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**A FULLY-MIXED FINITE ELEMENT
METHOD FOR THE COUPLING
OF THE NAVIER-STOKES AND
DARCY-FORCHHEIMER EQUATIONS**

POR

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A FULLY-MIXED FINITE ELEMENT METHOD FOR THE COUPLING OF THE NAVIER–STOKES AND DARCY–FORCHHEIMER EQUATIONS

UN MÉTODO DE ELEMENTOS FINITOS COMPLETAMENTE MIXTO
PARA EL ACOPLAMIENTO DE LAS ECUACIONES DE NAVIER–STOKES
Y DARCY–FORCHHEIMER

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Abstract

In this work we present and analyse a fully-mixed formulation for the nonlinear model given by the coupling of the Navier–Stokes and Darcy–Forchheimer equations with the Beavers–Joseph–Saffman condition on the interface. Our approach yields non-Hilbertian normed spaces and a twofold saddle point structure for the corresponding operator equation. Furthermore, since the convective term in the Navier–Stokes equation forces the velocity to live in a smaller space than usual, we augment the variational formulation with suitable Galerkin type terms. The resulting augmented scheme is then written equivalently as a fixed point equation, so that the well-known Schauder and Banach theorems, combined with classical results on nonlinear monotone operators, are applied to prove the unique solvability of the continuous and discrete systems. In particular, given an integer $k \geq 0$, Raviart–Thomas spaces of order k , continuous piecewise polynomials of degree $\leq k + 1$ and piecewise polynomials of degree $\leq k$ are employed in the fluid for approximating the pseudostress tensor, velocity and vorticity, respectively, whereas Raviart–Thomas spaces of order k and piecewise polynomials of degree $\leq k$ for the velocity and pressure, constitute a feasible choice in the porous medium. *A priori* error estimates and associated rates of convergence are derived, and several numerical examples illustrating the good performance of the method are reported.

Resumen

En este trabajo presentamos y analizamos una formulación completamente mixta para el modelo no lineal que se obtiene al acoplar las ecuaciones de Navier–Stokes y Darcy–Forchheimer con una condición de Beavers–Joseph–Saffman en la interfaz. Nuestro enfoque genera espacios normados no-Hilbert y una estructura de doble punto de silla para la ecuación del operador correspondiente. Además, dado que el término convectivo en la ecuación de Navier–Stokes obliga a la velocidad a vivir en un espacio más pequeño de lo habitual, aumentamos la formulación variacional con adecuados términos tipo Galerkin. El esquema resultante se escribe equivalentemente como una ecuación de punto fijo, de modo que los conocidos teoremas de Schauder y Banach, combinados con resultados clásicos de operadores monótonos no lineales, se utilizan para demostrar la unicidad de los problemas continuo y discreto. En particular, dado un entero $k \geq 0$, espacios de Raviart–Thomas de orden k , polinomios continuos a trozos de grado $\leq k + 1$ y polinomios a trozos de grado $\leq k$ se emplean en el fluido para aproximar el tensor de pseudoefuerzo, la velocidad y la vorticidad, respectivamente, mientras que espacios de Raviart–Thomas de orden k y polinomios a trozos de grado $\leq k$ para la velocidad y la presión, constituyen una opción viable en el medio poroso. Se derivan las estimaciones de error a priori y las razones de convergencia asociadas, y se proporcionan varios ejemplos numéricos que ilustran el buen desempeño del método.

CHAPTER 1

Introduction

The derivation of suitable mathematical and numerical models for the fluid flow between porous media and free-flow zones has been widely studied during the last decades, mostly due to its relevance in the fields of natural sciences, biology, and engineering branches. In particular, physical phenomena such as vuggy porous media appearing in petroleum extraction, groundwater system in karst aquifers, industrial filtrations, and blood motion in tumors and microvessels can be modelled by the Navier–Stokes/Darcy (or Stokes/Darcy) model (see, e.g., [5, 18, 34, 36]), which consists in a set of partial differential equations where the Navier–Stokes (or Stokes) problem is coupled with the Darcy model through a set of coupling equations acting on a common interface, which are given by mass conservation, balance of normal forces, and the so called Beavers–Joseph–Saffman condition. However, in applications such as the internal ventilation of a motorcycle helmet and reservoir wellbore (see, e.g., [9, 13, 3]), when the fluid velocity is higher and the porosity is nonuniform, which holds when the kinematic forces dominates over viscous forces, a better way to study this phenomenon is modifying the Darcy model in the porous medium by adding the Forch-

heimer term, which represents inertial effects, thus obtaining the Darcy–Forchheimer model (see [37, 35]).

In this context, and up to the authors’ knowledge, one of the first works in analysing the Navier–Stokes/Darcy–Forchheimer coupled problem is [3]. In there, the authors study the coupling of a 2D reservoir model with a 1.5D vertical wellbore model, both written in axisymmetric form. The physical problems are described by the Darcy–Forchheimer and the compressible Navier–Stokes equations, respectively, together with an exhaustive energy equation. Later on, motivated by the study of the internal ventilation of a motorcycle helmet, a penalization approach, for both 2D and 3D domains, was introduced and analysed in [13]. In particular, the authors consider the velocity and pressure in the whole domain as the main unknowns of the system, and the corresponding Galerkin approximation employs piecewise quadratic and linear elements for the velocity and pressure, respectively. More recently, in [10] a primal-mixed formulation of the Navier–Stokes/Darcy–Forchheimer system is analyzed by means of a fixed-point argument and classical results on nonlinear monotone operators (see [38, 39]). The corresponding mixed finite element scheme employs Bernardi–Raugel elements for the velocity in the free fluid region, Raviart–Thomas elements of lowest order for the filtration velocity in the porous media, and piecewise constant elements for the pressures and the Lagrange multiplier. Meanwhile, a fully-mixed finite element method is developed and analyzed for the coupling of the Stokes and Darcy–Forchheimer problems in [2]. This new approach yields non-Hilbertian normed spaces and a twofold saddle point structure for the corresponding operator equation, whose continuous and discrete solvabilities are analyzed by means of a suitable abstract theory developed for this purpose.

According to the above bibliographic discussion, the purpose of the present thesis is to extend the results obtained in [10] and [2] to the coupling of the Navier–Stokes and Darcy–Forchheimer problems with constant density and viscosity, but unlike [10], by considering now dual-mixed formulations in both domains. We introduce the pseudostress tensor as in [7] and subsequently eliminate the pressure unknown using the incompressibility condition. In addition, and in order to impose the symmetry of the pseudostress tensor, similarly to [22, 2], the vorticity is introduced as an additional unknown. The transmission conditions

consisting of mass conservation, balance of normal forces, and the Beavers–Joseph–Saffman law are imposed weakly, which yields the incorporation of additional Lagrange multipliers: the traces of the porous media pressure and the fluid velocity on the interface. Furthermore, the difficulty that the fluid velocity lives in H^1 instead of L^2 as usual, is resolved as in [7] by augmenting the variational formulation with residuals arising from the constitutive and equilibrium equations for the fluid flow, and the formula for the vorticity tensor. The resulting augmented variational system of equations is then ordered so that it shows a twofold saddle point structure. The well-posedness and uniqueness of both the continuous and discrete formulation is proved employing a fixed point argument and an abstract theory for twofold saddle point problems (see [20, 22, 2]). In particular, for the continuous formulation, and under a smallness data assumption, we prove existence and uniqueness of solution by means of a fixed-point strategy where the Schauder (for existence) and Banach (for uniqueness) fixed-point theorems are employed. In addition, an *a priori* error analysis is performed, and while it is possible to prove that the finite element method is convergent with a sub-optimal rate, the numerical results suggest that the method is optimally convergent provided the exact solutions are smooth enough. In particular, given an integer $k \geq 0$, we find that the interior Navier–Stokes variables: pseudostress tensor, velocity and vorticity, can be approximated using Raviart–Thomas spaces of order k , continuous piecewise polynomials of degree $\leq k + 1$ and piecewise polynomials of degree $\leq k$, respectively, while the interior Darcy–Forchheimer variables: velocity and pressure, can be approximated using Raviart–Thomas spaces of order k and piecewise polynomials of degree $\leq k$.

1.1 Outline

The rest of this thesis is organized as follows. The remainder of this chapter describes standard notations and functional spaces to be employed along the thesis. In Chapter 2 we introduce the modelling equations for the free-flow zone, the porous medium and the interface, to then in Chapter 3, derive an augmented fully-mixed variational formulation that will be written as a nonlinear operator equation with a twofold saddle point structure. In

Chapter 4 we develop an abstract theory for this type of problem, which includes the proper hypotheses on the spaces and involved operators to be imposed in order to guarantee the well-posedness of the continuous problem in rather general Banach spaces. Then, in Chapter 5 we use a fixed-point strategy to establish that our variational formulation is well posed. Next, in Chapter 6 we define the Galerkin scheme and derive general hypotheses on the discrete subspaces ensuring that, on the one hand, the discrete scheme becomes well posed, and on the other hand, it satisfies a Céa's estimate. A specific choice of finite element subspaces satisfying these assumptions as well as a sub-optimal rate of convergence are introduced in Chapter 7. Then, several numerical examples illustrating the performance of the method, confirming the theoretical sub-optimal order of convergence, but at the same time suggesting an optimal rate of convergence, are reported in Chapter 8. Finally, in Chapter 9 we name some conclusions and proposals for future work.

1.2 Preliminaries

Let $\mathcal{O} \subseteq \mathbb{R}^n$, $n \in \{2, 3\}$, denote a domain with Lipschitz boundary Γ . For $s \geq 0$ and $p \in [1, +\infty]$ we denote by $L^p(\mathcal{O})$ and $W^{s,p}(\mathcal{O})$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{L^p(\mathcal{O})}$ and $\|\cdot\|_{s,p;\mathcal{O}}$, respectively. Note that $W^{0,p}(\mathcal{O}) = L^p(\mathcal{O})$. If $p = 2$, we write $H^s(\mathcal{O})$ in place of $W^{s,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{s,\mathcal{O}}$. In addition, we denote by $W^{1/q,p}(\Gamma)$ the trace space of $W^{1,p}(\mathcal{O})$, and let $W^{-1/q,q}(\Gamma)$ be the dual space of $W^{1/q,p}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/q,p;\Gamma}$ and $\|\cdot\|_{-1/q,q;\Gamma}$, respectively, with $p, q \in (1, +\infty)$ satisfying $1/p + 1/q = 1$. By \mathbf{M} and \mathbb{M} we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M , and $\|\cdot\|$, with no subscripts, will stand for the natural norm of either an element or an operator in any product functional space. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,n}, \quad \operatorname{div} \mathbf{v} := \sum_{j=1}^n \frac{\partial v_j}{\partial x_j} \quad \text{and} \quad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,n}.$$

Furthermore, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\operatorname{div} \boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^t := (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^n \tau_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^n \tau_{ij} \zeta_{ij} \quad \text{and} \quad \boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau}) \mathbb{I},$$

where \mathbb{I} is the identity matrix in $\mathbb{R}^{n \times n}$. In what follows, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in \mathbb{R}^n or $\mathbb{R}^{n \times n}$. Additionally, we recall that

$$\mathbb{H}(\operatorname{div}; \mathcal{O}) := \left\{ \boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O}) : \operatorname{div} \boldsymbol{\tau} \in \mathbf{L}^2(\mathcal{O}) \right\},$$

equipped with the usual norm

$$\|\boldsymbol{\tau}\|_{\operatorname{div}; \mathcal{O}}^2 := \|\boldsymbol{\tau}\|_{0, \mathcal{O}}^2 + \|\operatorname{div} \boldsymbol{\tau}\|_{0, \mathcal{O}}^2,$$

is a standard Hilbert space in the realm of mixed problems. On the other hand, the following symbol for the $L^2(\Gamma)$ inner product

$$\langle \xi, \lambda \rangle_{\Gamma} := \int_{\Gamma} \xi \lambda \quad \forall \xi, \lambda \in L^2(\Gamma),$$

will also be employed for their respective extension as the duality parity between $W^{-1/q, q}(\Gamma)$ and $W^{1/q, p}(\Gamma)$. Furthermore, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^n$, $P_k(S)$ denotes the space of polynomial functions on S of degree $\leq k$. In addition, and coherently with previous notations, we set $\mathbf{P}_k(S) := [P_k(S)]^n$ and $\mathbb{P}_k(S) := [P_k(S)]^{n \times n}$. Finally, we end this chapter by mentioning that, throughout the rest of the thesis, we employ $\mathbf{0}$ to denote a generic null vector (or tensor), and use C and c , with or without subscripts, bars, tildes or hats, to denote generic constants independent of the discretization parameters, which may take different values at different places.

The model problem

In order to describe the geometry under consideration we let Ω_S and Ω_D be bounded and simply connected open polyhedral domains in \mathbb{R}^n , $n = \{2, 3\}$, such that $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \emptyset$ and $\Omega_S \cap \Omega_D = \emptyset$. Then, we let $\Gamma_S := \partial\Omega_S \setminus \bar{\Sigma}$, $\Gamma_D := \partial\Omega_D \setminus \bar{\Sigma}$, and denote by \mathbf{n} the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_S \cup \Sigma \cup \Omega_D$ and Ω_S (and hence inward to Ω_D when seen on Σ). On Σ we also consider a set of unit tangent vectors, which is given by $\mathbf{t} = \mathbf{t}_1$ when $n = 2$ (see Figure 2.1 below), and by $\{\mathbf{t}_1, \mathbf{t}_2\}$ when $n = 3$. The problem we are interested in consists of the movement of an incompressible viscous fluid occupying Ω_S which flows towards and from a porous medium Ω_D through Σ , where Ω_D is saturated with the same fluid. The mathematical model is defined by two separate groups of equations and by a set of coupling terms.

2.1 Navier–Stokes equations

In the free fluid domain Ω_S , the governing equations are those of the Navier–Stokes problem with constant density and viscosity, which are written in the following nonstandard pseudostress-velocity-pressure formulation:

$$\begin{aligned} \boldsymbol{\sigma}_S &= -p_S \mathbb{I} + 2\mu \mathbf{e}(\mathbf{u}_S) - \rho(\mathbf{u}_S \otimes \mathbf{u}_S) \quad \text{in } \Omega_S, \quad \operatorname{div} \mathbf{u}_S = 0 \quad \text{in } \Omega_S, \\ -\operatorname{div} \boldsymbol{\sigma}_S &= \mathbf{f}_S \quad \text{in } \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S, \end{aligned} \quad (2.1)$$

where $\boldsymbol{\sigma}_S$ is the nonlinear pseudostress tensor, \mathbf{u}_S is the fluid velocity and p_S is the pressure. In addition, $\mathbf{e}(\mathbf{u}_S) := \frac{1}{2} \{ \nabla \mathbf{u}_S + (\nabla \mathbf{u}_S)^t \}$ stands for the strain tensor of small deformations, μ is the viscosity of the fluid, ρ is the density, and $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$ is a given external force.

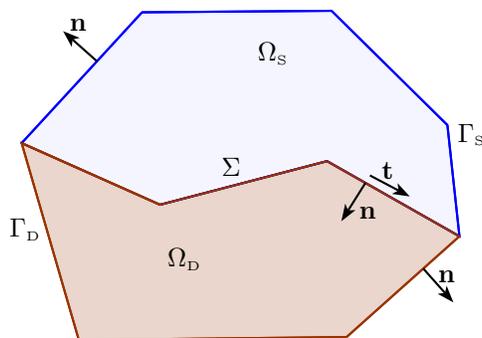


Figure 2.1: Sketch of a 2D geometry of our Navier–Stokes/Darcy–Forchheimer model

Now, in order to derive our fully-mixed formulation, we first observe, owing to the fact that $\operatorname{tr} \mathbf{e}(\mathbf{u}_S) = \operatorname{div} \mathbf{u}_S$, that the first two equations in (2.1) are equivalent to

$$\boldsymbol{\sigma}_S = -p_S \mathbb{I} + 2\mu \mathbf{e}(\mathbf{u}_S) - \rho(\mathbf{u}_S \otimes \mathbf{u}_S), \quad p_S = -\frac{1}{n} \operatorname{tr}(\boldsymbol{\sigma}_S + \rho(\mathbf{u}_S \otimes \mathbf{u}_S)) \quad \text{in } \Omega_S, \quad (2.2)$$

and hence, eliminating the pressure p_S (which anyway can be approximated later on by the post-processed formula suggested by the second equation of (2.2)), the Navier–Stokes problem (2.1) can be rewritten as

$$\boldsymbol{\sigma}_S^d = 2\mu \mathbf{e}(\mathbf{u}_S) - \rho(\mathbf{u}_S \otimes \mathbf{u}_S)^d \quad \text{in } \Omega_S, \quad -\operatorname{div} \boldsymbol{\sigma}_S = \mathbf{f}_S \quad \text{in } \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S. \quad (2.3)$$

Next, in order to impose weakly the symmetry of the pseudostress tensor and employ the integration by parts formula, we introduce the additional unknown

$$\gamma_S := \frac{1}{2} \left\{ \nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^t \right\} \quad \text{in } \Omega_S, \quad (2.4)$$

which represents the vorticity. In this way, instead of (2.3), in the sequel we consider the set of equations with unknowns $\boldsymbol{\sigma}_S$, γ_S and \mathbf{u}_S , given by

$$\frac{1}{2\mu} \boldsymbol{\sigma}_S^d = \nabla \mathbf{u}_S - \gamma_S - \frac{\rho}{2\mu} (\mathbf{u}_S \otimes \mathbf{u}_S)^d \quad \text{in } \Omega_S, \quad -\operatorname{div} \boldsymbol{\sigma}_S = \mathbf{f}_S \quad \text{in } \Omega_S, \quad \mathbf{u}_S = \mathbf{0} \quad \text{on } \Gamma_S, \quad (2.5)$$

where $\boldsymbol{\sigma}_S$ is a symmetric tensor in Ω_S .

2.2 Darcy–Forchheimer equations

In the porous medium Ω_D we consider a nonlinear version of the Darcy problem to approximate the velocity \mathbf{u}_D and the pressure p_D , which is considered when the fluid velocity is higher and the porosity is nonuniform. More precisely, we consider the Darcy–Forchheimer equations [37, 35]:

$$\frac{\mu}{\rho} \mathbf{K}^{-1} \mathbf{u}_D + \frac{F}{\rho} |\mathbf{u}_D| \mathbf{u}_D + \nabla p_D = \mathbf{f}_D \quad \text{in } \Omega_D, \quad \operatorname{div} \mathbf{u}_D = g_D \quad \text{in } \Omega_D, \quad \mathbf{u}_D \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_D, \quad (2.6)$$

where F represents the Forchheimer number of the porous medium, and $\mathbf{K} \in \mathbb{L}^\infty(\Omega_D)$ is a symmetric tensor in Ω_D representing the intrinsic permeability κ of the porous medium divided by the viscosity μ of the fluid. Throughout the thesis we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\mathbf{w} \cdot \mathbf{K}^{-1}(\mathbf{x}) \mathbf{w} \geq C_{\mathbf{K}} |\mathbf{w}|^2, \quad (2.7)$$

for almost all $\mathbf{x} \in \Omega_D$, and for all $\mathbf{w} \in \mathbb{R}^n$. In turn, as will be explained below, \mathbf{f}_D and g_D are given functions in $\mathbf{L}^{3/2}(\Omega_D)$ and $L^2(\Omega_D)$, respectively. In addition, according to the compressibility conditions, the boundary conditions on \mathbf{u}_D and \mathbf{u}_S , and the principle of mass

conservation (cf. (2.8) below), g_D must satisfy the compatibility condition:

$$\int_{\Omega_D} g_D = 0.$$

2.3 Transmission conditions

The transmission conditions that couple the Navier–Stokes and the Darcy–Forchheimer models through the interface Σ are given by

$$\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n} \quad \text{on } \Sigma \quad \text{and} \quad \boldsymbol{\sigma}_S \mathbf{n} + \sum_{i=1}^{n-1} \omega_i^{-1} (\mathbf{u}_S \cdot \mathbf{t}_i) \mathbf{t}_i = -p_D \mathbf{n} \quad \text{on } \Sigma, \quad (2.8)$$

where $\{\omega_1, \dots, \omega_{n-1}\}$ is a set of positive frictional constants that can be determined experimentally. The first equation in (2.8) corresponds to mass conservation on Σ , whereas the second one establishes the balance of the normal forces and a Beavers–Joseph–Saffman law.

The continuous formulation

In this chapter we proceed analogously to [10, Section 2] (see also [2, 11, 23, 22]) and derive a weak formulation of the coupled problem given by (2.5), (2.6), and (2.8).

3.1 Preliminaries

We first introduce further notations and definitions. In what follows, given $\star \in \{S, D\}$, we set

$$(p, q)_\star := \int_{\Omega_\star} p q, \quad (\mathbf{u}, \mathbf{v})_\star := \int_{\Omega_\star} \mathbf{u} \cdot \mathbf{v} \quad \text{and} \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_\star := \int_{\Omega_\star} \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

In addition, in the sequel we will employ the following Banach space,

$$\mathbf{H}^3(\text{div}; \Omega_D) := \left\{ \mathbf{v}_D \in \mathbf{L}^3(\Omega_D) : \text{div } \mathbf{v}_D \in L^2(\Omega_D) \right\},$$

endowed with the norm

$$\|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} := \left(\|\mathbf{v}_D\|_{\mathbf{L}^3(\Omega_D)}^3 + \|\text{div } \mathbf{v}_D\|_{0, \Omega_D}^3 \right)^{1/3}$$

and the following subspaces of $\mathbb{L}^2(\Omega_S)$, $\mathbf{H}^1(\Omega_S)$ and $\mathbf{H}^3(\text{div}; \Omega_D)$, respectively

$$\begin{aligned}\mathbb{L}_{\text{skew}}^2(\Omega_S) &:= \left\{ \boldsymbol{\eta}_S \in \mathbb{L}^2(\Omega_S) : \boldsymbol{\eta}_S^t = -\boldsymbol{\eta}_S \right\}, \\ \mathbf{H}_{\Gamma_S}^1(\Omega_S) &:= \left\{ \mathbf{v}_S \in \mathbf{H}^1(\Omega_S) : \mathbf{v}_S = \mathbf{0} \text{ on } \Gamma_S \right\}, \\ \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D) &:= \left\{ \mathbf{v}_D \in \mathbf{H}^3(\text{div}; \Omega_D) : \mathbf{v}_D \cdot \mathbf{n} = 0 \text{ on } \Gamma_D \right\}.\end{aligned}$$

Notice that $\mathbf{H}^3(\text{div}; \Omega_D) = \mathbf{H}(\text{div}; \Omega_D) \cap \mathbf{L}^3(\Omega_D)$, which guarantees that $\mathbf{v}_D \cdot \mathbf{n}$ is well defined for $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D)$ (see [10, Section 2.2] for details). In addition, analogously to [22] (see also [11]) we need to introduce the space of traces $\mathbf{H}_{00}^{1/2}(\Sigma) := \left[\mathbf{H}_{00}^{1/2}(\Sigma) \right]^n$, where

$$\mathbf{H}_{00}^{1/2}(\Sigma) := \left\{ v|_{\Sigma} : v \in \mathbf{H}_{\Gamma_S}^1(\Omega_S) \right\}.$$

Observe that, if $E_{0,S} : \mathbf{H}^{1/2}(\Sigma) \rightarrow L^2(\partial\Omega_S)$ is the extension operator defined by

$$E_{0,S}(\psi) := \begin{cases} \psi & \text{on } \Sigma \\ 0 & \text{on } \Gamma_S \end{cases} \quad \forall \psi \in \mathbf{H}^{1/2}(\Sigma),$$

we have that

$$\mathbf{H}_{00}^{1/2}(\Sigma) = \left\{ \psi \in \mathbf{H}^{1/2}(\Sigma) : E_{0,S}(\psi) \in \mathbf{H}^{1/2}(\partial\Omega_S) \right\},$$

which is endowed with the norm $\|\psi\|_{1/2,0,0;\Sigma} := \|E_{0,S}(\psi)\|_{1/2,\partial\Omega_S}$. The dual space of $\mathbf{H}_{00}^{1/2}(\Sigma)$ is denoted by $\mathbf{H}_{00}^{-1/2}(\Sigma)$.

3.2 The augmented fully-mixed variational formulation

We now proceed with the derivation of our augmented fully-mixed variational formulation for the Navier–Stokes/Darcy–Forchheimer coupled problem. To this end, we begin by introducing two additional unknowns on the coupling boundary

$$\boldsymbol{\varphi} := -\mathbf{u}_S|_{\Sigma} \in \mathbf{H}_{00}^{1/2}(\Sigma), \quad \lambda := p_D|_{\Sigma} \in W^{1/3,3/2}(\Sigma).$$

Then, similarly to [24, 22] and [10], we test the first equations of (2.5) and (2.6) with arbitrary $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}; \Omega_S)$ and $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\mathbf{div}; \Omega_D)$, respectively, integrate by parts, utilize the fact that $\boldsymbol{\sigma}_S^d : \boldsymbol{\tau}_S = \boldsymbol{\sigma}_S^d : \boldsymbol{\tau}_S^d$, and impose the remaining equations weakly, as well as the symmetry of $\boldsymbol{\sigma}_S$ and the transmission conditions (2.8) to obtain the variational problem: Find $\boldsymbol{\sigma}_S \in \mathbb{H}(\mathbf{div}; \Omega_S)$, $\boldsymbol{\gamma}_S \in \mathbb{L}_{\text{skew}}^2(\Omega_S)$, $\boldsymbol{\varphi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, $\mathbf{u}_D \in \mathbf{H}_{\Gamma_D}^3(\mathbf{div}; \Omega_D)$, $p_D \in L^2(\Omega_D)$, $\lambda \in W^{1/3, 3/2}(\Sigma)$ and \mathbf{u}_S in a suitable space (to be specified below), such that

$$\begin{aligned}
 \frac{1}{2\mu}(\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{u}_S, \mathbf{div} \boldsymbol{\tau}_S)_S + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma + (\boldsymbol{\gamma}_S, \boldsymbol{\tau}_S)_S + \frac{\rho}{2\mu}((\mathbf{u}_S \otimes \mathbf{u}_S)^d, \boldsymbol{\tau}_S)_S &= 0, \\
 \frac{\mu}{\rho}(\mathbf{K}^{-1} \mathbf{u}_D, \mathbf{v}_D)_D + \frac{\mathbf{F}}{\rho}(|\mathbf{u}_D| \mathbf{u}_D, \mathbf{v}_D)_D - (p_D, \mathbf{div} \mathbf{v}_D)_D - \langle \mathbf{v}_D \cdot \mathbf{n}, \lambda \rangle_\Sigma &= (\mathbf{f}_D, \mathbf{v}_D)_D, \\
 -(\mathbf{div} \boldsymbol{\sigma}_S, \mathbf{v}_S)_S &= (\mathbf{f}_S, \mathbf{v}_S)_S, \\
 (q_D, \mathbf{div} \mathbf{u}_D)_D &= (g_D, q_D)_D, \\
 (\boldsymbol{\sigma}_S, \boldsymbol{\eta}_S)_S &= 0, \\
 -\langle \boldsymbol{\varphi} \cdot \mathbf{n}, \xi \rangle_\Sigma - \langle \mathbf{u}_D \cdot \mathbf{n}, \xi \rangle_\Sigma &= 0, \\
 \langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{t, \Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma &= 0,
 \end{aligned} \tag{3.1}$$

for all $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}; \Omega_S)$, $\boldsymbol{\eta}_S \in \mathbb{L}_{\text{skew}}^2(\Omega_S)$, $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\mathbf{div}; \Omega_D)$, $q_D \in L^2(\Omega_D)$, $\xi \in W^{1/3, 3/2}(\Sigma)$ and $\mathbf{v}_S \in L^2(\Omega_S)$, where

$$\langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{t, \Sigma} := \sum_{i=1}^{n-1} \omega_i^{-1} \langle \boldsymbol{\varphi} \cdot \mathbf{t}_i, \boldsymbol{\psi} \cdot \mathbf{t}_i \rangle_\Sigma.$$

Notice here that the term $\langle \boldsymbol{\psi} \cdot \mathbf{n}, \xi \rangle_\Sigma$ is well-defined for all $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$ and for all $\xi \in W^{1/3, 3/2}(\Sigma)$ (see [2, Lemma 2.2] for details). Notice also that the fifth term in the first equation of (3.1) requires \mathbf{u}_S to live in a smaller space than $L^2(\Omega_S)$. In fact, by applying the Cauchy–Schwarz and Hölder inequalities and then the continuous injection \mathbf{i}_c of $\mathbf{H}^1(\Omega_S)$ into $L^4(\Omega_S)$ (see, e.g., [1, Theorem 6.3]), we find that there holds

$$|((\mathbf{u}_S \otimes \mathbf{w}_S)^d, \boldsymbol{\tau}_S)_S| \leq \|\mathbf{u}_S\|_{L^4(\Omega_S)} \|\mathbf{w}_S\|_{L^4(\Omega_S)} \|\boldsymbol{\tau}_S\|_{0, \Omega_S} \leq \|\mathbf{i}_c\|^2 \|\mathbf{u}_S\|_{1, \Omega_S} \|\mathbf{w}_S\|_{1, \Omega_S} \|\boldsymbol{\tau}_S\|_{\mathbf{div}; \Omega_S}, \tag{3.2}$$

for all $\mathbf{u}_S, \mathbf{w}_S \in \mathbf{H}^1(\Omega_S)$ and $\boldsymbol{\tau}_S \in \mathbb{H}(\mathbf{div}; \Omega_S)$. According to this, we propose to look for the unknown \mathbf{u}_S in $\mathbf{H}_{\Gamma_S}^1(\Omega_S)$ and to restrict the set of corresponding test functions \mathbf{v}_S to the same space.

Next, analogously to [22] (see also [11]), it is not difficult to see that the system (3.1) is not uniquely solvable since, given any solution $(\boldsymbol{\sigma}_S, \boldsymbol{\gamma}_S, \mathbf{u}_S, \boldsymbol{\varphi}, \mathbf{u}_D, p_D, \lambda)$ in the indicated spaces, and given any constant $c \in \mathbb{R}$, the vector defined by $(\boldsymbol{\sigma}_S - c\mathbb{I}, \boldsymbol{\gamma}_S, \mathbf{u}_S, \boldsymbol{\varphi}, \mathbf{u}_D, p_D + c, \lambda + c)$ also becomes a solution. As a consequence of the above, from now on we require the Darcy–Forchheimer pressure p_D to be in $L_0^2(\Omega_D)$, where

$$L_0^2(\Omega_D) := \left\{ q_D \in L^2(\Omega_D) : (q_D, 1)_D = 0 \right\}.$$

In turn, due to the decomposition $L^2(\Omega_D) = L_0^2(\Omega_D) \oplus \mathbb{R}$, the boundary conditions $\mathbf{u}_S = \mathbf{0}$ on Γ_S and $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D , the first transmission condition in (2.8), and the fact that $(q_D, 1)_D = 0$, guarantee that the fourth equation of (3.1) is equivalent to requiring it for all $q_D \in L_0^2(\Omega_D)$.

On the other hand, for convenience of the subsequent analysis, we consider the decomposition (see, for instance, [6, 19])

$$\mathbb{H}(\mathbf{div}; \Omega_S) = \mathbb{H}_0(\mathbf{div}; \Omega_S) \oplus \mathbb{R}\mathbb{I},$$

where

$$\mathbb{H}_0(\mathbf{div}; \Omega_S) := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega_S) : (\text{tr } \boldsymbol{\tau}, 1)_S = 0 \right\}$$

and redefine the pseudostress tensor as $\boldsymbol{\sigma}_S := \boldsymbol{\sigma}_S + \ell\mathbb{I}$, with the new unknowns $\boldsymbol{\sigma}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S)$ and $\ell \in \mathbb{R}$. In this way the first and the seventh equations of (3.1) are rewritten, equivalently, as

$$\begin{aligned} \frac{1}{2\mu} (\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + (\mathbf{u}_S, \mathbf{div} \boldsymbol{\tau}_S)_S + \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\varphi} \rangle_\Sigma + (\boldsymbol{\gamma}_S, \boldsymbol{\tau}_S)_S + \frac{\rho}{2\mu} ((\mathbf{u}_S \otimes \mathbf{u}_S)^d, \boldsymbol{\tau}_S)_S &= 0, \\ j \langle \boldsymbol{\varphi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= 0, \\ \langle \boldsymbol{\sigma}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \boldsymbol{\varphi}, \boldsymbol{\psi} \rangle_{t, \Sigma} + \langle \boldsymbol{\psi} \cdot \mathbf{n}, \lambda \rangle_\Sigma + \ell \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma &= 0, \end{aligned} \tag{3.3}$$

for all $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S)$, $j \in \mathbb{R}$ and $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma)$, respectively. Finally, we augment the resulting system through the incorporation of the following redundant Galerkin-type terms:

$$\begin{aligned} \kappa_1(\mathbf{div} \boldsymbol{\sigma}_S, \mathbf{div} \boldsymbol{\tau}_S)_S &= -\kappa_1(\mathbf{f}_S, \mathbf{div} \boldsymbol{\tau}_S)_S \quad \forall \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S), \\ \kappa_2 \left(\mathbf{e}(\mathbf{u}_S) - \frac{\rho}{2\mu}(\mathbf{u}_S \otimes \mathbf{u}_S)^d - \frac{1}{2\mu} \boldsymbol{\sigma}_S^d, \mathbf{e}(\mathbf{v}_S) \right)_S &= 0 \quad \forall \mathbf{v}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S), \\ \kappa_3 \left(\gamma_S - \frac{1}{2} \left\{ \nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^t \right\}, \boldsymbol{\eta}_S \right)_S &= 0 \quad \forall \boldsymbol{\eta}_S \in \mathbb{L}_{\text{skew}}^2(\Omega_S), \end{aligned} \quad (3.4)$$

where κ_1 , κ_2 and κ_3 are positive parameters to be specified later. Notice that the foregoing terms are nothing but consistent expressions, arising from the equilibrium and constitutive equations, and the definition of the vorticity in terms of the velocity gradient (cf. (2.4)). It is easy to see that each solution of the original system is also a solution of the resulting augmented one, and hence by solving the latter we find all the solutions of the former.

Now, it is clear that there are many different ways of ordering the augmented mixed variational formulation described above, but for the sake of the subsequent analysis, we proceed as in [25] (see also [22, 11]), and adopt one leading to a doubly-mixed structure. For this purpose, we group the spaces, unknowns, and test functions as follows:

$$\begin{aligned} \mathbf{X}_1 &:= \mathbb{H}_0(\mathbf{div}; \Omega_S) \times \mathbf{H}_{\Gamma_S}^1(\Omega_S) \times \mathbb{L}_{\text{skew}}^2(\Omega_S), \quad \mathbf{X}_2 := \mathbf{H}_{\Gamma_D}^3(\mathbf{div}; \Omega_D), \\ \mathbf{X} &:= \mathbf{X}_1 \times \mathbf{X}_2, \quad \mathbf{Y} := \mathbf{H}_{00}^{1/2}(\Sigma) \times W^{1/3, 3/2}(\Sigma), \\ \mathbb{H} &:= \mathbf{X} \times \mathbf{Y} \quad \text{and} \quad \mathbb{Q} := L_0^2(\Omega_D) \times \mathbb{R}, \\ \underline{\boldsymbol{\sigma}} &:= (\boldsymbol{\sigma}_S, \mathbf{u}_S, \gamma_S) \in \mathbf{X}_1, \quad \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}_S, \mathbf{v}_S, \boldsymbol{\eta}_S) \in \mathbf{X}_1, \\ \underline{\mathbf{t}} &:= (\underline{\boldsymbol{\sigma}}, \mathbf{u}_D) \in \mathbf{X}, \quad \underline{\boldsymbol{\varphi}} := (\boldsymbol{\varphi}, \lambda) \in \mathbf{Y}, \quad \underline{\mathbf{p}} = (p_D, \ell) \in \mathbb{Q}, \\ \underline{\mathbf{r}} &:= (\underline{\boldsymbol{\tau}}, \mathbf{v}_D) \in \mathbf{X}, \quad \underline{\boldsymbol{\psi}} := (\boldsymbol{\psi}, \xi) \in \mathbf{Y}, \quad \underline{\mathbf{q}} = (q_D, j) \in \mathbb{Q}, \end{aligned}$$

where \mathbf{X}_1 , \mathbf{X} , \mathbf{Y} , \mathbb{H} and \mathbb{Q} are respectively endowed with the norms

$$\begin{aligned} \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} &:= \|\boldsymbol{\tau}_S\|_{\text{div};\Omega_S} + \|\mathbf{v}_S\|_{1,\Omega_S} + \|\boldsymbol{\eta}_S\|_{0,\Omega_S}, \\ \|\underline{\mathbf{r}}\|_{\mathbf{X}} &:= \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} + \|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)}, \\ \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} &:= \|\boldsymbol{\psi}\|_{1/2,0,0;\Sigma} + \|\boldsymbol{\xi}\|_{1/3,3/2;\Sigma}, \\ \|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{H}} &:= \|\underline{\mathbf{r}}\|_{\mathbf{X}} + \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \\ \|\underline{\mathbf{q}}\|_{\mathbb{Q}} &:= \|q_D\|_{0,\Omega_D} + |J|. \end{aligned}$$

Hence, the augmented fully-mixed variational formulation for the system (3.1), (3.3) and (3.4) reads: Find $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ [\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \quad \forall \underline{\mathbf{q}} \in \mathbb{Q}, \end{aligned} \tag{3.5}$$

where, given $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$, the operator $\mathbf{A}(\mathbf{w}_S) : \mathbb{H} \rightarrow \mathbb{H}'$ is defined by

$$[\mathbf{A}(\mathbf{w}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := [\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}), \underline{\mathbf{r}}] + [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\varphi}}] + [\mathbf{b}(\underline{\mathbf{t}}), \underline{\boldsymbol{\psi}}] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}], \tag{3.6}$$

with the operator $\mathbf{a}(\mathbf{w}_S) : \mathbf{X} \rightarrow \mathbf{X}'$ given by

$$[\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}), \underline{\mathbf{r}}] := [\mathcal{A}_S(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] + [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] + [\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D], \tag{3.7}$$

with

$$\begin{aligned} [\mathcal{A}_S(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] &:= \frac{1}{2\mu} (\boldsymbol{\sigma}_S^d, \boldsymbol{\tau}_S^d)_S + \kappa_1 (\text{div} \boldsymbol{\sigma}_S, \text{div} \boldsymbol{\tau}_S)_S + (\mathbf{u}_S, \text{div} \boldsymbol{\tau}_S)_S - (\text{div} \boldsymbol{\sigma}_S, \mathbf{v}_S)_S \\ &\quad + (\gamma_S, \boldsymbol{\tau}_S)_S - (\boldsymbol{\sigma}_S, \boldsymbol{\eta}_S)_S + \kappa_2 \left(\mathbf{e}(\mathbf{u}_S) - \frac{1}{2\mu} \boldsymbol{\sigma}_S^d, \mathbf{e}(\mathbf{v}_S) \right)_S \\ &\quad + \kappa_3 \left(\gamma_S - \frac{1}{2} (\nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^t), \boldsymbol{\eta}_S \right)_S, \\ [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] &:= \frac{\rho}{2\mu} ((\mathbf{w}_S \otimes \mathbf{u}_S)^d, \boldsymbol{\tau}_S - \kappa_2 \mathbf{e}(\mathbf{v}_S))_S, \\ [\mathcal{A}_D(\mathbf{u}_D), \mathbf{v}_D] &:= \frac{\mu}{\rho} (\mathbf{K}^{-1} \mathbf{u}_D, \mathbf{v}_D)_D + \frac{\mathbf{F}}{\rho} (|\mathbf{u}_D| \mathbf{u}_D, \mathbf{v}_D)_D, \end{aligned} \tag{3.8}$$

whereas the operators $\mathbf{b} : \mathbf{X} \rightarrow \mathbf{Y}'$ and $\mathbf{c} : \mathbf{Y} \rightarrow \mathbf{Y}'$ are given, respectively, by

$$[\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}] := \langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_\Sigma - \langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma, \quad (3.9)$$

$$[\mathbf{c}(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}] := \langle \underline{\boldsymbol{\varphi}}, \underline{\boldsymbol{\psi}} \rangle_{\mathbf{t}, \Sigma} + \langle \underline{\boldsymbol{\varphi}} \cdot \mathbf{n}, \xi \rangle_\Sigma - \langle \underline{\boldsymbol{\psi}} \cdot \mathbf{n}, \lambda \rangle_\Sigma, \quad (3.10)$$

and the operator $\mathbf{B} : \mathbb{H} \rightarrow \mathbb{Q}'$ is defined by

$$[\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}] := -(q_D, \operatorname{div} \mathbf{v}_D)_D + j \langle \underline{\boldsymbol{\psi}} \cdot \mathbf{n}, 1 \rangle_\Sigma. \quad (3.11)$$

In turn, the functionals \mathbf{F} and \mathbf{G} are set as

$$[\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] := -\kappa_1 (\mathbf{f}_S, \operatorname{div} \boldsymbol{\tau}_S)_S + (\mathbf{f}_S, \mathbf{v}_S)_S + (\mathbf{f}_D, \mathbf{v}_D)_D \quad \text{and} \quad [\mathbf{G}, \underline{\mathbf{q}}] := -(g_D, q_D)_D. \quad (3.12)$$

In all the terms above, $[\cdot, \cdot]$ denotes the duality pairing induced by the corresponding operators. In addition, we let $\mathbf{b}' : \mathbf{Y} \rightarrow \mathbf{X}'$ and $\mathbf{B}' : \mathbb{Q} \rightarrow \mathbb{H}'$ be the adjoint of \mathbf{b} and \mathbf{B} , respectively, which satisfy $[\mathbf{b}'(\underline{\boldsymbol{\psi}}), (\underline{\mathbf{r}})] = [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]$ and $[\mathbf{B}'(\underline{\mathbf{q}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] = [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}]$ for all $\underline{\mathbf{r}} \in \mathbf{X}$, $\underline{\boldsymbol{\psi}} \in \mathbf{Y}$ and $\underline{\mathbf{q}} \in \mathbb{Q}$. Then, it is clear that (3.5) can be written equivalently as

$$\left[\begin{array}{cc|c} \mathbf{a}(\mathbf{u}_S) & \mathbf{b}' & \mathbf{B}' \\ \mathbf{b} & -\mathbf{c} & \\ \hline & \mathbf{B} & \mathbf{O} \end{array} \right] \left[\begin{array}{c} (\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) \\ \underline{\mathbf{p}} \end{array} \right] = \left[\begin{array}{c} \mathbf{F} \\ \mathbf{G} \end{array} \right],$$

from which the twofold saddle point structure is evident (see also Table 3.1).

	σ_S	\mathbf{u}_S	γ_S	\mathbf{u}_D	(φ, λ)	(p_D, ℓ)
τ_S	$\frac{1}{2\mu}(\sigma_S^d, \tau_S^d)_S$	$(\mathbf{u}_S, \operatorname{div} \tau_S)_S$	$(\gamma_S, \tau_S)_S$		$\langle \tau_S \mathbf{n}, \varphi \rangle_\Sigma$	
\mathbf{v}_S	$+\kappa_1(\operatorname{div} \sigma_S, \operatorname{div} \tau_S)_S$	$+\frac{\rho}{2\mu}((\mathbf{u}_S \otimes \mathbf{u}_S)^d, \tau_S)_S$				
η_S	$-\frac{\kappa_2}{2\mu}(\sigma_S^d, \mathbf{e}(\mathbf{v}_S))_S$	$\kappa_2(\mathbf{e}(\mathbf{u}_S), \mathbf{e}(\mathbf{v}_S))_S$				
\mathbf{v}_D	$-(\operatorname{div} \sigma_S, \mathbf{v}_S)_S$	$-\kappa_2 \frac{\rho}{2\mu}((\mathbf{u}_S \otimes \mathbf{u}_S)^d, \mathbf{e}(\mathbf{v}_S))_S$				
	$-(\sigma_S, \eta_S)_S$	$-\frac{\kappa_3}{2}(\nabla \mathbf{u}_S - (\nabla \mathbf{u}_S)^t, \eta_S)_S$	$\kappa_3(\gamma_S, \eta_S)_S$			
				$\frac{\mu}{\rho}(\mathbf{K}^{-1} \mathbf{u}_D, \mathbf{v}_D)_D$	$-(\mathbf{v}_D \cdot \mathbf{n}, \lambda)_\Sigma$	$-(p_D, \operatorname{div} \mathbf{v}_D)_D$
				$+\frac{F}{\rho}(\mathbf{u}_D \mathbf{u}_D, \mathbf{v}_D)_D$		
(ψ, ξ)	$\langle \sigma_S \mathbf{n}, \psi \rangle_\Sigma$			$-(\mathbf{u}_D \cdot \mathbf{n}, \xi)_\Sigma$	$-\langle \varphi, \psi \rangle_{t, \Sigma}$	$\ell \langle \psi \cdot \mathbf{n}, 1 \rangle_\Sigma$
					$+\langle \psi \cdot \mathbf{n}, \lambda \rangle_\Sigma$	
					$-\langle \varphi \cdot \mathbf{n}, \xi \rangle_\Sigma$	
(q_D, j)				$-(q_D, \operatorname{div} \mathbf{u}_D)_D$	$j \langle \varphi \cdot \mathbf{n}, 1 \rangle_\Sigma$	

Table 3.1: Twofold saddle point structure.

An abstract theory for twofold saddle point problems

In this chapter we develop and analyze an abstract theory motivated by the twofold saddle point problem (3.5). To this end, a modification of what was done in [2] will be employed (which is already a modification of what was done in [20] and [32]). First we introduce some definitions that will be utilized next. Let X and Y be reflexive Banach spaces. Then, we say that a nonlinear operator $T : X \rightarrow Y$ is bounded if $T(S)$ is bounded for each bounded set $S \subseteq X$. In addition, we say that a nonlinear operator $T : X \rightarrow X'$ is of *type M* if $u_n \rightharpoonup u$, $Tu_n \rightharpoonup f$ and $\limsup [Tu_n, u_n] \leq f(u)$ imply $Tu = f$. In turn, we say that T is coercive if

$$\frac{[Tu, u]}{\|u\|} \rightarrow \infty \quad \text{as} \quad \|u\| \rightarrow \infty.$$

Now, let X_1, X_2, Y, Q be separable and reflexive Banach spaces, set $X := X_1 \times X_2$ and $H := X \times Y$, and let $X'_1, X'_2, Y', Q', X' := X'_1 \times X'_2$, and $H' := X' \times Y'$, be their respective duals. Let $a : X \rightarrow X'$ be a nonlinear operator and $b : X \rightarrow Y', c : Y \rightarrow Y'$, and $B : H \rightarrow Q'$ be linear

bounded operators. We also let $b' : Y \rightarrow X'$ and $B' : Q \rightarrow H'$ be the corresponding adjoints, and define the nonlinear operator $A : H \rightarrow H'$ as:

$$A(\mathbf{r}, \boldsymbol{\psi}) := \begin{bmatrix} a & b' \\ b & -c \end{bmatrix} \begin{bmatrix} \mathbf{r} \\ \boldsymbol{\psi} \end{bmatrix} \in H' \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in H.$$

Then, we are interested in the following nonlinear variational problem: Given $(F, G) \in H' \times Q'$, find $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$ such that

$$\begin{bmatrix} A & B' \\ B & O \end{bmatrix} \begin{bmatrix} (\mathbf{t}, \boldsymbol{\varphi}) \\ \mathbf{p} \end{bmatrix} = \begin{bmatrix} F \\ G \end{bmatrix},$$

or, equivalently, such that

$$\begin{aligned} [A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] &= [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in H, \\ [B(\mathbf{t}, \boldsymbol{\varphi}), \mathbf{q}] &= [G, \mathbf{q}] \quad \forall \mathbf{q} \in Q. \end{aligned} \tag{4.1}$$

In what follows we proceed as in [2, 20, 22] to derive sufficient conditions under which (4.1) is well-posed. We first let V be the kernel of B , that is

$$V := \left\{ (\mathbf{r}, \boldsymbol{\psi}) \in H : [B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}] = 0 \quad \forall \mathbf{q} \in Q \right\},$$

and assume that:

(B₀) H is uniformly convex.

(B₁) $B : H \rightarrow Q'$ is surjective, which means that there exists $\beta > 0$ such that

$$\sup_{\substack{(\mathbf{r}, \boldsymbol{\psi}) \in H \\ (\mathbf{r}, \boldsymbol{\psi}) \neq 0}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_H} \geq \beta \|\mathbf{q}\|_Q \quad \forall \mathbf{q} \in Q.$$

Then, given $G \in Q'$ there exists a unique $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ such that (see [32, Lemma A.1] for

details):

$$B(\mathbf{t}_G, \boldsymbol{\varphi}_G) = G \quad \text{and} \quad \|(\mathbf{t}_G, \boldsymbol{\varphi}_G)\|_H = \|[\mathbf{t}_G, \boldsymbol{\varphi}_G]\|_{H/V} \leq \frac{1}{\beta} \|G\|_{Q'}, \quad (4.2)$$

where $[\mathbf{t}_G, \boldsymbol{\varphi}_G] := \{(\mathbf{r}, \boldsymbol{\psi}) \in H : (\mathbf{t}_G - \mathbf{r}, \boldsymbol{\varphi}_G - \boldsymbol{\psi}) \in V\}$ is the equivalence class in the quotient space H/V . Under the previous assumptions, we can show the following preliminary result.

Lemma 4.1 *Assume that hypotheses (B_0) and (B_1) hold. Then, the following problems are equivalent:*

(a) Find $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$ such that

$$[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in H. \quad (4.3)$$

(b) Find $(\mathbf{t}, \boldsymbol{\varphi}) \in H$ such that

$$[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in V. \quad (4.4)$$

More precisely, if $(\mathbf{t}, \boldsymbol{\varphi}) \in H$ is solution of (4.4), we can define $\mathbf{p} \in Q$ as the unique solution of the following problem: Find $\mathbf{p} \in Q$ such that

$$[B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] = [F - A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in H. \quad (4.5)$$

Then $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})$ is solution of (4.3). Conversely, if $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$ is a solution of (4.3), then $(\mathbf{t}, \boldsymbol{\varphi})$ is solution of (4.4) and \mathbf{p} is solution of (4.5).

Proof. First, let $(\mathbf{t}, \boldsymbol{\varphi}) \in H$ solution of (4.4) and notice that (4.5) has a unique solution. In fact, since $(\mathbf{t}, \boldsymbol{\varphi})$ is solution of (4.4), then $F - A(\mathbf{t}, \boldsymbol{\varphi}) \in {}^\circ V := \{G \in H' : G(\mathbf{r}, \boldsymbol{\psi}) = 0 \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in V\}$. Hence, since assumption (B_1) also guarantees that the adjoint operator B' is an isomorphism from Q into ${}^\circ V$, we deduce that there exists a unique $\mathbf{p} \in Q$ solution of (4.5) satisfying

$$\|\mathbf{p}\|_Q \leq \frac{1}{\beta} \|B'(\mathbf{p})\|_{H'} = \frac{1}{\beta} \|F - A(\mathbf{t}, \boldsymbol{\varphi})\|_{H'}, \quad (4.6)$$

and therefore $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})$ is solution of (4.3). The second assertion is evident. \square

Next, for the subsequent analysis, we let \tilde{X} and \tilde{Y} subspaces of X and Y , respectively,

such that $V := \tilde{X} \times \tilde{Y}$. In addition, since F is linear we are able to define two functionals, F_1 and F_2 , such that $[F, (\mathbf{r}, \boldsymbol{\psi})] = [F_1, \mathbf{r}] + [F_2, \boldsymbol{\psi}]$. In this way, we can state the following lemma.

Lemma 4.2 *Assume that hypotheses (B₀) and (B₁) hold, and let $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (4.2). Then, problem (4.1) is equivalent to: Find $(\mathbf{t}_0, \boldsymbol{\varphi}_0) \in V$ such that*

$$[A(\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), (\mathbf{r}, \boldsymbol{\psi})] = [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in V, \quad (4.7)$$

or equivalently, such that

$$\begin{aligned} [a(\mathbf{t}_0 + \mathbf{t}_G), \mathbf{r}] + [b'(\boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \mathbf{r}] &= [F_1, \mathbf{r}] \quad \forall \mathbf{r} \in \tilde{X}, \\ [b(\mathbf{t}_0 + \mathbf{t}_G), \boldsymbol{\psi}] - [c(\boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \boldsymbol{\psi}] &= [F_2, \boldsymbol{\psi}] \quad \forall \boldsymbol{\psi} \in \tilde{Y}. \end{aligned} \quad (4.8)$$

Moreover, the problem (4.1) has a unique solution if and only if the problem (4.7) has a unique solution.

Proof. The first assertion is evident. On the other hand, let $(\mathbf{t}_0, \boldsymbol{\varphi}_0) \in V$ be the solution of (4.7). Then we solve the problem (4.5) with $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_G$ and $\boldsymbol{\varphi} = \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G$. Hence, if \mathbf{p} is the unique solution of (4.5), it follows that $((\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \mathbf{p})$ is a solution of (4.1). Now, for the uniqueness of solution of (4.7), we set $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})$ as the unique solution of (4.1), and define $(\mathbf{t}_0, \boldsymbol{\varphi}_0) := (\mathbf{t}, \boldsymbol{\varphi}) - (\mathbf{t}_G, \boldsymbol{\varphi}_G)$ with $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (4.2). The uniform convexity of H insures that $(\mathbf{t}_0, \boldsymbol{\varphi}_0)$ is a unique solution of (4.7). \square

According to the previous analysis, we focus now on analyzing the solvability of (4.8). To that end, let us first assume the following assumptions:

(A₀) X_1, X_2 and Y are uniformly convex.

(A₁) there exists constant $\gamma > 0$ and $p_1, p_2 \geq 2$, such that

$$\|a(\mathbf{t}) - a(\mathbf{r})\|_{X'} \leq \gamma \sum_{j=1}^2 \left\{ \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} + \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} \left(\|\mathbf{t}_j\|_{X_j} + \|\mathbf{r}_j\|_{X_j} \right)^{p_j-2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in X$.

(A₂) for each $\mathbf{s} \in X$, the operator $a(\cdot + \mathbf{s}) : \tilde{X} \rightarrow \tilde{X}'$ is strictly monotone in the sense that there exist $\alpha > 0$ and $p_1, p_2 \geq 2$, such that

$$[a(\mathbf{t} + \mathbf{s}) - a(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \geq \alpha \left\{ \|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{p_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{p_2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in \tilde{X}$.

(A₃) there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{r} \in \tilde{X} \\ \mathbf{r} \neq 0}} \frac{[b(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_X} \geq \beta_1 \|\boldsymbol{\psi}\|_Y \quad \forall \boldsymbol{\psi} \in \tilde{Y}.$$

Notice that hypothesis (A₀) implies (B₀). Then, we can state the following preliminary lemma.

Lemma 4.3 *Assume that hypotheses (A₁) – (A₃) hold. Then, given $\boldsymbol{\psi} \in \tilde{Y}$ and $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (4.2) there exists a unique $\mathbf{t}_0(\boldsymbol{\psi}) \in \tilde{X}$, such that*

$$[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G), \mathbf{r}] = [F_1 - b'(\boldsymbol{\psi} + \boldsymbol{\varphi}_G), \mathbf{r}] \quad \forall \mathbf{r} \in \tilde{X}. \quad (4.9)$$

Moreover, there exists $C_1 > 0$, depending only on $\alpha, \beta, \gamma, p_1, p_2$ and $\|b'\|$, such that

$$\|\mathbf{t}_0(\boldsymbol{\psi})\|_X \leq C_1 \max_{i \in \{1,2\}} \left\{ \left(\|\boldsymbol{\psi}\|_Y + \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{p_1-1} + \|G\|_{Q'}^{p_2-1} + \|a(0)\|_{X'} \right)^{1/(p_i-1)} \right\}. \quad (4.10)$$

In addition, given $\boldsymbol{\psi}_1, \boldsymbol{\psi}_2 \in \tilde{Y}$ for which $\mathbf{t}_0(\boldsymbol{\psi}_1)$ and $\mathbf{t}_0(\boldsymbol{\psi}_2)$ satisfy (4.9), there exists $C_2 > 0$, depending only on α, p_1, p_2 and $\|b'\|$, such that

$$\|\mathbf{t}_0(\boldsymbol{\psi}_1) - \mathbf{t}_0(\boldsymbol{\psi}_2)\|_X \leq C_2 \max_{i \in \{1,2\}} \left\{ \|\boldsymbol{\psi}_1 - \boldsymbol{\psi}_2\|_Y^{1/(p_i-1)} \right\}. \quad (4.11)$$

Proof. We begin by noting that hypothesis (A₁) implies, in particular, that the nonlinear operator a is hemi-continuous, that is, for each $\mathbf{t}, \mathbf{r} \in X$, the mapping

$$J : \mathbb{R} \rightarrow \mathbb{R}, \quad z \mapsto J(z) := [a(\mathbf{t} + z\mathbf{r}), \mathbf{r}]$$

is continuous. In this way, as a consequence of hypotheses $(A_1) - (A_2)$ we deduce the well-posedness of the problem (4.9) (see [10, Theorem 3.1] for details). In turn, given $\psi_1, \psi_2 \in \tilde{Y}$ for which $\mathbf{t}_0(\psi_1)$ and $\mathbf{t}_0(\psi_2)$ satisfy (4.9), we deduce that

$$[a(\mathbf{t}_0(\psi_1) + \mathbf{t}_G) - a(\mathbf{t}_0(\psi_2) + \mathbf{t}_G), \mathbf{r}] = [b'(\psi_2 - \psi_1), \mathbf{r}] \quad \forall \mathbf{r} \in \tilde{X}. \quad (4.12)$$

Then, if we assume that $\mathbf{t}_0(\psi_1) = \mathbf{t}_0(\psi_2)$, hypotheses (A_1) and (A_3) , and (4.12), imply that $\psi_1 = \psi_2$. Equivalently, this shows that given $\psi_1, \psi_2 \in \tilde{Y}$ with $\psi_1 \neq \psi_2$ the solutions $\mathbf{t}_0(\psi_1)$ and $\mathbf{t}_0(\psi_2)$ of (4.9) are in fact different. Now, in order to obtain (4.10), we proceed similarly to [38, Proposition 2.3] (see also [10, Theorem 3.1]). In fact, given $\psi \in \tilde{Y}$, we take $\mathbf{r} = \mathbf{t}_0(\psi) \in \tilde{X}$ in (4.9), and observe that

$$[a(\mathbf{t}_0(\psi) + \mathbf{t}_G) - a(0 + \mathbf{t}_G), \mathbf{t}_0(\psi) - 0] = [F_1 - b'(\psi + \varphi_G) - a(\mathbf{t}_G), \mathbf{t}_0(\psi)].$$

Then, combining hypotheses $(A_1) - (A_2)$ and (4.2), it is clear that

$$\begin{aligned} \alpha \left\{ \|(\mathbf{t}_0(\psi))_1\|_{X_1}^{p_1} + \|(\mathbf{t}_0(\psi))_2\|_{X_2}^{p_2} \right\} &\leq \left\{ \|F_1\|_{X'} + \|b'(\psi + \varphi_G)\|_{X'} + \|a(\mathbf{t}_G)\|_{X'} \right\} \|\mathbf{t}_0(\psi)\|_X \\ &\leq c_1 \left\{ \|\psi\|_Y + \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{p_1-1} + \|G\|_{Q'}^{p_2-1} + \|a(0)\|_{X'} \right\} \|\mathbf{t}_0(\psi)\|_X, \end{aligned}$$

with $c_1 > 0$ depending only on γ, β, p_1, p_2 , and $\|b'\|$, which, after simple algebraic manipulation, yields (4.10) (see [2, Lemma 3.5] for details). In turn, in order to derive (4.11), we take $\mathbf{r} = \mathbf{t}_0(\psi_1) - \mathbf{t}_0(\psi_2)$ in (4.12), to obtain

$$[a(\mathbf{t}_0(\psi_1) + \mathbf{t}_G) - a(\mathbf{t}_0(\psi_2) + \mathbf{t}_G), \mathbf{t}_0(\psi_1) - \mathbf{t}_0(\psi_2)] = [b'(\psi_2 - \psi_1), \mathbf{t}_0(\psi_1) - \mathbf{t}_0(\psi_2)], \quad (4.13)$$

and proceeding analogously to (4.10), we find that

$$\|\mathbf{t}_0(\psi_1) - \mathbf{t}_0(\psi_2)\|_X \leq 2 \max_{i \in \{1,2\}} \left\{ \left(\frac{2}{\alpha} \|b'(\psi_2 - \psi_1)\|_{X'} \right)^{1/(p_i-1)} \right\},$$

which clearly implies (4.11) and completes the proof. □

According to the above, and given $(\mathbf{t}_G, \varphi_G) \in H$ satisfying (4.2), problem (4.8) is equiva-

lent to: find $\varphi_0 \in \tilde{Y}$ such that

$$[L(\varphi_0), \psi] := -[b(\mathbf{t}_0(\varphi_0)), \psi] + [c(\varphi_0), \psi] = [\tilde{F}_2, \psi] \quad \forall \psi \in \tilde{Y}, \quad (4.14)$$

where $\tilde{F}_2 := b(\mathbf{t}_G) - c(\varphi_G) - F_2$. More precisely, given $\mathbf{t}_0(\varphi_0) \in \tilde{X}$ solution of (4.9) with $\varphi_0 \in \tilde{Y}$ solution of (4.14), the vector $(\mathbf{t}_0, \varphi_0) := (\mathbf{t}_0(\varphi_0), \varphi_0) \in \tilde{X} \times \tilde{Y}$ solves (4.8). The conversely is straightforward. Hence, we now focus on proving that L is bijective. To that end, we assume one more hypothesis:

(A₄) c is positive semi-definite on \tilde{Y} , that is,

$$[c(\psi), \psi] \geq 0 \quad \forall \psi \in \tilde{Y}.$$

We begin with the injectivity of L .

Lemma 4.4 L is injective.

Proof. Let $\psi_1, \psi_2 \in \tilde{Y}$ such that $L(\psi_1) = L(\psi_2)$, that is $[L(\psi_1) - L(\psi_2), \psi] = 0$ for all $\psi \in \tilde{Y}$. Then, from the definition of L (cf. (4.14)) and taking in particular $\psi = \psi_2 - \psi_1$, we deduce that

$$[b(\mathbf{t}_0(\psi_2) - \mathbf{t}_0(\psi_1)), \psi_2 - \psi_1] - [c(\psi_2 - \psi_1), \psi_2 - \psi_1] = 0.$$

It follows, employing the identity (4.13) and hypothesis (A₄), that

$$[a(\mathbf{t}_0(\psi_1) + \mathbf{t}_G) - a(\mathbf{t}_0(\psi_2) + \mathbf{t}_G), \mathbf{t}_0(\psi_1) - \mathbf{t}_0(\psi_2)] \leq 0,$$

which, together with the assumption (A₂), yields $\mathbf{t}_0(\psi_1) = \mathbf{t}_0(\psi_2)$, and then, using similar arguments to those in Lemma 4.3, we deduce that $\psi_1 = \psi_2$, which concludes the proof. \square

Next, we show the surjectivity of L by means of classical results from nonlinear functional analysis. More precisely, we will see that, under the hypotheses that have been assumed for the solvability of (4.8), the operator L is continuous and monotone, and therefore, of *type M* (cf. [39, Lemma II.2.1]). Then, by also proving that L is bounded and coercive [39, Corollary II.2.2], the surjectivity of L is ensured. We begin with the *type M* property of L .

Lemma 4.5 *L is of type M.*

Proof. We first prove that L is continuous. Let $\{\psi_n\}_{n \in \mathbb{N}} \subseteq \tilde{Y}$ and $\psi \in \tilde{Y}$ such that $\|\psi_n - \psi\|_Y \rightarrow 0$ as $n \rightarrow \infty$. Thus, from the definition of L (cf. (4.14)), and using the continuity of b and c , and the estimate (4.11), we deduce that

$$\begin{aligned} \|\mathbf{L}(\psi_n) - \mathbf{L}(\psi)\| &\leq \|b\| \|\mathbf{t}_0(\psi) - \mathbf{t}_0(\psi_n)\|_X + \|c\| \|\psi_n - \psi\|_Y \\ &\leq C_2 \|b\| \max_{i \in \{1,2\}} \left\{ \|\psi_n - \psi\|_Y^{1/(p_i-1)} \right\} + \|c\| \|\psi_n - \psi\|_Y, \end{aligned}$$

where $p_i \geq 2$ for $i \in \{1, 2\}$, which gives $\|\mathbf{L}(\psi_n) - \mathbf{L}(\psi)\| \rightarrow 0$ as $n \rightarrow \infty$, thus proving the continuity of L. On the other hand, for the monotonicity, given $\psi_1, \psi_2 \in \tilde{Y}$, the definition of L (cf. (4.14)), assumption (A₄) and identity (4.13) imply

$$\begin{aligned} [\mathbf{L}(\psi_1) - \mathbf{L}(\psi_2), \psi_1 - \psi_2] &= [b(\mathbf{t}_0(\psi_2) - \mathbf{t}_0(\psi_1)), \psi_1 - \psi_2] + [c(\psi_1 - \psi_2), \psi_1 - \psi_2] \\ &\geq [a(\mathbf{t}_0(\psi_2) + \mathbf{t}_G) - a(\mathbf{t}_0(\psi_1) + \mathbf{t}_G), \mathbf{t}_0(\psi_2) - \mathbf{t}_0(\psi_1)], \end{aligned} \tag{4.15}$$

whence the monotonicity of L follows from (A₂). \square

Lemma 4.6 *L is bounded and coercive.*

Proof. Let $\psi \in \tilde{Y}$. Then, applying triangle inequality, the definition of L (cf. (4.14)), and the continuity of b and c , we have that

$$\|\mathbf{L}(\psi)\| \leq \|\mathbf{L}(\psi) - \mathbf{L}(0)\| + \|\mathbf{L}(0)\| \leq \|b\| \|\mathbf{t}_0(\psi) - \mathbf{t}_0(0)\|_X + \|c\| \|\psi\|_Y + \|b\| \|\mathbf{t}_0(0)\|_X.$$

Then, we notice from (4.10) that $\|\mathbf{t}_0(0)\|_X$ is bounded only in terms of data, which together with the estimate (4.11) applied to $\psi_1 = \psi$ and $\psi_2 = 0$, allow us to conclude that L is bounded. On the other hand, for the coercivity of L, we use again the definition of L (cf.

(4.14)), assumption (A₄) and identity (4.12), to find that

$$\begin{aligned} \frac{[L(\boldsymbol{\psi}), \boldsymbol{\psi}]}{\|\boldsymbol{\psi}\|_Y} &\geq \frac{[-b(\mathbf{t}_0(\boldsymbol{\psi})), \boldsymbol{\psi}]}{\|\boldsymbol{\psi}\|_Y} = \frac{[b'(0 - \boldsymbol{\psi}), \mathbf{t}_0(\boldsymbol{\psi})]}{\|\boldsymbol{\psi}\|_Y} \\ &= \frac{[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\psi})]}{\|\boldsymbol{\psi}\|_Y} = \tilde{T}(\boldsymbol{\psi}) + \widehat{T}(\boldsymbol{\psi}), \end{aligned} \quad (4.16)$$

where

$$\tilde{T}(\boldsymbol{\psi}) := \frac{[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G), \mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)]}{\|\boldsymbol{\psi}\|_Y}$$

and

$$\widehat{T}(\boldsymbol{\psi}) := \frac{[a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G), \mathbf{t}_0(0)]}{\|\boldsymbol{\psi}\|_Y}.$$

For estimating $\tilde{T}(\boldsymbol{\psi})$ we combine the hypothesis (A₃) and the identity (4.12), to obtain

$$\beta_1 \|\boldsymbol{\psi}\|_Y \leq \|a(\mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G)\|_{X'} \quad \forall \boldsymbol{\psi} \in \tilde{Y}. \quad (4.17)$$

Next, to simplify the notation, we set $\tilde{\mathbf{t}} := \mathbf{t}_0(\boldsymbol{\psi}) + \mathbf{t}_G$ and $\widehat{\mathbf{t}} := \mathbf{t}_0(0) + \mathbf{t}_G$. Then, from (4.17), hypothesis (A₁), and the inequality $(a + b)^q \leq C(q)(a^q + b^q)$, with $C(q)$ depending only on q , which is valid for all $q \geq 0$ and $a, b \geq 0$ [4, Lemma 2.2], we find that

$$\begin{aligned} \beta_1 \|\boldsymbol{\psi}\|_Y &\leq \gamma \sum_{i=1}^2 \left\{ \|(\mathbf{t}_0(\boldsymbol{\psi}))_i - (\mathbf{t}_0(0))_i\|_{X_i} + \|(\mathbf{t}_0(\boldsymbol{\psi}))_i - (\mathbf{t}_0(0))_i\|_{X_i} \left(\|\tilde{\mathbf{t}}_i\|_{X_i} + \|\widehat{\mathbf{t}}_i\|_{X_i} \right)^{p_i-2} \right\} \\ &\leq C \sum_{i=1}^2 \left\{ \left(1 + \|\widehat{\mathbf{t}}_i\|_{X_i}^{p_i-2} \right) \|(\mathbf{t}_0(\boldsymbol{\psi}))_i - (\mathbf{t}_0(0))_i\|_{X_i} + \|(\mathbf{t}_0(\boldsymbol{\psi}))_i - (\mathbf{t}_0(0))_i\|_{X_i}^{p_i-1} \right\}, \end{aligned}$$

which, together with the fact that $\|\widehat{\mathbf{t}}_i\|_{X_i}$, with $i \in \{1, 2\}$, is bounded (cf. (4.2) and (4.10)), yields

$$\|\boldsymbol{\psi}\|_Y \leq \widehat{C} \left\{ \|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X + \max_{i \in \{1, 2\}} \left\{ \|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X^{p_i-1} \right\} \right\}, \quad (4.18)$$

with $\widehat{C} > 0$ depending only on $\gamma, \beta_1, p_1, p_2, \|F_1\|_{X'}, \|G\|_{Q'}$, and $\|a(0)\|_{X'}$. Then, it is easy to see that $\|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X \rightarrow \infty$ as $\|\boldsymbol{\psi}\|_Y \rightarrow \infty$. In turn, from (4.18) and hypothesis (A₂), we

find that

$$\tilde{T}(\boldsymbol{\psi}) \geq \frac{\alpha}{\tilde{C}} \left(\frac{\min_{i \in \{1,2\}} \left\{ \|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X^{p_i-2} \right\}}{1 + \max_{i \in \{1,2\}} \left\{ \|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X^{p_i-2} \right\}} \right) \|\mathbf{t}_0(\boldsymbol{\psi}) - \mathbf{t}_0(0)\|_X, \quad (4.19)$$

where $p_i \geq 2$ for $i \in \{1, 2\}$, and hence $\tilde{T}(\boldsymbol{\psi}) \rightarrow \infty$ as $\|\boldsymbol{\psi}\|_Y \rightarrow \infty$. On the other hand, for $\hat{T}(\boldsymbol{\psi})$ it suffices to observe from (4.17) that

$$\hat{T}(\boldsymbol{\psi}) \geq -\beta_1 \|\mathbf{t}_0(0)\|_X, \quad (4.20)$$

and, employing again (4.10), conclude that $\hat{T}(\boldsymbol{\psi})$ is bounded independent of $\boldsymbol{\psi}$. Hence, replacing (4.19) and (4.20) back into (4.16), the coercivity of L is derived. \square

Therefore, the bijectivity of the operator L is a straightforward consequence of Lemmas 4.4, 4.5 and 4.6, which means, equivalently, that (4.14) has a unique solution $\boldsymbol{\varphi}_0 \in \tilde{Y}$. The main result of this chapter is established now.

Theorem 4.7 *Let X_1, X_2, Y, Q be separable and reflexive Banach spaces, being X_1, X_2 and Y uniformly convex, set $X := X_1 \times X_2$ and $H := X \times Y$, and let $X'_1, X'_2, Y', Q', X' := X'_1 \times X'_2$, and $H' := X' \times Y'$, be their respective duals. In addition, let $a : X \rightarrow X'$ be a nonlinear operator, and let $b : X \rightarrow Y', c : Y \rightarrow Y'$, and $B : H \rightarrow Q'$ be linear bounded operators. We also let $b' : Y \rightarrow X'$ and $B' : Q \rightarrow H'$ be the corresponding adjoints and define the nonlinear operator $A : H \rightarrow H'$ as*

$$[A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] := [a(\mathbf{t}), \mathbf{r}] + [b'(\boldsymbol{\varphi}), \mathbf{r}] + [b(\mathbf{t}), \boldsymbol{\psi}] - [c(\boldsymbol{\varphi}), \boldsymbol{\psi}] \quad \forall (\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi}) \in H.$$

In turn, let V be the kernel of B , that is

$$V := \left\{ (\mathbf{r}, \boldsymbol{\psi}) \in H : [B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}] = 0 \quad \forall \mathbf{q} \in Q \right\},$$

and let \tilde{X} and \tilde{Y} be subspaces of X and Y , respectively, such that $V = \tilde{X} \times \tilde{Y}$. Assume that

(i) *there exists constant $\gamma > 0$ and $p_1, p_2 \geq 2$, such that*

$$\|a(\mathbf{t}) - a(\mathbf{r})\|_{X'} \leq \gamma \sum_{j=1}^2 \left\{ \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} + \|\mathbf{t}_j - \mathbf{r}_j\|_{X_j} \left(\|\mathbf{t}_j\|_{X_j} + \|\mathbf{r}_j\|_{X_j} \right)^{p_j-2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in X$.

(ii) for each $\mathbf{s} \in X$, the operator $a(\cdot + \mathbf{s}) : \tilde{X} \rightarrow \tilde{X}'$ is strictly monotone in the sense that there exist $\alpha > 0$ and $p_1, p_2 \geq 2$, such that

$$[a(\mathbf{t} + \mathbf{s}) - a(\mathbf{r} + \mathbf{s}), \mathbf{t} - \mathbf{r}] \geq \alpha \left\{ \|\mathbf{t}_1 - \mathbf{r}_1\|_{X_1}^{p_1} + \|\mathbf{t}_2 - \mathbf{r}_2\|_{X_2}^{p_2} \right\},$$

for all $\mathbf{t} = (\mathbf{t}_1, \mathbf{t}_2), \mathbf{r} = (\mathbf{r}_1, \mathbf{r}_2) \in \tilde{X}$.

(iii) c is positive semi-definite on \tilde{Y} , that is,

$$[c(\boldsymbol{\psi}), \boldsymbol{\psi}] \geq 0 \quad \forall \boldsymbol{\psi} \in \tilde{Y}.$$

(iv) b satisfies an inf-sup condition on $\tilde{X} \times \tilde{Y}$, that is, there exists $\beta_1 > 0$ such that

$$\sup_{\substack{\mathbf{r} \in \tilde{X} \\ \mathbf{r} \neq \mathbf{0}}} \frac{[b(\mathbf{r}), \boldsymbol{\psi}]}{\|\mathbf{r}\|_X} \geq \beta_1 \|\boldsymbol{\psi}\|_Y \quad \forall \boldsymbol{\psi} \in \tilde{Y}.$$

(v) B satisfies an inf-sup condition on $H \times Q$, that is, there exists $\beta > 0$ such that

$$\sup_{\substack{(\mathbf{r}, \boldsymbol{\psi}) \in H \\ (\mathbf{r}, \boldsymbol{\psi}) \neq \mathbf{0}}} \frac{[B(\mathbf{r}, \boldsymbol{\psi}), \mathbf{q}]}{\|(\mathbf{r}, \boldsymbol{\psi})\|_H} \geq \beta \|\mathbf{q}\|_Q \quad \forall \mathbf{q} \in Q.$$

Then, for each $(F, G) \in H' \times Q'$ there exists a unique $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) \in H \times Q$, such that

$$\begin{aligned} [A(\mathbf{t}, \boldsymbol{\varphi}), (\mathbf{r}, \boldsymbol{\psi})] + [B'(\mathbf{p}), (\mathbf{r}, \boldsymbol{\psi})] &= [F, (\mathbf{r}, \boldsymbol{\psi})] \quad \forall (\mathbf{r}, \boldsymbol{\psi}) \in H, \\ [B(\mathbf{t}, \boldsymbol{\varphi}), \mathbf{q}] &= [G, \mathbf{q}] \quad \forall \mathbf{q} \in Q. \end{aligned} \tag{4.21}$$

Moreover, there exists $C > 0$, depending only on $\alpha, \gamma, \beta, \beta_1, p_1, p_2, \|b\|, \|b'\|$, and $\|c\|$ such that

$$\|((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p})\|_{H \times Q} \leq C \mathcal{M}(F, G), \tag{4.22}$$

where

$$\begin{aligned} \mathcal{M}(F, G) := \max \left\{ \mathcal{N}(F, G)^{1/(p_1-1)}, \mathcal{N}(F, G)^{1/(p_2-1)}, \mathcal{N}(F, G), \right. \\ \left. \mathcal{N}(F, G)^{(p_1-1)/(p_2-1)}, \mathcal{N}(F, G)^{(p_2-1)/(p_1-1)} \right\}, \end{aligned}$$

and $\mathcal{N}(F, G)$ is defined below in (4.25).

Proof. We begin by noting that the well-posedness of the problem (4.21) follows straightforwardly from Lemmas 4.1–4.6. Now, in order to obtain (4.22), we proceed similarly to [10, Theorem 3.1]. To that end, we first recall that $((\mathbf{t}, \boldsymbol{\varphi}), \mathbf{p}) = ((\mathbf{t}_0 + \mathbf{t}_G, \boldsymbol{\varphi}_0 + \boldsymbol{\varphi}_G), \mathbf{p}) \in H \times Q$ is solution of (4.21) with $(\mathbf{t}_0, \boldsymbol{\varphi}_0) \in V$ solution of (4.7) and $(\mathbf{t}_G, \boldsymbol{\varphi}_G) \in H$ satisfying (4.2). Next, applying (4.10) with $\boldsymbol{\psi} = 0$, we get

$$\|\mathbf{t}_0(0)\|_X \leq C_1 \max_{i \in \{1,2\}} \{\mathcal{N}_1(F, G)^{1/(p_i-1)}\}, \quad (4.23)$$

where $\mathcal{N}_1(F, G) := \|F_1\|_{X'} + \|G\|_{Q'} + \|G\|_{Q'}^{p_1-1} + \|G\|_{Q'}^{p_2-1} + \|a(0)\|_{X'}$. In turn, similarly to the monotonicity of L (cf. (4.15)), and using the fact that $\mathbf{t}_0 = \mathbf{t}_0(\boldsymbol{\varphi}_0)$, we deduce that

$$[a(\mathbf{t}_0 + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G), \mathbf{t}_0 - \mathbf{t}_0(0)] \leq [L(\boldsymbol{\varphi}_0) - L(0), \boldsymbol{\varphi}_0] = [\tilde{F}_2 - L(0), \boldsymbol{\varphi}_0],$$

where $\tilde{F}_2 = b(\mathbf{t}_G) - c(\boldsymbol{\varphi}_G) - F_2$. In this way, hypothesis (ii) and the definition of the operator L (cf. (4.14)), yield

$$\begin{aligned} & \alpha \left\{ \|(\mathbf{t}_0)_1 - (\mathbf{t}_0(0))_1\|_{X_1}^{p_1} + \|(\mathbf{t}_0)_2 - (\mathbf{t}_0(0))_2\|_{X_2}^{p_2} \right\} \\ & \leq \|\tilde{F}_2 - L(0)\|_{Y'} \|\boldsymbol{\varphi}_0\|_Y \leq C_2 \mathcal{N}_2(F, G) \|\boldsymbol{\varphi}_0\|_Y, \end{aligned}$$

where $\mathcal{N}_2(F, G) := \|F_2\|_{Y'} + \max_{i \in \{1,2\}} \{\mathcal{N}_1(F, G)^{1/(p_i-1)}\}$. Thus, it is clear that

$$\|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i} \leq C \mathcal{N}_2(F, G)^{1/p_i} \|\boldsymbol{\varphi}_0\|_Y^{1/p_i} \quad \text{for } i \in \{1, 2\}.$$

Then, from (4.17), the foregoing inequality, the fact that the upper bound in (4.23) can be bounded by $\mathcal{N}_2(F, G)$, and the hypothesis (i), we find that

$$\begin{aligned} \|\boldsymbol{\varphi}_0\|_Y & \leq \frac{1}{\beta_1} \|a(\mathbf{t}_0 + \mathbf{t}_G) - a(\mathbf{t}_0(0) + \mathbf{t}_G)\|_{X'} \\ & \leq c \sum_{i=1}^2 \left\{ (1 + \|(\mathbf{t}_0(0) + \mathbf{t}_G)_i\|_{X_i}^{p_i-2}) \|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i} + \|(\mathbf{t}_0)_i - (\mathbf{t}_0(0))_i\|_{X_i}^{p_i-1} \right\} \\ & \leq C \sum_{i=1}^2 \left\{ \left(\mathcal{N}_2(F, G)^{1/p_i} + \mathcal{N}_2(F, G)^{(p_i-1)^2/p_i} \right) \|\boldsymbol{\varphi}_0\|_Y^{1/p_i} + \mathcal{N}_2(F, G)^{(p_i-1)/p_i} \|\boldsymbol{\varphi}_0\|_Y^{(p_i-1)/p_i} \right\}, \end{aligned}$$

with C depending on $\alpha, \gamma, \beta, \beta_1, p_1, p_2, \|b\|, \|b'\|$, and $\|c\|$. In turn, applying Young's inequality conveniently allow us to deduce that

$$\|\varphi_0\|_Y \leq C_3 \mathcal{N}(F, G), \quad (4.24)$$

where

$$\mathcal{N}(F, G) := \max \left\{ \mathcal{N}_2(F, G)^{1/(p_1-1)}, \mathcal{N}_2(F, G)^{1/(p_2-1)}, \mathcal{N}_2(F, G)^{p_1-1}, \mathcal{N}_2(F, G)^{p_2-1} \right\}. \quad (4.25)$$

Therefore, using that $\varphi = \varphi_0 + \varphi_G$, and combining (4.2) and (4.24), we conclude that

$$\|\varphi\|_Y \leq \|\varphi_0\|_Y + \|\varphi_G\|_Y \leq c_1 \mathcal{N}(F, G), \quad (4.26)$$

with $c_1 > 0$ depending only on $\alpha, \gamma, \beta, \beta_1, p_1, p_2, \|b\|, \|b'\|$, and $\|c\|$. Similarly, due to $\mathbf{t} = \mathbf{t}_0 + \mathbf{t}_G$, and employing (4.2), (4.10), and (4.24), we conclude that

$$\|\mathbf{t}\|_X \leq \|\mathbf{t}_0\|_X + \|\mathbf{t}_G\|_X \leq c_2 \max_{i \in \{1, 2\}} \left\{ \mathcal{N}(F, G)^{1/(p_i-1)} \right\}, \quad (4.27)$$

with $c_2 > 0$ depending only on $\alpha, \gamma, \beta, \beta_1, p_1, p_2, \|b\|, \|b'\|$, and $\|c\|$. On the other hand, from (4.6) and (i), we deduce that

$$\|\mathbf{p}\|_Q \leq C \left\{ \|F\|_{H'} + \|\mathbf{t}\|_X + \|\varphi\|_Y + \|\mathbf{t}_1\|_{X_1}^{p_1-1} + \|\mathbf{t}_2\|_{X_2}^{p_2-1} + \|a(0)\|_{X'} \right\}, \quad (4.28)$$

which, together with (4.26) and (4.27), conclude the proof. \square

We remark that when $p_1 = p_2 = 2$ and $\|a(0)\|_{X'}$ is equal to zero, the previous analysis leads to the classical estimate

$$\|((\mathbf{t}, \varphi), \mathbf{p})\|_{H \times Q} \leq C \left\{ \|F\|_{H'} + \|G\|_{Q'} \right\},$$

with $C > 0$, depending only on $\alpha, \gamma, \beta, \beta_1, \|b\|, \|b'\|$, and $\|c\|$.

Analysis of the continuous formulation

In this chapter we analyze the well-posedness of the continuous problem (3.5) by using a fixed-point strategy and the abstract theory on twofold saddle point problems developed in Chapter 4. We begin by collecting some previous results and notations that will serve for the forthcoming analysis.

5.1 Preliminaries

Concerning the stability properties of the operators in (3.8), (3.9), (3.10), and (3.11), we first observe that \mathcal{A}_S , \mathbf{b} , \mathbf{c} , and \mathbf{B} are all continuous, that is there exist positive constants $C_{\mathcal{A}_S}$, $C_{\mathbf{b}}$, $C_{\mathbf{c}}$, and $C_{\mathbf{B}}$, such that

$$\begin{aligned} \left| [\mathcal{A}_S(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] \right| &\leq C_{\mathcal{A}_S} \|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1} \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}, & \left| [\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}] \right| &\leq C_{\mathbf{b}} \|\underline{\mathbf{r}}\|_{\mathbf{X}} \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, \\ \left| [\mathbf{c}(\underline{\boldsymbol{\varphi}}), \underline{\boldsymbol{\psi}}] \right| &\leq C_{\mathbf{c}} \|\underline{\boldsymbol{\varphi}}\|_{\mathbf{Y}} \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}}, & \left| [\mathbf{B}((\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})), \underline{\mathbf{q}}] \right| &\leq C_{\mathbf{B}} \|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{H}} \|\underline{\mathbf{q}}\|_{\mathbb{Q}}, \end{aligned} \tag{5.1}$$

whereas from the definition of \mathcal{B}_S (cf. (3.8)) and (3.2) we easily obtain that

$$\left| [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] \right| \leq \frac{\rho}{2\mu} (1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\| \|\mathbf{w}_S\|_{\mathbf{L}^4(\Omega_S)} \|\mathbf{u}_S\|_{1, \Omega_S} \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} \leq C_{\mathcal{B}_S} \|\mathbf{w}_S\|_{1, \Omega_S} \|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1} \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}, \quad (5.2)$$

with $C_{\mathcal{B}_S} := \frac{\rho}{2\mu} (1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|^2$. In turn, from the definition of \mathcal{A}_D (cf. (3.8)), (2.7), and the triangle and Hölder inequalities, we deduce that there exists $L_{\mathcal{A}_D} > 0$, depending only on $\mu, \rho, \mathbf{F}, \mathbf{K}$ and Ω_D , such that

$$\begin{aligned} & \|\mathcal{A}_D(\mathbf{u}_D) - \mathcal{A}_D(\mathbf{v}_D)\|_{(\mathbf{H}^3(\text{div}; \Omega_D))'} \\ & \leq L_{\mathcal{A}_D} \left\{ \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \left(\|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} + \|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \right) \right\}, \end{aligned} \quad (5.3)$$

for all $\mathbf{u}_D, \mathbf{v}_D \in \mathbf{H}^3(\text{div}; \Omega_D)$. In addition, using the Cauchy–Schwarz and Young’s inequalities, it is not difficult to see that \mathbf{F} and \mathbf{G} are bounded (cf. (3.12)), that is, there exist constants $C_{\mathbf{F}}, C_{\mathbf{G}} > 0$, such that

$$\|\mathbf{F}\|_{\mathbb{H}'} \leq C_{\mathbf{F}} \left\{ \|\mathbf{f}_S\|_{0, \Omega_S} + \|\mathbf{f}_D\|_{\mathbf{L}^{3/2}(\Omega_D)} \right\} \quad (5.4)$$

and

$$\|\mathbf{G}\|_{\mathcal{Q}'} \leq C_{\mathbf{G}} \|g_D\|_{0, \Omega_D}, \quad (5.5)$$

which confirm the announced smoothness of \mathbf{f}_D . On the other hand, from the definition of \mathcal{A}_D (cf. (3.8)), inequality (2.7) and [28, Lemma 5.1] (see [10, Section 2.3] for details), we deduce that there exists $\alpha_D > 0$, depending only on ρ, \mathbf{F} and Ω_D , such that for each $\mathbf{t}_D \in \mathbf{L}^3(\Omega_D)$ there holds

$$[\mathcal{A}_D(\mathbf{u}_D + \mathbf{t}_D) - \mathcal{A}_D(\mathbf{v}_D + \mathbf{t}_D), \mathbf{u}_D - \mathbf{v}_D] \geq \alpha_D \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{L}^3(\Omega_D)}^3 \quad \forall \mathbf{u}_D, \mathbf{v}_D \in \mathbf{L}^3(\Omega_D). \quad (5.6)$$

Finally, we recall that there exist positive constants $C_d(\Omega_S)$ and C_{K_o} , such that (see, [6, Proposition IV.3.1] and [6, 26], respectively, for details)

$$C_d(\Omega_S) \|\boldsymbol{\tau}_S\|_{0, \Omega_S}^2 \leq \|\boldsymbol{\tau}_S^d\|_{0, \Omega_S}^2 + \|\mathbf{div} \boldsymbol{\tau}_S\|_{0, \Omega_S}^2 \quad \forall \boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S) \quad (5.7)$$

and

$$C_{K_0} \|\mathbf{v}_S\|_{1,\Omega_S}^2 \leq \|\mathbf{e}(\mathbf{v}_S)\|_{0,\Omega_S}^2 \quad \forall \mathbf{v}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S). \quad (5.8)$$

Notice that, in particular, (5.8) is known as Korn's inequality. Then, we establish next the ellipticity of the operator \mathcal{A}_S .

Lemma 5.1 *Assume that for $\delta_1 \in (0, 4\mu)$ and $\delta_2 \in (0, 2)$ we choose*

$$\kappa_1 \in (0, +\infty), \quad \kappa_2 \in (0, 2\delta_1) \quad \text{and} \quad \kappa_3 \in \left(0, 2C_{K_0}\kappa_2\delta_2 \left(1 - \frac{\delta_1}{4\mu}\right)\right).$$

Then, there exists a constant $\alpha_S > 0$, such that there holds

$$[\mathcal{A}_S(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] \geq \alpha_S \|\underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}^2 \quad \forall \underline{\boldsymbol{\tau}} \in \mathbf{X}_1. \quad (5.9)$$

Proof. Let $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau}_S, \mathbf{v}_S, \boldsymbol{\eta}_S) \in \mathbf{X}_1$. Then from (3.8) we have that

$$\begin{aligned} [\mathcal{A}_S(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] &= \frac{1}{2\mu} \|\boldsymbol{\tau}_S^d\|_{0,\Omega_S}^2 + \kappa_1 \|\mathbf{div} \boldsymbol{\tau}_S\|_{0,\Omega_S}^2 + \kappa_2 \|\mathbf{e}(\mathbf{v}_S)\|_{0,\Omega_S}^2 + \kappa_3 \|\boldsymbol{\eta}_S\|_{0,\Omega_S}^2 \\ &\quad - \frac{\kappa_2}{2\mu} (\boldsymbol{\tau}_S^d, \mathbf{e}(\mathbf{v}_S))_S - \frac{\kappa_3}{2} (\nabla \mathbf{v}_S - (\nabla \mathbf{v}_S)^t, \boldsymbol{\eta}_S)_S. \end{aligned}$$

Hence, we proceed similarly to the proof of [7, Lemma 3.4] and utilize the Cauchy–Schwarz and Young inequalities to find that for any $\delta_1, \delta_2 > 0$, and for all $\underline{\boldsymbol{\tau}} \in \mathbf{X}_1$, there holds

$$\begin{aligned} [\mathcal{A}_S(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] &\geq \frac{1}{2\mu} \left(1 - \frac{\kappa_2}{2\delta_1}\right) \|\boldsymbol{\tau}_S^d\|_{0,\Omega_S}^2 + \kappa_1 \|\mathbf{div} \boldsymbol{\tau}_S\|_{0,\Omega_S}^2 \\ &\quad + \kappa_2 \left(1 - \frac{\delta_1}{4\mu}\right) \|\mathbf{e}(\mathbf{v}_S)\|_{0,\Omega_S}^2 - \frac{\kappa_3}{2\delta_2} \|\mathbf{v}_S\|_{1,\Omega_S}^2 + \kappa_3 \left(1 - \frac{\delta_2}{2}\right) \|\boldsymbol{\eta}_S\|_{0,\Omega_S}^2. \end{aligned}$$

Then, assuming the stipulated hypotheses on $\delta_1, \delta_2, \kappa_1, \kappa_2$, and κ_3 , and applying the inequalities (5.7) and (5.8), we can define the positive constants

$$\begin{aligned} \alpha_0(\Omega_S) &:= \min \left\{ \frac{1}{2\mu} \left(1 - \frac{\kappa_2}{2\delta_1}\right), \frac{\kappa_1}{2} \right\}, \quad \alpha_1(\Omega_S) := \min \left\{ C_d(\Omega_S) \alpha_0(\Omega_S), \frac{\kappa_1}{2} \right\} \\ \alpha_2(\Omega_S) &:= C_{K_0}\kappa_2 \left(1 - \frac{\delta_1}{4\mu}\right) - \frac{\kappa_3}{2\delta_2}, \quad \alpha_3(\Omega_S) := \kappa_3 \left(1 - \frac{\delta_2}{2}\right), \end{aligned}$$

which allows us to conclude that

$$[\mathcal{A}_S(\boldsymbol{\tau}), \boldsymbol{\tau}] \geq \alpha_S \|\boldsymbol{\tau}\|_{\mathbf{X}_1}^2 \quad \forall \boldsymbol{\tau} \in \mathbf{X}_1,$$

with $\alpha_S := \min \left\{ \alpha_1(\Omega_S), \alpha_2(\Omega_S), \alpha_3(\Omega_S) \right\}$. \square

Now, we recall some properties from [10, Section 3] that will be used later on to prove the inf-sup conditions for the operators \mathbf{b} and \mathbf{B} . Given $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D)$, the boundary condition $\mathbf{v}_D \cdot \mathbf{n} = 0$ on Γ_D means

$$\langle \mathbf{v}_D \cdot \mathbf{n}, E_{0,D}(\xi) \rangle_{\partial\Omega_D} = 0 \quad \forall \xi \in W^{1/3,3/2}(\Gamma_D),$$

where $E_{0,D} : W^{1/3,3/2}(\Gamma_D) \rightarrow W^{1/3,3/2}(\partial\Omega_D)$ is the extension operator defined by

$$E_{0,D}(\xi) := \begin{cases} \xi & \text{on } \Gamma_D \\ 0 & \text{on } \Sigma \end{cases} \quad \forall \xi \in W^{1/3,3/2}(\Gamma_D).$$

Additionally, we can identify the restriction of $\mathbf{v}_D \cdot \mathbf{n}$ to Σ with an element of $W^{-1/3,3}(\Sigma)$, namely

$$\langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_{\Sigma} := \langle \mathbf{v}_D \cdot \mathbf{n}, E_{\Sigma}(\xi) \rangle_{\partial\Omega_D} \quad \forall \xi \in W^{1/3,3/2}(\Sigma), \quad (5.10)$$

where $E_{\Sigma} : W^{1/3,3/2}(\Sigma) \rightarrow W^{1/3,3/2}(\partial\Omega_D)$ is any bounded extension operator. In turn, given $\psi \in W^{1/3,3/2}(\partial\Omega_D)$, there exist unique elements $\psi_{\Sigma} \in W^{1/3,3/2}(\Sigma)$ and $\psi_{\Gamma_D} \in W^{1/3,3/2}(\Gamma_D)$ such that

$$\psi = E_{\Sigma}(\psi_{\Sigma}) + E_{0,D}(\psi_{\Gamma_D}), \quad (5.11)$$

and there exist $C_1, C_2 > 0$, such that

$$C_1 \left\{ \|\psi_{\Sigma}\|_{1/3,3/2;\Sigma} + \|\psi_{\Gamma_D}\|_{1/3,3/2;\Gamma_D} \right\} \leq \|\psi\|_{1/3,3/2;\partial\Omega_D} \leq C_2 \left\{ \|\psi_{\Sigma}\|_{1/3,3/2;\Sigma} + \|\psi_{\Gamma_D}\|_{1/3,3/2;\Gamma_D} \right\}.$$

We end this section by remarking that the explicit expressions yielding the computation of the ellipticity constant α_S of \mathcal{A}_S , can be maximized by taking the parameters $\delta_1, \delta_2, \kappa_2$ and κ_3 as the middle points of their feasible ranges, and by choosing κ_1 so that it maximizes the

minimum defining $\alpha_0(\Omega_S)$. More precisely, we simply take

$$\begin{aligned} \delta_1 &= 2\mu, & \delta_2 &= 1, & \kappa_2 &= \delta_1 = 2\mu, \\ \kappa_3 &= C_{\text{Ko}}\kappa_2\delta_2 \left(1 - \frac{\delta_1}{4\mu}\right) = C_{\text{Ko}}\mu, & \kappa_1 &= \frac{1}{\mu} \left(1 - \frac{\kappa_2}{2\delta_1}\right) = \frac{1}{2\mu}, \end{aligned} \quad (5.12)$$

which yields

$$\alpha_0(\Omega_S) = \frac{1}{4\mu}, \quad \alpha_1(\Omega_S) = \frac{1}{4\mu} \min \{C_d(\Omega_S), 1\}, \quad \alpha_2(\Omega_S) = \alpha_3(\Omega_S) = \frac{C_{\text{Ko}}\mu}{2},$$

and hence

$$\alpha_S = \min \left\{ \frac{1}{4\mu} \min \{C_d(\Omega_S), 1\}, \frac{C_{\text{Ko}}\mu}{2} \right\}.$$

The explicit values of the stabilization parameters $\kappa_i, i \in \{1, 2, 3\}$, given by (5.12), will be employed in Chapter 8 for the corresponding numerical experiments.

5.2 A fixed-point approach

We begin the solvability analysis of (3.5) by defining the operator $\mathbf{T} : \mathbf{H}_{\Gamma_S}^1(\Omega_S) \rightarrow \mathbf{H}_{\Gamma_S}^1(\Omega_S)$ by

$$\mathbf{T}(\mathbf{w}_S) := \mathbf{u}_S \quad \forall \mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S), \quad (5.13)$$

where $\underline{\mathbf{t}} := (\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_S, \mathbf{u}_D)$ is the first component of the unique solution (to be confirmed below) of the nonlinear problem: Find $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{p}}] &= [\mathbf{F}, (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})] \quad \forall (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H}, \\ [\mathbf{B}(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{q}}] &= [\mathbf{G}, \underline{\mathbf{q}}] \quad \forall \underline{\mathbf{q}} \in \mathbb{Q}. \end{aligned} \quad (5.14)$$

Hence, it is not difficult to see that $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ is a solution of (3.5) if and only if $\mathbf{u}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$ is a fixed-point of \mathbf{T} , that is

$$\mathbf{T}(\mathbf{u}_S) = \mathbf{u}_S. \quad (5.15)$$

In this way, in what follows we focus on proving that \mathbf{T} possesses a unique fixed-point. However, we remark in advance that the definition of \mathbf{T} will make sense only in a closed ball of $\mathbf{H}_{\Gamma_S}^1(\Omega_S)$.

Before continuing with the solvability analysis of (5.15) (equivalently of (3.5)), we provide the hypotheses under which \mathbf{T} is well defined. To that end, we first note that, given $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$, the problem (5.14) has the same structure of the one in Theorem 4.7. Therefore, in what follows we apply this abstract result to establish the well-posedness of (5.14), or equivalently, the well-definiteness of \mathbf{T} . We begin by observing that, thanks to the uniform convexity and separability of $L^p(\Omega)$ for $p \in (1, +\infty)$, each space defining \mathbb{H} and \mathbb{Q} shares the same properties, which implies that \mathbb{H} and \mathbb{Q} are uniformly convex and separable as well.

We continue our analysis by proving that hypothesis (i) of Theorem 4.7 is verified with $p_1 = 2$ and $p_2 = 3$.

Lemma 5.2 *Let $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$. Then, there exists $\gamma > 0$, depending on C_{A_S}, C_{B_S} and L_{A_D} (cf. (5.1), (5.2), (5.3)), such that*

$$\begin{aligned} \|\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{r}})\|_{\mathbf{X}'} &\leq \gamma \left\{ \left(1 + \|\mathbf{w}_S\|_{1, \Omega_S}\right) \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1} + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \right. \\ &\quad \left. + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \left(\|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} + \|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \right) \right\}, \end{aligned} \quad (5.16)$$

for all $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_D), \underline{\mathbf{r}} = (\underline{\boldsymbol{\tau}}, \mathbf{v}_D) \in \mathbf{X}$.

Proof. The result follows straightforwardly from the definition of $\mathbf{a}(\mathbf{w}_S)$ (cf. (3.7)), the triangle inequality, and the stability properties (5.1), (5.2) and (5.3). We omit further details. \square

Now, let us look at the kernel of the operator \mathbf{B} (cf. (3.11)), which can be written, equivalently, as

$$\mathbb{V} = \left\{ (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H} : [\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}] = 0 \quad \forall \underline{\mathbf{q}} \in \mathbb{Q} \right\} = \tilde{\mathbf{X}} \times \tilde{\mathbf{Y}}, \quad (5.17)$$

where

$$\tilde{\mathbf{X}} = \mathbf{X}_1 \times \tilde{\mathbf{X}}_2 \quad \text{and} \quad \tilde{\mathbf{Y}} = \tilde{\mathbf{H}}_{00}^{1/2}(\Sigma) \times W^{1/3, 3/2}(\Sigma),$$

with

$$\tilde{\mathbf{X}}_2 = \left\{ \mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D) : \text{div}(\mathbf{v}_D) \in P_0(\Omega_D) \right\}$$

and

$$\tilde{\mathbf{H}}_{00}^{1/2}(\Sigma) := \left\{ \boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) : \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_{\Sigma} = 0 \right\}.$$

In addition, from a slight adaptation of [2, Lemma 4.2], we deduce that there exist a constant $C_{\text{div}} > 0$ such that

$$C_{\text{div}} \|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)}^3 \leq \|\mathbf{v}_D\|_{\mathbf{L}^3(\Omega_D)}^3 \quad \forall \mathbf{v}_D \in \tilde{\mathbf{X}}_2. \quad (5.18)$$

Thus, in the following result we provide the assumptions under which operator $\mathbf{a}(\mathbf{w}_S)$ satisfies hypothesis (ii) of Theorem 4.7.

Lemma 5.3 *Suppose that the parameters $\kappa_1, \kappa_2, \kappa_3$, satisfy the conditions required by Lemma 5.1, and let $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$ be such that $\|\mathbf{w}_S\|_{1, \Omega_S} \leq r$ with $r \in (0, r_0)$, and*

$$r_0 := \frac{\alpha_S \mu}{\rho(1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|^2}, \quad (5.19)$$

where $\|\mathbf{i}_c\|$ is the constant in (3.2) and α_S is the ellipticity constant of the operator \mathcal{A}_S (cf. (5.9)). Then, for each $\underline{\mathbf{s}} \in \mathbf{X}$, the nonlinear operator $\mathbf{a}(\mathbf{w}_S)(\cdot + \underline{\mathbf{s}})$ is strictly monotone on $\tilde{\mathbf{X}}$ (cf. (5.17)).

Proof. Let $\underline{\mathbf{s}} := (\underline{\boldsymbol{\zeta}}, \mathbf{s}_D) \in \mathbf{X}$ fixed, and let $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$ as indicated. Then, according to the definition of $\mathbf{a}(\mathbf{w}_S)$ (cf. (3.7)), the linearity of \mathcal{A}_S and $\mathcal{B}_S(\mathbf{w}_S)$, and combining (5.9), (5.6) and (5.18), we deduce that

$$\begin{aligned} [\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}} + \underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{r}} + \underline{\mathbf{s}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] &\geq \alpha_S \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}^2 \\ &+ \alpha_D C_{\text{div}} \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)}^3 + [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}), (\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}})], \end{aligned}$$

for all $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_D)$, $\underline{\mathbf{r}} = (\underline{\boldsymbol{\tau}}, \mathbf{v}_D) \in \tilde{\mathbf{X}}$. In addition, similarly to [11, Lemma 3.1], we know from the second inequality in (5.2) that

$$\left| [\mathcal{B}_S(\mathbf{w}_S)(\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}] \right| \leq \frac{\rho}{2\mu} (1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|^2 \|\mathbf{w}_S\|_{1, \Omega_S} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}^2,$$

which implies

$$\begin{aligned} &[\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}} + \underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{r}} + \underline{\mathbf{s}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \\ &\geq \left\{ \alpha_S - \frac{\rho}{2\mu} (1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|^2 \|\mathbf{w}_S\|_{1, \Omega_S} \right\} \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\mathbf{X}_1}^2 + \alpha_D C_{\text{div}} \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div}; \Omega_D)}^3. \end{aligned}$$

Consequently, by requiring $\|\mathbf{w}_S\|_{1,\Omega_S} \leq r_0$, with r_0 defined by (5.19), the foregoing inequality imply

$$[\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}} + \underline{\mathbf{s}}) - \mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{r}} + \underline{\mathbf{s}}), \underline{\mathbf{t}} - \underline{\mathbf{r}}] \geq \alpha \left\{ \|\underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\tau}}\|_{\tilde{\mathbf{X}}_1}^2 + \|\mathbf{u}_D - \mathbf{v}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)}^3 \right\}, \quad (5.20)$$

for all $\underline{\mathbf{t}}, \underline{\mathbf{r}} \in \tilde{\mathbf{X}}$, with $\alpha := \min \left\{ \frac{\alpha_S}{2}, \alpha_D C_{\text{div}} \right\}$ independent of \mathbf{w}_S . \square

We end the verification of the hypotheses of Theorem 4.7, with the positive semi-definiteness of \mathbf{c} and corresponding inf-sup conditions for the operators \mathbf{b} and \mathbf{B} , respectively.

Lemma 5.4 *There holds*

$$[\mathbf{c}(\underline{\boldsymbol{\psi}}), \underline{\boldsymbol{\psi}}] \geq 0 \quad \forall \underline{\boldsymbol{\psi}} \in \mathbf{Y}. \quad (5.21)$$

Proof. From the definition of operator \mathbf{c} , it readily follows that

$$[\mathbf{c}(\underline{\boldsymbol{\psi}}), \underline{\boldsymbol{\psi}}] := \sum_{i=1}^{n-1} \omega_i^{-1} \|\boldsymbol{\psi} \cdot \mathbf{t}_i\|_{0,\Sigma}^2 \geq 0 \quad \forall \underline{\boldsymbol{\psi}} \in \mathbf{Y},$$

which clearly confirms that \mathbf{c} is positive semi-definite. \square

Lemma 5.5 *There exists $\beta_1 > 0$, such that*

$$\sup_{\substack{\underline{\mathbf{r}} \in \tilde{\mathbf{X}} \\ \underline{\mathbf{r}} \neq \mathbf{0}}} \frac{[\mathbf{b}(\underline{\mathbf{r}}), \underline{\boldsymbol{\psi}}]}{\|\underline{\mathbf{r}}\|_{\mathbf{X}}} \geq \beta_1 \|\underline{\boldsymbol{\psi}}\|_{\mathbf{Y}} \quad \forall \underline{\boldsymbol{\psi}} \in \tilde{\mathbf{Y}}. \quad (5.22)$$

Proof. By using the diagonal character of the operator \mathbf{b} (cf. (3.9)), we find that (5.22) is equivalent to the following two independent inequalities,

$$\sup_{\substack{\boldsymbol{\tau}_S \in \mathbb{H}_0(\text{div};\Omega_S) \\ \boldsymbol{\tau}_S \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_S \mathbf{n}, \boldsymbol{\psi} \rangle_{\Sigma}}{\|\boldsymbol{\tau}_S\|_{\text{div};\Omega_S}} \geq \beta_{1,S} \|\boldsymbol{\psi}\|_{1/2,0;\Sigma} \quad \forall \boldsymbol{\psi} \in \tilde{\mathbf{H}}_{00}^{1/2}(\Sigma) \quad (5.23)$$

and

$$\sup_{\substack{\mathbf{v}_D \in \tilde{\mathbf{X}}_2 \\ \mathbf{v}_D \neq \mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \boldsymbol{\xi} \rangle_{\Sigma}}{\|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div};\Omega)}} \geq \beta_{1,D} \|\boldsymbol{\xi}\|_{1/3,3/2;\Sigma} \quad \forall \boldsymbol{\xi} \in W^{1/3,3/2}(\Sigma). \quad (5.24)$$

Similarly to [24, Lemma 3.6], we consider $\boldsymbol{\tau}_S = \boldsymbol{\tau}_{S,0} + c\mathbb{I} \in \mathbb{H}(\text{div};\Omega_S)$ with $\boldsymbol{\tau}_{S,0} \in \mathbb{H}_0(\text{div};\Omega_S)$

and $c \in P_0(\Omega_S)$. Thus, it is clear that $\langle \tau_S \mathbf{n}, \psi \rangle_\Sigma = \langle \tau_{S,0} \mathbf{n}, \psi \rangle_\Sigma$ for all $\psi \in \tilde{\mathbf{H}}_{00}^{1/2}(\Sigma)$, and $\|\tau_S\|_{\text{div};\Omega_S}^2 = \|\tau_{S,0}\|_{\text{div};\Omega_S}^2 + n c^2 |\Omega_S|$. Hence, it is enough to prove the supremum (5.23) considering τ_S living in $\mathbb{H}(\text{div}; \Omega_S)$ instead $\mathbb{H}_0(\text{div}; \Omega_S)$. In fact, given $\chi \in \mathbf{H}_{00}^{-1/2}(\Sigma)$ we let $\mathbf{z} \in \mathbf{H}^1(\Omega_S)$ as the unique solution of the boundary value problem

$$-\Delta \mathbf{z} = 0 \quad \text{in } \Omega_S, \quad \mathbf{z} = \mathbf{0} \quad \text{on } \Gamma_S, \quad \nabla \mathbf{z} \mathbf{n} = \chi \quad \text{on } \Sigma.$$

Next, we set $\tilde{\tau}_S = \nabla \mathbf{z}$ in Ω_S , and observe that $\text{div} \tilde{\tau}_S = \mathbf{0}$ in Ω_S , $\tilde{\tau}_S \mathbf{n} = \chi$ on Σ and $\|\tilde{\tau}_S\|_{\text{div};\Omega_S} \leq C \|\chi\|_{\mathbf{H}_{00}^{-1/2}(\Sigma)}$. It follows that

$$\sup_{\substack{\tau_S \in \mathbb{H}(\text{div};\Omega_S) \\ \tau_S \neq \mathbf{0}}} \frac{\langle \tau_S \mathbf{n}, \psi \rangle_\Sigma}{\|\tau_S\|_{\text{div};\Omega_S}} \geq \frac{\langle \tilde{\tau}_S \mathbf{n}, \psi \rangle_\Sigma}{\|\tilde{\tau}_S\|_{\text{div};\Omega_S}} \geq \frac{1}{C} \frac{\langle \chi, \psi \rangle_\Sigma}{\|\chi\|_{\mathbf{H}_{00}^{-1/2}(\Sigma)}},$$

which, considering that $\chi \in \mathbf{H}_{00}^{-1/2}(\Sigma)$ is arbitrary, yields (5.23) with $\beta_{1,S} = 1/C$. On the other hand, in order to obtain (5.24) we proceed analogously to [10, Lemma 3.5]. In fact, given $\phi \in W^{-1/3,3}(\Sigma)$, we define $\eta \in W^{-1/3,3}(\partial\Omega_D)$ as

$$\langle \eta, \psi \rangle_{\partial\Omega_D} = \langle \phi, \psi_\Sigma \rangle_\Sigma \quad \forall \psi \in W^{1/3,3/2}(\partial\Omega_D),$$

where $\psi_\Sigma \in W^{1/3,3/2}(\Sigma)$ is given according to the decomposition (5.11). It is not difficult to see that η is the null extension of ϕ outside Σ , that is

$$\begin{aligned} \langle \eta, E_{0,D}(\varrho) \rangle_{\partial\Omega_D} &= 0 \quad \forall \varrho \in W^{1/3,3/2}(\Gamma_D), \\ \langle \eta, E_\Sigma(\varrho) \rangle_{\partial\Omega_D} &= \langle \phi, \varrho \rangle_\Sigma \quad \forall \varrho \in W^{1/3,3/2}(\Sigma) \end{aligned} \tag{5.25}$$

and

$$\|\eta\|_{-1/3,3;\partial\Omega_D} \leq C \|\phi\|_{-1/3,3;\Sigma}. \tag{5.26}$$

Next, we set $\tilde{\mathbf{v}}_D := \nabla z \in \Omega_D$, with $z \in W^{1,3}(\Omega_D)$ being the unique solution of the boundary value problem

$$\Delta z = \frac{1}{|\Omega_D|} \langle \eta, 1 \rangle_{\partial\Omega_D} \quad \text{in } \Omega_D, \quad \nabla z \cdot \mathbf{n} = \eta \quad \text{on } \partial\Omega_D, \quad (z, 1)_D = 0.$$

It follows that $\text{div} \tilde{\mathbf{v}}_D = \frac{1}{|\Omega_D|} \langle \eta, 1 \rangle_{\partial\Omega_D} \in P_0(\Omega_D)$, $\tilde{\mathbf{v}}_D \cdot \mathbf{n} = \eta$ on $\partial\Omega_D$, and using (5.26) we find

that

$$\|\tilde{\mathbf{v}}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)} \leq c \|\eta\|_{-1/3,3;\partial\Omega_D} \leq \tilde{C} \|\phi\|_{-1/3,3;\Sigma}.$$

In addition, employing (5.10) and (5.25), we deduce that

$$\langle \tilde{\mathbf{v}}_D \cdot \mathbf{n}, \xi \rangle_\Sigma = \langle \tilde{\mathbf{v}}_D \cdot \mathbf{n}, E_\Sigma(\xi) \rangle_{\partial\Omega_D} = \langle \eta, E_\Sigma(\xi) \rangle_{\partial\Omega_D} = \langle \phi, \xi \rangle_\Sigma \quad \forall \xi \in W^{1/3,3/2}(\Sigma)$$

and

$$\langle \tilde{\mathbf{v}}_D \cdot \mathbf{n}, E_{0,D}(\varrho) \rangle_{\partial\Omega_D} = \langle \eta, E_{0,D}(\varrho) \rangle_{\partial\Omega_D} = 0 \quad \forall \varrho \in W^{1/3,3/2}(\Gamma_D),$$

the latter meaning that $\tilde{\mathbf{v}}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div};\Omega_D)$. Thus we obtain

$$\sup_{\substack{\mathbf{v}_D \in \tilde{\mathbf{X}}_2 \\ \mathbf{v}_D \neq \mathbf{0}}} \frac{\langle \mathbf{v}_D \cdot \mathbf{n}, \xi \rangle_\Sigma}{\|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div};\Omega)}} \geq \frac{|\langle \tilde{\mathbf{v}}_D \cdot \mathbf{n}, \xi \rangle_\Sigma|}{\|\tilde{\mathbf{v}}_D\|_{\mathbf{H}^3(\text{div};\Omega)}} \geq \frac{1}{\tilde{C}} \frac{|\langle \phi, \xi \rangle_\Sigma|}{\|\phi\|_{-1/3,3;\Sigma}}$$

for all $\phi \in W^{-1/3,3}(\Sigma)$, from which we conclude (5.24). \square

Lemma 5.6 *There exists $\beta > 0$ such that*

$$\sup_{\substack{(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \in \mathbb{H} \\ (\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}) \neq \mathbf{0}}} \frac{[\mathbf{B}(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}}), \underline{\mathbf{q}}]}{\|(\underline{\mathbf{r}}, \underline{\boldsymbol{\psi}})\|_{\mathbb{H}}} \geq \beta \|\underline{\mathbf{q}}\|_{\mathbb{Q}} \quad \forall \underline{\mathbf{q}} \in \mathbb{Q}. \quad (5.27)$$

Proof. Analogously to Lemma 5.5, we have that (5.27) is equivalent to:

$$\sup_{\substack{\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) \\ \boldsymbol{\psi} \neq \mathbf{0}}} \frac{j \langle \boldsymbol{\psi} \cdot \mathbf{n}, 1 \rangle_\Sigma}{\|\boldsymbol{\psi}\|_{1/2,00;\Sigma}} \geq \beta_j |j| \quad \forall j \in \mathbb{R}, \quad (5.28)$$

and

$$\sup_{\substack{\mathbf{v}_D \in \mathbf{X}_2 \\ \mathbf{v}_D \neq \mathbf{0}}} \frac{(\text{div } \mathbf{v}_D, q_D)_D}{\|\mathbf{v}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)}} \geq \beta_D \|q_D\|_{0,\Omega_D} \quad \forall q_D \in L_0^2(\Omega_D). \quad (5.29)$$

Notice that (5.28) reduces to proving that there exists $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ such that $\langle \boldsymbol{\psi}_0 \cdot \mathbf{n}, 1 \rangle_\Sigma \neq 0$. To that end, we proceed similarly to [24, Lemma 3.6] (see also [8, Lemma 3.2]). In fact, for the two dimensional case, and given x_1 and x_2 two points on $\partial\Omega_S$ such that the line $[x_1, x_2] \subseteq \Sigma$, we let x_m be the midpoint of $[x_1, x_2]$, and let x_3 be a point on $\partial\Omega_S$ or in the interior of Ω_S , in

such a way that two triangles T_1 and T_2 can be constructed. Then, we let v be continuous, linear on each side of Σ , equal to one in the vertex x_m , and zero on all other ones, and define $\psi_0 := v\mathbf{n}$, where \mathbf{n} is the normal vector on $[x_1, x_2]$. Similarly, for the three dimensional case, given x_1, x_2, x_3 three points on Σ defining a triangle with barycentric x_m , we let x_4 be a point in the interior of Ω_S or on an opposite piece of boundary, chosen in such a way that below x_4 three tetrahedra T_1, T_2, T_3 can be formed. Then, we let v be continuous, linear on each side of Σ , equal to one in the vertex x_m , and zero on all other ones, and define $\psi_0 := v\mathbf{n}$, where \mathbf{n} is the normal vector on $T_1 \cup T_2 \cup T_3$. Next, in order to prove (5.29), we proceed as in [10, Lemma 3.5] (see also [2, Lemma 4.3]). In fact, given $q_D \in L^2_0(\Omega_D)$, we set $\tilde{\mathbf{v}}_D := \nabla z$ in Ω_D , with $z \in W^{1,3}(\Omega_D)$ being the unique solution of the boundary value problem:

$$\Delta z = q_D \quad \text{in } \Omega_D, \quad \nabla z \cdot \mathbf{n} = 0 \quad \text{on } \partial\Omega_D \quad \text{and} \quad (z, 1)_D = 0.$$

It follows that $\operatorname{div} \tilde{\mathbf{v}}_D = q_D$ in Ω_D and $\tilde{\mathbf{v}}_D \cdot \mathbf{n} = 0$ on Γ_D , whence $\tilde{\mathbf{v}}_D \in \mathbf{H}^3_{\Gamma_D}(\operatorname{div}; \Omega_D)$ and satisfies

$$\|\tilde{\mathbf{v}}_D\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)} \leq C_D \|q_D\|_{0, \Omega_D}.$$

Thus, from the above arguments we deduce

$$\sup_{\substack{\mathbf{v}_D \in \mathbf{H}^3_{\Gamma_D}(\operatorname{div}; \Omega_D) \\ \mathbf{v}_D \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_D, q_D)_D}{\|\mathbf{v}_D\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)}} \geq \frac{(\operatorname{div} \tilde{\mathbf{v}}_D, q_D)_D}{\|\tilde{\mathbf{v}}_D\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)}} \geq \frac{1}{C_D} \|q_D\|_{0, \Omega_D},$$

which shows (5.29) with $\beta_D = 1/C_D$ and completes the proof. \square

We are now in position of establishing the well-definiteness of \mathbf{T} . To that end, and in order to simplify the subsequent analysis, we first note that, given $\mathbf{w}_S \in \mathbf{H}^1_{\Gamma_S}(\Omega_S)$, there holds $\|\mathbf{a}(\mathbf{w}_S)(\mathbf{0})\|_{\mathbf{X}'} = 0$. Then, by considering $p_1 = 2$ and $p_2 = 3$ in Theorem 4.7, we introduce the following notation

$$\begin{aligned} \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) := & \max \left\{ \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D)^{1/8}, \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D)^{1/4}, \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D)^{1/2}, \right. \\ & \left. \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D), \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D)^2, \mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D)^4 \right\}, \end{aligned}$$

with

$$\mathcal{N}(\mathbf{f}_S, \mathbf{f}_D, g_D) := \|\mathbf{f}_S\|_{0,\Omega_S} + \|\mathbf{f}_D\|_{\mathbf{L}^{3/2}(\Omega_D)} + \|g_D\|_{0,\Omega_D} + \|g_D\|_{0,\Omega_D}^2.$$

The main result of this section is established now.

Lemma 5.7 *Suppose that the parameters $\kappa_1, \kappa_2, \kappa_3$, satisfy the conditions required by Lemma 5.1. Let $r \in (0, r_0)$, with r_0 given by (5.19) and let $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$, $\mathbf{f}_D \in \mathbf{L}^{3/2}(\Omega_D)$ and $g_D \in L_0^2(\Omega_D)$. Then, the problem (5.14) has a unique solution $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ for each $\mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S)$ such that $\|\mathbf{w}_S\| \leq r$. Moreover, there exists a constant $c_T > 0$, independent of \mathbf{w}_S and the data $\mathbf{f}_S, \mathbf{f}_D$, and g_D , such that*

$$\|\mathbf{T}(\mathbf{w}_S)\|_{1,\Omega_S} = \|\mathbf{u}_S\|_{1,\Omega_S} \leq \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{H} \times \mathbb{Q}} \leq c_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \quad (5.30)$$

Proof. It follows from Lemmas 5.2–5.6 and a straightforward application of Theorem 4.7. In turn, estimate (5.30) is a direct consequence of (4.22) (cf. Theorem 4.7) and (5.4)-(5.5). \square

5.3 Solvability analysis of the fixed-point equation

In this section we proceed analogously to [7, Section 3.3] (see also [15, 11, 10]), and establish existence of a fixed point of the operator \mathbf{T} (cf. (5.13)) by means of the well known Schauder fixed-point theorem. The uniqueness can then be established by means of the Banach fixed-point theorem by utilizing the same estimates derived for the existence. We begin by recalling the first of the aforementioned results (see, e.g., [12, Theorem 9.12-1(b)]).

Theorem 5.8 *Let W be a closed and convex subset of a Banach space X , and let $T : W \rightarrow W$ be a continuous mapping such that $\overline{T(W)}$ is compact. Then T has at least one fixed point.*

The verification of the hypotheses of Theorem 5.8 is provided next.

Lemma 5.9 *Let $r \in (0, r_0)$, with r_0 given by (5.19), let \mathbf{W}_r be the closed ball defined by*

$$\mathbf{W}_r := \left\{ \mathbf{w}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S) : \|\mathbf{w}_S\|_{1,\Omega_S} \leq r \right\}, \quad (5.31)$$

and assume that the data satisfy

$$c_{\mathbf{T}} \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) \leq r, \quad (5.32)$$

with $c_{\mathbf{T}}$ the positive constant satisfying (5.30). Then there holds $\mathbf{T}(\mathbf{W}_r) \subseteq \mathbf{W}_r$.

Proof. It is straightforward consequence of Lemma 5.7. \square

We continue with the following results providing an estimate needed to derive the required continuity and compactness properties of the operator \mathbf{T} (cf. (5.13)).

Lemma 5.10 *Let $r \in (0, r_0)$, with r_0 given by (5.19), and let \mathbf{W}_r given by (5.31). Then there exists a positive constant $C_{\mathbf{T}}$, depending on κ_2 , $\|\mathbf{i}_c\|$, and α_S (cf. (5.9)), such that*

$$\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\tilde{\mathbf{w}}_S)\|_{1, \Omega_S} \leq C_{\mathbf{T}} \|\mathbf{T}(\tilde{\mathbf{w}}_S)\|_{1, \Omega_S} \|\mathbf{w}_S - \tilde{\mathbf{w}}_S\|_{\mathbf{L}^4(\Omega_S)} \quad \forall \mathbf{w}_S, \tilde{\mathbf{w}}_S \in \mathbf{W}_r. \quad (5.33)$$

Proof. Given $\mathbf{w}_S, \tilde{\mathbf{w}}_S \in \mathbf{W}_r$, we let $\mathbf{u}_S := \mathbf{T}(\mathbf{w}_S)$ and $\tilde{\mathbf{u}}_S := \mathbf{T}(\tilde{\mathbf{w}}_S)$. According to the definition of \mathbf{T} (cf. (5.14)), it follows that

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_S)(\underline{\mathbf{t}}, \underline{\varphi}) - \mathbf{A}(\tilde{\mathbf{w}}_S)(\tilde{\underline{\mathbf{t}}}, \tilde{\underline{\varphi}}), (\underline{\mathbf{r}}, \underline{\psi})] + [\mathbf{B}(\underline{\mathbf{r}}, \underline{\psi}), \underline{\mathbf{p}} - \tilde{\underline{\mathbf{p}}}] &= 0 \quad \forall (\underline{\mathbf{r}}, \underline{\psi}) \in \mathbb{H}, \\ [\mathbf{B}(\underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}, \underline{\varphi} - \tilde{\underline{\varphi}}), \underline{\mathbf{q}}] &= 0 \quad \forall \underline{\mathbf{q}} \in \mathbb{Q}. \end{aligned}$$

In particular, taking $\underline{\mathbf{r}} = \underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}$, $\underline{\psi} = \underline{\varphi} - \tilde{\underline{\varphi}}$, and $\underline{\mathbf{q}} = \underline{\mathbf{p}} - \tilde{\underline{\mathbf{p}}}$ in the latter system, and recalling the definition of \mathbf{A} (cf. (3.6)), we get

$$\begin{aligned} [\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\tilde{\mathbf{w}}_S)(\tilde{\underline{\mathbf{t}}}, \underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}) + [\mathbf{b}(\underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}), \underline{\varphi} - \tilde{\underline{\varphi}}] &= 0, \\ [\mathbf{b}(\underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}), \underline{\varphi} - \tilde{\underline{\varphi}}] - [\mathbf{c}(\underline{\varphi} - \tilde{\underline{\varphi}}), \underline{\varphi} - \tilde{\underline{\varphi}}] &= 0, \end{aligned}$$

which clearly implies

$$[\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\tilde{\mathbf{w}}_S)(\tilde{\underline{\mathbf{t}}}, \underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}) = -[\mathbf{c}(\underline{\varphi} - \tilde{\underline{\varphi}}), \underline{\varphi} - \tilde{\underline{\varphi}}], \quad (5.34)$$

where $\underline{\mathbf{t}} = (\underline{\boldsymbol{\sigma}}, \mathbf{u}_D)$ and $\tilde{\underline{\mathbf{t}}} = (\tilde{\boldsymbol{\sigma}}, \tilde{\mathbf{u}}_D)$. Hence, adding and subtracting $\mathcal{B}_S(\mathbf{w}_S)(\tilde{\boldsymbol{\sigma}})$ in the second term of the left-hand side of (5.34), noting that $\underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}} \in \tilde{\mathbf{X}}$, and using the strict monotonicity

of $\mathbf{a}(\mathbf{w}_S)$ (cf. (5.20)) and the fact that \mathbf{c} is positive semi-definite (cf. (5.21)), it follows that

$$\frac{\alpha_S}{2} \|\underline{\boldsymbol{\sigma}} - \tilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 \leq [\mathbf{a}(\mathbf{w}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{w}_S)(\tilde{\underline{\mathbf{t}}}), \underline{\mathbf{t}} - \tilde{\underline{\mathbf{t}}}] \leq [\mathcal{B}_S(\tilde{\mathbf{w}}_S - \mathbf{w}_S)(\tilde{\underline{\boldsymbol{\sigma}}}), \underline{\boldsymbol{\sigma}} - \tilde{\underline{\boldsymbol{\sigma}}}].$$

In this way, by applying the first inequality in (5.2) we deduce that

$$\|\underline{\boldsymbol{\sigma}} - \tilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 \leq \frac{\rho(1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|}{\alpha_S \mu} \|\tilde{\mathbf{u}}_S\|_{1, \Omega_S} \|\mathbf{w}_S - \tilde{\mathbf{w}}_S\|_{\mathbf{L}^4(\Omega_S)} \|\underline{\boldsymbol{\sigma}} - \tilde{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1},$$

which implies (5.33) with

$$C_{\mathbf{T}} := \frac{\rho(1 + \kappa_2^2)^{1/2} \|\mathbf{i}_c\|}{\alpha_S \mu}, \quad (5.35)$$

thus completing the proof. \square

Owing to the above analysis, we establish now the announced properties of the operator \mathbf{T} .

Lemma 5.11 *Given $r \in (0, r_0)$, with r_0 defined by (5.19), we let \mathbf{W}_r as in (5.31) and assume that the data $\mathbf{f}_S, \mathbf{f}_D$ and g_D satisfy (5.32). Then, $\mathbf{T} : \mathbf{W}_r \rightarrow \mathbf{W}_r$ is continuous and $\overline{\mathbf{T}(\mathbf{W}_r)}$ is compact.*

Proof. We proceed analogously to [7, Lemma 3.8]. In fact, from (5.33) and the continuity of the injection $\mathbf{i}_c : \mathbf{H}^1(\Omega_S) \rightarrow \mathbf{L}^4(\Omega_S)$, it follows that

$$\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\tilde{\mathbf{w}}_S)\|_{1, \Omega_S} \leq C_{\mathbf{T}} \|\mathbf{i}_c\| \|\mathbf{T}(\tilde{\mathbf{w}}_S)\|_{1, \Omega_S} \|\mathbf{w}_S - \tilde{\mathbf{w}}_S\|_{1, \Omega_S} \quad \forall \mathbf{w}_S, \tilde{\mathbf{w}}_S \in \mathbf{W}_r,$$

which proves the continuity of \mathbf{T} . Now, given a sequence $\{\mathbf{w}_{S,k}\}_{k \in \mathbb{R}}$ of \mathbf{W}_r , which is clearly bounded, there exists a subsequence $\{\mathbf{w}_{S,k}^{(1)}\}_{k \in \mathbb{R}} \subseteq \{\mathbf{w}_{S,k}\}_{k \in \mathbb{R}}$ and \mathbf{w}_S in $\mathbf{H}^1(\Omega_S)$ such that $\mathbf{w}_{S,k}^{(1)} \rightharpoonup \mathbf{w}_S$ in $\mathbf{H}^1(\Omega_S)$. In this way, thanks to the compactness of \mathbf{i}_c , we deduce that $\mathbf{w}_{S,k}^{(1)} \rightarrow \mathbf{w}_S$ in $\mathbf{L}^4(\Omega_S)$, which, combined with (5.33), implies that $\mathbf{T}(\mathbf{w}_{S,k}^{(1)}) \rightarrow \mathbf{T}(\mathbf{w}_S)$ in $\mathbf{H}^1(\Omega_S)$. This proves that $\overline{\mathbf{T}(\mathbf{W}_r)}$ is compact. \square

We are now in position of establishing the main result of this chapter.

Theorem 5.12 *Suppose that the parameters $\kappa_1, \kappa_2, \kappa_3$ satisfy the conditions required by Lemma 5.1. In addition, given $r \in (0, r_0)$, with r_0 defined by (5.19), we let \mathbf{W}_r as in (5.31), and assume that the*

data $\mathbf{f}_S, \mathbf{f}_D$ and g_D satisfy (5.32). Then, the augmented fully-mixed formulation (3.5) has a unique solution $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ with $\mathbf{u}_S \in \mathbf{W}_r$, and there holds

$$\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})\|_{\mathbb{H} \times \mathbb{Q}} \leq c_{\mathbf{T}} \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \quad (5.36)$$

Proof. The equivalence between (3.5) and the fixed-point equation (5.13), together with Lemmas 5.9 and 5.11, confirm the existence of solution of (3.5) as a direct application of the Schauder fixed-point Theorem 5.8. In addition, it is clear that the estimate (5.36) follows straightforwardly from (5.30). On the other hand, from (5.33), the continuity of the compact injection \mathbf{i}_c and the definition of $C_{\mathbf{T}}$ (cf. (5.35)), we obtain

$$\|\mathbf{T}(\mathbf{w}_S) - \mathbf{T}(\tilde{\mathbf{w}}_S)\|_{1, \Omega_S} \leq \frac{r}{r_0} \|\mathbf{w}_S - \tilde{\mathbf{w}}_S\|_{1, \Omega_S} \quad \forall \mathbf{w}_S, \tilde{\mathbf{w}}_S \in \mathbf{W}_r,$$

which, thanks to the Banach fixed-point theorem, implies that the solution is actually unique. □

The Galerkin scheme

In this chapter we introduce the Galerkin scheme of problem (3.5) and analyze its well-posedness by establishing suitable assumptions on the finite element subspaces involved.

6.1 Preliminaries

We first consider a set of arbitrary discrete subspaces, namely

$$\begin{aligned} \mathbf{H}_h(\Omega_S) &\subseteq \mathbf{H}(\operatorname{div}; \Omega_S), & \mathbf{H}_h^1(\Omega_S) &\subseteq \mathbf{H}^1(\Omega_S), & \mathbb{L}_h(\Omega_S) &\subseteq \mathbb{L}_{\text{skew}}^2(\Omega_S), \\ \mathbf{H}_h(\Omega_D) &\subseteq \mathbf{H}^3(\operatorname{div}; \Omega_D), & \Lambda_h^S(\Sigma) &\subseteq \mathbf{H}_{00}^{1/2}(\Sigma), & \Lambda_h^D(\Sigma) &\subseteq W^{1/3, 3/2}(\Sigma), & L_h(\Omega_D) &\subseteq L^2(\Omega_D) \end{aligned} \quad (6.1)$$

and set

$$\begin{aligned} \mathbb{H}_h(\Omega_S) &:= \left\{ \boldsymbol{\tau}_S \in \mathbb{H}(\operatorname{div}; \Omega_S) : \mathbf{c}^t \boldsymbol{\tau} \in \mathbf{H}_h(\Omega_S) \quad \forall \mathbf{c} \in \mathbb{R}^n \right\}, & \Lambda_h^S(\Sigma) &:= [\Lambda_h^S(\Sigma)]^n, \\ \mathbb{H}_{h,0}(\Omega_S) &:= \mathbb{H}_h(\Omega_S) \cap \mathbb{H}_0(\operatorname{div}; \Omega_S), & \mathbf{H}_{h,\Gamma_S}^1(\Omega_S) &:= \mathbf{H}_h^1(\Omega_S) \cap \mathbf{H}_{\Gamma_S}^1(\Omega_S), & (6.2) \\ \mathbf{H}_{h,\Gamma_D}(\Omega_D) &:= \mathbf{H}_h(\Omega_D) \cap \mathbf{H}_{\Gamma_D}^3(\operatorname{div}; \Omega_D), & L_{h,0}(\Omega_D) &:= L_h(\Omega_D) \cap L_0^2(\Omega_D). \end{aligned}$$

Then, defining the global subspaces, unknowns, and test functions as follows

$$\begin{aligned}
 \mathbf{X}_{h,1} &:= \mathbb{H}_{h,0}(\Omega_S) \times \mathbf{H}_{h,\Gamma_S}^1(\Omega_S) \times \mathbb{L}_h(\Omega_S), & \mathbf{X}_{h,2} &:= \mathbf{H}_{h,\Gamma_D}(\Omega_D), \\
 \mathbf{X}_h &:= \mathbf{X}_{h,1} \times \mathbf{X}_{h,2}, & \mathbf{Y}_h &:= \Lambda_h^S(\Sigma) \times \Lambda_h^D(\Sigma), \\
 \mathbb{H}_h &:= \mathbf{X}_h \times \mathbf{Y}_h & \text{and} & \quad \mathbb{Q}_h := L_{h,0}(\Omega_D) \times \mathbb{R},
 \end{aligned} \tag{6.3}$$

$$\begin{aligned}
 \underline{\boldsymbol{\sigma}}_h &:= (\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h}) \in \mathbf{X}_{h,1}, & \underline{\boldsymbol{\tau}}_h &:= (\boldsymbol{\tau}_{S,h}, \mathbf{v}_{S,h}, \boldsymbol{\eta}_{S,h}) \in \mathbf{X}_{h,1}, \\
 \underline{\mathbf{t}}_h &:= (\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_{D,h}) \in \mathbf{X}_h, & \underline{\boldsymbol{\varphi}}_h &:= (\boldsymbol{\varphi}_h, \lambda_h) \in \mathbf{Y}_h, & \underline{\mathbf{p}}_h &:= (p_{D,h}, \ell_h) \in \mathbb{Q}_h, \\
 \underline{\mathbf{r}}_h &:= (\underline{\boldsymbol{\tau}}_h, \mathbf{v}_{D,h}) \in \mathbf{X}_h, & \underline{\boldsymbol{\psi}}_h &:= (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{Y}_h, & \underline{\mathbf{q}}_h &:= (q_{D,h}, J_h) \in \mathbb{Q}_h,
 \end{aligned}$$

the Galerkin scheme associated with problem (3.5) reads: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned}
 [\mathbf{A}(\mathbf{u}_{S,h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h, \\
 [\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \quad \forall \underline{\mathbf{q}}_h \in \mathbb{Q}_h.
 \end{aligned} \tag{6.4}$$

Now, we proceed similarly to [24] (see also [22, 11]), and derive suitable hypotheses on the spaces (6.1) ensuring the well-posedness of problem (6.4). We begin by noticing that, in order to have meaningful subspaces $\mathbb{H}_{h,0}(\Omega_S)$ and $L_{h,0}(\Omega_D)$ we need to be able to eliminate multiples of the identity matrix and constant polynomials from $\mathbb{H}_h(\Omega_S)$ and $L_h(\Omega_D)$, respectively. This requirement is certainly satisfied if we assume:

$$\mathbf{(H.0)} \quad \mathbf{P}_0(\Omega_S) := [\mathbf{P}_0(\Omega_S)]^n \subseteq \mathbf{H}_h(\Omega_S) \text{ and } \mathbf{P}_0(\Omega_D) \subseteq L_h(\Omega_D).$$

In particular, it follows that $\mathbb{I} \in \mathbb{H}_h(\Omega_S)$ for all h , and hence there holds the decomposition

$$\mathbb{H}_h(\Omega_S) = \mathbb{H}_{h,0}(\Omega_S) \oplus \mathbf{P}_0(\Omega_S) \mathbb{I}.$$

Next, we look at the discrete kernel of \mathbf{B} , which is given by

$$\mathbb{V}_h = \left\{ (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h : [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{q}}_h] = 0 \quad \forall \underline{\mathbf{q}}_h \in \mathbb{Q}_h \right\}. \tag{6.5}$$

In order to have a more explicit definition of \mathbb{V}_h , we introduce the following assumption:

$$\mathbf{(H.1)} \quad \text{div } \mathbf{H}_h(\Omega_D) \subseteq L_h(\Omega_D).$$

Then, owing to **(H.1)** and recalling the definition of \mathbf{B} (cf. (3.11)), it follows that $\mathbb{V}_h = \tilde{\mathbf{X}}_h \times \tilde{\mathbf{Y}}_h$, where

$$\tilde{\mathbf{X}}_h = \mathbf{X}_{h,1} \times \tilde{\mathbf{X}}_{h,2} \quad \text{and} \quad \tilde{\mathbf{Y}}_h = \tilde{\Lambda}_h^S(\Sigma) \times \Lambda_h^D(\Sigma),$$

with

$$\tilde{\mathbf{X}}_{h,2} := \left\{ \mathbf{v}_{D,h} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D) : \operatorname{div}(\mathbf{v}_{D,h}) \in P_0(\Omega_D) \right\}$$

and

$$\tilde{\Lambda}_h^S(\Sigma) := \left\{ \boldsymbol{\psi}_h \in \Lambda_h^S(\Sigma) : \langle \boldsymbol{\psi}_h \cdot \mathbf{n}, 1 \rangle_\Sigma = 0 \right\}.$$

In particular, it readily follows that $\mathbb{V}_h \subseteq \mathbb{V}$.

On the other hand, for the subsequent analysis we need to ensure the discrete version of the inf-sup conditions (5.22) and (5.27) of \mathbf{b} and \mathbf{B} (cf. (3.9), (3.11)), respectively, namely the existence of constants $\tilde{\beta}_1, \tilde{\beta} > 0$, independent of h , such that

$$\sup_{\substack{\mathbf{r}_h \in \tilde{\mathbf{X}}_h \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[\mathbf{b}(\mathbf{r}_h), \boldsymbol{\psi}_h]}{\|\mathbf{r}_h\|_{\mathbf{X}}} \geq \tilde{\beta}_1 \|\boldsymbol{\psi}_h\|_{\mathbf{Y}} \quad \forall \boldsymbol{\psi}_h \in \tilde{\mathbf{Y}}_h \quad (6.6)$$

and

$$\sup_{\substack{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in \mathbb{H}_h \\ (\mathbf{r}_h, \boldsymbol{\psi}_h) \neq \mathbf{0}}} \frac{[\mathbf{B}(\mathbf{r}_h, \boldsymbol{\psi}_h), \mathbf{q}_h]}{\|(\mathbf{r}_h, \boldsymbol{\psi}_h)\|_{\mathbb{H}}} \geq \tilde{\beta} \|\mathbf{q}_h\|_{\mathbb{Q}} \quad \forall \mathbf{q}_h \in \mathbb{Q}_h. \quad (6.7)$$

For instance, applying the same diagonal argument utilized in [24, Section 3] (see also [22, 11, 10]), we deduce that \mathbf{b} satisfies the discrete inf-sup condition (6.6) if and only if the following hypotheses holds:

(H.2) There exist $\tilde{\beta}_{1,S}, \tilde{\beta}_{1,D} > 0$, independent of h , such that

$$\sup_{\substack{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S) \\ \boldsymbol{\tau}_{S,h} \neq \mathbf{0}}} \frac{\langle \boldsymbol{\tau}_{S,h} \mathbf{n}, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\tau}_{S,h}\|_{\operatorname{div}; \Omega_S}} \geq \tilde{\beta}_{1,S} \|\boldsymbol{\psi}_h\|_{1/2,0,0;\Sigma} \quad \forall \boldsymbol{\psi}_h \in \tilde{\Lambda}_h^S(\Sigma) \quad (6.8)$$

and

$$\sup_{\substack{\mathbf{v}_{D,h} \in \tilde{\mathbf{X}}_{h,2} \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{D,h} \cdot \mathbf{n}, \boldsymbol{\xi}_h \rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)}} \geq \tilde{\beta}_{1,D} \|\boldsymbol{\xi}_h\|_{1/3,3/2;\Sigma} \quad \forall \boldsymbol{\xi}_h \in \Lambda_h^D(\Sigma). \quad (6.9)$$

Similarly, employing the same arguments from [22, Section 5.2], we obtain that \mathbf{B} satisfies the discrete inf-sup condition (6.7) provided that the following hypothesis holds.

(H.3) There exist $\tilde{\beta}_D > 0$, independent of h , and $\psi_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$, such that

$$\psi_0 \in \Lambda_h^S(\Sigma) \quad \forall h \quad \text{and} \quad \langle \psi_0 \cdot \mathbf{n}, 1 \rangle_\Sigma \neq 0, \quad (6.10)$$

and

$$\sup_{\substack{\mathbf{v}_{D,h} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_{D,h}, q_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)}} \geq \tilde{\beta}_D \|q_{D,h}\|_{0,\Omega_D} \quad \forall q_{D,h} \in L_{h,0}(\Omega_D). \quad (6.11)$$

6.2 Solvability analysis of the discrete problem

In what follows, we assume that hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)** hold, and, analogously to the analysis of the continuous problem, we apply a fixed-point argument to prove the well-posedness of the Galerkin scheme (6.4). To that end, we now let $\mathbf{T}_h : \mathbf{H}_{h,\Gamma_S}^1(\Omega_S) \rightarrow \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)$ be the discrete operator defined by

$$\mathbf{T}_h(\mathbf{w}_{S,h}) := \mathbf{u}_{S,h} \quad \forall \mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S), \quad (6.12)$$

where $\underline{\mathbf{t}}_h := (\sigma_{S,h}, \mathbf{u}_{S,h}, \gamma_{S,h}, \mathbf{u}_{D,h})$ is the first component of the unique solution (to be confirmed below) of the discrete nonlinear problem: Find $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned} [\mathbf{A}(\mathbf{w}_{S,h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\mathbf{p}}_h] &= [\mathbf{F}, (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{H}_h, \\ [\mathbf{B}(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{q}}_h] &= [\mathbf{G}, \underline{\mathbf{q}}_h] \quad \forall \underline{\mathbf{q}}_h \in \mathbb{Q}_h. \end{aligned} \quad (6.13)$$

Then, the Galerkin scheme (6.4) can be rewritten, equivalently, as the fixed-point problem:

Find $\mathbf{u}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1$ such that

$$\mathbf{T}_h(\mathbf{u}_{S,h}) = \mathbf{u}_{S,h}. \quad (6.14)$$

Next, similarly to the analysis developed in Section 5.2, in what follows we provide suitable assumptions under which problem (6.13) is well posed or equivalently \mathbf{T}_h is well defined. For this purpose, we will require a discrete version of Theorem 4.7. In fact, we let

$X_{h,1}, X_{h,2}, Y_h$ and Q_h be finite dimensional subspaces of X_1, X_2, Y and Q , respectively, and set $X_h = X_{h,1} \times X_{h,2} \subseteq X = X_1 \times X_2$ and $H_h = X_h \times Y_h \subseteq H = X \times Y$. Let $a_h : X_h \rightarrow X'_h$ be the discrete version of the nonlinear operator a . Thus, we define the nonlinear operator $A_h : H_h \rightarrow H'_h$, as:

$$A_h(\mathbf{r}_h, \boldsymbol{\psi}_h) := \begin{bmatrix} a_h & b' \\ b & -c \end{bmatrix} \begin{bmatrix} \mathbf{r}_h \\ \boldsymbol{\psi}_h \end{bmatrix} \in H'_h \quad \forall (\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h.$$

Then, given $(F, G) \in H' \times Q'$, the Galerkin scheme reduces to: find $((\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{p}_h) \in H_h \times Q_h$ such that

$$\begin{aligned} [A_h(\mathbf{t}_h, \boldsymbol{\varphi}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] + [B'(\mathbf{p}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] &= [F, (\mathbf{r}_h, \boldsymbol{\psi}_h)] \quad \forall (\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h, \\ [B(\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{q}_h] &= [G, \mathbf{q}_h] \quad \forall \mathbf{q}_h \in Q_h. \end{aligned} \tag{6.15}$$

Next, we let V_h be the discrete kernel of B , that is,

$$V_h := \left\{ (\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h : [B(\mathbf{r}_h, \boldsymbol{\psi}_h), \mathbf{q}_h] = 0 \quad \forall \mathbf{q}_h \in Q_h \right\},$$

and let \tilde{X}_h and \tilde{Y}_h be subspaces of X_h and Y_h , respectively, such that $V_h = \tilde{X}_h \times \tilde{Y}_h$. The following result establishes the well-posedness of (6.15).

Theorem 6.1 *Assume that*

(i) *there exist constants $\tilde{\gamma} > 0$ and $p_1, p_2 \geq 2$, such that*

$$\|a_h(\mathbf{t}_h) - a_h(\mathbf{r}_h)\|_{X'} \leq \tilde{\gamma} \sum_{j=1}^2 \left\{ \|\mathbf{t}_{j,h} - \mathbf{r}_{j,h}\|_{X_j} + \|\mathbf{t}_{j,h} - \mathbf{r}_{j,h}\|_{X_j} \left(\|\mathbf{t}_{j,h}\|_{X_j} + \|\mathbf{r}_{j,h}\|_{X_j} \right)^{p_j-2} \right\},$$

for all $\mathbf{t}_h = (\mathbf{t}_{1,h}, \mathbf{t}_{2,h}), \mathbf{r}_h = (\mathbf{r}_{1,h}, \mathbf{r}_{2,h}) \in X_h$.

(ii) *for each $\mathbf{s}_h \in X_h$, the operator $a_h(\cdot + \mathbf{s}_h) : \tilde{X}_h \rightarrow \tilde{X}'_h$ is strictly monotone in the sense that there exist $\tilde{\alpha} > 0$ and $p_1, p_2 \geq 2$, such that*

$$[a_h(\mathbf{t}_h + \mathbf{s}_h) - a_h(\mathbf{r}_h + \mathbf{s}_h), \mathbf{t}_h - \mathbf{r}_h] \geq \tilde{\alpha} \left\{ \|\mathbf{t}_{1,h} - \mathbf{r}_{1,h}\|_{X_1}^{p_1} + \|\mathbf{t}_{2,h} - \mathbf{r}_{2,h}\|_{X_2}^{p_2} \right\},$$

for all $\mathbf{t}_h = (\mathbf{t}_{1,h}, \mathbf{t}_{2,h}), \mathbf{r}_h = (\mathbf{r}_{1,h}, \mathbf{r}_{2,h}) \in \tilde{X}_h$.

(iii) c is positive semi-definite on \tilde{Y}_h , that is,

$$[c(\boldsymbol{\psi}_h), \boldsymbol{\psi}_h] \geq 0 \quad \forall \boldsymbol{\psi}_h \in \tilde{Y}_h.$$

(iv) b satisfies an inf-sup condition on $\tilde{X}_h \times \tilde{Y}_h$, that is, there exists $\tilde{\beta}_1 > 0$ such that

$$\sup_{\substack{\mathbf{r}_h \in \tilde{X}_h \\ \mathbf{r}_h \neq \mathbf{0}}} \frac{[b(\mathbf{r}_h), \boldsymbol{\psi}_h]}{\|\mathbf{r}_h\|_X} \geq \tilde{\beta}_1 \|\boldsymbol{\psi}_h\|_Y \quad \forall \boldsymbol{\psi}_h \in \tilde{Y}_h.$$

(v) B satisfies an inf-sup condition on $H_h \times Q_h$, that is, there exists $\tilde{\beta} > 0$ such that

$$\sup_{\substack{(\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h \\ (\mathbf{r}_h, \boldsymbol{\psi}_h) \neq \mathbf{0}}} \frac{[B(\mathbf{r}_h, \boldsymbol{\psi}_h), \mathbf{q}_h]}{\|(\mathbf{r}_h, \boldsymbol{\psi}_h)\|_H} \geq \tilde{\beta} \|\mathbf{q}_h\|_Q \quad \forall \mathbf{q}_h \in Q_h.$$

Then, for each $(F, G) \in H' \times Q'$ there exists a unique $((\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{p}_h) \in H_h \times Q_h$, such that

$$\begin{aligned} [A_h(\mathbf{t}_h, \boldsymbol{\varphi}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] + [B'(\mathbf{p}_h), (\mathbf{r}_h, \boldsymbol{\psi}_h)] &= [F, (\mathbf{r}_h, \boldsymbol{\psi}_h)] \quad \forall (\mathbf{r}_h, \boldsymbol{\psi}_h) \in H_h, \\ [B(\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{q}_h] &= [G, \mathbf{q}_h] \quad \forall \mathbf{q}_h \in Q_h. \end{aligned}$$

Moreover, there exists $\tilde{C} > 0$, depending only on $\tilde{\alpha}, \tilde{\gamma}, \tilde{\beta}, \tilde{\beta}_1, p_1, p_2, \|b\|, \|b'\|$ and $\|c\|$, such that

$$\|((\mathbf{t}_h, \boldsymbol{\varphi}_h), \mathbf{p}_h)\|_{H \times Q} \leq \tilde{C} \mathcal{M}(F_h, G_h),$$

where

$$\begin{aligned} \mathcal{M}(F_h, G_h) &:= \max \left\{ \mathcal{N}(F_h, G_h)^{1/(p_1-1)}, \mathcal{N}(F_h, G_h)^{1/(p_2-1)}, \mathcal{N}(F_h, G_h), \right. \\ &\quad \left. \mathcal{N}(F_h, G_h)^{(p_1-1)/(p_2-1)}, \mathcal{N}(F_h, G_h)^{(p_2-1)/(p_1-1)} \right\} \end{aligned}$$

and $\mathcal{N}(F_h, G_h)$ is defined in (4.25), with $F_h := F|_{H_h}$ and $G_h := G|_{Q_h}$.

Proof. It reduces to a simple application of Theorem 4.7 to the present discrete setting. \square

Then, similarly to the continuous case we must verify that the operators defining the discrete problem satisfy the hypotheses of Theorem 6.1.

Lemma 6.2 *Let $\mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)$. Then, there exists $\tilde{\gamma} > 0$, depending on C_{A_S} , C_{B_S} , and L_{A_D} (cf. (5.1), (5.2), (5.3)), such that*

$$\begin{aligned} \|\mathbf{a}(\mathbf{w}_{S,h})(\underline{\mathbf{t}}_h) - \mathbf{a}(\mathbf{w}_{S,h})(\underline{\mathbf{r}}_h)\|_{\mathbf{X}'} &\leq \tilde{\gamma} \left\{ \left(1 + \|\mathbf{w}_{S,h}\|_{1,\Omega_S}\right) \|\underline{\boldsymbol{\sigma}}_h - \underline{\boldsymbol{\tau}}_h\|_{\mathbf{X}_1} + \|\mathbf{u}_{D,h} - \mathbf{v}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} \right. \\ &\quad \left. + \|\mathbf{u}_{D,h} - \mathbf{v}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} \left(\|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} + \|\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} \right) \right\}, \end{aligned}$$

for all $\underline{\mathbf{t}}_h = (\underline{\boldsymbol{\sigma}}_h, \mathbf{u}_{D,h})$, $\underline{\mathbf{r}}_h = (\underline{\boldsymbol{\tau}}_h, \mathbf{v}_{D,h}) \in \mathbf{X}_h$.

Proof. As for the continuous case, the result follows straightforwardly from the definition of $\mathbf{a}(\mathbf{w}_{S,h})$ (cf. (3.7)), the triangle inequality, and the stability properties (5.1), (5.2) and (5.3). We omit further details. \square

In the following result we provide the assumptions under which operator $\mathbf{a}(\mathbf{w}_{S,h})$ verifies hypothesis (ii) of Theorem 6.1.

Lemma 6.3 *Suppose that the parameters $\kappa_1, \kappa_2, \kappa_3$, satisfy the conditions required by Lemma 5.1. Let $\mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{1,\Omega_S} \leq r$ with $r \in (0, r_0)$, and r_0 defined by (5.19). Then, for each $\underline{\mathbf{s}}_h \in \mathbf{X}_h$, the nonlinear operator $\mathbf{a}(\mathbf{w}_{S,h})(\cdot + \underline{\mathbf{s}}_h)$ is strictly monotone on $\tilde{\mathbf{X}}_h$ (cf. (6.5)).*

Proof. The proof follows analogously to the proof of Lemma 5.3. Further details are omitted. \square

The following lemma establishes the well-definiteness of operator \mathbf{T}_h (cf. (6.12)).

Lemma 6.4 *Assume that hypotheses (H.0), (H.1), (H.2) and (H.3) hold. Assume further that the parameters $\kappa_1, \kappa_2, \kappa_3$, satisfy the conditions required by Lemma 5.1. Let $r \in (0, r_0)$, with r_0 defined by (5.19), and let $\mathbf{f}_S \in \mathbf{L}^2(\Omega_S)$, $\mathbf{f}_D \in \mathbf{L}^{3/2}(\Omega_D)$ and $g_D \in L_0^2(\Omega_S)$. Then, problem (6.13) has a unique solution $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ for each $\mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{1,\Omega_S} \leq r$. Moreover, there exists a constant \tilde{c}_T independent of $\mathbf{w}_{S,h}$ and the data, such that*

$$\|\mathbf{T}_h(\mathbf{w}_{S,h})\|_{1,\Omega_S} \leq \|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H} \times \mathbb{Q}} \leq \tilde{c}_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \quad (6.16)$$

Proof. Let $\mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)$ such that $\|\mathbf{w}_{S,h}\|_{1,\Omega_S} \leq r$. Recalling that $\mathbb{H}_h \subseteq \mathbb{H}$, $\mathbb{Q}_h \subseteq \mathbb{Q}$ and $\mathbb{V}_h \subseteq \mathbb{V}$, a straightforward application of Lemmas 6.2, 6.3 and 5.4, implies, respectively,

that hypotheses (i), (ii) and (iii) in Theorem 6.1, hold. In turn, the inf-sup conditions (iv) and (v) follow from hypotheses **(H.2)** and **(H.3)**, respectively. Therefore, according to the above, a direct application of Theorem 6.1 allows us to conclude that there exists a unique $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ solution to (6.13) which satisfies (6.16), whit \tilde{c}_T independent of $\mathbf{w}_{S,h}$ and h . \square

We are now in position of establishing the well posedness of (6.4)

Theorem 6.5 *Assume that hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)** hold. Assume further that the parameters $\kappa_1, \kappa_2, \kappa_3$ satisfy the conditions required by Lemma 5.1. In addition, given $r \in (0, r_0)$, with r_0 defined by (5.19), we let $\mathbf{W}_r^h := \left\{ \mathbf{w}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S) : \|\mathbf{w}_{S,h}\|_{1,\Omega_S} \leq r \right\}$, and assume that the data $\mathbf{f}_S, \mathbf{f}_D$ and g_D satisfy*

$$\tilde{c}_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) \leq r, \quad (6.17)$$

with $\tilde{c}_T > 0$ the constant in (6.16). Then, there exists a unique $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ solution to (6.4), which satisfies $\mathbf{u}_{S,h} \in \mathbf{W}_r^h$, and

$$\|((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H} \times \mathbb{Q}} \leq \tilde{c}_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D). \quad (6.18)$$

Proof. We first observe, owing to (6.16), that the assumption (6.17) guarantees that $\mathbf{T}_h(\mathbf{W}_r^h) \subseteq \mathbf{W}_r^h$. Next, analogously to the proof of Lemma 5.10, that is, applying the strict monotonicity of $\mathbf{a}(\mathbf{w}_{S,h})$, for each $\mathbf{w}_{S,h} \in \mathbf{W}_r^h$, we find that

$$\|\mathbf{T}_h(\mathbf{w}_{S,h}) - \mathbf{T}_h(\tilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S} \leq C_T \|\mathbf{i}_c\| \|\mathbf{T}_h(\tilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S} \|\mathbf{w}_{S,h} - \tilde{\mathbf{w}}_{S,h}\|_{1,\Omega_S} \quad \forall \mathbf{w}_{S,h}, \tilde{\mathbf{w}}_{S,h} \in \mathbf{W}_r^h,$$

which, together with (5.35), (6.16), (6.17) and (5.19), implies

$$\|\mathbf{T}_h(\mathbf{w}_{S,h}) - \mathbf{T}_h(\tilde{\mathbf{w}}_{S,h})\|_{1,\Omega_S} \leq \frac{r}{r_0} \|\mathbf{w}_{S,h} - \tilde{\mathbf{w}}_{S,h}\|_{1,\Omega_S} \quad \forall \mathbf{w}_{S,h}, \tilde{\mathbf{w}}_{S,h} \in \mathbf{W}_r^h,$$

thus confirming that $\mathbf{T}_h : \mathbf{W}_r^h \rightarrow \mathbf{W}_r^h$ is a contraction mapping. Then, the Banach fixed-point theorem and the equivalence between (6.4) and (6.14) imply the well-posedness of (6.4). In turn, the estimate (6.18) follows directly from (6.16). \square

6.3 *A priori* error analysis

In this section we establish the corresponding Céa estimate. For this purpose, we first introduce some notations and state a couple of previous results. We begin by recalling the discrete inf-sup condition of \mathbf{B} (cf. (6.7)), and a classical result on mixed methods (see, for instance [19, Theorem 2.6]) ensuring the existence of a constant $c > 0$, independent of h , such that:

$$\text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h) \leq c \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{H}_h). \quad (6.19)$$

In turn, in order to simplify the subsequent analysis, we write $\mathbf{e}_\sigma = \underline{\boldsymbol{\sigma}} - \underline{\boldsymbol{\sigma}}_h$, $\mathbf{e}_{\mathbf{u}_D} = \mathbf{u}_D - \mathbf{u}_{D,h}$, $\mathbf{e}_\varphi = \underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}_h$ and $\mathbf{e}_\mathbf{p} = \underline{\mathbf{p}} - \underline{\mathbf{p}}_h$. Then, proceeding similarly to [2, Section 3.3], we consider the unique decompositions $\underline{\boldsymbol{\sigma}}_h = \tilde{\boldsymbol{\sigma}}_h + \tilde{\boldsymbol{\sigma}}_h^\perp$, $\mathbf{u}_{D,h} = \tilde{\mathbf{u}}_{D,h} + \tilde{\mathbf{u}}_{D,h}^\perp$ and $\underline{\boldsymbol{\varphi}}_h = \tilde{\boldsymbol{\varphi}}_h + \tilde{\boldsymbol{\varphi}}_h^\perp$, with $((\tilde{\boldsymbol{\sigma}}_h, \tilde{\mathbf{u}}_{D,h}), \tilde{\boldsymbol{\varphi}}_h) \in \mathbb{V}_h$ and $((\tilde{\boldsymbol{\sigma}}_h^\perp, \tilde{\mathbf{u}}_{D,h}^\perp), \tilde{\boldsymbol{\varphi}}_h^\perp) \in \tilde{\mathbf{X}}_h^\perp \times \tilde{\mathbf{Y}}_h^\perp =: \mathbb{V}_h^\perp$, where

$$\mathbb{V}_h^\perp = \left\{ (\underline{\mathbf{s}}_h, \underline{\boldsymbol{\phi}}_h) \in \mathbb{H}_h : \left\langle (\underline{\mathbf{s}}_h, \underline{\boldsymbol{\phi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \right\rangle = 0 \quad \forall (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) \in \mathbb{V}_h \right\}.$$

Next, given arbitrary $((\tilde{\boldsymbol{\tau}}_h, \tilde{\mathbf{v}}_{D,h}), \tilde{\boldsymbol{\psi}}_h) \in \mathbb{V}_h$ and $\tilde{\mathbf{q}}_h \in \mathbb{Q}_h$, we decompose the errors into

$$\mathbf{e}_\sigma = \delta_\sigma - \tilde{\boldsymbol{\sigma}}_h^\perp + \boldsymbol{\eta}_\sigma, \quad \mathbf{e}_{\mathbf{u}_D} = \delta_{\mathbf{u}_D} - \tilde{\mathbf{u}}_{D,h}^\perp + \boldsymbol{\eta}_{\mathbf{u}_D}, \quad \mathbf{e}_\varphi = \delta_\varphi - \tilde{\boldsymbol{\varphi}}_h^\perp + \boldsymbol{\eta}_\varphi \quad \text{and} \quad \mathbf{e}_\mathbf{p} = \delta_\mathbf{p} + \boldsymbol{\eta}_\mathbf{p}, \quad (6.20)$$

with

$$\begin{aligned} \delta_\sigma &= \underline{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\tau}}_h, & \boldsymbol{\eta}_\sigma &= \tilde{\boldsymbol{\tau}}_h - \tilde{\boldsymbol{\sigma}}_h, & \delta_{\mathbf{u}_D} &= \mathbf{u}_D - \tilde{\mathbf{v}}_{D,h}, & \boldsymbol{\eta}_{\mathbf{u}_D} &= \tilde{\mathbf{v}}_{D,h} - \tilde{\mathbf{u}}_{D,h}, \\ \delta_\varphi &= \underline{\boldsymbol{\varphi}} - \tilde{\boldsymbol{\psi}}_h, & \boldsymbol{\eta}_\varphi &= \tilde{\boldsymbol{\psi}}_h - \tilde{\boldsymbol{\varphi}}_h, & \delta_\mathbf{p} &= \underline{\mathbf{p}} - \tilde{\mathbf{q}}_h, & \boldsymbol{\eta}_\mathbf{p} &= \tilde{\mathbf{q}}_h - \underline{\mathbf{p}}_h. \end{aligned} \quad (6.21)$$

Consequently, the following Galerkin orthogonality property holds:

$$\begin{aligned} [\mathbf{A}(\mathbf{u}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) - \mathbf{A}(\mathbf{u}_{S,h})(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \mathbf{e}_\mathbf{p}] &= 0, \\ [\mathbf{B}((\mathbf{e}_\sigma, \mathbf{e}_{\mathbf{u}_D}), \mathbf{e}_\varphi), \underline{\mathbf{q}}_h] &= 0, \end{aligned} \quad (6.22)$$

for all $(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) := ((\tilde{\boldsymbol{\tau}}_h, \mathbf{v}_{D,h}), \tilde{\boldsymbol{\psi}}_h) \in \mathbb{H}_h$ and $\underline{\mathbf{q}}_h \in \mathbb{Q}_h$.

We now establish the main result of this section.

Theorem 6.6 *Assume that the hypotheses (H.0), (H.1), (H.2), and (H.3), as well as the conditions on $\kappa_1, \kappa_2, \kappa_3$ required by Lemma 5.1, hold. Let $r \in (0, r_0)$ with r_0 defined by (5.19), and assume further that the data $\mathbf{f}_S, \mathbf{f}_D$, and g_D satisfy*

$$c_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D) \leq \frac{r}{2}, \quad (6.23)$$

with c_T the constant satisfying (5.30). Let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) := ((\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}_D, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ with $\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_S, \mathbf{u}_S, \boldsymbol{\gamma}_S)$ and $\mathbf{u}_S \in \mathbf{W}_r$, and $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) := ((\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_{D,h}, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ with $\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{S,h}, \mathbf{u}_{S,h}, \boldsymbol{\gamma}_{S,h})$ and $\mathbf{u}_{S,h} \in \mathbf{W}_r^h$, be the unique solutions of problems (3.5) and (6.4), respectively. Then there exists $C > 0$, independent of h and the continuous and discrete solutions, such that

$$\begin{aligned} & \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H} \times \mathbb{Q}} \\ & \leq C \left\{ \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{H}_h)^{1/2} + \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{H}_h) + \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{H}_h)^2 + \text{dist}(\underline{\mathbf{p}}, \mathbb{Q}_h) \right\}. \end{aligned} \quad (6.24)$$

Proof. Given $((\tilde{\underline{\mathbf{t}}}_h, \tilde{\underline{\mathbf{v}}}_{D,h}), \tilde{\underline{\boldsymbol{\psi}}}_h) \in \mathbb{V}_h$, and $\tilde{\underline{\mathbf{q}}}_h \in \mathbb{Q}_h$, we define $\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\underline{\mathbf{u}}_D}, \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}, \boldsymbol{\delta}_{\underline{\mathbf{p}}}, \boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\underline{\mathbf{u}}_D}, \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}$, and $\boldsymbol{\eta}_{\underline{\mathbf{p}}}$, as in (6.21). In turn, since $((\tilde{\underline{\boldsymbol{\sigma}}}_h, \tilde{\underline{\mathbf{u}}}_{D,h}), \tilde{\underline{\boldsymbol{\varphi}}}_h) \in \mathbb{V}_h$, it follows that $((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}) \in \mathbb{V}_h$. Thus, from the second equation in (6.22), (6.20), and the fact that $[\mathbf{B}((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}), \underline{\mathbf{q}}_h] = 0$, it follows that

$$[\mathbf{B}((\tilde{\underline{\boldsymbol{\sigma}}}_h^\perp, \tilde{\underline{\mathbf{u}}}_{D,h}^\perp), \tilde{\underline{\boldsymbol{\varphi}}}_h^\perp), \underline{\mathbf{q}}_h] = [\mathbf{B}((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}), \underline{\mathbf{q}}_h],$$

which together with the continuity and discrete inf-sup condition of \mathbf{B} (cf. (5.1) and (6.7)), (6.20), (6.21), and the triangle inequality, yields

$$\|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}}) - ((\tilde{\underline{\boldsymbol{\sigma}}}_h^\perp, \tilde{\underline{\mathbf{u}}}_{D,h}^\perp), \tilde{\underline{\boldsymbol{\varphi}}}_h^\perp)\|_{\mathbb{H}} \leq \left(1 + \frac{C_{\mathbf{B}}}{\tilde{\beta}}\right) \|((\boldsymbol{\delta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\delta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\delta}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}. \quad (6.25)$$

On the other hand, taking $(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h) := ((\boldsymbol{\eta}_{\underline{\boldsymbol{\sigma}}}, \boldsymbol{\eta}_{\underline{\mathbf{u}}_D}), \boldsymbol{\eta}_{\underline{\boldsymbol{\varphi}}}) \in \mathbb{V}_h \subset \mathbb{H}_h$ in the first row of (6.22), recalling the definition of the operator $\mathbf{A}(\mathbf{u}_S)$ (cf. (3.6)), observing that thanks to (H.1) we

have that $\mathbb{V}_h \subseteq \mathbb{V}$, we deduce

$$\begin{aligned} [\mathbf{a}(\mathbf{u}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{u}_{S,h})(\underline{\mathbf{t}}_h), (\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D})] + [\mathbf{b}(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D}), \mathbf{e}_\varphi] &= 0, \\ [\mathbf{b}(\mathbf{e}_\sigma, \mathbf{e}_{\mathbf{u}_D}), \boldsymbol{\eta}_\varphi] - [\mathbf{c}(\mathbf{e}_\varphi), \boldsymbol{\eta}_\varphi] &= 0. \end{aligned} \quad (6.26)$$

In this way, from (6.20) and the first equation of (6.26), we find that

$$\begin{aligned} [\mathbf{b}(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D}), \boldsymbol{\eta}_\varphi] &= - \left\{ [\mathbf{a}(\mathbf{u}_S)(\underline{\mathbf{t}}) - \mathbf{a}(\mathbf{u}_S)(\underline{\mathbf{t}}_h), (\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D})] + [\mathcal{B}_S(\mathbf{u}_S - \mathbf{u}_{S,h})(\underline{\boldsymbol{\sigma}}_h), \boldsymbol{\eta}_\sigma] \right. \\ &\quad \left. + [\mathbf{b}(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D}), \boldsymbol{\delta}_\varphi - \tilde{\varphi}_h^\perp] \right\}. \end{aligned}$$

Hence, noting that $(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D}) \in \tilde{\mathbf{X}}_h$, employing the discrete inf-sup condition of \mathbf{b} (cf. (6.6)), inequality (5.16), and the continuity of \mathbf{b} and \mathbf{B} (cf. (5.1)), and then applying the first inequality in (5.2) and bounding $\|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{\mathbf{L}^4(\Omega_S)}$ by $\|\mathbf{i}_c\| \|\mathbf{e}_\sigma\|_{\mathbf{X}_1}$, we deduce that there exist a constant $C_1 > 0$, independent of h , such that

$$\begin{aligned} \tilde{\beta}_1 \|\boldsymbol{\eta}_\varphi\|_{\mathbf{Y}} &\leq C_1 \left\{ \left(1 + \|\mathbf{u}_S\|_{1,\Omega_S} + \|\mathbf{u}_{S,h}\|_{1,\Omega_S} \right) \|\mathbf{e}_\sigma\|_{\mathbf{X}_1} \right. \\ &\quad \left. + \left(1 + \|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)} + \|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} \right) \|\mathbf{e}_{\mathbf{u}_D}\|_{\mathbf{H}^3(\text{div};\Omega_D)} + \|\boldsymbol{\delta}_\varphi - \tilde{\varphi}_h^\perp\|_{\mathbf{Y}} \right\}. \end{aligned} \quad (6.27)$$

Then, recalling that both $\|\mathbf{u}_S\|_{1,\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{1,\Omega_S}$ are bounded by r_0 (cf. (5.19)), as well as that both $\|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)}$ and $\|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)}$ are bounded by data (cf. Theorems 5.12 and 6.5), the estimate (6.27) together with (6.20), (6.21), and (6.25) allow us to conclude that

$$\|\boldsymbol{\eta}_\varphi\|_{\mathbf{Y}} \leq C_2 \left\{ \|((\boldsymbol{\delta}_\sigma, \boldsymbol{\delta}_{\mathbf{u}_D}), \boldsymbol{\delta}_\varphi)\|_{\mathbb{H}} + \|(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D})\|_{\mathbf{X}} \right\}, \quad (6.28)$$

with $C_2 > 0$ depending only on parameters, data and other constants, all of them independent of h . In turn, noting that $\underline{\mathbf{t}}_h = \tilde{\mathbf{t}}_h + \tilde{\mathbf{t}}_h^\perp$ with $\tilde{\mathbf{t}}_h = (\tilde{\boldsymbol{\sigma}}_h, \tilde{\mathbf{u}}_{D,h}) \in \tilde{\mathbf{X}}_h$ and $\tilde{\mathbf{t}}_h^\perp = (\tilde{\boldsymbol{\sigma}}_h^\perp, \tilde{\mathbf{u}}_{D,h}^\perp) \in \tilde{\mathbf{X}}_h^\perp$, and combining the first and second equation of (6.26), we are able to find that

$$\begin{aligned} [\mathbf{a}(\mathbf{u}_{S,h})(\tilde{\mathbf{r}}_h + \tilde{\mathbf{t}}_h^\perp) - \mathbf{a}(\mathbf{u}_{S,h})(\tilde{\mathbf{t}}_h + \tilde{\mathbf{t}}_h^\perp), (\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D})] &= [\mathbf{a}(\mathbf{u}_{S,h})(\tilde{\mathbf{r}}_h + \tilde{\mathbf{t}}_h^\perp) - \mathbf{a}(\mathbf{u}_{S,h})(\underline{\mathbf{t}}), (\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D})] \\ &\quad - [\mathcal{B}_S(\mathbf{u}_S - \mathbf{u}_{S,h})(\underline{\boldsymbol{\sigma}}), \boldsymbol{\eta}_\sigma] - [\mathbf{b}(\boldsymbol{\eta}_\sigma, \boldsymbol{\eta}_{\mathbf{u}_D}), \mathbf{e}_\varphi], \end{aligned}$$

where

$$\begin{aligned} [\mathbf{b}(\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\eta}}_{\mathbf{u}_D}), \underline{\mathbf{e}}_{\underline{\boldsymbol{\varphi}}}] &= [\mathbf{b}(\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\eta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}} - \underline{\tilde{\boldsymbol{\varphi}}}_h^\perp] - [\mathbf{b}((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}) - (\underline{\tilde{\boldsymbol{\sigma}}}_h^\perp, \underline{\tilde{\mathbf{u}}}_{D,h}^\perp), \underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}}] \\ &\quad + [\mathbf{c}(\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}} - \underline{\tilde{\boldsymbol{\varphi}}}_h^\perp), \underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}}] + [\mathbf{c}(\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}}), \underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}}]. \end{aligned}$$

Next, using the second inequality in (5.2), bounding $\|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{1,\Omega_S}$ by $\|\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}} - \underline{\tilde{\boldsymbol{\sigma}}}_h^\perp\|_{\mathbf{X}_1} + \|\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}$, recalling from (5.36) that $\|\underline{\boldsymbol{\sigma}}\|_{\mathbf{X}_1} \leq c_T \mathcal{M}(\mathbf{f}_S, \mathbf{f}_D, g_D)$, and employing assumption (6.23), we have

$$[\mathcal{B}_S(\mathbf{u}_S - \mathbf{u}_{S,h})(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}] \leq \frac{\alpha_S}{4} \left\{ \|\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 + \|\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}} - \underline{\tilde{\boldsymbol{\sigma}}}_h^\perp\|_{\mathbf{X}_1} \|\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1} \right\}.$$

Hence, using the strict monotonicity of $\mathbf{a}(\mathbf{u}_{S,h})$ (cf. Lemma 6.3), the continuity of \mathbf{b} , \mathbf{c} , and \mathbf{B} (cf. (5.1)), the inequalities (5.16) and (6.25), and the positive semi-definiteness property of \mathbf{c} (cf. (5.21)), we deduce that there exist $C_3, C_4 > 0$ independents of h , such that

$$\begin{aligned} &\frac{\alpha_S}{4} \|\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}\|_{\mathbf{X}_1}^2 + \alpha_D C_{\text{div}} \|\underline{\boldsymbol{\eta}}_{\mathbf{u}_D}\|_{\mathbf{H}^3(\text{div}; \Omega_D)}^3 \\ &\leq C_3 \left\{ \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}^2 \right\} \|(\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\eta}}_{\mathbf{u}_D})\|_{\mathbf{X}} \\ &\quad + C_4 \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} \|\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}}\|_{\mathbf{Y}}, \end{aligned}$$

which together with (6.28), Young's inequality and simple algebraic manipulations, yield

$$\|(\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\eta}}_{\mathbf{u}_D})\|_{\mathbf{X}} \leq \tilde{c} \left\{ \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}^{1/2} + \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}^2 \right\}. \quad (6.29)$$

In this way, employing (6.20), (6.25), (6.28), (6.29), and the triangle inequality, we obtain

$$\begin{aligned} \|((\underline{\mathbf{e}}_{\underline{\boldsymbol{\sigma}}}, \underline{\mathbf{e}}_{\mathbf{u}_D}), \underline{\mathbf{e}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} &\leq \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}}) - ((\underline{\tilde{\boldsymbol{\sigma}}}_h^\perp, \underline{\tilde{\mathbf{u}}}_{D,h}^\perp), \underline{\tilde{\boldsymbol{\varphi}}}_h^\perp)\|_{\mathbb{H}} + \|((\underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\eta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\eta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} \\ &\leq \tilde{C} \left\{ \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}^{1/2} + \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}} + \|((\underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\sigma}}}, \underline{\boldsymbol{\delta}}_{\mathbf{u}_D}), \underline{\boldsymbol{\delta}}_{\underline{\boldsymbol{\varphi}}})\|_{\mathbb{H}}^2 \right\}. \end{aligned} \quad (6.30)$$

In turn, in order to estimate \mathbf{e}_P , we first observe from (6.20) and the first row of (6.22), that

$$\begin{aligned} [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\boldsymbol{\eta}}_P] &= - \left\{ [\mathbf{A}(\mathbf{u}_S)(\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}) - \mathbf{A}(\mathbf{u}_S)(\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), (\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h)] \right. \\ &\quad \left. + [\mathcal{B}_S(\mathbf{u}_S - \mathbf{u}_{S,h})(\underline{\boldsymbol{\sigma}}_h), \underline{\boldsymbol{\tau}}_h] + [\mathbf{B}(\underline{\mathbf{r}}_h, \underline{\boldsymbol{\psi}}_h), \underline{\boldsymbol{\delta}}_P] \right\}. \end{aligned}$$

Then, proceeding similarly to (6.27), we employ again the discrete inf-sup condition of \mathbf{B} (cf. (6.7)), the definition of $\mathbf{A}(\mathbf{u}_S)$ (cf. (3.6)), the inequality (5.16), and the continuity of \mathbf{b} , \mathbf{c} , \mathbf{B} , and \mathcal{B}_S (cf. (5.1), (5.2)), to obtain

$$\begin{aligned} \tilde{\beta} \|\boldsymbol{\eta}_{\mathbf{p}}\|_{\mathbb{Q}} &\leq C_5 \left\{ \left(1 + \|\mathbf{u}_S\|_{1,\Omega_S} + \|\mathbf{u}_{S,h}\|_{1,\Omega_{S,h}} \right) \|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{X}_1} \right. \\ &\quad \left. + \left(1 + \|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)} + \|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)} \right) \|\mathbf{e}_{\mathbf{u}_D}\|_{\mathbf{H}^3(\text{div};\Omega_D)} + \|\mathbf{e}_{\boldsymbol{\varphi}}\|_{\mathbf{Y}} + \|\boldsymbol{\delta}_{\mathbf{p}}\|_{\mathbb{Q}} \right\}. \end{aligned}$$

Thus, using again that both $\|\mathbf{u}_S\|_{1,\Omega_S}$ and $\|\mathbf{u}_{S,h}\|_{1,\Omega_S}$ are bounded by r_0 (cf. (5.19)), as well as that both $\|\mathbf{u}_D\|_{\mathbf{H}^3(\text{div};\Omega_D)}$ and $\|\mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div};\Omega_D)}$ are bounded by data (cf. Theorems 5.12 and 6.5), we find that there exist $C_6 > 0$ depending only on parameters, data and other constants, all of them independent of h , such that

$$\|\boldsymbol{\eta}_{\mathbf{p}}\|_{\mathbb{Q}} \leq C_6 \left\{ \|((\mathbf{e}_{\boldsymbol{\sigma}}, \mathbf{e}_{\mathbf{u}_D}), \mathbf{e}_{\boldsymbol{\varphi}})\|_{\mathbb{H}} + \|\boldsymbol{\delta}_{\mathbf{p}}\|_{\mathbb{Q}} \right\}.$$

In this way, from (6.20), the triangle inequality, the foregoing bound and (6.30), we conclude that

$$\begin{aligned} \|\mathbf{e}_{\mathbf{p}}\|_{\mathbb{Q}} &\leq \|\boldsymbol{\delta}_{\mathbf{p}}\|_{\mathbb{Q}} + \|\boldsymbol{\eta}_{\mathbf{p}}\|_{\mathbb{Q}} \\ &\leq \widehat{C} \left\{ \|((\boldsymbol{\delta}_{\boldsymbol{\sigma}}, \boldsymbol{\delta}_{\mathbf{u}_D}), \boldsymbol{\delta}_{\boldsymbol{\varphi}})\|_{\mathbb{H}}^{1/2} + \|((\boldsymbol{\delta}_{\boldsymbol{\sigma}}, \boldsymbol{\delta}_{\mathbf{u}_D}), \boldsymbol{\delta}_{\boldsymbol{\varphi}})\|_{\mathbb{H}} + \|((\boldsymbol{\delta}_{\boldsymbol{\sigma}}, \boldsymbol{\delta}_{\mathbf{u}_D}), \boldsymbol{\delta}_{\boldsymbol{\varphi}})\|_{\mathbb{H}}^2 + \|\boldsymbol{\delta}_{\mathbf{p}}\|_{\mathbb{Q}} \right\}. \end{aligned} \quad (6.31)$$

Finally, recalling that $((\tilde{\boldsymbol{\tau}}_h, \tilde{\mathbf{v}}_{D,h}), \tilde{\boldsymbol{\psi}}_h) \in \mathbb{V}_h$, and $\tilde{\mathbf{q}}_h \in \mathbb{Q}_h$ are arbitrary, (6.30) and (6.31) give

$$\begin{aligned} &\|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H} \times \mathbb{Q}} \\ &\leq C \left\{ \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h)^{1/2} + \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h) + \text{dist}((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \mathbb{V}_h)^2 + \text{dist}(\underline{\mathbf{p}}, \mathbb{Q}_h) \right\}, \end{aligned}$$

which together with (6.19), concludes the proof. \square

A particular choice of finite element subspaces

We now introduce specific discrete spaces satisfying hypotheses (H.0), (H.1), (H.2), and (H.3) in 2D and 3D. To this end, we let \mathcal{T}_h^S and \mathcal{T}_h^D be respective triangulations of the domains Ω_S and Ω_D , which are formed by shape-regular triangles (in \mathbb{R}^2) or tetrahedra (in \mathbb{R}^3), and assume that they match in Σ so that $\mathcal{T}_h^S \cup \mathcal{T}_h^D$ is a triangulation of $\Omega_S \cup \Sigma \cup \Omega_D$. We also let Σ_h be the partition of Σ inherited from \mathcal{T}_h^S (or \mathcal{T}_h^D). Then for each $T \in \mathcal{T}_h^S \cup \mathcal{T}_h^D$ we set the local Raviart–Thomas space of order k as

$$\text{RT}_k(T) := \mathbf{P}_k(T) \oplus \mathbf{P}_k(T)\mathbf{x},$$

where $\mathbf{x} := (x_1, \dots, x_n)^t$ is a generic vector of \mathbb{R}^n .

7.1 Raviart–Thomas elements in 2D

We define the discrete subspaces in (6.1) as follows:

$$\begin{aligned}
 \mathbf{H}_h(\Omega_S) &:= \left\{ \tau_{S,h} \in \mathbf{H}(\operatorname{div}; \Omega_S) : \tau_{S,h}|_T \in \mathbf{RT}_k(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\
 \mathbf{H}_h^1(\Omega_S) &:= \left\{ \mathbf{v}_{S,h} \in [\mathcal{C}(\overline{\Omega_S})]^2 : \mathbf{v}_{S,h}|_T \in \mathbf{P}_{k+1}(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\
 \mathbb{L}_h(\Omega_S) &:= \left\{ \boldsymbol{\eta}_{S,h} \in \mathbb{L}_{\text{skew}}^2(\Omega_S) : \boldsymbol{\eta}_{S,h}|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\
 \mathbf{H}_h(\Omega_D) &:= \left\{ \mathbf{v}_{D,h} \in \mathbf{H}^3(\operatorname{div}; \Omega_D) : \mathbf{v}_{D,h}|_T \in \mathbf{RT}_k(T) \quad \forall T \in \mathcal{T}_h^D \right\}, \\
 \mathbb{L}_h(\Omega_D) &:= \left\{ q_{D,h} \in L^2(\Omega_D) : q_{D,h}|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^D \right\}.
 \end{aligned} \tag{7.1}$$

In addition, in order to introduce the particular subspaces $\Lambda_h^S(\Sigma)$ and $\Lambda_h^D(\Sigma)$, we follow the simplest approach suggested in [24] and [10], respectively. In fact, we first assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ that arises by joining pairs of adjacent edges of Σ_h , and denote the resulting edges still by e . Since Σ_h is automatically of bounded variation (that is, the ratio of lengths of adjacent edges is bounded) and, therefore, so is Σ_{2h} . Now, if the number of edges of Σ_h is odd, we simply reduce it to even case by joining any pair of two adjacent elements, and then construct Σ_{2h} from this modified partition. Hence, denoting by x_0 and x_N the extreme points of Σ , we define

$$\Lambda_h^S(\Sigma) := \left\{ \psi_h \in \mathcal{C}(\Sigma) : \psi_h|_e \in \mathbb{P}_{k+1}(e) \quad \forall \text{ edge } e \in \Sigma_{2h}, \quad \psi_h(x_0) = \psi_h(x_N) = 0 \right\}. \tag{7.2}$$

In turn, since the space $\prod_{e \in \Sigma_h} W^{1-1/p, p}(e)$ coincides with $W^{1-1/p, p}(\Sigma)$, without extra conditions when $1 < p < 2$ (in this case $p = 3/2$) [30, Theorem 1.5.2.3-(a)], it can be readily seen that a conforming finite element subspace for $W^{1/3, 3/2}(\Sigma)$ can be defined by

$$\Lambda_h^D(\Sigma) := \left\{ \xi_h : \Sigma \rightarrow \mathbb{R} : \xi_h|_e \in \mathbb{P}_k(e) \quad \forall \text{ edge } e \in \Sigma_h \right\}. \tag{7.3}$$

Then, we define the global spaces \mathbb{H}_h and \mathbb{Q}_h (cf. (6.3)), by combining (6.2), (6.3), (7.1), (7.2), and (7.3).

Now, let $\Pi_D : \mathbf{H}^3(\text{div}; \Omega_D) \rightarrow \mathbf{H}_h(\Omega_D)$ be the Raviart–Thomas operator on Ω_D , which is well defined for all $\mathbf{v}_D \in \mathbf{H}^3(\text{div}; \Omega_D)$ (see [10, Section 4.1] and [17, Remark 3.1] for details), and characterized by

$$\int_e (\mathbf{v}_D - \Pi_D(\mathbf{v}_D)) \cdot \mathbf{n} \psi = 0 \quad \forall \psi \in P_k(e), \forall \text{edge } e \in \mathcal{T}_h^D$$

and

$$\int_T (\mathbf{v}_D - \Pi_D(\mathbf{v}_D)) \cdot \boldsymbol{\eta} = 0 \quad \forall \boldsymbol{\eta} \in \mathbf{P}_{k-1}(T).$$

Furthermore, if \mathcal{P}_D denotes the $L^2(\Omega_D)$ -orthogonal projection, there holds

$$\text{div}(\Pi_D(\mathbf{v}_D)) = \mathcal{P}_D(\text{div } \mathbf{v}_D) \quad \forall \mathbf{v}_D \in \mathbf{H}^3(\text{div}; \Omega_D). \quad (7.4)$$

Next, concerning hypotheses (H.0), (H.1), (H.2) and (H.3), we start mentioning that (H.0), and (H.1) follow straightforwardly from the definitions in (7.1). Thus, for the Navier–Stokes terms of (H.2) and (H.3), the existence of $\boldsymbol{\psi}_0 \in \mathbf{H}_{00}^{1/2}(\Sigma)$ satisfying (6.10) follows as explained in Lemma 5.6. In turn, similarly to [22, Section 5.3.1], we define

$$\Phi_h(\Sigma) := \left\{ \boldsymbol{\phi}_h : \Sigma \rightarrow \mathbb{R} : \quad \boldsymbol{\phi}_h|_e \in P_k(e) \quad \forall \text{edge } e \in \Sigma_h \right\},$$

and employing [33, Theorem A.1] and [24, Lemma 4.1], we deduce that (6.8) is equivalent to the existence of $\widehat{\beta}_{1,S} > 0$, independent of h , such that,

$$\sup_{\substack{\boldsymbol{\phi}_h \in \Phi_h(\Sigma) \\ \boldsymbol{\phi}_h \neq \mathbf{0}}} \frac{\langle \boldsymbol{\phi}_h, \boldsymbol{\psi}_h \rangle_\Sigma}{\|\boldsymbol{\phi}_h\|_{-1/2, \Sigma}} \geq \widehat{\beta}_{1,S} \|\boldsymbol{\psi}_h\|_{1/2, 0; \Sigma} \quad \forall \boldsymbol{\psi}_h \in \widetilde{\Lambda}_h^S(\Sigma),$$

which follows from [24, Lemma 5.2] and the additional hypotheses of quasi-uniformity in a neighborhood Ω_Σ^S of the interface Σ on the Ω_S -side. Finally, for the Darcy–Forchheimer terms in (H.2) and (H.3) (cf. (6.9) and (6.11), respectively) we establish the following lemmas.

Lemma 7.1 *There exists $\widetilde{\beta}_{1,D} > 0$ independent of h , such that*

$$\sup_{\substack{\mathbf{v}_{D,h} \in \widetilde{\mathbf{X}}_{h,2} \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{\langle \mathbf{v}_{D,h} \cdot \mathbf{n}, \boldsymbol{\xi}_h \rangle_\Sigma}{\|\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\text{div}; \Omega_D)}} \geq \widetilde{\beta}_{1,D} \|\boldsymbol{\xi}_h\|_{1/3, 3/2; \Sigma} \quad \forall \boldsymbol{\xi}_h \in \Lambda_h^D(\Sigma).$$

Proof. See [10, Lemma 4.5] for details. □

Lemma 7.2 *There exists $\tilde{\beta}_D > 0$, independent of h , such that*

$$\sup_{\substack{\mathbf{v}_{D,h} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D) \\ \mathbf{v}_{D,h} \neq \mathbf{0}}} \frac{(\operatorname{div} \mathbf{v}_{D,h}, q_{D,h})_D}{\|\mathbf{v}_{D,h}\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)}} \geq \tilde{\beta}_D \|q_{D,h}\|_{0,\Omega_D} \quad \forall q_{D,h} \in L_{h,0}(\Omega_D). \quad (7.5)$$

Proof. We proceed as in [10, Lemma 4.6]. In fact, let $q_{D,h} \in L_{h,0}(\Omega_D) \subseteq L_0^2(\Omega_D)$ and apply [27, Corollary 2.4] to deduce that there exists $\mathbf{z}_D \in \mathbf{H}_0^1(\Omega_D)$ such that

$$\operatorname{div} \mathbf{z}_D = -q_{D,h} \in \Omega_D \quad \text{and} \quad \|\mathbf{z}_D\|_{1,\Omega_D} \leq C \|q_{D,h}\|_{0,\Omega_D}. \quad (7.6)$$

Now, defining $\mathbf{z}_{D,h} = \Pi_D(\mathbf{z}_D) \in \mathbf{H}_h(\Omega_D)$, employing the continuity of Π_D and (7.4), we deduce that

$$\|\Pi_D(\mathbf{z}_D)\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)} \leq C_D \|\mathbf{z}_D\|_{\mathbf{H}^3(\operatorname{div}; \Omega_D)} \quad \text{and} \quad \operatorname{div} \mathbf{z}_{D,h} = -q_{D,h} \in \Omega_D. \quad (7.7)$$

Therefore, (7.6), (7.7) and the continuous embedding from $H^1(\Omega_D)$ into $L^3(\Omega_D)$ implies (7.5) and conclude the proof. \square

7.2 Raviart–Thomas elements in 3D

Let us now consider the discrete spaces:

$$\begin{aligned} \mathbf{H}_h(\Omega_S) &:= \left\{ \boldsymbol{\tau}_{S,h} \in \mathbf{H}(\operatorname{div}; \Omega_S) : \boldsymbol{\tau}_{S,h}|_T \in \mathbf{RT}_k(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\ \mathbf{H}_h^1(\Omega_S) &:= \left\{ \mathbf{v}_{S,h} \in [\mathcal{C}(\overline{\Omega_S})]^3 : \mathbf{v}_{S,h}|_T \in \mathbf{P}_{k+1}(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\ \mathbb{L}_h(\Omega_S) &:= \left\{ \boldsymbol{\eta}_{S,h} \in \mathbb{L}_{\text{skew}}^2(\Omega_S) : \boldsymbol{\eta}_{S,h}|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^S \right\}, \\ \mathbf{H}_h(\Omega_D) &:= \left\{ \mathbf{v}_{D,h} \in \mathbf{H}^3(\operatorname{div}; \Omega_D) : \mathbf{v}_{D,h}|_T \in \mathbf{RT}_k(T) \quad \forall T \in \mathcal{T}_h^D \right\}, \\ \mathbb{L}_h(\Omega_D) &:= \left\{ q_{D,h} \in L^2(\Omega_D) : q_{D,h}|_T \in \mathbb{P}_k(T) \quad \forall T \in \mathcal{T}_h^D \right\}. \end{aligned} \quad (7.8)$$

Now, proceeding analogously to the 2D case, we introduce an independent triangulation $\Sigma_{\hat{h}}$ of Σ , by triangles K of diameter \hat{h} , and define $\hat{h}_\Sigma := \{\hat{h}_K : K \in \Sigma_{\hat{h}}\}$. Then, denoting by $\partial\Sigma$

the polygonal boundary of Σ , we define

$$\Lambda_h^S(\Sigma) := \left\{ \psi_h \in \mathcal{C}(\Sigma) : \psi_h|_K \in P_{k+1}(K) \quad \forall K \in \Sigma_{\widehat{h}}, \quad \psi_h = 0 \quad \text{on} \quad \partial\Sigma \right\}. \quad (7.9)$$

In turn, similarly to (7.3), we deduce now from [29, Section 2] that a conforming finite element subspace for $W^{1/3,3/2}(\Sigma)$ can be defined by

$$\Lambda_h^D(\Sigma) := \left\{ \xi_h : \Sigma \rightarrow \mathbb{R} : \xi_h|_e \in P_k(K) \quad \forall \text{face } K \in \Sigma_h \right\}. \quad (7.10)$$

Then, we define the global spaces \mathbb{H}_h and \mathbb{Q}_h (cf. (6.3)), by combining (6.2), (6.3), (7.8), (7.9) and (7.10). Now, concerning hypotheses **(H.0)**, **(H.1)**, **(H.2)** and **(H.3)**, we first observe that applying the same arguments as for the 2D case, it follows that **(H.0)**, **(H.1)**, and **(H.3)** hold. However, for the inf-sup conditions in **(H.2)** we employ [21, Lemma 7.5] to conclude that there exists $C_0 \in (0, 1)$ such that for each pair $(h_\Sigma, \widehat{h}_\Sigma)$ verifying $h_\Sigma \leq C_0 \widehat{h}_\Sigma$, the inf-sup condition (6.8) hold, whereas (6.9) follows from Lemma 7.1 taking in account now (7.10).

7.3 Rate of convergence

Now, for both cases 2D and 3D domains, we derive the theoretical rate of convergence of our discrete scheme (6.4). To that end, we first recall recall from [10, Section 5], [19, 26] and [17], the approximation properties of the finite element subspaces involved, which are named after the unknowns to which they are applied later on.

(AP $_{h}^{\sigma_S}$) For each $\delta \in (0, k + 1]$ and for each $\boldsymbol{\tau}_S \in \mathbb{H}_0(\mathbf{div}; \Omega_S) \cap \mathbb{H}^\delta(\Omega_S)$ with $\mathbf{div} \boldsymbol{\tau}_S \in \mathbf{H}^\delta(\Omega_S)$, there holds

$$\text{dist} \left(\boldsymbol{\tau}_S, \mathbb{H}_{h,0}(\Omega_S) \right) := \inf_{\boldsymbol{\tau}_{S,h} \in \mathbb{H}_{h,0}(\Omega_S)} \|\boldsymbol{\tau}_S - \boldsymbol{\tau}_{S,h}\|_{\mathbf{div}; \Omega_S} \leq C h^\delta \left\{ \|\boldsymbol{\tau}_S\|_{\delta, \Omega_S} + \|\mathbf{div} \boldsymbol{\tau}_S\|_{\delta, \Omega_S} \right\}.$$

(AP $_{h}^{\mathbf{u}_S}$) For each $\delta \in (0, k + 1]$ and for each $\mathbf{v}_S \in \mathbf{H}_{\Gamma_S}^1(\Omega_S) \cap \mathbf{H}^{1+\delta}(\Omega_S)$, there holds

$$\text{dist} \left(\mathbf{v}_S, \mathbf{H}_{h,\Gamma_S}^1(\Omega_S) \right) := \inf_{\mathbf{v}_{S,h} \in \mathbf{H}_{h,\Gamma_S}^1(\Omega_S)} \|\mathbf{v}_S - \mathbf{v}_{S,h}\|_{1, \Omega_S} \leq C h^\delta \|\mathbf{v}_S\|_{1+\delta, \Omega_S}.$$

($\mathbf{AP}_h^{\gamma_S}$) For each $\delta \in (0, k + 1]$ and for each $\boldsymbol{\eta}_S \in \mathbb{L}_{\text{skew}}^2(\Omega_S) \cap \mathbb{H}^\delta(\Omega_S)$, there holds

$$\text{dist} \left(\boldsymbol{\eta}_S, \mathbb{L}_h(\Omega_S) \right) := \inf_{\boldsymbol{\eta}_{S,h} \in \mathbb{L}_h(\Omega_S)} \|\boldsymbol{\eta}_S - \boldsymbol{\eta}_{S,h}\|_{0,\Omega_S} \leq C h^\delta \|\boldsymbol{\eta}_S\|_{\delta,\Omega_S}.$$

($\mathbf{AP}_h^{\mathbf{u}_D}$) For each $\delta \in (0, k + 1]$ and for each $\mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D) \cap \mathbf{W}^{\delta,3}(\Omega_D)$ with $\text{div } \mathbf{v}_D \in \mathbf{H}^\delta(\Omega_D)$, there holds

$$\text{dist} \left(\mathbf{v}_D, \mathbf{H}_{h,\Gamma_D}(\Omega_D) \right) := \inf_{\mathbf{v}_{D,h} \in \mathbf{H}_{h,\Gamma_D}(\Omega_D)} \|\mathbf{v}_D - \mathbf{v}_{D,h}\|_{\mathbf{H}^3(\text{div}; \Omega_D)} \leq C h^\delta \left\{ \|\mathbf{v}_D\|_{\delta,3;\Omega_D} + \|\text{div } \mathbf{v}_D\|_{\delta,\Omega_D} \right\}.$$

($\mathbf{AP}_h^{\boldsymbol{\psi}}$) For each $\delta \in (0, k + 1]$ and for each $\boldsymbol{\psi} \in \mathbf{H}_{00}^{1/2}(\Sigma) \cap \mathbf{H}^{1/2+\delta}(\Sigma)$, there holds

$$\text{dist} \left(\boldsymbol{\psi}, \boldsymbol{\Lambda}_h^S(\Sigma) \right) := \inf_{\boldsymbol{\psi}_h \in \boldsymbol{\Lambda}_h^S(\Sigma)} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1/2,00;\Sigma} \leq C h^\delta \|\boldsymbol{\psi}\|_{1/2+\delta,\Sigma}.$$

(\mathbf{AP}_h^λ) For each $\delta \in (0, k + 1]$ and for each $\xi \in W^{\delta,3/2}(\Sigma)$, there holds

$$\text{dist} \left(\xi, \Lambda_h^D(\Sigma) \right) := \inf_{\xi_h \in \Lambda_h^D(\Sigma)} \|\xi - \xi_h\|_{1/3,3/2;\Sigma} \leq C h^{\delta-1/3} \|\xi\|_{\delta,3/2;\Sigma}.$$

($\mathbf{AP}_h^{p_D}$) For each $\delta \in (0, k + 1]$ and for each $q_D \in L_0^2(\Omega_D) \cap \mathbf{H}^\delta(\Omega_D)$, there holds

$$\text{dist} \left(q_D, L_{h,0}(\Omega_D) \right) := \inf_{q_{D,h} \in L_{h,0}(\Omega_D)} \|q_D - q_{D,h}\|_{0,\Omega_D} \leq C h^\delta \|q_D\|_{\delta,\Omega_S}.$$

We remark here, similarly to [10, Section 5], that the sub-optimal approximation property (\mathbf{AP}_h^λ) follows from the fact that $W^{1/3,3/2}(\Sigma)$ is the interpolation space with index $1/(3\delta)$ between $W^{\delta,3/2}(\Sigma)$ and $L^{3/2}(\Sigma)$, and from the estimate $\|\xi - \xi_h\|_{L^{3/2}(\Sigma)} \leq C h^\delta \|\xi\|_{\delta,3/2;\Sigma}$, which is valid for all $\xi \in W^{\delta,3/2}(\Sigma)$ and $\xi_h := \mathcal{P}_\Sigma(\xi)$, with \mathcal{P}_Σ being the $L^2(\Sigma)$ -orthogonal projection onto $\Lambda_h^D(\Sigma)$ (cf. [16, Proposition 1.135]). In fact, given $\xi \in W^{\delta,3/2}(\Sigma)$ there exists a constant $C > 0$, depending on Σ , such that

$$\|\xi - \xi_h\|_{1/3,3/2;\Sigma} \leq c \|\xi - \xi_h\|_{L^{3/2}(\Sigma)}^{1-1/(3\delta)} \|\xi\|_{\delta,3/2;\Sigma}^{1/(3\delta)} \leq C h^{\delta-1/3} \|\xi\|_{\delta,3/2;\Sigma},$$

where we have used the fact that ξ_h is piecewise polynomial of degree $\leq k$ and then for each $\delta \in (0, k + 1]$ there holds $\|\xi - \xi_h\|_{\delta,3/2;\Sigma} \leq C \|\xi\|_{\delta,3/2;\Sigma}$.

It follows that there exist positive constants $C(\underline{\mathbf{t}})$, $C(\underline{\boldsymbol{\varphi}})$, and $C(\underline{\mathbf{p}})$, depending on the extra regularity assumptions for $\underline{\mathbf{t}}$, $\underline{\boldsymbol{\varphi}}$, and $\underline{\mathbf{p}}$, respectively, and whose explicit expressions are obtained from the right-hand side of the foregoing approximation properties, such that

$$\text{dist}(\underline{\mathbf{t}}, \mathbf{X}_h) \leq C(\underline{\mathbf{t}}) h^\delta, \quad \text{dist}(\underline{\boldsymbol{\varphi}}, \mathbf{Y}_h) \leq C(\underline{\boldsymbol{\varphi}}) h^{\delta-1/3} \quad \text{and} \quad \text{dist}(\underline{\mathbf{p}}, \mathbf{Q}_h) \leq C(\underline{\mathbf{p}}) h^\delta.$$

Then, we establish the theoretical rate of convergence of our Galerkin scheme (6.4). Notice that, defining $s := \min\{\delta, k+1\}$, at least a sub-optimal rate of convergence of order $O(h^{(s-1/3)/2})$ is confirmed.

Theorem 7.3 *Assume that the hypotheses of Theorem 6.6 hold. Let $((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \in \mathbb{H} \times \mathbb{Q}$ with $\mathbf{u}_S \in \mathbf{W}_r$ and $((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ with $\mathbf{u}_{S,h} \in \mathbf{W}_r^h$ be the unique solutions of the continuous and discrete problems (3.5) and (6.4), respectively. Assume further that there exists $\delta > 0$, such that $\boldsymbol{\sigma}_S \in \mathbb{H}^\delta(\Omega_S)$, $\text{div} \boldsymbol{\sigma}_S \in \mathbf{H}^\delta(\Omega_S)$, $\mathbf{u}_S \in \mathbf{H}^{1+\delta}(\Omega_S)$, $\boldsymbol{\gamma}_S \in \mathbb{H}^\delta(\Omega_S)$, $\mathbf{u}_D \in \mathbf{W}^{\delta,3}(\Omega_D)$, $\text{div} \mathbf{u}_D \in \mathbf{H}^\delta(\Omega_D)$, $\boldsymbol{\varphi} \in \mathbf{H}^{1/2+\delta}(\Sigma)$, $\lambda \in \mathbf{W}^{\delta,3/2}(\Sigma)$ and $p_D \in \mathbf{H}^\delta(\Omega_D)$. Then, defining $s := \min\{\delta, k+1\}$, there exists a positive constant $C((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}})$ depending on $C(\underline{\mathbf{t}})$, $C(\underline{\boldsymbol{\varphi}})$, and $C(\underline{\mathbf{p}})$, all them independent of h and the continuous and discrete solutions, such that*

$$\begin{aligned} & \|((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) - ((\underline{\mathbf{t}}_h, \underline{\boldsymbol{\varphi}}_h), \underline{\mathbf{p}}_h)\|_{\mathbb{H} \times \mathbb{Q}} \\ & \leq C((\underline{\mathbf{t}}, \underline{\boldsymbol{\varphi}}), \underline{\mathbf{p}}) \{h^{(s-1/3)/2} + h^{s/2} + h^{s-1/3} + h^s + h^{2(s-1/3)} + h^{2s}\}. \end{aligned}$$

Proof. It follows from a direct application of Theorem 6.6 and the approximation properties of the discrete subspaces. We omit further details. \square

Numerical Results

In this chapter we present some examples illustrating the performance of our augmented mixed finite element scheme (6.4) on a set of quasi-uniform triangulations of the corresponding domains. Our implementation is based on a FreeFem++ code [31], in conjunction with the direct linear solver UMFPACK [14].

In order to solve the nonlinear problem (6.4), given $\mathbf{0} \neq \mathbf{w}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D)$ we introduce the Gâteaux derivate associated to \mathcal{A}_D (cf. (3.8)):

$$\mathcal{D}\mathcal{A}_D(\mathbf{w}_D)(\mathbf{u}_D, \mathbf{v}_D) := \frac{\mu}{\rho}(\mathbf{K}^{-1}\mathbf{u}_D, \mathbf{v}_D)_D + \frac{F}{\rho}(|\mathbf{w}_D|\mathbf{u}_D, \mathbf{v}_D)_D + \frac{F}{\rho} \left(\frac{\mathbf{w}_D \cdot \mathbf{u}_D}{|\mathbf{w}_D|}, \mathbf{w}_D \cdot \mathbf{v}_D \right)_D,$$

for all $\mathbf{u}_D, \mathbf{v}_D \in \mathbf{H}_{\Gamma_D}^3(\text{div}; \Omega_D)$. In this way, we propose the Newton-type strategy: Given

$\mathbf{t}_h^0 := (\underline{\boldsymbol{\sigma}}_h^0, \mathbf{u}_{D,h}^0) \in \mathbf{X}_h$ with $\mathbf{u}_{D,h}^0 \neq \mathbf{0}$, for $m \geq 1$, find $((\mathbf{t}_h^m, \underline{\boldsymbol{\varphi}}_h^m), \mathbf{p}_h^m) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned}
 & [\mathcal{A}_S(\underline{\boldsymbol{\sigma}}_{S,h}^m), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}_S(\mathbf{u}_{S,h}^{m-1})(\underline{\boldsymbol{\sigma}}_{S,h}^m), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}_S(\mathbf{u}_{S,h}^m)(\underline{\boldsymbol{\sigma}}_{S,h}^{m-1}), \underline{\boldsymbol{\tau}}_h] + \mathcal{D}\mathcal{A}_D(\mathbf{u}_{D,h}^{m-1})(\mathbf{u}_{D,h}^m, \mathbf{v}_{D,h}) \\
 & + [\mathbf{b}(\mathbf{r}_h), \underline{\boldsymbol{\varphi}}_h^m] + [\mathbf{b}(\mathbf{t}_h^m), \underline{\boldsymbol{\psi}}_h] - [\mathbf{c}(\underline{\boldsymbol{\varphi}}_h^m), \underline{\boldsymbol{\psi}}_h] + [\mathbf{B}(\mathbf{r}_h, \underline{\boldsymbol{\psi}}_h), \mathbf{p}_h^m] \\
 & = [\mathcal{B}_S(\mathbf{u}_{S,h}^{m-1})(\underline{\boldsymbol{\sigma}}_{S,h}^{m-1}), \underline{\boldsymbol{\tau}}_h] + \frac{\mathbf{F}}{\rho} (|\mathbf{u}_{D,h}^{m-1}| \mathbf{u}_{D,h}^{m-1}, \mathbf{v}_{D,h})_D + [\mathbf{F}, (\mathbf{r}_h, \underline{\boldsymbol{\psi}}_h)], \\
 & [\mathbf{B}(\mathbf{t}_h^m, \underline{\boldsymbol{\varphi}}_h), \mathbf{q}_h] = [\mathbf{G}, \mathbf{q}_h],
 \end{aligned} \tag{8.1}$$

for all $((\mathbf{r}_h, \underline{\boldsymbol{\psi}}_h), \mathbf{q}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$.

In all the numerical experiments below, the iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, i.e.,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{l^2}}{\|\mathbf{coeff}^{m+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ is the standard l^2 -norm in \mathbb{R}^N , with N denoting the total number of degrees of freedom defining the finite element subspaces \mathbb{H}_h and \mathbb{Q}_h , and tol is a fixed tolerance chosen as $tol = 1E - 06$. As usual, the individual errors are denoted by:

$$\begin{aligned}
 e(\boldsymbol{\sigma}_S) &:= \|\boldsymbol{\sigma}_S - \boldsymbol{\sigma}_{S,h}\|_{\text{div}, \Omega_S}, & e(\mathbf{u}_S) &:= \|\mathbf{u}_S - \mathbf{u}_{S,h}\|_{1, \Omega_S}, & e(\boldsymbol{\gamma}_S) &:= \|\boldsymbol{\gamma}_S - \boldsymbol{\gamma}_{S,h}\|_{0, \Omega_S}, \\
 e(p_S) &:= \|p_S - p_{S,h}\|_{0, \Omega_S}, & e(\mathbf{u}_D) &:= \|\mathbf{u}_D - \mathbf{u}_{D,h}\|_{\mathbf{H}^3(\text{div}; \Omega_D)}, & e(p_D) &:= \|p_D - p_{D,h}\|_{0, \Omega_D}, \\
 e(\boldsymbol{\varphi}) &:= \|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{(0,1), \Sigma}, & e(\lambda) &:= \|\lambda - \lambda_h\|_{L^{3/2}(\Sigma)},
 \end{aligned}$$

where $p_{S,h}$ is the postprocessed pressure given by

$$p_{S,h} := -\frac{1}{n} \text{tr}(\boldsymbol{\sigma}_{S,h} + (\mathbf{u}_{S,h} \otimes \mathbf{u}_{S,h})) - \ell_h \quad \text{in } \Omega_S.$$

Notice that, since the natural norms to measure the error of the interface unknowns $\|\lambda - \lambda_h\|_{1/3, 3/2; \Sigma}$ and $\|\boldsymbol{\varphi} - \boldsymbol{\varphi}_h\|_{1/2, 0; \Sigma}$ are not computable, we have decided to replace them respectively by $\|\cdot\|_{L^{3/2}(\Sigma)}$ and $\|\cdot\|_{(0,1), \Sigma}$, where the last one is defined based on the fact that

$\mathbf{H}^{1/2}(\Sigma)$ is the interpolation space with index $1/2$ between $\mathbf{H}^1(\Sigma)$ and $\mathbf{L}^2(\Sigma)$:

$$\|\boldsymbol{\psi}\|_{(0,1),\Sigma} := \|\boldsymbol{\psi}\|_{0,\Sigma}^{1/2} \|\boldsymbol{\psi}\|_{1,\Sigma}^{1/2} \quad \forall \boldsymbol{\psi} \in \mathbf{H}^1(\Sigma).$$

Next, we define the experimental rates of convergence

$$\begin{aligned} r(\boldsymbol{\sigma}_S) &:= \frac{\log(e(\boldsymbol{\sigma}_S)/e'(\boldsymbol{\sigma}_S))}{\log(h_S/h'_S)}, & r(\mathbf{u}_S) &:= \frac{\log(e(\mathbf{u}_S)/e'(\mathbf{u}_S))}{\log(h_S/h'_S)}, & r(\boldsymbol{\gamma}_S) &:= \frac{\log(e(\boldsymbol{\gamma}_S)/e'(\boldsymbol{\gamma}_S))}{\log(h_S/h'_S)}, \\ r(p_S) &:= \frac{\log(e(p_S)/e'(p_S))}{\log(h_S/h'_S)}, & r(\mathbf{u}_D) &:= \frac{\log(e(\mathbf{u}_D)/e'(\mathbf{u}_D))}{\log(h_D/h'_D)}, & r(p_D) &:= \frac{\log(e(p_D)/e'(p_D))}{\log(h_D/h'_D)}, \\ r(\boldsymbol{\varphi}) &:= \frac{\log(e(\boldsymbol{\varphi})/e'(\boldsymbol{\varphi}))}{\log(\hat{h}_\Sigma/\hat{h}'_\Sigma)}, & r(\lambda) &:= \frac{\log(e(\lambda)/e'(\lambda))}{\log(h_\Sigma/h'_\Sigma)}, \end{aligned}$$

where h_\star and h'_\star ($\star \in \{S, D, \Sigma\}$) denote two consecutive mesh sizes with their respective errors e and e' , respectively. In turn, we take \hat{h}_Σ as two times h_Σ , which comes from the restriction on the mesh sizes $h_\Sigma \leq C_0 \hat{h}_\Sigma$ when considering the constant $C_0 = 1/2$. The numeric results confirm that this choice is suitable. The examples to be considered in this chapter are described next. In all of them, for the sake of simplicity, we choose the parameters $\mu = 1$, $\rho = 1$, $\omega = 1$ and $\mathbf{K} = \mathbb{I}$, and according to (5.12), the stabilization parameters are taken as $\kappa_1 = 1/(2\mu)$, $\kappa_2 = 2\mu$ and $\kappa_3 = C_{K_0}\mu$, where, similarly to [11, Section 7] we choose heuristically $C_{K_0} = 1/2$. Additionally, regarding the conditions $(\text{tr } \boldsymbol{\sigma}_{S,h}, 1)_S = 0$ and $(p_{D,h}, 1)_D = 0$, these are imposed via a penalization strategy.

8.1 Example 1: Tombstone-shaped domain without source in the porous media.

In our first example we consider a semi-disk-shaped fluid domain coupled with a porous square, i.e., $\Omega_S := \{(x_1, x_2) : x_1^2 + (x_2 - 0.5)^2 < 0.5^2, x_2 > 0.5\}$, $\Omega_D := (-0.5, 0.5)^2$ and $\Sigma := (-0.5, 0.5) \times \{0.5\}$. We consider the Forchheimer number $F = 1$ and the data \mathbf{f}_S , \mathbf{f}_D , and g_D , are adjusted so that the exact solution in the tombstone-shaped domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$ is given by the smooth functions

$$\begin{aligned} \mathbf{u}_S(x_1, x_2) &= \begin{pmatrix} \cos(\pi x_1) \sin(\pi x_2) \\ -\sin(\pi x_1) \cos(\pi x_2) \end{pmatrix} && \text{in } \Omega_S, \\ \mathbf{u}_D(x_1, x_2) &= \begin{pmatrix} \pi \exp(x_1) \sin(\pi x_2) \\ \exp(x_1) \cos(\pi x_2) \end{pmatrix} && \text{in } \Omega_D, \\ p_\star(x_1, x_2) &= \sin(\pi x_1) \sin(\pi x_2) && \text{in } \Omega_\star, \quad \text{with } \star \in \{S, D\}. \end{aligned}$$

Notice that the source of the porous media is $g_D = 0$. Notice also that this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ . However, the Beavers–Joseph–Saffman condition (cf. (2.8)) is not satisfied, the Dirichlet boundary condition on Γ_S and the Neumann boundary condition on Γ_D are both non-homogeneous. In this way, the right-hand side of the resulting system must be modified accordingly.

DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$r(\boldsymbol{\gamma}_S)$	$e(p_S)$	$r(p_S)$
230	0.330	2.5485	–	1.4696	–	0.5276	–	0.3729	–
800	0.191	1.2766	1.256	0.4464	2.165	0.2901	1.087	0.1590	1.549
2872	0.091	0.6185	0.983	0.1657	1.344	0.1442	0.948	0.0762	0.997
10991	0.049	0.3071	1.111	0.0774	1.209	0.0729	1.082	0.0387	1.077
43462	0.024	0.1511	1.019	0.0365	1.081	0.0336	1.112	0.0188	1.035
171401	0.013	0.0763	1.155	0.0184	1.159	0.0177	1.083	0.0094	1.175

h_D	\widehat{h}_Σ	h_Σ	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\lambda)$	iter
0.373	1/2	1/4	0.7464	–	0.1748	–	0.5282	–	0.1554	–	6
0.190	1/4	1/8	0.3480	1.133	0.0639	1.495	0.2294	1.203	0.0708	1.135	7
0.095	1/8	1/16	0.1678	1.053	0.0305	1.067	0.1075	1.094	0.0351	1.012	7
0.054	1/16	1/32	0.0856	1.172	0.0151	1.219	0.0529	1.023	0.0175	1.008	8
0.025	1/32	1/64	0.0427	0.908	0.0075	0.916	0.0251	1.073	0.0087	1.004	8
0.015	1/64	1/128	0.0214	1.280	0.0037	1.293	0.0128	0.971	0.0043	1.001	8

Table 8.1: Example 1, Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 1$.

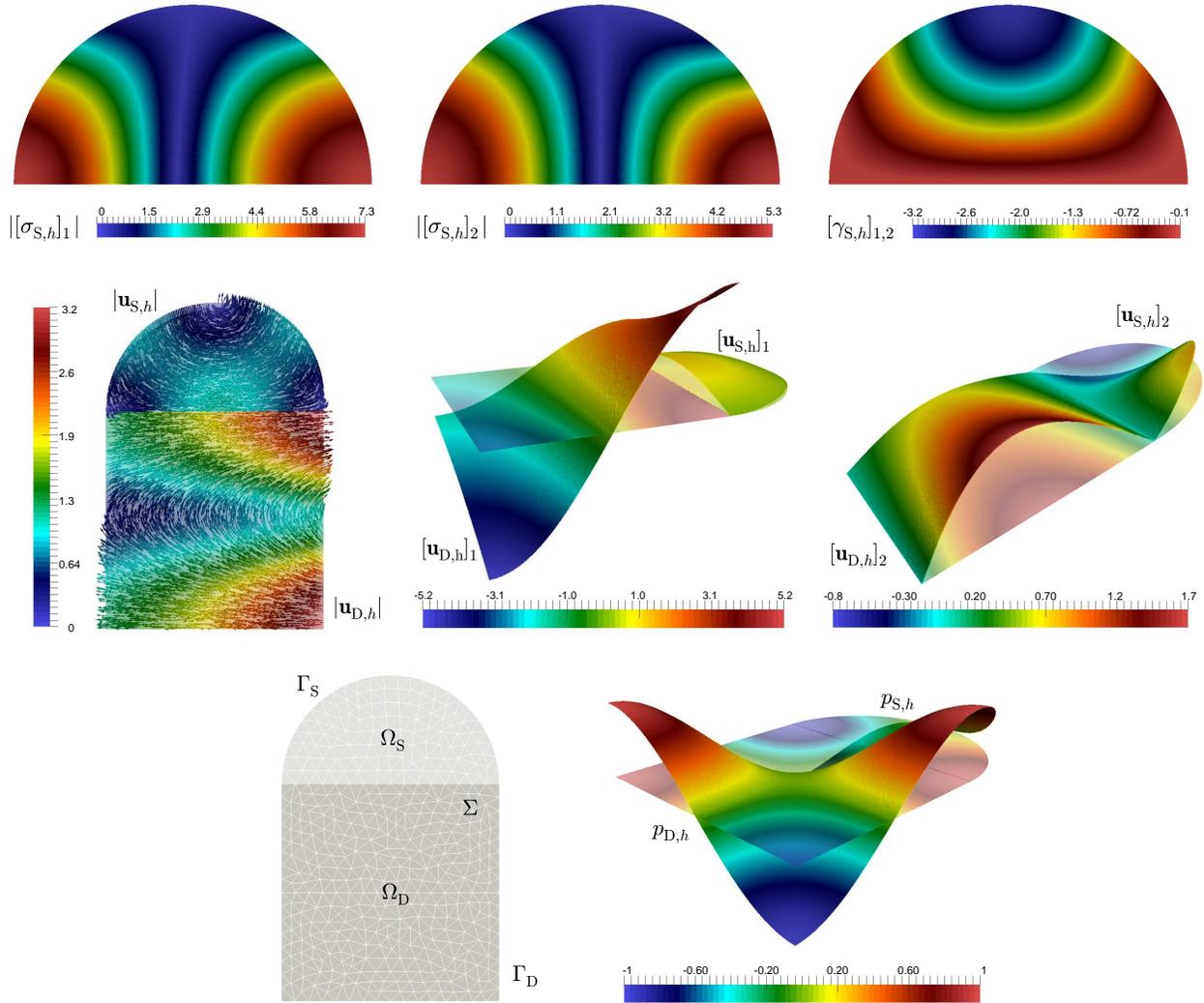


Figure 8.1: Example 1. Approximated spectral norm of the pseudostress tensor components and the skew-symmetric part of the Navier–Stokes velocity gradient (top panel), velocity streamlines and velocity components on the whole domain (middle panel), and geometry configuration and pressure field in the whole domain (bottom panel).

8.2 Example 2: Inverted-L-shaped domain coupled with a square domain.

In our second example, we consider an inverted-L-shaped domain coupled with a square, which yields a porous medium partially surrounded by a fluid. More precisely, we consider the domain $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, with $\Omega_D := (-1, 0)^2$, $\Omega_S := (-1, 1)^2 \setminus \Omega_D$ and $\Sigma := (-1, 0) \times \{0\} \cup \{0\} \times (-1, 0)$. The Forchheimer number is chosen as $F = 1$ and the data \mathbf{f}_S , \mathbf{f}_D , and g_D , are adjusted so that the exact solution in the square Ω is given by the smooth functions

$$\begin{aligned} \mathbf{u}_S(x_1, x_2) &= \begin{pmatrix} -\pi \sin(\pi x_1) \cos(\pi x_2) \\ \pi \cos(\pi x_1) \sin(\pi x_2) \end{pmatrix} \quad \text{in } \Omega_S, \\ \mathbf{u}_D(x_1, x_2) &= \begin{pmatrix} \sin(\pi x_1) \exp(x_2) \\ \exp(x_1) \sin(\pi x_2) \end{pmatrix} \quad \text{in } \Omega_D, \\ p_\star(x_1, x_2) &= \cos(\pi x_1) \cos(\pi x_2) \quad \text{in } \Omega_\star, \quad \text{with } \star \in \{S, D\}. \end{aligned}$$

Notice that, this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ and $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D . However, the Beavers–Joseph–Saffman condition (cf. (2.8)) is not satisfied and the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_S is non-homogeneous and therefore the right-hand side of the resulting system must be modified accordingly.

DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$r(\boldsymbol{\gamma}_S)$	$e(p_S)$	$r(p_S)$
756	0.375	23.6030	–	5.3592	–	4.6970	–	3.3911	–
2847	0.195	11.3037	1.223	2.7705	1.006	2.5019	0.961	1.7970	0.969
10644	0.096	5.6685	0.977	1.3917	0.974	1.3373	0.886	0.8695	1.027
42043	0.052	2.7945	1.152	0.6885	1.146	0.6898	1.078	0.4231	1.173
165156	0.029	1.4040	1.198	0.3452	1.201	0.3468	1.197	0.2097	1.221
657612	0.015	0.7003	0.993	0.1728	0.988	0.1723	0.998	0.1057	0.978

h_D	\widehat{h}_Σ	h_Σ	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\lambda)$	iter
0.373	1/2	1/4	0.4384	–	0.1273	–	2.0677	–	0.2560	–	4
0.190	1/4	1/8	0.1974	1.185	0.0583	1.159	1.1721	0.819	0.1151	1.154	4
0.093	1/8	1/16	0.1002	0.946	0.0306	0.900	0.5560	1.076	0.0556	1.049	4
0.051	1/16	1/32	0.0491	1.183	0.0150	1.184	0.2832	0.973	0.0277	1.007	4
0.025	1/32	1/64	0.0247	0.982	0.0074	1.004	0.1418	0.997	0.0138	1.003	4
0.014	1/64	1/128	0.0124	1.153	0.0037	1.147	0.0701	1.017	0.0069	1.001	4

Table 8.2: Example 2, Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 1$.

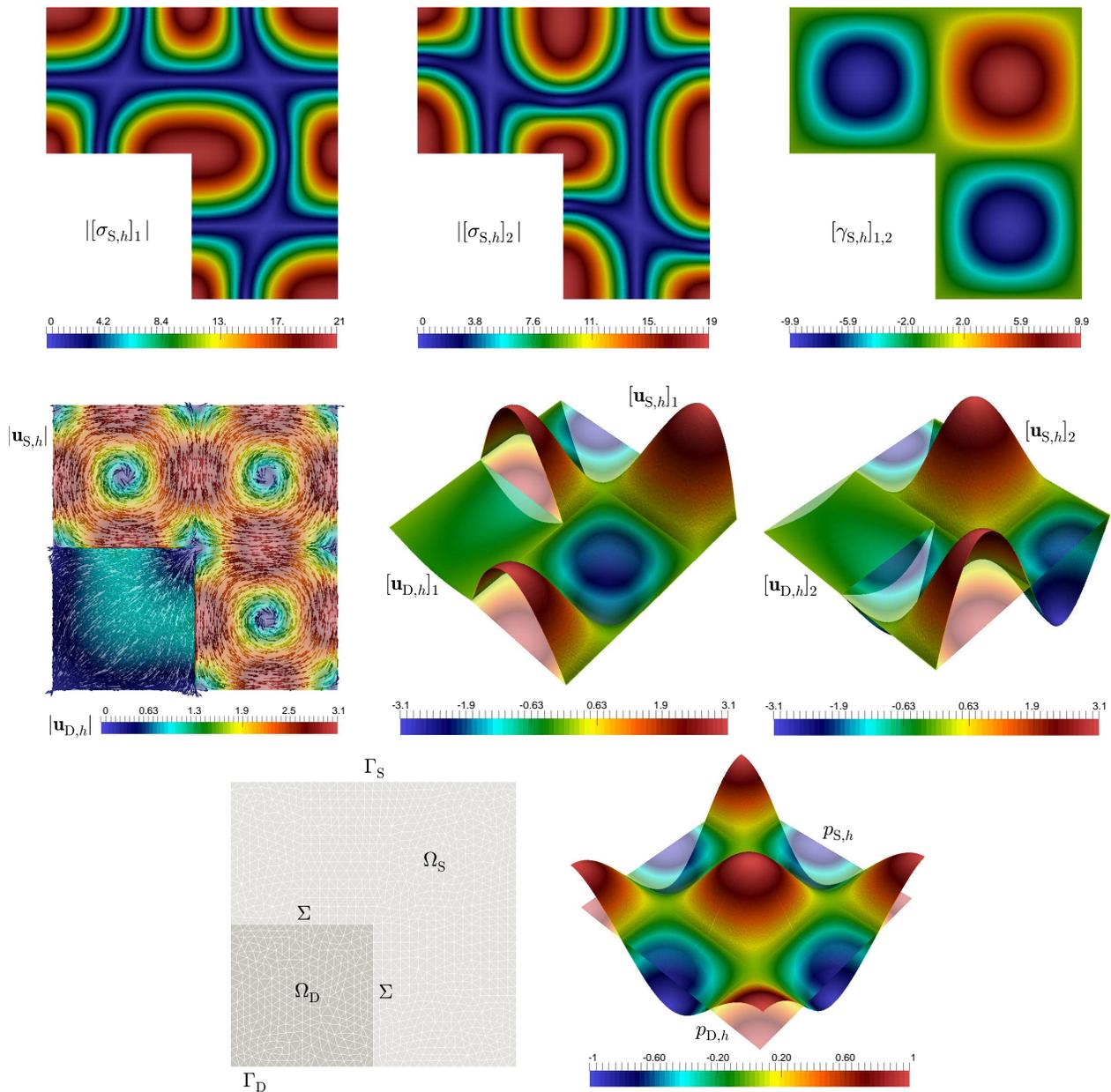


Figure 8.2: Example 2. Approximated spectral norm of the pseudostress tensor components and the skew-symmetric part of the Navier–Stokes velocity gradient (top panel), velocity streamlines and velocity components on the whole domain (middle panel), and geometry configuration and pressure field in the whole domain (bottom panel).

8.3 Example 3: 2D helmet-shaped domain with different Forchheimer numbers.

In our last example, and inspired by [10], we focus on the performance of the numerical method (8.1) with respect to the number of Newton iterations required to achieve certain tolerance given different Forchheimer numbers. Hence, we consider

$$F \in \{0, 10^0, 10^1, 10^2, 10^3, 10^4, 10^5\},$$

the 2D helmet-shaped domain described by $\Omega = \Omega_S \cup \Sigma \cup \Omega_D$, where $\Omega_D := (-1, 1) \times (-0.5, 0)$, $\Sigma := (-1, 1) \times \{0\}$, and $\Omega_S := (-1, 1) \times (0, 1.25) \setminus \tilde{\Omega}_S$ with $\tilde{\Omega}_S := (-0.75, 0.75) \times (0.25, 1.25)$. The data \mathbf{f}_S , \mathbf{f}_D , and g_D , are adjusted so that the exact solution in the 2D helmet-shaped domain Ω is given by the smooth functions

$$\begin{aligned} \mathbf{u}_S(x_1, x_2) &= \begin{pmatrix} -\sin(2\pi x_1) \cos(2\pi x_2) \\ \cos(2\pi x_1) \sin(2\pi x_2) \end{pmatrix} \quad \text{in } \Omega_S, \\ \mathbf{u}_D(x_1, x_2) &= \begin{pmatrix} \sin(2\pi x_1) \exp(x_2) \\ \exp(x_1) \sin(2\pi x_2) \end{pmatrix} \quad \text{in } \Omega_D, \\ p_\star(x_1, x_2) &= \sin(\pi x_1) \exp(x_2) \quad \text{in } \Omega_\star, \quad \text{with } \star \in \{S, D\}. \end{aligned}$$

Notice that, this solution satisfies $\mathbf{u}_S \cdot \mathbf{n} = \mathbf{u}_D \cdot \mathbf{n}$ on Σ and $\mathbf{u}_D \cdot \mathbf{n} = 0$ on Γ_D . However, the Beavers–Joseph–Saffman condition (cf. (2.8)) is not satisfied and the Dirichlet boundary condition for the Navier–Stokes velocity on Γ_S is non-homogeneous and therefore the right-hand side of the resulting system must be modified accordingly.

$F \backslash h$	0.200	0.100	0.050	0.026	0.014	0.007
0	4	3	3	3	3	3
10^0	5	5	5	5	5	5
10^1	7	8	8	9	9	9
10^2	8	9	10	10	11	11
10^3	9	9	10	11	11	12
10^4	9	9	10	11	12	12
10^5	9	9	10	11	12	12

Table 8.3: Example 3, Performance of the iterative method (number of Newton iterations) upon variations of the Forchheimer number F .

DOF	h_S	$e(\boldsymbol{\sigma}_S)$	$r(\boldsymbol{\sigma}_S)$	$e(\mathbf{u}_S)$	$r(\mathbf{u}_S)$	$e(\boldsymbol{\gamma}_S)$	$r(\boldsymbol{\gamma}_S)$	$e(p_S)$	$r(p_S)$
1264	0.188	14.7840	–	2.1893	–	1.5363	–	1.4364	–
4833	0.100	7.1648	1.153	1.0943	1.104	0.7868	1.065	0.6581	1.243
17586	0.050	3.6068	0.991	0.5431	1.011	0.4149	0.923	0.3027	1.120
69327	0.026	1.7936	1.054	0.2718	1.045	0.2087	1.038	0.1477	1.083
269604	0.014	0.8992	1.183	0.1350	1.200	0.1044	1.187	0.0738	1.190
1076768	0.007	0.4490	0.932	0.0674	0.932	0.0523	0.926	0.0368	0.935

h_D	\widehat{h}_Σ	h_Σ	$e(\mathbf{u}_D)$	$r(\mathbf{u}_D)$	$e(p_D)$	$r(p_D)$	$e(\boldsymbol{\varphi})$	$r(\boldsymbol{\varphi})$	$e(\lambda)$	$r(\lambda)$	iter
0.200	1/4	1/8	1.2761	–	0.1131	–	1.1300	–	0.2569	–	7
0.095	1/8	1/16	0.6135	0.984	0.0388	1.438	0.4481	1.335	0.0744	1.787	8
0.049	1/16	1/32	0.3115	1.037	0.0151	1.447	0.2129	1.074	0.0302	1.300	8
0.026	1/32	1/64	0.1566	1.081	0.0067	1.283	0.1003	1.086	0.0142	1.089	9
0.013	1/64	1/128	0.0784	0.968	0.0033	0.999	0.0502	0.997	0.0070	1.026	9
0.007	1/128	1/256	0.0393	1.204	0.0016	1.221	0.0249	1.011	0.0035	1.010	9

Table 8.4: Example 3, Degrees of freedom, mesh sizes, errors, convergence history and Newton iteration count for the approximation of the Navier–Stokes/Darcy–Forchheimer problem with $F = 10$.

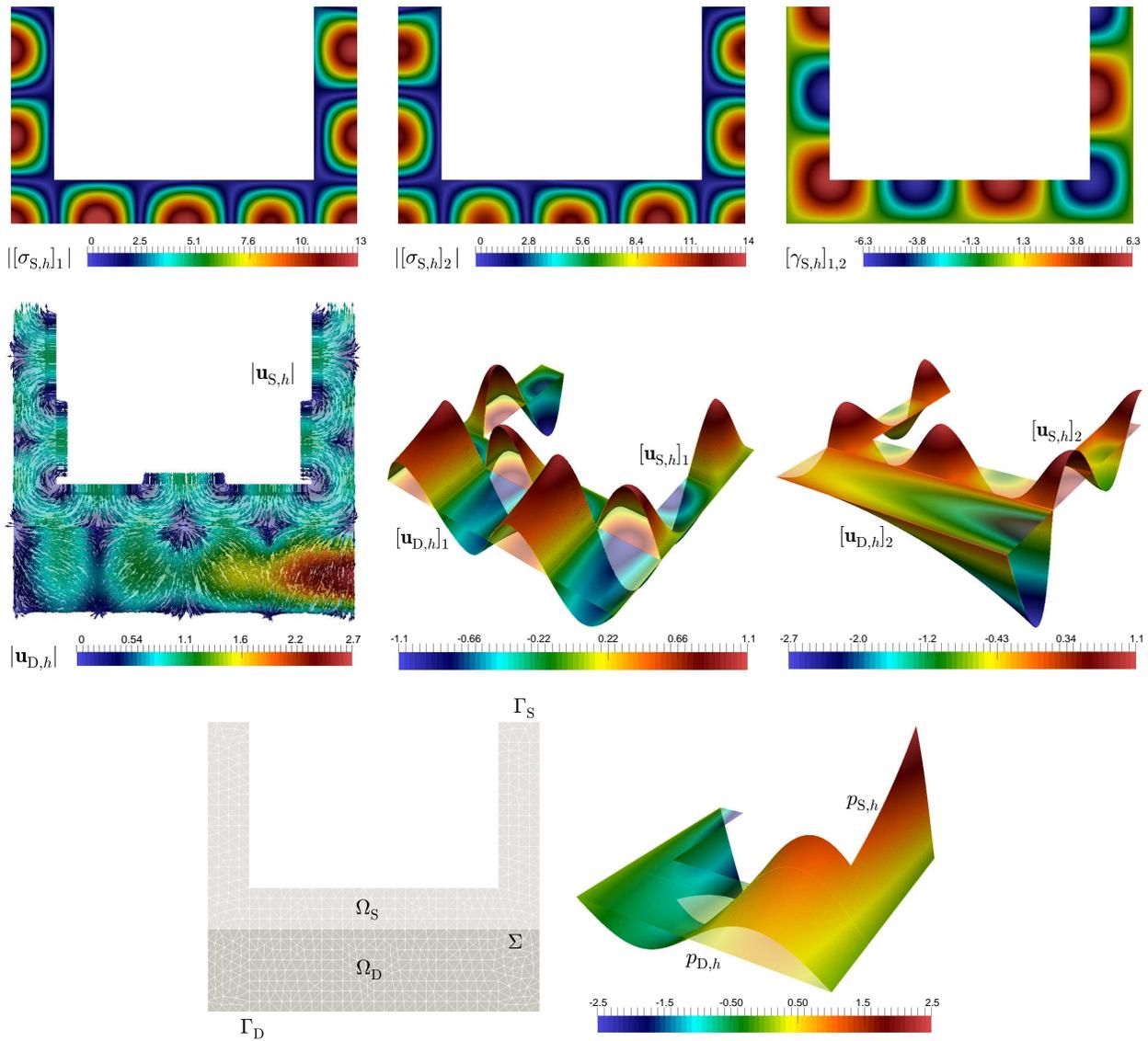


Figure 8.3: Example 3. Approximated spectral norm of the pseudostress tensor components and the skew-symmetric part of the Navier–Stokes velocity gradient (top panel), velocity streamlines and velocity components on the whole domain (middle panel), and geometry configuration and pressure field in the whole domain (bottom panel).

8.4 Summary of examples

In Tables 8.1, 8.2 and 8.4 we summarize the convergence history for a sequence of quasi-uniform triangulations, considering the finite element spaces introduced in Section 7.1 with $k = 0$, and solving the nonlinear problem (8.1), which require around eight, four and nine Newton iterations for the Examples 1, 2 and 3, respectively. We observe that the sub-optimal rate of convergence $O(h^{(k+2/3)/2})$ provided by Theorem 7.3 (when $\delta = k + 1$) is attained in all the cases (with $k = 0$). Even more, the numerical result suggest that there exist a way to prove optimal rate of convergence $O(h^{k+1})$. In Table 8.3 we show the behaviour of the iterative method (8.1) as a function of the Forchheimer number F , considering different mesh sizes $h := \max\{h_S, h_D\}$, and a tolerance $tol = 1E - 06$. Here we observe that the higher the parameter F the higher the number of iterations as it occurs also in the Newton method for the Navier–Stokes/Darcy–Forchheimer coupled problem. Notice also that when $F = 0$ the Darcy–Forchheimer equations reduce to the classical linear Darcy equations and as expected the iterative Newton method (8.1) is faster.

On the other hand, the approximated spectral norm of the pseudostress tensor components, the skew-symmetric part of the Navier–Stokes velocity gradient, the velocity streamlines, the velocity components on the whole domain, the geometry configuration and the pressure field in the whole domain of the approximate solutions for the three examples are displayed in Figures 8.1, 8.2 and 8.3. All the figures were obtained with 171401, 657612, and 1076768 degrees of freedom for the Examples 1, 2, and 3, respectively. In particular, we can observe in Figures 1 and 3 that the second components of \mathbf{u}_S and \mathbf{u}_D coincide on Σ as expected, and hence, the continuity of the normal components of the velocities on Σ is preserved. In turn, we can see that the velocity streamlines are higher in the Darcy–Forchheimer domain. Moreover, it can be seen that the pressure is continuous in the whole domain and preserves the sinusoidal behaviour. Finally, similarly to Figures 1 and 3, in Figure 2 we can also observe the continuity of the normal components of the velocities on Σ since their first component of \mathbf{u}_S and \mathbf{u}_D coincide on $\{0\} \times (-1, 0)$, whereas their second components coincide on $(-1, 0) \times \{0\}$.

Conclusions and future work

9.1 Conclusions

In this thesis we developed an augmented fully-mixed finite element method for the coupling of the Navier–Stokes and Darcy–Forchheimer equations. We proved solvability of both continuous and discrete problems as well as their convergence, all illustrated by means of examples and numerical simulations. The main conclusions of this work are:

1. We obtained an extension of the works done in [2] and [10]. Unlike [2] we consider the Navier–Stokes equations in the free fluid, and a mixed formulation in this region instead of the standard velocity-pressure formulation as in [10].
2. A fully-mixed augmented formulation was developed for the problem as well as sufficient conditions to guarantee its continuous and discrete well-posedness by utilizing classical theory of monotone operators and fixed-point arguments.
3. The fully-mixed finite element method proposed here was shown to be sub-optimally

convergent. However, the numerical examples presented shows an optimal convergence suggesting that there should exist a way to prove such result.

9.2 Future work

The methods developed and the results obtained in this thesis have motivated several ongoing and future projects. Some of them are described below:

1. As a natural continuation, we are interested in carrying out an a posteriori error analysis for the fully-mixed formulation of the Navier–Stokes/Darcy–Forchheimer coupled problem in order to improve its robustness in the presence of complex geometries or solutions with high gradients and singularities.
2. As a complement and motivated by [11], we are interested in analyzing the coupling of the Navier–Stokes and Darcy Forchheimer equations assuming non-linear viscosity for the fluid.
3. Numerical experiments with $k \geq 1$ for both cases 2D and 3D, are also planned.
4. The extension of this finite element method to the unsteady state case will be considering in the future as well.

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