \neq j, \\ \operatorname{Sym}_i \mathcal{D}_j^{(\gamma)} & \text{if } i = j. \end{cases} \tag{2.2.18}$$

Also, as

$$x_j \sigma_i^* q = \begin{cases} \sigma_i^*(x_j q) & \text{if } i \neq j, \\ -\sigma_j^*(x_j q) & \text{if } i = j, \end{cases} \tag{2.2.19}$$

we further have

$$x_j \operatorname{Sym}_i q = \begin{cases} \operatorname{Sym}_i(x_j q) & \text{if } i \neq j, \\ \operatorname{Skew}_i(x_j q) & \text{if } i = j \end{cases} \quad \text{and} \quad x_j \operatorname{Skew}_i q = \begin{cases} \operatorname{Skew}_i(x_j q) & \text{if } i \neq j, \\ \operatorname{Sym}_i(x_j q) & \text{if } i = j. \end{cases} \tag{2.2.20}$$

Because of the commutation property (2.2.16), we can unambiguously use the multi-index notation to express compositions of Dunkl operators; hence, given $a \in [\mathbb{N}_0]^d$, we shall write $\mathcal{D}_a^{(\gamma)} := (\mathcal{D}_1^{(\gamma)})^{a_1} \circ \cdots \circ (\mathcal{D}_d^{(\gamma)})^{a_d}$. We can now compactly express the following consequence of Proposition 2.2.1: For all multi-indices $a \in [\mathbb{N}_0]^d$ and $f \in \mathrm{C}^{|a|}(\overline{B^d})$,

$$\left\| D_a^{(\gamma)} f \right\|_\infty \leq \prod_{i=1}^{d} (1 + |\gamma_i|)^{a_i} \, \|\partial_a f\|_\infty . \tag{2.2.21}$$

We define the Dunkl gradient by $\mathcal{D}^{(\gamma)} f := \sum_{j=1}^{d} \mathcal{D}_j^{(\gamma)}(f)\, e_j$. Given $m \in \mathbb{N}_0$ we define the Sobolev-type inner product $\langle \cdot, \cdot \rangle_{\alpha,\gamma;m} \colon \mathrm{C}^m(\overline{B^d}) \times \mathrm{C}^m(\overline{B^d}) \to \mathbb{R}$ by

$$(\forall\, p, q \in \mathrm{C}^m(\overline{B^d})) \quad \langle p, q \rangle_{\alpha,\gamma;m} := \sum_{k=0}^{m} \left\langle (\mathcal{D}^{(\gamma)})^k p, (\mathcal{D}^{(\gamma)})^k q \right\rangle_{\alpha,\gamma}, \tag{2.2.22}$$

where $(\mathcal{D}^{(\gamma)})^k$ is the k-fold Dunkl gradient. Using the multi-index notation, this inner product can also be expressed as $(p, q) \mapsto \sum_{k=0}^{m} \sum_{|a|=k} \binom{k}{a} \langle \mathcal{D}_a^{(\gamma)} p, \mathcal{D}_a^{(\gamma)} q \rangle_{\alpha,\gamma}$ (here $\binom{k}{a} = \frac{k!}{a_1! \cdots a_d!}$ is the number of times $\mathcal{D}_a^{(\gamma)} p$ with $|a| = k$ appears in the k-dimensional array-valued $(\mathcal{D}^{(\gamma)})^k p$) and is of course bounded from above and below by positive-constant multiples of $(p, q) \mapsto$

$\sum_{|a|\le m}\langle \mathcal{D}_a^{(\gamma)} p, \mathcal{D}_a^{(\gamma)} q\rangle_{\alpha,\gamma}$.

We define now in some detail the function spaces involved in our main result Theorem 2.1.1.

Definition 2.2.2. Given $m \in \mathbb{N}_0$, we define $\mathrm{H}^m_{\alpha,\gamma}$ as the topological completion of $(\mathrm{C}^m(\overline{B^d}), \|\cdot\|_{\alpha,\gamma;m})$.

That is, up to isometry, $\mathrm{H}^m_{\alpha,\gamma}$ is the space of equivalence classes of Cauchy sequences of $(\mathrm{C}^m(\overline{B^d}), \|\cdot\|_{\alpha,\gamma;m})$ with respect to the equivalence relation $\sim$ defined by $(x_n)_{n\in\mathbb{N}} \sim (y_n)_{n\in\mathbb{N}} \iff \lim_{n\to\infty} \|x_n - y_n\|_{\alpha,\gamma;m} = 0$, equipped with the metric $(x, y) \mapsto \lim_{n\to\infty} \|x_n - y_n\|_{\alpha,\gamma;m}$, where $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are any representatives of the equivalence classes x and y, respectively, which makes it a complete metric space. Identifying each $f \in \mathrm{C}^m(\overline{B^d})$ with the equivalence class of the constant sequence $(f)_{n\in\mathbb{N}}$, $\mathrm{C}^m(\overline{B^d})$ is a dense subset of $\mathrm{H}^m_{\alpha,\gamma}$ [32, Th. III.33.VII], [18, Th. 4.3.19].

It is easily checked that the map $(x, y) \mapsto \lim_{n\to\infty}\langle x_n, y_n\rangle_{\alpha,\gamma;m}$, where again $(x_n)_{n\in\mathbb{N}}$ and $(y_n)_{n\in\mathbb{N}}$ are any representatives of the equivalence classes x and y, respectively, is a well defined inner product that induces the above metric, whence $\mathrm{H}^m_{\alpha,\gamma}$ is a Hilbert space. We denote that inner product by $\langle\cdot,\cdot\rangle_{\alpha,\gamma;m}$ as well.

Proposition 2.2.3. *Polynomials are dense in* $\mathrm{H}^m_{\alpha,\gamma}$.

Proof. Let $f \in \mathrm{H}^m_{\alpha,\gamma}$ and $\epsilon > 0$. By the characterization of $\mathrm{H}^m_{\alpha,\gamma}$ as a topological completion in Definition 2.2.2, there exists $g \in \mathrm{C}^m(\overline{B^d})$ such that $\|f - g\|_{\alpha,\gamma;m} < \epsilon/2$. Now, g can be extended to a $\mathrm{C}^m(\mathbb{R}^d)$ function $\tilde{g}$ [46], which, by smooth truncation if necessary, can be assumed to have its support contained in the ball $B(0, 2)$. By [20, Cor. 3], there exists a polynomial p such that

$$\sum_{|a|\le m} \sup_{B^d} |\partial_a g - \partial_a p| = \sum_{|a|\le m} \sup_{B^d} |\partial_a \tilde{g} - \partial_a p| < \frac{\epsilon}{2\, c_{d,m}},$$

where $c_{d,m} = \|1\|^{1/2}_{\alpha,\gamma} \max_{|\alpha|\le m} \binom{|a|}{a}^{1/2} \max_{|\alpha|\le m} \prod_{i=1}^d (1+|\gamma_i|)^{\alpha_i}$ (this constant is finite on account of (2.1.3)). Thus, by (2.2.21) and the definition (2.2.22),

$$\|g - p\|_{\alpha,\gamma;m} \le \|1\|^{1/2}_{\alpha,\gamma} \max_{|\alpha|\le m} \binom{|a|}{a}^{1/2} \left(\sum_{|\alpha|\le m} \left\| D_a^{(\gamma)} g - D_a^{(\gamma)} p \right\|_\infty^2 \right)^{1/2}$$

$$\leq \|1\|_{\alpha,\gamma}^{1/2} \max_{|\alpha|\leq m} \binom{|a|}{a}^{1/2} \max_{|\alpha|\leq m} \prod_{i=1}^{d} (1+|\gamma_i|)^{\alpha_i} \sum_{|\alpha|\leq m} \|\partial_a g - \partial_a p\|_\infty < \frac{\epsilon}{2}.$$

□

Remark 2.2.4. We define our Dunkl–Sobolev spaces as topological completions of strongly differentiable functions with respect to the chosen norm; that is, 'H' spaces in the nomenclature of Meyers & Serrin [36]. One might also define Dunkl–Sobolev spaces intrinsically, as spaces of (classes of equivalence of) $L^2_{\alpha,\gamma}$ functions whose Dunkl operators up to a certain order still belong to $L^2_{\alpha,\gamma}$; i.e., 'W' spaces in the nomenclature of [36]. To the latter end distributional generalizations of the Dunkl operators (see, e.g., [45, Th. 4.4]) might be required to properly define their action on non-differentiable functions. However, we do not know if such 'W' spaces would be appropriate substitutes for (perhaps even identical to) our 'H' spaces.

2.3 Orthogonal polynomial spaces

Let $\mathcal{V}_k^{(\alpha,\gamma)}$ be the space of orthogonal polynomials of degree k with respect to the weight $W_{\alpha,\gamma}$; i.e.,

$$\mathcal{V}_k^{(\alpha,\gamma)} := \left\{ p \in \Pi_k^d \mid (\forall\, q \in \Pi_{k-1}^d)\ \langle p, q\rangle_{\alpha,\gamma} = 0 \right\}. \tag{2.3.1}$$

If $k < 0$ we adopt the convention $\Pi_k^d = \{0\}$ and so $\mathcal{V}_k^{(\alpha,\gamma)} = \{0\}$. As $W_{\alpha,\gamma}$ is centrally symmetric, it transpires from [17, Th. 3.3.11] that, for all $k \in \mathbb{N}_0 = \{0, 1, 2, \dots\}$, there holds the following parity relation:

$$(\forall\, p_k \in \mathcal{V}_k^{(\alpha,\gamma)})\ (\forall\, x \in B^d) \quad p_k(-x) = (-1)^k p_k(x). \tag{2.3.2}$$

There holds (cf. [17, Sec. 3.1])

$$\dim(\mathcal{V}_n^{\alpha,\gamma}) = \dim(\Pi_n^d) - \dim(\Pi_{n-1}^d) = \binom{n+d-1}{n}. \tag{2.3.3}$$

Let $\mathrm{proj}_k^{(\alpha,\gamma)}$ denote the orthogonal projection from $L^2_{\alpha,\gamma}$ onto $\mathcal{V}_k^{(\alpha,\gamma)}$. From [17, Th. 3.2.18],

$\Pi_n^d = \bigoplus_{k=0}^n \mathcal{V}_k^{(\alpha,\gamma)}$ and $\mathrm{L}^2_{\alpha,\gamma} = \bigoplus_{k=0}^\infty \mathcal{V}_k^{(\alpha,\gamma)}$, whence

$$(\forall\, n \in \mathbb{N}_0) \quad S_n^{(\alpha,\gamma)} = \sum_{k=0}^n \mathrm{proj}_k^{(\alpha,\gamma)} \quad \text{and} \quad (\forall\, u \in \mathrm{L}^2_{\alpha,\gamma})\ u = \sum_{k=0}^\infty \mathrm{proj}_k^{(\alpha,\gamma)}(u). \tag{2.3.4}$$

We mention in passing that we will denote the entrywise application of $S_n^{(\alpha,\gamma)}$ to $\mathrm{L}^2_{\alpha,\gamma}$ vectors and higher-order tensors by $S_n^{(\alpha,\gamma)}$ as well. Parseval's identity takes the form

$$\left(\forall\, u \in \mathrm{L}^2_{\alpha,\gamma}\right) \quad \|u\|^2_{\alpha,\gamma} = \sum_{k=0}^\infty \left\|\mathrm{proj}_k^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma}. \tag{2.3.5}$$

The following proposition, analogous to [22, Prop. 3.1], collects relations between orthogonal polynomial spaces and projectors onto them that do not involve Dunkl operators.

Proposition 2.3.1. *Let $\alpha \in (-1,\infty)$ and $\gamma \in (-1,\infty)^d$.*

(i) Let $p_k \in \mathcal{V}_k^{(\alpha+1,\gamma)}$. Then, $(1-\|\cdot\|^2)p_k \in \mathcal{V}_k^{(\alpha,\gamma)} \oplus \mathcal{V}_{k+2}^{(\alpha,\gamma)}$.

(ii) Let $q_k \in \mathcal{V}_k^{(\alpha,\gamma)}$. Then, $q_k = \mathrm{proj}_{k-2}^{(\alpha+1,\gamma)}(q_k) + \mathrm{proj}_k^{(\alpha+1,\gamma)}(q_k)$.

(iii) Let $u \in \mathrm{L}^2_{\alpha,\gamma}$. Then, $\mathrm{proj}_k^{(\alpha+1,\gamma)}(u) = \mathrm{proj}_k^{(\alpha+1,\gamma)}\left(\mathrm{proj}_k^{(\alpha,\gamma)}(u) + \mathrm{proj}_{k+2}^{(\alpha,\gamma)}(u)\right)$.

(iv) Let $u \in \mathrm{L}^2_{\alpha,\gamma}$. Then,

$$\mathrm{proj}_k^{(\alpha+1,\gamma)}(u) = \mathrm{proj}_k^{(\alpha,\gamma)}(u) + \mathrm{proj}_k^{(\alpha+1,\gamma)} \circ \mathrm{proj}_{k+2}^{(\alpha,\gamma)}(u) - \mathrm{proj}_{k-2}^{(\alpha+1,\gamma)} \circ \mathrm{proj}_k^{(\alpha,\gamma)}(u).$$

Proof. Given $q \in \Pi_{k-1}^d$, $\langle (1-\|\cdot\|^2)p_k, q\rangle_{\alpha,\gamma} = \langle p_k, q\rangle_{\alpha+1,\gamma} = 0$ by definition (2.3.1). Also, by the parity relation (2.3.2), $(1-\|\cdot\|^2)p_k \perp_{\alpha,\gamma} \mathcal{V}_{k+1}^{(\alpha,\gamma)}$. Therefore part (i) stems from (2.3.4). An analogous argument accounts for part (ii). Part (iii) comes from the fact that given $p_k \in \mathcal{V}_k^{(\alpha+1,\gamma)}$,

$$\begin{aligned}\langle \mathrm{proj}_k^{(\alpha+1,\gamma)}(u), p_k\rangle_{\alpha+1,\gamma} &= \langle u, p_k\rangle_{\alpha+1,\gamma} = \langle u, (1-\|\cdot\|^2)p_k\rangle_{\alpha,\gamma} \\ &\overset{\text{(i)}}{=} \langle \mathrm{proj}_k^{(\alpha,\gamma)}(u) + \mathrm{proj}_{k+2}^{(\alpha,\gamma)}(u), (1-\|\cdot\|^2)p_k\rangle_{\alpha,\gamma} = \langle \mathrm{proj}_k^{(\alpha,\gamma)}(u) + \mathrm{proj}_{k+2}^{(\alpha,\gamma)}(u), p_k\rangle_{\alpha+1,\gamma}.\end{aligned}$$

Part (iv) is obtained from adding and substracting $\mathrm{proj}_{k-2}^{(\alpha+1,\gamma)}(\mathrm{proj}_k^{(\alpha,\gamma)}(u))$ to the right-hand side of part (iii) and using part (ii). □

Proposition 2.3.2. *Let $\alpha \in (-1,\infty)$ and $\gamma \in (-1,\infty)^d$.*

(i) Let $f \in \mathrm{L}^2_{\alpha,\gamma}$ be σ_j-odd. Then, $\int_{B^d} f(x)\, W_{\alpha,\gamma}(x)\,\mathrm{d}x = 0$.

(ii) Given $k \in \mathbb{N}_0$, $j \in \{1,\dots,d\}$ and $p_k \in \mathcal{V}_k^{(\alpha,\gamma)}$, $p_k \circ \sigma_j \in \mathcal{V}_k^{(\alpha,\gamma)}$ as well.

Proof. Because of the invariance of the Lebesgue measure with respect to reflections, $\int_{B^d} f(x)\, W_{\alpha,\gamma}(x)\,\mathrm{d}x = \int_{B^d} f(\sigma_j x)\, W_{\alpha,\gamma}(\sigma_j(x))\,\mathrm{d}x$. As $W_{\alpha,\gamma}$ is σ_j-invariant, part (i) follows.

Part (ii) is proven similarly, using additionally the fact that the composition with σ_j preserves the degree of a polynomial. □

Given any $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$ we introduce the differential-difference operators $\mathcal{D}_j^{(\alpha,\gamma;\star)}$, $j \in \{1,\dots,d\}$, by

$$\begin{aligned}\mathcal{D}_j^{(\alpha,\gamma;\star)} q(x) &:= -(1-\|x\|^2)^{-\alpha}\, \mathcal{D}_j^{(\gamma)}\Big((1-\|x\|^2)^{\alpha+1} q(x)\Big)\\ &= -(1-\|x\|^2)\, \mathcal{D}_j^{(\gamma)} q(x) + 2(\alpha+1) x_j q(x). \end{aligned} \tag{2.3.6}$$

From the inclusions in (2.2.14) they inherit

$$\mathcal{D}_j^{(\alpha,\gamma;\star)}\left(\mathrm{C}^m(\overline{B^d})\right) \subseteq \mathrm{C}^{m-1}(\overline{B^d}) \quad \text{and} \quad \mathcal{D}_j^{(\alpha,\gamma;\star)}\left(\Pi^d_m\right) \subseteq \Pi^d_{m+1} \tag{2.3.7}$$

for $m \in \mathbb{N}$ and $m \in \mathbb{N}_0$, respectively. Also, from (2.2.18) and (2.2.20),

$$\mathcal{D}_j^{(\alpha,\gamma;\star)}\,\mathrm{Sym}_i = \begin{cases}\mathrm{Sym}_i\, \mathcal{D}_j^{(\alpha,\gamma;\star)} & \text{if } i \neq j,\\ \mathrm{Skew}_i\, \mathcal{D}_j^{(\alpha,\gamma;\star)} & \text{if } i = j\end{cases} \quad \text{and}$$
$$\mathcal{D}_j^{(\alpha,\gamma;\star)}\,\mathrm{Skew}_i = \begin{cases}\mathrm{Skew}_i\, \mathcal{D}_j^{(\alpha,\gamma;\star)} & \text{if } i \neq j,\\ \mathrm{Sym}_i\, \mathcal{D}_j^{(\alpha,\gamma;\star)} & \text{if } i = j.\end{cases} \tag{2.3.8}$$

As its notation suggests, the $\mathcal{D}_j^{(\alpha,\gamma;\star)}$ operator is indeed adjoint to the Dunkl operator $\mathcal{D}_j^{(\gamma)}$, to the extent allowed by the first part of the following proposition, analogous to [22, Prop. 3.2], that also goes on to show that $\mathcal{D}_j^{(\alpha,\gamma;\star)}$ is a parameter-lowering and degree-raising operator, that $\mathcal{D}_j^{(\gamma)}$ is a parameter-raising and degree-lowering operator and a useful commutation relation between projections onto orthogonal polynomials spaces and a Dunkl operator.

Proposition 2.3.3. *Let $\alpha \in (-1,\infty)$, $\gamma \in (-1,\infty)^d$ and $j \in \{1,\dots,d\}$.*

(i) Let $p, q \in \mathrm{C}^1(\overline{B^d})$. Then, $\langle \mathcal{D}_j^{(\gamma)} p, q\rangle_{\alpha+1,\gamma} = \langle p, \mathcal{D}_j^{(\alpha,\gamma;\star)} q\rangle_{\alpha,\gamma}$.

(ii) Let $r_k \in \mathcal{V}_k^{\alpha+1,\gamma}$. Then, $\mathcal{D}_j^{(\alpha,\gamma;\star)} r_k \in \mathcal{V}_{k+1}^{\alpha,\gamma}$.

(iii) Let $p_k \in \mathcal{V}_k^{\alpha,\gamma}$. Then, $\mathcal{D}_j^{(\gamma)} p_k \in \mathcal{V}_{k-1}^{\alpha+1,\gamma}$.

(iv) Let $u \in \mathrm{C}^1(\overline{B^d})$. Then, $\mathcal{D}_j^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u) = \operatorname{proj}_{k-1}^{(\alpha+1,\gamma)}(\mathcal{D}_j^{(\gamma)} u)$.

Proof. As both $\mathcal{D}_j^{(\gamma)}$ and $\mathcal{D}_j^{(\alpha,\gamma;\star)}$ flip σ_j-symmetry into σ_j-antisymmetry and vice versa (cf. (2.2.18) and (2.3.8)), per part (i) of Proposition 2.3.2, it is enough to prove (i) in the special cases where p and q are either σ_j-even and σ_j-odd or σ_j-odd and σ_j-even, respectively. Let us define, for $\delta, \varepsilon > 0$, the set $X_{\delta,\varepsilon} := \{x \in B^d \mid |x_j| > \delta \ \wedge \ (\forall\, i \in \{1,\dots,d\}) \setminus \{j\} \ \ |x_i| > \varepsilon\}$. By integration by parts,

$$\begin{aligned}\int_{X_{\delta,\varepsilon}} \partial_j p(x) q(x) W_{\alpha+1,\gamma}(x)\,\mathrm{d}x \\ = \underbrace{\int_{\partial X_{\delta,\varepsilon}} p(x) q(x) W_{\alpha+1,\gamma}(x)\nu_j(x)\,\mathrm{d}S(x)}_{:=b_{\delta,\varepsilon}} - \int_{X_{\delta,\varepsilon}} p(x)\partial_j(q(x)W_{\alpha+1,\gamma}(x))\,\mathrm{d}x, \quad (2.3.9)\end{aligned}$$

where ν is the outher normal vector field defined almost anywhere (with respect to the surface measure) on $\partial X_{\delta,\varepsilon}$. Now, for every $x \in X_{\delta,\varepsilon}$, by direct computation

$$\begin{aligned}\partial_j(q(x)W_{\alpha+1,\gamma}(x)) \\ = \left(\partial_j q(x)(1-\|x\|^2) - 2(\alpha+1)x_j q(x)\right) W_{\alpha,\gamma}(x) + \frac{\gamma_j}{x_j} q(x) W_{\alpha+1,\gamma}(x). \quad (2.3.10)\end{aligned}$$

From the definition (2.2.12) of $\mathcal{D}_j^{(\gamma)}$, (2.3.9) and (2.3.10),

$$\int_{X_{\delta,\varepsilon}} \mathcal{D}_j^{(\gamma)} p(x)\, q(x)\, W_{\alpha+1,\gamma}(x)\, \mathrm{d}x$$
$$= b_{\delta,\varepsilon} - \int_{X_{\delta,\varepsilon}} p(x) \left(\partial_j q(x)\,(1-\|x\|^2) - 2(\alpha+1) x_j\, q(x)\right) W_{\alpha,\gamma}(x)\, \mathrm{d}x$$
$$- \frac{\gamma_j}{2} \int_{X_{\delta,\varepsilon}} \frac{p(x)+p(\sigma_j x)}{x_j} q(x)\, W_{\alpha+1,\gamma}(x)\, \mathrm{d}x. \quad (2.3.11)$$

As $X_{\delta,\varepsilon}$ and $W_{\alpha,\gamma}$ are σ_j-invariant, a simple computation shows that

$$\int_{X_{\delta,\varepsilon}} \frac{p(x)+p(\sigma_j x)}{x_j} q(x)\, W_{\alpha+1,\gamma}(x)\, \mathrm{d}x = \int_{X_{\delta,\varepsilon}} p(x) \frac{q(x)-q(\sigma_j x)}{x_j}\, W_{\alpha+1,\gamma}(x)\, \mathrm{d}x,$$

which, substituted into (2.3.11), results in (cf. (2.3.6))

$$\int_{X_{\delta,\varepsilon}} \mathcal{D}_j^{(\gamma)} p(x)\, q(x)\, W_{\alpha+1,\gamma}(x)\, \mathrm{d}x = b_{\delta,\varepsilon} + \int_{X_{\delta,\varepsilon}} p(x)\, \mathcal{D}_j^{(\alpha,\gamma;\star)} q(x)\, W_{\alpha,\gamma}(x)\, \mathrm{d}x. \quad (2.3.12)$$

As $W_{\alpha+1,\gamma}$ vanishes on $\partial X_{\delta,\varepsilon} \cap \mathbb{S}^{d-1}$ and ν_j vanishes almost everywhere on each of the sets $\{x \in \partial X_{\delta,\varepsilon} \mid |x_i| = \varepsilon\}$ for $i \in \{1,\ldots,d\} \setminus \{j\}$, the boundary integral in (2.3.9), (2.3.11) and (2.3.12) can be written as

$$b_{\delta,\varepsilon} = \int_{\left\{x \in \partial X_{\delta,\varepsilon} \mid |x_j| = \delta\right\}} p(x)\, q(x)\, W_{\alpha+1,\gamma}(x)\, \mathrm{sign}(x_j)\, \mathrm{d}S(x)$$
$$= \int_{\left\{x \in \partial X_{\delta,\varepsilon} \mid |x_j| = \delta\right\}} \frac{p(x)\, q(x)}{x_j}\, W_{\alpha+1,\gamma+e_j}(x)\, \mathrm{d}S(x).$$

Since pq is σ_j-odd, we infer from Proposition 2.2.1 that $x \mapsto p(x)q(x)/x_j = \rho_j(pq)/2$ belongs to $\mathrm{C}(\overline{B^d})$. Also, as $\alpha+1>0$, $(1-\|x\|^2)^{\alpha+1} \le 1$ for all x in the integration domain above. Additionally, said integration domain is contained in $\{x \in [-1,1]^d \mid |x_j| = \delta\}$. Thus,

$$|b_{\delta,\varepsilon}| \le \delta^{\gamma_j+1} \max_{x \in \overline{B^d}} \left| \frac{p(x)q(x)}{x_j} \right| \prod_{\substack{i=1 \\ i \ne j}}^{d} \int_{[-1,1]} |x_i|^{\gamma_i}\, \mathrm{d}x_i.$$

Then, as $\gamma_i > -1$ for $i \in \{1,\ldots,d\}$, for every fixed ε, $\lim_{\delta \to 0^+} b_{\varepsilon,\delta} = 0$. Then, (i) follows from (2.3.12) by first taking the limit as $\delta \to 0^+$ (which makes the boundary integral disappear) and then the limit as $\varepsilon \to 0^+$ (the volume integrals over $X_{\delta,\epsilon}$ converging to the corresponding ones over B^d by the dominated convergence theorem)

Given $r_k \in \mathcal{V}_k^{\alpha+1,\gamma}$, by (2.3.7), $\mathcal{D}_j^{(\alpha,\gamma;\star)} r_k \in \Pi_{k+1}^d$, and, on account of part (i), the latter is $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal to Π_k^d, whence part (ii). An analogous argument accounts for part (iii).

Given $u \in \mathrm{C}^1(\overline{B^d})$, by part (iii), $\mathcal{D}_j^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u) \in \mathcal{V}_{k-1}^{(\alpha+1,\gamma)}$. Part (iv) then comes about from the fact that for all $r \in \mathcal{V}_{k-1}^{(\alpha+1,\gamma)}$,

$$\langle \mathcal{D}_j^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u), r\rangle_{\alpha+1,\gamma} \overset{\text{(i)}}{=} \langle \operatorname{proj}_k^{(\alpha,\gamma)}(u), \mathcal{D}_j^{(\alpha,\gamma;\star)} r\rangle_{\alpha,\gamma} \overset{\text{(ii)}}{=} \langle u, \mathcal{D}_j^{(\alpha,\gamma;\star)} r\rangle_{\alpha,\gamma} \overset{\text{(i)}}{=} \langle \mathcal{D}_j^{(\gamma)} u, r\rangle_{\alpha+1,\gamma}.$$

□

Given $\gamma \in \mathbb{R}^d$ we introduce the differential-difference operators $\mathcal{D}_{i,j}^{(\gamma)}$, $i, j \in \{1, \ldots, d\}$, by

$$\mathcal{D}_{i,j}^{(\gamma)} := x_i \, \mathcal{D}_j^{(\gamma)} - x_j \, \mathcal{D}_i^{(\gamma)}. \tag{2.3.13}$$

Under this definition, the $\mathcal{D}_{i,i}^{(\gamma)}$ operators are simply the null operator. If $\gamma = 0$ and $i < j$, the $\mathcal{D}_{i,j}^{(\gamma)}$ operators are angular derivatives [13, Sec. 1.8].

The following proposition shows that this operator is minus its adjoint in a certain sense, that the orthogonal polynomials spaces are parameter- and degree-invariant with respect to this operator and a commutation relation involving this operator and projectors onto the same orthogonal polynomial spaces.

Proposition 2.3.4. *Let $\alpha \in (-1, \infty)$, $\gamma \in (-1, \infty)^d$, $i, j \in \{1, \ldots, d\}$.*

(i) Let $p, q \in \mathrm{C}^1(\overline{B^d})$. Then, $\langle \mathcal{D}_{i,j}^{(\gamma)} p, q\rangle_{\alpha,\gamma} = -\langle p, \mathcal{D}_{i,j}^{(\gamma)} q\rangle_{\alpha,\gamma}$.

(ii) Let $p_k \in \mathcal{V}_k^{(\alpha,\gamma)}$. Then, $\mathcal{D}_{i,j}^{(\gamma)} p_k \in \mathcal{V}_k^{(\alpha,\gamma)}$.

(iii) Let $u \in \mathrm{C}^1(\overline{B^d})$. Then, $\mathcal{D}_{i,j}^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u) = \operatorname{proj}_k^{(\alpha,\gamma)}(\mathcal{D}_{i,j}^{(\gamma)} u)$.

Proof. In the non-trivial case $i \neq j$, we infer from the commutation relations (2.2.18) and (2.2.20) and the definition (2.3.13) that the operator $\mathcal{D}_{i,j}^{(\gamma)}$ flips both the σ_i-parity and the σ_j parity of each term in the four-way decomposition (2.2.4) of p. Then, by part (i) of Proposition 2.3.2,

$$\langle \mathcal{D}_{i,j}^{(\gamma)} p, q\rangle_{\alpha,\gamma} = \langle \mathcal{D}_{i,j}^{(\gamma)} (\operatorname{Sym}_i \operatorname{Sym}_j p), \operatorname{Skew}_i \operatorname{Skew}_j q\rangle_{\alpha,\gamma} + \langle \mathcal{D}_{i,j}^{(\gamma)} (\operatorname{Sym}_i \operatorname{Skew}_j p), \operatorname{Skew}_i \operatorname{Sym}_j q\rangle_{\alpha,\gamma}$$
$$+ \langle \mathcal{D}_{i,j}^{(\gamma)} (\operatorname{Skew}_i \operatorname{Sym}_j p), \operatorname{Sym}_i \operatorname{Skew}_j q\rangle_{\alpha,\gamma} + \langle \mathcal{D}_{i,j}^{(\gamma)} (\operatorname{Skew}_i \operatorname{Skew}_j p), \operatorname{Sym}_i \operatorname{Sym}_j q\rangle_{\alpha,\gamma}.$$

Thus, it is enough to consider the special cases in which p and q are simultaneously of opposite σ_i- and σ_j-parity. Those cases, in turn, are covered by the supposition that pq is simultaneously σ_i-odd and σ_j-odd, which we adopt from now on.

By direct computation it is rapidly checked that,

$$\begin{aligned}\langle \mathcal{D}_{i,j}^{(\gamma)} p, q\rangle_{\alpha,\gamma} + \langle p, \mathcal{D}_{i,j}^{(\gamma)} q\rangle_{\alpha,\gamma} &= \langle \mathcal{D}_{i,j}^{(0)} p, q\rangle_{\alpha,\gamma} + \langle p, \mathcal{D}_{i,j}^{(0)} q\rangle_{\alpha,\gamma} \\ &+ \int_{B^d} \left(\frac{\gamma_j}{2} x_i \frac{p(x) - p(\sigma_j x)}{x_j} - \frac{\gamma_i}{2} x_j \frac{p(x) - p(\sigma_i x)}{x_i} \right) q(x) W_{\alpha,\gamma}(x)\, \mathrm{d}x \\ &+ \int_{B^d} p(x) \left(\frac{\gamma_j}{2} x_i \frac{q(x) - q(\sigma_j x)}{x_j} - \frac{\gamma_i}{2} x_j \frac{q(x) - q(\sigma_i x)}{x_i} \right) W_{\alpha,\gamma}(x)\, \mathrm{d}x. \quad (2.3.14)\end{aligned}$$

As the purely differential operator $\mathcal{D}_{i,j}^{(0)} = x_i \partial_j - x_j \partial_i$ satisfies the relation $\mathcal{D}_{i,j}^{(0)}(pq) = \mathcal{D}_{i,j}^{(0)}(p)\, q + p\, \mathcal{D}_{i,j}^{(0)}(q)$ and vanishes on radial functions,

$$\langle \mathcal{D}_{i,j}^{(0)} p, q\rangle_{\alpha,\gamma} + \langle p, \mathcal{D}_{i,j}^{(0)}(q)\rangle_{\alpha,\gamma} = \int_{B^d} \operatorname{div} \Big(p(x) q(x) (1 - \|x\|^2)^\alpha (x_i e_j - x_j e_i) \Big) \prod_{k=1}^{d} |x_k|^{\gamma_k}\, \mathrm{d}x. \quad (2.3.15)$$

Let us define, for $\varepsilon > 0$ and $0 < r < 1$, the set $X_{r,\varepsilon} := \{x \in rB^d \mid (\forall\, k \in \{1, \dots, d\})\ |x_k| > \varepsilon\}$. By the Lebesgue dominated convergence theorem and integration by parts,

$$\begin{aligned}&\langle \mathcal{D}_{i,j}^{(0)} p, q\rangle_{\alpha,\gamma} + \langle p, \mathcal{D}_{i,j}^{(0)}(q)\rangle_{\alpha,\gamma} \\ &\qquad = \lim_{\substack{r \to 1^- \\ \varepsilon \to 0^+}} \Bigg(\int_{\partial X_{r,\varepsilon}} p(x) q(x) W_{\alpha,\gamma}(x) (x_i e_j - x_j e_i) \cdot \nu(x)\, \mathrm{d}S(x) \\ &\qquad\qquad - \underbrace{\int_{X_{r,\varepsilon}} p(x) q(x) W_{\alpha,\gamma}(x) (x_i e_j - x_j e_i) \cdot \sum_{l=1}^{d} \Big(x_l^{-1} \gamma_l\, e_l \Big)\, \mathrm{d}x}_{:= v_{r,\varepsilon}} \Bigg), \quad (2.3.16)\end{aligned}$$

where ν is the outer unit normal vector field defined almost anywhere (with respect to the surface measure we have denoted by S) on $\partial X_{r,\varepsilon}$. For $k \in \{1, \dots, d\}$, let us define the subsurfaces

$A_{r,\varepsilon,k} := \{x \in \partial X_{r,\varepsilon} \mid |x_k| = \varepsilon\}$. Then, the union $(r\mathbb{S}^{d-1} \cap \partial X_{r,\varepsilon}) \cup \bigcup_{k=1}^{d} A_{r,\varepsilon,k}$ is a decomposition of $\partial X_{r,\varepsilon}$ in sets whose pairwise intersections have zero S-measure. Now, for S-almost every $x \in r\mathbb{S}^{d-1} \cap \partial X_{r,\varepsilon}$, $\nu(x) = r^{-1}x$, which is orthogonal to $x_i e_j - x_j e_i$, and for $k \in \{1, \ldots, d\}$, for S-almost every $x \in A_{r,\varepsilon,k}$, $\nu(x) = -\operatorname{sign}(x_k)\, e_k$, which is again orthogonal to $x_i e_j - x_j e_i$ if $k \notin \{i, j\}$. Hence, on defining

$$\begin{aligned} I_{r,\varepsilon,j} &:= -\int_{A_{r,\varepsilon,j}} p(x)q(x)x_i \operatorname{sign}(x_j) W_{\alpha,\gamma}(x)\, \mathrm{d}S(x), \\ I_{r,\varepsilon,i} &:= \int_{A_{r,\varepsilon,i}} p(x)q(x)x_j \operatorname{sign}(x_i) W_{\alpha,\gamma}(x)\, \mathrm{d}S(x), \end{aligned}$$

we can express (2.3.16) as

$$\langle \mathcal{D}_{i,j}^{(0)} p, q\rangle_{\alpha,\gamma} + \langle p, \mathcal{D}_{i,j}^{(0)} q\rangle_{\alpha,\gamma} = \lim_{\substack{r\to 1^- \\ \varepsilon \to 0^+}} \left(I_{r,\varepsilon,j} + I_{r,\varepsilon,i} - v_{r,\varepsilon}\right). \tag{2.3.17}$$

As pq is σ_j-odd, by Proposition 2.2.1, $x \mapsto p(x)q(x)/x_j = \rho_j(pq)/2$ belongs to $\mathrm{C}(\overline{B^d})$. Also, for all $x \in A_{r,\varepsilon,j}$, $\|x\| \le r < 1$, which in turn implies that $(1 - \|x\|^2)^\alpha$ is bounded by $(1 - r^2)^\alpha$ if $\alpha < 0$ and by 1 if $\alpha \ge 0$. Further, $A_{r,\varepsilon,j}$ is contained in $\{x \in [-1, 1]^d \mid |x_j| = \varepsilon\}$. Thus,

$$|I_{r,\varepsilon,j}| \le \varepsilon^{\gamma_j+1} \sup_{x\in\overline{B^d}} \left|\frac{p(x)q(x)}{x_j}\right| r \begin{cases} (1-r^2)^\alpha & \text{if } \alpha < 0 \\ 1 & \text{if } \alpha \ge 0 \end{cases} \times \prod_{\substack{k=1 \\ k\ne j}}^{d} \int_{[-1,1]} |x_k|^{\gamma_k}\, \mathrm{d}x_k.$$

As all the entries of γ are greater than -1, the integrals over $[-1, 1]$ above are finite, so we can conclude that, for all $r \in (0, 1)$, $\lim_{\varepsilon\to 0^+} I_{r,\varepsilon,j} = 0$. The same argument holds for $I_{r,\varepsilon,i}$, so for all $r \in (0, 1)$, $\lim_{\varepsilon\to 0^+} I_{r,\varepsilon,i} = 0$.

By expanding the dot product in the integral in $v_{r,\varepsilon}$ (cf. (2.3.16)), judiciously expanding, say, $p = \mathrm{Sym}_i(p) + \mathrm{Skew}_i(p)$ or $p = \mathrm{Sym}_j(p) + \mathrm{Skew}_j(p)$ and changing variable through σ_i or σ_j where necessary to make $\mathrm{Sym}_i(p)$ and $\mathrm{Sym}_j(p)$ disappear and $\mathrm{Skew}_i(q)$ and $\mathrm{Skew}_j(q)$ appear, we find that

$$v_{r,\varepsilon} = \int_{X_{r,\varepsilon}} \left(\frac{\gamma_j}{2} x_i \frac{p(x) - p(\sigma_j x)}{x_j} - \frac{\gamma_i}{2} x_j \frac{p(x) - p(\sigma_i x)}{x_i}\right) q(x) W_{\alpha,\gamma}(x)\, \mathrm{d}x$$

$$+ \int_{X_{r,\varepsilon}} p(x) \left(\frac{\gamma_j}{2} x_i \frac{q(x) - q(\sigma_j x)}{x_j} - \frac{\gamma_i}{2} x_j \frac{q(x) - q(\sigma_i x)}{x_i} \right) W_{\alpha,\gamma}(x) \, \mathrm{d}x. \quad (2.3.18)$$

Therefore, substituting (2.3.18) into (2.3.17) and the result, in turn, into (2.3.14), yields (i).

Let $p_k \in \mathcal{V}_k^{(\alpha,\gamma)}$. By (2.2.14), $\mathcal{D}_{i,j}^{(\gamma)} p_k \in \Pi_k^d$, and, on account of part (i), the latter is $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal to Π_{k-1}^d, whence part (ii).

Given $u \in \mathrm{C}^1(\overline{B^d})$, by part (ii), $\mathcal{D}_{i,j}^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u) \in \mathcal{V}_k^{(\alpha,\gamma)}$. Part (iii) then follows from the fact that for all $r \in \mathcal{V}_k^{(\alpha,\gamma)}$,

$$\langle \mathcal{D}_{i,j}^{(\gamma)} \operatorname{proj}_k^{(\alpha,\gamma)}(u), r \rangle_{\alpha,\gamma} \overset{\text{(i)}}{=} -\langle \operatorname{proj}_k^{(\alpha,\gamma)}(u), \mathcal{D}_{i,j}^{(\gamma)} r \rangle_{\alpha,\gamma} \overset{\text{(ii)}}{=} -\langle u, \mathcal{D}_{i,j}^{(\gamma)} r \rangle_{\alpha,\gamma} \overset{\text{(i)}}{=} \langle \mathcal{D}_{i,j}^{(\gamma)} u, r \rangle_{\alpha,\gamma}.$$

□

2.4 Sturm–Liouville problems and approximation results

In rough terms, we will infer from the regularity of a function being approximated the weighted summability of the squared norms of its projectors onto a sequence of orthogonal polynomial spaces. In turn, this will lead to information about the approximation quality of the truncation projection $S_N^{(\alpha,\gamma)}$. In this endeavor, the characterization of orthogonal polynomial spaces as eigenspaces of a Sturm–Liouville-type operator will be essential.

From [17, Th. 8.1.3], if $\alpha > -1$ and $\gamma \in (-1,\infty)^d$, every $p_n \in \mathcal{V}_n^{\alpha,\gamma}$ satisfies

$$\mathcal{L}^{(\alpha,\gamma)}(p_n) := \left(-\Delta^{(\gamma)} + (x \cdot \nabla)^2 + 2\lambda^{\alpha,\gamma} x \cdot \nabla \right) p_n = n(n + 2\lambda^{\alpha,\gamma}) p_n, \quad (2.4.1)$$

where

$$\Delta^{(\gamma)} = \sum_{i=1}^{d} (\mathcal{D}_i^{(\gamma)})^2 \qquad \text{and} \qquad \lambda^{\alpha,\gamma} = \alpha + \frac{1}{2} \sum_{i=1}^{d} \gamma_i + \frac{d}{2}. \quad (2.4.2)$$

We will now put the operator $\mathcal{L}^{(\alpha,\gamma)}$ of (2.4.1) into a form that we can test, treat with integration-by-parts substitutes (part (i) of Proposition 2.3.3 and part (i) of Proposition 2.3.4) and turn into a transparently self-adjoint weak form.

Taking into account the second characterization in (2.3.6) defining $\mathcal{D}_j^{(\alpha,\gamma;\star)}$, it is readily checked that

$$\sum_{i=1}^{d} \mathcal{D}_i^{(\alpha,\gamma;\star)}(\mathcal{D}_i^{(\gamma)} p) = -(1-\|x\|^2)\Delta^{(\gamma)} p + 2(\alpha+1) x\cdot\nabla p + 2(\alpha+1)\sum_{i=1}^{d} \gamma_i \operatorname{Skew}_i(p). \tag{2.4.3}$$

Also, from the definition (2.3.13) and (2.2.17), for all $i, j \in \{1, \dots, d\}$ with $i \neq j$,

$$\begin{aligned}(\mathcal{D}_{i,j}^{(\gamma)})^2 = (x_i^2(\mathcal{D}_j^{(\gamma)})^2 + x_j^2(\mathcal{D}_i^{(\gamma)})^2) - 2x_i x_j\, \mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)} \\ - (x_i\, \mathcal{D}_i^{(\gamma)} + x_j\, \mathcal{D}_j^{(\gamma)}) - (\gamma_i x_j \sigma_i^* \mathcal{D}_j^{(\gamma)} + \gamma_j x_i \sigma_j^* \mathcal{D}_i^{(\gamma)}).\end{aligned} \tag{2.4.4}$$

Then, as a direct consequence of (2.4.4), we can write

$$\begin{aligned}\sum_{1\le i<j\le d} (\mathcal{D}_{i,j}^{(\gamma)})^2 &= \frac{1}{2}\sum_{\substack{1\le i,j\le d\\ i\neq j}} (\mathcal{D}_{i,j}^{(\gamma)})^2 \\ &= \|x\|^2\,\Delta^{(\gamma)} - \sum_{1\le i,j\le d} x_i x_j\, \mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)} - (d-1)\sum_{1\le i\le d} x_i\, \mathcal{D}_i^{(\gamma)} - \sum_{\substack{1\le i,j\le d\\ i\neq j}} \gamma_i x_j \sigma_i^* \mathcal{D}_j^{(\gamma)}.\end{aligned} \tag{2.4.5}$$

Considering the easily verifiable identities

$$(x\cdot\nabla)^2 = \sum_{1\le i,j\le d} x_i x_j \partial_i\partial_j + (x\cdot\nabla), \tag{2.4.6}$$

$$x_i^2(\mathcal{D}_i^{(\gamma)})^2 = x_i^2\partial_i^2 + \gamma_i x_i \partial_i - \gamma_i \operatorname{Skew}_i \tag{2.4.7}$$

and

$$(x_i\,\mathcal{D}_i^{(\gamma)})(x_j\,\mathcal{D}_j^{(\gamma)}) = x_i x_j \partial_i\partial_j + (\gamma_j x_i \partial_i \operatorname{Skew}_j + \gamma_i x_j \partial_j \operatorname{Skew}_i) + \gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j, \tag{2.4.8}$$

for $i \neq j$; we can readily write

$$\begin{aligned}\sum_{1\le i,j\le d} x_i x_j\, \mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)} &= \sum_{1\le i\le d} x_i^2(\mathcal{D}_i^{(\gamma)})^2 + \sum_{\substack{1\le i,j\le d\\ i\neq j}} x_i x_j \mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)} \\ &= (x\cdot\nabla)^2 - (x\cdot\nabla) + \sum_{1\le i\le d} \gamma_i x_i \partial_i + 2\sum_{\substack{1\le i,j\le d\\ i\neq j}} \gamma_i x_j \partial_j \operatorname{Skew}_i\end{aligned}$$

$$- \sum_{1\le i\le d} \gamma_i \operatorname{Skew}_i + \sum_{\substack{1\le i,j\le d\\ i\ne j}} \gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j\,. \quad (2.4.9)$$

Then, replacing (2.4.9) in (2.4.5) and using the fact that $x_j\,\mathcal{D}_j^{(\gamma)} = x_j\partial_j + \gamma_j \operatorname{Skew}_j$, we get

$$\begin{aligned}\sum_{1\le i<j\le d} (\mathcal{D}_{i,j}^{(\gamma)})^2 = \|x\|^2\,\Delta^{(\gamma)} - (x\cdot\nabla)^2 - (d-2)(x\cdot\nabla) - (d-2)\sum_{1\le i\le d}\gamma_i \operatorname{Skew}_i\\ - \sum_{\substack{1\le i,j\le d\\ i\ne j}} \gamma_i\sigma_i^* x_j\partial_j - \sum_{\substack{1\le i,j\le d\\ i\ne j}} \gamma_i\gamma_j\sigma_i^* \operatorname{Skew}_j - \sum_{1\le i\le d}\gamma_i x_i\partial_i\\ - 2\sum_{\substack{1\le i,j\le d\\ i\ne j}} \gamma_i x_j\partial_j \operatorname{Skew}_i - \sum_{\substack{1\le i,j\le d\\ i\ne j}} \gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j\,. \quad (2.4.10)\end{aligned}$$

Lastly, considering the identity $\sum_{1\le i\le d}\gamma_i x_i\partial_i = (\sum_{1\le i\le d}\gamma_i)(x\cdot\nabla) - \sum_{\substack{1\le i,j\le d\\ i\ne j}}\gamma_i x_j\partial_j$, adding and substracting the term $(\sum_{1\le i\le d}\gamma_i)\sum_{i\le i\le d}\gamma_i \operatorname{Skew}_i$, considering the identity $\operatorname{Skew}_j = \operatorname{Id} - \sigma_j^* - \operatorname{Skew}_j$, and simplifying, we can readily obtain

$$\begin{aligned}\sum_{1\le i<j\le d} (\mathcal{D}_{i,j}^{(\gamma)})^2 = \|x\|^2\,\Delta^{(\gamma)} - (x\cdot\nabla)^2 - \left(d-2+\sum_{i=1}^d \gamma_i\right)(x\cdot\nabla)\\ - \left(d-2+\sum_{i=1}^d\gamma_i\right)\sum_{i=1}^d \gamma_i \operatorname{Skew}_i + \sum_{i=1}^d\sum_{j=1}^d \gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j\,. \quad (2.4.11)\end{aligned}$$

Thus, substracting (2.4.11) from (2.4.3) to then note the appearance of the operator $\mathcal{L}^{(\alpha,\gamma)}$ of (2.4.1) we can conclude that it can also be expressed as

$$\begin{aligned}\mathcal{L}^{(\alpha,\gamma)}(p) = \sum_{i=1}^d \mathcal{D}_i^{(\alpha,\gamma;\star)}(\mathcal{D}_i^{(\gamma)}p) - \sum_{1\le i<j\le d}(\mathcal{D}_{i,j}^{(\gamma)})^2 p\\ - 2\lambda^{\alpha,\gamma}\sum_{i=1}^d \gamma_i \operatorname{Skew}_i(p) + \sum_{i=1}^d\sum_{j=1}^d \gamma_i\gamma_j \operatorname{Skew}_i(\operatorname{Skew}_j(p)). \quad (2.4.12)\end{aligned}$$

Using part (i) of Proposition 2.3.2, part (i) of Proposition 2.3.3 and part (i) of Proposition 2.3.4, we find that

$$\left(\forall\, p\in \mathrm{C}^2(\overline{B^d})\right)\ \left(\forall\, q\in \mathrm{C}^1(\overline{B^d})\right)\quad \langle \mathcal{L}^{(\alpha,\gamma)}(p), q\rangle_{\alpha,\gamma} = B(p,q), \quad (2.4.13)$$

where the symmetric bilinear form $B\colon \mathrm{C}^1(\overline{B^d}) \times \mathrm{C}^1(\overline{B^d}) \to \mathbb{R}$ is defined by

$$\begin{aligned} B(u,v) := \sum_{i=1}^{d} \langle \mathcal{D}_i^{(\gamma)} u, \mathcal{D}_i^{(\gamma)} v\rangle_{\alpha+1,\gamma} + \sum_{1\le i<j\le d} \langle \mathcal{D}_{i,j}^{(\gamma)} u, \mathcal{D}_{i,j}^{(\gamma)} v\rangle_{\alpha,\gamma} \\ - 2\lambda^{\alpha,\gamma} \sum_{i=1}^{d} \gamma_i \langle \mathrm{Skew}_i(u), \mathrm{Skew}_i(v)\rangle_{\alpha,\gamma} \\ + \sum_{i=1}^{d}\sum_{j=1}^{d} \gamma_i\gamma_j \langle \mathrm{Skew}_i(\mathrm{Skew}_j(u)), \mathrm{Skew}_i(\mathrm{Skew}_j(v))\rangle_{\alpha,\gamma}. \end{aligned} \tag{2.4.14}$$

Through (2.4.13) the eigenvalue (Sturm–Liouville) problem (2.4.1) satisfied by the $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomials can be expressed in the weak form

$$\left(\forall\, p_n \in \mathcal{V}_n^{(\alpha,\gamma)}\right) \ \left(\forall\, q \in \mathrm{C}^1(\overline{B^d})\right) \quad B(p_n, q) = n(n+2\lambda^{\alpha,\gamma})\langle p_n, q\rangle_{\alpha,\gamma}, \tag{2.4.15}$$

Directly from the definition (2.4.14) and standard inequalities follows the bound

$$\left(\forall\, u, v \in \mathrm{C}^1(\overline{B^d})\right) \quad |B(u,v)| \le C_B \left\|u\right\|_{\alpha,\gamma;1} \left\|v\right\|_{\alpha,\gamma;1} \tag{2.4.16}$$

for some $C_B = C_B(\alpha,\gamma) > 0$. Given any polynomial $p \in \Pi^d$, it follows from (2.4.15) and (2.3.5) that

$$B(p,p) = \sum_{n=0}^{\mathrm{degree}(p)} n(n+2\lambda^{\alpha,\gamma}) \left\|\mathrm{proj}_n^{(\alpha,\gamma)}(p)\right\|_{\alpha,\gamma}^2 \ge \inf_{n\in\mathbb{N}_0} \left(n(n+2\lambda^{\alpha,\gamma})\right) \left\|p\right\|_{\alpha,\gamma}^2.$$

From the definition of $\lambda^{\alpha,\gamma}$ in (2.4.2) and the fact that $\alpha, \gamma_1, \dots, \gamma_d > -1$ it follows that the above infimum is $\min(0, 1+2\lambda^{\alpha,\gamma})$. Also, because of the bound (2.4.16) and the density of polynomials in $\mathrm{H}^1_{\alpha,\gamma} \supseteq \mathrm{C}^1(\overline{B^d})$ (cf. Proposition 2.2.3), the above inequality can be extended to $\mathrm{C}^1(\overline{B^d})$ functions. Thus, choosing any $K > \max(0, -1-2\lambda^{\alpha,\gamma})$, the shifted bilinear form $\tilde{B}\colon \mathrm{C}^1(\overline{B^d}) \times \mathrm{C}^1(\overline{B^d}) \to \mathbb{R}$, defined by

$$\tilde{B}(p,q) := B(p,q) + K\langle p, q\rangle_{\alpha,\gamma}, \tag{2.4.17}$$

is an inner product in $\mathrm{C}^1(\overline{B^d})$; we denote the induced norm by $\left\|\cdot\right\|_{\tilde{B}}$. This allows for defining an *ad hoc* function space in very much the same vein of Definition 2.2.2.

Definition 2.4.1. We define $\mathrm{H}_{\tilde{B}}$ as the topological completion of $(\mathrm{C}^1(\overline{B^d}), \|\cdot\|_{\tilde{B}})$.

Proposition 2.4.2. *There holds the inclusion* $\mathrm{H}^1_{\alpha,\gamma} \subseteq \mathrm{H}_{\tilde{B}}$ *and*

$$(\forall\, u \in \mathrm{H}^1_{\alpha,\gamma}) \quad \|u\|_{\tilde{B}} \leq (C_B + K)^{1/2} \, \|u\|_{\alpha,\gamma;1}\,;$$

that is, $\mathrm{H}^1_{\alpha,\gamma}$ *is continuously embedded in* $\mathrm{H}_{\tilde{B}}$*.*

Proof. From Definition 2.2.2, every $u \in \mathrm{H}^1_{\alpha,\gamma}$ is (a class of equivalence of) a Cauchy sequence $(u_n)_{n\in\mathbb{N}}$ of $\mathrm{C}^1(\overline{B^d})$ functions with respect to the norm $\|\cdot\|_{\alpha,\gamma;1}$ of (2.2.22). By (2.4.16), $\|u_m - u_n\|_{\tilde{B}} \leq (C_B + K)^{1/2} \, \|u_m - u_n\|_{\alpha,\gamma;1} \xrightarrow{m,n\to\infty} 0$, so $u \in \mathrm{H}_{\tilde{B}}$ according to Definition 2.4.1, and $\|u\|_{\tilde{B}} = \lim_{n\to\infty} \|u_n\|_{\tilde{B}} \leq (C_B + K)^{1/2} \lim_{n\to\infty} \|u_n\|_{\alpha,\gamma;1} = (C_B + K)^{1/2} \, \|u\|_{\alpha,\gamma;1}$. □

In the sequence of results Lemma 2.4.3, Lemma 2.4.4 and Corollary 2.4.5 below, we will exploit the Sturm–Liouville-type equations satisfied by our orthogonal polynomial spaces, both in its strong ($\mathcal{L}^{(\alpha,\gamma)}$-based) and weak ($B$ and $\tilde{B}$-based) forms, to prove that Dunkl–Sobolev regularity implies convergence rates of our truncation projector, with the error measured in $\mathrm{L}^2_{\alpha,\gamma}$. See [23, Lem. 2.2, Lem. 2.3 and Cor. 2.4] for the corresponding results in the $\gamma = 0$ case.

Lemma 2.4.3. *Let* $\alpha \in (-1,\infty)$ *and* $\gamma \in (-1,\infty)^d$*. For all* $u \in \mathrm{H}_{\tilde{B}}$*, the series* $\sum_{n=0}^{\infty} \mathrm{proj}_n^{(\alpha,\gamma)}(u)$ *(cf. (2.3.4)) converges in* $\mathrm{H}_{\tilde{B}}$ *as well. There also holds the Parseval identity*

$$(\forall\, u \in \mathrm{H}_{\tilde{B}}) \quad \|u\|^2_{\tilde{B}} = \sum_{n=0}^{\infty} \left(n(n + 2\lambda^{\alpha,\gamma}) + K\right) \left\|\mathrm{proj}_n^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma}.$$

Proof. By density (cf. Definition 2.4.1), (2.4.15) extends to $q \in \mathrm{H}_{\tilde{B}}$. Adding $K\langle p_n, q\rangle_{\alpha,\gamma}$ to both sides we obtain

$$(\forall\, p_n \in \mathcal{V}_n^{(\alpha,\gamma)})\ (\forall\, q \in \mathrm{H}_{\tilde{B}}) \quad \tilde{B}(p_n, q) = \left(n(n + 2\lambda^{\alpha,\gamma}) + K\right) \langle p_n, q\rangle_{\alpha,\gamma}.$$

Polynomials are dense in $\mathrm{H}_{\tilde{B}}$. Indeed, if $s \in \mathrm{H}_{\tilde{B}}$ is $\mathrm{H}_{\tilde{B}}$-orthogonal to Π^d, by the above equality and the fact that $n(n+2\lambda^{\alpha,\gamma}) + K > 0$ for all $n \in \mathbb{N}_0$, it follows that s is $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal to Π^d as well; i.e, $s = 0$. Now, as the $\mathcal{V}_n^{(\alpha,\gamma)}$ are finite-dimensional (cf. (2.3.3)), there exists a Hilbert basis of $\mathrm{L}^2_{\alpha,\gamma}$ consisting of $\mathrm{L}^2_{\alpha,\gamma}$-orthonormal polynomials. Such a basis can be renormalized to obtain a Hilbert basis of the closure of polynomials in $\mathrm{H}_{\tilde{B}}$; i.e., $\mathrm{H}_{\tilde{B}}$ itself. The desired results then stem from the basic properties of Hilbert bases; see, e.g., [7, Corollary 5.10]. □

Lemma 2.4.4. *Let $\alpha \in (-1,\infty)$, $\gamma \in (-1,\infty)^d$ and $l \in \mathbb{N}_0$. Then, there exists $C = C(\alpha,\gamma,l) > 0$ such that*

$$\left(\forall\, u \in \mathrm{H}^l_{\alpha,\gamma}\right) \quad \sum_{n=0}^{\infty} (n(n+2\lambda^{\alpha,\gamma}) + K)^l \left\|\mathrm{proj}_n^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma} \leq C \left\|u\right\|^2_{\alpha,\gamma;l}.$$

Proof. The $l = 0$ case is simply (2.3.5). From Proposition 2.4.2 and Lemma 2.4.3, for all $u \in \mathrm{H}^1_{\alpha,\gamma}$,

$$\sum_{n=0}^{\infty} (n(n+2\lambda^{\alpha,\gamma}) + K) \left\|\mathrm{proj}_n^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma} = \|u\|^2_{\tilde{B}} \leq (C_B + K) \|u\|^2_{\alpha,\gamma;1}, \tag{2.4.18}$$

which accounts for the $l = 1$ case.

Particularizing (2.4.13) to $p \in \mathrm{C}^2(\overline{B^d})$ and $q \in \Pi^d$ and using the symmetry of the bilinear form B and the inner product of $\mathrm{L}^2_{\alpha,\gamma}$, we find that

$$(\forall\, p \in \mathrm{C}^2(\overline{B^d}))\ (\forall\, q \in \Pi^d) \quad \langle \mathcal{L}^{(\alpha,\gamma)}(p), q\rangle_{\alpha,\gamma} = \langle p, \mathcal{L}^{(\alpha,\gamma)}(q)\rangle_{\alpha,\gamma}. \tag{2.4.19}$$

Now, by virtue of the bound (2.2.21) and the definitions (2.3.6) and (2.3.13), the operators $\mathcal{D}_j^{(\gamma)}$, $\mathcal{D}_j^{(\alpha,\gamma;\star)}$ and $\mathcal{D}_{i,j}^{(\gamma)}$ are bounded operators between $\mathrm{C}^m(\overline{B^d})$ and $\mathrm{C}^{m-1}(\overline{B^d})$, $m \geq 1$. From Definition 2.2.2 they extend to bounded operators from $\mathrm{H}^m_{\alpha,\gamma}$ and $\mathrm{H}^{m-1}_{\alpha,\gamma}$. Using these extended first-order operators in the definition of $\mathcal{L}^{(\alpha,\gamma)}$ in (2.4.12), the resulting extended $\mathcal{L}^{(\alpha,\gamma)}$ and $\mathcal{L}^{(\alpha,\gamma)} + K\,I$ operators are bounded maps between $\mathrm{H}^m_{\alpha,\gamma}$ to $\mathrm{H}^{m-2}_{\alpha,\gamma}$, $m \geq 2$. The $m = 2$ case allows for extending (2.4.19) to

$$(\forall\, u \in \mathrm{H}^2_{\alpha,\gamma})\ (\forall\, q \in \Pi^d) \quad \langle \mathcal{L}^{(\alpha,\gamma)}(u), q\rangle_{\alpha,\gamma} = \langle u, \mathcal{L}^{(\alpha,\gamma)}(q)\rangle_{\alpha,\gamma}. \tag{2.4.20}$$

Then, for all $u \in \mathrm{H}^2_{\alpha,\gamma}$ and $q \in \mathcal{V}_n^{(\alpha,\gamma)}$,

$$\langle \mathrm{proj}_n^{(\alpha,\gamma)}(\mathcal{L}_n^{(\alpha,\gamma)}(u)), q\rangle_{\alpha,\gamma} = \langle \mathcal{L}_n^{(\alpha,\gamma)}(u), q\rangle_{\alpha,\gamma} \overset{(2.4.20)}{=} \langle u, \mathcal{L}_n^{(\alpha,\gamma)}(q)\rangle_{\alpha,\gamma}$$

$$\overset{(2.4.1)}{=} n(n+2\lambda^{\alpha,\gamma})\langle u,q\rangle_{\alpha,\gamma} = n(n+2\lambda^{\alpha,\gamma})\langle \operatorname{proj}_n^{(\alpha,\gamma)}(u),q\rangle_{\alpha,\gamma},$$

whence

$$(\forall\, u \in \mathrm{H}^2_{\alpha,\gamma}) \quad \operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{L}^{(\alpha,\gamma)}(u)) = n(n+2\lambda^{\alpha,\gamma})\operatorname{proj}_n^{(\alpha,\gamma)}(u). \tag{2.4.21}$$

Therefore, if $l \geq 2$ is even, our desired result stems from

$$\begin{aligned}\sum_{n=0}^{\infty}(n(n+2\lambda^{\alpha,\gamma})+K)^l\left\|\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma} &\overset{(2.4.21)}{=} \sum_{n=0}^{\infty}\left\|\operatorname{proj}_n^{(\alpha,\gamma)}\left((\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{l/2}(u)\right)\right\|^2_{\alpha,\gamma}\\ &= \left\|(\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{l/2}(u)\right\|^2_{\alpha,\gamma} \leq \left\|(\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{l/2}\right\|^2_{\mathcal{L}(\mathrm{H}^l_{\alpha,\gamma},\mathrm{L}^2_{\alpha,\gamma})}\|u\|^2_{\alpha,\gamma;l}\,.\end{aligned}$$

Finally, if $l \geq 3$ is odd,

$$\begin{aligned}&\sum_{n=0}^{\infty}(n(n+2\lambda^{\alpha,\gamma})+K)^l\left\|\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma}\\ &\overset{(2.4.21)}{=} \sum_{n=0}^{\infty}(n(n+2\lambda^{\alpha,\gamma})+K)\left\|\operatorname{proj}_n^{(\alpha,\gamma)}\left((\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{(l-1)/2}(u)\right)\right\|^2_{\alpha,\gamma}\\ &\overset{(2.4.18)}{=} (C_B+K)\left\|(\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{(l-1)/2}(u)\right\|^2_{\alpha,\gamma;1}\\ &\leq (C_B+K)\left\|(\mathcal{L}^{(\alpha,\gamma)}+K\,I)^{(l-1)/2}\right\|^2_{\mathcal{L}(\mathrm{H}^l_{\alpha,\gamma},\mathrm{H}^1_{\alpha,\gamma})}\|u\|^2_{\alpha,\gamma;l}\,.\end{aligned}$$

□

Corollary 2.4.5. *For all $\alpha \in (-1,\infty)$, $d \in \mathbb{N}$, $\gamma \in (-1,\infty)^d$ and $l \in \mathbb{N}_0$, there exists $C = C(\alpha,\gamma,l)$ such that*

$$(\forall\, N \in \mathbb{N}_0)\,(\forall\, u \in \mathrm{H}^l_{\alpha,\gamma}) \quad \left\|u - S_N^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma} \leq C(N+1)^{-l}\|u\|_{\alpha,\gamma;l}\,.$$

Proof. This is a direct consequence of the Parseval identity (2.3.5), Lemma 2.4.4 and the fact that $n(n+2\lambda^{\alpha,\gamma})+K$ depends quadratically on n. □

Proposition 2.4.7 below allows for quantifying the $\mathrm{L}^2_{\alpha,\gamma}$ norm of a member of $\mathcal{V}_k^{(\alpha+1,\gamma)}$ with respect to its $\mathrm{L}^2_{\alpha+1,\gamma}$ norm, thus containing the seed of the quantification of the price to be paid in our main result Theorem 2.1.1 because of the mismatch of the orthogonal projector there and the norm the approximation error is measured with; its third part is a Dunkl variant of the Markov brothers' inequality. However, we need the following technical proposition first.

Proposition 2.4.6. *Let* $\alpha \in (-1,\infty)$ *and* $\gamma \in (-1,\infty)^d$*. Then, there exists* $M_{\alpha,\gamma} > 0$ *such that*

$$(\forall\, p \in \mathrm{L}^2_{\alpha,\gamma}) \qquad -2\lambda^{\alpha,\gamma}\sum_{i=1}^{d}\gamma_i \left\|\mathrm{Skew}_i(p)\right\|^2_{\alpha,\gamma} + \sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j \left\|\mathrm{Skew}_i(\mathrm{Skew}_j(p))\right\|^2_{\alpha,\gamma} \geq -M_{\alpha,\gamma}\left\|p\right\|^2_{\alpha,\gamma}.$$

Proof. This comes from the fact that the Skew_j operators are bounded in $\mathrm{L}^2_{\alpha,\gamma}$. □

Proposition 2.4.7. *Let* $\alpha \in (-1,\infty)$ *and* $\gamma \in (-1,\infty)^d$.

(i) For all $p, q \in \mathcal{V}_k^{(\alpha+1,\gamma)}$,

$$\langle p, q\rangle_{\alpha,\gamma} = \left(\frac{k + d/2 + \sum_{j=1}^{d}\gamma_j/2}{\alpha+1} + 1\right)\langle p, q\rangle_{\alpha+1,\gamma}.$$

(ii) Let $k \in \mathbb{N}_0$*. Then, for all* $r \in \mathcal{V}_k^{(\alpha,\gamma)}$,

$$\|\mathcal{D}^{(\gamma)} r\|_{\alpha,\gamma} \leq \left(\frac{(k(k+2\lambda^{\alpha,\gamma}) + M_{\alpha,\gamma})(k+\lambda^{\alpha,\gamma})}{\alpha+1}\right)^{1/2}\|r\|_{\alpha,\gamma},$$

where $M_{\alpha,\gamma} > 0$ *is that of Proposition 2.4.6. If* r *is, additionally, a radial function, this inequality turns into an equality by replacing* $M_{\alpha,\gamma}$ *with* 0.

(iii) There exists a constant $C = C(\alpha,\gamma) > 0$ *such that, for all* $n \in \mathbb{N}_0$ *and* $p \in \Pi_n^d$,

$$\|\mathcal{D}^{(\gamma)} p\|_{\alpha,\gamma} \leq C n^2 \|p\|_{\alpha,\gamma}.$$

Proof. On homogeneous polynomials of degree k, $k \in \mathbb{N}_0$, there holds $x \cdot \nabla = k\,I$. As a first consequence, $x \cdot \nabla$ maps Π_n^d into itself, for every $n \in \mathbb{N}_0$.

Let $p, q \in \mathcal{V}_k^{(\alpha+1,\gamma)}$. As every member of $\mathcal{V}_k^{(\alpha+1,\gamma)}$ is a linear combination of homogeneous polynomials of degree ranging from 0 to k, there exists a homogeneous polynomial s_p of degree

k such that $p - s_p \in \Pi^d_{k-1}$ and hence $x \cdot \nabla p - x \cdot \nabla s_p \in \Pi^d_{k-1}$. Thus,

$$\langle x \cdot \nabla p, q\rangle_{\alpha+1,\gamma} = \langle x \cdot \nabla s_p, q\rangle_{\alpha+1,\gamma} = k\langle s_p, q\rangle_{\alpha+1,\gamma} = k\langle p, q\rangle_{\alpha+1,\gamma}. \tag{2.4.22}$$

Using the fact that $\operatorname{div}(x) = d$ and (2.4.22) (which is still valid if the roles of p and q are interchanged),

$$\begin{aligned}(2k+d)\langle p, q\rangle_{\alpha+1,\gamma} &= \langle x \cdot \nabla p, q\rangle_{\alpha+1,\gamma} + \langle p, x \cdot \nabla q\rangle_{\alpha+1,\gamma} + d\langle p, q\rangle_{\alpha+1,\gamma} \\ &= \int_{B^d} \operatorname{div}(p(x)q(x)x) W_{\alpha+1,\gamma}(x)\,\mathrm{d}x. \end{aligned}\tag{2.4.23}$$

Now,

$$\begin{aligned}\int_{B^d} \operatorname{div}(p(x)q(x)x)\, W_{\alpha+1,\gamma}(x)\,\mathrm{d}x + \sum_{j=1}^{d} \gamma_j \langle p, q\rangle_{\alpha+1,\gamma} &= \sum_{j=1}^{d} \langle \mathcal{D}^{(\gamma)}_j (x_j pq), 1\rangle_{\alpha+1,\gamma} \\ = \sum_{j=1}^{d} \langle x_j pq, \mathcal{D}^{(\alpha,\gamma;\star)}_j (1)\rangle_{\alpha,\gamma} &= 2(\alpha+1) \int_{B^d} p(x)q(x)\, \|x\|^2\, W_{\alpha,\gamma}(x)\,\mathrm{d}x,\end{aligned}$$

where the first equality comes from the definition (2.2.12) and part (i) of Proposition 2.3.2, the second from part (i) of Proposition 2.3.3 and the third from the definition (2.3.6). Substituting this into (2.4.23), yields

$$(2k+d)\langle p, q\rangle_{\alpha+1,\gamma} = 2(\alpha+1) \int_{B^d} p(x)q(x)\, \|x\|^2\, W_{\alpha,\gamma}(x)\,\mathrm{d}x - \sum_{j=1}^{d} \gamma_j \langle p, q\rangle_{\alpha+1,\gamma}.$$

Part (i) then follows from the fact that $W_{\alpha,\gamma}(x) = \|x\|^2\, W_{\alpha,\gamma}(x) + W_{\alpha+1,\gamma}(x)$.

Part (ii) is obviously true if $k = 0$; otherwise, from part (iii) of Proposition 2.3.3 and part (i) above,

$$(\forall\, r \in \mathcal{V}^{(\alpha,\gamma)}_k) \quad \|\mathcal{D}^{(\gamma)} r\|^2_{\alpha,\gamma} = \frac{k + \lambda^{\alpha,\gamma}}{\alpha+1} \|\mathcal{D}^{(\gamma)} r\|^2_{\alpha+1,\gamma}. \tag{2.4.24}$$

On the other hand, from (2.4.14) and (2.4.15) (with p_n and q there both set as r),

$$\|\mathcal{D}^{(\gamma)} r\|^2_{\alpha+1,\gamma} + \sum_{1 \le i < j \le d} \|\mathcal{D}^{(\gamma)}_{i,j} r\|^2_{\alpha,\gamma} - 2\lambda^{\alpha,\gamma} \sum_{i=1}^{d} \gamma_i\, \|\mathrm{Skew}_i(r)\|^2_{\alpha,\gamma}$$

$$+\sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j \left\|\mathrm{Skew}_i(\mathrm{Skew}_j(r))\right\|_{\alpha,\gamma}^2 + M_{\alpha,\gamma}\left\|r\right\|_{\alpha,\gamma}^2$$
$$= \left(k(k+2\lambda^{\alpha,\gamma}) + M_{\alpha,\gamma}\right)\left\|r\right\|_{\alpha,\gamma}^2 .$$

Per Proposition 2.4.6, dropping the second, third, fourth and fifth terms from the left-hand side of the above equality, the remaining first term will be bounded from above by the right-hand side. Combining the resulting inequality with (2.4.24) and taking square roots results in the generic case of part (ii). If r is radial, the second, third and fourth terms on the left-hand side above vanish, and $M_{\alpha,\gamma}$ can be canceled from both sides; what now remains an equality can also be combined with (2.4.24).

Given $n \in \mathbb{N}_0$ and $p \in \Pi_n^d$, from (2.3.4), part (ii) above, and the Cauchy–Schwarz inequality,

$$\begin{aligned}\|\mathcal{D}^{(\gamma)}p\|_{\alpha,\gamma} &\leq \sum_{k=0}^{n}\|\mathcal{D}^{(\gamma)}\operatorname{proj}_k^{(\alpha,\gamma)}(p)\|_{\alpha,\gamma}\\ &\leq \left(\sum_{k=0}^{n}\frac{(k(k+2\lambda^{\alpha,\gamma})+M_{\alpha,\gamma})(k+\lambda^{\alpha,\gamma})}{\alpha+1}\right)^{1/2}\left(\sum_{k=0}^{n}\left\|\operatorname{proj}_k^{(\alpha,\gamma)}(p)\right\|_{\alpha,\gamma}^2\right)^{1/2}\\ &= \left(\frac{(n+1)(n+2\lambda^{\alpha,\gamma})(n^2+2\lambda^{\alpha,\gamma}n+n+2M_{\alpha,\gamma})}{4(\alpha+1)}\right)^{1/2}\|p\|_{\alpha,\gamma}.\end{aligned}$$

Part (iii) then follows after realizing that there exists a positive constant C depending on α and γ only such that $\frac{(n+1)(n+2\lambda^{\alpha,\gamma})(n^2+2\lambda^{\alpha,\gamma}n+n+2M_{\alpha,\gamma})}{4(\alpha+1)} \leq C^2n^4$ for all $n \in \mathbb{N}_0$. □

Now we prove a lemma with the core of the main result, a bridging corollary and then, finally, the main result itself.

Lemma 2.4.8. *Let $\alpha \in (-1,\infty)$, $\gamma \in (-1,\infty)^d$ and $l \in \mathbb{N}$. Then, there exists $C = C(\alpha,\gamma,l) > 0$ such that for all $u \in \mathrm{H}_{\alpha,\gamma}^l$, $n \in \mathbb{N}$ and $j \in \{1,\dots,d\}$,*

$$\left\|\mathcal{D}_j^{(\gamma)}S_n^{(\alpha,\gamma)}(u) - S_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\right\|_{\alpha,\gamma} \leq C\,n^{3/2-l}\left\|\mathcal{D}_j^{(\gamma)}u\right\|_{\alpha,\gamma;l-1}.$$

Proof. Let us first assume that $u \in \mathrm{C}^l(\overline{B^d})$. Combining part (iv) of Proposition 2.3.1 and part (iv) of Proposition 2.3.3, we obtain

$$\begin{aligned}\mathcal{D}_j^{(\gamma)}\operatorname{proj}_{k+1}^{(\alpha,\gamma)}(u)-\operatorname{proj}_k^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)&\\ =\operatorname{proj}_k^{(\alpha+1,\gamma)}\circ\operatorname{proj}_{k+2}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)&-\operatorname{proj}_{k-2}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_k^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\end{aligned}\tag{2.4.25}$$

Using (2.3.4) to express $S_n^{(\alpha,\gamma)}$ in terms of the $\operatorname{proj}_k^{(\alpha,\gamma)}$, using (2.4.25), noticing that a telescoping sum results and using part (ii) of Proposition 2.3.1 to expand an appearance of $\operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\in\mathcal{V}_n^{(\alpha,\gamma)}$,

$$\begin{aligned}\mathcal{D}_j^{(\gamma)}S_n^{(\alpha,\gamma)}(u)-S_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)&=\sum_{k=0}^{n}\mathcal{D}_j^{(\gamma)}\operatorname{proj}_k^{(\alpha,\gamma)}(u)-\sum_{k=0}^{n}\operatorname{proj}_k^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\\ &=\sum_{k=0}^{n-1}\Big(\mathcal{D}_j^{(\gamma)}\operatorname{proj}_{k+1}^{(\alpha,\gamma)}(u)-\operatorname{proj}_k^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\Big)-\operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\\ =\operatorname{proj}_{n-2}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)&+\operatorname{proj}_{n-1}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_{n+1}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)-\operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\\ =\operatorname{proj}_{n-1}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_{n+1}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)&-\operatorname{proj}_n^{(\alpha+1,\gamma)}\circ\operatorname{proj}_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u).\end{aligned}\tag{2.4.26}$$

Now, by part (i) of Proposition 2.4.7, the fact that $\|\operatorname{proj}_{n-1}^{(\alpha+1,\gamma)}\|_{\mathcal{L}(\mathrm{L}^2_{\alpha+1,\gamma})}\le 1$ and the fact that $\|\cdot\|_{\alpha+1,\gamma}\le\|\cdot\|_{\alpha,\gamma}$ in $\mathrm{L}^2_{\alpha,\gamma}$ (because $W_{\alpha+1,\gamma}\le W_{\alpha,\gamma}$) we have that, for all $n\ge 1$,

$$\|\operatorname{proj}_{n-1}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_{n+1}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2\le\frac{n+d/2+\sum_{j=1}^d\gamma_j/2+\alpha}{\alpha+1}\|\operatorname{proj}_{n+1}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2.\tag{2.4.27}$$

Analogous arguments show that, for all $n\in\mathbb{N}$,

$$\|\operatorname{proj}_{n}^{(\alpha+1,\gamma)}\circ\operatorname{proj}_{n}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2\le\frac{n+1+d/2+\sum_{j=1}^d\gamma_j/2+\alpha}{\alpha+1}\|\operatorname{proj}_{n}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2.\tag{2.4.28}$$

Taking the squared $\mathrm{L}^2_{\alpha,\gamma}$ norm of both ends of (2.4.26), exploiting the $\mathrm{L}^2_{\alpha,\gamma}$ orthogonality of $\mathcal{V}_{n-1}^{(\alpha+1,\gamma)}$ and $\mathcal{V}_n^{(\alpha+1,\gamma)}$ (a consequence of the parity relation (2.3.2)) and the bounds (2.4.27) and (2.4.28) we observe that

$$\|\mathcal{D}_j^{(\gamma)}S_n^{(\alpha,\gamma)}(u)-S_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2\le\frac{n+1+d/2+\sum_{j=1}^d\gamma_j/2+\alpha}{\alpha+1}\|\mathcal{D}_j^{(\gamma)}u-S_{n-1}^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)}u)\|_{\alpha,\gamma}^2.$$

As $\mathcal{D}_j^{(\gamma)}u\in\mathrm{C}^{l-1}(\overline{B^d})$ (cf. Proposition 2.2.1), we can appeal to Corollary 2.4.5 to obtain the desired result for $u\in\mathrm{C}^l(\overline{B^d})$ after realizing that there exists a constant $\tilde{C}$ depending only on

α, γ and l such that $\frac{n+1+d/2+\sum_{j=1}^{d}\gamma_j/2+\alpha}{\alpha+1}(n^{-(l-1)})^2 \leq \tilde{C}\, n^{3-2l}$ for all $n \in \mathbb{N}$. The general result then follows via density of $\mathrm{C}^l(\overline{B^d})$ in $\mathrm{H}^l_{\alpha,\gamma}$ (Definition 2.2.2). □

Corollary 2.4.9. *Let $\alpha \in (-1,\infty)$, $\gamma \in (-1,\infty)^d$ and $r, l \in \mathbb{N}$ with $r \leq l$. Then, there exists $C = C(\alpha,\gamma,l,r) > 0$ such that, for all $u \in \mathrm{H}^l_{\alpha,\gamma}$ and $n \in \mathbb{N}$,*

$$\left\|(\mathcal{D}^{(\gamma)})^r S_n^{(\alpha,\gamma)}(u) - S_n^{(\alpha,\gamma)}((\mathcal{D}^{(\gamma)})^r u)\right\|_{\alpha,\gamma} \leq C\, n^{2r-1/2-l}\, \|u\|_{\alpha,\gamma;l}\,.$$

Proof. Let us first note that iterating part (iii) of Proposition 2.4.7 we find that for all $r \in \mathbb{N}$ there exists $C > 0$ depending on α, γ, and r such that

$$(\forall\, n \in \mathbb{N}_0)\ (\forall\, p \in \Pi_n^d) \quad \|(\mathcal{D}^{(\gamma)})^r p\|_{\alpha,\gamma} \leq C\, n^{2r}\, \|p\|_{\alpha,\gamma}\,. \tag{2.4.29}$$

We will now operate by induction on r. Taking the square root of the sum with respect to j of the square of both sides of the inequality in Lemma 2.4.8 the case $r = 1$ follows almost immediately. Let us suppose now that our desired result holds for some $r \in \{1,\ldots,l\}$ and that $r+1 \leq l$. Then, for all $j \in \{1,\ldots,d\}$, by the triangle inequality,

$$\begin{aligned}&\left\|(\mathcal{D}^{(\gamma)})^r \mathcal{D}_j^{(\gamma)} S_n^{(\alpha,\gamma)}(u) - S_n^{(\alpha,\gamma)}((\mathcal{D}^{(\gamma)})^r \mathcal{D}_j^{(\gamma)} u)\right\|_{\alpha,\gamma}\\ &\leq \left\|(\mathcal{D}^{(\gamma)})^r \mathcal{D}_j^{(\gamma)} S_n^{(\alpha,\gamma)}(u) - (\mathcal{D}^{(\gamma)})^r S_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)} u)\right\|_{\alpha,\gamma} + \left\|(\mathcal{D}^{(\gamma)})^r S_n^{(\alpha,\gamma)}(\mathcal{D}_j^{(\gamma)} u) - S_n^{(\alpha,\gamma)}((\mathcal{D}^{(\gamma)})^r \mathcal{D}_j^{(\gamma)} u)\right\|_{\alpha,\gamma}.\end{aligned}$$

By (2.4.29) and Lemma 2.4.8, the first term is bounded by an appropriate constant times $n^{2r} n^{3/2-l}\|\mathcal{D}_j^{(\gamma)} u\|_{\alpha,\gamma;l-1}$. By the induction hypothesis and the fact that $\mathcal{D}_j^{(\gamma)} u \in \mathrm{H}^{l-1}_{\alpha,\gamma}$, the second term is bounded by an appropriate constant times $n^{2r-1/2-(l-1)}\|\mathcal{D}_j^{(\gamma)} u\|_{\alpha,\gamma;l-1}$. Then, the desired result in the $r+1$ case follows from summing up with respect to j and standard inequalities connecting vector 1- and 2-norms. □

Proof of Theorem 2.1.1. For every $k \in \{1,\ldots,r\}$,

$$\begin{aligned}&\left\|(\mathcal{D}^{(\gamma)})^k u - (\mathcal{D}^{(\gamma)})^k S_N^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2\\ &\qquad \leq 2\left\|(\mathcal{D}^{(\gamma)})^k u - S_N^{(\alpha,\gamma)}((\mathcal{D}^{(\gamma)})^k u)\right\|_{\alpha,\gamma}^2 + 2\left\|S_N^{(\alpha,\gamma)}((\mathcal{D}^{(\gamma)})^k u) - (\mathcal{D}^{(\gamma)})^k S_N^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2\end{aligned}$$

$$\leq C_1\,(N+1)^{-2(l-k)} \sum_{|\beta|=k} \binom{k}{\beta} \|\mathcal{D}_\beta^{(\gamma)} u\|^2_{\alpha,\gamma;l-k} + C_2\,N^{4k-1-2l}\,\|u\|^2_{\alpha,\gamma;l} \leq C_3\,N^{4r-1-2l}\,\|u\|^2_{\alpha,\gamma;l}\,,$$

where we have used Corollary 2.4.5, Corollary 2.4.9 and C_1 and C_2 depend on α, γ, l and k only and C_3 depends on α, γ, l and r only. Thus,

$$\left\|u - S_N^{(\alpha,\gamma)}(u)\right\|^2_{\alpha,\gamma;r} \leq \left(C_4\,(N+1)^{-2l} + r\,C_3\,N^{4r-1-2l}\right)\|u\|^2_{\alpha,\gamma;l} \leq C_5\,N^{4r-1-2l}\,\|u\|^2_{\alpha,\gamma;l}\,,$$

where we have again used Corollary 2.4.5, C_4 depends on α, γ, and l only and C_5 depends on α, γ, l and r only. □

2.5 On the sharpness of the main result

We will say that our main result, Theorem 2.1.1 is *sharp* if the power on the truncation degree N appearing there cannot be lowered. We refer to [22, Sec. 5] for an account of sharpness results for previous incarnations of our main result, to which we should add that the one-dimensional, Jacobi-weighted variant of [48, Th. 2.6] comes with its own proof of sharpness (for the cases in which, in our notation, $r = l$).

We will prove the sharpness of our main result for all dimensions $d \in \mathbb{N}$, natural singularity parameters $\alpha > -1$ and $\gamma \in (-1,\infty)^d$, but restricted to $l = r = 1$.

We will find it easier to work with an alternative norm, equivalent to that of $\mathrm{H}^1_{\alpha,\gamma}$, as proved in Proposition 2.5.3 (see [23, Lem. 2.6] for the corresponding result in the $\gamma = 0$ case). However, we first need to show that differentiable functions with vanishing Dunkl gradient are constant in B^d.

Proposition 2.5.1. *Let $\gamma > -1$, $L > 0$, and $p \in \mathrm{C}^1(-L,L)$ such that*

$$\mathcal{D}_1^{(\gamma)} p = 0 \quad \text{in } (-L,L). \tag{2.5.1}$$

Then, p is constant in $(-L,L)$.

Proof. As $\frac{\mathrm{Skew}\,p}{x}$ is a always an even function and so is 0, directly from the definition (2.2.12) of $\mathcal{D}_1^{(\gamma)}$, it follows that p' is an even function. Therefore, p can be expressed as the sum of a constant and an odd function which also belongs to $\mathrm{C}^1(-L, L)$. Hence, $y := \mathrm{Skew}(p)|_{(0,L)}$ satisfies the Cauchy–Euler differential equation

$$x\, y'(x) + \gamma\, y(x) = 0,$$

whence it has the form

$$y(x) = C\, x^{-\gamma}.$$

As y extends to a $\mathrm{C}^1(-1, 1)$ function, C has to vanish. □

Proposition 2.5.2. *Let $\gamma \in (-1, \infty)^d$ and $p \in \mathrm{C}^1(B^d)$ such that*

$$\mathcal{D}^{(\gamma)} p = 0 \quad \textit{in } B^d.$$

Then, p is constant in B^d.

Proof. Given two points in B^d, they can be connected via a polygonal path consisting exclusively of segments that are parallel to a coordinate axis. By applying Proposition 2.5.1 in every segment, it transpires that p is constant along this polygonal path and, in particular, the evaluations of p at the original two points coincide. □

Proposition 2.5.3. *The following is an equivalent inner product for $(\mathrm{C}^1(\overline{B^d}), \langle\cdot,\cdot\rangle_{\alpha,\gamma;1})$.*

$$\langle u, v\rangle_{\alpha,\gamma;1,\mathrm{P}} := \langle \mathcal{D}^{(\gamma)} u, \mathcal{D}^{(\gamma)} v\rangle_{\alpha,\gamma} + \langle S_0^{\alpha,\gamma}(u), S_0^{\alpha,\gamma}(v)\rangle_{\alpha,\gamma}. \tag{2.5.2}$$

Therefore the topological completion of $(\mathrm{C}^1(\overline{B^d}), \langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}})$ equals $\mathrm{H}^1_{\alpha,\gamma}$, with the extension of $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$ to $\mathrm{H}^1_{\alpha,\gamma}$ (cf. Definition 2.2.2) being an equivalent inner product.

Proof. $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$ being an inner product is a direct consequence of Proposition 2.5.2. Clearly, $\|\cdot\|_{\alpha,\gamma;1,\mathrm{P}} \leq \|\cdot\|_{\alpha,\gamma,1}$.

We will now prove the converse bound. Let $u \in \mathrm{C}^1(\overline{B^d})$. Given $N \in \mathbb{N}$, by Parseval's identity (2.3.5),

$$\|u\|_{\alpha,\gamma}^2 = \left\|S_N^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 + \sum_{n=N+1}^{\infty}\left\|\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2. \tag{2.5.3}$$

As Π_N^d is finite dimensional, there exists a positive constant $C>0$, depending only on N, α and γ, such that

$$\left(\forall\, p\in\Pi_N^d\right)\quad \|p\|_{\alpha,\gamma}^2 \le C\left(\left\|S_0^{(\alpha,\gamma)}p\right\|_{\alpha,\gamma}^2 + \left\|\mathcal{D}^{(\gamma)}p\right\|_{\alpha+1,\gamma}^2\right).$$

In particular, with $p=S_N^{(\alpha,\gamma)}u$ and using part (iv) of Proposition 2.3.3, we have

$$\begin{aligned}\left\|S_N^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 &\le C\left(\left\|S_0^{(\alpha,\gamma)}S_N^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 + \left\|\mathcal{D}^{(\gamma)}S_N^{(\alpha,\gamma)}u\right\|_{\alpha+1,\gamma}^2\right)\\ &= C\left(\left\|S_0^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 + \left\|S_{N-1}^{(\alpha+1,\gamma)}\mathcal{D}^{(\gamma)}u\right\|_{\alpha+1,\gamma}^2\right) \le C\left(\left\|S_0^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 + \left\|\mathcal{D}^{(\gamma)}u\right\|_{\alpha+1,\gamma}^2\right). \end{aligned}\tag{2.5.4}$$

In turn, as $\operatorname{proj}_n^{(\alpha,\gamma)}(u)\in\mathcal{V}_n^{(\alpha,\gamma)}$, by (2.4.15), (2.4.14), part (iv) of Proposition 2.3.3, part (iii) of Proposition 2.3.4 and taking into account that $\|\operatorname{Skew}_i\cdot\|_{\alpha,\gamma}\le\|\cdot\|_{\alpha,\gamma}$ for all $i\in\{1,\dots,d\}$, we obtain

$$\begin{aligned} n(n+2\lambda^{\alpha,\gamma})\left\|\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2 &= B(\operatorname{proj}_n^{(\alpha,\gamma)}(u),\operatorname{proj}_n^{(\alpha,\gamma)}(u))\\ &= \left\|\mathcal{D}^{(\gamma)}\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha+1,\gamma}^2 + \sum_{1\le i<j\le d}\left\|\mathcal{D}_{i,j}^{(\gamma)}\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2\\ &\quad - 2\lambda^{\alpha,\gamma}\sum_{i=1}^d\gamma_i\left\|\operatorname{Skew}_i\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2 + \sum_{i,j=1}^d\gamma_i\gamma_j\left\|\operatorname{Skew}_i\operatorname{Skew}_j\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2\\ &\le \left\|\operatorname{proj}_{n-1}^{(\alpha+1,\gamma)}\mathcal{D}^{(\gamma)}(u)\right\|_{\alpha+1,\gamma}^2 + \sum_{1\le i<j\le d}\left\|\operatorname{proj}_n^{(\alpha,\gamma)}\mathcal{D}_{i,j}^{(\gamma)}(u)\right\|_{\alpha,\gamma}^2 + \tilde{C}\left\|\operatorname{proj}_n^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}^2, \end{aligned}\tag{2.5.5}$$

where $\tilde{C}=\tilde{C}(\alpha,\gamma):=2\left|\lambda^{\alpha,\gamma}\right|\sum_{i=1}^d|\gamma_i| + \sum_{i,j=1}^d|\gamma_i\gamma_j|$. Let us now fix $N\in\mathbb{N}$ to any value which ensures that $\tilde{C}<n(n+2\lambda^{\alpha,\gamma})$ for all $n>N$. Then, combining (2.5.3), (2.5.4) and (2.5.5) and using Parseval's identity (2.3.5) again, we obtain

$$\begin{aligned}\|u\|_{\alpha,\gamma}^2 &\le C\left(\left\|S_0^{(\alpha,\gamma)}u\right\|_{\alpha,\gamma}^2 + \left\|\mathcal{D}^{(\gamma)}u\right\|_{\alpha+1,\gamma}^2\right)\\ &\quad + \sup_{n>N}\frac{1}{n(n+2\lambda^{\alpha,\gamma})-\tilde{C}}\left[\left\|\mathcal{D}^{(\gamma)}u\right\|_{\alpha+1,\gamma}^2 + \sum_{1\le i<j\le d}\left\|\mathcal{D}_{i,j}^{(\gamma)}u\right\|_{\alpha,\gamma}^2\right].\end{aligned}$$

The result follows upon using the bounds $\|\cdot\|_{\alpha+1,\gamma} \le \|\cdot\|_{\alpha,\gamma}$ and $\left\|\mathcal{D}_{i,j}^{(\gamma)}\cdot\right\|_{\alpha,\gamma}^2 \le 2\left\|\mathcal{D}^{(\gamma)}\cdot\right\|_{\alpha,\gamma}^2$. □

We can now prove our sharpness result.

Theorem 2.5.4. *For all $\alpha > -1$ and $\gamma \in (-1,\infty)^d$, Theorem 2.1.1 is sharp in the case $l = r = 1$.*

Proof. Let $P_n^{(\alpha,\beta)}$ denote the Jacobi polynomial of parameter (α,β) and degree n [42, Ch. IV]. From [42, Eqs. (4.21.7) and (4.3.3)] and [1, Eq. (6.4.21)],

$$P_n^{(\alpha,\beta)\prime}(x) = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}(x), \tag{2.5.6}$$

$$\begin{aligned} h_n^{(\alpha,\beta)} &:= \int_{-1}^{1} \left|P_n^{(\alpha,\beta)}(x)\right|^2 (1-x)^\alpha (1+x)^\beta \,\mathrm{d}x \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1} \frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+1)\Gamma(n+\alpha+\beta+1)}, \end{aligned} \tag{2.5.7}$$

$$P_n^{(\alpha,\beta)}(x) = \frac{n+\alpha+\beta+1}{2n+\alpha+\beta+1} P_n^{(\alpha+1,\beta)}(x) - \frac{n+\beta}{2n+\alpha+\beta+1} P_{n-1}^{(\alpha+1,\beta)}(x); \tag{2.5.8}$$

the last expression in (2.5.7) must be modified if $n = 0$. Let us adopt the abbreviation $s(\gamma) = \sum_{j=1}^d \gamma_j$. Given $n \in \mathbb{N}$, we define $t_{\alpha,\gamma,n} \in \Pi_{2n}^d$ by

$$\begin{aligned} t_{\alpha,\gamma,n}(x) := \frac{2n+2\lambda^{\alpha,\gamma}-2}{4n+2\lambda^{\alpha,\gamma}-2} P_n^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}(2\|x\|^2-1) \\ - \frac{2n+s(\gamma)+d-2}{4n+2\lambda^{\alpha,\gamma}-2} P_{n-1}^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}(2\|x\|^2-1). \end{aligned} \tag{2.5.9}$$

From [17, Prop. 8.1.5], we learn that the first term defining $t_{\alpha,\gamma,n}$ in (2.5.9) is a member of $\mathcal{V}_{2n}^{(\alpha,\gamma)}$ and the second is a member of $\mathcal{V}_{2n-2}^{(\alpha,\gamma)}$. Therefore

$$R_{\alpha,\gamma,n}(x) := t_{\alpha,\gamma,n} - S_{2n-1}^{(\alpha,\gamma)}(t_{\alpha,\gamma,n})(x) = \frac{2n+2\lambda^{\alpha,\gamma}-2}{4n+2\lambda^{\alpha,\gamma}-2} P_n^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}(2\|x\|^2-1). \tag{2.5.10}$$

As $R_{\alpha,\gamma,n}$ is a radial member of $\mathcal{V}_{2n}^{(\alpha,\gamma)}$, from part (ii) of Proposition 2.4.7,

$$\left\|\mathcal{D}^{(\gamma)} R_{\alpha,\gamma,n}\right\|_{\alpha,\gamma}^2 = \frac{2n(2n+2\lambda^{\alpha,\gamma})(2n+\lambda^{\alpha,\gamma})}{\alpha+1} \|R_{\alpha,\gamma,n}\|_{\alpha,\gamma}^2. \tag{2.5.11}$$

Also,

$$\begin{aligned}\|R_{\alpha,\gamma,n}\|_{\alpha,\gamma}^2 &= \frac{(2n+2\lambda^{\alpha,\gamma}-2)^2}{(4n+2\lambda^{\alpha,\gamma}-2)^2}\int_{B^d}\left|P_n^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}(2\|x\|^2-1)\right|^2 W_{\alpha,\gamma}(x)\,\mathrm{d}x \\ &= \frac{(2n+2\lambda^{\alpha,\gamma}-2)^2}{(4n+2\lambda^{\alpha,\gamma}-2)^2}2^{-(2+\alpha+\frac{1}{2}s(\gamma)+\frac{d-2}{2})}h_n^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}\left|\mathbb{S}^{d-1}\right|_\gamma,\end{aligned} \tag{2.5.12}$$

where $\left|\mathbb{S}^{d-1}\right|_\gamma := \int_{\mathbb{S}^{d-1}} W_{0,\gamma}(x)\,\mathrm{d}S(x)$; the integral was computed by first switching to generalized spherical coordinates and then performing the change of variable $t = 2r^2-1$. Given $j \in \{1,\ldots,d\}$,

$$\begin{aligned}\mathcal{D}_j^{(\gamma)}t_{\alpha,\gamma,n}(x) &\overset{(2.5.6)}{=} \frac{2n+2\lambda^{\alpha,\gamma}-2}{4n+2\lambda^{\alpha,\gamma}-2}x_j\Big[(2n+2\lambda^{\alpha,\gamma})P_{n-1}^{(\alpha+1,\frac{1}{2}s(\gamma)+\frac{d}{2})}(2\|x\|^2-1) \\ &\qquad - (2n+s(\gamma)+d-2)P_{n-2}^{(\alpha+1,\frac{1}{2}s(\gamma)+\frac{d}{2})}(2\|x\|^2-1)\Big] \\ &\overset{(2.5.8)}{=} (2n+2\lambda^{\alpha,\gamma}-2)\,x_j\,P_{n-1}^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d}{2})}(2\|x\|^2-1).\end{aligned}$$

Hence,

$$\begin{aligned}\left\|\mathcal{D}^{(\gamma)}t_{\alpha,\gamma,n}\right\|_{\alpha,\gamma}^2 &= (2n+2\lambda^{\alpha,\gamma}-2)^2\int_{B^d}\|x\|^2\left|P_{n-1}^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d}{2})}(2\|x\|^2-1)\right|^2 W_{\alpha,\gamma}(x)\,\mathrm{d}x \\ &= (2n+2\lambda^{\alpha,\gamma}-2)^2 2^{-(2+\alpha+\frac{1}{2}s(\gamma)+\frac{d}{2})}h_{n-1}^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d}{2})}\left|\mathbb{S}^{d-1}\right|_\gamma,\end{aligned} \tag{2.5.13}$$

where the integral over B^d was computed similarly to that in (2.5.12). Therefore, for $n \geq 2$,

$$\begin{aligned}\frac{\left\|\mathcal{D}^{(\gamma)}R_{\alpha,\gamma,n}\right\|_{\alpha,\gamma}^2}{\|\mathcal{D}^{(\gamma)}t_{\alpha,\gamma,n}\|_{\alpha,\gamma}^2} &\overset{(2.5.11)}{=} \frac{2n(2n+2\lambda^{\alpha,\gamma})(2n+\lambda^{\alpha,\gamma})}{\alpha+1}\frac{\|R_{\alpha,\gamma,n}\|_{\alpha,\gamma}^2}{\|\mathcal{D}^{(\gamma)}t_{\alpha,\gamma,n}\|_{\alpha,\gamma}^2} \\ &\overset{(2.5.12),(2.5.13)}{=} \frac{2n(2n+2\lambda^{\alpha,\gamma})(2n+\lambda^{\alpha,\gamma})}{\alpha+1}\frac{2h_n^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d-2}{2})}}{(4n+2\lambda^{\alpha,\gamma}-2)^2 h_{n-1}^{(\alpha,\frac{1}{2}s(\gamma)+\frac{d}{2})}} \\ &\overset{(2.5.7)}{=} \frac{4n(2n+2\lambda^{\alpha,\gamma})(2n+\lambda^{\alpha,\gamma})}{(\alpha+1)(4n+2\lambda^{\alpha,\gamma}-2)^2}\frac{(2n+\lambda^{\alpha,\gamma}-1)\Gamma(n+\alpha+1)\Gamma(n)}{2(2n+\lambda^{\alpha,\gamma})\Gamma(n+\alpha)\Gamma(n+1)} \\ &= \frac{(2n+2\lambda^{\alpha,\gamma})(n+\alpha)}{(\alpha+1)(4n+2\lambda^{\alpha,\gamma}-2)} \sim \frac{2n-1}{4(\alpha+1)} \quad \text{as } n\to\infty,\end{aligned} \tag{2.5.14}$$

where we have exploited the identity $\Gamma(z+1) = z\Gamma(z)$ and we use $\sim$ to denote that the ratio of two expressions thus linked tends to 1. As $u \mapsto \left\|\mathcal{D}^{(\gamma)}u\right\|_{\alpha,\gamma} + \left\|S_0^{(\alpha,\gamma)}(u)\right\|_{\alpha,\gamma}$ is an equivalent norm for $\mathrm{H}^1_{\alpha,\gamma}$ (cf. Proposition 2.5.3) and both $t_{\alpha,\gamma,n}$ and $R_{\alpha,\gamma,n}$ are $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal to $\mathcal{V}_0^{(\alpha,\gamma)}$ if $n \geq 2$, we infer from (2.5.14) that there exists a positive constant C depending on d, α and γ only such that

$$\lim_{n\to\infty} \frac{\left\|t_{\alpha,\gamma,n} - S_{2n-1}^{(\alpha,\gamma)}(t_{\alpha,\gamma,n})\right\|_{\alpha,\gamma;1}}{\left\|t_{\alpha,\gamma,n}\right\|_{\alpha,\gamma;1}(2n-1)^{1/2}} = C.$$

Thus, the $l = r = 1$ instance of Theorem 2.1.1 is sharp, because otherwise the left-hand side limit would vanish. □

CHAPTER 3

Characterization of Dunkl–Sobolev orthogonal polynomials

3.1 Introduction

In Chapter 2, we proved that the orthogonal projector $S_N^{(\alpha,\gamma)}$ satisfies the bound

Theorem 2.1.1. *For all integers $1 \le r \le l$, $\alpha \in (-1,\infty)$ and $\gamma \in (-1,\infty)^d$, there exists $C = C(\alpha,\gamma,l,r) > 0$ such that*

$$(\forall\, u \in \mathrm{H}^l_{\alpha,\gamma}) \quad \left\| u - S_N^{(\alpha,\gamma)}(u) \right\|_{\alpha,\gamma;r} \le C\, N^{-1/2+2r-l} \left\| u \right\|_{\alpha,\gamma;l} .$$

As it was mentioned in Section 2.1, the mismatch between the orthogonality that defines the projection operator $S_N^{(\alpha,\gamma)}$ and the Hilbert norm in which the error is measured makes us expect Theorem 2.1.1 to be non-optimal for general l and r.

Obviously, considering the orthogonal projector defined by the Dunkl–Sobolev inner product

$\langle\cdot,\cdot\rangle_{\alpha,\gamma;r}$ (or any other equivalent inner product) would lead to the optimal approximation rate with respect to N. In order to learn what this optimal approximation rate is, in the light of the arguments of Chapter 2, we expect it will be useful to characterize orthogonal polynomial spaces with respect to the aforementioned inner product as eigenspaces of suitable weak Sturm–Liouville problems.

The purpose of this chapter is studying and characterizing the orthogonal polynomials spaces associated to a suitable first order Dunkl–Sobolev inner product equivalent to $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1}$ (cf. (2.2.22)) in terms of the orthogonal polynomials $\mathcal{V}_n^{\alpha,\gamma}$ defined in Chapter 2, culminating in the characterization as eigenspaces of explicit self-adjoint Sturm–Liouville problems. We expect that this characterization will lead to quasi-optimal approximation results via arguments much in the vein of those of Section 2.4. In particular, we expect to be able to smoothly readapt the arguments used to obtain Corollary 2.4.5 to deduce its analogue in this context; that is, a bound of the $\mathrm{H}^1_{\alpha,\gamma}$-orthogonal projector error measured in $\|\cdot\|_{\alpha,\gamma;1}$ in terms of powers on the degree of approximation.

The work relies heavily in previous results obtained in Chapter 2—specifically the parameter (non-) shifting properties found in Proposition 2.3.3 and Proposition 2.3.4—and commutator properties involving Dunkl operators and their derived operators.

The outline of this chapter is as follows. In Section 3.2 we introduce additional notation and prove basic related results that will be used later. In Section 3.3 we define the $\mathrm{H}^1_{\alpha,\gamma}$-orthogonal polynomial spaces associated with a certain inner product equivalent to $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1}$ and state some of their basic properties. In Section 3.4 we study some commutators between operators derived from Dunkl operators and use them to decompose the relevant Dunkl–Sobolev orthogonal polynomial spaces in terms of $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomial spaces. In Section 3.5 we prove that orthogonal polynomials with respect to the aforementioned equivalent inner product satisfy two Sturm–Liouville problems strongly related to the one satisfied by $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomials (cf. (2.4.15)).

3.2 Preliminary definitions and results

The operator $\Delta^{(\gamma)}$ (cf. (2.4.2)) is called the h-Laplacian operator associated with the weight $h_\gamma(x) := \prod_{j=1}^d |x_j|^{\gamma_j}$. We say that a polynomial p is h-harmonic if $\Delta^{(\gamma)} p = 0$. Given $n \in \mathbb{N}_0$, we denote the space of h-harmonic homogeneous polynomials of degree n by $\mathcal{H}_n^d(h_\gamma)$. If $n < 0$ we adopt the convention $\mathcal{H}_n^d(h_\gamma) = \{0\}$. From [17, Th. 7.1.6] we know that the h-harmonic homogeneous polynomials are orthogonal polynomials with respect to the inner product $(p, q) \mapsto \int_{\mathbb{S}^{d-1}} p\, q\, h_\gamma \,\mathrm{d}S$. Moreover, [17, Th. 7.1.7] gives us an explicit expression for the dimension of $\mathcal{H}_n^d(h_\gamma)$,

$$\dim\left(\mathcal{H}_n^d(h_\gamma)\right) = \binom{n+d-1}{n} - \binom{n+d-3}{n-2}. \tag{3.2.1}$$

For $n \in \mathbb{N}_0$, let $\{Y_\nu^n\}_{\nu=1}^{\dim(\mathcal{H}_n^d(h_\gamma))}$ be an orthonormal basis of $\mathcal{H}_n^d(h_\gamma)$.

Proposition 3.2.1 ([17, Prop. 8.1.5])**.** *Let $\alpha > -1$, $\gamma \in (-1, \infty)^d$ and $n \in \mathbb{N}_0$. Then, the polynomials $P_{j,\nu}^n$, where $j \in \{0, \dots, \lfloor n/2 \rfloor\}$ and $\nu \in \{1, \dots, \dim(\mathcal{H}_{n-2j}^d(h_\gamma))\}$, defined by*

$$P_{j,\nu}^n(x) := P_j^{\left(\alpha, n-2j+\sum_{i=1}^d \frac{\gamma_i}{2} + \frac{d-2}{2}\right)}(2\,\|x\|^2 - 1)\, Y_\nu^{n-2j}(x),$$

form an $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal basis of $\mathcal{V}_n^{\alpha,\gamma}$.

A consequence of Proposition 3.2.1 is that, for all $\alpha > -1$ and $\gamma \in (-1, \infty)^d$,

$$\mathcal{V}_0^{\alpha,\gamma} = \operatorname{span}(\{1\}) \quad \text{and} \quad \mathcal{V}_1^{\alpha,\gamma} = \operatorname{span}\left(\{x \mapsto x_i\}_{i=1}^d\right). \tag{3.2.2}$$

The following commutator property will help us prove that $\mathcal{D}_{i,j}^{(\gamma)}$ maps $\mathcal{H}_n^d(h_\gamma)$ to itself.

Proposition 3.2.2. *Let $i, k, l \in \{1, \dots, d\}$, $\alpha \in \mathbb{R}$ and $\gamma \in \mathbb{R}^d$. Then,*

$$\begin{aligned} \mathcal{D}_i^{(\gamma)}\, \mathcal{D}_{k,l}^{(\gamma)} - \mathcal{D}_{k,l}^{(\gamma)}\, \mathcal{D}_i^{(\gamma)} &= \delta_{i,k}(I + \gamma_i \sigma_i^*)\mathcal{D}_l^{(\gamma)} - \delta_{i,l}(I + \gamma_i \sigma_i^*)\mathcal{D}_k^{(\gamma)} \\ &= \delta_{i,k}\mathcal{D}_l^{(\gamma)}(I + \gamma_i \sigma_i^*) - \delta_{i,l}\mathcal{D}_k^{(\gamma)}(I + \gamma_i \sigma_i^*) \end{aligned} \tag{3.2.3}$$

and

$$(\mathcal{D}_i^{(\gamma)})^2 \mathcal{D}_{k,l}^{(\gamma)} - \mathcal{D}_{k,l}^{(\gamma)} (\mathcal{D}_i^{(\gamma)})^2 = 2(\delta_{i,k}\mathcal{D}_l^{(\gamma)} - \delta_{i,l}\mathcal{D}_k^{(\gamma)})\mathcal{D}_i^{(\gamma)} \tag{3.2.4}$$

Proof. (3.2.3) is a direct consequence of the definitions of the operators $\mathcal{D}_{k,l}^{(\gamma)}$ and $\mathcal{D}_i^{(\gamma)}$, the identity $\mathcal{D}_i^{(\gamma)}(x_l f) = x_l \mathcal{D}_i^{(\gamma)} f + \delta_{i,l}(I + \gamma_i \sigma_i^*) f$ (cf. (2.2.17)) and the fact that the Dunkl operators $\mathcal{D}_i^{(\gamma)}$ commute with each other. (3.2.4) is obtained using (3.2.3) twice and the identity $\mathcal{D}_i^{(\gamma)}\sigma_i^* = -\sigma_i^*\mathcal{D}_i^{(\gamma)}$. □

Summing over $i \in \{1, \dots, d\}$ in (3.2.4), we conclude that

$$\Delta^{(\gamma)}\mathcal{D}_{k,l}^{(\gamma)} = \mathcal{D}_{k,l}^{(\gamma)}\Delta^{(\gamma)} \tag{3.2.5}$$

and therefore

$$\mathcal{D}_{i,j}^{(\gamma)}\mathcal{H}_n^d(h_\gamma) \subset \mathcal{H}_n^d(h_\gamma). \tag{3.2.6}$$

3.3 Definition of Dunkl–Sobolev orthogonal polynomial spaces

In Proposition 2.5.3 we proved that the following is an equivalent inner product for $(\mathrm{C}^1(\overline{B^d}), \langle\cdot,\cdot\rangle_{\alpha,\gamma;1})$

$$\langle u, v\rangle_{\alpha,\gamma;1,\mathrm{P}} := \langle \mathcal{D}^{(\gamma)}u, \mathcal{D}^{(\gamma)}v\rangle_{\alpha,\gamma} + \langle S_0^{\alpha,\gamma}(u), S_0^{\alpha,\gamma}(v)\rangle_{\alpha,\gamma}. \tag{3.3.1}$$

Therefore the topological completion of $(\mathrm{C}^1(\overline{B^d}), \langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}})$ equals $\mathrm{H}^1_{\alpha,\gamma}$, with the extension of $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$ to $\mathrm{H}^1_{\alpha,\gamma}$ (cf. Definition 2.2.2) being an equivalent inner product.

Given $n \in \mathbb{N}_0$, we denote by $\mathcal{V}_n^{\alpha,\gamma,1}$ the space of $\mathrm{H}^1_{\alpha,\gamma}$-orthogonal polynomials of degree n with respect to the inner product $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$; that is,

$$\mathcal{V}_n^{\alpha,\gamma,1} := \{p \in \Pi_n^d \mid (\forall\, q \in \Pi_{n-1}^d)\ \langle p, q\rangle_{\alpha,\gamma;1,\mathrm{P}} = 0\}. \tag{3.3.2}$$

By the usual arguments (cf. [17, Sec. 3.1]),

$$\dim\left(\mathcal{V}_n^{\alpha,\gamma,1}\right) = \dim(\Pi_n^d) - \dim(\Pi_{n-1}^d) = \binom{n+d-1}{n}. \tag{3.3.3}$$

Note that, as $\mathcal{D}^{(\gamma)}1 = 0$, given $n \geq 1$ and $p_n \in \mathcal{V}_n^{\alpha,\gamma,1}$, $0 = \langle p_n, 1\rangle_{\alpha,\gamma;1,\mathrm{P}} = \langle S_0^{\alpha,\gamma}(p_n), S_0^{\alpha,\gamma}(1)\rangle_{\alpha,\gamma} =$

$\langle S_0^{\alpha,\gamma}(p_n), 1\rangle_{\alpha,\gamma}$, so $\mathcal{V}_n^{\alpha,\gamma,1} \perp_{\alpha,\gamma} 1$. Thus, we have the following analogue of (3.2.2): For all $\alpha > -1$ and $\gamma \in (-1,+\infty)^d$,

$$\mathcal{V}_0^{\alpha,\gamma,1} = \operatorname{span}(\{1\}) \quad \text{and} \quad \mathcal{V}_1^{\alpha,\gamma,1} = \operatorname{span}\left((x \mapsto x_i)_{i=1}^d\right). \tag{3.3.4}$$

On account of part (iii) of Proposition 2.3.3, one might expect that, roughly speaking, $\mathcal{V}_n^{\alpha,\gamma,1}$ must be related to $\mathcal{V}_n^{\alpha-1,\gamma}$. The following result states that this is indeed the case, with equality, if $\alpha > 0$ and $n \neq 2$.

Proposition 3.3.1. *Let $\alpha > 0$, $\gamma \in (-1,+\infty)^d$ and $n \in \mathbb{N}_0 \setminus \{2\}$. Then, $\mathcal{V}_n^{\alpha,\gamma,1} = \mathcal{V}_n^{\alpha-1,\gamma}$.*

Proof. If $n = 0$ or $n = 1$ this comes from (3.2.2) and (3.3.4). Let us suppose now that $n \geq 3$ and let $p_n \in \mathcal{V}_n^{\alpha-1,\gamma}$. Then, $\langle p_n, 1\rangle_{\alpha,\gamma} = \langle p_n, x \mapsto (1-\|x\|^2)\rangle_{\alpha-1,\gamma} = 0$, so $S_0^{\alpha,\gamma}(p_n) = 0$. Hence, given $q \in \Pi_{n-1}^d$, $\langle p_n, q\rangle_{\alpha,\gamma;1,\mathrm{P}} = \langle \mathcal{D}^{(\gamma)} p_n, \mathcal{D}^{(\gamma)} q\rangle_{\alpha,\gamma} \overset{\text{Prop. 2.3.3(iii)}}{=} 0$. This establishes that $\mathcal{V}_n^{\alpha-1,\gamma}$ is a subspace of $\mathcal{V}_n^{\alpha,\gamma,1}$. As, per (2.3.3) and (3.3.3), $\dim(\mathcal{V}_n^{\alpha-1,\gamma}) = \dim(\mathcal{V}_n^{\alpha,\gamma,1})$, we obtain the desired equality. □

The rest of this chapter can be seen as an effort to extend the consequences of the above result to the whole natural range for α, namely $(-1,\infty)$.

3.4 Decomposition of Dunkl–Sobolev orthogonal polynomial spaces

If $\alpha > 0$, from part (ii) of Proposition 2.3.3, $\mathcal{D}_j^{(\alpha-1,\gamma;\star)} \mathcal{V}_n^{\alpha,\gamma} \subset \mathcal{V}_{n+1}^{\alpha-1,\gamma}$. Combining this with part (iii) of Proposition 2.3.3, it is immediate that $\mathcal{D}_i^{(\gamma)} \mathcal{D}_j^{(\alpha-1,\gamma;\star)} \mathcal{V}_n^{\alpha,\gamma} \subset \mathcal{V}_n^{\alpha,\gamma}$. Even though the inclusion $\mathcal{D}_j^{(\alpha-1,\gamma;\star)} \mathcal{V}_n^{\alpha,\gamma} \subset \mathcal{V}_{n+1}^{\alpha-1,\gamma}$ cannot be extended to $\alpha > -1$, its combination with part (iii) of Proposition 2.3.3 can, as we prove below in Proposition 3.4.2. First, however, we need the following commutation relation that will be later use in combination with (3.2.3).

Proposition 3.4.1. *Let $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$ and $j \in \{1,\ldots,d\}$. Then,*

$$\mathcal{D}_i^{(\gamma)} \mathcal{D}_j^{(\alpha-1,\gamma;\star)} - \mathcal{D}_j^{(\alpha,\gamma;\star)} \mathcal{D}_i^{(\gamma)} = 2\mathcal{D}_{i,j}^{(\gamma)} + 2\alpha\delta_{i,j}(I + \gamma_i \sigma_i^*), \tag{3.4.1}$$

Proof. It is a direct consequence of the definition of the operators $\mathcal{D}_i^{(\gamma)}$, $\mathcal{D}_j^{(\alpha,\gamma;\star)}$ and $\mathcal{D}_{i,j}^{(\gamma)}$ in (2.2.12), (2.3.6) and (2.3.13), and the identity $\mathcal{D}_i^{(\gamma)}(x_j f) = x_j \mathcal{D}_i^{(\gamma)} f + \delta_{i,j}(I+\gamma_i\sigma_i^*)f$ (cf. (2.2.17)) □

Proposition 3.4.2. *Let $\alpha > -1$, $\gamma \in (-1,+\infty)^d$ and $n \in \mathbb{N}_0$.*

(i) Let $i, j \in \{1,\ldots,d\}$. Then,

$$\mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{V}_n^{\alpha,\gamma} \subset \mathcal{V}_n^{\alpha,\gamma}.$$

(ii) Let $i, j, k, l \in \{1,\ldots,d\}$. Then,

$$\mathcal{D}_i^{(\gamma)}\mathcal{D}_{k,l}^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{V}_n^{\alpha,\gamma} \subset \mathcal{V}_n^{\alpha,\gamma}.$$

Proof. Let $p_n \in \mathcal{V}_n^{\alpha,\gamma}$. By (3.4.1) in Proposition 3.4.1, $\mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}p_n$ coincides with $\mathcal{D}_j^{(\alpha,\gamma;\star)}\mathcal{D}_i^{(\gamma)}p_n + 2\mathcal{D}_{i,j}^{(\gamma)}p_n + 2\alpha\delta_{i,j}(I+\gamma_i\sigma_i^*)p_n$. The first term belongs to $\mathcal{V}_n^{\alpha,\gamma}$ because of part (iii) of Proposition 2.3.3 and part (ii) of Proposition 2.3.3, the second because of part (ii) of Proposition 2.3.4 and the third because of part (ii) of Proposition 2.3.2. Thus, we have proved part (i). Similarly, by (3.2.3) in Proposition 3.2.2, $\mathcal{D}_i^{(\gamma)}\mathcal{D}_{k,l}^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}p_n$ coincides with $\mathcal{D}_{k,l}^{(\gamma)}\mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}p_n + \delta_{i,k}(I+\gamma_i\sigma_i^*)\mathcal{D}_l^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}p_n - \delta_{i,l}(I+\gamma_i\sigma_i^*)\mathcal{D}_k^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}p_n$. Each of the resulting three terms belongs to $\mathcal{V}_n^{\alpha,\gamma}$ because of part (i) with the help of part (ii) of Proposition 2.3.4 in the case of the first and of part (ii) of Proposition 2.3.2 for both the second and third terms. This accounts for part (ii). □

Proposition 3.4.3. *Let $\alpha > -1$, $\gamma \in (-1,+\infty)^d$ and $\beta \in [\mathbb{N}_0]^d$. Then, $\mathcal{D}_\beta^{(\gamma)}\mathcal{H}_n^d(h_\gamma) \subset \mathcal{V}_{n-|\beta|}^{\alpha,\gamma}$.*

Proof. Let $h_n \in \mathcal{H}_n^d(h_\gamma)$. Then, using (2.2.16), $\Delta^{(\gamma)}(\mathcal{D}_i^{(\gamma)}h_n) = \mathcal{D}_i^{(\gamma)}\Delta^{(\gamma)}h_n = 0$. Also, given any $x \in B^d$ and $s > 0$, $\mathcal{D}_i^{(\gamma)}h_n(sx) = s^{-1}\mathcal{D}_i^{(\gamma)}(h_n(sx)) = s^{-1}\mathcal{D}_i^{(\gamma)}(s^n h_n(x)) = s^{n-1}\mathcal{D}_i^{(\gamma)}h_n(x)$. Thus, $\mathcal{D}_i^{(\gamma)}h_n \in \mathcal{H}_{n-1}^d(h_\gamma)$. The desired result then follows from induction on $|\beta|$ and Proposition 3.2.1. □

Let $\mathcal{M}^{\alpha,\gamma}$ denote the second order differential-difference operator defined by

$$\begin{aligned}\mathcal{M}^{\alpha,\gamma}(u) &:= (1-\|\cdot\|^2)^{1-\alpha}\Delta^{(\gamma)}\left((1-\|\cdot\|^2)^{1+\alpha}u\right)\\ &\overset{(2.3.6)}{=} \sum_{j=1}^{d}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\alpha,\gamma;\star)}u.\end{aligned} \tag{3.4.2}$$

Proposition 3.4.4. *Let $\alpha\in(-1,\infty)$, $\gamma\in(-1,\infty)^d$ and $j\in\{1,\ldots,d\}$. Then, for all $u,v\in$ $\mathrm{C}^2(\overline{B^d})$*

$$\begin{aligned}&\langle\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\alpha,\gamma;\star)}u,v\rangle_{\alpha,\gamma}\\ &\quad=-\left\langle\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)^{\alpha+1}u\right),\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)v\right)\right\rangle_{0,\gamma}=\left\langle u,\mathcal{D}_j^{(\gamma)}\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)v\right)\right\rangle_{\alpha+1,\gamma}.\end{aligned} \tag{3.4.3}$$

Proof. The first and second equalities of (3.4.3) can be respectively rewritten as

$$\left\langle\mathcal{D}_j^{(\gamma)}\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)^{\alpha+1}u\right),v\right\rangle_{1,\gamma}=\left\langle\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)^{\alpha+1}u\right),\mathcal{D}_j^{(0,\gamma;\star)}(v)\right\rangle_{0,\gamma}, \tag{3.4.4}$$

and

$$\left\langle\mathcal{D}_j^{(\alpha,\gamma;\star)}(u),\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)v\right)\right\rangle_{\alpha,\gamma}=\left\langle u,\mathcal{D}_j^{(\gamma)}\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)v\right)\right\rangle_{\alpha+1,\gamma}, \tag{3.4.5}$$

therefore, while (3.4.5) is an instance of part (i) of Proposition 2.3.3, (3.4.4) can be seen as a formal application of the same result for non-regular enough functions; the proof of the latter can be easily obtained by readapting the one of part (i) of Proposition 2.3.3. □

Proposition 3.4.5. *Let $\alpha\in(-1,\infty)$ and $\gamma\in(-1,\infty)^d$ such that $\alpha>-1/2$ or $\gamma_i\geq 0$ for all $i\in\{1,\ldots,d\}$. Then $\mathcal{M}^{\alpha,\gamma}:\mathrm{C}^2(\overline{B^d})\to\mathrm{C}(\overline{B^d})$ is injective.*

Proof. Let $u\in\mathrm{C}^2(\overline{B^d})$ such that $\mathcal{M}^{\alpha,\gamma}u=0$ in B^d. Suppose at first that $\alpha>-1/2$, it can be proved in a similar way as the omitted proof of the first equality of (3.4.3) that for all $j\in\{1,\ldots,d\}$

$$\langle\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\alpha,\gamma;\star)}u,(1-\|\cdot\|^2)^{\alpha}u\rangle_{\alpha,\gamma}=-\left\|\mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)^{\alpha+1}u\right)\right\|_{0,\gamma}^2,$$

both terms making sense because of the restriction on α. Summing up with respect to j, we find that $\mathcal{D}^{(\gamma)}\left((1-\|\cdot\|^2)^{1+\alpha}\,u\right)=0$ in B^d. Therefore by Proposition 2.5.2 and $(1-\|\cdot\|^2)^{1+\alpha}\,u$ vanishing on the sphere, u must be the null function.

On the other hand, if $\gamma_i \geq 0$ for all $i \in \{1,\dots,d\}$, then the condition $\mathcal{M}^{\alpha,\gamma}u = 0$ in B^d implies that the function $(1-\|\cdot\|^2)^{1+\alpha}\,u$ is h-harmonic in the unit ball and vanishes on the unit sphere. By the maximum principle [41, Th. 4.2], it must be the null function. □

Proposition 3.4.6. *Let $\alpha > -1$, $\gamma \in (-1,+\infty)^d$, $h \in \mathrm{C}^3(\overline{B^d})$ be h-harmonic and $p \in \mathrm{C}^3(\overline{B^d})$. Then,*

$$\langle \mathcal{D}^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(p), \mathcal{D}^{(\gamma)}h\rangle_{\alpha,\gamma} = 0.$$

Proof. Denote by $\mathcal{D}^{(\alpha,\gamma;\star)}$ the corresponding gradient associated with $\mathcal{D}_j^{(\alpha,\gamma;\star)}$, that is, $\mathcal{D}^{(\alpha,\gamma;\star)}f := \sum_{j=1}^d \mathcal{D}_j^{(\alpha,\gamma;\star)}(f)\,e_j$. Let $i \in \{1,\dots,d\}$. Given $q \in \mathrm{C}^2(\overline{B^d})$, using the definition (2.3.6) and the expansion of the Dunkl operator on a product in (2.2.13) we obtain

$$\begin{aligned}
\mathcal{D}^{(\gamma)}(\mathcal{D}_i^{(\alpha-1,\gamma;\star)}q(x)) &= \mathcal{D}^{(\gamma)}\left(-(1-\|x\|^2)\mathcal{D}_i^{(\gamma)}q(x) + 2\alpha x_i\,q(x)\right)\\
&= 2\mathcal{D}_i^{(\gamma)}q(x)\,x - (1-\|x\|^2)\mathcal{D}^{(\gamma)}(\mathcal{D}_i^{(\gamma)}q(x)) + 2\alpha x_i\mathcal{D}^{(\gamma)}q(x) + 2\alpha(q(x)+\gamma_i q(\sigma_i x))\,e_i\\
&= 2(\alpha+1)\mathcal{D}_i^{(\gamma)}q(x)x - (1-\|x\|^2)\mathcal{D}^{(\gamma)}(\mathcal{D}_i^{(\gamma)}q(x)) + 2\alpha(x_i\mathcal{D}^{(\gamma)}q(x) - \mathcal{D}_i^{(\gamma)}q(x)x + (q(x)+\gamma_i q(\sigma_i x))\,e_i)\\
&= \mathcal{D}^{(\alpha,\gamma;\star)}q(x) + 2\alpha(x_i\mathcal{D}^{(\gamma)}q(x) - \mathcal{D}_i^{(\gamma)}q(x)x + (q(x)+\gamma_i q(\sigma_i x))\,e_i).
\end{aligned}$$

Setting $q = \mathcal{D}_i^{(\alpha,\gamma;\star)}p$, we find after exploiting the fact that σ_i^* is $\mathrm{L}^2_{\alpha,\gamma}$-self adjoint, that

$$\begin{aligned}
&\langle \mathcal{D}^{(\gamma)}(\mathcal{D}_i^{(\alpha-1,\gamma;\star)}\mathcal{D}_i^{(\alpha,\gamma;\star)}p), \mathcal{D}^{(\gamma)}h\rangle_{\alpha,\gamma}\\
&\quad= \langle \mathcal{D}^{(\alpha,\gamma;\star)}\mathcal{D}_i^{(\alpha,\gamma;\star)}p, \mathcal{D}^{(\gamma)}h\rangle_{\alpha,\gamma} + 2\,\alpha\langle \mathcal{D}_i^{(\alpha,\gamma;\star)}p, (I+\gamma_i\sigma_i^*)\mathcal{D}_i^{(\gamma)}h\rangle_{\alpha,\gamma}\\
&\qquad+ 2\,\alpha\int_{B^d}\left(x_i\mathcal{D}^{(\gamma)}\mathcal{D}_i^{(\alpha,\gamma;\star)}p(x) - \mathcal{D}_i^{(\gamma)}\mathcal{D}_i^{(\alpha,\gamma;\star)}p(x)x\right)\cdot\mathcal{D}^{(\gamma)}h(x)\,W_{\alpha,\gamma}(x)\,\mathrm{d}x.
\end{aligned}$$

Due to part (i) of Proposition 2.3.3 and the fact that h is h-harmonic, the first term in the right-hand side above vanishes. Thus, using the second form of the operator $\mathcal{M}^{\alpha,\gamma}$ given in (3.4.2), expressing dot products as sums making the $\mathcal{D}_{i,j}^{(\gamma)}$ operators appear, using part (i) of

Proposition 2.3.4 and part (i) of Proposition 2.3.3 again,

$$\begin{aligned}
&\langle \mathcal{D}^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(p), \mathcal{D}^{(\gamma)}h\rangle_{\alpha,\gamma} - 2\alpha\sum_{i=1}^{d}\langle p, \mathcal{D}_i^{(\gamma)}(I+\gamma_i\sigma_i^*)\mathcal{D}_i^{(\gamma)}h\rangle_{\alpha+1,\gamma}\\
&\qquad = 2\,\alpha\sum_{i,j=1}^{d}\langle \mathcal{D}_{i,j}^{(\gamma)}(\mathcal{D}_i^{(\alpha,\gamma;\star)}p), \mathcal{D}_j^{(\gamma)}h\rangle_{\alpha,\gamma} = -2\,\alpha\sum_{i,j=1}^{d}\langle \mathcal{D}_i^{(\alpha,\gamma;\star)}p, \mathcal{D}_{i,j}^{(\gamma)}\mathcal{D}_j^{(\gamma)}h\rangle_{\alpha,\gamma}\\
&\qquad = -2\,\alpha\sum_{i,j=1}^{d}\langle p, \mathcal{D}_i^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)}\mathcal{D}_j^{(\gamma)}h\rangle_{\alpha+1,\gamma}.
\end{aligned}$$

Now, by direct computation, $\mathcal{D}_{i,j}^{(\gamma)}(\mathcal{D}_j^{(\gamma)}h) = x_i\mathcal{D}_j^{(\gamma)}\mathcal{D}_j^{(\gamma)}h + \delta_{i,j}(I+\gamma_i\sigma_i^*)\mathcal{D}_j^{(\gamma)}h - \mathcal{D}_i^{(\gamma)}(x_j\mathcal{D}_j^{(\gamma)}h)$, whence

$$\begin{aligned}
&\langle \mathcal{D}^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(p), \mathcal{D}^{(\gamma)}h\rangle_{\alpha,\gamma} - 2\alpha\sum_{i=1}^{d}\langle p, \mathcal{D}_i^{(\gamma)}(I+\gamma_i\sigma_i^*)\mathcal{D}_i^{(\gamma)}h\rangle_{\alpha,\gamma}\\
&= -2\alpha\left(\sum_{i=1}^{d}\langle p, \mathcal{D}_i^{(\gamma)}(x_i\Delta^{(\gamma)})\rangle_{\alpha+1,\gamma} + \sum_{i=1}^{d}\langle p, \mathcal{D}_i^{(\gamma)}(I+\gamma_i\sigma_i^*)\mathcal{D}_i^{(\gamma)}h\rangle_{\alpha+1,\gamma} + \langle p, \Delta^{(\gamma)}(x\cdot\mathcal{D}^{(\gamma)})h\rangle_{\alpha+1,\gamma}\right).
\end{aligned}$$

Lastly, using the identities $\mathcal{D}_i^{(\gamma)}(x_jf) = x_j\mathcal{D}_i^{(\gamma)}f + \delta_{i,j}(I+\gamma_i\sigma_i^*)f$ (cf. (2.2.17)) and (2.2.15) we obtain for all $i,j\in\{1,\dots,d\}$

$$\mathcal{D}_i^{(\gamma)}\mathcal{D}_i^{(\gamma)}(x_j\mathcal{D}_j^{(\gamma)}h) = x_j\mathcal{D}_i^{(\gamma)}\mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)}h + 2\delta_{i,j}\mathcal{D}_i^{(\gamma)}\mathcal{D}_j^{(\gamma)}h,$$

from where it becomes apparent that, being h an h-harmonic function, so is $x\cdot\mathcal{D}^{(\gamma)}h$. The result then follows. □

Proposition 3.4.7. *Let $\alpha>-1$, $\gamma\in(-1,+\infty)^d$ and $n\in\mathbb{N}_0$. Then, $\mathcal{M}^{\alpha,\gamma}(\mathcal{V}_n^{\alpha+1,\gamma})\subset\mathcal{V}_n^{\alpha,\gamma}\oplus_{\alpha,\gamma}\mathcal{V}_{n+2}^{\alpha,\gamma}$.*

Proof. Let $p_n\in\mathcal{V}_n^{\alpha+1,\gamma}$. From the second form of $\mathcal{M}^{\alpha,\gamma}$ given in (3.4.2), $\mathcal{M}^{\alpha,\gamma}(p_n)\in\Pi_{n+2}^d$.

Now, from all $q\in\Pi_{n-1}^d$,

$$\langle\mathcal{M}^{\alpha,\gamma}(p_n), q\rangle_{\alpha,\gamma} \overset{(3.4.3)}{=} \langle p_n, \Delta^{(\gamma)}((1-\|\cdot\|^2)\,q)\rangle_{\alpha+1,\gamma} = 0,$$

where the vanishing of the latter term comes about because $\Delta^{(\gamma)}((1-\|\cdot\|^2)\,q)$ is a polynomial of degree equal or less than $n-1$. Thus, $\mathcal{M}^{\alpha,\gamma}(p_n) \perp_{\alpha,\gamma} \Pi^d_{n-1}$. As the h-Laplacian operator and multiplication by centrally symmetric functions preserve the parity of a function, $\mathcal{M}^{\alpha,\gamma}(p_n)$ inherits the parity of p_n given in (2.3.2), which in turn is the opposite of that of $\mathcal{V}^{\alpha,\gamma}_{n+1}$, whence $\mathcal{M}^{\alpha,\gamma}(p_n) \perp_{\alpha,\gamma} \mathcal{V}^{\alpha,\gamma}_{n+1}$. □

Lemma 3.4.8. *Let $\alpha > -1$ and $\gamma \in (-1,+\infty)^d$ such that $\alpha > -1/2$ or $\gamma_i \geq 0$ for all $i \in \{1,\dots,d\}$. Then,*

$$\mathcal{V}^{\alpha,\gamma,1}_n = \begin{cases} \mathcal{V}^{\alpha,\gamma}_n & \text{if } n \leq 2, \\ \mathcal{H}^d_n(h_\gamma) \oplus_{\alpha,\gamma,1} \mathcal{M}^{\alpha,\gamma}(\mathcal{V}^{\alpha+1,\gamma}_{n-2}) & \text{if } n \geq 3. \end{cases}$$

Proof. Directly from the definition (3.3.2), $\mathcal{V}^{\alpha,\gamma,1}_0 = \Pi^d_0 = \mathcal{V}^{\alpha,\gamma}_0$. The case $n=1$ follows from the fact that for all $p_1 \in \mathcal{V}^{\alpha,\gamma}_1$ and $p_0 \in \Pi^d_0$, $\mathcal{D}^{(\gamma)} p_0 = 0$ and $\operatorname{proj}^{\alpha,\gamma}_0(p_1) = 0$. Members of $\mathcal{V}^{\alpha,\gamma}_2$ are $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$-orthogonal to $\mathcal{V}^{\alpha,\gamma}_0$ for the same reason and $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$-orthogonal to $\mathcal{V}^{\alpha,\gamma}_1$ because, using (3.2.2) and part (iii) of Proposition 2.3.3, for each $i \in \{1,\dots,d\}$, $\mathcal{D}^{(\gamma)}_i \mathcal{V}^{\alpha,\gamma}_2 \subset \mathcal{V}^{\alpha,\gamma}_1$.

Let us suppose from now on that $n \geq 3$. Let $h_n \in \mathcal{H}^d_n(h_\gamma)$. From Proposition 3.2.1, $\operatorname{proj}^{\alpha,\gamma}_0(h_n) = 0$. Hence, for all $q \in \Pi^d_{n-1}$, $\langle h_n, q\rangle_{\alpha,\gamma;1,\mathrm{P}} = \sum_{i=1}^d \langle \mathcal{D}^{(\gamma)}_i h_n, \mathcal{D}^{(\gamma)}_i q\rangle_{\alpha,\gamma} = 0$, the latter equality following from Proposition 3.4.3 on account of each of the $\mathcal{D}^{(\gamma)}_i q$ belonging to Π^d_{n-2}. Therefore, $h_n \in \mathcal{V}^{\alpha,\gamma,1}_n$.

Let $p_{n-2} \in \mathcal{V}^{\alpha+1,\gamma}_{n-2}$. Then, $\mathcal{M}^{\alpha,\gamma}(p_{n-2}) \in \Pi^d_n$ and, from Proposition 3.4.7, $\operatorname{proj}^{\alpha,\gamma}_0(\mathcal{M}^{\alpha,\gamma}(p_n)) = 0$. Also, from part (ii) of Proposition 2.3.3, for every $j \in \{1,\dots,d\}$, $\mathcal{D}^{(\alpha,\gamma;\star)}_j p_{n-2} \in \mathcal{V}^{\alpha,\gamma}_{n-1}$. Thus, for all $q \in \Pi^d_{n-1}$,

$$\begin{aligned}\langle \mathcal{M}^{\alpha,\gamma}(p_{n-2}), q\rangle_{\alpha,\gamma;1,\mathrm{P}} &= \sum_{i=1}^d \langle \mathcal{D}^{(\gamma)}_i \mathcal{M}^{\alpha,\gamma}(p_{n-2}), \mathcal{D}^{(\gamma)}_i q\rangle_{\alpha,\gamma} \\ &\overset{(3.4.2)}{=} \sum_{j=1}^d \sum_{i=1}^d \left\langle \mathcal{D}^{(\gamma)}_i \mathcal{D}^{(\alpha-1,\gamma;\star)}_j \mathcal{D}^{(\alpha,\gamma;\star)}_j p_{n-2}, \mathcal{D}^{(\gamma)}_i q\right\rangle_{\alpha,\gamma}.\end{aligned}$$

As, per Proposition 3.4.2, each of the $\mathcal{D}^{(\gamma)}_i \mathcal{D}^{(\alpha-1,\gamma;\star)}_j$ maps $\mathcal{V}^{\alpha,\gamma}_{n-1}$ into itself and all the $\mathcal{D}^{(\gamma)}_i q$ belong to Π^d_{n-2}, we find that $\mathcal{M}^{\alpha,\gamma}(p_{n-2}) \in \mathcal{V}^{\alpha,\gamma,1}_n$.

The injectiveness of $\mathcal{M}^{\alpha,\gamma}$ (cf. Proposition 3.4.5) assures that $\dim\left(\mathcal{M}^{\alpha,\gamma}(\mathcal{V}^{\alpha+1,\gamma}_{n-2})\right) = \dim(\mathcal{V}^{\alpha+1,\gamma}_{n-2})$.

From Proposition 3.4.6, the vector space $\mathcal{M}^{\alpha,\gamma}(\mathcal{V}_{n-2}^{\alpha+1,\gamma})$ is $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$-orthogonal to $\mathcal{H}_n^d(h_\gamma)$. Hence,

$$\begin{aligned}\dim\left(\mathcal{H}_n^d(h_\gamma)\oplus\mathcal{M}^{\alpha,\gamma}(\mathcal{V}_{n-2}^{\alpha+1,\gamma})\right)&=\dim\left(\mathcal{H}_n^d(h_\gamma)\right)+\dim\left(\mathcal{M}^{\alpha,\gamma}(\mathcal{V}_{n-2}^{\alpha+1,\gamma})\right)\\&\overset{(3.2.1)}{=}\binom{n+d-1}{n}-\binom{n+d-3}{n-2}+\binom{n+d-3}{n-2}=\binom{n+d-1}{n}\overset{(3.3.3)}{=}\dim\left(\mathcal{V}_n^{\alpha,\gamma,1}\right).\end{aligned}$$

This completes the proof. □

Remark 3.4.9. By direct computation, $\mathcal{M}^{\alpha,\gamma}(1)=2(\alpha+1)\left((2\alpha+\sum_{j=1}^d\gamma_j+d)\left\|x\right\|^2-(\sum_{j=1}^d\gamma_j+d)\right)$, which is not $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,\mathrm{P}}$-orthogonal to 1. Thus, $\mathcal{V}_2^{\alpha,\gamma,1}$ cannot obey the decomposition the $\mathcal{V}_n^{\alpha,\gamma,1}$ do for $n\geq 3$.

3.5 Sturm–Liouville problems satisfied by $\mathrm{H}^1_{\alpha,\gamma}$-orthogonal polynomials

Proposition 3.5.1. *Let $\alpha>-1$, $\gamma\in(-1,+\infty)^d$ and $n\in\mathbb{N}_0$ such that $\alpha>-1/2$ or $\gamma_i\geq 0$ for all $i\in\{1,\dots,d\}$. Then for all $k\in\{1,\dots,d\}$, $\mathcal{D}_k^{(\gamma)}\mathcal{V}_n^{\alpha,\gamma,1}\subset\mathcal{V}_{n-1}^{\alpha,\gamma}$.*

Proof. If $n=0$ or $n=1$ this is immediate. Otherwise, let $p_n\in\mathcal{V}_n^{\alpha,\gamma,1}$. If $n=2$, Lemma 3.4.8 states that $p_n\in\mathcal{V}_2^{\alpha,\gamma}$, so by (3.2.2) and (iii) of Proposition 2.3.3, $\mathcal{D}_k^{(\gamma)}p_n\in\mathcal{V}_1^{\alpha,\gamma}$. If $n\geq 3$, it transpires from Lemma 3.4.8 that there exist $h_n\in\mathcal{H}_n^d(h_\gamma)$ and $r_{n-2}\in\mathcal{V}_{n-2}^{\alpha+1,\gamma}$ such that $p_n=h_n+\mathcal{M}^{\alpha,\gamma}(r_{n-2})$. By Proposition 3.4.3, $\mathcal{D}_k^{(\gamma)}h_n\in\mathcal{V}_{n-1}^{\alpha,\gamma}$. On the other hand, by the second form of $\mathcal{M}^{\alpha,\gamma}$ given in (3.4.2), $\mathcal{D}_k^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(r_{n-2})=\sum_{l=1}^d\mathcal{D}_k^{(\gamma)}\mathcal{D}_l^{(\alpha-1,\gamma;\star)}\mathcal{D}_l^{(\alpha,\gamma;\star)}r_{n-2}$, so through part (ii) of Proposition 2.3.3 and Proposition 3.4.2 we infer that $\mathcal{D}_l^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(r_{n-2})\in\mathcal{V}_{n-1}^{\alpha,\gamma}$. □

Theorem 3.5.2. *Let $\alpha>-1$, $\gamma\in(-1,+\infty)^d$ such that $\alpha>-1/2$ or $\gamma_i\geq 0$ for all $i\in\{1,\dots,d\}$. Let $n\in\mathbb{N}_0$ and $p_n\in\mathcal{V}_n^{\alpha,\gamma,1}$. Then,*

$$\begin{aligned}(\forall\, q\in\mathrm{C}^2(\overline{B^d}))\quad&\langle(\mathcal{D}^{(\gamma)})^2p_n,(\mathcal{D}^{(\gamma)})^2q\rangle_{\alpha+1,\gamma}+\sum_{1\leq i<j\leq d}\langle\mathcal{D}_{i,j}^{(\gamma)}\mathcal{D}^{(\gamma)}p_n,\mathcal{D}_{i,j}^{(\gamma)}\mathcal{D}^{(\gamma)}q\rangle_{\alpha,\gamma}\\&-2\lambda^{\alpha,\gamma}\sum_{i=1}^d\gamma_i\langle\operatorname{Skew}_i\mathcal{D}^{(\gamma)}p_n,\operatorname{Skew}_i\mathcal{D}^{(\gamma)}q\rangle_{\alpha,\gamma}\end{aligned}$$

$$+\sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j\langle \operatorname{Skew}_i \operatorname{Skew}_j \mathcal{D}^{(\gamma)}p_n, \operatorname{Skew}_i \operatorname{Skew}_j \mathcal{D}^{(\gamma)}q\rangle_{\alpha,\gamma} = \lambda_n^{\alpha,\gamma,1}\langle p_n, q\rangle_{\alpha,\gamma;1,\mathrm{P}}, \quad (3.5.1)$$

where

$$\lambda_n^{\alpha,\gamma,1} = \begin{cases} 0 & \text{if } n \le 1, \\ (n-1)(n+2\lambda^{\alpha,\gamma}-1) & \text{if } n \ge 2. \end{cases} \quad (3.5.2)$$

Proof. If $n \le 1$, all $(\mathcal{D}^{(\gamma)})^2 p_n$ and, for all admissible i and j, $\mathcal{D}_{i,j}^{(\gamma)}\mathcal{D}^{(\gamma)}p_n$ and $\operatorname{Skew}_i \mathcal{D}^{(\gamma)}p_n$ vanish and the desired result immediately follows. From now on we suppose that $n \ge 2$.

We first note that $S_0^{\alpha,\gamma}(p_n) = 0$. Indeed, $0 = \langle p_n, 1\rangle_{\alpha,\gamma;1,\mathrm{P}} = \langle S_0^{\alpha,\gamma}(p_n), 1\rangle_{\alpha,\gamma}$. From Proposition 3.5.1, for every $k \in \{1,\dots,d\}$, $\mathcal{D}_k^{(\gamma)}p_n \in \mathcal{V}_{n-1}^{\alpha,\gamma}$. As, for every $q \in \mathrm{C}^2(\overline{B^d})$, $\mathcal{D}_k^{(\gamma)}q \in \mathrm{C}^1(\overline{B^d})$, we can substitute $n \leftarrow n-1$, $p_n \leftarrow \mathcal{D}_k^{(\gamma)}p_n$ and $q \leftarrow \mathcal{D}_k^{(\gamma)}q$ in (2.4.15), sum up with respect to k and obtain the desired result upon realizing that, as $S_0^{\alpha,\gamma}(p_n) = 0$, $\langle \mathcal{D}^{(\gamma)}p_n, \mathcal{D}^{(\gamma)}q\rangle_{\alpha,\gamma} = \langle p_n, q\rangle_{\alpha,\gamma;1,\mathrm{P}}$. □

Next we state analogues of part (ii) of Proposition 2.3.3, part (iii) of Proposition 2.3.3 and part (ii) of Proposition 2.3.4 for first-order Dunkl–Sobolev orthogonal polynomial spaces.

Proposition 3.5.3. *Let $\alpha > -1$ and $\gamma \in (-1,+\infty)^d$ such that $\alpha > -1/2$ or $\gamma_i \ge 0$ for all $i \in \{1,\dots,d\}$. Let $i, j \in \{1,\dots,d\}$. Then,*

(i) For all $n \ge 3$,

$$\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{V}_n^{\alpha+1,\gamma,1} \subset \mathcal{V}_{n+1}^{\alpha,\gamma,1}.$$

(ii) For all $n \in \mathbb{N}_0$,

$$\mathcal{D}_j^{(\gamma)}\mathcal{V}_n^{\alpha,\gamma,1} \subset \mathcal{V}_{n-1}^{\alpha+1,\gamma,1}.$$

(iii) For all $n \in \mathbb{N}_0$,

$$\mathcal{D}_{i,j}^{(\gamma)}\mathcal{V}_n^{\alpha,\gamma,1} \subset \mathcal{V}_n^{\alpha,\gamma,1}.$$

Proof. Let $n \ge 3$ and $p_n \in \mathcal{V}_n^{\alpha+1,\gamma,1}$. As $\alpha + 1 > 0$, from Proposition 3.3.1, $p_n \in \mathcal{V}_n^{\alpha,\gamma}$, and therefore, by (iii) of Proposition 2.3.3, $\mathcal{D}_j^{(\gamma)}p_n \in \mathcal{V}_{n-1}^{(\alpha+1,\gamma)}$ Hence, by the definition of the operator

$\mathcal{D}_j^{(\alpha-1,\gamma;\star)}$ in (2.3.6),

$$\langle \mathcal{D}_j^{(\alpha-1,\gamma;\star)} p_n, 1\rangle_{\alpha,\gamma} = -\langle \mathcal{D}_j^{(\gamma)} p_n, 1\rangle_{\alpha+1,\gamma} + 2\alpha\langle p_n, x_j\rangle_{\alpha,\gamma} = 0,$$

so $S_0^{\alpha,\gamma}(\mathcal{D}_j^{(\alpha-1,\gamma;\star)} p_n) = 0$. Another consequence of the fact that $p_n \in \mathcal{V}_n^{\alpha,\gamma}$ is that, by Proposition 3.4.2, $\mathcal{D}^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)} p_n \in [\mathcal{V}_n^{\alpha,\gamma}]^d$. Thus, given any $q \in \Pi_n^d$, $\langle \mathcal{D}_j^{(\alpha-1,\gamma;\star)} p_n, q\rangle_{\alpha,\gamma;1,\mathrm{P}} = \langle \mathcal{D}^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)} p_n, \mathcal{D}^{(\gamma)} q\rangle_{\alpha,\gamma} = 0$ and we have proved part (i).

Part (ii) is immediate if $n = 0$ or $n = 1$. Now, let $n \geq 2$ and $p_n \in \mathcal{V}_n^{\alpha,\gamma,1}$. From Proposition 3.5.1, $\mathcal{D}_j^{(\gamma)} p_n \in \mathcal{V}_{n-1}^{\alpha,\gamma}$, so $S_0^{\alpha,\gamma}(\mathcal{D}_j^{(\gamma)} p_n) = 0$. Also, from part (iii) of Proposition 2.3.3, $\mathcal{D}^{(\gamma)}\mathcal{D}_j^{(\gamma)} p_n \in [\mathcal{V}_{n-2}^{\alpha+1,\gamma}]^d$. Then, given any $q \in \Pi_{n-2}^d$, $\mathcal{D}^{(\gamma)} q \in [\Pi_{n-3}^d]^d$, $\langle \mathcal{D}_j^{(\gamma)} p_n, q\rangle_{\alpha+1,\gamma;1,\mathrm{P}} = \langle \mathcal{D}^{(\gamma)}\mathcal{D}_j^{(\gamma)} p_n, \mathcal{D}^{(\gamma)} q\rangle_{\alpha+1,\gamma} = 0$.

If $n \in \{0,1,2\}$, Lemma 3.4.8 states that $\mathcal{V}_n^{\alpha,\gamma,1} = \mathcal{V}_n^{\alpha,\gamma}$ so part (iii) is inherited directly from (ii) of Proposition 2.3.4. Now, let $n \geq 3$. We recall that, as expressed in (3.2.5), $\mathcal{D}_{i,j}^{(\gamma)}$ and the h-Laplacian $\Delta^{(\gamma)}$ commute. As the $\mathcal{M}^{\alpha,\gamma}$ operator (3.4.2) can be expressed as the pre- and post-composition of the Laplacian with multiplication with certain radial functions, if follows that $\mathcal{D}_{i,j}^{(\gamma)}$ also commutes with $\mathcal{M}^{\alpha,\gamma}$. Hence, on account of part (ii) of Proposition 2.3.4, $\mathcal{D}_{i,j}^{(\gamma)}\mathcal{M}^{\alpha,\gamma}(\mathcal{V}_{n-2}^{\alpha+1,\gamma}) \subset \mathcal{M}^{\alpha,\gamma}(\mathcal{V}_{n-2}^{\alpha+1,\gamma})$. Also, from (3.2.6), $\mathcal{D}_{i,j}^{(\gamma)}\mathcal{H}_n^d(h_\gamma) \subset \mathcal{H}_n^d(h_\gamma)$. Thus, by the decomposition in Lemma 3.4.8, the remaining cases of (iii) follow. □

Now we show that Dunkl–Sobolev-orthogonal polynomials in $\mathcal{V}_n^{\alpha,\gamma,1}$ satisfy a second-order Sturm–Liouville problem in strong form. To that end we first deduce the following commutator property between Sturm–Liouville and Dunkl operators that will prove useful.

Proposition 3.5.4. *Let $\alpha \in \mathbb{R}$, $\gamma \in \mathbb{R}^d$ and $k \in \{1,\ldots,d\}$. Then,*

$$\mathcal{D}_k^{(\gamma)}\mathcal{L}^{(\alpha-1,\gamma)} - \mathcal{L}^{(\alpha,\gamma)}\mathcal{D}_k^{(\gamma)} = (2\lambda^{\alpha,\gamma} - 1)\mathcal{D}_k^{(\gamma)}. \tag{3.5.3}$$

Proof. By repeated application of (3.2.3) of Proposition 3.2.2 and taking into account the identity $\sigma_i^*\mathcal{D}_{i,j}^{(\gamma)} = -\mathcal{D}_{i,j}^{(\gamma)}\sigma_i^*$ (cf. (2.2.15)), we have that for all i,j satisfying $1 \leq i < j \leq d$ and $k \in \{1,\ldots,d\}$,

$$
\begin{aligned}
(\mathcal{D}_{i,j}^{(\gamma)})^2\mathcal{D}_k^{(\gamma)} &= \mathcal{D}_{i,j}^{(\gamma)}\left(\mathcal{D}_k^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)} - \delta_{k,i}\mathcal{D}_j^{(\gamma)}(I+\gamma_k\sigma_k^*) + \delta_{k,j}\mathcal{D}_i^{(\gamma)}(I+\gamma_k\sigma_k^*)\right)\\
&= \mathcal{D}_k^{(\gamma)}(\mathcal{D}_{i,j}^{(\gamma)})^2 - \delta_{k,i}\mathcal{D}_j^{(\gamma)}(I+\gamma_k\sigma_k^*)\mathcal{D}_{i,j}^{(\gamma)} + \delta_{k,j}\mathcal{D}_i^{(\gamma)}(I+\gamma_k\sigma_k^*)\mathcal{D}_{i,j}^{(\gamma)}\\
&- \delta_{k,i}(\mathcal{D}_j^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)} + (I+\gamma_j\sigma_j^*)\mathcal{D}_i^{(\gamma)})(I+\gamma_k\sigma_k^*) + \delta_{k,j}(\mathcal{D}_i^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)} - (I+\gamma_i\sigma_i^*)\mathcal{D}_j^{(\gamma)})(I+\gamma_k\sigma_k^*)\\
&= \mathcal{D}_k^{(\gamma)}(\mathcal{D}_{i,j}^{(\gamma)})^2 - 2\delta_{k,i}\mathcal{D}_j^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)} + 2\delta_{k,j}\mathcal{D}_i^{(\gamma)}\mathcal{D}_{i,j}^{(\gamma)} - \delta_{k,i}(I+\gamma_j\sigma_j^*)\mathcal{D}_i^{(\gamma)}(I+\gamma_k\sigma_k^*) - \delta_{k,j}(I+\gamma_i\sigma_i^*)\mathcal{D}_j^{(\gamma)}(I+\gamma_k\sigma_k^*),
\end{aligned}
\tag{3.5.4}
$$

so

$$
\begin{aligned}
-\sum_{1\le i<j\le d}(\mathcal{D}_{i,j}^{(\gamma)})^2\mathcal{D}_k^{(\gamma)} &= -\mathcal{D}_k^{(\gamma)}\sum_{1\le i<j\le d}(\mathcal{D}_{i,j}^{(\gamma)})^2 + 2\sum_{j=k+1}^{d}\mathcal{D}_j^{(\gamma)}\mathcal{D}_{k,j}^{(\gamma)} - 2\sum_{i=1}^{k-1}\mathcal{D}_i^{(\gamma)}\mathcal{D}_{i,k}^{(\gamma)}\\
&+ \sum_{j=k+1}^{d}(I+\gamma_j\sigma_j^*)\mathcal{D}_k^{(\gamma)}(I+\gamma_k\sigma_k^*) + \sum_{i=1}^{k-1}(I+\gamma_i\sigma_i^*)\mathcal{D}_k^{(\gamma)}(I+\gamma_k\sigma_k^*)\\
&= -\mathcal{D}_k^{(\gamma)}\sum_{1\le i<j\le d}(\mathcal{D}_{i,j}^{(\gamma)})^2 + 2\sum_{j=1}^{d}\mathcal{D}_j^{(\gamma)}\mathcal{D}_{k,j}^{(\gamma)} + \mathcal{D}_k^{(\gamma)}\left(\sum_{\substack{j=1\\ j\neq k}}^{d}(I+\gamma_j\sigma_j^*)\right)(I+\gamma_k\sigma_k^*).
\end{aligned}
\tag{3.5.5}
$$

By (3.2.3) and (3.4.1) of Proposition 3.2.2 and the fact that $\mathcal{D}_j^{(\gamma)}$ and $\mathcal{D}_k^{(\gamma)}$ commute, we find that, for all $j,k\in\{1,\dots,d\}$,

$$
\begin{aligned}
\mathcal{D}_j^{(\alpha,\gamma;\star)}\mathcal{D}_j^{(\gamma)}\mathcal{D}_k^{(\gamma)} &= \mathcal{D}_k^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\gamma)} - 2\mathcal{D}_{k,j}^{(\gamma)}\mathcal{D}_j^{(\gamma)} - 2\alpha\delta_{k,j}(I+\gamma_k\sigma_k^*)\mathcal{D}_j^{(\gamma)}\\
&= \mathcal{D}_k^{(\gamma)}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\gamma)} - 2(\mathcal{D}_j^{(\gamma)}\mathcal{D}_{k,j}^{(\gamma)} - \delta_{j,k}(I+\gamma_j\sigma_j^*)\mathcal{D}_j^{(\gamma)} + (I+\gamma_j\sigma_j^*)\mathcal{D}_k^{(\gamma)}) - 2\alpha\delta_{k,j}(I+\gamma_k\sigma_k^*)\mathcal{D}_j^{(\gamma)},
\end{aligned}
\tag{3.5.6}
$$

whence

$$
\begin{aligned}
&\sum_{j=1}^{d}\mathcal{D}_j^{(\alpha,\gamma;\star)}\mathcal{D}_j^{(\gamma)}\mathcal{D}_k^{(\gamma)}\\
&= \mathcal{D}_k^{(\gamma)}\sum_{j=1}^{d}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\gamma)} - 2\sum_{j=1}^{d}\mathcal{D}_j^{(\gamma)}\mathcal{D}_{k,j}^{(\gamma)} - 2\left(\sum_{j=1}^{d}(I+\gamma_j\sigma_j^*) + (\alpha-1)(I+\gamma_k\sigma_k^*)\right)\mathcal{D}_k^{(\gamma)}\\
&= \mathcal{D}_k^{(\gamma)}\sum_{j=1}^{d}\mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\gamma)} - 2\sum_{j=1}^{d}\mathcal{D}_j^{(\gamma)}\mathcal{D}_{k,j}^{(\gamma)} - 2\mathcal{D}_k^{(\gamma)}\left(\sum_{\substack{j=1\\ j\neq k}}^{d}(I+\gamma_j\sigma_j^*) + \alpha(I-\gamma_k\sigma_k^*)\right).
\end{aligned}
\tag{3.5.7}
$$

By (2.2.18)

$$
\begin{aligned}
&-2\lambda^{\alpha,\gamma}\sum_{j=1}^{d}\gamma_j \operatorname{Skew}_j \mathcal{D}_k^{(\gamma)} + \sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j \mathcal{D}_k^{(\gamma)} \\
&\qquad = \mathcal{D}_k^{(\gamma)}\Bigg(-2\lambda^{\alpha,\gamma}\sum_{\substack{j=1\\ j\neq k}}^{d}\gamma_j \operatorname{Skew}_j - 2\lambda^{\alpha,\gamma}\gamma_k \operatorname{Sym}_k + \sum_{\substack{i=1\\ i\neq k}}^{d}\sum_{\substack{j=1\\ j\neq k}}^{d}\gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j \\
&\qquad\qquad + 2\sum_{\substack{j=1\\ j\neq k}}\gamma_j\gamma_k \operatorname{Sym}_k \operatorname{Skew}_j + \gamma_k^2 \operatorname{Sym}_k\Bigg)
\end{aligned} \tag{3.5.8}
$$

Summing (3.5.5), (3.5.7) and (3.5.8), and taking into account the easily verifiable identity

$$
\begin{aligned}
&\sum_{\substack{j=1\\ j\neq k}}^{d}(I+\gamma_j\sigma_j^*)(I+\gamma_k\sigma_k^*) - 2\sum_{\substack{j=1\\ j\neq k}}^{d}(I+\gamma_j\sigma_j^*) - 2\alpha(I-\gamma_k\sigma_k^*) - 2\lambda^{\alpha,\gamma}\sum_{\substack{j=1\\ j\neq k}}^{d}\gamma_j \operatorname{Skew}_j \\
&\quad - 2\lambda^{\alpha,\gamma}\gamma_k \operatorname{Sym}_k + \sum_{\substack{i=1\\ i\neq k}}^{d}\sum_{\substack{j=1\\ j\neq k}}^{d}\gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j + 2\sum_{\substack{j=1\\ j\neq k}}^{d}\gamma_j\gamma_k \operatorname{Sym}_k \operatorname{Skew}_j + \gamma_k^2 \operatorname{Sym}_k \\
&\qquad = -2\lambda^{\alpha-1,\gamma}\sum_{j=1}^{d}\gamma_j \operatorname{Skew}_j + \sum_{i=1}^{d}\sum_{j=1}^{d}\gamma_i\gamma_j \operatorname{Skew}_i \operatorname{Skew}_j - (2\lambda^{\alpha,\gamma}-1)I
\end{aligned}
$$

we conclude (3.5.3). □

Theorem 3.5.5. *Let $\alpha > -1$, $\gamma \in (-1,+\infty)^d$ such that $\alpha > -1/2$ or $\gamma_i \geq 0$ for all $i \in \{1,\dots,d\}$. Let $n \in \mathbb{N}_0 \setminus \{2\}$ and $p_n \in \mathcal{V}_n^{\alpha,\gamma,1}$. Then,*

$$
\mathcal{L}^{(\alpha-1,\gamma)}(p_n) = n(n+2\lambda^{\alpha-1,\gamma})\,p_n.
$$

Proof. Let $p_n \in \mathcal{V}_n^{\alpha,\gamma,1}$, $n \in \mathbb{N}_0 \setminus \{2\}$. Given $k \in \{1,\dots,d\}$, by Proposition 3.5.1, $\mathcal{D}_k^{(\gamma)} p_n \in \mathcal{V}_{n-1}^{\alpha,\gamma}$. Then, by (2.4.1) and (2.4.12), we find that for all $k \in \{1,\dots,d\}$

$$
(n-1)(n+2\lambda^{\alpha,\gamma}-1)\mathcal{D}_k^{(\gamma)}p_n = \mathcal{L}^{(\alpha,\gamma)}\mathcal{D}_k^{(\gamma)}p_n \overset{(3.5.3)}{=} \mathcal{D}_k^{(\gamma)}\mathcal{L}^{(\alpha-1,\gamma)}p_n - (2\lambda^{\alpha,\gamma}-1)\mathcal{D}_k^{(\gamma)}p_n;
$$

that is

$$
(\forall\, k \in \{1,\dots,d\}) \quad \mathcal{D}_k^{(\gamma)}\left(\mathcal{L}^{(\alpha-1,\gamma)}(p_n) - n(n+2\lambda^{\alpha-1,\gamma})p_n\right) = 0, \tag{3.5.9}
$$

whence, by Proposition 2.5.2, $\mathcal{L}^{(\alpha-1,\gamma)}(p_n) - n(n+2\lambda^{\alpha-1,\gamma})p_n \in \Pi_0^d$. In order to conclude we will prove that

$$S_0^{\alpha,\gamma}\left(\mathcal{L}^{(\alpha-1,\gamma)}(p_n) - n(n+2\lambda^{\alpha-1,\gamma})p_n\right) = 0. \tag{3.5.10}$$

Indeed, if $n=0$, p_n is constant so $\mathcal{L}^{(\alpha-1,\gamma)}(p_n)$ vanishes and so does $n(n+2\lambda^{\alpha-1,\gamma})$. If $n=1$, $\mathcal{V}_n^{\alpha,\gamma,1} = \mathcal{V}_n^{\alpha,\gamma}$ (cf. Lemma 3.4.8) consists of polynomials of the form $x \mapsto v_1x_1 + \ldots + v_dx_d$, $v \in \mathbb{R}^d$ (cf. (3.2.2) and (3.3.4)); by direct computation it is readily checked that $\mathcal{L}^{(\alpha-1,\gamma)}$ applied to such a polynomial results in another such polynomial, so in this case, $S_0^{\alpha,\gamma}(\mathcal{L}^{(\alpha-1,\gamma)}(p_n)) = S_0^{\alpha,\gamma}(p_n) = 0$. If $n \geq 3$, $\langle S_0^{\alpha,\gamma}(p_n), 1\rangle_{\alpha,\gamma} = \langle p_n, 1\rangle_{\alpha,\gamma;1,\mathrm{P}} = 0$, so $S_0^{\alpha,\gamma}(p_n) = 0$. From part (iii) of Proposition 3.5.3, $\Delta_0^{(\gamma)}p_n := \sum_{1\leq i<j\leq d}(\mathcal{D}_{i,j}^{(\gamma)})^2 p_n \in \mathcal{V}_n^{\alpha,\gamma,1}$, so, by the same argument, $S_0^{\alpha,\gamma}(\Delta_0^{(\gamma)}p_n) = 0$. Further,

$$\begin{aligned}\left\langle S_0^{\alpha,\gamma}\left(\sum_{j=1}^d \mathcal{D}_j^{(\alpha-1,\gamma;\star)}\mathcal{D}_j^{(\gamma)}p_n\right), 1\right\rangle_{\alpha,\gamma} &\overset{(2.3.6)}{=} -\int_{B^d}\sum_{j=1}^d \mathcal{D}_j^{(\gamma)}\left((1-\|\cdot\|^2)^\alpha \mathcal{D}_j^{(\gamma)}p_n\right)W_{1,\gamma} \\ &\overset{(2.2.13)}{=} -\langle \Delta^{(\gamma)}p_n, 1\rangle_{\alpha+1,\gamma} + 2\alpha\sum_{j=1}^d \langle \mathcal{D}_j^{(\gamma)}p_n, x_j\rangle_{\alpha,\gamma} = 0.\end{aligned}$$

where the last equality follows from the fact that, per Proposition 3.5.1 and (iii) of Proposition 2.3.3, $\Delta^{(\gamma)}p_n \in \mathcal{V}_{n-2}^{\alpha+1,\gamma}$ and $\mathcal{D}_j^{(\gamma)}p_n \in \mathcal{V}_{n-1}^{\alpha,\gamma}$ for all $j \in \{1,\ldots,d\}$.

Finally, we obtain (3.5.10) after realizing that $S_0^{(\alpha,\gamma)}$ vanishes on every function that is skew-symmetric in at least one of its variables. □

CHAPTER 4

Connection relations of a 2D base

4.1 Introduction

One of the main drivers of the study of orthogonal polynomials is their application to numerical approximation of solutions of differential equations in what are usually known as spectral methods (see, for instance, [5, 8, 44]). To that end, knowing explicit bases of the orthogonal polynomials spaces to approximate the unknown solution (and, possibly, test some kind of equation residual) and how the operators encoding the problem act on these polynomials is very important.

Motivated by this fact, in this chapter we focus on studying specific bases of bivariate $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal polynomials. Namely, we are interested in studying connection relations of polynomials of the form $P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2\,\|x\|^2-1)\,Y_m^{(\gamma)}(x)$ (we call them *Dunkl–Zernike polynomials*), where $Y_m^{(\gamma)}$ is a h-harmonic homogeneous polynomial of degree m (cf. Section 3.2) whose explicit expression will be specified within this work. We provide explicit incarnations of some connection relations obtained in Chapter 2 and cast these results into relations connecting expansion coefficients of functions with respect to those bases with the corresponding coefficients of operators on these functions.

We are aware of the length and complexity of our relations compared with the elegance of those of Zernike polynomials found in [47]. We conjecture, based on extensive computational experimentation, that there is no simultaneous recombination of Dunkl operators and of the Dunkl–Zernike polynomials that makes, for example, the spectral differentiation relation simple.

The work of this chapter follows an inductive presentation mainly motivated by the structure of the polynomials $Y_m^{(\gamma)}$, which are defined in terms of Generalized Gegenbauer polynomials, which in turn, are defined in terms of Jacobi polynomials. We start by presenting basic properties of Jacobi polynomials (Section 4.2), then we study Generalized Gegenbauer polynomials (Section 4.3), then, in turn, h-harmonic homogeneous polynomials (Section 4.4), culminating with the construction and connection relations of Dunkl–Zernike polynomials (Section 4.5) and the corresponding relations among expansion coefficients in terms of the latter (Section 4.6).

4.2 Jacobi polynomials

The Jacobi polynomials $P_n^{(\alpha,\beta)}$ are polynomials orthogonal on $(-1,1)$ with respect to the weight $x \mapsto (1-x)^\alpha(1+x)^\beta$ and normalized according to [42, Eq. (4.1.1)]

$$P_n^{(\alpha,\beta)}(1) = \binom{n+\alpha}{n}. \tag{4.2.1}$$

Their weighted square norm is given by [42, Eq. (4.3.3)]

$$\begin{aligned}\left\|P_n^{(\alpha,\beta)}\right\|^2_{\mathrm{jac};\alpha,\beta} &:= \int_{-1}^{1} P_n^{(\alpha,\beta)}(x)^2(1-x)^\alpha(1+x)^\beta \,\mathrm{d}x \\ &= \frac{2^{\alpha+\beta+1}}{2n+\alpha+\beta+1}\frac{\Gamma(n+\alpha+1)\Gamma(n+\beta+1)}{\Gamma(n+\alpha+\beta+1)n!},\end{aligned} \tag{4.2.2}$$

for $n=0$ the product $(2n+\alpha+\beta+1)\Gamma(n+\alpha+\beta+1)$ must be replaced by $\Gamma(\alpha+\beta+2)$.

The identity $\Gamma(z+1) = z\Gamma(z)$ for $z \in \mathbb{C}\setminus\{0,-1,-2,-3,\dots\}$, joined with simple arithmetic manipulations, lets us obtain the following recurrence relations for the weighted square norm of Jacobi polynomials:

$$\left\|P_{n+1}^{(\alpha,\beta)}\right\|_{\mathrm{jac};\alpha,\beta}^2 = \frac{2n+\alpha+\beta+1}{2n+\alpha+\beta+3}\frac{n+\alpha+1}{n+\alpha+\beta+1}\frac{n+\beta+1}{n+1}\left\|P_n^{(\alpha,\beta)}\right\|_{\mathrm{jac};\alpha,\beta}^2 \tag{4.2.3}$$

and

$$\left\|P_n^{(\alpha,\beta+1)}\right\|_{\mathrm{jac};\alpha,\beta+1}^2 = 2\frac{2n+\alpha+\beta+1}{2n+\alpha+\beta+2}\frac{n+\beta+1}{n+\alpha+\beta+1}\left\|P_n^{(\alpha,\beta)}\right\|_{\mathrm{jac};\alpha,\beta}^2; \tag{4.2.4}$$

for $n=0$, the quotient $\frac{2n+\alpha+\beta+1}{n+\alpha+\beta+1}$ must be replaced by 1.

In equations (4.1.3), (4.1.4), (4.21.7) and (4.5.4) of [42] we find the identities

$$P_n^{(\alpha,\beta)}(-\,\cdot\,) = (-1)^n P_n^{(\beta,\alpha)}, \tag{4.2.5}$$

$$P_n^{(\alpha,\beta)}(-1) = (-1)^n \binom{n+\beta}{n}, \tag{4.2.6}$$

$$P_n^{(\alpha,\beta)\prime} = \frac{n+\alpha+\beta+1}{2} P_{n-1}^{(\alpha+1,\beta+1)}, \tag{4.2.7}$$

and

$$(1-\cdot)P_n^{(\alpha+1,\beta)} = \frac{2}{2n+\alpha+\beta+2}\left((n+\alpha+1)P_n^{(\alpha,\beta)} - (n+1)P_{n+1}^{(\alpha,\beta)}\right), \tag{4.2.8}$$

$$(1+\cdot)P_n^{(\alpha,\beta+1)} = \frac{2}{2n+\alpha+\beta+2}\left((n+\beta+1)P_n^{(\alpha,\beta)} + (n+1)P_{n+1}^{(\alpha,\beta)}\right), \tag{4.2.9}$$

respectively. In [1, Eq. (6.4.21)] we find the identity

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)} = (n+\alpha+\beta+1)P_n^{(\alpha+1,\beta)} - (n+\beta)P_{n-1}^{(\alpha+1,\beta)}. \tag{4.2.10}$$

Combining (4.2.10) with the parity relation (4.2.5) we obtain

$$(2n+\alpha+\beta+1)P_n^{(\alpha,\beta)} = (n+\alpha+\beta+1)P_n^{(\alpha,\beta+1)} + (n+\alpha)P_{n-1}^{(\alpha,\beta+1)}. \tag{4.2.11}$$

Taking the product of $\frac{n+\alpha+1}{2n+\alpha+\beta+2}$ and (4.2.10) with β shifted upwards by 1 and adding to the result the product of $\frac{n+\beta+1}{2n+\alpha+\beta+2}$ and (4.2.11) with α shifted upwards by 1 we obtain[1]

$$(\alpha+\beta+n+2)P_n^{(\alpha+1,\beta+1)} = (\alpha+n+1)P_n^{(\alpha,\beta+1)} + (\beta+n+1)P_n^{(\alpha+1,\beta)}. \tag{4.2.12}$$

Proposition 4.2.1.

$$(1+\cdot)P_n^{(\alpha,\beta+1)\prime} = (\beta+n+1)P_n^{(\alpha+1,\beta)} - (\beta+1)P_n^{(\alpha,\beta+1)}.$$

Proof. This equality follows from using (4.2.7) to substitute the derivative in its left-hand side, using (4.2.9) to expand the resulting product of $x \mapsto (1+x)$ and a Jacobi polynomial, using (4.2.10) with $\beta \leftarrow \beta+1$ to substitute $P_{n-1}^{(\alpha+1,\beta+1)}$ and then using (4.2.12) to substitute $P_n^{(\alpha+1,\beta+1)}$. □

4.3 Generalized Gegenbauer polynomials

For $\lambda > -\frac{1}{2}$, $\mu > -\frac{1}{2}$ and $n \in \mathbb{N}_0$, the generalized Gegenbauer polynomials $C_n^{(\lambda,\mu)}$ are defined by

$$C_n^{(\lambda,\mu)}(x) = \begin{cases} P_{\frac{n}{2}}^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})}(2x^2-1), & \text{if } n \text{ is even,} \\ x\, P_{\frac{n-1}{2}}^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})}(2x^2-1), & \text{if } n \text{ is odd.} \end{cases} \tag{4.3.1}$$

They are orthogonal polynomials on $-1 < x < 1$ with respect to the weight function $|x|^{2\mu}(1-x^2)^{\lambda-\frac{1}{2}}$ and are connected with the $S_n^{(\alpha,\beta)}$ of [2, Ex. 2] through

$$\frac{(\lambda+\mu)_{\lceil \frac{n}{2} \rceil}}{(\mu+\frac{1}{2})_{\lceil \frac{n}{2} \rceil}} C_n^{(\lambda,\mu)} = S_n^{(\lambda-1/2,\mu-1/2)}. \tag{4.3.2}$$

[1] A shifted, scaled and rearranged version of this identity appears without proof at http://functions.wolfram.com/Polynomials/JacobiP/17/02/02/0006/.

Also, their weighted square norms are given by

$$\left\|C_n^{(\lambda,\mu)}\right\|^2_{\text{gg};\lambda,\mu} := \int_{-1}^{1} C_n^{(\lambda,\mu)}(x)^2\, |x|^{2\mu}\, (1-x^2)^{\lambda-\frac{1}{2}}\, \mathrm{d}x$$
$$= \begin{cases} 2^{-(\lambda+\mu)} \left\| P_{\frac{n}{2}}^{(\lambda-\frac{1}{2},\mu-\frac{1}{2})} \right\|^2_{\text{jac};\lambda-\frac{1}{2},\mu-\frac{1}{2}}, & \text{if } n \text{ is even,} \\ 2^{-(\lambda+\mu+1)} \left\| P_{\frac{n-1}{2}}^{(\lambda-\frac{1}{2},\mu+\frac{1}{2})} \right\|^2_{\text{jac};\lambda-\frac{1}{2},\mu+\frac{1}{2}}, & \text{if } n \text{ is odd,} \end{cases} \tag{4.3.3}$$

where the integral was rewritten, by exploiting the symmetry of the integrand and performing the change of variable $t = 2x^2 - 1$.

By using (4.2.3) we can obtain the following recurrence relation for the weighted square norm of Generalized Gegenbauer polynomials,

$$\frac{\left\|C_{n+2}^{(\lambda,\mu)}\right\|^2_{\text{gg};\lambda,\mu}}{\left\|C_n^{(\lambda,\mu)}\right\|^2_{\text{gg};\lambda,\mu}} = \begin{cases} \frac{n+\lambda+\mu}{n+\lambda+\mu+2} \frac{n+2\lambda+1}{n+2\lambda+2\mu} \frac{n+2\mu+1}{n+2}, & \text{if } n \text{ is even,} \\ \frac{n+\lambda+\mu}{n+\lambda+\mu+2} \frac{n+2\lambda}{n+2\lambda+2\mu+1} \frac{n+2\mu+2}{n+1}, & \text{if } n \text{ is odd,} \end{cases} \tag{4.3.4}$$

for $n = 0$, the quotient $\frac{n+\lambda+\mu}{n+2\lambda+2\mu}$ must be replaced by $\frac{1}{2}$.

We remark that our choice of normalization is justified by our desire of extending the range of the parameter μ in the results found in [2] and [17, Def. 1.5.5] concerning this family of orthogonal polynomials. Indeed, our normalization ensures that for $n \in \mathbb{N}_0$, $x \in \mathbb{R}$ and $\lambda > -\frac{1}{2}$ fixed, then $\mu \mapsto C_n^{(\lambda,\mu)}(x)$ is a polynomial. As a consequence, analytic continuation can be used to extend the range of validity of the aforementioned results as it will be detailed below.

The following two results are the unidimensional incarnations of (ii) and (i) of Proposition 2.3.1, respectively, in the present context.

Proposition 4.3.1.

$$2(\lambda+\mu+n)\, C_n^{(\lambda,\mu)} = \begin{cases} (2\lambda+2\mu+n)\, C_n^{(\lambda+1,\mu)}(x) - (2\mu+n-1)\, C_{n-2}^{(\lambda+1,\mu)}(x), & \textit{if } n \textit{ is even,} \\ (2\lambda+2\mu+n+1)\, C_n^{(\lambda+1,\mu)}(x) - (2\mu+n)\, C_{n-2}^{(\lambda+1,\mu)}(x), & \textit{if } n \textit{ is odd.} \end{cases} \tag{4.3.5}$$

Proof.

$$
\begin{aligned}
&2(\lambda+\mu+n)\,C_n^{(\lambda,\mu)}(x)\\
&\overset{(4.2.10)}{=}\begin{cases}(2\lambda+2\mu+n)\,P_{\frac{n}{2}}^{(\lambda+\frac12,\mu-\frac12)}(2x^2-1)-(2\mu+n-1)P_{\frac{n-2}{2}}^{(\lambda+\frac12,\mu-\frac12)}(2x^2-1), & \text{if } n \text{ is even},\\ (2\lambda+2\mu+n+1)\,x\,P_{\frac{n-1}{2}}^{(\lambda+\frac12,\mu+\frac12)}(2x^2-1)-(2\mu+n)\,x\,P_{\frac{(n-2)-1}{2}}^{(\lambda+\frac12,\mu+\frac12)}(2x^2-1), & \text{if } n \text{ is odd}.\end{cases}\\
&\overset{(4.3.1)}{=}\begin{cases}(2\lambda+2\mu+n)\,C_n^{(\lambda+1,\mu)}(x)-(2\mu+n-1)\,C_{n-2}^{(\lambda+1,\mu)}(x), & \text{if } n \text{ is even},\\ (2\lambda+2\mu+n+1)\,C_n^{(\lambda+1,\mu)}(x)-(2\mu+n)\,C_{n-2}^{(\lambda+1,\mu)}(x), & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}
$$

□

Proposition 4.3.2.

$$
2(n+\lambda+\mu+1)\,(1-\cdot^2)\,C_n^{(\lambda+1,\mu)}=\begin{cases}(n+2\lambda+1)\,C_n^{(\lambda,\mu)}-(n+2)\,C_{n+2}^{(\lambda,\mu)}, & n \ \textit{even},\\ (n+2\lambda)\,C_n^{(\lambda,\mu)}-(n+1)\,C_{n+2}^{(\lambda,\mu)}, & n \ \textit{odd}.\end{cases} \tag{4.3.6}
$$

Proof.

$$
\begin{aligned}
(1-x^2)\,C_n^{(\lambda+1,\mu)}(x) &\overset{(4.3.1)}{=}\begin{cases}(1-x^2)\,P_{\frac{n}{2}}^{(\lambda+\frac12,\mu-\frac12)}(2x^2-1), & \text{if } n \text{ is even},\\ x\,(1-x^2)\,P_{\frac{n-1}{2}}^{(\lambda+\frac12,\mu+\frac12)}(2x^2-1), & \text{if } n \text{ is odd}.\end{cases}\\
&=\begin{cases}\frac12(1-(2x^2-1))\,P_{\frac{n}{2}}^{(\lambda+\frac12,\mu-\frac12)}(2x^2-1), & \text{if } n \text{ is even},\\ \frac12 x\,(1-(2x^2-1))\,P_{\frac{n-1}{2}}^{(\lambda+\frac12,\mu+\frac12)}(2x^2-1), & \text{if } n \text{ is odd}.\end{cases}\\
&\overset{(4.2.8)}{=}\begin{cases}\frac{1}{n+\lambda+\mu+1}\left((\frac{n+1}{2}+\lambda)P_{\frac{n}{2}}^{(\lambda-\frac12,\mu-\frac12)}(2x^2-1)-\frac{n+2}{2}P_{\frac{n+2}{2}}^{(\lambda-\frac12,\mu-\frac12)}(2x^2-1)\right), & \text{if } n \text{ is even},\\ \frac{1}{n+\lambda+\mu+1}\left((\frac{n}{2}+\lambda)x\,P_{\frac{n-1}{2}}^{(\lambda-\frac12,\mu+\frac12)}(2x^2-1)-\frac{n+1}{2}x\,P_{\frac{n+1}{2}}^{(\lambda-\frac12,\mu+\frac12)}(2x^2-1)\right), & \text{if } n \text{ is odd}.\end{cases}\\
&\overset{(4.3.1)}{=}\begin{cases}\frac{1}{n+\lambda+\mu+1}\left(\left(\frac{n+1}{2}+\lambda\right)C_n^{(\lambda,\mu)}(x)-\frac{n+2}{2}C_{n+2}^{(\lambda,\mu)}(x)\right), & \text{if } n \text{ is even},\\ \frac{1}{n+\lambda+\mu+1}\left(\left(\frac{n}{2}+\lambda\right)C_n^{(\lambda,\mu)}(x)-\frac{n+1}{2}C_{n+2}^{(\lambda,\mu)}(x)\right), & \text{if } n \text{ is odd}.\end{cases}
\end{aligned}
$$

□

The following corresponds to the three-term recurrence for Generalized Gegenbauer polynomials.

Proposition 4.3.3.

$$C_n^{(\lambda,\mu)}(x) = \begin{cases} \frac{2(\lambda+\mu+n-1)}{n} x\, C_{n-1}^{(\lambda,\mu)}(x) - \frac{2\mu+n-1}{n} C_{n-2}^{(\lambda,\mu)}(x), & \text{if } n \text{ is even}, \\ \frac{2(\lambda+\mu+n-1)}{2\lambda+2\mu+n-1} x\, C_{n-1}^{(\lambda,\mu)}(x) - \frac{2\lambda+n-2}{2\lambda+2\mu+n-1} C_{n-2}^{(\lambda,\mu)}(x), & \text{if } n \text{ is odd}. \end{cases} \tag{4.3.7}$$

Note that if $n = 1$ and $\lambda + \mu = 0$ the terms $\frac{2(\lambda+\mu+n-1)}{2\lambda+2\mu+n-1}$ and $\frac{2\lambda+n-2}{2\lambda+2\mu+n-1}$ should be interpreted as 1 and 0, respectively.

Proof. For $\mu \geq 0$, this can be found in [17, Def. 1.5.5] renormalized according to (4.3.2). To extend the range of μ to $(-1/2, 0)$, it is enough to multiply (4.3.7) by the common denominator appearing in the right hand side of the equation, to then interpret the result as an equality of polynomials with respect to μ which can be easily extended via analytic continuation. □

Proposition 4.3.4.

$$C_n^{(\lambda,\mu)\prime}(x) + \mu \frac{C_n^{(\lambda,\mu)}(x) - C_n^{(\lambda,\mu)}(-x)}{x} = \begin{cases} (n + 2\lambda + 2\mu)\, C_{n-1}^{(\lambda+1,\mu)}(x), & \text{if } n \text{ is even}, \\ (n + 2\mu)\, C_{n-1}^{(\lambda+1,\mu)}(x), & \text{if } n \text{ is odd}. \end{cases} \tag{4.3.8}$$

Proof. For $\mu \geq 0$, this is [2, Lem. 3.2] renormalized according to (4.3.2). For $-1/2 < \mu < 0$, proceed similarly as in the proof Proposition 4.3.3. □

4.4 A basis of homogeneous h-harmonic polynomials

In [17, Thm. 7.5.1] an explicit orthogonal basis for $\mathcal{H}_m^2(h_\gamma)$ is given. We recall that $h_\gamma(x) = |x_1|^{\gamma_1} |x_2|^{\gamma_2}$ (cf. Section 3.2).

Theorem 4.4.1. *Let $\gamma_1, \gamma_2 > -1$. Then, a mutually $\mathrm{L}^2(\mathbb{S}^1, h_\gamma)$-orthogonal basis for $\mathcal{H}_m^2(h_\gamma)$ is given by $\{Y_m^{(\gamma;\mathrm{even})}, Y_m^{(\gamma;\mathrm{odd})}\}$ if $m \geq 1$ and $\{Y_0^{(\gamma;\mathrm{even})}\}$ if $m = 0$, where*

$$\begin{aligned} Y_m^{(\gamma;\mathrm{even})}(x) &= r^m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta), \\ Y_m^{(\gamma;\mathrm{odd})}(x) &= r^m \sin\theta\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta); \end{aligned} \tag{4.4.1}$$

here we use the polar coordinates $x = r(\cos\theta, \sin\theta)$. We adopt the convention $Y_0^{(\gamma;\mathrm{odd})} \equiv 0$.

The labels ‘even’ and ‘odd’ in (4.4.1) allude to even and odd σ_2^*-parity, respectively.

Their weighted square norm is given by

$$\left\|Y_m^{(\gamma;l)}\right\|_{h_\gamma}^2 := \int_{\mathbb{S}^1} Y_m^{(\gamma;l)}(x)^2 h_\gamma(x)\,\mathrm{d}S(x) = \begin{cases} 2\left\|C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}\right\|_{\mathrm{gg};\frac{\gamma_2}{2},\frac{\gamma_1}{2}}^2, & \text{if } l = \text{even},\\ 2\left\|C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}\right\|_{\mathrm{gg};\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2}}^2, & \text{if } l = \text{odd}, \end{cases} \tag{4.4.2}$$

where the integral was rewritten by first switching to polar coordinates, then exploiting the symmetry of the emerging integrand and finally performing the change of variable $t = \cos(\theta)$.

By using (4.3.4), we can obtain the following recurrence relation for the weighted square norm of homogeneous h-harmonic polynomials,

$$\frac{\left\|Y_{m+2}^{(\gamma;l)}\right\|_{h_\gamma}^2}{\left\|Y_m^{(\gamma;l)}\right\|_{h_\gamma}^2} = \frac{2m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2+4}\begin{cases} \frac{m+\gamma_2+1}{m+\gamma_1+\gamma_2}\frac{m+\gamma_1+1}{m+2}, & \text{if } l = \text{even}, m \text{ is even},\\ \frac{m+\gamma_2}{m+\gamma_1+\gamma_2+1}\frac{m+\gamma_1+2}{m+1}, & \text{if } l = \text{even}, m \text{ is odd},\\ \frac{m+\gamma_2+1}{m+\gamma_1+\gamma_2+2}\frac{m+\gamma_1+1}{m}, & \text{if } l = \text{odd}, m \text{ is even},\\ \frac{m+\gamma_2+2}{m+\gamma_1+\gamma_2+1}\frac{m+\gamma_1}{m+1}, & \text{if } l = \text{odd}, m \text{ is odd}, \end{cases} \tag{4.4.3}$$

for $l = \text{even}$ and $m = 0$, the quotient $\frac{2m+\gamma_1+\gamma_2}{m+\gamma_1+\gamma_2}$ must be replaced by 1.

Given the definition (4.4.1) in polar coordinates, it will be useful to express Dunkl operators in this coordinate system,

$$(\mathcal{D}_1^{(\gamma)}u)(r,\theta) = \cos(\theta)\partial_r u - \frac{\sin(\theta)}{r}\partial_\theta u + \frac{\gamma_1}{2}\frac{u(r,\theta)-u(r,\pi-\theta)}{r\cos(\theta)}, \tag{4.4.4}$$

$$(\mathcal{D}_2^{(\gamma)}u)(r,\theta) = \sin(\theta)\partial_r u + \frac{\cos(\theta)}{r}\partial_\theta u + \frac{\gamma_2}{2}\frac{u(r,\theta)-u(r,-\theta)}{r\sin(\theta)}. \tag{4.4.5}$$

We already know from the proof of Proposition 3.4.3 that Dunkl operators map $\mathcal{H}_m^d(h_\gamma)$ to $\mathcal{H}_{m-1}^d(h_\gamma)$. The following result shows how these operators act on this specific basis of $\mathcal{H}_m^d(h_\gamma)$.

Proposition 4.4.2.

$$\mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{even})} = \begin{cases} (m+\gamma_2-1)\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is even}, \\ (m+\gamma_1)\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.4.6}$$

$$\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\text{even})} = \begin{cases} -(m+\gamma_1-1)\, Y_{m-1}^{(\gamma;\text{odd})}, & \textit{if } m \textit{ is even}, \\ -(m+\gamma_1)\, Y_{m-1}^{(\gamma;\text{odd})}, & \textit{if } m \textit{ is odd}. \end{cases}. \tag{4.4.7}$$

$$\mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{odd})} = \begin{cases} (m+\gamma_1-1)\, Y_{m-1}^{(\gamma;\text{odd})}, & \textit{if } m \textit{ is even}, \\ (m+\gamma_2)\, Y_{m-1}^{(\gamma;\text{odd})}, & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.4.8}$$

$$\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\text{odd})} = \begin{cases} (m+\gamma_2-1)\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is even}, \\ (m+\gamma_2)\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.4.9}$$

Proof. In order to not lose legibility we will only proceed assuming that m is even. We state in advance that all the procedures and techniques that will be used hereunder can be easily readapted for the opposite parity case.

Using the polar form of the Dunkl operators in (4.4.4) and (4.4.5) and Proposition 4.3.4 in the even case with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$,

$$\begin{aligned} \mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{even})} &= \mathcal{D}_1^{(\gamma)} \left(r^m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) \right) \\ &= m\, r^{m-1} \cos(\theta)\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) + \sin^2(\theta) r^{m-1} C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})\prime}(\cos\theta) + \frac{\gamma_1}{2} r^{m-1} \frac{C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(-\cos\theta)}{\cos\theta} \\ &\overset{(4.3.8)}{=} r^{m-1} \Bigg(m\cos(\theta)\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) + (m+\gamma_1+\gamma_2)(1-\cos^2(\theta))\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\ &\qquad + \frac{\gamma_1}{2}\cos(\theta) \left(C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(-\cos\theta) \right) \Bigg). \end{aligned} \tag{4.4.10}$$

Using Proposition 4.3.3 in the odd case with $n \leftarrow m+1$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$ and Proposition 4.3.2 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$; (4.4.10) rewrites as

$$\mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{even})} = r^{m-1}\left(\frac{m(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_{m+1}^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) + \frac{m(m+\gamma_2-1)}{2m+\gamma_1+\gamma_2} C_{m-1}^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta)\right.$$
$$\left. + \frac{(m+\gamma_2-1)(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_{m-1}^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - \frac{m(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_{m+1}^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta)\right)$$
$$= (m+\gamma_2-1) Y_{m-1}^{(\gamma;\text{even})},$$

hence (4.4.6) in the m-even case.

Now, for (4.4.7), using Proposition 4.3.4 in the even case with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$,

$$\begin{aligned}
\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\text{even})} &= \mathcal{D}_2^{(\gamma)}\left(r^m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta)\right) \\
&= m\, r^{m-1}\sin(\theta)\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - \sin(\theta)\cos(\theta) r^{m-1} C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})\prime}(\cos\theta) \\
&\quad + \frac{\gamma_2}{2} r^m \frac{C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos(-\theta))}{r\sin\theta} \\
&= r^{m-1}\sin(\theta)\left(m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - \cos(\theta)\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})\prime}(\cos\theta)\right) \\
&\overset{(4.3.8)}{=} r^{m-1}\sin(\theta)\left(m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - (m+\gamma_1+\gamma_2)\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\right. \\
&\quad \left. + \frac{\gamma_1}{2}\left(C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(-\cos\theta)\right)\right).
\end{aligned} \tag{4.4.11}$$

Using Proposition 4.3.1 with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$ and Proposition 4.3.3 in the even case with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$; (4.4.11) rewrites as,

$$\begin{aligned}
\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\text{even})} &= r^{m-1}\sin(\theta)\left(m\, C_m^{(\frac{\gamma_2}{2},\frac{\gamma_1}{2})}(\cos\theta) - (m+\gamma_1+\gamma_2)\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\right) \\
&= r^{m-1}\sin(\theta)\left(\frac{m(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_m^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \frac{m(m+\gamma_1-1)}{2m+\gamma_1+\gamma_2} C_{m-2}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\right. \\
&\quad \left. - \frac{m(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_m^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \frac{(m+\gamma_1-1)(m+\gamma_1+\gamma_2)}{2m+\gamma_1+\gamma_2} C_{m-2}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\right)
\end{aligned}$$

$$= -(m+\gamma_1-1)\,Y_{m-1}^{(\gamma;\text{odd})}, \quad (4.4.12)$$

proving (4.4.7) in the m-even case. For (4.4.8), using Proposition 4.3.4 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$,

$$\begin{aligned}
\mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{odd})} &= \mathcal{D}_1^{(\gamma)}\left(r^m \sin(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\right) \\
&= m r^{m-1}\cos(\theta)\sin(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&\quad - \sin(\theta) r^{m-1}\left(\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})\prime}(\cos\theta)\right) \\
&\quad + \frac{\gamma_1}{2} r^{m-1}\sin(\theta)\frac{C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(-\cos\theta)}{\cos(\theta)} \\
&= r^{m-1}\sin(\theta)\Bigg((m-1)\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})\prime}(\cos\theta) \\
&\quad + \frac{\gamma_1}{2}\frac{C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(-\cos\theta)}{\cos(\theta)}\Bigg) \\
&\overset{(4.3.8)}{=} r^{m-1}\sin(\theta)\Bigg((m-1)\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&\quad + (m+\gamma_1-1)(1-\cos^2(\theta)) C_{m-2}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta) \\
&\quad + \frac{\gamma_1}{2}\cos(\theta)\left(C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(-\cos\theta)\right)\Bigg). \quad (4.4.13)
\end{aligned}$$

Using Proposition 4.3.3 in the even case with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$; and Proposition 4.3.2 in the even case with $n \leftarrow m-2$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$, and $x \leftarrow \cos(\theta)$,

$$\begin{aligned}
\mathcal{D}_1^{(\gamma)} Y_m^{(\gamma;\text{odd})} &= r^{m-1}\sin(\theta)\Bigg((m+\gamma_1-1)\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&\quad + (m+\gamma_1-1)(1-\cos^2(\theta)) C_{m-2}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta)\Bigg) \\
&= r^{m-1}\sin(\theta)\Bigg(\frac{m(m+\gamma_1-1)}{2m+\gamma_1+\gamma_2} C_m^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + \frac{(m+\gamma_1-1)(m+\gamma_1-1)}{2m+\gamma_1+\gamma_2} C_{m-2}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&\quad + \frac{(m+\gamma_2+1)(m+\gamma_1-1)}{2m+\gamma_1+\gamma_2} C_{m-2}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \frac{m(m+\gamma_1-1)}{2m+\gamma_1+\gamma_2} C_m^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\Bigg)
\end{aligned}$$

$$= (m+\gamma_1-1)\, Y_{m-1}^{(\gamma;\mathrm{odd})},$$

which concludes (4.4.8) in the m-even case.

For (4.4.9), using Proposition 4.3.4 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$,

$$\begin{aligned}
&\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\mathrm{odd})} = \mathcal{D}_2^{(\gamma)} \left(r^m \sin(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \right) \\
&= m r^{m-1} \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + r^{m-1}\cos(\theta) \left(\cos(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})\prime}(\cos\theta) \right) \\
&\qquad + \gamma_2 r^{m-1} C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&= r^{m-1} \Bigg(m \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + \cos^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - \cos(\theta)\sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})\prime}(\cos\theta) \\
&\qquad + \gamma_2\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \Bigg) \\
&\overset{(4.3.8)}{=} r^{m-1} \Bigg(m \sin^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + \cos^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&- (m+\gamma_1-1)\cos(\theta)\sin^2(\theta)\, C_{m-2}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta) + \frac{\gamma_1}{2}\sin^2(\theta) \left(C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) - C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(-\cos\theta) \right) \\
&\qquad + \gamma_2\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \Bigg). \qquad (4.4.14)
\end{aligned}$$

Using simultaneously Proposition 4.3.1 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$; and Proposition 4.3.3 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}+2$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$; to then apply Proposition 4.3.2 in the odd case with $n \leftarrow m-3$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$, and finally make use of Proposition 4.3.1 in the odd case with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$ and $x \leftarrow \cos(\theta)$, we get

$$\begin{aligned}
\mathcal{D}_2^{(\gamma)} Y_m^{(\gamma;\mathrm{odd})} = r^{m-1} \Bigg(&(m+\gamma_1)(1-\cos^2(\theta))\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) + \cos^2(\theta)\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \\
&- (m+\gamma_1-1)\cos(\theta)(1-\cos^2(\theta))\, C_{m-2}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta) + \gamma_2\, C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta) \Bigg)
\end{aligned}$$

$$= r^{m-1}\Bigg((m+\gamma_1-1)(1-\cos^2(\theta))\left[C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)-\cos(\theta)\,C_{m-2}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta)\right]$$

$$+(\gamma_2+1)\,C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\Bigg)$$

$$\overset{(4.3.5),(4.3.7)}{=} r^{m-1}\Bigg(-(m+\gamma_1-1)(1-\cos^2(\theta))\,C_{m-3}^{(\frac{\gamma_2}{2}+2,\frac{\gamma_1}{2})}(\cos\theta)+(\gamma_2+1)\,C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\Bigg)$$

$$\overset{(4.3.6)}{=} r^{m-1}\Bigg(-\frac{(m+\gamma_1-1)(m+\gamma_2-1)}{2m+\gamma_1+\gamma_2-2}\,C_{m-3}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)+\frac{(m+\gamma_1-1)(m-2)}{2m+\gamma_1+\gamma_2-2}\,C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)$$

$$+(\gamma_2+1)\,C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\Bigg)$$

$$=(m+\gamma_2-1)\,r^{m-1}\Bigg(\frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2-2}\,C_{m-1}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)-\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2-2}\,C_{m-3}^{(\frac{\gamma_2}{2}+1,\frac{\gamma_1}{2})}(\cos\theta)\Bigg)$$

$$\overset{(4.3.5)}{=}(m+\gamma_2-1)\,Y_{m-1}^{(\gamma;\text{even})}(x),$$

hence (4.4.9) in the m-even case. □

The following will serve in the proof of Proposition 4.5.2.

Proposition 4.4.3.

$$x_1 Y_m^{(\gamma;\text{even})}(x) = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{even})}(x)+\frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{even})}(x), & \text{if } m \text{ is even,} \\ \frac{m+1}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{even})}(x)+\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{even})}(x), & \text{if } m \text{ is odd.} \end{cases} \tag{4.4.15}$$

$$x_2 Y_m^{(\gamma;\text{even})}(x) = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{odd})}(x)-\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{odd})}(x), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{odd})}(x)-\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{odd})}(x), & \text{if } m \text{ is odd.} \end{cases} \tag{4.4.16}$$

$$x_1 Y_m^{(\gamma;\text{odd})}(x) = \begin{cases} \frac{m}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{odd})}(x)+\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{odd})}(x), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{odd})}(x)+\frac{m+\gamma_2}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{odd})}(x), & \text{if } m \text{ is odd.} \end{cases} \tag{4.4.17}$$

$$x_2 Y_m^{(\gamma;\text{odd})}(x) = \begin{cases} -\frac{m}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{even})} + \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is even}, \\ -\frac{m+1}{2m+\gamma_1+\gamma_2} Y_{m+1}^{(\gamma;\text{even})} + \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2} r^2\, Y_{m-1}^{(\gamma;\text{even})}, & \textit{if } m \textit{ is odd.} \end{cases} \tag{4.4.18}$$

Proof. Write $x_j Y_m^{(\gamma;l)}$ in polar form for $j \in \{1,2\}$ and $l \in \{\text{even}, \text{odd}\}$. Then, (4.4.15), (4.4.16), (4.4.17) and (4.4.18) are direct consequences of Proposition 4.3.3 (with $n \leftarrow m+1$, $\lambda \leftarrow \frac{\gamma_2}{2}$, $\mu \leftarrow \frac{\gamma_1}{2}$), Proposition 4.3.1 (with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}$ and $\mu \leftarrow \frac{\gamma_1}{2}$), Proposition 4.3.3 (with $n \leftarrow m$, $\lambda \leftarrow \frac{\gamma_2}{2}+1$ and $\mu \leftarrow \frac{\gamma_1}{2}$) and Proposition 4.3.2 (with $n \leftarrow m-1$, $\lambda \leftarrow \frac{\gamma_2}{2}$ and $\mu \leftarrow \frac{\gamma_1}{2}$), respectively. □

4.5 Dunkl–Zernike polynomials

Given $m, n \in \mathbb{N}_0$, we define

$$\begin{aligned} Z_{m,n}^{(\alpha,\gamma;\text{even})}(x) &:= P_n^{(\alpha, m+\frac{\gamma_1+\gamma_2}{2})}(2\,\|x\|^2 - 1)\, Y_m^{(\gamma;\text{even})}(x), \\ Z_{m,n}^{(\alpha,\gamma;\text{odd})}(x) &:= P_n^{(\alpha, m+\frac{\gamma_1+\gamma_2}{2})}(2\,\|x\|^2 - 1)\, Y_m^{(\gamma;\text{odd})}(x). \end{aligned} \tag{4.5.1}$$

The $Z_{m,n}^{(\alpha,\gamma;\text{even})}$ and the $Z_{m,n}^{(\alpha,\gamma;\text{odd})}$ are bivariate polynomials of total degree $m+2n$, except the $Z_{0,n}^{(\alpha,\gamma;\text{odd})}$, which are null. We adopt the convention that if $m < 0$ or $n < 0$ then $Z_{m,n}^{(\alpha;\text{even})} \equiv 0$ and $Z_{m,n}^{(\alpha;\text{odd})} \equiv 0$.

Given $N \in \mathbb{N}_0$, from Proposition 3.2.1 we know that $\{Z_{m,n}^{(\alpha,\gamma;\text{even})}\}_{\substack{m\ge 0, n\ge 0\\ m+2n=N}} \cup \{Z_{m,n}^{(\alpha,\gamma;\text{odd})}\}_{\substack{m\ge 1, n\ge 0\\ m+2n=N}}$ is an $\mathrm{L}^2_{\alpha,\gamma}$-orthogonal basis of $\mathcal{V}_N^{\alpha,\gamma}$.

For $l \in \{\text{even}, \text{odd}\}$, the weighted square norm is given by

$$\begin{aligned} \left\| Z_{m,n}^{(\alpha,\gamma;l)} \right\|_{\alpha,\gamma}^2 &:= \int_{B^2} Z_{m,n}^{(\alpha,\gamma;l)}(x)^2 (1 - \|x\|^2)^\alpha h_\gamma(x)\, \mathrm{d}x \\ &= 2^{-(m+\alpha+\frac{\gamma_1+\gamma_2}{2}+2)} \left\| P_n^{(\alpha, m+\frac{\gamma_1+\gamma_2}{2})} \right\|_{\text{jac};\alpha, m+\frac{\gamma_1+\gamma_2}{2}}^2 \left\| Y_m^{(\gamma;l)} \right\|_{h_\gamma}^2, \end{aligned} \tag{4.5.2}$$

where the integral was computed by switching to polar coordinates, exploiting the homogeneity of $Y_m^{(\gamma;l)}$ and performing the change of variable $t = 2r^2 - 1$ in the appearing radial integral.

Relations (4.2.3), (4.2.4) and (4.4.3) help us obtain the following recurrence relations for the weighted square norm of Dunkl–Zernike polynomials,

$$\frac{\left\|Z_{m,n+1}^{(\alpha,\gamma;l)}\right\|_{\alpha,\gamma}^2}{\left\|Z_{m,n}^{(\alpha,\gamma;l)}\right\|_{\alpha,\gamma}^2} = \frac{\left\|P_{n+1}^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}\right\|_{\mathrm{jac};\alpha,m+\frac{\gamma_1+\gamma_2}{2}}^2}{\left\|P_{n}^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}\right\|_{\mathrm{jac};\alpha,m+\frac{\gamma_1+\gamma_2}{2}}^2}$$
$$= \frac{2m+4n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+6}\,\frac{2m+2n+\gamma_1+\gamma_2+2}{2m+2n+2\alpha+\gamma_1+\gamma_2+2}\,\frac{n+\alpha+1}{n+1} \tag{4.5.3}$$

and

$$\frac{\left\|Z_{m+2,n}^{(\alpha,\gamma;l)}\right\|_{\alpha,\gamma}^2}{\left\|Z_{m,n}^{(\alpha,\gamma;l)}\right\|_{\alpha,\gamma}^2} = \frac{1}{4}\,\frac{\left\|P_{n}^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2}+2)}\right\|_{\mathrm{jac};\alpha,m+\frac{\gamma_1+\gamma_2}{2}+2}^2}{\left\|P_{n}^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}\right\|_{\mathrm{jac};\alpha,m+\frac{\gamma_1+\gamma_2}{2}}^2}\,\frac{\left\|Y_{m+2}^{(\gamma;l)}\right\|_{h_\gamma}^2}{\left\|Y_{m}^{(\gamma;l)}\right\|_{h_\gamma}^2}, \tag{4.5.4}$$

for $n=0$, the quotient $\frac{2m+4n+2\alpha+\gamma_1+\gamma_2+2}{2m+2n+2\alpha+\gamma_1+\gamma_2+2}$ must be replaced by 1. The last relation's explicit form has been omitted due to its great length.

The following proposition corresponds to the incarnation of part (ii) of Proposition 2.3.1 in the present context.

Proposition 4.5.1.

$$\left(m+2n+\alpha+\frac{\gamma_1+\gamma_2}{2}+1\right) Z_{m,n}^{(\alpha,\gamma;\mathrm{even})}$$
$$= \left(m+n+\alpha+\frac{\gamma_1+\gamma_2}{2}+1\right) Z_{m,n}^{(\alpha+1,\gamma;\mathrm{even})} - \left(m+n+\frac{\gamma_1+\gamma_2}{2}\right) Z_{m,n-1}^{(\alpha+1,\gamma;\mathrm{even})},$$

$$\left(m+2n+\alpha+\frac{\gamma_1+\gamma_2}{2}+1\right) Z_{m,n}^{(\alpha,\gamma;\mathrm{odd})}$$
$$= \left(m+n+\alpha+\frac{\gamma_1+\gamma_2}{2}+1\right) Z_{m,n}^{(\alpha+1,\gamma;\mathrm{odd})} - \left(m+n+\frac{\gamma_1+\gamma_2}{2}\right) Z_{m,n-1}^{(\alpha+1,\gamma;\mathrm{odd})}.$$

Proof. These follow directly from the definitions in (4.5.1) and the identity (4.2.10). □

The following corresponds to the three-term recurrence [17, Sec. 1.3.2] in our setting.

Proposition 4.5.2.

$$x_1\, Z_{m,n}^{(\alpha,\gamma;\mathrm{even})}$$
$$= \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{even})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{even})}\right) \\ +\frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{even})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{even})}\right), & \textit{if } m \textit{ is even}, \\ \frac{m+1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{even})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{even})}\right) \\ +\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{even})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{even})}\right), & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.5.5}$$

$$x_2\, Z_{m,n}^{(\alpha,\gamma;\mathrm{even})}$$
$$= \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{odd})}\right) \\ -\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{odd})}\right), & \textit{if } m \textit{ is even}, \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{odd})}\right) \\ -\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{odd})}\right), & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.5.6}$$

$$x_1\, Z_{m,n}^{(\alpha,\gamma;\mathrm{odd})}$$
$$= \begin{cases} \frac{m}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{odd})}\right) \\ +\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{odd})}\right), & \textit{if } m \textit{ is even}, \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m+1,n-1}^{(\alpha,\gamma;\mathrm{odd})}\right) \\ +\frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n}^{(\alpha,\gamma;\mathrm{odd})} + \frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2} Z_{m-1,n+1}^{(\alpha,\gamma;\mathrm{odd})}\right), & \textit{if } m \textit{ is odd}. \end{cases} \tag{4.5.7}$$

$$x_2\, Z_{m,n}^{(\alpha,\gamma;\mathrm{odd})}$$

$$= \begin{cases} \left.\begin{aligned} &-\frac{m}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m+1,n}+\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m+1,n-1}\right)\\ &+\frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m-1,n}+\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m-1,n+1}\right)\end{aligned}\right., & \textit{if } m \textit{ is even},\\ \left.\begin{aligned} &-\frac{m+1}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m+1,n}+\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m+1,n-1}\right)\\ &+\frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\left(\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m-1,n}+\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}Z^{(\alpha,\gamma;\text{even})}_{m-1,n+1}\right)\end{aligned}\right., & \textit{if } m \textit{ is odd}.\end{cases} \tag{4.5.8}$$

Proof. The result is obtained after using Proposition 4.4.3 to expand $x_jY_m^{(\gamma;l)}$, to then consider (4.2.9) with $\beta \leftarrow m-1+\frac{\gamma_1+\gamma_2}{2}$ and (4.2.11) with $\beta \leftarrow m+\frac{\gamma_1+\gamma_2}{2}$ to expand respectively $(1+(2r^2-1))P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2r^2-1)$ and the remaining instance of $P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2r^2-1)$. □

In order to prove the next result we will make use of the product rule for Dunkl operators (2.2.13), which we reproduce here for convenience.

$$\mathcal{D}_j^{(\gamma)}(fg)(x) = g(x)\mathcal{D}_j^{(\gamma)}f(x) + f(x)\partial_j g(x) + \frac{\gamma_j}{2}f(\sigma_j x)\frac{g(x)-g(\sigma_j x)}{x_j}. \tag{4.5.9}$$

In particular, setting $f \leftarrow Y_m^{(\gamma;l)}$, with $l \in \{\text{even}, \text{odd}\}$ and $m \geq 1$, and $g \leftarrow P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2\left\|\cdot\right\|^2-1)$, we get

$$\begin{aligned}&\mathcal{D}_j^{(\gamma)}(P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2\left\|\cdot\right\|^2-1)Y_m^{(\gamma;l)})(x)\\ &\quad= P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}(2\left\|x\right\|^2-1)\mathcal{D}_j^{(\gamma)}Y_m^l(x) + 4x_jY_m^{(\gamma;l)}(x){P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}}'(2\left\|x\right\|^2-1).\end{aligned} \tag{4.5.10}$$

Then, taking into account Proposition 4.4.2 and using Proposition 4.4.3 to expand $x_jY_m^{(\gamma;l)}$, to then consider Proposition 4.2.1 with $\beta \leftarrow m+\frac{\gamma_1+\gamma_2}{2}-1$ to expand $(1+(2r^2-1)){P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}}'(2r^2-1)$ and (4.2.7) to expand the remaining instance of ${P_n^{(\alpha,m+\frac{\gamma_1+\gamma_2}{2})}}'(2r^2-1)$, we obtain the incarnation of part (iii) of Proposition 2.3.3 in our setting.

Proposition 4.5.3. *If* $m \geq 1$,

$$\mathcal{D}_1^{(\gamma)}Z_{m,n}^{(\alpha,\gamma;\text{even})}$$

$$= \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} \\ +\frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2)\, Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, & \textit{if } m \textit{ is even} \\ \frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} \\ +\frac{m+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2)\, Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, & \textit{if } m \textit{ is odd.} \end{cases} \tag{4.5.11}$$

$$\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})}$$
$$= \begin{cases} -\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} \\ +\frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, & \textit{if } m \textit{ is even} \\ -\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} \\ +\frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, & \textit{if } m \textit{ is odd.} \end{cases} \tag{4.5.12}$$

$$\mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{odd})}$$
$$= \begin{cases} \frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2) Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} \\ +\frac{m}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, & \textit{if } m \textit{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2) Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} \\ +\frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, & \textit{if } m \textit{ is odd.} \end{cases} \tag{4.5.13}$$

$$\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{odd})}$$
$$= \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} \\ -\frac{m}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, & \textit{if } m \textit{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2)\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} \\ -\frac{m+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2) Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, & \textit{if } m \textit{ is odd.} \end{cases} \tag{4.5.14}$$

$$\mathcal{D}_1^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})} = (2n+2\alpha+\gamma_1+\gamma_2+2) Z_{1,n-1}^{(\alpha+1,\gamma;\text{even})}. \tag{4.5.15}$$

$$\mathcal{D}_2^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})} = (2n + 2\alpha + \gamma_1 + \gamma_2 + 2)\, Z_{1,n-1}^{(\alpha+1,\gamma;\text{odd})}. \tag{4.5.16}$$

Proof. The only thing left to observe is that the identities (4.5.15) and (4.5.16) are a straightforward consequence of (4.2.7). □

4.6 Connection relations for Dunkl–Zernike expansion coefficients

Let $\alpha, \gamma_1, \gamma_2 \in (-1, +\infty)$. In this section we study how the connecting relations presented above let us know how the Dunkl–Zernike expansion coefficients of a polynomial under the mappings involved can be computed in terms of the Dunkl–Zernike expansion (potentially with different parameters) of the polynomial itself.

Namely, in Subsection 4.6.1 we show how to compute the expansion of a polynomial in a $(\alpha+1, \gamma)$-Dunkl–Zernike basis in terms of its (α, γ)-Dunkl–Zernike expansion. In Subsection 4.6.2 we obtain the Dunkl analogue of the spectral differentiation formula, more specifically, we show how to compute the expansion of Dunkl operators applied on polynomials in the $(\alpha + 1, \gamma)$-Dunkl–Zernike basis in terms of the (α, γ)-Dunkl–Zernike expansion of the polynomial itself. Finally, in Subsection 4.6.3 we show how to compute the the expansion of polynomials multiplied by x_i in the (α, γ)-Dunkl–Zernike basis in terms of the Dunkl–Zernike expansion of the polynomial itself in the same basis.

4.6.1 Raising parameter α

Given $k, m, n \in \mathbb{N}_0$ let

$$A_k = k + \alpha + \frac{\gamma_1 + \gamma_2}{2} + 1, \quad B_{m,n} = m + n + \alpha + \frac{\gamma_1 + \gamma_2}{2} + 1, \quad C_{m,n} = m + n + \frac{\gamma_1 + \gamma_2}{2}.$$

Then, from Proposition 4.5.1

$$A_{m+2n} Z_{m,n}^{(\alpha,\gamma;\text{even})} = B_{m,n} Z_{m,n}^{(\alpha+1,\gamma;\text{even})} - C_{m,n} Z_{m,n-1}^{(\alpha+1,\gamma;\text{even})}, \tag{4.6.1}$$

$$A_{m+2n} Z_{m,n}^{(\alpha,\gamma;\text{odd})} = B_{m,n} Z_{m,n}^{(\alpha+1,\gamma;\text{odd})} - C_{m,n} Z_{m,n-1}^{(\alpha+1,\gamma;\text{odd})}. \tag{4.6.2}$$

The relation (4.6.1) holds for all $m, n \in \mathbb{N}_0$, yet when $n = 0$ the second term in its right-hand side should be ignored because of the convention $Z_{m,-1}^{(\alpha+1,\gamma;\text{even})} \equiv 0$. Note that if $m = n = 0$ and $\alpha + \frac{\gamma_1+\gamma_2}{2} = -1$, (4.6.1) turns into $0 = 0$. The relation (4.6.2) holds for all $m \geq 1$ and $n \in \mathbb{N}_0$, yet when $n = 0$, the second term in its right-hand side should be ignored because of the convention $Z_{m,-1}^{(\alpha+1,\gamma;\text{odd})} \equiv 0$.

Let $N \in \mathbb{N}_0$ and $u \in \Pi_N^2$. Then, u admits a unique expansion with respect to the non-identically null Dunkl–Zernike polynomials of singularity parameter (α, γ) of degree less than or equal to N:

$$u = \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} u_{m,n;\text{even}}\, Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} u_{m,n;\text{odd}}\, Z_{m,n}^{(\alpha,\gamma;\text{odd})} \tag{4.6.3}$$

Similarly, u admits a unique expansion with respect to the non-identically null Dunkl–Zernike polynomials of singularity parameter $(\alpha + 1, \gamma)$ of degree less than or equal to N:

$$u = \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} \tilde{u}_{m,n;\text{even}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})} + \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} \tilde{u}_{m,n;\text{odd}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{odd})} \tag{4.6.4}$$

Now,

$$\begin{aligned}
&\sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} u_{m,n;\text{even}}\, Z_{m,n}^{(\alpha,\gamma;\text{even})} \\
&\overset{(4.6.1)}{=} \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\text{even}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})} - \sum_{\substack{m\geq 0, n\geq 1\\ m+2n\leq N}} \frac{C_{m,n}}{A_{m+2n}} u_{m,n;\text{even}}\, Z_{m,n-1}^{(\alpha+1,\gamma;\text{even})} \\
&= \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\text{even}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})} - \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N-2}} \frac{C_{m,n+1}}{A_{m+2n+2}} u_{m,n+1;\text{even}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})}.
\end{aligned} \tag{4.6.5}$$

In the first resulting sum double sum, when $m = n = 0$ and $\alpha + \frac{\gamma_1+\gamma_2}{2} = -1$, the ratio $B_{m,n}/A_{m+2n}$ is undefined and should be replaced by 1. Also,

$$\begin{aligned}
&\sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} u_{m,n;\mathrm{odd}}\, Z_{m,n}^{(\alpha,\gamma;\mathrm{odd})}\\
&\overset{(4.6.2)}{=} \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\mathrm{odd}}\, Z_{m,n}^{(\alpha+1,\gamma;\mathrm{odd})} - \sum_{\substack{m\geq 1, n\geq 1\\ m+2n\leq N}} \frac{C_{m,n}}{A_{m+2n}} u_{m,n;\mathrm{odd}}\, Z_{m,n-1}^{(\alpha+1,\gamma;\mathrm{odd})}\\
&= \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\mathrm{odd}}\, Z_{m,n}^{(\alpha+1,\gamma;\mathrm{odd})} - \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N-2}} \frac{C_{m,n+1}}{A_{m+2n+2}} u_{m,n+1;\mathrm{odd}}\, Z_{m,n}^{(\alpha+1,\gamma;\mathrm{odd})}.
\end{aligned} \tag{4.6.6}$$

We can use (4.6.5) and (4.6.6) to compare (4.6.3) with (4.6.4) term by term. As those (m, n) that partake in the first double sum in the last expression of (4.6.5) and (4.6.6) but are excluded from the corresponding second double sum in each case can be characterized by $m+2n \geq N-1$, we have

$$\tilde{u}_{m,n;\mathrm{even}} = \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\mathrm{even}} - \begin{cases} \frac{C_{m,n+1}}{A_{m+2n+2}} u_{m,n+1;\mathrm{even}} & \text{if } m+2n \leq N-2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.6.7}$$

(here, when $m = n = 0$ and $\alpha + \frac{\gamma_1+\gamma_2}{2} = -1$, the ratio $B_{m,n}/A_{m+2n}$ is undefined and should be replaced by 1) and

$$\tilde{u}_{m,n;\mathrm{odd}} = \frac{B_{m,n}}{A_{m+2n}} u_{m,n;\mathrm{odd}} - \begin{cases} \frac{C_{m,n+1}}{A_{m+2n+2}} u_{m,n+1;\mathrm{odd}} & \text{if } m+2n \leq N-2,\\ 0 & \text{otherwise.}\end{cases} \tag{4.6.8}$$

4.6.2 Dunkl spectral differentiation

Given $m \geq 1$ and $n \in \mathbb{N}_0$, let

$$D_{m,n}^{(1,\mathrm{even})} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is even,}\\ \frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is odd.}\end{cases}$$

$$E_{m,n}^{(1,\text{even})} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is even,} \\ \frac{m+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is odd.} \end{cases}$$

$$D_{m,n}^{(2,\text{even})} = \begin{cases} -\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is even,} \\ -\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is odd.} \end{cases},$$

$$E_{m,n}^{(2,\text{even})} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is odd.} \end{cases},$$

$$D_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is odd.} \end{cases}$$

$$E_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is odd} \end{cases}$$

$$D_{m,n}^{(2,\text{odd})} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}(2m+2n+\gamma_1+\gamma_2), & \text{if } m \text{ is odd} \end{cases}$$

$$E_{m,n}^{(2,\text{odd})} = \begin{cases} -\frac{m}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is even,} \\ -\frac{m+1}{2m+\gamma_1+\gamma_2}(2m+2n+2\alpha+\gamma_1+\gamma_2+2), & \text{if } m \text{ is odd} \end{cases}$$

$$F_n^{(1,\text{even})} = 2n+2\alpha+\gamma_1+\gamma_2+2,$$

$$F_n^{(2,\text{even})} = 2n+2\alpha+\gamma_1+\gamma_2+2.$$

Then, from Proposition 4.5.3

$$\mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})} = D_{m,n}^{(1,\text{even})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} + E_{m,n}^{(1,\text{even})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, \tag{4.6.9}$$

$$\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})} = D_{m,n}^{(2,\text{even})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} + E_{m,n}^{(2,\text{even})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, \tag{4.6.10}$$

$$\mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{odd})} = D_{m,n}^{(1,\text{odd})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} + E_{m,n}^{(1,\text{odd})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}, \tag{4.6.11}$$

$$\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{odd})} = D_{m,n}^{(2,\text{odd})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} + E_{m,n}^{(2,\text{odd})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})}, \tag{4.6.12}$$

$$\mathcal{D}_1^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})} = F_n^{(1,\text{even})} Z_{1,n-1}^{(\alpha+1,\gamma;\text{even})}, \tag{4.6.13}$$

$$\mathcal{D}_2^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})} = F_n^{(2,\text{even})} Z_{1,n-1}^{(\alpha+1,\gamma;\text{odd})}. \tag{4.6.14}$$

The relations (4.6.9), (4.6.10) , (4.6.11) and (4.6.12) hold for all $m \geq 1$ and $n \in \mathbb{N}_0$, yet when $n = 0$ the second terms in their right-hand side should be ignored because of the convention $Z_{m,-1}^{(\alpha+1,\gamma;\text{even})} \equiv 0 \equiv Z_{m,-1}^{(\alpha+1,\gamma;\text{odd})}$.

Let $N \in \mathbb{N}_0$ and $u \in \Pi_N^2$. As it is stated in (4.6.3), u is uniquely expanded in terms of the non-identically null Dunkl–Zernike polynomials of parameters (α, γ) of degree less or equal to N.

Similarly, being $\mathcal{D}_j^{(\gamma)} u \in \Pi_{N-1}^2$, it admits a unique expansion with respect to the non-identically null Dunkl–Zernike polynomials of parameters $(\alpha + 1, \gamma)$ of degree less or equal to $N - 1$

$$\mathcal{D}_j^{(\gamma)} u = \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N-1}} \hat{u}_{m,n;j,\text{even}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})} + \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N-1}} \hat{u}_{m,n;j,\text{odd}}\, Z_{m,n}^{(\alpha+1,\gamma;\text{odd})}. \tag{4.6.15}$$

Now,

$$\begin{aligned}
&\sum_{\substack{m\geq 0, n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\, \mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})} \\
&\qquad = \sum_{\substack{m\geq 1, n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\, \mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{n\geq 0\\ 2n\leq N}} u_{0,n;\text{even}}\, \mathcal{D}_1^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})} \\
&\qquad \overset{(4.6.9),(4.6.13)}{=} \sum_{\substack{m\geq 1, n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} D_{m,n}^{(1,\text{even})}\, Z_{m-1,n}^{(\alpha+1,\gamma;\text{even})} + \sum_{\substack{m\geq 1, n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{even}} E_{m,n}^{(1,\text{even})}\, Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{even})} \\
&\qquad\qquad + \sum_{\substack{n\geq 1\\ 2n\leq N}} u_{0,n;\text{even}}\, F_n^{(1,\text{even})}\, Z_{1,n-1}^{(\alpha+1,\gamma;\text{even})} \\
&= \sum_{\substack{m\geq 0, n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{even}} D_{m+1,n}^{(1,\text{even})}\, Z_{m,n}^{(\alpha+1,\gamma;\text{even})} + \sum_{\substack{m\geq 2, n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{even}} E_{m-1,n+1}^{(1,\text{even})} Z_{m,n}^{(\alpha+1,\gamma;\text{even})} \\
&\qquad + \sum_{\substack{n\geq 0\\ 2n\leq N-2}} u_{0,n+1;\text{even}}\, F_{n+1}^{(1,\text{even})}\, Z_{1,n}^{(\alpha+1,\gamma;\text{even})}.
\end{aligned} \tag{4.6.16}$$

Also,

$$\sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}}\,\mathcal{D}_1^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{odd})}$$

$$\overset{(4.6.11)}{=} \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} D_{m,n}^{(1,\text{odd})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{odd}} E_{m,n}^{(1,\text{odd})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}$$

$$= \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{odd}} D_{m+1,n}^{(1,\text{odd})} Z_{m,n}^{(\alpha+1,\gamma;\text{odd})} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{odd}} E_{m-1,n+1}^{(1,\text{odd})} Z_{m,n}^{(\alpha+1,\gamma;\text{odd})}. \tag{4.6.17}$$

We can use (4.6.16) and (4.6.17) to compare the result of applying $\mathcal{D}_1^{(\gamma)}$ in (4.6.3) with (4.6.15) term by term obtaining,

$$\hat{u}_{m,n;1;\text{even}} = \begin{cases} u_{m+1,n;\text{even}} D_{m+1,n}^{(1,\text{even})}, & \text{if } m = 0, \\ u_{m+1,n;\text{even}} D_{m+1,n}^{(1,\text{even})} + u_{0,n+1;\text{even}} F_{n+1}^{(1;\text{even})}, & \text{if } m = 1\ , \\ u_{m+1,n;\text{even}} D_{m+1,n}^{(1,\text{even})} + u_{m-1,n+1;\text{even}} E_{m-1,n+1}^{(1,\text{even})}, & \text{if } m \geq 2. \end{cases} \tag{4.6.18}$$

$$\hat{u}_{m,n;1,\text{odd}} = \begin{cases} u_{m+1,n;\text{odd}} D_{m+1,n}^{(1;\text{odd})}, & \text{if } m = 1, \\ u_{m+1,n;\text{odd}} D_{m+1,n}^{(1;\text{odd})} + u_{m-1,n+1;\text{odd}} E_{m-1,n+1}^{(1;\text{odd})}, & \text{if } m \geq 2. \end{cases} \tag{4.6.19}$$

On the other hand we have

$$\sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\,\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})}$$

$$= \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\,\mathcal{D}_2^{(\gamma)} Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{n\geq 0\\ 2n\leq N}} u_{0,n;\text{even}}\,\mathcal{D}_2^{(\gamma)} Z_{0,n}^{(\alpha,\gamma;\text{even})}$$

$$\overset{(4.6.10),(4.6.14)}{=} \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} D_{m,n}^{(2,\text{even})} Z_{m-1,n}^{(\alpha+1,\gamma;\text{odd})} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{even}} E_{m,n}^{(2,\text{even})} Z_{m+1,n-1}^{(\alpha+1,\gamma;\text{odd})}$$

$$+ \sum_{\substack{n\geq 1\\ 2n\leq N}} u_{0,n;\text{even}}\, F_n^{(2,\text{even})} Z_{1,n-1}^{(\alpha+1,\gamma;\text{odd})}$$

$$= \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{even}} D_{m+1,n}^{(2,\text{even})} Z_{m,n}^{(\alpha+1,\gamma;\text{odd})} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{even}} E_{m-1,n+1}^{(2,\text{even})} Z_{m,n}^{(\alpha+1,\gamma;\text{odd})}$$

$$+\sum_{\substack{n\geq 0\\ 2n\leq N-2}} u_{0,n+1;\text{even}}\, F^{(2,\text{even})}_{n+1} Z^{(\alpha+1,\gamma;\text{odd})}_{1,n}, \quad (4.6.20)$$

and

$$\sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}}\, \mathcal{D}^{(\gamma)}_2 Z^{(\alpha,\gamma;\text{odd})}_{m,n}$$
$$\stackrel{(4.6.12)}{=} \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} D^{(2,\text{odd})}_{m,n} Z^{(\alpha+1,\gamma;\text{even})}_{m-1,n} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{odd}} E^{(2,\text{odd})}_{m,n} Z^{(\alpha+1,\gamma;\text{even})}_{m+1,n-1}$$
$$= \sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{odd}} D^{(2,\text{odd})}_{m+1,n} Z^{(\alpha+1,\gamma;\text{even})}_{m,n} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{odd}} E^{(2,\text{odd})}_{m-1,n+1} Z^{(\alpha+1,\gamma;\text{even})}_{m,n}. \quad (4.6.21)$$

Utilizing (4.6.20) and (4.6.21) to compare the result of applying $\mathcal{D}^{(\gamma)}_2$ in (4.6.3) with (4.6.15) we obtain,

$$\hat{u}_{m,n;2,\text{even}} = \begin{cases} u_{m+1,n;\text{odd}} D^{(2,\text{odd})}_{m+1,n}, & \text{if } 0\leq m\leq 1, \\ u_{m+1,n;\text{odd}} D^{(2,\text{odd})}_{m+1,n} + u_{m-1,n+1;\text{odd}} E^{(2,\text{odd})}_{m-1,n+1}, & \text{if } m\geq 2. \end{cases} \quad (4.6.22)$$

$$\hat{u}_{m,n;2,\text{odd}} = \begin{cases} u_{m+1,n;\text{even}} D^{(2,\text{even})}_{m+1,n} + u_{0,n+1;\text{even}} F^{(2,\text{even})}_{n+1}, & \text{if } m=1, \\ u_{m+1,n;\text{even}} D^{(2,\text{even})}_{m+1,n} + u_{m-1,n+1;\text{even}} E^{(2,\text{even})}_{m-1,n+1}, & \text{if } m\geq 2. \end{cases} \quad (4.6.23)$$

4.6.3 Multiplication by x_i

Given $m\geq 0$ and $n\in\mathbb{N}_0$, let

$$G^{(1,\text{even})}_{m,n} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2} \frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even}, \\ \frac{m+1}{2m+\gamma_1+\gamma_2} \frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd}. \end{cases}$$

$$H^{(1,\text{even})}_{m,n} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2} \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even}, \\ \frac{m+1}{2m+\gamma_1+\gamma_2} \frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd}. \end{cases}$$

$$I_{m,n}^{(1,\text{even})} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$J_{m,n}^{(1,\text{even})} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$G_{m,n}^{(2,\text{even})} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$H_{m,n}^{(2,\text{even})} = \begin{cases} \frac{m+\gamma_1+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$I_{m,n}^{(2,\text{even})} = \begin{cases} -\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ -\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$J_{m,n}^{(2,\text{even})} = \begin{cases} -\frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ -\frac{m+\gamma_1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$G_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$H_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_1+\gamma_2+1}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$I_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$J_{m,n}^{(1,\text{odd})} = \begin{cases} \frac{m+\gamma_1-1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$G_{m,n}^{(2,\text{odd})} = \begin{cases} -\frac{m}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ -\frac{m+1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+2\alpha+\gamma_1+\gamma_2+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$H^{(2,\text{odd})}_{m,n} = \begin{cases} -\frac{m}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ -\frac{m+1}{2m+\gamma_1+\gamma_2}\frac{2n+2\alpha}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$I^{(2,\text{odd})}_{m,n} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2m+2n+\gamma_1+\gamma_2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

$$J^{(2,\text{odd})}_{m,n} = \begin{cases} \frac{m+\gamma_2-1}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is even,} \\ \frac{m+\gamma_2}{2m+\gamma_1+\gamma_2}\frac{2n+2}{2m+4n+2\alpha+\gamma_1+\gamma_2+2}, & \text{if } m \text{ is odd.} \end{cases}$$

Then, from Proposition 4.5.2

$$x_1\, Z^{(\alpha,\gamma;\text{even})}_{m,n} = G^{(1,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m+1,n} + H^{(1,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m+1,n-1} + I^{(1,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m-1,n} + J^{(1,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m-1,n+1}, \tag{4.6.24}$$

$$x_2\, Z^{(\alpha,\gamma;\text{even})}_{m,n} = G^{(2,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m+1,n} + H^{(2,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m+1,n-1} + I^{(2,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m-1,n} + J^{(2,\text{even})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m-1,n+1}, \tag{4.6.25}$$

$$x_1\, Z^{(\alpha,\gamma;\text{odd})}_{m,n} = G^{(1,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m+1,n} + H^{(1,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m+1,n-1} + I^{(1,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m-1,n} + J^{(1,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{odd})}_{m-1,n+1}, \tag{4.6.26}$$

$$x_2\, Z^{(\alpha,\gamma;\text{odd})}_{m,n} = G^{(2,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m+1,n} + H^{(2,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m+1,n-1} + I^{(2,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m-1,n} + J^{(2,\text{odd})}_{m,n}\, Z^{(\alpha,\gamma;\text{even})}_{m-1,n+1}. \tag{4.6.27}$$

Let $N \in \mathbb{N}_0$ and $u \in \Pi^2_N$. As it was stated in (4.6.3), u is uniquely expanded in terms of the non-identically null Dunkl–Zernike polynomials of parameters (α, γ) of degree less or equal to N.

Similarly, being $x_j\, u \in \Pi^2_{N+1}$, it admits a unique expansion with respect to the non-identically null Dunkl–Zernike polynomials of (α, γ) of degree less or equal to $N + 1$

$$x_j\, u = \sum_{\substack{m\geq 0, n\geq 0 \\ m+2n\leq N+1}} \bar{u}_{m,n;j,\text{even}}\, Z^{(\alpha,\gamma;\text{even})}_{m,n} + \sum_{\substack{m\geq 1, n\geq 0 \\ m+2n\leq N+1}} \bar{u}_{m,n;j,\text{odd}}\, Z^{(\alpha,\gamma;\text{odd})}_{m,n}. \tag{4.6.28}$$

We have

$$\begin{aligned}
&\sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\, x_1\, Z_{m,n}^{(\alpha,\gamma;\text{even})}\\
&\qquad\overset{(4.6.24)}{=} \sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} G_{m,n}^{(1,\text{even})} Z_{m+1,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 0,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{even}} H_{m,n}^{(1,\text{even})} Z_{m+1,n-1}^{(\alpha,\gamma;\text{even})}\\
&\qquad\quad + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} I_{m,n}^{(1,\text{even})} Z_{m-1,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} J_{m,n}^{(1,\text{even})} Z_{m-1,n+1}^{(\alpha,\gamma;\text{even})}\\
&\quad = \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N+1}} u_{m-1,n;\text{even}} G_{m-1,n}^{(1,\text{even})} Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{even}} H_{m-1,n+1}^{(1,\text{even})} Z_{m,n}^{(\alpha,\gamma;\text{even})}\\
&\quad + \sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{even}} I_{m+1,n}^{(1,\text{even})} Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 0,n\geq 1,\\ m+2n\leq N+1}} u_{m+1,n-1;\text{even}} J_{m+1,n-1}^{(1,\text{even})} Z_{m,n}^{(\alpha,\gamma;\text{even})}
\end{aligned} \tag{4.6.29}$$

and

$$\begin{aligned}
&\sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}}\, x_1\, Z_{m,n}^{(\alpha,\gamma;\text{odd})}\\
&\qquad\overset{(4.6.26)}{=} \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} G_{m,n}^{(1,\text{odd})} Z_{m+1,n}^{(\alpha,\gamma;\text{odd})} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{odd}} H_{m,n}^{(1,\text{odd})} Z_{m+1,n-1}^{(\alpha,\gamma;\text{odd})}\\
&\qquad\quad + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} I_{m,n}^{(1,\text{odd})} Z_{m-1,n}^{(\alpha,\gamma;\text{odd})} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} J_{m,n}^{(1,\text{odd})} Z_{m-1,n+1}^{(\alpha,\gamma;\text{odd})}\\
&\quad = \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N+1}} u_{m-1,n;\text{odd}} G_{m-1,n}^{(1,\text{odd})} Z_{m,n}^{(\alpha,\gamma;\text{odd})} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{odd}} H_{m-1,n+1}^{(1,\text{odd})} Z_{m,n}^{(\alpha,\gamma;\text{odd})}\\
&\quad + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{odd}} I_{m+1,n}^{(1,\text{odd})} Z_{m,n}^{(\alpha,\gamma;\text{odd})} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N+1}} u_{m+1,n-1;\text{odd}} J_{m+1,n-1}^{(1,\text{odd})} Z_{m,n}^{(\alpha,\gamma;\text{odd})}
\end{aligned} \tag{4.6.30}$$

We can use (4.6.29) and (4.6.30) to compare (4.6.28) with the result of multiplying (4.6.3) by

x_1, obtaining

$$
\begin{aligned}
&\bar{u}_{m,n;1;\mathrm{even}} \\
&= \begin{cases}
u_{m+1,n;\mathrm{even}} I^{(1,\mathrm{even})}_{m+1,n}, & \text{if } m=0,\ n=0,\ m+2n \leq N-1 \\
u_{m-1,n;\mathrm{even}} G^{(1,\mathrm{even})}_{m-1,n} + u_{m-1,n+1,\mathrm{even}} H^{(1,\mathrm{even})}_{m-1,n+1} \\ \qquad + u_{m+1,n;\mathrm{even}} I^{(1,\mathrm{even})}_{m+1,n}, & \text{if } m \geq 1,\ n=0,\ m+2n \leq N-1, \\
u_{m+1,n;\mathrm{even}} I^{(1,\mathrm{even})}_{m+1,n} + u_{m+1,n-1;\mathrm{even}} J^{(1,\mathrm{even})}_{m+1,n-1}, & \text{if } m=0,\ n \geq 1,\ m+2n \leq N-1, \\
u_{m-1,n;\mathrm{even}} G^{(1,\mathrm{even})}_{m-1,n} + u_{m-1,n+1;\mathrm{even}} H^{(1,\mathrm{even})}_{m-1,n+1} \\ + u_{m+1,n;\mathrm{even}} I^{(1,\mathrm{even})}_{m+1,n} + u_{m+1,n-1;\mathrm{even}} J^{(1,\mathrm{even})}_{m+1,n-1}, & \text{if } m,n \geq 1,\ m+2n \leq N-1, \\
0, & \text{if } m=0,\ n=0,\ m+2n \geq N, \\
u_{m-1,n;\mathrm{even}} G^{(1,\mathrm{even})}_{m-1,n}, & \text{if } m \geq 1,\ n=0,\ m+2n \geq N, \\
u_{m+1,n-1,\mathrm{even}} J^{(1,\mathrm{even})}_{m+1,n-1}, & \text{if } m=0,\ n \geq 1,\ m+2n \geq N \\
u_{m-1,n;\mathrm{even}} G^{(1,\mathrm{even})}_{m-1,n} + u_{m+1,n-1;\mathrm{even}} J^{(1,\mathrm{even})}_{m+1,n-1}, & \text{if } m,n \geq 1,\ m+2n \geq N.
\end{cases} \\
&\bar{u}_{m,n;1;\mathrm{odd}} \\
&= \begin{cases}
u_{m+1,n;\mathrm{odd}} I^{(1,\mathrm{odd})}_{m+1,n}, & \text{if } m=1,\ n=0,\ m+2n \leq N-1 \\
u_{m-1,n;\mathrm{odd}} G^{(1,\mathrm{odd})}_{m-1,n} + u_{m-1,n+1,\mathrm{odd}} H^{(1,\mathrm{odd})}_{m-1,n+1} \\ \qquad + u_{m+1,n;\mathrm{odd}} I^{(1,\mathrm{odd})}_{m+1,n}, & \text{if } m \geq 2,\ n=0,\ m+2n \leq N-1, \\
u_{m+1,n;\mathrm{odd}} I^{(1,\mathrm{odd})}_{m+1,n} + u_{m+1,n-1;\mathrm{odd}} J^{(1,\mathrm{odd})}_{m+1,n-1}, & \text{if } m=1,\ n \geq 1,\ m+2n \leq N-1, \\
u_{m-1,n;\mathrm{odd}} G^{(1,\mathrm{odd})}_{m-1,n} + u_{m-1,n+1;\mathrm{odd}} H^{(1,\mathrm{odd})}_{m-1,n+1} \\ + u_{m+1,n;\mathrm{odd}} I^{(1,\mathrm{odd})}_{m+1,n} + u_{m+1,n-1;\mathrm{odd}} J^{(1,\mathrm{odd})}_{m+1,n-1}, & \text{if } m \geq 2,\ n \geq 1,\ m+2n \leq N-1, \\
0, & \text{if } m=1,\ n=0,\ m+2n \geq N, \\
u_{m-1,n;\mathrm{odd}} G^{(1,\mathrm{odd})}_{m-1,n}, & \text{if } m \geq 2,\ n=0,\ m+2n \geq N, \\
u_{m+1,n-1,\mathrm{odd}} J^{(1,\mathrm{odd})}_{m+1,n-1}, & \text{if } m=1,\ n \geq 1,\ m+2n \geq N \\
u_{m-1,n;\mathrm{odd}} G^{(1,\mathrm{odd})}_{m-1,n} + u_{m+1,n-1;\mathrm{odd}} J^{(1,\mathrm{odd})}_{m+1,n-1}, & \text{if } m \geq 2,\ n \geq 1,\ m+2n \geq N.
\end{cases}
\end{aligned}
\tag{4.6.31}
$$

On the other hand, we have

$$\begin{aligned}
&\sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}}\, x_2\, Z^{(\alpha,\gamma;\text{even})}_{m,n} \\
&\overset{(4.6.25)}{=} \sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} G^{(2,\text{even})}_{m,n} Z^{(\alpha,\gamma;\text{odd})}_{m+1,n} + \sum_{\substack{m\geq 0,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{even}} H^{(2,\text{even})}_{m,n} Z^{(\alpha,\gamma;\text{odd})}_{m+1,n-1} \\
&\quad + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} I^{(2,\text{even})}_{m,n} Z^{(\alpha,\gamma;\text{odd})}_{m-1,n} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{even}} J^{(2,\text{even})}_{m,n} Z^{(\alpha,\gamma;\text{odd})}_{m-1,n+1} \\
&= \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N+1}} u_{m-1,n;\text{even}} G^{(2,\text{even})}_{m-1,n} Z^{(\alpha,\gamma;\text{odd})}_{m,n} + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{even}} H^{(2,\text{even})}_{m-1,n+1} Z^{(\alpha,\gamma;\text{odd})}_{m,n} \\
&+ \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{even}} I^{(2,\text{even})}_{m+1,n} Z^{(\alpha,\gamma;\text{odd})}_{m,n} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N+1}} u_{m+1,n-1;\text{even}} J^{(2,\text{even})}_{m+1,n-1} Z^{(\alpha,\gamma;\text{odd})}_{m,n} \qquad (4.6.32)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}}\, x_2\, Z^{(\alpha,\gamma;\text{odd})}_{m,n} \\
&\overset{(4.6.27)}{=} \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} G^{(2,\text{odd})}_{m,n} Z^{(\alpha,\gamma;\text{even})}_{m+1,n} + \sum_{\substack{m\geq 1,n\geq 1,\\ m+2n\leq N}} u_{m,n;\text{odd}} H^{(2,\text{odd})}_{m,n} Z^{(\alpha,\gamma;\text{even})}_{m+1,n-1} \\
&\quad + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} I^{(2,\text{odd})}_{m,n} Z^{(\alpha,\gamma;\text{even})}_{m-1,n} + \sum_{\substack{m\geq 1,n\geq 0,\\ m+2n\leq N}} u_{m,n;\text{odd}} J^{(2,\text{odd})}_{m,n} Z^{(\alpha,\gamma;\text{even})}_{m-1,n+1} \\
&= \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N+1}} u_{m-1,n;\text{odd}} G^{(2,\text{odd})}_{m-1,n} Z^{(\alpha,\gamma;\text{even})}_{m,n} + \sum_{\substack{m\geq 2,n\geq 0,\\ m+2n\leq N-1}} u_{m-1,n+1;\text{odd}} H^{(2,\text{odd})}_{m-1,n+1} Z^{(\alpha,\gamma;\text{even})}_{m,n} \\
&+ \sum_{\substack{m\geq 0,n\geq 0,\\ m+2n\leq N-1}} u_{m+1,n;\text{odd}} I^{(2,\text{odd})}_{m+1,n} Z^{(\alpha,\gamma;\text{even})}_{m,n} + \sum_{\substack{m\geq 0,n\geq 1,\\ m+2n\leq N+1}} u_{m+1,n-1;\text{odd}} J^{(2,\text{odd})}_{m+1,n-1} Z^{(\alpha,\gamma;\text{even})}_{m,n} \qquad (4.6.33)
\end{aligned}$$

Utilizing (4.6.32) and (4.6.33) to compare (4.6.28) with the result of multiplying (4.6.3) by x_2, we obtain

$$\bar{u}_{m,n;2;\text{even}} = \begin{cases} u_{m+1,n;\text{odd}} I^{(2,\text{odd})}_{m+1,n}, & \text{if } 0 \le m \le 1,\ n=0,\ m+2n \le N-1 \\ u_{m-1,n;\text{odd}} G^{(2,\text{odd})}_{m-1,n} + u_{m-1,n+1,\text{odd}} H^{(2,\text{odd})}_{m-1,n+1} \\ \qquad + u_{m+1,n;\text{odd}} I^{(2,\text{odd})}_{m+1,n}, & \text{if } m \ge 2,\ n=0,\ m+2n \le N-1, \\ u_{m+1,n;\text{odd}} I^{(2,\text{odd})}_{m+1,n} + u_{m+1,n-1;\text{odd}} J^{(2,\text{odd})}_{m+1,n-1}, & \text{if } 0 \le m \le 1,\ n \ge 1,\ m+2n \le N-1, \\ u_{m-1,n;\text{odd}} G^{(2,\text{odd})}_{m-1,n} + u_{m-1,n+1;\text{odd}} H^{(2,\text{odd})}_{m-1,n+1} \\ + u_{m+1,n;\text{odd}} I^{(2,\text{odd})}_{m+1,n} + u_{m+1,n-1;\text{odd}} J^{(2,\text{odd})}_{m+1,n-1}, & \text{if } m \ge 2,\ n \ge 1,\ m+2n \le N-1, \\ 0, & \text{if } 0 \le m \le 1,\ n=0,\ m+2n \ge N, \\ u_{m-1,n;\text{odd}} G^{(2,\text{odd})}_{m-1,n}, & \text{if } m \ge 2,\ n=0,\ m+2n \ge N, \\ u_{m+1,n-1,\text{odd}} J^{(2,\text{odd})}_{m+1,n-1}, & \text{if } 0 \le m \le 1,\ n \ge 1,\ m+2n \ge N \\ u_{m-1,n;\text{odd}} G^{(2,\text{odd})}_{m-1,n} + u_{m+1,n-1;\text{odd}} J^{(2,\text{odd})}_{m+1,n-1}, & \text{if } m \ge 2,\ n \ge 1,\ m+2n \ge N. \end{cases}$$

$$\bar{u}_{m,n;2;\text{odd}} = \begin{cases} u_{m-1,n;\text{even}} G^{(2,\text{even})}_{m-1,n} + u_{m-1,n+1;\text{even}} H^{(2,\text{even})}_{m-1,n+1} \\ \qquad + u_{m+1,n;\text{even}} I^{(2,\text{even})}_{m+1,n}, & \text{if } n=0,\ m+2n \le N-1, \\ u_{m-1,n;\text{even}} G^{(2,\text{even})}_{m-1,n} + u_{m-1,n+1;\text{even}} H^{(2,\text{even})}_{m-1,n+1} \\ + u_{m+1,n;\text{even}} I^{(2,\text{even})}_{m+1,n} + u_{m+1,n-1;\text{even}} J^{(2,\text{even})}_{m+1,n-1}, & \text{if } n \ge 1,\ m+2n \le N-1, \\ u_{m-1,n;\text{even}} G^{(2,\text{even})}_{m-1,n}, & \text{if } n=0,\ m+2n \ge N, \\ u_{m-1,n;\text{even}} G^{(2,\text{even})}_{m-1,n} + u_{m+1,n-1;\text{even}} J^{(2,\text{even})}_{m+1,n-1}, & \text{if } n \ge 1,\ m+2n \ge N. \end{cases} \tag{4.6.34}$$

CHAPTER 5

Tools for numerical computations with orthogonal polynomials

5.1 Introduction

In this chapter we present and describe the basic mechanisms behind a recently developed Julia 1.2.0 package DunklZernikeExpansions [3] which exploits the connection relations obtained in Chapter 4 in order to perform basic computations with bivariate polynomials in the unit ball expressed as a linear combination of Dunkl–Zernike polynomials.

This package was developed inspired by some traits of the Chebfun [10, 40] and ApproxFun [29, 37] Julia packages; it can be considered the successor of ZernikeSuite [21], although the latter exploits the elegance of the basis found in [47].

We used our DunklZernikeExpansions package to verify identities of Chapter 2 and Chapter 3 considerably faster than it is possible with symbolic computation.

5.2 The code

The code that will be presented in the following sections corresponds to the file `DunklZernikeExpansions.jl` in the Git repository of the package DunklZernikeExpansions [3] in the exact version retrievable as its *commit* `da6d16f`.

5.2.1 Basic constructions

As it was stated in (4.6.3), given u a bivariate polynomial of degree N, it admits a unique expansion with respect to the non-identically null Dunkl–Zernike polynomials of parameter (α, γ) of degree less than or equal to N

$$u = \sum_{\substack{m\geq 0, n\geq 0\\ m+2n\leq N}} u_{m,n;\text{even}}\, Z_{m,n}^{(\alpha,\gamma;\text{even})} + \sum_{\substack{m\geq 1, n\geq 0\\ m+2n\leq N}} u_{m,n;\text{odd}}\, Z_{m,n}^{(\alpha,\gamma;\text{odd})} \tag{5.2.1}$$

That is, we can uniquely represent every bivariate polynomial by its expansion coefficients and the parameter (α, γ); this is the main and basic idea behind this package.

One of the main obstacles in storing the expansion coefficients is dealing with their triple, non-tensorial, indexation. To address this problem we implemented some auxiliary functions that allow us, for example, to know which position in the coefficients vector (unidimensional array) corresponds to the coefficient $u_{m,n,\text{even}}$.

Listing 5.1: Basic configuration of the package

```
module DunklZernikeExpansions

import Base: +, -, *, /, ==, isapprox
import Jacobi:jacobi
import SpecialFunctions:gamma

export DZFun, DZParam, DZPoly, wip, evalDZ, mbx1, mbx2, symx1, symx2, skewx1, skewx2, Dunklx1, Dunklx2,
    DunklAngular, project, mbr, adjointDunklx1, adjointDunklx2
```

In Listing 5.1 we name the module (package), import some functions from other packages—

arithmetic operators, Jacobi polynomials and the Gamma function—that will be used later, and specify which Julia objects of the package will be exported; i.e., directly available to the user.

Listing 5.2: Deduce the degree of a polynomial from the number of expansion coefficients

```
function inferDegree(l::Int64)
    # Given l it returns two integers; the first one is the lowest integer n such that (n+1)(n+2)÷2 ≥ l;
    # the second one is the residual (n+1)(n+2)÷2 - l
    n = (-3 + sqrt(1+8*l))/2 # This will be a float
    cn = convert(Int64, ceil(n))
    cn, (cn+1)*(cn+2)÷2-l
end
```

In Listing 5.2 we implement a function that determines the degree of a polynomial from the number of expansion coefficients.

Listing 5.3: Dimension of Π_N^2 (number of coefficients to uniquely expand its members)

```
polyDim(deg::Int64) = (deg+1)*(deg+2)÷2
```

Listing 5.4: Parameter structure

```
struct DZParam
    γ1::Float64
    γ2::Float64
    α::Float64
    function DZParam(γ1, γ2, α)
        @assert γ1 > -1 && γ2 > -1 && α > -1
        new(γ1, γ2, α)
    end
end

function isapprox(κ1::DZParam, κ2::DZParam)
    a = 1.0e-12
    isapprox(κ1.γ1,κ2.γ1;atol=a) && isapprox(κ1.γ2,κ2.γ2;atol=a) && isapprox(κ1.α,κ2.α;atol=a)
```

```
end
```

In Listing 5.4 we define a structure called DZParam which represents the parameter of the expanding Dunkl–Zernike polynomials. Also, we define a binary operation between DZParam elements which allows us to determine when two of them are approximately equal.

Listing 5.5: Dunkl–Zernike expansion structure

```
struct DZFun
    κ::DZParam
    degree::Int64
    coefficients::Vector{Float64}
    # The coefficients of a polynomial are ordered by degree first;
    # within each degree, the coefficients accompanying generalized cosines appear in the odd-indexed positions and generalized sines in the even-indexed positions;
    # within the odd (resp. even) positions the coefficients accompanying Dunkl-Zernike polynomials involving spherical harmonics of higher degree appear first
    function DZFun(κ, degree, coefficients)
        @assert 2*length(coefficients) == (degree+1)*(degree+2)
        new(κ, degree, coefficients)
    end
end

function DZFun(κ::Tuple{T1,T2,T3}, degree::Int64, coefficients::Vector{Float64}) where {T1<:Real, T2<:Real, T3<:Real}
    param = DZParam(κ...)
    DZFun(param, degree, coefficients)
end

function DZFun(κ::Vector{T}, degree::Int64, coefficients::Vector{Float64}) where T<:Real
    @assert length(κ) == 3 "If the parameter is given as a vector it must be of length 3"
    param = DZParam(κ...)
    DZFun(param, degree, coefficients)
end

function DZFun(κ, coefficients::Vector{T}) where T<:Real
    n, res = inferDegree(length(coefficients))
    cl = polyDim(n)
    newcoefficients = zeros(Float64, cl)
    newcoefficients[1:length(coefficients)] = coefficients
    DZFun(κ, n, newcoefficients)
```

```
end
```

In Listing 5.5 we find the core of this package. Here we define a structure called `DZFun` which represents the Dunkl–Zernike expansions. To initialize a `DZFun` one only needs the parameter (as a three-element vector or `DZParam`) of the expanding Dunkl–Zernike polynomials, and the expansion coefficients. Optionally, one can provide the degree of the polynomial being expanded to assert that the inputted coefficients are actually expanding a polynomial of that degree. As it is mentioned in lines 38–40, the coefficients of the polynomial are order by degree first; within each degree, the coefficients accompanying $Z_{m,n}^{(\alpha,\gamma;\text{even})}$ appear in the odd-indexed positions and the ones accompanying $Z_{m,n}^{(\alpha,\gamma;\text{odd})}$ in the even-indexed positions; within the odd (resp. even) positions, the coefficients accompanying Dunkl–Zernike polynomials involving h-harmonics of higher degree appear first.

Listing 5.6: Basic operations with `DZFun`

```
function ==(f::DZFun, g::DZFun)
    equalκ = f.κ == g.κ
    fl = length(f.coefficients)
    gl = length(g.coefficients)
    maxl = max(fl,gl)
    equalc = [f.coefficients;zeros(maxl-fl)] == [g.coefficients;zeros(maxl-gl)]
    equalκ && equalc
end

function isapprox(f::DZFun, g::DZFun)
    equalκ = f.κ ≈ g.κ
    fl = length(f.coefficients)
    gl = length(g.coefficients)
    maxl = max(fl,gl)
    equalc = [f.coefficients;zeros(maxl-fl)] ≈ [g.coefficients;zeros(maxl-gl)]
    equalκ && equalc
end

# Unary operations
-(f::DZFun) = DZFun(f.κ, f.degree, -f.coefficients)

# Binary operations
```

```julia
for op = (:+, :-)
    @eval begin
        function ($op)(f::DZFun, g::DZFun)
            @assert f.κ ≈ g.κ
            fl = length(f.coefficients)
            gl = length(g.coefficients)
            retl = max(fl, gl)
            retd = max(f.degree, g.degree)
            retcoefficients = zeros(Float64, retl)
            retcoefficients[1:fl] = f.coefficients;
            retcoefficients[1:gl] = ($op)(retcoefficients[1:gl], g.coefficients);
            DZFun(f.κ, retd, retcoefficients)
        end
    end
end

# Operations with scalars
for op = (:+, :-)
    @eval begin
        function ($op)(f::DZFun, a::Number)
            ($op)(f, DZFun(f.κ, 0, [a]))
        end
    end
end
for op = (:*, :/)
    @eval begin
        function ($op)(f::DZFun, a::Number)
            DZFun(f.κ, f.degree, ($op)(f.coefficients, a))
        end
    end
end
for op = (:+, :*)
    @eval begin
        ($op)(a::Number, f::DZFun) = ($op)(f, a)
    end
end
-(a::Number, f::DZFun) = a + (-f)
```

In Listing 5.6 we implement basic operations with DZFun, including comparative operators $==$ and $\approx$, sum, subtraction and multiplication by scalar.

Listing 5.7: Bijection between triple indexation and position in coefficients vector

```
# Position range of coefficients of given degree
positionRange(deg::Integer) = (polyDim(deg-1)+1):polyDim(deg)

function pairing(m::Int64, n::Int64, even::Bool)
    @assert m≥0 && n≥0
    @assert m>0 || even
    deg = m+2n
    1+polyDim(deg-1)+(~even)+2*n
end

function inversepairing(i::Int64)
    @assert i≥0
    deg, res = inferDegree(i)
    n = (deg-res)÷2
    m = deg-2*n
    even = ~Bool((deg-res)%2)
    (m,n,even)
end
```

The functions `pairing` and `inversepairing` establish the bijection between the triple indexation of the expansion coefficients appearing in (5.2.1) and their positions in the coefficients vector defining a `DZFun`.

Listing 5.8: Single Dunkl–Zernike polynomial as `DZFun`

```
# Dunkl-Zernike polynomials
function DZPoly(κ::DZParam, m::Int64, n::Int64, even::Bool)
    i = pairing(m, n, even)
    v = zeros(Float64, i)
    v[i] = 1.0
    DZFun(κ, v)
end

function DZPoly(κ::Tuple{T1,T2,T3}, m::Int64, n::Int64, even::Bool) where {T1<:Real, T2<:Real, T3<:Real}
    param = DZParam(κ...)
    DZPoly(param, m, n, even)
end

function DZPoly(κ::Vector{T}, m::Int64, n::Int64, even::Bool) where T<:Real
    @assert length(κ) == 3 "If the parameter is given as a vector it must be of length 3"
```

```
    param = DZParam(κ...)
    DZPoly(param, m, n, even)
end
```

DZPoly lets us represent a single Dunkl–Zernike polynomial by a DZFun structure.

5.2.2 Computations with DunklZernikeExpansions

Listing 5.9: Computation of $S_N^{(\alpha,\gamma)}$

```
function project(f::DZFun, N::Int64)
    if f.degree≤N
        f
    else
        DZFun(f.κ,f.coefficients[1:polyDim(N)])
    end
end
```

The function project expresses the result of applying $S_N^{(\alpha,\gamma)}$ on a polynomial represented by a DZFun in a new DZFun with the same parameter.

Listing 5.10: Raising parameter α

```
"""
Express a DZFun in a base with α raised by 1
"""
function raise(f::DZFun)
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    α = f.κ.α
    N = f.degree
    outκ = DZParam(γ1, γ2, α+1)
    outcoefs = zeros(Float64, length(f.coefficients))
    for n = 0:N÷2
        poscos = pairing(0,n,true)
        if n == 0
            outcoefs[poscos] = f.coefficients[poscos]
```

```
        else
            outcoefs[poscos] = (n+α+(γ1+γ2)/2+1)/(2n+α+(γ1+γ2)/2+1)*f.coefficients[poscos]
        end
        if 2n≤N-2
            poscosup = pairing(0,n+1,true)
            outcoefs[poscos] -= (n+1+(γ1+γ2)/2)/(2n+α+3+(γ1+γ2)/2)*f.coefficients[poscosup]
        end
        for m = 1:N - 2n
            poscos = pairing(m,n,true)
            possin = pairing(m,n,false)
            outcoefs[poscos] = (m+n+α+(γ1+γ2)/2+1)/(m+2n+α+(γ1+γ2)/2+1)*f.coefficients[poscos]
            outcoefs[possin] = (m+n+α+(γ1+γ2)/2+1)/(m+2n+α+(γ1+γ2)/2+1)*f.coefficients[possin]
            if m+2n≤N-2
                poscosup = pairing(m,n+1,true)
                possinup = pairing(m,n+1,false)
                outcoefs[poscos] -= (m+n+1+(γ1+γ2)/2)/(m+2n+α+3+(γ1+γ2)/2)*f.coefficients[poscosup]
                outcoefs[possin] -= (m+n+1+(γ1+γ2)/2)/(m+2n+α+3+(γ1+γ2)/2)*f.coefficients[possinup]
            end
        end
    end
    DZFun(outκ, N, outcoefs)
end
```

The function `raise` reexpress a polynomial represented by a `DZFun` in a new `DZFun` with α raised by 1. Its implementation is based on Subsection 4.6.1.

Listing 5.11: Lowering parameter α

```
"""
Express a DZFun in a base with α lowered by 1
"""
function lower(f::DZFun)
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    α = f.κ.α - 1 #New α
    @assert α>-1
    N = f.degree
    outκ = DZParam(γ1, γ2, α)
    origcoefs = f.coefficients
    outcoefs = zeros(Float64, length(origcoefs))
    for n = N÷2:-1:0
```

```
        m = 0
        poscos = pairing(m,n,true)
        if n == 0
            if m+2n>N-2
                outcoefs[poscos] = origcoefs[poscos]
            else
                poscosup = pairing(m,n+1,true)
                outcoefs[poscos] = origcoefs[poscos] + (m+n+1+(γ1+γ2)/2)/(m+2n+3+α+(γ1+γ2)/2)*outcoefs[poscosup]
            end
        else
            if m+2n>N-2
                outcoefs[poscos] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*origcoefs[poscos]
            else
                poscosup = pairing(m,n+1,true)
                outcoefs[poscos] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*( origcoefs[poscos] + (m+n+1+(γ1+γ2)/2)/(m+2n+3+α+(γ1+γ2)/2)*outcoefs[poscosup] )
            end
        end
        for m = 1:N-2n
            if m+2n>N-2
                poscos = pairing(m,n,true)
                possin = pairing(m,n,false)
                outcoefs[poscos] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*origcoefs[poscos]
                outcoefs[possin] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*origcoefs[possin]
            else
                poscos = pairing(m,n,true)
                possin = pairing(m,n,false)
                poscosup = pairing(m,n+1,true)
                possinup = pairing(m,n+1,false)
                outcoefs[poscos] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*( origcoefs[poscos] + (m+n+1+(γ1+γ2)/2)/(m+2n+3+α+(γ1+γ2)/2)*outcoefs[poscosup] )
                outcoefs[possin] = (m+2n+α+(γ1+γ2)/2+1)/(m+n+α+(γ1+γ2)/2+1)*( origcoefs[possin] + (m+n+1+(γ1+γ2)/2)/(m+2n+3+α+(γ1+γ2)/2)*outcoefs[possinup] )
            end
        end
    end
    DZFun(outκ, N, outcoefs)
end
```

The function `lower` corresponds to the inverse function of `raise`.

Listing 5.12: Auxiliary functions for point evaluation and computation of weighted inner product of DZFun

```
"""
Evaluate Generalized Gegenbauer
"""
function genGeg(x::Number,n::Integer,lam::Number,mu::Number)
    if iseven(n)
        return jacobi(2x^2-1,n÷2,lam-0.5,mu-0.5)
    else
        return x*jacobi(2x^2-1,(n-1)÷2,lam-0.5,mu+0.5)
    end
end

"""
Square norm of a Jacobi polynomial
"""
function jacsqn(n::Integer,α::Float64,β::Float64)
    if n == 0 && α+β+1≈0
        2^(α+β+1)*gamma(α+1)*gamma(β+1)
    else
        (2^(α+β+1)/(2n+α+β+1))*( (gamma(n+α+1)*gamma(n+β+1))/(gamma(n+α+β+1)*factorial(n)) )
    end
end

"""
Square norm of a Generalized Gegenbauer polynomial
"""
function ggsqn(n::Integer,lam::Number,mu::Number)
    if iseven(n)
        jacsqn(n÷2,lam-0.5,mu-0.5)/2^(lam+mu)
    else
        jacsqn((n-1)÷2,lam-0.5,mu+0.5)/2^(lam+mu+1)
    end
end

"""
Square norm (on the circle) of hharmonic
"""
function hhsqn(m::Integer,γ1::Float64,γ2::Float64,even::Bool)
    if even
        2*ggsqn(m,γ2/2,γ1/2)
    else
        2*ggsqn(m-1,γ2/2+1,γ1/2)
    end
```

```
end

"""
Square norm of an element of a Dunkl-Zernike polynomial
"""
function DZsqn(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64,even::Bool)
    jacsqn(n,α,m+(γ1+γ2)/2)/2^(m+α+(γ1+γ2)/2+2)*hhsqn(m,γ1,γ2,even)
end

"""
Compute the ratio between the weighted square norms of two consecutive Jacobi polynomials of same parameters
"""
JacDegreeRatio(n::Integer,α::Float64,β::Float64) = ((2n+α+β+1)/(2n+α+β+3))*((n+α+1)/(n+α+β+1))*((n+β+1)/(n+1))

"""
Compute the ratio between twe weighted square norm of two Jacobi polynoamials of same degree and first parameter but differing in its second parameter in two units
"""
JacParameterRatio(n::Integer,α::Float64,β::Float64) = 4*((2n+α+β+1)/(2n+α+β+3))*((n+β+2)/(n+α+β+2))*((n+β+1)/(n+α+β+1))

"""
Compute the ratio between the weighted square norm of two Generalized Gegenbauer polynomials of same parameters but differing in the degree in two units.
"""
function GGRatio(n::Integer,λ::Float64,μ::Float64)
    if iseven(n)
        JacDegreeRatio(n÷2,λ-0.5,μ-0.5)
    else
        JacDegreeRatio((n-1)÷2,λ-0.5,μ+0.5)
    end
end

"""
Compute the ratio between the weighted square norm of two h-harmonic polynomials of same parameters but differing in the degree in two units
"""
function hhRatio(m::Integer,γ1::Float64,γ2::Float64,even::Bool)
    if even
        GGRatio(m,γ2/2,γ1/2)
    else
        GGRatio(m-1,γ2/2+1,γ1/2)
    end
```

```
end

"""
Compute the ratio between the weighted square norm of two DZ polynomials of same parameters but differing
    in n in one unit
"""
DZnRatio(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64,even::Bool) = JacDegreeRatio(n,α,m+(γ1+γ2
    )/2)

"""
Compute the ratio between the weighted square norm of two DZ polynomials of same parameters but differing
    in m in two units
"""
DZmRatio(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64,even::Bool) = .25*JacParameterRatio(n,α,m
    +(γ1+γ2)/2)*hhRatio(m,γ1,γ2,even)
```

Listing 5.13: Computation of weighted inner product between two `DZFun`

```
"""
Compute weighted inner product between two DZFun with the same parameters
"""
function wip(f::DZFun,g::DZFun)
    @assert f.κ ≈ g.κ
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    α = f.κ.α
    vf = f.coefficients
    vg = g.coefficients
    N = min(f.degree,g.degree)
    out = 0.0

    for even=[true,false]
        pivot1 = DZsqn(1-even,0,α,γ1,γ2,even)
        pivot2 = DZsqn(2-even,0,α,γ1,γ2,even)
        for m=(1-even):N
            aux = pivot1
            for n=0:(N-m)÷2
                ix = pairing(m,n,even)
                out += vf[ix]*vg[ix]*aux
                aux = DZnRatio(m,n,α,γ1,γ2,even)*aux
            end

```

```
            (pivot1,pivot2) = (pivot2,DZmRatio(m,0,α,γ1,γ2,even)*pivot1)
        end
    end
    out
end
```

The function `wip` computes the weighted inner product $\langle\cdot,\cdot\rangle_{\alpha,\gamma}$ between two polynomials represented by `DZFun` with parameter (α,γ). Its implementation avoids computing the weighted square norms of Dunkl–Zernike polynomials via (4.5.2) (except for a few low degree ones), but takes advantage of the recurrence relations (4.5.3) and (4.5.4). This way, we avoid the very present risk of numerical under- and overflow, while ensuring good speed performance.

Listing 5.14: Evaluate `DZFun`

```
"""
Evaluate DZFun
"""
function evalDZ(f::DZFun,x::Number,y::Number)
    out = 0.0
    coefficients = f.coefficients
    α = f.κ.α
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    r2 = x^2+y^2
    t = atan(y,x)
    for j = 1:length(coefficients)
        (m,n,even) = inversepairing(j)
        if even
            out += coefficients[j]*r2^(m/2)*genGeg(cos(t),m,γ2/2,γ1/2)*jacobi(2r2-1,n,α,m+(γ1+γ2)/2)
        else
            out += coefficients[j]*r2^(m/2)*sin(t)*genGeg(cos(t),m-1,γ2/2+1,γ1/2)*jacobi(2r2-1,n,α,m+(γ1+γ2)/2)
        end
    end
    out
end
```

The function `evalDZ` let us evaluate a polynomial represented by a `DZFun` at a given point

(x, y).

Listing 5.15: Auxiliary functions for Dunkl spectral differentiation

```
function D1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        (m+γ2-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    else
        (m+γ1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    end
end
function E1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        (m+γ1+γ2)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    else
        (m+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    end
end
function D2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        -(m+γ1-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    else
        -(m+γ1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    end
end
function E2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        (m+γ1+γ2)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    else
        (m+γ1+γ2+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    end
end
function D1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        (m+γ1-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    else
        (m+γ2)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    end
end
function E1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        m*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    else
```

```
        (m+γ1+γ2+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    end
end
function D2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        (m+γ2-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    else
        (m+γ2)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)
    end
end
function E2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if iseven(m)
        -m*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    else
        -(m+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)
    end
end
F1even(n::Integer,α::Float64,γ1::Float64,γ2::Float64) = 2n+2α+γ1+γ2+2
F2even(n::Integer,α::Float64,γ1::Float64,γ2::Float64) = 2n+2α+γ1+γ2+2
```

The functions in Listing 5.15 compute the constants appearing at the beginning of Subsection 4.6.2.

Listing 5.16: Shifted Dunkl operators acting in DZFun

```
"""
Dunkl-x1 operator with shift
"""
function DunklShiftx1(f::DZFun)
    OrigCoeff = f.coefficients
    α = f.κ.α
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    N = f.degree

    OutDegree = max(0,N-1)
    OutCoeff = zeros(polyDim(OutDegree))

    m = 0
    for n=0:fld(N-1,2)
        ixMN = pairing(m,n,true) # Index associated to (0,n,Even)
```

```
        ixMpN = pairing(m+1,n,true) # Index associated to (1,n,Even)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D1even(m+1,n,α,γ1,γ2)
    end

    m = 1
    for n=0:fld(N-1-m,2)
        ixMN = pairing(m,n,true) # Index associated to (1,n,Even)
        ixMpN = pairing(m+1,n,true) # Index associated to (2,n,Even)
        ixMmNp = pairing(m-1,n+1,true) # Index associated to (0,n+1,Even)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D1even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*F1even(n+1,α,γ1,γ2)

        ixMN = pairing(m,n,false) # Index associated to (1,n,Odd)
        ixMpN = pairing(m+1,n,false) # Index associated to (2,n,Odd)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D1odd(m+1,n,α,γ1,γ2)
    end
    for m=2:(N-1)
        for n=0:(N-1-m)÷2
            ixMN = pairing(m,n,true) # Index associated to (m,n,Even)
            ixMpN = pairing(m+1,n,true) # Index associated to (m+1,n,Even)
            ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,n+1,Even)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*D1even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*E1even(m-1,n+1,α,γ1
                ,γ2)

            ixMN = pairing(m,n,false) # Index associated to (m,n,Odd)
            ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
            ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,n+1,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*D1odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*E1odd(m-1,n+1,α,γ1,
                γ2)
        end
    end
    DZFun([γ1,γ2,α+1],OutDegree,OutCoeff)
end

"""
Dunkl-x2 operator with shift
"""
function DunklShiftx2(f::DZFun)
    OrigCoeff = f.coefficients
    α = f.κ.α
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    N = f.degree

    OutDegree = max(0,N-1)
    OutCoeff = zeros(polyDim(OutDegree))
```

```julia

    m = 0
    for n=0:fld(N-1,2)
        ixMN = pairing(m,n,true) # Index associated to (0,n,Even)
        ixMpN = pairing(m+1,n,false) # Index associated to (1,n,Odd)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D2odd(m+1,n,α,γ1,γ2)
    end

    m = 1
    for n=0:fld(N-1-m,2)
        ixMN = pairing(m,n,true) # Index associated to (1,n,Even)
        ixMpN = pairing(m+1,n,false) # Index associated to (2,n,odd)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D2odd(m+1,n,α,γ1,γ2)

        ixMN = pairing(m,n,false) # Index associated to (1,n,Odd)
        ixMpN = pairing(m+1,n,true) # Index associated to (2,n,Even)
        ixMmNp = pairing(m-1,n+1,true) # Index assoaciated to (0,n+1,Even)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*D2even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*F2even(n+1,α,γ1,γ2)
    end
    for m=2:(N-1)
        for n=0:(N-1-m)÷2
            ixMN = pairing(m,n,true) # Index associated to (m,n,Even)
            ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
            ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,n+1,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*D2odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*E2odd(m-1,n+1,α,γ1,
                γ2)

            ixMN = pairing(m,n,false) # Index associated to (m,n,Odd)
            ixMpN = pairing(m+1,n,true) # Index associated to (m+1,n,Even)
            ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,n+1,Even)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*D2even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*E2even(m-1,n+1,α,γ1
                ,γ2)
        end
    end
    DZFun([γ1,γ2,α+1],OutDegree,OutCoeff)
end
```

The function `DunklShiftx1` (resp. `DunklShiftx2`) expresses the result of applying the Dunkl operator $\mathcal{D}_1^\gamma$ (resp. $\mathcal{D}_2^\gamma$) on a polynomial represented by a `DZFun` in a new `DZFun` with α raised by 1. The implementation is based on Subsection 4.6.2.

Listing 5.17: Unshifted Dunkl operators acting in DZFun

```
"""
Unshifted Dunkl operators
"""
Dunklx1(f::DZFun) = lower(DunklShiftx1(f))
Dunklx2(f::DZFun) = lower(DunklShiftx2(f))
```

The function Dunklx1 (resp. Dunklx2) expresses the result of applying the Dunkl operator $\mathcal{D}_1^\gamma$ (resp. $\mathcal{D}_2^\gamma$) on a polynomial represented by a DZFun in a new DZFun with the same parameter.

Listing 5.18: Auxiliary functions for multiplication by x_i

```
function G1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif m > 0
        (m+γ1+γ2)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif n > 0
        (2m+2n+2α+γ1+γ2+2)/(2m+4n+2α+γ1+γ2+2)
    else
        1.0
    end
end

function H1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+1)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif m > 0
        (m+γ1+γ2)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (2n+2α)/(2m+4n+2α+γ1+γ2+2)
    end
end

function I1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ2-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end
```

```

function J1even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ2-1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function G2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1+γ2+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif m > 0
        (m+γ1+γ2)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif n > 0
        (2m+2n+2α+γ1+γ2+2)/(2m+4n+2α+γ1+γ2+2)
    else
        1.0
    end
end

function H2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1+γ2+1)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    elseif m > 0
        (m+γ1+γ2)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (2n+2α)/(2m+4n+2α+γ1+γ2+2)
    end
end

function I2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        -(m+γ1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        -(m+γ1-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function J2even(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        -(m+γ1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        -(m+γ1-1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
```

```
end

function G1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1+γ2+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        m*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function H1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ1+γ2+1)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        m*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function I1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ2)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ1-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function J1odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ2)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ1-1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function G2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        -(m+1)*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        -m*(2m+2n+2α+γ1+γ2+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function H2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        -(m+1)*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
```

```
    else
        -m*(2n+2α)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function I2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ2)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ2-1)*(2m+2n+γ1+γ2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end

function J2odd(m::Integer,n::Integer,α::Float64,γ1::Float64,γ2::Float64)
    if isodd(m)
        (m+γ2)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    else
        (m+γ2-1)*(2n+2)/(2m+γ1+γ2)/(2m+4n+2α+γ1+γ2+2)
    end
end
```

The functions in Listing 5.18 compute the constants appearing at the beginning of Subsection 4.6.3.

Listing 5.19: Multiplication by x_i

```
"""
Compute the result of multiplying a DZFun by x1
"""
function mbx1(f::DZFun)
    OrigCoeff = f.coefficients
    α = f.κ.α
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    N = f.degree

    OutCoeff = zeros(polyDim(N+1))

    # Even part

    m = 0
```

```
    n = 0
    ixMN = pairing(m,n,true) # Index associated to (0,0,Even)
    if m+2n≤N-1
        ixMpN = pairing(m+1,n,true) # Index assoacited to (1,0,Even)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*I1even(m+1,n,α,γ1,γ2)
    else
        OutCoeff[ixMN] = 0
    end

    n = 0
    for m = 1:N+1-2n
        ixMN = pairing(m,n,true) # Index associated to (m,0,Even)
        ixMmN = pairing(m-1,n,true) # Index associated to (m-1,0,Even)
        if m+2n≤N-1
            ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,1,Even)
            ixMpN = pairing(m+1,n,true) # Index associated to (m+1,n,Even)
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H1even(m-1,n+1,α,γ1
                ,γ2) + OrigCoeff[ixMpN]*I1even(m+1,n,α,γ1,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1even(m-1,n,α,γ1,γ2)
        end
    end

    m = 0
    for n = 1:(N+1-m)÷2
        ixMN = pairing(m,n,true) # Index associated to (0,n,Even)
        ixMpNm = pairing(m+1,n-1,true) # Index associated to (1,n-1,Even)
        if m+2n≤N-1
            ixMpN = pairing(m+1,n,true) # Index associated to (1,n,Even)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*I1even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1even(m+1,n-1,α,γ1
                ,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMpNm]*J1even(m+1,n-1,α,γ1,γ2)
        end
    end

    for n = 1:(N+1)÷2
        for m = 1:N+1-2n
            ixMN = pairing(m,n,true) # Index associated to (m,n,Even)
            ixMmN = pairing(m-1,n,true) # Index associated to (m-1,n,Even)
            ixMpNm = pairing(m+1,n-1,true) # Index associated to (m+1,n-1,Even)
            if m+2n≤N-1
                ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,n+1,Even)
                ixMpN = pairing(m+1,n,true) # Index associated to (m+1,n,Even)
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H1even(m-1,n+1,
```

```
                α,γ1,γ2) + OrigCoeff[ixMpN]*I1even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1even(m+1,n-1,α,γ1,γ2)
            else
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1even(m+1,n-1,α,γ1,γ2)
            end
        end
    end

    # Odd part

    m = 1
    n = 0
    ixMN = pairing(m,n,false) # Index associated to (1,0,Odd)
    if m+2n≤N-1
        ixMpN = pairing(m+1,n,false) # Index assoacited to (2,0,Odd)
        OutCoeff[ixMN] = OrigCoeff[ixMpN]*I1odd(m+1,n,α,γ1,γ2)
    else
        OutCoeff[ixMN] = 0
    end

    n = 0
    for m = 2:N+1-2n
        ixMN = pairing(m,n,false) # Index associated to (m,0,Odd)
        ixMmN = pairing(m-1,n,false) # Index associated to (m-1,0,Odd)
        if m+2n≤N-1
            ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,1,Odd)
            ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H1odd(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I1odd(m+1,n,α,γ1,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1odd(m-1,n,α,γ1,γ2)
        end
    end

    m = 1
    for n = 1:(N+1-m)÷2
        ixMN = pairing(m,n,false) # Index associated to (1,n,Odd)
        ixMpNm = pairing(m+1,n-1,false) # Index associated to (2,n-1,Odd)
        if m+2n≤N-1
            ixMpN = pairing(m+1,n,false) # Index associated to (2,n,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*I1odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1odd(m+1,n-1,α,γ1,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMpNm]*J1odd(m+1,n-1,α,γ1,γ2)
```

```
        end
    end

    for n = 1:(N+1)÷2
        for m = 2:N+1-2n
            ixMN = pairing(m,n,false) # Index associated to (m,n,Odd)
            ixMmN = pairing(m-1,n,false) # Index associated to (m-1,n,Odd)
            ixMpNm = pairing(m+1,n-1,false) # Index associated to (m+1,n-1,Odd)
            if m+2n≤N-1
                ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,n+1,Odd)
                ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H1odd(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I1odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1odd(m+1,n-1,α,γ1,γ2)
            else
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G1odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J1odd(m+1,n-1,α,γ1,γ2)
            end
        end
    end
    DZFun([γ1,γ2,α],N+1,OutCoeff)
end

"""
Compute the result of multiplying a DZFun by x2
"""
function mbx2(f::DZFun)
    OrigCoeff = f.coefficients
    α = f.κ.α
    γ1 = f.κ.γ1
    γ2 = f.κ.γ2
    N = f.degree

    OutCoeff = zeros(polyDim(N+1))

    # Even part

    n = 0
    for m = 0:1
        ixMN = pairing(m,n,true) # Index associated to (m,0,Even)
        if m+2n≤N-1
            ixMpN = pairing(m+1,n,false) # Index associated to (m+1,0,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMpN]*I2odd(m+1,n,α,γ1,γ2)
        else
            OutCoeff[ixMN] = 0
```

```
        end
    end

    n = 0
    for m = 2:N+1-2n
        ixMN = pairing(m,n,true) # Index associated to (m,0,Even)
        ixMmN = pairing(m-1,n,false) # Index associated to (m-1,0,Odd)
        if m+2n≤N-1
            ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,1,Odd)
            ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H2odd(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I2odd(m+1,n,α,γ1,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2odd(m-1,n,α,γ1,γ2)
        end
    end

    for m = 0:1
        for n = 1:(N+1-m)÷2
            ixMN = pairing(m,n,true) # Index associated to (m,n,Even)
            ixMpNm = pairing(m+1,n-1,false) # Index associated to (m+1,n-1,Odd)
            if m+2n≤N-1
                ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
                OutCoeff[ixMN] = OrigCoeff[ixMpN]*I2odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J2odd(m+1,n-1,α,γ1,γ2)
            else
                OutCoeff[ixMN] = OrigCoeff[ixMpNm]*J2odd(m+1,n-1,α,γ1,γ2)
            end
        end
    end

    for n = 1:(N+1)÷2
        for m = 2:N+1-2n
            ixMN = pairing(m,n,true) # Index associated to (m,n,Even)
            ixMmN = pairing(m-1,n,false) # Index associated to (m-1,n,Odd)
            ixMpNm = pairing(m+1,n-1,false) # Index associated to (m+1,n-1,Odd)
            if m+2n≤N-1
                ixMmNp = pairing(m-1,n+1,false) # Index associated to (m-1,n+1,Odd)
                ixMpN = pairing(m+1,n,false) # Index associated to (m+1,n,Odd)
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H2odd(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I2odd(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J2odd(m+1,n-1,α,γ1,γ2)
            else
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2odd(m-1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J2odd(m+1,n-1,α,γ1,γ2)
```

```
            end
        end
    end

    # Odd part

    for m = 1:N+1
        n = 0
        ixMN = pairing(m,n,false) # Index associated to (m,0,Odd)
        ixMmN = pairing(m-1,n,true) # Index associated to (m-1,0,Even)
        if m+2n≤N-1
            ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,1,Even)
            ixMpN = pairing(m+1,n,true) # Index associated to (m+1,0,Even)
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H2even(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I2even(m+1,n,α,γ1,γ2)
        else
            OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2even(m-1,n,α,γ1,γ2)
        end
        for n = 1:(N+1-m)÷2
            ixMN = pairing(m,n,false) # Index associated to (m,n,Odd)
            ixMmN = pairing(m-1,n,true) # Index associated to (m-1,n,Even)
            ixMpNm = pairing(m+1,n-1,true) # Index associated to (m+1,n-1,Even)
            if m+2n≤N-1
                ixMmNp = pairing(m-1,n+1,true) # Index associated to (m-1,n+1,Even)
                ixMpN = pairing(m+1,n,true) # Index associated to (m+1,n,Even)
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMmNp]*H2even(m-1,n+1,α,γ1,γ2) + OrigCoeff[ixMpN]*I2even(m+1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J2even(m+1,n-1,α,γ1,γ2)
            else
                OutCoeff[ixMN] = OrigCoeff[ixMmN]*G2even(m-1,n,α,γ1,γ2) + OrigCoeff[ixMpNm]*J2even(m+1,n-1,α,γ1,γ2)
            end
        end
    end
    DZFun([γ1,γ2,α],N+1,OutCoeff)
end
```

The function `mbx1` (resp. `mbx2`) expresses the result of multiplying a polynomial, represented by a `DZFun`, by x_1 (resp. x_2) in a new `DZFun` with the same parameter. The implementation is based on Subsection 4.6.3.

Listing 5.20: Other useful operators

```
function symx1(f::DZFun)
    outcoefs = deepcopy(f.coefficients)
    for i = 1:polyDim(f.degree)
        (m,n,even) = inversepairing(i)
        meven = iseven(m)
        if xor(even,meven)
            outcoefs[i] = 0.0
        end
    end
    DZFun(f.κ, f.degree, outcoefs)
end

function skewx1(f::DZFun)
    outcoefs = deepcopy(f.coefficients)
    for i = 1:polyDim(f.degree)
        (m,n,even) = inversepairing(i)
        meven = iseven(m)
        if ~xor(even,meven)
            outcoefs[i] = 0.0
        end
    end
    DZFun(f.κ, f.degree, outcoefs)
end

function symx2(f::DZFun)
    outcoefs = deepcopy(f.coefficients)
    for i = 1:polyDim(f.degree)
        (m,n,even) = inversepairing(i)
        if ~even
            outcoefs[i] = 0.0
        end
    end
    DZFun(f.κ, f.degree, outcoefs)
end

function skewx2(f::DZFun)
    outcoefs = deepcopy(f.coefficients)
    for i = 1:polyDim(f.degree)
        (m,n,even) = inversepairing(i)
        if even
            outcoefs[i] = 0.0
        end
    end
```

```
    DZFun(f.κ, f.degree, outcoefs)
end

"""
Compute the result of applying the angular Dunkl operator D_{12} to a DZFun without shifting parameters.
"""
DunklAngular(f::DZFun) = mbx1(Dunklx2(f)) - mbx2(Dunklx1(f))

"""
Compute the result of multiplying a DZFun by (1-x1^2-x2^2)
"""
mbr(f::DZFun) = f-mbx1(mbx1(f))-mbx2(mbx2(f))

"""
Compute the (α,γ)-adjoint of the Dunkl operator applied in a DZFun
"""
adjointDunklx1(f::DZFun,α::Float64) = -mbr(Dunklx1(f)) + 2*(α+1)*mbx1(f)
adjointDunklx2(f::DZFun,α::Float64) = -mbr(Dunklx2(f)) + 2*(α+1)*mbx2(f)
end # module
```

The functions `symx1`, `symx2`, `skewx1` and `skewx2` express the result of applying Sym_1, Sym_2, Skew_1 and Skew_2, respectively, on a polynomial represented by a `DZFun` in a new `DZFun` with the same parameter. The function `mbr` computes the action of multiplying by $1-\|x\|^2$ via `mbx1` and `mbx2`. `adjointDunklx1` and `adjointDunklx2` computes the action of applying $\mathcal{D}_1^{(\alpha,\gamma;\star)}$ and $\mathcal{D}_2^{(\alpha,\gamma;\star)}$, respectively.

CHAPTER 6

Conclusions and future work

6.1 Conclusions

In Chapter 2 we proved our mismatched approximation result Theorem 2.1.1 and its sharpness for special values of the regularity parameters of the function being approximated and the norm used to measure the error. On the way, we developed a suite of auxiliary results connecting Dunkl operators and $L^2_{\alpha,\gamma}$-orthogonal polynomials.

In Chapter 3 we characterized, for restricted parameters, $H^1_{\alpha,\gamma}$-orthogonal polynomials with respect to $\langle\cdot,\cdot\rangle_{\alpha,\gamma;1,P}$ in terms of h-harmonic polynomials and $L^2_{\alpha+1,\gamma}$-orthogonal polynomials. Moreover, we showed that these orthogonal polynomials satisfy the same second order Sturm–Liouville problem satisfied by $L^2_{\alpha-1,\gamma}$-orthogonal polynomials, even when $\alpha < 0$. Along the way, we developed auxiliary results connecting Dunkl operators and $H^1_{\alpha,\gamma}$-orthogonal polynomials and proved that Dunkl operators map $H^1_{\alpha,\gamma}$-orthogonal polynomials to $L^2_{\alpha,\gamma}$-orthogonal polynomials.

In Chapter 4 we obtained connecting relations between specific bases of bivariate $L^2_{\alpha,\gamma}$-orthogonal polynomials; more precisely, we obtained explicit incarnations of the three-term-recurrence, part (i) of Proposition 2.3.1 and part (iii) of Proposition 2.3.3. Moreover, we generalized this relations to arbitrary expansions of polynomials in terms of these bases.

In Chapter 5 we described DunklZernikeExpansions [3], a recently developed Julia 1.2.0 package by Gonzalo A. Benavides and Leonardo E. Figueroa which implements the connecting relations mentioned in the previous paragraph and allows for easy and fast numerical computation with polynomials expressed in terms of Dunkl–Zernike polynomials.

6.2 Future work

Starting from this work, some avenues of further work that we detect are:

(i) Adapt our arguments used in Chapter 2 and Chapter 3 to weights invariant with respect to other reflection groups.

(ii) Explore how Theorem 2.1.1 fares under polynomial-preserving mappings onto other domains, simplices foremost.

(iii) Find analogues of Dunkl operators that raise or lower components of γ instead of α.

(iv) Study orthogonal polynomials spaces with respect to equivalent inner products for $\mathrm{H}^m_{\alpha,\gamma}$ with $m \geq 2$, and decompose them in terms of lower order Dunkl–Sobolev orthogonal polynomials.

(v) Take advantage of the characterization of Dunkl–Sobolev orthogonal polynomial spaces as eigenspaces of the Sturm–Liouville problems Theorem 3.5.2 and Theorem 3.5.5 to obtain a suitable analogue of Theorem 2.1.1 by considering the orthogonal polynomial projector associated to $\langle \cdot, \cdot \rangle_{\alpha,\gamma;1,\mathrm{P}}$.

(vi) Get rid of the restriction $\alpha > -1/2$ or $\gamma_i \geq 0$ for all $i \in \{1, \ldots, d\}$ in Proposition 3.4.5 and its consequences.

(vii) Construct quadrature rules and efficient interpolation procedures from the bases studied in Chapter 4 and implement them in the Julia package DunklZernikeExpansions described in Chapter 5 in order to enable eventual numerical resolution of bivariate differential-difference equations.

(viii) Confirm or falsify the conjecture that there is no nice Wünsche-like basis of Dunkl–Zernike polynomials (cf. Section 4.1).

Bibliography

[1] George E. Andrews, Richard Askey, and Ranjan Roy, *Special functions*, Encyclopedia of Mathematics and its Applications, vol. 71, Cambridge University Press, Cambridge, 1999. MR 1688958 (2000g:33001)

[2] Y. Ben Cheikh and M. Gaied, *Characterization of the Dunkl-classical symmetric orthogonal polynomials*, Appl. Math. Comput. **187** (2007), no. 1, 105–114. MR 2323560

[3] Gonzalo A. Benavides and Leonardo E. Figueroa, *DunklZernikeExpansions Julia package*, https://github.com/lfiguero/DunklZernikeExpansions.

[4] Gonzalo A. Benavides and Leonardo E. Figueroa, *Orthogonal polynomial projection error in Dunkl-Sobolev norms in the ball*, arXiv e-prints (2020), arXiv:2002.01638.

[5] Christine Bernardi and Yvon Maday, *Spectral methods*, Handbook of numerical analysis, Vol. V, Handb. Numer. Anal., V, North-Holland, Amsterdam, 1997, pp. 209–485. MR 1470226

[6] V. I. Bogachev, *Measure theory. Volumes I and II*, Springer-Verlag, Berlin, Heidelberg, 2007. MR 2267655 (2008g:28002)

[7] Haim Brezis, *Functional analysis, Sobolev spaces and partial differential equations*, Universitext, Springer, New York, 2011. MR 2759829

[8] C. Canuto, M. Y. Hussaini, A. Quarteroni, and T. A. Zang, *Spectral methods*, Scientific Computation, Springer-Verlag, Berlin, 2006, Fundamentals in single domains. MR 2223552 (2007c:65001)

[9] C. Canuto and A. Quarteroni, *Approximation results for orthogonal polynomials in Sobolev spaces*, Math. Comp. **38** (1982), no. 157, 67–86. MR 637287 (82m:41003)

[10] Chebfun, *Chebfun Julia package*, https://github.com/chebfun/chebfun.

[11] T. S. Chihara, *An introduction to orthogonal polynomials*, Gordon and Breach Science Publishers, New York-London-Paris, 1978, Mathematics and its Applications, Vol. 13. MR 0481884

[12] T. S. Chihara, *45 years of orthogonal polynomials: a view from the wings*, Proceedings of the Fifth International Symposium on Orthogonal Polynomials, Special Functions and their Applications (Patras, 1999), vol. 133, 2001, pp. 13–21. MR 1858267

[13] Feng Dai and Yuan Xu, *Approximation theory and harmonic analysis on spheres and balls*, Springer Monographs in Mathematics, Springer, New York, 2013. MR 3060033

[14] Feng Dai and Yuan Xu, *Analysis on h-harmonics and Dunkl transforms*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser/Springer, Basel, 2015, Edited by Sergey Tikhonov. MR 3309987

[15] Charles F. Dunkl, *Differential-difference operators associated to reflection groups*, Trans. Amer. Math. Soc. **311** (1989), no. 1, 167–183. MR 951883

[16] Charles F. Dunkl, *Hankel transforms associated to finite reflection groups*, Hypergeometric functions on domains of positivity, Jack polynomials, and applications (Tampa, FL, 1991), Contemp. Math., vol. 138, Amer. Math. Soc., Providence, RI, 1992, pp. 123–138. MR 1199124

[17] Charles F. Dunkl and Yuan Xu, *Orthogonal polynomials of several variables*, second ed., Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 2014. MR 3289583

[18] Ryszard Engelking, *General topology*, second ed., Sigma Series in Pure Mathematics, vol. 6, Heldermann Verlag, Berlin, 1989, Translated from the Polish by the author. MR 1039321 (91c:54001)

[19] Arthur Erdélyi, Wilhelm Magnus, Fritz Oberhettinger, and Francesco G. Tricomi, *Higher transcendental functions.*, McGraw-Hill Book Company, Inc., New York-Toronto-London, 1953, Based, in part, on notes left by Harry Bateman. MR 0058756

[20] J.-Cl. Evard and F. Jafari, *Direct computation of the simultaneous Stone-Weierstrass approximation of a function and its partial derivatives in Banach spaces, and combination with Hermite interpolation*, J. Approx. Theory **78** (1994), no. 3, 351–363. MR 1292966

[21] Leonardo E. Figueroa, *ZernikeSuite Julia package*, https://github.com/lfiguero/ZernikeSuite.

[22] Leonardo E. Figueroa, *Orthogonal polynomial projection error measured in Sobolev norms in the unit ball*, J. Approx. Theory **220** (2017), 31–43. MR 3659787

[23] Leonardo E. Figueroa, *Orthogonal polynomial projection error measured in Sobolev norms in the unit disk*, Constr. Approx. **46** (2017), no. 1, 171–197. MR 3668633

[24] F. Finkel, D. Gómez-Ullate, A. González-López, M. A. Rodriguez, and R. Zhdanov, *Dunkl operators and Calogero-Sutherland models*, New trends in integrability and partial solvability, NATO Sci. Ser. II Math. Phys. Chem., vol. 132, Kluwer Acad. Publ., Dordrecht, 2004, pp. 157–189. MR 2153337

[25] V. L. Goncharov, *The theory of best approximation of functions*, J. Approx. Theory **106** (2000), no. 1, 2–57, Translated from the 1945 Russian original by Olga Holtz and Vladimir Yegorov. MR 1778073

[26] Ben-Yu Guo, *Gegenbauer approximation in certain Hilbert spaces and its applications to singular differential equations*, SIAM J. Numer. Anal. **37** (2000), no. 2, 621–645. MR 1740765 (2000k:65216)

[27] Mourad E. H. Ismail and Ruiming Zhang, *A review of multivariate orthogonal polynomials*, J. Egyptian Math. Soc. **25** (2017), no. 2, 91–110. MR 3629382

[28] Dunham Jackson, *Formal properties of orthogonal polynomials in two variables*, Duke Math. J. **2** (1936), no. 3, 423–434. MR 1545933

[29] JuliaApproximation, *ApproxFun Julia package*, https://github.com/JuliaApproximation/ApproxFun.jl.

[30] Tom Koornwinder, *Two-variable analogues of the classical orthogonal polynomials*, Theory and application of special functions (Proc. Advanced Sem., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1975), 1975, pp. 435–495. Math. Res. Center, Univ. Wisconsin, Publ. No. 35. MR 0402146

[31] H. L. Krall and I. M. Sheffer, *Orthogonal polynomials in two variables*, Ann. Mat. Pura Appl. (4) **76** (1967), 325–376. MR 228920

[32] K. Kuratowski, *Topology. Vol. I*, New edition, revised and augmented. Translated from the French by J. Jaworowski, Academic Press, New York-London; Państwowe Wydawnictwo Naukowe, Warsaw, 1966. MR 0217751

[33] P.-S. Laplace, *Théorie analytique des probabilités*, Collected in Œuvres complètes VII, 1812.

[34] A.-M Legendre, *Recherches sur l'attraction des sphéroïdes homogènes*, Mémoires de Mathématiques et de Physique, présentés à l'Académie Royale des Sciences, par divers savans, et lus dans ses Assemblées, 1785.

[35] Francisco Marcellán and Yuan Xu, *On Sobolev orthogonal polynomials*, Expo. Math. **33** (2015), no. 3, 308–352. MR 3360352

[36] Norman G. Meyers and James Serrin, $H = W$, Proc. Nat. Acad. Sci. U.S.A. **51** (1964), 1055–1056. MR 0164252 (29 #1551)

[37] S. Olver and A. Townsend, *A practical framework for infinite-dimensional linear algebra*, 2014 First Workshop for High Performance Technical Computing in Dynamic Languages, Nov 2014, pp. 57–62.

[38] Eric M. Opdam, *Harmonic analysis for certain representations of graded Hecke algebras*, Acta Math. **175** (1995), no. 1, 75–121. MR 1353018

[39] I. G. Petrovski, *Ordinary differential equations*, Prentice-Hall, Inc., Englewood Cliffs, N.J., 1966, Revised English edition. Translated from the Russian and edited by Richard A. Silverman. MR 0193298

[40] R. B. Platte and L. N. Trefethen, *Chebfun: a new kind of numerical computing*, Progress in industrial mathematics at ECMI 2008, Math. Ind., vol. 15, Springer, Heidelberg, 2010, pp. 69–87. MR 2767023

[41] Margit Rösler, *Generalized Hermite polynomials and the heat equation for Dunkl operators*, Comm. Math. Phys. **192** (1998), no. 3, 519–542. MR 1620515

[42] Gábor Szegő, *Orthogonal polynomials*, fourth ed., American Mathematical Society, Providence, R.I., 1975, American Mathematical Society, Colloquium Publications, Vol. XXIII. MR 0372517 (51 #8724)

[43] P.L. Tchebychev, *Théorie des mécanismes connus sous le nom de parallélogrammes*, Imprimerie de l'Académie impériale des sciences, 1853.

[44] Lloyd N. Trefethen, *Spectral methods in MATLAB*, Software, Environments, and Tools, vol. 10, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2000. MR 1776072

[45] Khalifa Trimèche, *The Dunkl intertwining operator on spaces of functions and distributions and integral representation of its dual*, Integral Transform. Spec. Funct. **12** (2001), no. 4, 349–374. MR 1872375

[46] Hassler Whitney, *Functions differentiable on the boundaries of regions*, Ann. of Math. (2) **35** (1934), no. 3, 482–485. MR 1503174

[47] Alfred Wünsche, *Generalized Zernike or disc polynomials*, J. Comput. Appl. Math. **174** (2005), no. 1, 135–163. MR 2102653 (2005g:42065)

[48] Yuan Xu, *Approximation by polynomials in Sobolev spaces with Jacobi weight*, J. Fourier Anal. Appl. **24** (2018), no. 6, 1438–1459. MR 3881837