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TESIS PREGRADO

UN MÉTODO DE GALERKIN DISCONTINUO PARA EL PROBLEMA DEL BIARMÓNICO

(A DISCONTINUOUS GALERKIN METHOD FOR THE BIHARMONIC PROBLEM)

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A Discontinuous Galerkin Method for the Biharmonic Problem

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Cord-cordis es la palabra latina que denomina al corazón. Y de Cord-cordis nace la palabra cordel aquello que amarra sin soltar.
Una cordada, pues, es ese grupo de personas unidas por un corazón, que se confirma con la mera cuerda.

Dedicado a mi familia.

Abstract

A Hybridizable Discontinuous Galerkin (HDG) method for solving the biharmonic problem $\Delta^2 u = f$ is proposed and analyzed in this work. More precisely, we employ a Discontinuous Galerkin (DG) method based on a system of first-order equations, which we propose to approximate u, ∇u, $\mathcal{H}(u)$ and $\nabla \cdot \mathcal{H}(u)$ simultaneously, where $\mathcal{H}$ corresponds to the Hessian matrix. This method allows us to eliminate all the interior variables locally to obtain a global system for $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$ that approximate u and ∇u, respectively, on the interfaces of the triangulation. As a consequence the only globally coupled degrees of freedom are those of the approximations of u and ∇u on the faces of the elements. We also carry out an priori error analysis using the orthogonal L^2-projection and conclude that the orders of convergence for the errors in the approximation of $\mathcal{H}(u)$, $\nabla \cdot \mathcal{H}(u)$, ∇u and u are $k + 1/2$, $k - 1/2$, k, and $k + 1$, respectively, where $k \geq 1$ is the polynomial degree of the discrete spaces. Our numerical results suggest that the approximations of $\mathcal{H}(u)$, ∇u and u converge with optimal order $k + 1$ and the approximation of $\nabla \cdot \mathcal{H}(u)$ converge with suboptimal order k.

Resumen

Proponemos y analizamos un método de Galerkin Discontinuo hibridizable (HDG por sus siglas en inglés) para resolver el problema del biarmónico $\Delta^2 u = f$. Más precisamente, utilizamos un método HDG basado en un sistema de ecuaciones de primer orden, el cual sugiere aproximar u, ∇u, $\mathcal{H}(u)$ and $\nabla \cdot \mathcal{H}(u)$ simultáneamente, donde $\mathcal{H}$ denota a la matriz Hessiana. Este método nos permite eliminar todas las variables interiores localmente para obtener un sistema global para $\hat{u}_h$ y $\hat{\boldsymbol{q}}_h$, incógnitas que aproximan a u y ∇u respectivamente sobre el esqueleto de toda la triangulación . Como consecuencia los únicos grados de libertad que son acoplados globalmente son aquellos asociados a las aproximaciones de u y ∇u sobre las caras de los elementos. También realizamos un análisis de error a priori usando el proyector ortogonal L^2 y concluimos que los órdenes de convergencia para las aproximaciones de $\mathcal{H}(u)$, $\nabla \cdot \mathcal{H}(u)$, ∇u y u son $k+1/2$, $k-1/2$, k y $k+1$ respectivamente, donde $k \geq 1$ es el grado polinomial de los espacios discretos. Nuestros resultados numéricos sugieren que las aproximaciones de $\mathcal{H}(u)$, ∇u y u convergen con orden óptimo $k+1$, y la aproximación de $\nabla \cdot \mathcal{H}(u)$ converge con orden subóptimo k.

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CHAPTER 1

Introduction

Discontinuous Galerkin(DG) methods are one of the most commonly used family of numerical methods to approximate the solution of partial deferential equations. The first DG method was introduced in 1973 by Reed and Hill in the context of neutron transport equation which corresponds to a time independent linear hyperbolic equation [1]. Subsequently, it was extended to the compressible Navier-Stokes equations and to second-order elliptic equations [2]. Later many DG methods applied to second-order elliptic problems have been proposed and a historical review can be found in [3]. A comparison of these DG methods has been maden in an unified framework [4]. However, when the **Continuous Galerkin (CG) methods** were compared with DG methods, these last were criticized for having too many degrees of freedom and for not being as easy to implement. As answer to the critics, hybridization of DG method was proposed. In particular, **Hybridizable Discontinuous Galerkin (HDG) methods** were introduced, for diffusion problems, in a unified framework [5].

During the last decade, **HDG** methods have been proposed and analyzed when applied to different types of equations, for instance, diffusion problems [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], convection-diffusion equations [16], [17], [18], [19], [20], the wave equation [21], [22], [23], Stokes flow [24], [25], [25], [26], [27], [28], [29], [30], Oseen and Brinkman equations [31], [32], Navier-Stokes equations [33], [34], [35], nolinear conservation laws [36], [37], [38], [39], linear and non-linear elasticity [40], [41], [42], Timonshenko Beams [43], [44], among others.

Our aim in this thesis is to propose and implement an **HDG** method for numerically solving the biharmonic problem: find u such that

$$\begin{array}{rcll} \Delta^2 u & = & f & \text{in} \quad \Omega\,, \\ u & = & 0 & \text{on} \quad \partial\Omega\,, \\ \nabla u\cdot\boldsymbol{n} & = & 0 & \text{on} \quad \partial\Omega\,, \end{array} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^d$ is polygonal/polyhedral domain with boundary $\Gamma := \partial\Omega$, $d \in \{2,3\}$ and data $f \in \mathrm{L}^2(\Omega)$. We introduce $\boldsymbol{q} = \nabla u$, $\underline{\boldsymbol{z}} = \nabla\boldsymbol{q}$, $\boldsymbol{\sigma} = \nabla\cdot\underline{\boldsymbol{z}}$ as unknowns and the problem (1.1) is rewritten as

$$\begin{array}{rclrcll} \boldsymbol{q} & = & \nabla u\,, & \underline{\boldsymbol{z}} & = & \nabla\boldsymbol{q} & \text{in} \quad \Omega\,, \\ \boldsymbol{\sigma} & = & \nabla\cdot\underline{\boldsymbol{z}}\,, & \nabla\cdot\boldsymbol{\sigma} & = & f & \text{in} \quad \Omega\,, \\ u & = & 0\,, & \boldsymbol{q}\cdot\boldsymbol{n} & = & 0 & \text{on} \quad \partial\Omega. \end{array} \tag{1.2}$$

where $(\nabla\boldsymbol{q})_{ij} := \partial x_j(q_i)$ for $1 \le i,j \le d$, q_i is the i-th component of $\boldsymbol{q}$ and $(\nabla\cdot\underline{\boldsymbol{z}})_i := \sum_{j=1}^{d} \partial x_j z_{ij}$, z_{ij} is the ij-entry of $\underline{\boldsymbol{z}}$.

One of the motivations to solve this problem lies in the existing relationship between Reissner-Mindlin plate (R-M) and biharmonic problems. In fact, the Mindlin-Reissner theory of plates is an extension of Kirchoff-Love plates theory that takes into account shear deformations through-the-thickness of a plate. Then the R-M model is a problem of the structural engineering, specifically, it arises from studying the mechanics of deformable plates, understanding that a plate is a solid that geometrically can be approximated by a bidimendional surface Ω.

The authors in [45] developed a mixed finite element method for the Reissner-Mindlin plate model:

$$\begin{array}{rcll} -\nabla\cdot(\mathcal{C}\underline{\boldsymbol{\epsilon}}(\boldsymbol{r})) - \lambda t^{-2}(\nabla u - \boldsymbol{r}) &=& 0 & \text{in } \tilde{\Omega}\,, \\ -\lambda t^{-2}\nabla\cdot(\nabla u - \boldsymbol{r}) &=& f & \text{in } \tilde{\Omega}\,, \\ & & & \\ u &=& 0 & \text{on } \partial\tilde{\Omega}\,, \\ \boldsymbol{r} &=& 0 & \text{on } \partial\tilde{\Omega}\,, \end{array} \tag{1.3}$$

where $\tilde{\Omega} \in \mathbb{R}^2$ is a polygonal domain and $f \in L^2(\tilde{\Omega})$. Here t is the thickness of the plate and λ is fixed positive parameter and $\underline{\boldsymbol{\epsilon}}(\boldsymbol{r}) := \frac{1}{2}\left(\nabla u + (\nabla u)^T\right)$. Moreover, the tensor $\mathcal{C}$ is defined to be

$$\mathcal{C}\underline{\boldsymbol{\tau}} = \frac{E}{12(1-\nu^2)}((1-\nu)\underline{\boldsymbol{\tau}} + \nu\,\mathrm{tr}(\underline{\boldsymbol{\tau}})\mathbb{I})\,,$$

where ν is the Poisson's ratio, $E = \frac{2(1+\nu)\lambda}{l}$ is the Young's modulus and l is the shear correction factor. The variable u correspond to the transverse displacement and $\boldsymbol{r}$ the rotation.

The method developed in [45] is based on the following non-dimensionalized formulation of the problem (1.3)

$$\begin{array}{lclclcll} \boldsymbol{q} &=& \nabla u &,& \underline{\boldsymbol{\rho}} &=& \frac{1}{2}(\nabla\boldsymbol{r} - (\nabla\boldsymbol{r})^T) & \text{in } \tilde{\Omega}\,, \\ \mathcal{A}(\underline{\boldsymbol{z}}) &=& \nabla\boldsymbol{r} - \underline{\boldsymbol{\rho}} &,& \boldsymbol{\sigma} &=& \nabla\cdot\underline{\boldsymbol{z}} & \text{in } \tilde{\Omega}\,, \\ \boldsymbol{r} - \boldsymbol{q} - \hat{t}^2\boldsymbol{\sigma} &=& 0 &,& \nabla\cdot\boldsymbol{\sigma} &=& f & \text{in } \tilde{\Omega}\,, \\ u &=& 0 &,& \boldsymbol{r} &=& 0 & \text{on } \partial\tilde{\Omega}. \end{array} \tag{1.4}$$

Here, $\hat{t} := \frac{t}{\sqrt{\lambda}}$ and $\mathcal{A}$ denotes the inverse of $\mathcal{C}$, i.e.,

$$\mathcal{A}(\underline{\boldsymbol{s}}) = \frac{12(1+\nu)\underline{\boldsymbol{s}}}{E} - \frac{12\nu(1-\nu)\mathrm{tr}(\underline{\boldsymbol{s}})\mathbb{I}}{E(1+\nu)}\,. \tag{1.5}$$

The variables $(\nabla\boldsymbol{q})_{ij}$ for $1 \le i,j \le d$ where q_i is the i-th component of $\boldsymbol{q}$ and $(\nabla\cdot\underline{\boldsymbol{z}})_i$ where the z_{ij} is the ij-entry of $\underline{\boldsymbol{z}}$ are defined in the same way as in (1.2).

Now, since the original structure (3D) of the model (1.3) was approximated $\tilde{\Omega}$ by a two-dimensional domain Ω, it makes sense approach t to zero in the formulation (1.4). In this case, we obtain the following equations

$$\begin{array}{lclclcll} \boldsymbol{q} &=& \nabla u &,& \underline{\boldsymbol{\rho}} &=& \frac{1}{2}(\nabla\boldsymbol{q} - (\nabla\boldsymbol{q})^T) & \text{in } \Omega\,, \\ \mathcal{A}(\underline{\boldsymbol{z}}) &=& \nabla\boldsymbol{q} - \underline{\boldsymbol{\rho}} &,& \boldsymbol{\sigma} &=& \nabla\cdot\underline{\boldsymbol{z}} & \text{in } \Omega\,, \\ \boldsymbol{r} &=& \boldsymbol{q} &,& \nabla\cdot\boldsymbol{\sigma} &=& f & \text{in } \Omega\,. \end{array}$$

Later using the definition (1.5), assuming that $\underline{\boldsymbol{z}}$ is symmetric, Young's modulus have a value of $E = 12$ and that the longitudinal deformations of the plate are larger than the transverse deformations, that is, Poisson's modulus is zero ($\nu = 0$), so we obtain that $\underline{\boldsymbol{z}} = \nabla \boldsymbol{q}$ in Ω and $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on $\partial\Omega$. Therefore, the R-M problem can be written as (1.2).

Another application where the biharmonic operator appears, arises from fluid dynamics. In particular, in the stream function formulation of the two-dimensional incompressible Navier-Stokes equation [46]. In fact, if u is the velocity of a incompressible fluid, γ the pressure and ν the viscosity, then,

$$\begin{aligned} \frac{\partial}{\partial t}\Delta\psi + [\nabla^{\perp}\cdot\psi]\Delta\psi &= \nu\,\Delta^2\psi \quad \text{in} \quad \Omega\,, &(1.6)\\ \nabla^{\perp}\psi &= U \quad \text{on} \quad \partial\Omega &(1.7)\end{aligned}$$

where ψ is the scalar stream function such that $u = \nabla^{\perp}\psi = \left(-\frac{\partial\psi}{\partial y}, \frac{\partial\psi}{\partial x}\right)$. Thinking in a bounded domain Ω enclosed by rigid walls, the impermeability of the walls and the no-slip condition imply $u(x,t) = U(x,t)$ for $x \in \partial\Omega$ and $t > 0$, where U is the prescribed velocity in the wall.

We will now focus on revising the literature related to numerical methods to solve (1.1).

In the literature we can find, in general, two different ways to rewrite the problem (1.1) as a second or first order system. One of them introduces $z = \Delta u$ as unknown and thus, from the first equation in (1.1) we have

$$\begin{aligned} -\Delta z &= f\,,\\ -\Delta u &= z\,. \end{aligned} \tag{1.8}$$

A method to solve (1.8) is the **Ciarlet and Raviart (C-R) mixed method** [47]. It chooses as unknowns u and $z = -\Delta u$ to obtain a coupled system of Poisson problems. The error analysis of the C-R method shows that the convergence rates for the approximation of u and z are k and $k-1$, respectively, if polynomials of degree $k \geq 2$ are used.

Other examples of mixed methods are the **Hellan-Herrmann-Jhonson (HHJ) method** and **Herrmann-Miyoshi (HM) method**. In 1980, I. Babuška, J. Osborn and J. Pitkaranta studied the convergence results of Brezzi and Babuška [48] to analyse three examples previously proposed in the literature. These examples were the **C-R**, **HHJ** and **HM** methods mentioned above. For the case of **C-R** method, the authors in [48] introduced the auxiliary variable $z = -\Delta u$ and rewrite the biharmonic problem (1.1) as the second order system (1.8) with boundary conditions $u = \nabla u = 0$ on $\partial\Omega$. In addition, they assumed that $f \in \mathrm{H}^{-1}(\Omega)$. Later, in their error analysis, they established: If $u \in \mathrm{H}^r(\Omega)$, $r \geq 3$ and $k \geq 2$ then $||z - z_h||_{0,\mathcal{T}_h} \leq Ch^{s-2}||u||_s$, where $s = \min\{r, k+1\}$. They also estimated $||u - u_h||_{1,\mathcal{T}_h}$ using a duality argument and concluded that $||u - u_h||_{1,\mathcal{T}_h} \leq Ch^{s-1}||u||_s$, where $s = \min\{r, k+1\}$. Here, u_h and z_h are the approximations of z and u delivered by their method.

For the **HM** method the auxiliary variable is the matrix of second-order partial derivatives of u ($\underline{\boldsymbol{z}} = \mathcal{H}(u)$ where $\mathcal{H}$ is the hessian matrix of u). The authors in [48] concluded that if $u \in \mathrm{H}^r(\Omega)$, $r \geq 3$ y $k \geq 2$ then $||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h} \leq Ch^{s-2}||u||_s$ and $||u - u_h||_1 \leq Ch^{s-1}||u||_s$, where $s = \min\{r, k+1\}$.

By last, the **HHJ** method uses the same formulation as the **HM** method, but the choice of finite dimensional spaces is different. In [48], it is concluded that if $u \in \mathrm{H}^r(\Omega)$, $r \geq 3$ and $k \geq 2$ then $||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h} \leq Ch^{s}||u||_{s+2}$ with $s = \min\{k, r-2\}$. In addition, if $k = 1$ they obtained

$$\begin{aligned} ||u - u_h||_{0,\mathcal{T}_h} &\leq Ch^2||u||_4\,,\\ ||u - u_h||_{1,\mathcal{T}_h} &\leq Ch||u||_3\,, \end{aligned}$$

and, if $k \geq 2$,

$$
\begin{aligned}
||u-u_h||_{0,\mathcal{T}_h} &\leq Ch^s||u||_{s+1} \quad \text{where } s=\min\{k+1,r-1\}\ ,\\
||u-u_h||_{1,\mathcal{T}_h} &\leq Ch^{s-1}||u||_s \quad \text{where } s=\min\{r,k+1\}\ .
\end{aligned}
$$

In the context of the **HDG** methods for the formulation (1.8), [43] developed an **HDG** method, called **Single-Face-Hybridizable (SFH) method**, which rewrites (1.8) as a first order system of equations. The authors in [43] introduced the additional variables $\boldsymbol{q}=\nabla u$ and $\boldsymbol{\sigma}=\nabla z$ and then the problem (1.8) is rewritten as

$$
\begin{array}{rclcrcll}
\boldsymbol{\sigma}+\nabla z &=& 0 &,& \nabla\cdot\boldsymbol{\sigma} &=& f & \text{in } \Omega\,,\\
\boldsymbol{q}+\nabla u &=& 0 &,& \nabla\cdot\boldsymbol{q} &=& z & \text{in } \Omega\,,\\
u &=& 0 &,& \boldsymbol{q}\cdot\boldsymbol{n} &=& 0 & \text{on } \partial\Omega\,.
\end{array}
\tag{1.9}
$$

Notice that in this formulation u, z are scalar unknowns and $\boldsymbol{\sigma}$, $\boldsymbol{q}$ are vector unknowns.

The **SFH** method is a particular case of an **HDG** scheme where the penalty function τ is defined on each simplex $T\in\mathcal{T}_h$ as,

$$
\tau=\begin{cases} 0\,, & \text{on } \partial T\setminus e_T^\tau\,,\\ \tau_T>0\,, & \text{on } e_T^\tau\,,\end{cases}
$$

where e_T^τ is an arbitrary but fixed face of T. We recall that, in the case of a general **HDG** method, the penalty function can be non zero on each face of T. The authors in [43] chose as traces unknowns $\hat{u}$ and $\hat{z}$, and define the numerical fluxes $\hat{\boldsymbol{q}}$, $\hat{\boldsymbol{\sigma}}$ in terms of the penalty function and the other variables. This helps to obtain error estimates in coherence with the numerical results. They also proved under the assumption of extra regularity for the unknown $\boldsymbol{q}$, this is $\boldsymbol{q}\in W^{k+1,\infty}(\mathcal{T}_h)$, that when polynomials of degree at most $k\geq 1$ are used on all the variables, optimal convergence rates are obtained for u and $\boldsymbol{q}$; the approximations to $z=\Delta u$ and $\boldsymbol{\sigma}=\nabla z$ converge with order $k+1/2$ and $k-1/2$, respectively. In addition, they showed that it is possible to locally devise a new approximation of u that superconverges with order $k+2$ for $k\geq 2$ and with order $5/2$ for $k=1$. Finally, although they predicted that the converge rates for z and $\boldsymbol{\sigma}$ are suboptimal with rates $k+1/2$ and $k-1/2$, respectively, their numerical experiments showed that the convergence rates are optimal in any fixed interior subdomian of Ω.

A second alternative for the problem (1.1) is to study this problem rewritten as in (1.2). Notice that in this case, there is only one scalar unknown, two vector unknowns and one tensor unknown, the variables $(\nabla\boldsymbol{q})_{i,j}$ and $(\nabla\cdot\underline{\boldsymbol{z}})_i$ are defined in the same way as in (1.2).

In [49] a mixed method is studied for the formulation (1.2), which is based on a system of first-order equations. The authors in [49] also introduced a hybrid form of their method which allowed them to reduce the globally coupled degrees of freedom to only those associated to the Lagrange multipliers that approximate the solution and its derivative at the faces of the triangulation. On the other hand, their error analysis indicated that if $k\geq 1$, the approximations u, $\boldsymbol{q}$, $\underline{\boldsymbol{z}}$ have optimal convergence rate while $\boldsymbol{\sigma}$ has suboptimal convergence rate. For $k=0$ they got that the approximations of u, $\boldsymbol{q}$, $\underline{\boldsymbol{z}}$ converge with order 1 and for $\boldsymbol{\sigma}$ the theory was not conclusive. Their numerical results showed that, for smooth solutions, the order of convergence of the errors for u, $\boldsymbol{q}$, $\underline{\boldsymbol{z}}$ are optimal and that the order of convergence for the error in $\boldsymbol{\sigma}$ is suboptimal.

In the context of **HDG** method, [44] studied an **HDG** method for Timoshenko beams, which can be seen as the biharmonic (formulation (1.2)) in one dimension. Their error analysis is based on the use of a projection especially designed to fit the structure of the numerical traces of the **HDG** method. This property allowed them to prove, in a very concise manner, that the projection of the errors is bounded in terms of the distance between the exact solution and its projection. They considered polynomials of degree $k \geq 0$ for the local spaces of the unknowns, and showed that the projection of the error in each of them superconverges with order $k+2$ when $k \geq 1$ and converges with order 1 for $k \geq 0$. As a result, they showed that the **HDG** method converge with optimal order $k+1$ for all the unknowns defined in each $T \in \mathcal{T}_h$. In addition, they obtain that the numerical traces converge with order $2k+1$.

We present in this manuscript an **HDG** method to solve (1.2), which, to the best of our knownledge, has not been studied before. Next, we summarize the main contribution of our work. In Chapter 2, we give details of how to build an **HDG** scheme for the biharmonic problem, introduce and give sense to the numerical fluxes. We also explain the properties of consistency and conservative of these numerical fluxes. Next, we deduce our **HDG** scheme applied to biharmonic problem and show that it has unique solution. In addition we introduce the *local solvers* with the aim of explaining the advantage of the **HDG** scheme studied in this thesis.

In Chapter 3, we carry out an priori error analysis using orthogonal L^2-projectors and conclude that the orders of convergence for $\underline{\boldsymbol{z}}$, $\boldsymbol{\sigma}$, $\boldsymbol{q}$ and u are $k+1/2$, $k-1/2$, k and $k+1$, respectively, with $k \geq 1$ being the polynomial degree of the local discrete spaces. Finally, in Chapter 4, we show two numerical examples which suggest that the method converges with better orders than those predicted by the theory. That is, when we approximate $\underline{\boldsymbol{z}}$, $\boldsymbol{\sigma}$, $\boldsymbol{q}$ or u by polynomials of degree $k \geq 1$, the numerical results show that $\underline{\boldsymbol{z}}_h$, $\boldsymbol{q}_h$ and u_h converge with optimal order $k+1$, while the function $\boldsymbol{\sigma}_h$ converges with suboptimal order k.

CHAPTER 2

An HDG method applied to the biharmonic problem

2.1 Notation and Preliminaries

In order to study of the proposed HDG method applied to the formulation (1.2). It is necessary to set the notation that we will use. Given $h > 0$, we denote by $\mathcal{T}_h$ a uniformly shaped regular triangulation of $\bar{\Omega}$ made of simplices. Given $T \in \mathcal{T}_h$, we denote by h_T its diameter and $\boldsymbol{n}_T$ its unit outward normal. If there is no confusion, we will write $\boldsymbol{n}$ instead of $\boldsymbol{n}_T$. Moreover the set of interior edges or faces of $\mathcal{T}_h$ is denoted by $\mathcal{E}_h^i$, the set of boundary edges or faces is denoted by $\mathcal{E}_h^\Gamma$ and set of all faces is denoted by $\mathcal{E}_h$. We define by $\partial\mathcal{T}_h$ the union of the boundary of the elements $T \in \mathcal{T}_h$. Also, we define

$$(\cdot,\cdot)_{\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} (\cdot,\cdot)_T \quad , \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h} := \sum_{T\in\mathcal{T}_h} \langle\cdot,\cdot\rangle_{\partial T} \quad \text{and} \quad \langle\cdot,\cdot\rangle_{\partial\mathcal{T}_h\setminus\Gamma} := \sum_{e\in\mathcal{E}_h^i} \langle\cdot,\cdot\rangle_e \,.$$

where $(\cdot,\cdot)_T$, $\langle\cdot,\cdot\rangle_{\partial T}$ and $\langle\cdot,\cdot\rangle_e$ are the standard L^2 inner products over an element T, its boundary ∂T and face e, respectively. In addition, given $k \in \mathbb{Z}_0^+$, we denote by $\mathbb{P}_k(T)$ the space of the polynomials of degree at most k defined on T. We also say that the space denoted by $\mathbb{P}_k(e)$ consists of polynomials of degree k defined on a face e of a element $T \in \mathcal{T}_h$.

The space $\mathrm{H}^l(\mathcal{T}_h)$, with $l \geq 0$, denotes the space of functions defined in Ω whose restriction to each element T belongs to the Sobolev space $\mathrm{H}^l(T)$ and the traces of functions in $\mathrm{H}^1(\mathcal{T}_h)$ belong to $\mathrm{M}(\partial\mathcal{T}_h) := \prod_{T\in\mathcal{T}_h} L^2(\partial T)$.

In addition, boldface variables indicate vector-valued functions and an underline boldface variables indicate tensor-valued function.

In Chapter 3, the error analysis is based on the L^2-orthogonal projection and we will denote by P, $\overline{P}$ and $\mathcal{P}$ the L^2-projectors into the spaces $\mathbb{P}_k(\mathcal{T}_h)$, $[\mathbb{P}_k(\mathcal{T}_h)]^d$ and $[\mathbb{P}_k(\mathcal{T}_h)]^{d\times d}$, respectively, where $\mathbb{P}_k(\mathcal{T}_h)$ is the space of functions whose restriction to each element T belongs to $\mathbb{P}_k(T)$.

It is well-known (see Lemma 1.59 in [3]), that P satisfies, on each $T \in \mathcal{T}_h$, the following property: Given $\eta \in \mathrm{H}^1(T)$, there exists $C > 0$, independent of h_T, for all $e \in \partial T$ such that

$$||\eta - P\eta||_{0,e} \leq C h_T^{s-1/2} |\eta|_{s,T} \quad , \quad s \geq 1. \tag{2.1}$$

Additionally, it is also known that P satisfies the following: Let k and m non negative integers and $r \in \mathbb{R}$ such that $0 < r \leq k$. There exists a constant $C := C(k,d) > 0$ such that

$$||D^m(\eta - P\eta)||_{0,T} \leq C h_T^{r+1-m} ||\eta||_{r+1,T} \quad \forall\, \eta \in \mathrm{H}^{r+1}(T)\,. \tag{2.2}$$

Moreover, the following discrete trace inequality (see Lemma 1.52 in [3]) holds: Let $T \in \mathcal{T}_h$ and $e \in \partial T$. Then, there exist a constant, independent of h_T, $C_{tr} > 0$ such that

$$h_T^{1/2} ||\eta||_{0,e} \leq C_{tr} ||\eta||_{0,T} \,\forall\, \eta \in \mathbb{P}_k(T). \tag{2.3}$$

In addition, we also define the L^2-projection, denoted by P_∂, as follows. Given a face $e \in \mathcal{E}_h$ and $\eta \in L^2(e)$, $P_\partial \eta$ is defined as the unique element of $\mathbb{P}_k(e)$ that satisfies

$$\langle P_\partial \eta - \eta, w \rangle_e = 0 \quad \forall\, w \in \mathbb{P}_k(e)\,. \tag{2.4}$$

Remark 1. *If $u \in H_0^1(\Omega)$ and $\boldsymbol{q} = \nabla u$ such that $\boldsymbol{q} \cdot \boldsymbol{n} = 0$ on Γ then $\boldsymbol{q} = 0$ on Γ.*

2.2 The HDG method

We recall the problem that we are interested in: Find u such that

$$\begin{array}{rcll} \Delta^2 u & = & f & \text{in} \ \ \Omega\,, \\ u & = & 0 & \text{on} \ \ \Gamma\,, \\ \nabla\, u \cdot \boldsymbol{n} & = & 0 & \text{on} \ \ \Gamma\,. \end{array} \tag{2.5}$$

We introduce the auxiliary variables $\boldsymbol{q} := \nabla u$, $\underline{\boldsymbol{z}} := -\nabla\, \boldsymbol{q}$ and $\boldsymbol{\sigma} := -\nabla \cdot \underline{\boldsymbol{z}}$. Using these variables, the problem (2.5) is rewritten as:

$$\begin{array}{lccl} \boldsymbol{q} & = & \nabla\, u & \text{in} \ \ \Omega\,, \\ \underline{\boldsymbol{z}} & = & -\nabla\, \boldsymbol{q} & \text{in} \ \ \Omega\,, \\ \boldsymbol{\sigma} & = & -\nabla \cdot \underline{\boldsymbol{z}} & \text{in} \ \ \Omega\,, \\ \nabla \cdot \boldsymbol{\sigma} & = & f & \text{in} \ \ \Omega\,, \\ \boldsymbol{q} \cdot \boldsymbol{n} & = & 0 & \text{in} \ \ \Gamma\,, \\ u & = & 0 & \text{in} \ \ \Gamma\,. \end{array} \tag{2.6}$$

In order to devise our HDG method, on each element T of the mesh $\mathcal{T}_h$, we derive a weak formulation.

Testing with $\boldsymbol{v} \in [\mathrm{H}(\mathrm{div}; T)]^d$ and integrating by parts in the first equation of (2.6), we obtain

$$\int_T \boldsymbol{q} \cdot \boldsymbol{v} + \int_T (\nabla \cdot \boldsymbol{v}) u - \langle \boldsymbol{v} \cdot \boldsymbol{n}, u \rangle_{\partial T} = 0 \quad \forall\, \boldsymbol{v} \in [\mathrm{H}(\mathrm{div}; T)]^d\,. \tag{2.7}$$

Then, testing with $\underline{\boldsymbol{s}} \in [\mathrm{H}(\mathrm{div}; T)]^{d \times d}$ the second equation of (2.6) and integrating by part, we have

$$\int_T \underline{\boldsymbol{z}} : \underline{\boldsymbol{s}} - \int_T (\nabla \cdot \underline{\boldsymbol{s}}) \cdot \boldsymbol{q} + \langle \underline{\boldsymbol{s}}\, \boldsymbol{n}, \boldsymbol{q} \rangle_{\partial T} = 0 \quad \forall\, \underline{\boldsymbol{s}} \in [\mathrm{H}(\mathrm{div}; T)]^{d \times d} \tag{2.8}$$

Moreover, we test the third and fourth equations in (2.6) with $\boldsymbol{m} \in [\mathrm{H}^1(T)]^d$ and $w \in \mathrm{H}^1(\mathrm{T})$, respectively, and obtain

$$\begin{array}{rcl} \displaystyle\int_T \boldsymbol{\sigma} \cdot \boldsymbol{m} - \int_T \underline{\boldsymbol{z}} : \nabla \boldsymbol{m} + \langle \underline{\boldsymbol{z}}\, \boldsymbol{n}, \boldsymbol{m} \rangle_{\partial T} & = & 0 \quad \forall\, \boldsymbol{m} \in [\mathrm{H}^1(T)]^d\,, \\ \displaystyle\int_T \nabla w \cdot \boldsymbol{\sigma} - \langle \boldsymbol{\sigma} \cdot \boldsymbol{n}, w \rangle_{\partial T} + \int_T f\, w & = & 0 \quad \forall\, w \in \mathrm{H}^1(\mathrm{T})\,. \end{array} \tag{2.9}$$

The proposed HDG method applied to biharmonic problem yields an approximation u_h of $u \in \mathrm{H}^1(\mathcal{T}_h)$, approximations $\boldsymbol{\sigma}_h$ and $\boldsymbol{q}_h$ of $\boldsymbol{\sigma} \in [\mathrm{H}(\mathrm{div};\mathcal{T}_h)]^d$ and $\boldsymbol{q} \in [\mathrm{H}^1(\mathcal{T}_h)]^d$, respectively, and an approximation $\underline{\boldsymbol{z}}_h$ of $\underline{\boldsymbol{z}} \in [\mathrm{H}(\mathrm{div};\mathcal{T}_h)]^{d\times d}$. It also provides a single valued approximation $\hat{u}_h$ to the trace of u and a single valued approximation $\hat{\boldsymbol{q}}_h$ of the trace of $\boldsymbol{q}$ on elements boundaries. We introduce the following discrete spaces associated to the triangulation $\mathcal{T}_h$.

$$\begin{aligned}
V_h &:= \left\{v \in \mathrm{H}^1(\mathcal{T}_h) : v|_T \in V(T)\right\} ,\\
Q_h &:= \left\{\boldsymbol{m} \in [\mathrm{H}^1(\mathcal{T}_h)]^d : \boldsymbol{m}|_T \in Q(T)\right\} ,\\
\Sigma_h &:= \left\{\boldsymbol{\eta} \in [\mathrm{H}(\mathrm{div};\mathcal{T}_h)]^d : \boldsymbol{\eta}|_T \in \Sigma(T)\right\} ,\\
Z_h &:= \left\{\underline{\boldsymbol{s}} \in [\mathrm{H}(\mathrm{div};\mathcal{T}_h)]^{d\times d} : \underline{\boldsymbol{s}}|_T \in Z(T)\right\} ,
\end{aligned}$$

where $V(T)$, $Q(T)$, $\Sigma(T)$ and $Z(T)$ are finite dimensional spaces on an element T. We recall that the general idea in the DG methods (see [4], Section 3) is to introduce unknowns on mesh skeleton called *numerical fluxes*. In our case we shall add $[\hat{u}_{h,T}, \hat{\boldsymbol{q}}_{h,T}]$ and $\left[\hat{\underline{\boldsymbol{z}}}_{h,T}, \hat{\boldsymbol{\sigma}}_{h,T}\right]$ as new unknowns on $\mathcal{E}_h$. The *numerical fluxes* $\hat{u}_{h,T}, \hat{\boldsymbol{q}}_{h,T}, \hat{\underline{\boldsymbol{z}}}_{h,T}$ and $\hat{\boldsymbol{\sigma}}_{h,T}$ are approximations to u, ∇u, $\mathcal{H}(u)$ and $\nabla\cdot\mathcal{H}(u)$, respectively, on the boundary of T. So, we consider the following local formulation: Find $\underline{\boldsymbol{z}}_h \in Z_h$, $\boldsymbol{\sigma}_h \in \Sigma_h$, $\boldsymbol{q}_h \in Q_h$ and $u_h \in V_h$ such that for all $T \in \mathcal{T}_h$ we have

$$\begin{array}{lcll}
(\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}})_T - (\nabla\cdot\underline{\boldsymbol{s}}, \boldsymbol{q}_h)_T + \left\langle \underline{\boldsymbol{s}}\,\boldsymbol{n}, \hat{\boldsymbol{q}}_{h,T}\right\rangle_{\partial T} &=& 0 & \forall\, \underline{\boldsymbol{s}} \in Z(T) \quad ,\\
(\boldsymbol{q}_h, \boldsymbol{v})_T + (\nabla\cdot\boldsymbol{v}, u_h)_T - \langle \boldsymbol{v}\cdot\boldsymbol{n}, \hat{u}_{h,T}\rangle_{\partial T} &=& 0 & \forall\, \boldsymbol{v} \in \Sigma(T) \quad ,\\
(\boldsymbol{\sigma}_h, \boldsymbol{m})_T - (\underline{\boldsymbol{z}}_h, \nabla\,\boldsymbol{m})_T + \left\langle \hat{\underline{\boldsymbol{z}}}_{h,T}\boldsymbol{n}, \boldsymbol{m}\right\rangle_{\partial T} &=& 0 & \forall\, \boldsymbol{m} \in Q(T) \quad ,\\
(\boldsymbol{\sigma}_h, \nabla w)_T - \langle \hat{\boldsymbol{\sigma}}_{h,T}\cdot\boldsymbol{n}, w\rangle_{\partial T} + (f, w)_T &=& 0 & \forall\, w \in V(T) \quad .
\end{array} \tag{2.10}$$

To complete the specification of the HDG method we must express the *numerical fluxes* in terms of u_h, $\boldsymbol{q}_h$, $\boldsymbol{\sigma}_h$ and $\underline{\boldsymbol{z}}_h$ and in terms of the boundary conditions. Then, we first give sense to numerical fluxes and traces unknowns, so we must prevent functions in $\mathrm{M}(\partial\mathcal{T}_h)$ to be double-valued on $\mathcal{E}_h \setminus \Gamma$. In Γ there is no problem because they are single-valued. The properties of *consistency* and *conservativity* of the *numerical fluxes* will allow us to identify the traces as elements of $\mathrm{M}(\partial\mathcal{T}_h)$ such that the two values coincide on all internal edge.

We take the scalar and vector numerical fluxes $\hat{u}_h = (\hat{u}_{h,T})_{T\in\mathcal{T}_h}$, $\hat{\boldsymbol{q}}_h = (\hat{\boldsymbol{q}}_{h,T})_{T\in\mathcal{T}_h}$ and tensor and vector numerical fluxes $\hat{\underline{\boldsymbol{z}}}_h = (\hat{\underline{\boldsymbol{z}}}_{h,T})_{T\in\mathcal{T}_h}$ and $\hat{\boldsymbol{\sigma}}_h = (\hat{\boldsymbol{\sigma}}_{h,T})_{T\in\mathcal{T}_h}$ to be linear functions

$$\begin{array}{lcl c lcl}
\hat{u}_h : \mathrm{H}^1(\mathcal{T}_h) &\to& \mathrm{M}(\partial\mathcal{T}_h) &;& \hat{\underline{\boldsymbol{z}}}_h : \mathrm{H}^3(\mathcal{T}_h)\times[\mathrm{H}^1(\mathcal{T}_h)]^{d\times d} &\to& [\mathrm{M}(\partial\mathcal{T}_h)]^{d\times d}\\
\hat{\boldsymbol{q}}_h : \mathrm{H}^2(\mathcal{T}_h)\times[\mathrm{H}^1(\mathcal{T}_h)]^{d} &\to& [\mathrm{M}(\partial\mathcal{T}_h)]^{d} &;& \hat{\boldsymbol{\sigma}}_h : \mathrm{H}^4(\mathcal{T}_h)\times[\mathrm{H}^1(\mathcal{T}_h)]^{d} &\to& [\mathrm{M}(\partial\mathcal{T}_h)]^{d}\,.
\end{array}$$

We say that the numerical fluxes are *consistent* if

$$\begin{array}{lcl c lcl}
\hat{u}_h(v) &=& v|_{\partial T} &;& \hat{\underline{\boldsymbol{z}}}_h(v, \mathcal{H}(v)) &=& \mathcal{H}(v)|_{\partial T}\\
\hat{\boldsymbol{q}}_h(v, \nabla v) &=& \nabla v|_{\partial T} &;& \hat{\boldsymbol{\sigma}}_h(v, \nabla\cdot\mathcal{H}(v)) &=& \nabla\cdot\mathcal{H}(v)|_{\partial T}\ ,
\end{array}$$

for any smooth function v that satisfies the boundary conditions. We also say that the numerical fluxes are *conservative* if they are single-valued on $\partial\mathcal{T}_h$.

As we have mentioned before we express the numerical fluxes in terms of $u_h \in V_h$, $\boldsymbol{q}_h \in Q_h$, $\boldsymbol{\sigma}_h \in \Sigma_h$, $\underline{\boldsymbol{z}}_h \in Z_h$.To that end, we look for $\hat{\boldsymbol{q}}_h$ and $\hat{u}_h$ in the spaces Φ_h and W_h defined by

$$\begin{aligned}
\Phi_h &:= \left\{\hat{\boldsymbol{\tau}} \in [L^2(\mathcal{E}_h)]^d : \hat{\boldsymbol{\tau}}|_e \in \Phi(e) \quad \forall\, e \in \mathcal{E}_h\right\} ,\\
W_h &:= \left\{\hat{v} \in L^2(\mathcal{E}_h) : \hat{v}|_e \in W(e) \quad \forall\, e \in \mathcal{E}_h\right\} .
\end{aligned}$$

In addition, on each $T \in \mathcal{T}_h$ we define the normal components for $\hat{\boldsymbol{\sigma}}_h$ and $\hat{\underline{\boldsymbol{z}}}_h$ as follow

$$
\begin{aligned}
\hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n} &= \underline{\boldsymbol{z}}_h \boldsymbol{n} + \tau_3 (u_h - \hat{u}_h)\boldsymbol{n} + \tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h) && \text{for each} \quad e \in \partial T\,, \\
\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} &= \boldsymbol{\sigma}_h \cdot \boldsymbol{n} + \tau_1 (u_h - \hat{u}_h) + \tau_2(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)\cdot \boldsymbol{n} && \text{for each} \quad e \in \partial T\,,
\end{aligned} \tag{2.11}
$$

where τ_i, $i = 1, 2, 3, 4$ are stabilization parameters defined on $\partial\mathcal{T}_h$ which we will assume to be constant on each face of the triangulation. Properties of consistent and conservation for numerical fluxes and boundary conditions can be summarized in the next four equations

$$
\begin{aligned}
\langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{\beta}_h \rangle_{\partial\mathcal{T}_h \setminus \Gamma} &= 0\,, \\
\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \alpha_h \rangle_{\partial\mathcal{T}_h \setminus \Gamma} &= 0\,, \\
\langle \hat{\boldsymbol{q}}_h, \boldsymbol{\beta}_h \rangle_{\Gamma} &= 0\,, \\
\langle \hat{u}_h, \alpha_h \rangle_{\Gamma} &= 0\,,
\end{aligned} \tag{2.12}
$$

for each $(\boldsymbol{\beta}_h, \alpha_h) \in \Phi_h \times W_h$. From now on we call *numerical traces* the unknowns $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$, and *numerical fluxes* $\hat{\underline{\boldsymbol{z}}}_h$ and $\hat{\boldsymbol{\sigma}}_h$.

2.3 The discrete scheme

We study an HDG method for the case where the local spaces $V(T)$, $Q(T)$, $\Sigma(T)$, $Z(T)$, $\Phi(e)$ and $W(e)$ are polynomial spaces. Defined as follows:

$$
\begin{aligned}
&V(T) := \mathbb{P}_k(T)\,, \quad Q(T) := [\mathbb{P}_k(T)]^d\,, \quad \Sigma(T) := [\mathbb{P}_k(T)]^d\,, \\
&W(e) := \mathbb{P}_k(e)\,, \quad \Phi(e) := [\mathbb{P}_k(e)]^d\,, \quad Z(T) := \{\underline{\boldsymbol{s}}|_T : \text{each row of } \underline{\boldsymbol{s}}|_T \text{ belongs to } Q(T)\}
\end{aligned} \tag{2.13}
$$

Then, the HDG scheme that we analyse reads:

Find $(\underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h, \hat{u}_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times \Phi_h \times W_h$ such that

$$(\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h)_{\mathcal{T}_h} - (\nabla_h \cdot \underline{\boldsymbol{s}}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{s}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} = 0\,, \tag{2.14a}$$

$$(\boldsymbol{q}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\nabla_h \cdot \boldsymbol{v}_h, u_h)_{\mathcal{T}_h} - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial\mathcal{T}_h} = 0\,, \tag{2.14b}$$

$$(\boldsymbol{\sigma}_h, \boldsymbol{m}_h)_{\mathcal{T}_h} - (\underline{\boldsymbol{z}}_h, \nabla_h \boldsymbol{m}_h)_{\mathcal{T}_h} + \langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{m}_h \rangle_{\partial\mathcal{T}_h} = 0\,, \tag{2.14c}$$

$$(\boldsymbol{\sigma}_h, \nabla_h w_h)_{\mathcal{T}_h} - \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, w_h \rangle_{\partial\mathcal{T}_h} + (f, w_h)_{\mathcal{T}_h} = 0\,, \tag{2.14d}$$

$$\langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{\beta}_h \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0\,, \tag{2.14e}$$

$$\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \alpha_h \rangle_{\partial\mathcal{T}_h \setminus \Gamma} = 0\,, \tag{2.14f}$$

$$\langle \hat{\boldsymbol{q}}_h, \boldsymbol{\beta}_h \rangle_{\Gamma} = 0\,, \tag{2.14g}$$

$$\langle \hat{u}_h, \alpha_h \rangle_{\Gamma} = 0\,, \tag{2.14h}$$

for all $(\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h, \boldsymbol{\beta}_h, \alpha_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times \Phi_h \times W_h$. Where we recall that defined the *numerical fluxes* as

$$\hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n} \quad = \quad \underline{\boldsymbol{z}}_h \boldsymbol{n} + \tau_3(u_h - \hat{u}_h)\boldsymbol{n} + \tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h) \qquad \text{on} \qquad \partial\mathcal{T}_h$$

and

$$\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} \quad = \quad \boldsymbol{\sigma}_h \cdot \boldsymbol{n} + \tau_1(u_h - \hat{u}_h) + \tau_2(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h) \cdot \boldsymbol{n} \qquad \text{on} \qquad \partial\mathcal{T}_h \,.$$

Later, in the next section, we will explain that from the scheme (2.14) we can locally eliminate all the interior variables in terms of the numerical traces. The following result shows that the previous scheme has a unique solution.

Theorem 2.1. *Let τ_1, τ_2, τ_3, τ_4 be reals numbers such that τ_1, τ_4 are positive and $\tau_3 + \tau_2 = 0$. Then, the scheme* (2.14) *is well-posed.*

Proof. Since (2.14) is a linear system, it is enough to show that the homogeneous problem ($f \equiv 0$) has only the trivial solution.

Since the number of unknowns is equal to the number of equations, then the system is compatible.

Now, considering $f \equiv 0$, testing the equation (2.14c) with $\boldsymbol{m}_h = \boldsymbol{q}_h$ and integrating by parts, we obtain

$$(\boldsymbol{\sigma}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} + (\boldsymbol{q}_h, \nabla_h \cdot \underline{\boldsymbol{z}}_h)_{\mathcal{T}_h} + \langle (\hat{\underline{\boldsymbol{z}}}_h - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q}_h \rangle_{\partial\mathcal{T}_h} = 0 \,. \tag{2.15}$$

Testing (2.14a) with $\underline{\boldsymbol{s}}_h = \underline{\boldsymbol{z}}_h$, and (2.14b) with $\boldsymbol{v}_h = \boldsymbol{\sigma}_h$, we get

$$||\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} - (\nabla_h \cdot \underline{\boldsymbol{z}}_h, \boldsymbol{q}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{z}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} = 0 \,, \tag{2.16}$$

and

$$(\boldsymbol{q}_h, \boldsymbol{\sigma}_h)_{\mathcal{T}_h} = -(\nabla_h \cdot \boldsymbol{\sigma}_h, u_h)_{\mathcal{T}_h} + \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial\mathcal{T}_h} \,. \tag{2.17}$$

Then, adding (2.15) and (2.16), and replacing $(\boldsymbol{q}_h, \boldsymbol{\sigma}_h)$ from (2.17), it holds

$$-(\nabla_h \cdot \boldsymbol{\sigma}_h, u_h)_{\mathcal{T}_h} + \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial\mathcal{T}_h} + \langle \underline{\boldsymbol{z}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h - \boldsymbol{q}_h \rangle_{\partial\mathcal{T}_h} + \langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{q}_h \rangle_{\partial\mathcal{T}_h} + ||\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} = 0 \,. \tag{2.18}$$

Also, testing equation (2.14d) with $w_h = u_h$ and integrating by parts, we obtain that

$$(\nabla_h \cdot \boldsymbol{\sigma}_h, u_h)_{\mathcal{T}_h} - \langle (\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, u_h \rangle_{\partial\mathcal{T}_h} = 0 \,. \tag{2.19}$$

Then, we add (2.18) and (2.19) and obtain

$$||\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} + \langle \underline{\boldsymbol{z}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h - \boldsymbol{q}_h \rangle_{\partial\mathcal{T}_h} + \langle \boldsymbol{\sigma}_h \cdot \boldsymbol{n}, \hat{u}_h - u_h \rangle_{\partial\mathcal{T}_h} + \langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{q}_h \rangle_{\partial\mathcal{T}_h} + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, u_h \rangle_{\partial\mathcal{T}_h} = 0 \,. \tag{2.20}$$

By last, taking $\boldsymbol{\beta}_h = \begin{cases} \hat{\boldsymbol{q}}_h & \text{on} \quad \partial\mathcal{T}_h/\Gamma \\ \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n} & \text{on} \quad \Gamma \end{cases}$ in (2.14e) and (2.14g) and $\alpha_h = \begin{cases} \hat{u}_h & \text{on} \quad \partial\mathcal{T}_h/\Gamma \\ \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} & \text{on} \quad \Gamma \end{cases}$ in (2.14f) and (2.15), we have that

$$-\langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} = 0 \quad \text{and} \quad -\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial\mathcal{T}_h} = 0 \,.$$

Adding these to terms to (2.20), replacing the definitions of $\hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}$ and $\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}$ and using the fact $\tau_2 + \tau_3 = 0$, it holds

$$||\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} + ||(\tau_4)^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||^2_{\partial\mathcal{T}_h} + ||(\tau_1)^{1/2}(u_h - \hat{u}_h)||^2_{\partial\mathcal{T}_h} = 0 \tag{2.21}$$

which implies that $\underline{\boldsymbol{z}}_h = 0$ in $\mathcal{T}_h$, $\boldsymbol{q}_h = \hat{\boldsymbol{q}}_h$ in $\partial\mathcal{T}_h$ and $u_h = \hat{u}_h$ on $\partial\mathcal{T}_h$. Then, by (2.11), $\hat{\underline{\boldsymbol{z}}}_h\boldsymbol{n} = 0$ and $\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} = \boldsymbol{\sigma}_h \cdot \boldsymbol{n}$ on $\partial\mathcal{T}_h$.

Testing (2.14c) with $\boldsymbol{m}_h = \boldsymbol{\sigma}_h$ and using the fact that $\underline{\boldsymbol{z}}_h = 0$ and $\hat{\underline{\boldsymbol{z}}}_h\boldsymbol{n} = 0$, we obtain $\boldsymbol{\sigma}_h = 0$ in $\mathcal{T}_h$ and hence $\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} = \boldsymbol{\sigma}_h \cdot \boldsymbol{n} = 0$ on $\partial\mathcal{T}_h$.

We notice that integrating by part equation (2.14a) and testing with $\underline{\boldsymbol{s}}_h = \nabla_h \boldsymbol{q}_h$, we conclude that $||\nabla_h \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} = 0$. Thus, $\boldsymbol{q}_h$ is constant in $\mathcal{T}_h$. Since $\boldsymbol{q}_h = \hat{\boldsymbol{q}}_h =$ constant on $\partial\mathcal{T}_h$, from (2.14g) we obtain that $\boldsymbol{q}_h = 0$ in T and then $\boldsymbol{q}_h = 0$ in $\mathcal{T}_h$.
Finally, since $\boldsymbol{\sigma}_h = 0$ and $u_h = \hat{u}_h$ on $\partial\mathcal{T}_h$, integrating by parts (2.14b) and taking $\boldsymbol{v}_h = \nabla u_h$, we obtain $||\nabla_h u_h||^2_{0,\mathcal{T}_h} = 0$. Consequently, u_h is constant in $\mathcal{T}_h$ and using the condition (2.14h), we have that $u_h = 0$ on $\mathcal{T}_h$. □

2.4 Local solvers

The formulation (2.14) corresponds to a hybrid scheme since it was constructed by relaxing the continuity requirements of the spaces

$$\begin{aligned}\tilde{\Sigma}_h &= \left\{\boldsymbol{\eta} \in [\mathrm{H}(\mathrm{div};\Omega)]^d : \boldsymbol{\eta}|_T \in \Sigma(T)\right\}\\ \tilde{Z}_h &= \left\{\underline{\boldsymbol{s}} \in [\mathrm{H}(\mathrm{div};\Omega)]^{d\times d} : \underline{\boldsymbol{s}}|_T \in Z(T)\right\}.\end{aligned}$$

As consequence we have had to introduce the unknowns $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$ to approximate u and $\boldsymbol{q}$ on the interfaces of $\mathcal{T}_h$.

In this section we show that only the globally coupled degrees of freedom are those associated with the unknowns $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$ in a global system. The other unknowns can be recovered element by element. For doing that, we start by writing an auxiliary scheme inspired in the hybrid formulation (2.14), where the *numerical traces* are given. In other words, given $\lambda \in W_h$, $\boldsymbol{\zeta} \in \Phi_h$ and $f \in L^2(\Omega)$ we look for $(\underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)}, u_h^{(\lambda,\boldsymbol{\zeta},f)}) \in Z_h \times \Sigma_h \times Q_h \times V_h$ such that, on each $T \in \mathcal{T}_h$, it satisfies

$$(\underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)}, \underline{\boldsymbol{s}}_h)_T - (\nabla_h \cdot \underline{\boldsymbol{s}}_h, \boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)})_T = \langle \underline{\boldsymbol{s}}_h\boldsymbol{n}, \boldsymbol{\zeta}\rangle_{\partial T}\,, \tag{2.22a}$$

$$(\boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{v}_h)_T + (\nabla_h \cdot \boldsymbol{v}_h, u_h^{(\lambda,\boldsymbol{\zeta},f)})_T = \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \lambda\rangle_{\partial T}\,, \tag{2.22b}$$

$$\begin{aligned}(\boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{m}_h)_T + (\boldsymbol{m}_h, \nabla_h \cdot \underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)})_T + \left\langle \tau_3 u_h^{(\lambda,\boldsymbol{\zeta},f)}\boldsymbol{n} + \tau_4 \boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{m}_h\right\rangle_{\partial T} &= \langle \tau_3 \lambda \boldsymbol{n}, \boldsymbol{m}_h\rangle_{\partial T} + \\ &\quad + \langle \tau_4 \boldsymbol{\zeta}, \boldsymbol{m}_h\rangle_{\partial T}\,,\end{aligned} \tag{2.22c}$$

$$\begin{aligned}(\nabla_h \cdot \boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)}, w_h)_T + \left\langle \tau_1 u_h^{(\lambda,\boldsymbol{\zeta},f)} + \tau_2 \boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)} \cdot \boldsymbol{n}, w_h\right\rangle_{\partial T} &= (f, w_h)_T \; + \\ &\quad + \langle \tau_1 \lambda + \tau_2 \boldsymbol{\zeta} \cdot \boldsymbol{n}, w_h\rangle_{\partial T}\end{aligned} \tag{2.22d}$$

for each $(\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h) \in Z(T) \times \Sigma(T) \times Q(T) \times V(T)$.

The system (2.22) allows us to notice that, if we know the values of λ, $\boldsymbol{\zeta}$ and f on each $T \in \mathcal{T}_h$, then $(\underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)}, \boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)}, u_h^{(\lambda,\boldsymbol{\zeta},f)})$ is the unique solution of (2.22). This is summarized in next theorem.

Theorem 2.2. *Given $\lambda \in W_h$, $\boldsymbol{\zeta} \in \Phi_h$ and $f \in L^2(\Omega)$, if $\tau_1 > 0$, $\tau_4 > 0$, $\tau_2 + \tau_3 = 0$, then the system* (2.22) *has unique solution.*

Proof. The proof follows as in the proof of Theorem 2.1 $\square$

Now, we use the linearity of the scheme (2.22) to introduce the *local solvers*. First, for $\lambda \in W_h$ let $(\underline{\boldsymbol{Z}}_1(\lambda), \boldsymbol{S}_1(\lambda), \boldsymbol{Q}_1(\lambda), U_1(\lambda)) \in Z_h \times \Sigma_h \times Q_h \times V_h$ such that, on each $T \in \mathcal{T}_h$, it solves

$$
\begin{aligned}
(\underline{\boldsymbol{Z}}_1(\lambda), \underline{\boldsymbol{s}}_h)_T - (\nabla_h \cdot \underline{\boldsymbol{s}}_h, \boldsymbol{Q}_1(\lambda))_T &= 0\,, \\
(\boldsymbol{Q}_1(\lambda), \boldsymbol{v}_h)_T + (\nabla_h \cdot \boldsymbol{v}_h, U_1(\lambda))_T &= \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \lambda \rangle_{\partial T}\,, \\
(\boldsymbol{S}_1(\lambda), \boldsymbol{m}_h)_T + (\nabla_h \cdot \underline{\boldsymbol{Z}}_1(\lambda), \boldsymbol{m}_h)_T + \langle \tau_3\, U_1(\lambda)\, \boldsymbol{n} + \tau_4 \boldsymbol{Q}_1(\lambda), \boldsymbol{m}_h \rangle_{\partial T} &= \langle \tau_3 \lambda, \boldsymbol{m}_h \cdot \boldsymbol{n} \rangle_{\partial T}\,, \\
(\nabla_h \cdot \boldsymbol{S}_1(\lambda), w_h)_T + \langle \tau_1\, U_1(\lambda) + \tau_2\, \boldsymbol{Q}_1(\lambda) \cdot \boldsymbol{n}, w_h \rangle_{\partial T} &= \langle \tau_1\, \lambda, w_h \rangle_{\partial T}
\end{aligned}
\tag{2.23}
$$

$\forall\, (\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h) \in Z(T) \times \Sigma(T) \times Q(T) \times V(T)$.

Similarly, for $\boldsymbol{\zeta} \in \Phi_h$ let $(\underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta}), \boldsymbol{S}_2(\boldsymbol{\zeta}), \boldsymbol{Q}_2(\boldsymbol{\zeta}), U_2(\boldsymbol{\zeta})) \in Z_h \times \Sigma_h \times Q_h \times V_h$ such that, on each $T \in \mathcal{T}_h$, it satisfies

$$
\begin{aligned}
(\underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta}), \underline{\boldsymbol{s}}_h)_T - (\nabla_h \cdot \underline{\boldsymbol{s}}_h, \boldsymbol{Q}_2(\boldsymbol{\zeta}))_T &= \langle \underline{\boldsymbol{s}}_h \boldsymbol{n}, \boldsymbol{\zeta} \rangle_{\partial T}\,, \\
(\boldsymbol{Q}_2(\boldsymbol{\zeta}), \boldsymbol{v}_h)_T + (\nabla_h \cdot \boldsymbol{v}_h, U_2(\boldsymbol{\zeta}))_T &= 0\,, \\
(\boldsymbol{S}_2(\boldsymbol{\zeta}), \boldsymbol{m}_h)_T + (\nabla_h \cdot \boldsymbol{Z}_2(\boldsymbol{\zeta}), \boldsymbol{m}_h)_T + \langle \tau_3\, U_2(\boldsymbol{\zeta})\, \boldsymbol{n} + \tau_4\, \boldsymbol{Q}_2(\boldsymbol{\zeta}), \boldsymbol{m}_h \rangle_{\partial T} &= \langle \tau_4\, \boldsymbol{\zeta}, \boldsymbol{m}_h \rangle_{\partial T}\,, \\
(\nabla_h \cdot \boldsymbol{S}_2(\boldsymbol{\zeta}), w_h)_T + \langle \tau_1\, U_2(\boldsymbol{\zeta}) + \tau_2\, \boldsymbol{Q}_2(\boldsymbol{\zeta}) \cdot \boldsymbol{n}, w_h \rangle_{\partial T} &= \langle \tau_2\, \boldsymbol{\zeta} \cdot \boldsymbol{n}, w_h \rangle_{\partial T}
\end{aligned}
\tag{2.24}
$$

$\forall\, (\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h) \in Z(T) \times \Sigma(T) \times Q(T) \times V(T)$.

By last, for the data $f \in L^2(\Omega)$ restricted to each element, let $(\underline{\boldsymbol{Z}}_3(f), \boldsymbol{S}_3(f), \boldsymbol{Q}_3(f), U_3(f)) \in Z_h \times \Sigma_h \times Q_h \times V_h$ such that, on each $T \in \mathcal{T}_h$, it solves

$$
\begin{aligned}
(\boldsymbol{Z}_3(f), \underline{\boldsymbol{s}}_h)_T - \langle \nabla_h \cdot \underline{\boldsymbol{s}}_h, \boldsymbol{Q}_3(f) \rangle_{\partial T} &= 0\,, \\
(\boldsymbol{Q}_3(f), \boldsymbol{v}_h)_T + (\nabla_h \cdot \boldsymbol{v}_h, U_3(f))_T &= 0\,, \\
(\boldsymbol{S}_3(f), \boldsymbol{m}_h)_T + (\boldsymbol{m}_h, \nabla_h \cdot \boldsymbol{Z}_3(f))_T + \langle \tau_3\, U_3(f)\, \boldsymbol{n} + \tau_4\, \boldsymbol{Q}_3(f), \boldsymbol{m}_h \rangle_{\partial T} &= 0\,, \\
(\nabla_h \cdot \boldsymbol{S}_3(f), w_h)_T)_T + \langle \tau_1\, U_3(f) + \tau_2\, \boldsymbol{Q}_3(f) \cdot \boldsymbol{n}, w_h \rangle_{\partial T} &= (f, w_h)_T
\end{aligned}
\tag{2.25}
$$

$\forall\, (\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h) \in Z(T) \times \Sigma(T) \times Q(T) \times V(T)$.

It is clear that the problems (2.23), (2.24), (2.25) have a unique solution since $(\underline{\boldsymbol{Z}}_1(\lambda), \boldsymbol{S}_1(\lambda), \boldsymbol{Q}_1(\lambda), U_1(\lambda))$, $(\underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta}), \boldsymbol{S}_2(\boldsymbol{\zeta}), \boldsymbol{Q}_2(\boldsymbol{\zeta}), U_2(\boldsymbol{\zeta}))$ and $(\underline{\boldsymbol{Z}}_3(f), \boldsymbol{S}_3(f), \boldsymbol{Q}_3(f), U_3(f))$ are the solutions of the system (2.22) with $(\boldsymbol{\zeta} = 0, \lambda \neq 0, f = 0)$, $(\boldsymbol{\zeta} \neq 0, \lambda = 0, f = 0)$ and $(\boldsymbol{\zeta} = 0, \lambda = 0, f \in L^2(\Omega))$

respectively. Consequently we can define the linear operators:

$$\begin{aligned}
\mathcal{L}_1 : W_h &\to Z_h \times \Sigma_h \times Q_h \times V_h\\
\lambda &\mapsto \mathcal{L}_1(\lambda) = (Z_1(\lambda), S_1(\lambda), Q_1(\lambda), U_1(\lambda)) \,,\\
\mathcal{L}_2 : \Phi_h &\to Z_h \times \Sigma_h \times Q_h \times V_h\\
\boldsymbol{\zeta} &\mapsto \mathcal{L}_2(\boldsymbol{\zeta}) = (Z_2(\boldsymbol{\zeta}), S_2(\boldsymbol{\zeta}), Q_2(\boldsymbol{\zeta}), U_2(\boldsymbol{\zeta})) \,,\\
\mathcal{L}_3 : L^2(\Omega) &\to Z_h \times \Sigma_h \times Q_h \times V_h\\
f &\mapsto \mathcal{L}_3(f) = (Z_3(f), S_3(f), Q_3(f), U_3(f)) \,,
\end{aligned}$$

associated to the problems (2.23), (2.24) and (2.25) respectively. The operators $\mathcal{L}_i$ with $i = 1, 2, 3$ are called *local solvers*, and they establish a relation between functions defined on ∂T and the solutions defined on T.

Now, since we have introduced the *local solvers*, the local solution of (2.22) can be written as

$$\begin{aligned}
\underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)} &= \underline{\boldsymbol{Z}}_1(\lambda) + \underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta}) + \underline{\boldsymbol{Z}}_3(f) \,,\\
\boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)} &= \boldsymbol{S}_1(\lambda) + \boldsymbol{S}_2(\boldsymbol{\zeta}) + \boldsymbol{S}_3(f) \,,\\
\boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)} &= \boldsymbol{Q}_1(\lambda) + \boldsymbol{Q}_2(\boldsymbol{\zeta}) + \boldsymbol{Q}_3(f) \,,\\
u_h^{(\lambda,\boldsymbol{\zeta},f)} &= U_1(\lambda) + U_2(\boldsymbol{\zeta}) + U_3(f) \,.
\end{aligned} \tag{2.26}$$

On the other hand, given the fact that the *numerical traces* λ and $\boldsymbol{\zeta}$ are chosen, then we set them to satisfy the equations $\langle \boldsymbol{\zeta}, \boldsymbol{\beta}_h \rangle_\Gamma = 0$ and $\langle \lambda, \alpha_h \rangle_\Gamma = 0$.

At the same time, the *numerical fluxes* $\hat{\underline{\boldsymbol{z}}}_h^{(\lambda,\boldsymbol{\zeta},f)}$ and $\hat{u}_h^{(\lambda,\boldsymbol{\zeta},f)}$ are defined as

$$\hat{\underline{\boldsymbol{z}}}_h^{(\lambda,\boldsymbol{\zeta},f)}\, \boldsymbol{n} = \underline{\boldsymbol{z}}_h^{(\lambda,\boldsymbol{\zeta},f)}\, \boldsymbol{n} + \tau_3 (u_h^{(\lambda,\boldsymbol{\zeta},f)} - \lambda) + \tau_4 (\boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)} - \boldsymbol{\zeta}) \,,$$

and

$$\hat{\boldsymbol{\sigma}}_h^{(\lambda,\boldsymbol{\zeta},f)} \cdot \boldsymbol{n} = \boldsymbol{\sigma}_h^{(\lambda,\boldsymbol{\zeta},f)} \cdot \boldsymbol{n} + \tau_1 (u_h^{(\lambda,\boldsymbol{\zeta},f)} - \lambda) + \tau_2 (\boldsymbol{q}_h^{(\lambda,\boldsymbol{\zeta},f)} - \boldsymbol{\zeta}) \cdot \boldsymbol{n} \,.$$

Additionally, we ask $\hat{\underline{\boldsymbol{z}}}_h^{(\lambda,\boldsymbol{\zeta},f)}$ and $\hat{\boldsymbol{\sigma}}_h^{(\lambda,\boldsymbol{\zeta},f)}$ to satisfy the compatibility conditions

$$\begin{aligned}
\left\langle \hat{\underline{\boldsymbol{z}}}_h^{(\lambda,\boldsymbol{\zeta},f)}\, \boldsymbol{n}, \boldsymbol{\beta}_h \right\rangle_{\partial\mathcal{T}_h \setminus \Gamma} &= 0\\
\left\langle \hat{\boldsymbol{\sigma}}_h^{(\lambda,\boldsymbol{\zeta},f)} \cdot \boldsymbol{n}, \alpha_h \right\rangle_{\partial\mathcal{T}_h \setminus \Gamma} &= 0 \,.
\end{aligned} \tag{2.27}$$

Next, replacing (2.26) in the equations (2.27), we define the following global problem which will allow us to find the unknowns λ and $\boldsymbol{\zeta}$. Let $(\boldsymbol{\zeta}, \lambda) \in \Phi_h^0 \times W_h^0$ solve

$$\mathcal{A}((\boldsymbol{\zeta}, \lambda), (\boldsymbol{\beta}_h, \alpha_h)) = b(\boldsymbol{\beta}_h, \alpha_h) \tag{2.28a}$$

for all $(\boldsymbol{\beta}_h, \alpha_h) \in \Phi_h^0 \times W_h^0$, where $\mathcal{A} : (\Phi_h^0 \times W_h^0) \times (\Phi_h^0 \times W_h^0) \longrightarrow \mathbb{R}$ is a bilinear form and $b : \Phi_h^0 \times W_h^0 \longrightarrow \mathbb{R}$ is a linear form defined as follow: for $(\boldsymbol{\gamma}, \xi) \in \Phi_h^0 \times W_h^0$ and $(\boldsymbol{\mu}, \eta) \in \Phi_h^0 \times W_h^0$

$$\begin{aligned}
\mathcal{A}((\boldsymbol{\gamma}, \xi), (\boldsymbol{\mu}, \eta)) := {} & \langle (\underline{\boldsymbol{Z}}_1(\xi) + \underline{\boldsymbol{Z}}_2(\boldsymbol{\gamma}))\boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} + \langle \tau_3 \,(U_1(\xi) + U_2(\boldsymbol{\gamma}) - \xi)\boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} + \\
& + \langle \tau_4 \,(\boldsymbol{Q}_1(\xi) + \boldsymbol{Q}_2(\boldsymbol{\gamma}) - \boldsymbol{\gamma}), \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma} + \langle (\boldsymbol{S}_1(\xi) + \boldsymbol{S}_2(\boldsymbol{\gamma})) \cdot \boldsymbol{n}, \eta \rangle_{\partial\mathcal{T}_h \setminus \Gamma}\\
& + \langle \tau_1 \,(U_1(\xi) + U_2(\boldsymbol{\gamma}) - \xi), \eta \rangle_{\partial\mathcal{T}_h \setminus \Gamma} + \langle \tau_2 \,(\boldsymbol{Q}_1(\xi) + \boldsymbol{Q}_2(\boldsymbol{\gamma}) - \boldsymbol{\gamma}) \cdot \boldsymbol{n}, \eta \rangle_{\partial\mathcal{T}_h \setminus \Gamma} \,,
\end{aligned}$$

$$\begin{aligned}
b(\boldsymbol{\mu}, \eta) := {} & - \langle \tau_3 \, U_3(f)\boldsymbol{n} + \tau_4 \, \boldsymbol{Q}_3(f) + \underline{\boldsymbol{Z}}_3(f)\boldsymbol{n}, \boldsymbol{\mu} \rangle_{\partial\mathcal{T}_h \setminus \Gamma}\\
& - \langle \tau_1 \, U_3(f) + \tau_2 \, \boldsymbol{Q}_3(f) \cdot \boldsymbol{n} + \boldsymbol{S}_3(f) \cdot \boldsymbol{n}, \eta \rangle_{\partial\mathcal{T}_h \setminus \Gamma} \,.
\end{aligned}$$

Where $\Phi_h^0 := \{\boldsymbol{\chi} \in \Phi_h : \boldsymbol{\chi}|_\Gamma = 0\}$ and $W_h^0 := \{\omega \in W_h : \omega|_\Gamma = 0\}$.

Theorem 2.3 is the main result of this section and shows that to solve (2.14) we only need to solve the problem (2.28) for $\hat{\boldsymbol{q}}_h$ and $\hat{u}_h$.

Theorem 2.3. *The problem* (2.28) *is well defined. Moreover, if* $(\hat{\boldsymbol{q}}_h, \hat{u}_h) \in \Phi_h \times W_h$ *solves* (2.28) *and if*

$$\begin{aligned} \underline{\boldsymbol{z}}_h &= \underline{\boldsymbol{Z}}_1(\hat{u}_h) + \underline{\boldsymbol{Z}}_2(\hat{\boldsymbol{q}}_h) + \underline{\boldsymbol{Z}}_3(f)\,, \\ \boldsymbol{\sigma}_h &= \boldsymbol{S}_1(\hat{u}_h) + \boldsymbol{S}_2(\hat{\boldsymbol{q}}_h) + \boldsymbol{S}_3(f)\,, \\ \boldsymbol{q}_h &= \boldsymbol{Q}_1(\hat{u}_h) + \boldsymbol{Q}_2(\hat{\boldsymbol{q}}_h) + \boldsymbol{Q}_3(f)\,, \\ u_h &= U_1(\hat{u}_h) + U_2(\hat{\boldsymbol{q}}_h) + U_3(f)\,, \end{aligned} \tag{2.29}$$

then $(\underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h, \hat{u}_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times \Phi_h \times W_h$ *solves* (2.14).

Proof. We start by showing that the problem (2.28) is well defined. It is clear that the problem (2.28) is a square system, since $\dim\{\Phi_h\} = |\mathcal{E}_h^i| \dim([\mathbb{P}_k(e)]^d) = d|\mathcal{E}_h^i|\binom{d-1+k}{k}$ and $\dim\{W_h\} = |\mathcal{E}_h^i| \dim(\mathbb{P}_k(e))$, thus the problem (2.28) is a linear system with $(d+1)|\mathcal{E}_h^i|\binom{d-1+k}{k}$ unknowns and equations.

Next, we consider the homogeneous system (2.28) ($f \equiv 0$) and the solutions of the local solvers $\mathcal{L}_1(\hat{u}_h)$ and $\mathcal{L}_2(\hat{\boldsymbol{q}}_h)$ in the following way. Since $f \equiv 0$, then $\mathcal{L}_3(f) = (\underline{\boldsymbol{0}}, \boldsymbol{0}, \boldsymbol{0}, 0)$. This implies that the right hand side of (2.28) is also zero.

Then, the system (2.28) is rewritten as:

$$\begin{aligned} \left\langle \hat{\mathbf{Z}}\,\boldsymbol{n}, \boldsymbol{\beta}_h \right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0\,, \\ \left\langle \hat{\mathbf{S}}\cdot\boldsymbol{n}, \alpha_h \right\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0\,, \end{aligned} \tag{2.30}$$

where $\hat{\mathbf{Z}}\,\boldsymbol{n} := (\underline{\boldsymbol{Z}}_1(\lambda) + \underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta}))\,\boldsymbol{n} + \tau_3(U_1(\lambda) + U_2(\boldsymbol{\zeta}) - \lambda) + \tau_4(\boldsymbol{Q}_1(\lambda) + \boldsymbol{Q}_2(\boldsymbol{\zeta}) - \boldsymbol{\zeta})$ and $\hat{\mathbf{S}}\cdot\boldsymbol{n} = (\boldsymbol{S}_1(\lambda) + \boldsymbol{S}_2(\boldsymbol{\zeta}))\cdot\boldsymbol{n} + \tau_1(U_1(\lambda) + U_2(\boldsymbol{\zeta}) - \lambda) + \tau_2(\boldsymbol{Q}_1(\lambda) + \boldsymbol{Q}_2(\boldsymbol{\zeta}) - \boldsymbol{\zeta})$.

On the other hand, when we add the systems associated to $\mathcal{L}_1(\hat{u}_h)$ and $\mathcal{L}_2(\hat{\boldsymbol{q}}_h)$ we get

$$(\tilde{\underline{\boldsymbol{z}}}, \underline{\boldsymbol{s}}_h)_{\mathcal{T}_h} - (\nabla_h\cdot\underline{\boldsymbol{s}}_h, \tilde{\boldsymbol{q}})_{\mathcal{T}_h} = -\left\langle \underline{\boldsymbol{s}}_h\boldsymbol{n}, \boldsymbol{\zeta}\right\rangle_{\partial\mathcal{T}_h}\,, \tag{2.31a}$$

$$(\tilde{\boldsymbol{q}}, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\nabla_h\cdot\boldsymbol{v}_h, \tilde{u})_{\mathcal{T}_h} = \left\langle \boldsymbol{v}_h\cdot\boldsymbol{n}, \lambda\right\rangle_{\partial\mathcal{T}_h}\,, \tag{2.31b}$$

$$(\tilde{\boldsymbol{\sigma}}, \boldsymbol{m}_h)_{\mathcal{T}_h} + (\nabla_h\cdot\tilde{\underline{\boldsymbol{z}}}, \boldsymbol{m}_h)_{\mathcal{T}_h} + \left\langle \hat{\mathbf{Z}}\,\boldsymbol{n} - \tilde{\underline{\boldsymbol{z}}}\,\boldsymbol{n}, \boldsymbol{m}_h\right\rangle_{\partial\mathcal{T}_h} = 0\,, \tag{2.31c}$$

$$(\nabla_h\cdot\tilde{\boldsymbol{\sigma}}, w_h)_{\mathcal{T}_h} + \left\langle \hat{\mathbf{S}}\cdot\boldsymbol{n} - \tilde{\underline{\boldsymbol{s}}}\cdot\boldsymbol{n}, w_h\right\rangle_{\partial\mathcal{T}_h} = 0\,. \tag{2.31d}$$

Where $\tilde{\underline{\boldsymbol{z}}} = \underline{\boldsymbol{Z}}_1(\lambda) + \underline{\boldsymbol{Z}}_2(\boldsymbol{\zeta})$, $\tilde{\boldsymbol{q}} = \boldsymbol{Q}_1(\lambda) + \boldsymbol{Q}_2(\boldsymbol{\zeta})$, $\tilde{u} = U_1(\lambda) + U_2(\boldsymbol{\zeta})$, $\tilde{\boldsymbol{\sigma}} = \boldsymbol{S}_1(\lambda) + \boldsymbol{S}_2(\boldsymbol{\zeta})$. Now, using similar arguments as in Theorem 2.1 we obtain

$$||\tilde{\underline{\boldsymbol{z}}}||_{0,\mathcal{T}_h}^2 + ||(\tau_4)^{1/2}(\tilde{\boldsymbol{q}} - \boldsymbol{\zeta})||_{0,\partial\mathcal{T}_h}^2 + ||(\tau_1)^{1/2}(\tilde{u} - \lambda)||_{0,\partial\mathcal{T}_h}^2 = 0\,. \tag{2.32}$$

From the equation (2.32), we conclude that $\tilde{\underline{\boldsymbol{z}}} = 0$ in $\mathcal{T}_h$, $\tilde{\boldsymbol{q}} = \boldsymbol{\zeta}$ on $\partial\mathcal{T}_h$ and $\tilde{u} = \lambda$ on $\partial\mathcal{T}_h$. Similarly to the proof of Theorem 2.1, since $\boldsymbol{\zeta}|_\Gamma = 0$ and $\lambda|_\Gamma = 0$, we have $\tilde{\boldsymbol{q}} = 0$ and $\tilde{u} = 0$ in $\mathcal{T}_h$. Hence $\boldsymbol{\zeta} = 0$ and $\lambda = 0$ on $\partial\mathcal{T}_h$, and the problem (2.28) had only one solution.

By last, we show that $(\underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h, \hat{u}_h)$ defined in (2.29), is the unique solution of (2.14). To that end we only need to consider the three local solvers and add the system associated to each to them. Looking the variables in (2.29) we see that we have rebuilt the scheme (2.14). Then Theorem 2.1 showed that $(\underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, u_h, \hat{\boldsymbol{q}}_h, \hat{u}_h)$ is the unique solution of (2.14). $\square$

To close this section we comment that Theorem 2.2, the local solvers and Theorem 2.3 allow us to justify the main feature of the hybrid scheme (2.14). That is, we can eliminate all the interior variables locally to obtain a global system, (2.28), for the $\hat{\boldsymbol{q}}_h$ and $\hat{u}_h$ that approximate u and $\boldsymbol{q}$ on the interfaces of the triangulation. This characteristic represents a computational advantage because the only globally coupled degrees of freedom are those associated with the unknowns $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$.

CHAPTER 3

A priori error analysis

3.1 Error equations and energy argument

The aim now is to obtain the error estimates related to the proposed HDG method. In particular, we bound the interpolation and the projection of errors. Let $(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u) \in [\mathrm{H}(\mathrm{div};\Omega)]^{d\times d} \times [\mathrm{H}(\mathrm{div};\Omega)]^{d} \times [\mathrm{H}^1_0(\Omega)]^d \times \mathrm{H}^1_0(\Omega)$ the weak solution of (2.6), then using (2.14) we can write the following equations for the errors $\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h$, $\boldsymbol{q} - \boldsymbol{q}_h$, $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$ and $u - u_h$:

$$
\begin{aligned}
(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h)_{\mathcal{T}_h} - (\nabla\cdot \underline{\boldsymbol{s}}_h, \boldsymbol{q} - \boldsymbol{q}_h)_{\mathcal{T}_h} + \langle \underline{\boldsymbol{s}}_h\, \boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} &= 0\,, &\text{(3.1a)}\\
(\boldsymbol{q} - \boldsymbol{q}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\nabla\cdot \boldsymbol{v}_h, u - u_h)_{\mathcal{T}_h} - \langle \boldsymbol{v}_h\cdot\boldsymbol{n}, u - \hat{u}_h\rangle_{\partial\mathcal{T}_h} &= 0\,, &\text{(3.1b)}\\
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{m}_h)_{\mathcal{T}_h} - (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \nabla\, \boldsymbol{m}_h)_{\mathcal{T}_h} + \langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{m}_h\rangle_{\partial\mathcal{T}_h} &= 0\,, &\text{(3.1c)}\\
(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla\, w_h)_{\mathcal{T}_h} - \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, w_h\rangle_{\partial\mathcal{T}_h} &= 0\,, &\text{(3.1d)}\\
\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{\beta}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0\,, &\text{(3.1e)}\\
\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, \alpha_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} &= 0\,, &\text{(3.1f)}\\
\langle \boldsymbol{q} - \hat{\boldsymbol{q}}_h, \boldsymbol{\beta}_h\rangle_{\Gamma} &= 0\,, &\text{(3.1g)}\\
\langle u - \hat{u}_h, \alpha_h\rangle_{\Gamma} &= 0\,, &\text{(3.1h)}
\end{aligned}
$$

for all $(\underline{\boldsymbol{s}}_h, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h, \boldsymbol{\beta}_h, \alpha_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times \Phi_h \times W_h$.

First, we consider the following identify for $||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h}$

Lemma 3.1. *If τ_1 and τ_4 are positive and $\tau_2 = \tau_3 = 0$, then*

$$
\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} = \;& \big\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma})\cdot\boldsymbol{n}, Pu - u\big\rangle_{\partial\mathcal{T}_h} + \Big\langle \tau_1^{1/2}(u_h - \hat{u}_h), \tau_1^{-1/2}(Pu - u)\Big\rangle_{\partial\mathcal{T}_h}\\
& + \big\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}\big\rangle_{\partial\mathcal{T}_h} + \Big\langle \tau_4^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h), \tau_4^{-1/2}(\overline{P}\boldsymbol{q} - \boldsymbol{q})\Big\rangle_{\partial\mathcal{T}_h}\\
& - ||\tau_1^{1/2}(u_h - \hat{u}_h)||^2_{0,\partial\mathcal{T}_h} - \Big\langle \tau_1^{-1/2}(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma})\cdot\boldsymbol{n}, \tau_1^{1/2}(u_h - \hat{u}_h)\Big\rangle_{\partial\mathcal{T}_h}\\
& - ||\tau_4^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h} - \Big\langle \tau_4^{-1/2}(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \tau_4^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)\Big\rangle_{\partial\mathcal{T}_h}
\end{aligned}
\qquad (3.2)
$$

Proof. Considering the definition of $\mathcal{P}$ (see, Section 2.1) and (3.1a) with $\underline{\boldsymbol{s}}_h = \mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h$, we obtain

$$
\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)_{\mathcal{T}_h}\\
&= (\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \boldsymbol{q} - \boldsymbol{q}_h)_{\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Using the definition of $\overline{P}$ and integrating by parts previous expression, we obtain

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= (\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \overline{P}\boldsymbol{q} - \boldsymbol{q}_h)_{\mathcal{T}_h} - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} \\
&= -(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \nabla(\overline{P}\boldsymbol{q} - \boldsymbol{q}_h))_{\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} .
\end{aligned}$$

By definition of $\mathcal{P}$ and (3.1c) with $\boldsymbol{m}_h = \overline{P}\boldsymbol{q} - \boldsymbol{q}_h$, we have

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \nabla(\overline{P}\boldsymbol{q} - \boldsymbol{q}_h))_{\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} \\
&= -(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h)_{\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} .
\end{aligned}$$

By definition of $\overline{P}$ and then using (3.1b) with $\boldsymbol{v}_h = \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, there holds

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{q} - \boldsymbol{q}_h)_{\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\,\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} \\
&= +(\nabla\cdot(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h), u - u_h)_{\mathcal{T}_h} - \left\langle(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\boldsymbol{n}, u - \hat{u}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} .
\end{aligned}$$

By definition of P, integrating by parts, and then using (3.1d) with $w_h = Pu - u_h$ we have

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \nabla(Pu - u_h))_{\mathcal{T}_h} + \left\langle(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{n}, Pu - u_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{n}, u - \hat{u}_h\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} \\
&= -\langle(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, Pu - u_h\rangle_{\partial\mathcal{T}_h} + \left\langle(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{n}, Pu - u_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\cdot\boldsymbol{n}, u - \hat{u}_h\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h} - \langle(\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h} .
\end{aligned}$$

On the other hand, in (3.1f) and (3.1h) we take $\alpha_h = \begin{cases} -\hat{u}_h\,, & \text{on} \quad \partial\mathcal{T}_h\setminus\Gamma \\ P_\partial(\boldsymbol{\sigma}\cdot\boldsymbol{n}) - \hat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}\,, & \text{on} \quad \Gamma \end{cases}$, and obtain, after adding both equations,

$$\langle(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, -\hat{u}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \langle P_\partial(\boldsymbol{\sigma}\cdot\boldsymbol{n}) - \hat{\boldsymbol{\sigma}}_h\cdot\boldsymbol{n}, u - \hat{u}_h\rangle_\Gamma = 0\,.$$

Since $u \in \mathrm{H}^1_0(\Omega)$ and using the definition of P_∂, we have

$$\langle(\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n}, u - \hat{u}_h\rangle_{\partial\mathcal{T}_h} = 0\,.$$

Similarly, in (3.1e) and (3.1g) we take $\boldsymbol{\beta}_h = \begin{cases} -\hat{\boldsymbol{q}}_h\,, & \text{on} \quad \partial\mathcal{T}_h\setminus\Gamma \\ \overline{P}_\partial(\underline{\boldsymbol{z}}\,\boldsymbol{n}) - \hat{\underline{\boldsymbol{z}}}_h\,\boldsymbol{n}\,, & \text{on} \quad \Gamma \end{cases}$, and obtain

$$\langle(\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\,\boldsymbol{n}, -\hat{\boldsymbol{q}}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma} + \left\langle\overline{P}_\partial(\underline{\boldsymbol{z}}\,\boldsymbol{n}) - \hat{\underline{\boldsymbol{z}}}_h\,\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h\right\rangle_\Gamma = 0\,.$$

Since $\boldsymbol{q} \in [\mathrm{H}_0^1(\Omega)]^d$ and using the definition of $\overline{P}_\partial$, we have

$$\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\, \boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} = 0\,.$$

Thus,

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, Pu - u_h \rangle_{\partial\mathcal{T}_h} + \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, Pu - u_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, u - \hat{u}_h \right\rangle_{\partial\mathcal{T}_h} - \left\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h \right\rangle_{\partial\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad - \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, \hat{u}_h - u \rangle_{\partial\mathcal{T}_h} - \langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \hat{\boldsymbol{q}}_h - \boldsymbol{q} \rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

Rearranging terms in a convenient way, we have

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, Pu - u \rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, u_h - \hat{u}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, Pu - u_h \right\rangle_{\partial\mathcal{T}_h} - \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, u - \hat{u}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} + \langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q}_h \right\rangle_{\partial\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q} - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} \\
&= -\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, Pu - u \rangle_{\partial\mathcal{T}_h} + \langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, u_h - \hat{u}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, Pu - u \rangle_{\partial\mathcal{T}_h} - \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, u_h - \hat{u}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} + \langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

Rearranging terms again conveniently, we get

$$\begin{aligned}
||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} &= -\langle (\boldsymbol{\sigma} - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, Pu - u \rangle_{\partial\mathcal{T}_h} - \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot \boldsymbol{n}, u_h - \hat{u}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle (\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, u_h - \hat{u}_h \rangle_{\partial\mathcal{T}_h} + \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{n}, Pu - u \right\rangle_{\partial\mathcal{T}_h} \\
&\quad - \left\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle (\underline{\boldsymbol{z}}_h - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} \\
&= \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot \boldsymbol{n}, Pu - u \right\rangle_{\partial\mathcal{T}_h} - \langle (\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, Pu - u \rangle_{\partial\mathcal{T}_h} \\
&\quad + \langle (\boldsymbol{\sigma}_h - \hat{\boldsymbol{\sigma}}_h) \cdot \boldsymbol{n}, u_h - \hat{u}_h \rangle_{\partial\mathcal{T}_h} - \left\langle (\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}) \cdot \boldsymbol{n}, u_h - \hat{u}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} - \left\langle (\underline{\boldsymbol{z}}_h - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{q} - \boldsymbol{q} \right\rangle_{\partial\mathcal{T}_h} \\
&\quad \langle (\underline{\boldsymbol{z}}_h - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h} - \langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \boldsymbol{q}_h - \hat{\boldsymbol{q}}_h \rangle_{\partial\mathcal{T}_h}
\end{aligned}$$

Finally, by replacing the definitions of numerical traces (2.11) we finish the proof. □

The next two lemmas will allow us to find a expression for $||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h}$ and $||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h}$ in a similar way as we proceeded in previous lemma.

Lemma 3.2. *If τ_4 is positive and $\tau_3 = 0$, then,*

$$\begin{aligned}
||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &= -(\nabla \cdot (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h), \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)_{\mathcal{T}_h} + \left\langle (\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}})\boldsymbol{n}, \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\rangle_{\partial\mathcal{T}_h} \\
&\quad + \left\langle \tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h), \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned} \tag{3.3}$$

Proof. Considering the definition of $\overline{P}$ and using (3.1c) with $\boldsymbol{m}_h = \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h$, we obtain

$$\begin{aligned}
||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &= (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)_{\mathcal{T}_h} \\
&= (\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, \nabla(\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h))_{\mathcal{T}_h} - \left\langle (\underline{\boldsymbol{z}} - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n}, \overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h \right\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

Integrating by parts and rearranging terms, we have

$$\begin{aligned}
||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &= -(\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)_{\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad -\left\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}\\
&= -(\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)_{\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}})\boldsymbol{n},\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad +\left\langle(\hat{\underline{\boldsymbol{z}}}_h-\underline{\boldsymbol{z}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

Finally, replacing the definition of numerical traces (2.11) with $\tau_3=0$, we conclude

$$\begin{aligned}
||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &= -(\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h),\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)_{\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}})\boldsymbol{n},\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad +\left\langle\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h),\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

□

Lemma 3.3. *There holds*

$$\begin{aligned}
||\overline{P}\boldsymbol{q}-\boldsymbol{q}_h||^2_{0,\mathcal{T}_h} &= (\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla(Pu-u_h))_{\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad -\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},Pu-u\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned} \tag{3.4}$$

Proof. From equation (3.1b) with $\boldsymbol{v}_h=\overline{P}\boldsymbol{q}-\boldsymbol{q}_h$, we obtain

$$\begin{aligned}
||\overline{P}\boldsymbol{q}-\boldsymbol{q}_h||^2_{0,\mathcal{T}_h} &= (\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)_{\mathcal{T}_h}\\
&= -(\nabla\cdot(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h),u-u_h)_{\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},u-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

Integrating by parts, using the definition of $\overline{P}$ and rearranging terms properly, we conclude

$$\begin{aligned}
||\overline{P}\boldsymbol{q}-\boldsymbol{q}_h||^2_{0,\mathcal{T}_h} &= (\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla(Pu-u_h))_{\mathcal{T}_h}-\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},Pu-u_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad +\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},u-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\
&= (\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla(Pu-u_h))_{\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\
&\quad -\left\langle(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h)\cdot\boldsymbol{n},Pu-u\right\rangle_{\partial\mathcal{T}_h}.
\end{aligned}$$

□

Next result will allow us to bound the norm of the projection of the error in the variables $\underline{\boldsymbol{z}}$, $\boldsymbol{\sigma}$ and $\boldsymbol{q}$. In the proof we will use Lemmas 3.1, 3.2 and 3.3. Inequalities (2.2) and (2.1) will be also employed.

Theorem 3.4. *Let τ_1, τ_2, τ_3 and τ_4 real numbers such that $\tau_1>0$, $\tau_4>0$ and $\tau_2=\tau_3=0$. Let us assume $k\geq 1$ and $\underline{\boldsymbol{z}}\in[H^{r+1}(\mathcal{T}_h)]^{d\times d}$, $\boldsymbol{\sigma}\in[H^{r+1}(\mathcal{T}_h)]^d$, $\boldsymbol{q}\in[H^{r+1}(\mathcal{T}_h)]^d$ and $u\in H^{r+1}(\mathcal{T}_h)$. Then there exist positive constants $C_1=C(k,d,\tau_1,\tau_4)$, $C_2=C(k,d,\tau_4)$, $C_3=C(k,d,\tau_1)$, such that*

a) $||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h}+\frac{1}{2}||\tau_1^{1/2}(u_h-\hat{u}_h)||^2_{0,\partial\mathcal{T}_h}+\frac{1}{2}||\tau_4^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h}\leq C_1h^{2r+1}||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||^2_{r+1,\mathcal{T}_h}$

b) $||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||_{0,\mathcal{T}_h}\leq C_2h^{r-1/2}||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||_{r+1,\mathcal{T}_h}$

c) $||\overline{P}\boldsymbol{q}-\boldsymbol{q}_h||^2_{0,\mathcal{T}_h} \leq C_{inv}h^{-2}||Pu-u_h||^2_{0,\mathcal{T}_h} + C_3 h^{2r}||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||^2_{r+1,\mathcal{T}_h}$

with $0<r\leq k$ *and* $||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||^2_{r+1,\mathcal{T}_h} := ||\boldsymbol{\sigma}||^2_{r+1,\mathcal{T}_h} + ||\boldsymbol{q}||^2_{r+1,\mathcal{T}_h} + ||\underline{\boldsymbol{z}}||^2_{r+1,\mathcal{T}_h} + ||u||^2_{r+1,\mathcal{T}_h}$.

Proof. Let us begin with the first inequality. We consider Lemma 3.1 and apply Cauchy-Schwarz inequality to obtain

$$
\begin{aligned}
&||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} + ||(\tau_1)^{1/2}(u_h-\hat{u}_h)||^2_{0,\partial\mathcal{T}_h} + ||(\tau_4)^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h} \\
&\leq\ ||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}||_{\partial\mathcal{T}_h}||Pu-u||_{\partial\mathcal{T}_h} + C_{\tau_1}||Pu-u||_{0,\partial\mathcal{T}_h}||\tau_1^{1/2}(u_h-\hat{u}_h)||_{0,\partial\mathcal{T}_h} \\
&\ +||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||_{0,\partial\mathcal{T}_h}||\overline{P}\boldsymbol{q}-\boldsymbol{q}||_{0,\partial\mathcal{T}_h} + C_{\tau_4}||\overline{P}\boldsymbol{q}-\boldsymbol{q}||_{0,\partial\mathcal{T}_h}||\tau_4^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||_{0,\partial\mathcal{T}_h} \\
&\ +C_{\tau_1}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}||_{0,\partial\mathcal{T}_h}||\tau_1^{1/2}(u_h-\hat{u}_h)||_{0,\partial\mathcal{T}_h} + C_{\tau_4}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||_{0,\partial\mathcal{T}_h}||\tau_4^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||_{0,\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Using Young's inequality and arranging terms properly, we have

$$
\begin{aligned}
&||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} + \tfrac{1}{2}||(\tau_1)^{1/2}(u_h-\hat{u}_h)||^2_{0,\partial\mathcal{T}_h} + \tfrac{1}{2}||(\tau_4)^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h} \\
&\leq\ C_{\tau_1}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}||^2_{0,\partial\mathcal{T}_h} + C_{\tau_1}||Pu-u||^2_{\partial\mathcal{T}_h} + C_{\tau_4}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||^2_{0,\partial\mathcal{T}_h} + C_{\tau_4}||\overline{P}\boldsymbol{q}-\boldsymbol{q}||^2_{0,\partial\mathcal{T}_h}\,.
\end{aligned}
$$

By last, we use inequality (2.1) with $s=r+1$ and we finish the proof as follow

$$
||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} + \tfrac{1}{2}||(\tau_1)^{1/2}(u_h-\hat{u}_h)||^2_{0,\partial\mathcal{T}_h} + \tfrac{1}{2}||(\tau_4)^{1/2}(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h} \leq Ch^{2r+1}||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||^2_{r+1,\mathcal{T}_h}\,.
$$

For second inequality, we use Cauchy-Schwarz and Young's inequalities in Lemma 3.2 to get, for any $\epsilon>0$, that

$$
\begin{aligned}
||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} \ &\leq\ \frac{1}{4}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{\mathcal{T}_h} + ||\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}}-z)||^2_{\mathcal{T}_h} \\
&\quad +\frac{1}{2\epsilon}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{\epsilon}{2}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||^2_{0,\partial\mathcal{T}_h} \\
&\quad \frac{1}{2\epsilon}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{\epsilon}{2}||\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Now, we choose $\epsilon = 2h^{-1}C^2_{tr}$ and replace

$$
\begin{aligned}
||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} \ &\leq\ \frac{1}{4}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} + ||\nabla\cdot(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)||^2_{0,\mathcal{T}_h} \\
&\quad +\frac{h}{4C^2_{tr}}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C^2_{tr}}{h}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||^2_{0,\partial\mathcal{T}_h} \\
&\quad +\frac{h}{4C^2_{tr}}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C^2_{tr}}{h}||\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Using the inverse inequality

$$
\begin{aligned}
||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} \ &\leq\ \frac{1}{4}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} + C_{inv}h^{-2}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} \\
&\quad +\frac{h}{4C^2_{tr}}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C^2_{tr}}{h}||\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}||^2_{0,\partial\mathcal{T}_h} \\
&\quad +\frac{h}{4C^2_{tr}}||\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C^2_{tr}}{h}||\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h}\,.
\end{aligned}
$$

Using the inequalities (2.1) with $s = r+1$

$$\begin{aligned} ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &\leq \frac{1}{4}||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} + C_{inv}h^{-2}||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h)||^2_{0,\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2\, C\, h^{2r+1}}{h}||\underline{\boldsymbol{z}}||^2_{r+1,\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2}{h}||\tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

Use inequality (2.3) we obtain

$$\begin{aligned} ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &\leq \frac{3}{4}||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} + C_{inv}h^{-2}||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||^2_{0,\mathcal{T}_h} \\ &+ C_{tr}^2\, C\, h^{2r}||\underline{\boldsymbol{z}}||^2_{r+1,\mathcal{T}_h} + \frac{C_{tr}^2}{h}||\tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||^2_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

Finally, using the first inequality of the Theorem 3.4 we conclude

$$\frac{1}{4}||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} \leq C_2 h^{2r-1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||^2_{r+1,\mathcal{T}_h}\,.$$

For the third inequality, we consider Lemma 3.3, Cauchy-Schwarz and Young's inequality

$$\begin{aligned} ||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} &\leq \frac{1}{4}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} + ||\nabla(Pu - u_h)||^2_{0,\mathcal{T}_h} \\ &+ \frac{1}{2\epsilon}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{\epsilon}{2}||u_h - \hat{u}_h||^2_{0,\partial\mathcal{T}_h} \\ &+ \frac{1}{2\epsilon}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{\epsilon}{2}||Pu - u||^2_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

We choose $\epsilon = \frac{2C_{tr}^2}{h}$

$$\begin{aligned} ||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} &\leq \frac{1}{4}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} + ||\nabla(Pu - u_h)||^2_{0,\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2}{h}||u_h - \hat{u}_h||^2_{0,\partial\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2}{h}||Pu - u||^2_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

Using the inverse inequality

$$\begin{aligned} ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &\leq \frac{1}{4}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} + C_{inv}h^{-2}||Pu - u_h||^2_{0,\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2}{h}||u_h - \hat{u}_h||^2_{0,\partial\mathcal{T}_h} \\ &+ \frac{h}{4C_{tr}^2}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\partial\mathcal{T}_h} + \frac{C_{tr}^2}{h}||Pu - u||^2_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

Using inequality (2.3) and inequality (2.1) with $s = r+1$ we get

$$\begin{aligned} ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||^2_{0,\mathcal{T}_h} &\leq \frac{3}{4}||\overline{P}\boldsymbol{q} - \boldsymbol{q}_h||^2_{0,\mathcal{T}_h} + C_{inv}h^{-2}||Pu - u_h||^2_{0,\mathcal{T}_h} \\ &+ \frac{C_{tr}^2}{h}C_{\tau_1}||(\tau_1)^{1/2}(u_h - \hat{u}_h)||^2_{0,\partial\mathcal{T}_h} + h^{-1}C_{tr}^2 C h^{2r+1}||u||^2_{r+1,\mathcal{T}_h}\,. \end{aligned}$$

Arranging terms and using the first inequality of the Theorem 3.4, we conclude

$$\frac{1}{4}||\overline{P}\boldsymbol{q}-\boldsymbol{q}_h||^2_{0,\mathcal{T}_h} \leq C_{inv}h^{-2}||Pu-u_h||^2_{0,\mathcal{T}_h}+C_3h^{2r}||(\underline{\boldsymbol{z}},\boldsymbol{\sigma},\boldsymbol{q},u)||^2_{r+1,\mathcal{T}_h}\,,$$

where $C_3 := C_{\tau_1}$.

□

3.2 Duality argument

In this section we estimate $||Pu-u_h||_{0,\Omega}$ via a duality argument. Given $\gamma \in L^2(\Omega)$ we consider the dual problem

$$\begin{aligned}
\boldsymbol{\phi}-\nabla\psi &= 0 \quad \text{in} \quad \Omega\,,\\
\underline{\boldsymbol{\varphi}}+\nabla\boldsymbol{\phi} &= 0 \quad \text{in} \quad \Omega\,,\\
\boldsymbol{\eta}+\nabla\cdot\underline{\boldsymbol{\varphi}} &= 0 \quad \text{in} \quad \Omega\,,\\
\nabla\cdot\boldsymbol{\eta} &= \gamma \quad \text{in} \quad \Omega\,,\\
\\
\boldsymbol{\phi}\cdot\boldsymbol{n} &= 0 \quad \text{in} \quad \Gamma\,,\\
\psi &= 0 \quad \text{in} \quad \Gamma\,.
\end{aligned} \tag{3.5}$$

We assume that this boundary value problem have the following regularity estimate

$$||\psi||_{4,\Omega}+||\boldsymbol{\phi}||_{3,\Omega}+||\underline{\boldsymbol{\varphi}}||_{2,\Omega}+||\boldsymbol{\eta}||_{1,\Omega} \leq C_{reg}||\gamma||_{0,\Omega}\,. \tag{3.6}$$

The regularity assumption (3.6) holds, for example, for polygonal domains with inner-angle conditions [50].

Lemma 3.5. *Let $\gamma \in L^2(\Omega)$, $\tau_2=\tau_3=0$ and $(\psi,\boldsymbol{\phi},\underline{\boldsymbol{\varphi}},\boldsymbol{\eta})$ the corresponding solution of (3.5). Then,*

$$\begin{aligned}
(Pu-u_h,\gamma) = & -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n},Pu-u\right\rangle_{\partial\mathcal{T}_h}-\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\
&+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n},u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\
&+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}})\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}+\left\langle\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h),\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}\\
&+\left\langle(\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma})\cdot\boldsymbol{n},P\psi-\psi\right\rangle_{\partial\mathcal{T}_h}+\left\langle\tau_1(u_h-\hat{u}_h),P\psi-\psi\right\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned} \tag{3.7}$$

Proof. We use the fact that $\nabla\cdot\boldsymbol{\eta}=\gamma$, add and subtract $\nabla\cdot\overline{P}\boldsymbol{\eta}$ to get

$$\begin{aligned}
(Pu-u_h,\gamma)_{\mathcal{T}_h} &= (Pu-u_h,\nabla\cdot\boldsymbol{\eta})_{\mathcal{T}_h}\\
&= (Pu-u_h,\nabla\cdot\boldsymbol{\eta}-\nabla\cdot\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}+(Pu-u_h,\nabla\cdot\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}\,.
\end{aligned}$$

Using the definition of P and integrating by parts, we obtain

$$\begin{aligned}
(Pu-u_h,\gamma)_{\mathcal{T}_h} &= (Pu-u_h,\nabla\cdot\boldsymbol{\eta}-\nabla\cdot\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}+(u-u_h,\nabla\cdot\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}\\
&= \left\langle(\boldsymbol{\eta}-\overline{P}\boldsymbol{\eta})\cdot\boldsymbol{n},Pu-u_h\right\rangle_{\partial\mathcal{T}_h}+(u-u_h,\nabla\cdot\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}\,.
\end{aligned}$$

Testing equation (3.1b) with $\boldsymbol{v}_h=\overline{P}\boldsymbol{\eta}$ and replacing properly, we have

$$\begin{aligned}
(Pu-u_h,\gamma)_{\mathcal{T}_h} = & \left\langle(\boldsymbol{\eta}-\overline{P}\boldsymbol{\eta})\cdot\boldsymbol{n},Pu-u_h\right\rangle_{\partial\mathcal{T}_h}-(\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta})_{\mathcal{T}_h}\\
&+\left\langle\overline{P}\boldsymbol{\eta}\cdot\boldsymbol{n},u-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\,.
\end{aligned}$$

We use definition of $\overline{P}$, add and subtract $\boldsymbol{\eta}$, add the term $-\langle \boldsymbol{\eta}\cdot\boldsymbol{n}, u-\hat{u}_h\rangle_{\partial\mathcal{T}_h}=0$, consider that $\boldsymbol{\eta}\in \mathrm{H}^1(\Omega)$, $u\in \mathrm{H}^1_0(\Omega)$ and $\hat{u}_h$ is single-valued, using the fact that $\boldsymbol{\eta}=-\nabla\cdot\underline{\boldsymbol{\varphi}}$, we obtain

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= \left\langle(\boldsymbol{\eta}-\overline{P}\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u_h\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla\cdot\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\,.\end{aligned}$$

Rearranging terms and adding and subtracting $\nabla\cdot\mathcal{P}\underline{\boldsymbol{\varphi}}$, we have

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla\cdot\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\[1em] &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla\cdot\underline{\boldsymbol{\varphi}}-\nabla\cdot\mathcal{P}\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &\quad +(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\nabla\cdot\mathcal{P}\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\,.\end{aligned}$$

Integrating by parts, using the definition of $\mathcal{P}$, $\overline{P}$ and equation (3.1a) with $\underline{\boldsymbol{s}}_h=\mathcal{P}\underline{\boldsymbol{\varphi}}$, it holds

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}-\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\overline{P}\boldsymbol{q}-\boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad +(\boldsymbol{q}-\boldsymbol{q}_h,\nabla\cdot\mathcal{P}\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\[1em] &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+\left\langle(\underline{\boldsymbol{\varphi}}-\mathcal{P}\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\overline{P}\boldsymbol{q}-\boldsymbol{q}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad +(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}+\left\langle\mathcal{P}\underline{\boldsymbol{\varphi}}\,\boldsymbol{n},\boldsymbol{q}-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\,.\end{aligned}$$

Next, since $\underline{\boldsymbol{\varphi}}\in[\mathrm{H}^2(\Omega)]^{d\times d}$, $\boldsymbol{q}\in[\mathrm{H}^1_0(\Omega)]^d$ and $\left\langle\underline{\boldsymbol{\varphi}}\,\boldsymbol{n},\boldsymbol{q}-\boldsymbol{q}_h\right\rangle_\Gamma=\left\langle\overline{P}_\partial(\underline{\boldsymbol{\varphi}}\,\boldsymbol{n}),\boldsymbol{q}-\boldsymbol{q}_h\right\rangle_\Gamma=0$ by (3.1g) we add the term $-\left\langle\underline{\boldsymbol{\varphi}}\,\boldsymbol{n},\boldsymbol{q}-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}=0$ and arranging similar terms:

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\boldsymbol{q}-\overline{P}\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &\quad +\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}+(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\,.\end{aligned}$$

We use definition of $\mathcal{P}$, add and subtract $\underline{\boldsymbol{\varphi}}$ and we use the fact of $\underline{\boldsymbol{\varphi}}=-\nabla\boldsymbol{\phi}$,

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}-\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &\quad +\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &\quad -(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\nabla\boldsymbol{\phi})_{\mathcal{T}_h}\,.\end{aligned}$$

Add and subtract $\nabla\overline{P}\boldsymbol{\phi}$, integrating by parts and using the definition of $\overline{P}$ and $\mathcal{P}$, we get

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} &= -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h}-\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &\quad +\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h}+\left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n},\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &\quad -(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h}+(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &\quad +\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}-(\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\nabla\overline{P}\boldsymbol{\phi})_{\mathcal{T}_h}\,.\end{aligned}$$

Using equation (3.1c) with $\boldsymbol{m}_h = \overline{P}\boldsymbol{\phi}$, we have

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} \;=\;& -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &-(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h} + (\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\overline{P}\boldsymbol{\phi})_{\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

we use definition of $\overline{P}$, add and subtract $\boldsymbol{\phi}$ and replace $\boldsymbol{\phi}$ for $\nabla\psi$,

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} \;=\;& -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &-(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h} + (\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h} - (\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi})_{\mathcal{T}_h}\\ &-(\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\nabla\psi)_{\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Add and subtract $\nabla P\psi$, integrating by parts and using the definition of P and $\overline{P}$, we obtain

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} \;=\;& -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\boldsymbol{n}, \overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\varphi}-\underline{\varphi})\,\boldsymbol{n}, \boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &-(\overline{P}\boldsymbol{q}-\boldsymbol{q}_h,\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})_{\mathcal{T}_h} + (\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h,\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})_{\mathcal{T}_h}\\ &+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h} - (\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi})_{\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{n},P\psi-\psi\right\rangle_{\partial\mathcal{T}_h} - (\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\nabla P\psi)_{\mathcal{T}_h}\\ &-\left\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Using the definition of L^2-projector, testing (3.1d) with $w_h = P\psi$. The previous expression is reduced to

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} \;=\;& -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\boldsymbol{n}, \overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\boldsymbol{n}, \boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(P\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h} - \langle(P\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\cdot\boldsymbol{n},P\psi-\psi\rangle_{\partial\mathcal{T}_h}\\ &-\langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},P\psi\rangle_{\partial\mathcal{T}_h} - \left\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n},\overline{P}\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}.\end{aligned}$$

Notice that using the boundary conditions $\psi = 0$ in Γ and $\boldsymbol{\phi}\cdot\boldsymbol{n} = 0$ in Γ, the fact of $\boldsymbol{\sigma} \in [\mathrm{H}(\mathrm{div};\Omega)]^d$, $\underline{\boldsymbol{z}} \in [\mathrm{H}(\mathrm{div};\Omega)]^{d\times d}$ and the properly of conservative on $\hat{z}_h$ and $\hat{\boldsymbol{\sigma}}_h$ we have

$$\langle(\underline{\boldsymbol{z}}-\hat{\underline{\boldsymbol{z}}}_h)\,\boldsymbol{n},\boldsymbol{\phi}\rangle_{\partial\mathcal{T}_h} = 0 \quad\text{and}\quad \langle(\boldsymbol{\sigma}-\hat{\boldsymbol{\sigma}}_h)\cdot\boldsymbol{n},\psi\rangle_{\partial\mathcal{T}_h} = 0 \tag{3.8}$$

Finally, using the definition of the L^2-projection, adding the equations (3.8) and replacing the definition (2.11) we get

$$\begin{aligned}(Pu-u_h,\gamma)_{\mathcal{T}_h} \;=\;& -\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, Pu-u\right\rangle_{\partial\mathcal{T}_h} - \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \overline{P}\boldsymbol{q}-\boldsymbol{q}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\eta}-\boldsymbol{\eta})\cdot\boldsymbol{n}, u_h-\hat{u}_h\right\rangle_{\partial\mathcal{T}_h} + \left\langle(\mathcal{P}\underline{\boldsymbol{\varphi}}-\underline{\boldsymbol{\varphi}})\,\boldsymbol{n}, \boldsymbol{q}_h-\hat{\boldsymbol{q}}_h\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\mathcal{P}\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}})\,\boldsymbol{n},\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h} + \left\langle\tau_4(\boldsymbol{q}_h-\hat{\boldsymbol{q}}_h),\overline{P}\boldsymbol{\phi}-\boldsymbol{\phi}\right\rangle_{\partial\mathcal{T}_h}\\ &+\left\langle(\overline{P}\boldsymbol{\sigma}-\boldsymbol{\sigma})\cdot\boldsymbol{n},P\psi-\psi\right\rangle_{\partial\mathcal{T}_h} + \langle\tau_1(u_h-\hat{u}_h),P\psi-\psi\rangle_{\partial\mathcal{T}_h}\end{aligned}$$

□

Theorem 3.6. *Let us assume that* (3.6) *holds. Then, there exists a constant* $C := C(k, d, \tau_1, \tau_4)$ *such that*

$$||Pu - u_h||_{0,\mathcal{T}_h} \leq Ch^{r+1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h} \tag{3.9}$$

Proof. Applying the Cauchy-Schwarz inequality in the Lemma 3.5, we obtain

$$\begin{aligned} |(Pu - u_h, \gamma)| \leq\ & ||\overline{P}\boldsymbol{\eta} - \boldsymbol{\eta}||_{0,\partial\mathcal{T}_h}||Pu - u||_{0,\partial\mathcal{T}_h} + ||\mathcal{P}\underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}||_{0,\partial\mathcal{T}_h}||\overline{P}\boldsymbol{q} - \boldsymbol{q}||_{0,\partial\mathcal{T}_h} \\ & + C_{\tau_1}||\overline{P}\boldsymbol{\eta} - \boldsymbol{\eta}||_{0,\partial\mathcal{T}_h}||\tau_1^{1/2}(u_h - \hat{u}_h)||_{0,\partial\mathcal{T}_h} \\ & + C_{\tau_4}||\mathcal{P}\underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}||_{0,\partial\mathcal{T}_h}||\tau_4^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||_{0,\partial\mathcal{T}_h} + ||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}||_{0,\partial\mathcal{T}_h}||\overline{P}\boldsymbol{\phi} - \boldsymbol{\phi}||_{0,\partial\mathcal{T}_h} \\ & + C_{\tau_4}||\tau_4(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||_{0,\partial\mathcal{T}_h}||\overline{P}\boldsymbol{\phi} - \boldsymbol{\phi}||_{0,\partial\mathcal{T}_h} + ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}||_{0,\partial\mathcal{T}_h}||P\psi - \psi||_{0,\partial\mathcal{T}_h} \\ & + C_{\tau_1}||P\psi - \psi||_{0,\partial\mathcal{T}_h}||\tau_1^{1/2}(u_h - \hat{u}_h)||_{0,\partial\mathcal{T}_h}\,. \end{aligned}$$

We use properties of the projections $\overline{P}$, $\mathcal{P}$ and P (see inequality (2.1)) on each variable of dual problem and deduce that

$$\begin{aligned} ||\overline{P}\boldsymbol{\eta} - \boldsymbol{\eta}||_{0,\partial\mathcal{T}_h} &\leq Ch^{1/2}||\gamma||_{0,\mathcal{T}_h}\,, \\ ||\mathcal{P}\underline{\boldsymbol{\varphi}} - \underline{\boldsymbol{\varphi}}||_{0,\partial\mathcal{T}_h} &\leq Ch^{3/2}||\gamma||_{0,\mathcal{T}_h}\,, \\ ||\overline{P}\boldsymbol{\phi} - \boldsymbol{\phi}||_{0,\partial\mathcal{T}_h} &\leq Ch^{5/2}||\gamma||_{0,\mathcal{T}_h}\,, \\ ||P\psi - \psi||_{0,\partial\mathcal{T}_h} &\leq Ch^{7/2}||\gamma||_{0,\mathcal{T}_h}\,. \end{aligned}$$

By replacing this in the above inequality, we obtain

$$\begin{aligned} ||Pu - u_h||_{0,\mathcal{T}_h} &= \sup_{0\neq\gamma\in L^2(\Omega)} \frac{(Pu - u_h, \gamma)}{||\gamma||_{0,\mathcal{T}_h}} \\ &\leq Ch^{1/2}\Big\{||Pu - u||_{0,\partial\mathcal{T}_h} + ||\overline{P}\boldsymbol{q} - \boldsymbol{q}||_{0,\partial\mathcal{T}_h} + ||\tau_1^{1/2}(u_h - \hat{u}_h)||_{0,\partial\mathcal{T}_h} \\ &\quad + ||\tau_4^{1/2}(\boldsymbol{q}_h - \hat{\boldsymbol{q}}_h)||_{0,\partial\mathcal{T}_h} + ||\mathcal{P}\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}||_{0,\partial\mathcal{T}_h} + ||\overline{P}\boldsymbol{\sigma} - \boldsymbol{\sigma}||_{0,\partial\mathcal{T}_h}\Big\} \\ &\leq Ch^{r+1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h} \end{aligned}$$

□

3.3 Error estimates

To summarize all the previous results we state then in a single theorem.

Theorem 3.7. *Let us assume that hypotheses of Theorems 3.4 and 3.6 Then, there exists a constant* $C > 0$*, independent of the mesh size, such that:*

$$\begin{aligned} ||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h} &\leq Ch^{r+1/2}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,, \\ ||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\mathcal{T}_h} &\leq Ch^{r-1/2}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,, \\ ||\boldsymbol{q} - \boldsymbol{q}_h||_{0,\mathcal{T}_h} &\leq Ch^{r}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,, \\ ||u - u_h||_{0,\mathcal{T}_h} &\leq Ch^{r+1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,. \end{aligned}$$

with $0 < r \leq k$.

Proof. It follows directly from Theorems 3.4 and 3.6. □

3.4 Errors associated to the numerical traces

In this section, we obtain estimates for the projection of the errors associated to the numerical traces $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$. We introduce the norm $|||\cdot|||_h$ defined by $|||\mu|||_h^2 = \sum_{T\in\mathcal{T}_h} h_T||\mu||^2_{0,\partial T}$ for any function $\mu \in L^2(\partial\mathcal{T}_h) := \prod_{T\in\mathcal{T}_h} L^2(\partial T)$. We proceed as in the proof of Theorem 4.1 in [6]. The authors of [6] use a local argument, which can be found in [51].

Theorem 3.8. *Let us assume that the hypotheses of Theorems 3.4 and 3.6 hold. Then exists $C > 0$, independent of the mesh size, such that*

$$|||P_\partial u - \hat{u}_h|||_h \;\leq\; Ch^{r+1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h} \tag{3.10}$$

$$|||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h|||_h \;\leq\; Ch^{r}||(\underline{\boldsymbol{z}}, \boldsymbol{q}, \boldsymbol{\sigma}, u)||_{r+1,\mathcal{T}_h} \tag{3.11}$$

with $0 < r \leq k$.

Proof. Let $T \in \mathcal{T}_h$, for $k \geq 1$. We consider functions $\boldsymbol{r}$ and $\underline{\boldsymbol{r}}$ in $[\mathbb{P}_k(T)]^d$ and $[\mathbb{P}_k(T)]^{d\times d}$, respectively, such that $\boldsymbol{r}\cdot\boldsymbol{n} = P_\partial u - \hat{u}_h$, $\underline{\boldsymbol{r}}\boldsymbol{n} = \underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h$ and $||\boldsymbol{r}||_T \leq Ch_T^{1/2}||P_\partial u - \hat{u}_h||_{\partial T}$, $||\underline{\boldsymbol{r}}||_T \leq Ch_T^{1/2}||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h||_{\partial T}$. Testing the equations (3.1a) and (3.1b) with $\underline{\boldsymbol{s}}_h = h_T\underline{\boldsymbol{r}}$ and $\boldsymbol{v}_h = h_T\boldsymbol{r}$ we get

$$\begin{aligned} h_T||P_\partial u - \hat{u}_h||^2_{0,\partial T} &= -(\boldsymbol{q} - \boldsymbol{q}_h, h_T\boldsymbol{r})_T - (\nabla\cdot h_T\boldsymbol{r}, u - u_h)_T\,, \\ h_T||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h||^2_{0,\partial T} &= -(\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h, h_T\underline{\boldsymbol{r}})_T + (\nabla\cdot h_T\underline{\boldsymbol{r}}, \boldsymbol{q} - \boldsymbol{q}_h)_T\,. \end{aligned}$$

Using Cauchy-Schwarz and inverse inequalities, we have

$$\begin{aligned} h_T||P_\partial u - \hat{u}_h||_{0,\partial T} &\leq Ch_T^{3/2}||\boldsymbol{q} - \boldsymbol{q}_h||_{0,T} + Ch_T^{1/2}||u - u_h||_{0,T}\,, \\ h_T||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h||_{0,\partial T} &\leq Ch_T^{3/2}||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,T} + Ch_T^{1/2}||\boldsymbol{q} - \boldsymbol{q}_h||_{0,T}\,. \end{aligned}$$

Multiplying by $h_T^{-1/2}$ on both sides and adding over $T \in \mathcal{T}_h$ we get

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} h_T^{1/2}||P_\partial u - \hat{u}_h||_{0,\partial T} &\leq Ch||\boldsymbol{q} - \boldsymbol{q}_h||_{0,\mathcal{T}_h} + C||u - u_h||_{0,\mathcal{T}_h}\,, \\ \sum_{T\in\mathcal{T}_h} h_T^{1/2}||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h||_{0,\partial T} &\leq Ch||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h} + C||\boldsymbol{q} - \boldsymbol{q}_h||_{0,\mathcal{T}_h}\,. \end{aligned}$$

Using the inequalities of the Theorem 3.7, we obtain

$$\begin{aligned} \sum_{T\in\mathcal{T}_h} h_T^{1/2}||P_\partial u - \hat{u}_h||_{0,\partial T} &\leq Ch^{r+1}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,, \\ \sum_{T\in\mathcal{T}_h} h_T^{1/2}||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h||_{0,\partial\mathcal{T}_h} &\leq Ch_T^{r+3/2}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h} + Ch^r||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,. \end{aligned}$$

Thus, we finish the proof concluding that

$$\begin{aligned} |||P_\partial u - \hat{u}_h|||_h^2 &\leq Ch^{2r+2}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,, \\ |||\underline{P_\partial}\boldsymbol{q} - \hat{\boldsymbol{q}}_h|||_h^2 &\leq Ch^{2r}||(\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}, u)||_{r+1,\mathcal{T}_h}\,. \end{aligned}$$

$\square$

To finish this chapter we comment that the interpolation errors for $\underline{\boldsymbol{z}}, \boldsymbol{\sigma}, \boldsymbol{q}$ and u are deduced using the equation (2.2). That is, if $\underline{\boldsymbol{z}}$, $\boldsymbol{\sigma}$, $\boldsymbol{q}$ and u belong to $\mathrm{H}^{r+1}(\mathcal{T}_h)$ then the orders of convergence for all of them is $O(h^{r+1})$.

CHAPTER 4

Numerical result

In this chapter we illustrate the performance of the HDG method with two numerical tests. For the first example, we consider as exact solution $u(x,y) = 10(y-1)^3y^3(x-1)^2x^2$ for $(x,y) \in \Omega :=]0,1[^2$ and f, $\boldsymbol{q}$, $\boldsymbol{\sigma}$ and $\underline{\boldsymbol{z}}$ are calculated accordingly. For the second example we consider a sinusoidal function defined on the same domain. This is, $u(x,y) = -\sin(y)\sin(x)$ for (x,y) in Ω, and obtain the exact expressions for the functions f $\boldsymbol{q}$, $\boldsymbol{\sigma}$ and $\underline{\boldsymbol{z}}$.
The penalty functions τ_1, τ_2, τ_3, τ_4 defined on $\partial\mathcal{T}_h$ are assumed constants such that $\tau_2 = \tau_3 = 0$ and $\tau_1 = \tau_4 = 1$, as suggested by Theorems 2.1 and 3.7. In addition the domain is partitioned into uniform meshes of size $\frac{1}{2^{(i+1)}}$ with $i = 1,2,\ldots,14$ and denote the mesh with size $\frac{1}{2^{(i+1)}}$ by "mesh i".

On the other hand, the convergence of the approximate solutions is assessed by computing errors in the respective norms and experimental rates, that we define as usual

$$\begin{aligned}
e(u) &:= ||u-u_h||_{0,\Omega}\,, \quad e(\boldsymbol{q}) := ||\boldsymbol{q}-\boldsymbol{q}_h||_{0,\Omega}\,, \quad e(\boldsymbol{\sigma}) := ||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||_{0,\Omega}\,,\\
e(\underline{\boldsymbol{z}}) &:= ||\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||_{0,\Omega}\,, \quad e(\hat{u}) := ||u-\hat{u}_h||_{0,\partial\mathcal{T}_h}\,, \quad e(\hat{\boldsymbol{q}}) := ||\boldsymbol{q}-\hat{\boldsymbol{q}}_h||_{0,\partial\mathcal{T}_h}\,,\\
r(u) &:= \frac{\log(e(u)/\tilde{e}(u))}{\log(h/\tilde{h})}\,, \quad r(\boldsymbol{q}) := \frac{\log(e(\boldsymbol{q})/\tilde{e}(\boldsymbol{q})}{\log(h/\tilde{h})}\,, \quad r(\boldsymbol{\sigma}) := \frac{\log(e(\boldsymbol{\sigma})/\tilde{e}(\boldsymbol{\sigma}))}{\log(h/\tilde{h})}\,,\\
r(\underline{\boldsymbol{z}}) &:= \frac{\log(e(\underline{\boldsymbol{z}})/\tilde{e}(\underline{\boldsymbol{z}}))}{\log(h/\tilde{h})}\,, \quad r(\hat{u}) := \frac{\log(e(\hat{u})/\tilde{e}(\hat{u}))}{\log(h/\tilde{h})}\,, \quad r(\hat{\boldsymbol{q}}) := \frac{\log(e(\hat{\boldsymbol{q}})/\tilde{e}(\hat{\boldsymbol{q}}))}{\log(h/\tilde{h})}\,,
\end{aligned}$$

where e and $\tilde{e}$ denote errors computed on two consecutive meshes of sizes h and $\tilde{h}$, respectively.

4.1 First example

In Tables 4.1 and 4.2, we display the history of convergence corresponding to $u(x,y) = 10(y-1)^3y^3(x-1)^2x^2$ on the unit square. We observe that, the error in the variable u converges optimally with order h^{k+1}, as is predicted by Theorem 3.7. The error in $\boldsymbol{q}$, also converges to zero with order h^{k+1}, which is one power of h more than our theoretical estimate in Theorem 3.7. In the case of the variable $\boldsymbol{\sigma}$, Tables 4.1 and 4.2 suggest that the order of convergence is h^k, in contrast to the estimate in Theorem 3.7 that predicts an order $h^{k-1/2}$. These tables also indicate that, for $k = 1,2,3$, the error in $\underline{\boldsymbol{z}}$ is of order h^{k+1}, which is half a power of h more than the theoretical rate in Theorem 3.7. For $k=0$, $\underline{\boldsymbol{z}}$ converges with order between 1/2 and 1, which is a slightly higher than the order provided in Theorem 3.7.

Figure 4.1 shows the graphics of error in logarithmic scale in the following order; upper left is the graph corresponding to $k = 0$, upper right is the graph for $k = 1$, lower left is the graph for $k = 2$ and lower right is the graph for $k = 3$.

In the Figure 4.2 we display the behavior of the errors $||u-u_h||_{0,\mathcal{T}_h}$, $||\boldsymbol{q}-\boldsymbol{q}_h||_{0,\mathcal{T}_h}$, $||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||_{0,\mathcal{T}_h}$ and $||\underline{\boldsymbol{z}}-\underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h}$ when we increase the polynomial degree. We observe that, for $k > 7$, the errors are affected by round-off errors.

In addition, Table 4.3 shows the order of convergence for the errors associated with $\hat{\boldsymbol{q}}_h$ and $\hat{u}_h$. We can see that for $|||P_\partial u - \hat{u}_h|||_h$ the order of convergence is $k+2$ and in the case $|||P_\partial \boldsymbol{q} - \hat{\boldsymbol{q}}_h|||_h$ is $k + 1/2$ if $k = 1, 2, 3$, which is slightly better than the result established by our the theory. In fact, the estimates in Theorem 3.8 is predict an order of $k + 1$ and k for the error associated to $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$, respectively. For $k = 0$ the convergence of $\hat{u}_h$ is optimal with order 1, and for $\hat{\boldsymbol{q}}_h$, the error seems to converge with very low order. Comparing this with the theoretical results obtained in Section 2.4, we conclude that the numerical result are slightly better than theoretical results.

k	Mesh i	h	N	$e(u)$	$r(u)$	$e(\boldsymbol{q})$	$r(\boldsymbol{q})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\underline{\boldsymbol{z}})$	$r(\underline{\boldsymbol{z}})$
	4	0.1508	44	4.38e-01	0.37	3.75e-02	0.12	7.53e-01	0.88	4.98e-02	0.64
	5	0.0995	101	2.97e-01	0.94	2.64e-02	0.84	5.10e-01	0.94	4.09e-02	0.48
	6	0.0682	215	2.17e-01	0.83	1.85e-02	0.94	4.02e-01	0.63	3.19e-02	0.66
	7	0.0499	401	1.57e-01	1.04	1.36e-02	0.98	3.45e-01	0.49	2.57e-02	0.70
	8	0.0354	800	1.12e-01	0.98	9.91e-03	0.93	2.99e-01	0.41	2.10e-02	0.58
0	9	0.0251	1586	7.96e-02	0.99	7.14e-03	0.96	2.65e-01	0.36	1.72e-02	0.59
	10	0.0177	3190	5.50e-02	1.06	5.09e-03	0.97	2.36e-01	0.33	1.35e-02	0.70
	11	0.0125	6367	3.95e-02	0.96	3.67e-03	0.95	2.11e-01	0.32	1.06e-02	0.69
	12	0.0089	12737	2.80e-02	0.99	2.62e-03	0.97	1.92e-01	0.27	8.20e-03	0.75
	13	0.0063	25497	1.97e-02	1.00	1.86e-03	0.98	1.76e-01	0.25	6.32e-03	0.75
	14	0.0044	50917	1.40e-02	1.00	1.33e-03	0.98	1.65e-01	0.19	4.85e-03	0.77
	4	0.1508	44	6.73e-02	2.26	5.11e-03	1.47	5.35e-01	0.27	2.17e-02	1.04
	5	0.0995	101	2.98e-02	1.96	2.25e-03	1.97	3.75e-01	0.85	9.06e-03	2.10
	6	0.0682	215	1.45e-02	1.91	1.17e-03	1.74	2.86e-01	0.72	4.96e-03	1.60
	7	0.0499	401	7.35e-03	2.18	6.09e-04	2.09	2.16e-01	0.90	2.70e-03	1.94
	8	0.0354	800	3.70e-03	1.99	3.12e-04	1.94	1.51e-01	1.04	1.32e-03	2.07
1	9	0.0251	1586	1.90e-03	1.94	1.60e-04	1.96	1.17e-01	0.73	7.44e-04	1.68
	10	0.0177	3190	9.28e-04	2.05	7.60e-05	2.12	8.01e-02	1.10	3.54e-04	2.13
	11	0.0125	6367	4.70e-04	1.97	3.92e-05	1.91	5.94e-02	0.86	1.87e-04	1.84
	12	0.0089	12737	2.33e-04	2.02	1.96e-05	2.00	4.17e-02	1.02	9.31e-05	2.02
	13	0.0063	25497	1.17e-04	1.98	9.82e-06	2.00	2.99e-02	0.96	4.75e-05	1.94
	14	0.0044	50917	5.89e-05	1.99	4.91e-06	2.00	2.12e-02	0.99	2.38e-05	1.99

Table 4.1: Example 1 : Errors for first example and $k = 0, 1$. N correspond to number of elements of the mesh i with $h \approx \frac{1}{\sqrt{N}}$.

k	Mesh i	h	N	$e(u)$	$r(u)$	$e(\boldsymbol{q})$	$r(\boldsymbol{q})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\underline{\boldsymbol{z}})$	$r(\underline{\boldsymbol{z}})$
	4	0.1508	44	6.97e-03	1.80	7.43e-04	3.02	1.28e-01	0.86	4.87e-03	2.05
	5	0.0995	101	2.17e-03	2.81	2.26e-04	2.87	5.61e-02	1.99	1.20e-03	3.37
	6	0.0682	215	7.40e-04	2.84	7.76e-05	2.82	2.76e-02	1.88	4.66e-04	2.50
	7	0.0499	401	3.48e-04	2.42	2.87e-05	3.19	1.50e-02	1.95	1.74e-04	3.15
	8	0.0354	800	1.23e-04	3.02	1.03e-05	2.98	8.31e-03	1.71	7.18e-05	2.57
2	9	0.0251	1586	3.77e-05	3.45	3.85e-06	2.87	4.19e-03	2.00	2.55e-05	3.02
	10	0.0177	3190	1.25e-05	3.16	1.30e-06	3.10	2.02e-03	2.09	8.89e-06	3.02
	11	0.0125	6367	4.43e-06	3.00	4.74e-07	2.93	1.04e-03	1.92	3.25e-06	2.92
	12	0.0089	12737	1.57e-06	2.99	1.66e-07	3.03	5.17e-04	2.02	1.15e-06	3.00
	13	0.0063	25497	5.63e-07	2.96	5.90e-08	2.97	2.61e-04	1.96	4.11e-07	2.96
	14	0.0044	50917	1.98e-07	3.02	2.10e-08	2.98	1.31e-04	2.00	1.47e-07	2.97
	4	0.1508	44	2.21e-03	2.62	1.14e-04	2.85	2.71e-02	1.32	7.85e-04	1.90
	5	0.0995	101	2.99e-04	4.82	1.96e-05	4.23	6.68e-03	3.37	1.15e-04	4.62
	6	0.0682	215	6.23e-05	4.15	4.89e-06	3.68	2.08e-03	3.09	2.98e-05	3.58
	7	0.0499	401	2.32e-05	3.16	1.65e-06	3.48	8.49e-04	2.87	9.83e-06	3.56
	8	0.0354	800	6.00e-06	3.92	4.08e-07	4.05	4.17e-04	2.06	3.16e-06	3.29
3	9	0.0251	1586	1.11e-06	4.93	9.48e-08	4.27	9.88e-05	4.21	5.39e-07	5.17
	10	0.0177	3190	2.74e-07	4.01	2.19e-08	4.20	3.29e-05	3.14	1.29e-07	4.10
	11	0.0125	6367	6.65e-08	4.10	5.56e-09	3.97	1.23e-05	2.85	3.30e-08	3.94
	12	0.0089	12737	1.65e-08	4.02	1.40e-09	3.98	4.33e-06	3.01	8.49e-09	3.92
	13	0.0063	25497	4.24e-09	3.92	3.64e-10	3.88	1.50e-06	3.05	2.20e-09	3.89

Table 4.2: Example 1 : Errors for first example and $k = 2, 3$. N correspond to number of elements of the mesh i with $h \approx \frac{1}{\sqrt{N}}$.

4.2 Second example

In Table 4.4, we display the history of convergence corresponding to $u(x, y) = -\sin y \sin(x)$ on the unit square. We observe that, the error in the variable u converges optimally with order h^{k+1}, as predicted by Theorem 3.7. The error in $\boldsymbol{q}$, also converges to zero with order h^{k+1}, which is one power of h more than our theoretical estimate in Theorem 3.7. In the case of the variable $\boldsymbol{\sigma}$, Table 4.4 suggest that the order of convergence is h^k, in contrast to the estimate in Theorem 3.7 that predicts an order $h^{k-1/2}$. These tables also indicate that, for $k = 1, 2, 3$, the error in $\underline{\boldsymbol{z}}$ is of order h^{k+1}, which is half a power of h more than the theoretical rate in Theorem 3.7. For $k = 0$, $\underline{\boldsymbol{z}}$ converges with order between 1/2 and 1, which is a slightly higher than the order provided in Theorem 3.7.

Figure 4.3 shows the graphics of error in logarithmic scale in the following order; upper left is the graph corresponding to $k = 0$, upper right is the graph for $k = 1$, lower left is the graph for $k = 2$ and lower right is the graph for $k = 3$.

In the same way as in first example, in Figure 4.4 shows the behavior of the errors $||u - \hat{u}_h||_{0,\mathcal{T}_h}$, $||\boldsymbol{q} - \hat{\boldsymbol{q}}_h||_{0,\mathcal{T}_h}$, $||\boldsymbol{\sigma} - \boldsymbol{\sigma}_h||_{0,\mathcal{T}_h}$ and $||\underline{\boldsymbol{z}} - \underline{\boldsymbol{z}}_h||_{0,\mathcal{T}_h}$ when we increase the polynomial degree. We observe that for polynomials of degree $k > 5$, the errors are affected by round-off errors.

By last, we can observe in Table 4.5 the order of convergence of the errors for $\hat{\boldsymbol{q}}$ and $\hat{u}$. We can see that for $|||P_\partial u - \hat{u}_h|||_h$ the order of convergence is $k+2$ and in the case $|||\boldsymbol{q} - \hat{\boldsymbol{q}}_h|||_h$ the order of convergence is $k + 1/2$ if $k = 1, 2, 3$, which is slightly better than established by the theory. In

k	Mesh i	h	N	$e(\hat{u})$	$r(\hat{u})$	$e(\hat{\boldsymbol{q}})$	$r(\hat{\boldsymbol{q}})$
0	4	0.1508	44	1.57e-02	0.22	2.60e-02	1.41
	5	0.0995	101	1.07e-02	0.92	2.25e-02	0.35
	6	0.0682	215	7.26e-03	1.04	2.06e-02	0.22
	7	0.0499	401	5.55e-03	0.86	2.07e-02	-0.00
	8	0.0354	800	3.99e-03	0.96	2.02e-02	0.07
	9	0.0251	1586	2.87e-03	0.96	1.98e-02	0.06
	10	0.0177	3190	2.09e-03	0.91	1.80e-02	0.26
	11	0.0125	6367	1.50e-03	0.96	1.66e-02	0.24
	12	0.0089	12737	1.07e-03	0.97	1.48e-02	0.32
	13	0.0063	25497	7.66e-04	0.97	1.31e-02	0.35
	14	0.0044	50917	5.48e-04	0.97	1.15e-02	0.38
1	4	0.1508	44	4.85e-04	2.80	5.37e-03	1.54
	5	0.0995	101	1.15e-04	3.46	1.67e-03	2.81
	6	0.0682	215	3.16e-05	3.42	8.26e-04	1.87
	7	0.0499	401	1.22e-05	3.06	3.66e-04	2.61
	8	0.0354	800	4.94e-06	2.61	1.51e-04	2.56
	9	0.0251	1586	1.92e-06	2.76	7.19e-05	2.18
	10	0.0177	3190	5.92e-07	3.37	2.96e-05	2.54
	11	0.0125	6367	2.22e-07	2.84	1.30e-05	2.38
	12	0.0089	12737	7.67e-08	3.06	5.53e-06	2.47
	13	0.0063	25497	2.76e-08	2.94	2.37e-06	2.44
	14	0.0044	50917	9.63e-09	3.05	1.01e-06	2.46
2	4	0.1508	44	6.11e-05	3.91	7.37e-04	2.33
	5	0.0995	101	8.94e-06	4.62	1.51e-04	3.81
	6	0.0682	215	1.97e-06	4.00	5.17e-05	2.84
	7	0.0499	401	5.30e-07	4.22	1.59e-05	3.79
	8	0.0354	800	1.41e-07	3.84	5.77e-06	2.92
	9	0.0251	1586	4.18e-08	3.55	1.70e-06	3.58
	10	0.0177	3190	9.38e-09	4.28	4.94e-07	3.53
	11	0.0125	6367	2.54e-09	3.78	1.51e-07	3.43
	12	0.0089	12737	6.21e-10	4.07	4.55e-08	3.46
	13	0.0063	25497	1.56e-10	3.98	1.37e-08	3.45
3	4	0.1508	44	5.37e-06	4.27	1.06e-04	1.80
	5	0.0995	101	5.10e-07	5.67	8.79e-06	5.98
	6	0.0682	215	9.10e-08	4.56	2.16e-06	3.72
	7	0.0499	401	2.46e-08	4.20	4.88e-07	4.78
	8	0.0354	800	5.61e-09	4.28	1.56e-07	3.30
	9	0.0251	1586	7.92e-10	5.72	2.08e-08	5.90
	10	0.0177	3190	1.23e-10	5.33	4.26e-09	4.53
	11	0.0125	6367	2.49e-11	4.62	9.39e-10	4.38

Table 4.3: Example 1 : Errors numerical traces. N correspond to number of elements of the mesh i with $h \approx \frac{1}{\sqrt{N}}$.

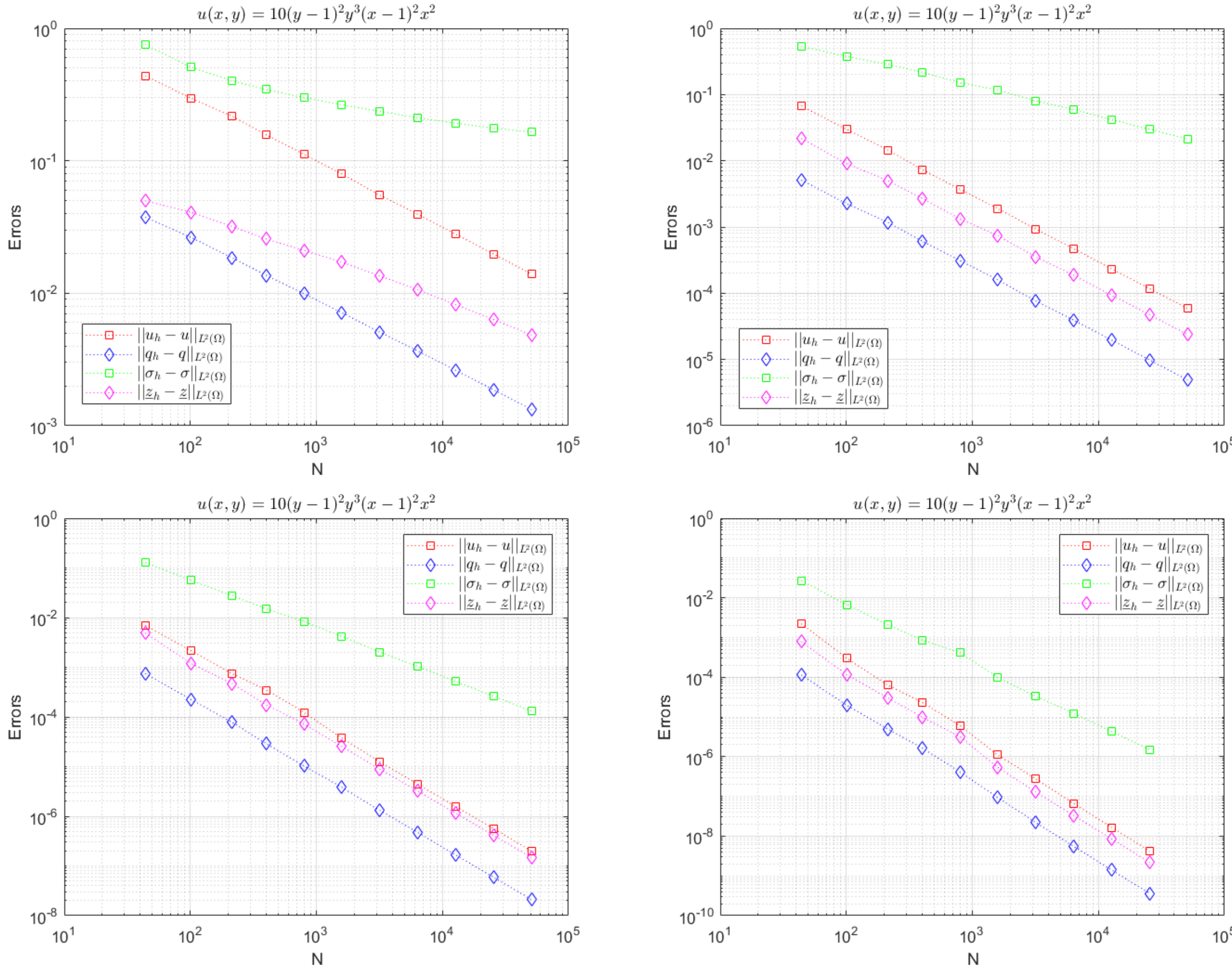

Figure 4.1: Example 1 : Errors in logarithmic scale $k = 0, 1, 2, 3$

fact, the estimates in Theorem 3.8 is predict an order of $k+1$ and k for the error associated to $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$, respectively. For $k = 0$ the convergence of $\hat{u}_h$ is optimal with order 1, and for $\hat{\boldsymbol{q}}_h$, the errors seems to converge with very low order. Comparing this with the theoretical results obtained in Section 2.4, we conclude that numerical result are slightly better than theoretical results.

The Figure 4.5 shows performance of HDG method for two different mesh. If left column corresponds to mesh with 215 elements and the right column corresponds to mesh with 1620 elements. Each row has associated a different polynomial degree in the following order. The first row shows the graphic of the approximate $\boldsymbol{\sigma}_h$ when polynomials of degree $k = 1$ are used, the second row shows the graphic when polynomials of degree $k = 2$ are used and the last row shows the performance when polynomials of degree $k = 3$ are used. Finally, Figure 4.6 shows the exact solution and a uniform triangulation with 215 elements.

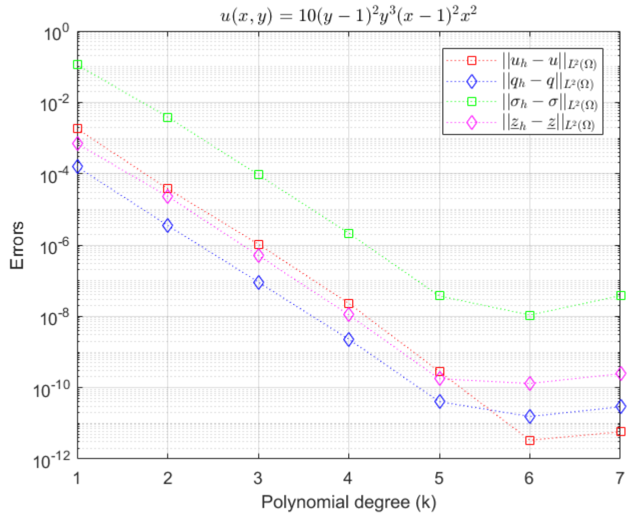

Figure 4.2: Example 1: With a fixed mesh of 1620 elements, the graphics show the behavior of the errors associated with u, , and for different polynomial degrees.

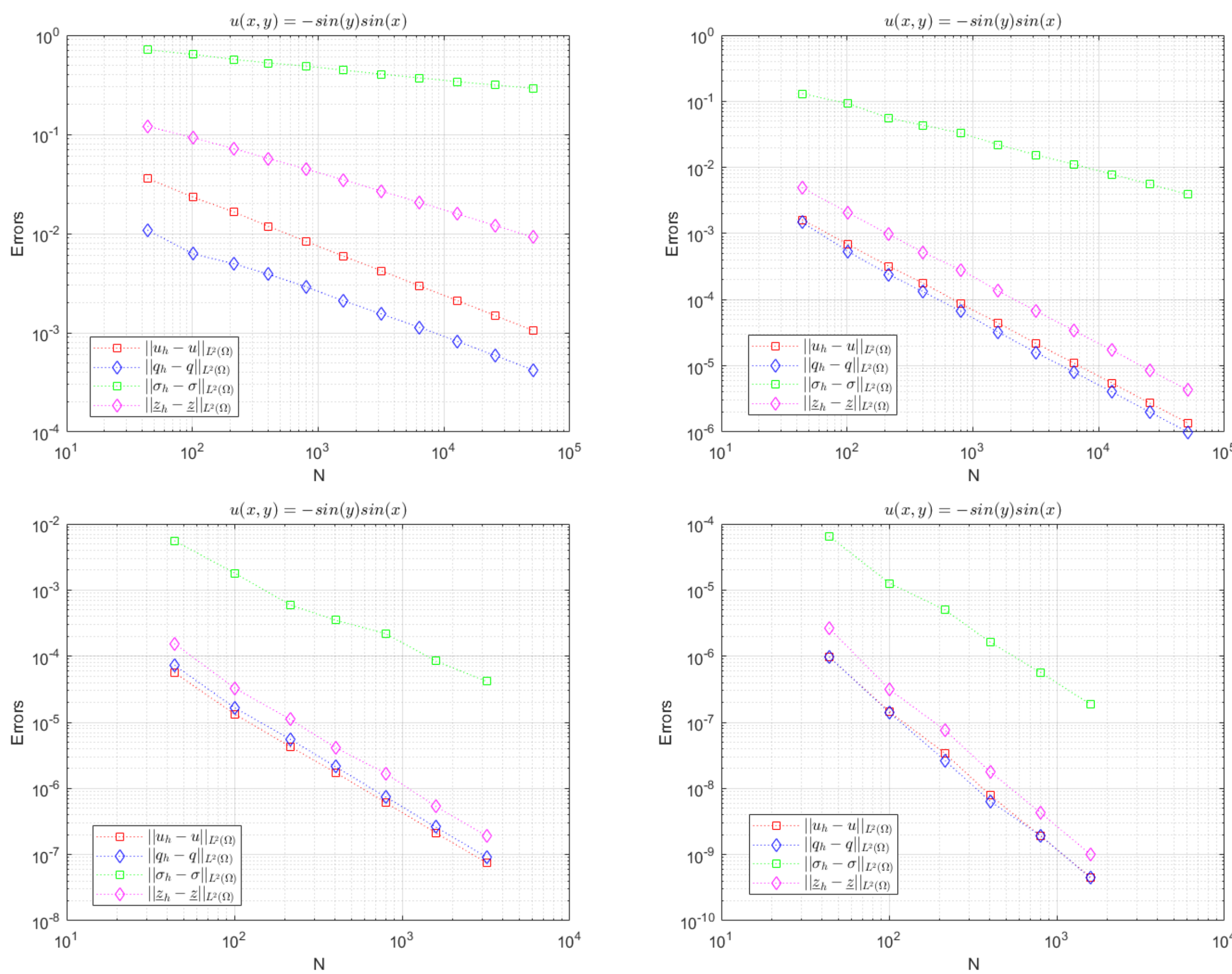

Figure 4.3: Example 2 : Errors in logarithmic scale $k = 0, 1, 2, 3$

k	Mesh i	h	N	$e(u)$	$r(u)$	$e(\boldsymbol{q})$	$r(\boldsymbol{q})$	$e(\boldsymbol{\sigma})$	$r(\boldsymbol{\sigma})$	$e(\underline{\boldsymbol{z}})$	$r(\underline{\boldsymbol{z}})$
	4	0.1508	44	3.60e-02	0.89	1.08e-02	0.12	7.13e-01	0.10	1.20e-01	0.70
	5	0.0995	101	2.33e-02	1.05	6.25e-03	1.33	6.39e-01	0.27	9.26e-02	0.63
	6	0.0682	215	1.65e-02	0.92	4.94e-03	0.62	5.70e-01	0.30	7.16e-02	0.68
	7	0.0499	401	1.19e-02	1.05	3.90e-03	0.76	5.24e-01	0.27	5.71e-02	0.73
	8	0.0354	800	8.34e-03	1.02	2.91e-03	0.85	4.88e-01	0.21	4.47e-02	0.71
0	9	0.0251	1586	5.92e-03	1.00	2.10e-03	0.95	4.44e-01	0.28	3.48e-02	0.73
	10	0.0177	3190	4.21e-03	0.97	1.54e-03	0.88	4.04e-01	0.27	2.67e-02	0.75
	11	0.0125	6367	2.97e-03	1.01	1.14e-03	0.89	3.70e-01	0.25	2.06e-02	0.76
	12	0.0089	12737	2.11e-03	0.98	8.20e-04	0.94	3.40e-01	0.24	1.58e-02	0.77
	13	0.0063	25497	1.49e-03	1.01	5.89e-04	0.96	3.14e-01	0.23	1.20e-02	0.78
	14	0.0044	50917	1.06e-03	1.00	4.22e-04	0.96	2.91e-01	0.22	9.20e-03	0.78
	4	0.1508	44	1.62e-03	1.73	1.48e-03	1.63	1.30e-01	0.47	4.94e-03	1.65
	5	0.0995	101	6.92e-04	2.04	5.40e-04	2.43	9.25e-02	0.83	2.08e-03	2.08
	6	0.0682	215	3.19e-04	2.05	2.40e-04	2.14	5.60e-02	1.33	9.68e-04	2.03
	7	0.0499	401	1.78e-04	1.87	1.32e-04	1.93	4.34e-02	0.81	5.20e-04	1.99
	8	0.0354	800	8.71e-05	2.07	6.71e-05	1.95	3.31e-02	0.79	2.83e-04	1.76
1	9	0.0251	1586	4.43e-05	1.97	3.23e-05	2.14	2.21e-02	1.17	1.36e-04	2.15
	10	0.0177	3190	2.16e-05	2.05	1.58e-05	2.05	1.54e-02	1.04	6.69e-05	2.03
	11	0.0125	6367	1.09e-05	1.98	7.99e-06	1.98	1.11e-02	0.95	3.40e-05	1.96
	12	0.0089	12737	5.48e-06	2.00	4.03e-06	1.97	7.87e-03	0.99	1.72e-05	1.96
	13	0.0063	25497	2.74e-06	2.00	1.99e-06	2.03	5.55e-03	1.00	8.57e-06	2.01
	14	0.0044	50917	1.37e-06	2.00	9.97e-07	2.00	3.92e-03	1.01	4.30e-06	1.99
	4	0.1508	44	5.63e-05	2.09	7.28e-05	2.15	5.55e-03	0.05	1.54e-04	1.57
	5	0.0995	101	1.32e-05	3.49	1.64e-05	3.58	1.79e-03	2.73	3.26e-05	3.74
	6	0.0682	215	4.30e-06	2.97	5.56e-06	2.87	5.98e-04	2.90	1.11e-05	2.85
2	7	0.0499	401	1.74e-06	2.90	2.13e-06	3.08	3.49e-04	1.73	4.14e-06	3.17
	8	0.0354	800	6.10e-07	3.04	7.40e-07	3.07	2.18e-04	1.36	1.69e-06	2.60
	9	0.0251	1586	2.14e-07	3.06	2.62e-07	3.04	8.45e-05	2.78	5.36e-07	3.35
	10	0.0177	3190	7.51e-08	3.00	9.11e-08	3.02	4.15e-05	2.03	1.91e-07	2.96
	4	0.1508	44	9.71e-07	2.96	9.79e-07	3.02	6.52e-05	2.07	2.65e-06	2.78
	5	0.0995	101	1.44e-07	4.60	1.38e-07	4.71	1.26e-05	3.95	3.11e-07	5.16
	6	0.0682	215	3.41e-08	3.80	2.66e-08	4.36	5.02e-06	2.44	7.65e-08	3.71
3	7	0.0499	401	7.88e-09	4.70	6.39e-09	4.57	1.63e-06	3.61	1.79e-08	4.66
	8	0.0354	800	1.91e-09	4.11	1.94e-09	3.45	5.67e-07	3.05	4.30e-09	4.13
	9	0.0251	1586	4.55e-10	4.19	4.44e-10	4.31	1.88e-07	3.23	1.00e-09	4.25

Table 4.4: Example 2 : Errors for second example and $k = 0, 1, 2, 3$. N correspond to number of elements of the mesh i with $h \approx \frac{1}{\sqrt{N}}$.

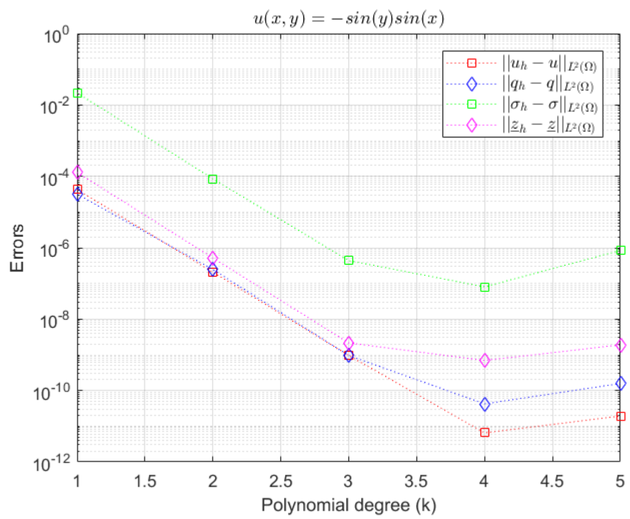

Figure 4.4: Example 2: With a fixed mesh of 1620 elements, the graphics show the behavior of the errors associated to $\underline{\boldsymbol{z}}_h$, $\boldsymbol{\sigma}_h$, $\boldsymbol{q}_h$ and u_h for different polynomial degrees.

k	Mesh i	h	N	$e(\hat{u})$	$r(\hat{u})$	$e(\hat{q})$	$r(\hat{q})$
0	4	0.1508	44	6.78e-03	1.20	9.87e-02	0.30
	5	0.0995	101	3.22e-03	1.79	9.54e-02	0.08
	6	0.0682	215	2.17e-03	1.04	8.82e-02	0.21
	7	0.0499	401	1.60e-03	0.97	8.03e-02	0.30
	8	0.0354	800	1.16e-03	0.95	7.24e-02	0.30
	9	0.0251	1586	8.09e-04	1.05	6.43e-02	0.35
	10	0.0177	3190	5.72e-04	0.99	5.66e-02	0.37
	11	0.0125	6367	4.20e-04	0.90	4.93e-02	0.40
	12	0.0089	12737	2.99e-04	0.98	4.26e-02	0.42
	13	0.0063	25497	2.14e-04	0.97	3.67e-02	0.44
	14	0.0044	50917	1.51e-04	1.01	3.14e-02	0.45
1	4	0.1508	44	2.38e-04	1.58	1.32e-03	1.58
	5	0.0995	101	4.59e-05	3.96	4.10e-04	2.82
	6	0.0682	215	1.55e-05	2.88	1.76e-04	2.23
	7	0.0499	401	5.99e-06	3.05	7.44e-05	2.77
	8	0.0354	800	2.28e-06	2.80	3.80e-05	1.95
	9	0.0251	1586	7.89e-07	3.10	1.38e-05	2.96
	10	0.0177	3190	2.72e-07	3.04	5.97e-06	2.40
	11	0.0125	6367	9.87e-08	2.94	2.54e-06	2.47
	12	0.0089	12737	3.58e-08	2.93	1.11e-06	2.40
	13	0.0063	25497	1.25e-08	3.04	4.58e-07	2.54
	14	0.0044	50917	4.44e-09	2.99	1.95e-07	2.47
2	4	0.1508	44	6.65e-06	2.20	3.33e-05	1.56
	5	0.0995	101	7.85e-07	5.14	4.66e-06	4.73
	6	0.0682	215	2.21e-07	3.36	1.49e-06	3.03
	7	0.0499	401	5.99e-08	4.18	4.09e-07	4.14
	8	0.0354	800	1.40e-08	4.22	1.60e-07	2.72
	9	0.0251	1586	3.77e-09	3.83	3.84e-08	4.17
	10	0.0177	3190	9.42e-10	3.97	1.16e-08	3.43
	11	0.0125	6367	2.38e-10	3.98	3.62e-09	3.37
3	4	0.1508	44	1.36e-07	2.48	7.72e-07	1.95
	5	0.0995	101	9.76e-09	6.34	6.14e-08	6.09
	6	0.0682	215	1.76e-09	4.53	1.43e-08	3.85
	7	0.0499	401	3.44e-10	5.25	2.32e-09	5.84
	8	0.0354	800	6.68e-11	4.74	1.05e-09	2.29
	9	0.0251	1586	1.55e-11	4.26	2.80e-10	3.87

Table 4.5: Example 2 : Errors numerical traces. N correspond to number of elements of the mesh i with $h \approx \frac{1}{\sqrt{N}}$.

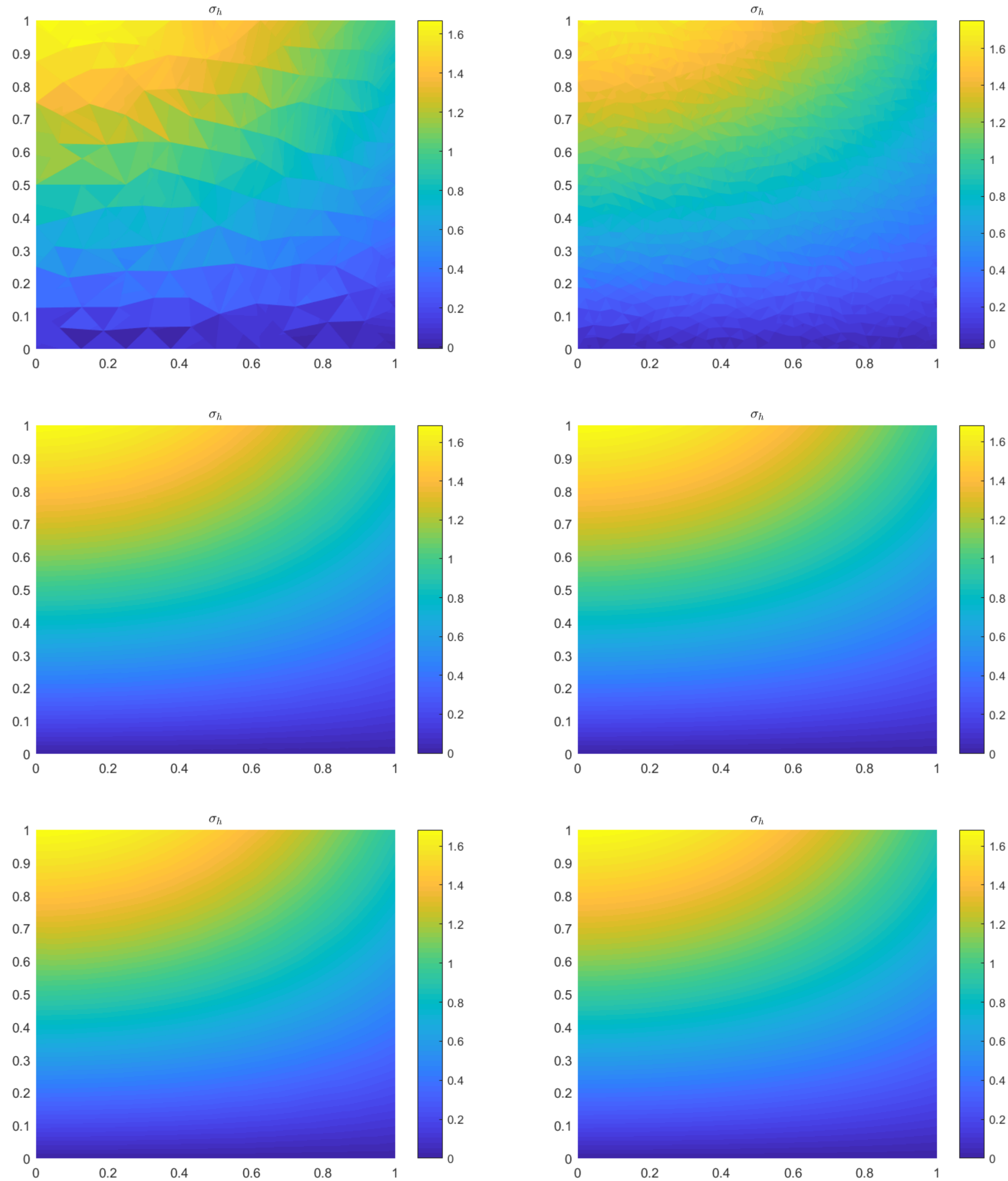

Figure 4.5: Example 2 : HDG performance for different size mesh and different polynomial degree, $k = 1$, $k = 2$ and $k = 3$ to first, second and third rows respectively. Left pictures were made with 215 elements and right pictures were made with 1620 elements.

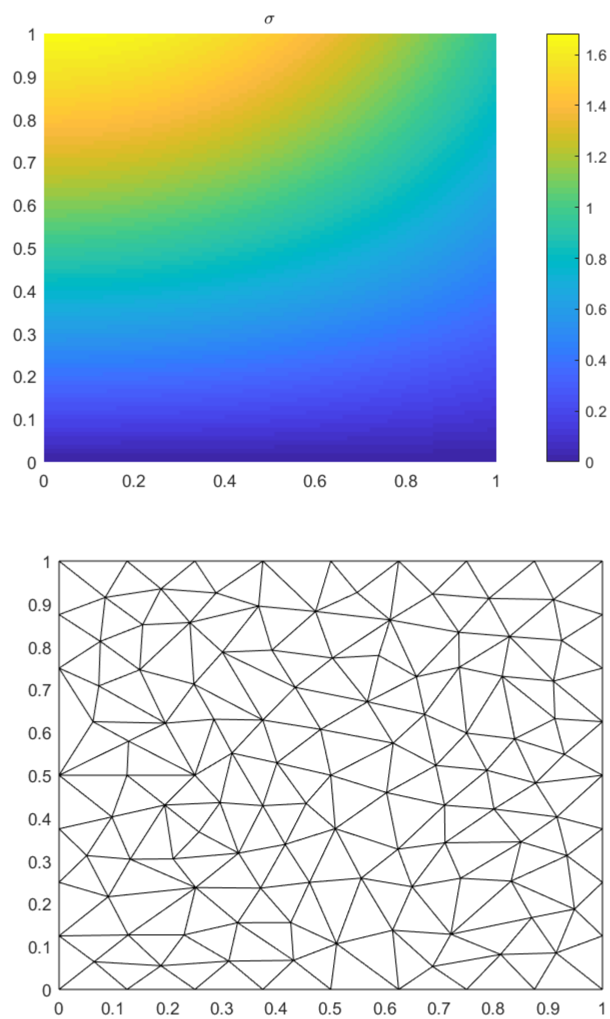

Figure 4.6: Example 2 : Top picture correspond to exact solution of first component of the vector $\boldsymbol{\sigma}$ for second example and bottom picture shows a uniform mesh with 215 elements.

CHAPTER 5

Conclusions and Future work

Conclusions: The HDG methods proposed has unique solution and it is numerically feasible, getting good results of convergence for the variables u, ∇u and $\mathcal{H}(u)$. The method allowed us to eliminate all the interior variables locally to obtain a global system for $\hat{u}_h$ and $\hat{\boldsymbol{q}}_h$ that approximate u and ∇u, respectively, on the interfaces of the triangulation. As a consequence the only globally coupled degrees of freedom are those of the approximations of u and ∇u on the faces of the elements. By last, when we carried out a priori analysis using the orthogonal L^2-projection we concluded that the orders of convergence for the errors in the approximation of $\mathcal{H}(u)$, $\nabla \cdot \mathcal{H}(u)$, ∇u and u are $k+1/2$, $k-1/2$, k y $k+1$ respectively ($k \geq 1$). However, our numerical results suggest that the approximations of $\mathcal{H}(u)$, $\nabla \cdot \mathcal{H}(u)$,∇u and u are slightly better than theoretical results. Namely the approximations of $\mathcal{H}(u)$, ∇u and u converge with optimal order $k+1$ and the approximation of $\nabla \cdot \mathcal{H}(u)$ converge with suboptimal order k.

Future works:

1. As the numerical results obtained were better that those predicted by the theory. The future task will be to improve the error estimates.

2. If the previous point is possible, then the future task will be to extend the method applied to the biharmonic problem in domains curve.

3. Given the difficulties we had to obtain the error estimates for the HDG scheme that is studied in this thesis, we decide to propose the next HDG scheme.

   Let V_h, Q_h, Σ_h, Z_h and Φ_h like in (2.13) and define W_h^0 as follow

$$W_h^0 := \left\{ \hat{v} \in L^2(\mathcal{E}_h) : \hat{v}\big|_e \in \mathbb{P}_k(e) \wedge \hat{v}|_{\partial\Omega} = 0 \right\} .$$

   Then the proposed HDG scheme is: Find $(\underline{\boldsymbol{z}}_h, \boldsymbol{\sigma}_h, \boldsymbol{q}_h, u_h, \hat{u}_h, \hat{\underline{\boldsymbol{z}}}_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times W_h^0 \times \Phi_h$ such that

$$\begin{aligned}
(\boldsymbol{\sigma}_h, \boldsymbol{m}_h)_{\mathcal{T}_h} - (\underline{\boldsymbol{z}}_h, \nabla_h \boldsymbol{m}_h)_{\mathcal{T}_h} + \langle \hat{\underline{\boldsymbol{z}}}_h \boldsymbol{n}, \boldsymbol{m}_h \rangle_{\partial \mathcal{T}_h} &= 0\,, && \text{(5.1a)}\\
(\boldsymbol{\sigma}_h, \nabla_h w_h)_{\mathcal{T}_h} - \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, w_h \rangle_{\partial \mathcal{T}_h} + (f, w_h)_{\mathcal{T}_h} &= 0\,, && \text{(5.1b)}\\
(\boldsymbol{q}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + (\nabla_h \cdot \boldsymbol{v}_h, u_h)_{\mathcal{T}_h} - \langle \boldsymbol{v}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial \mathcal{T}_h} &= 0\,, && \text{(5.1c)}\\
(\boldsymbol{q}_h, \nabla_h \cdot \underline{\boldsymbol{s}}_h)_{\mathcal{T}_h} - \langle \underline{\boldsymbol{s}}_h \boldsymbol{n}, \hat{\boldsymbol{q}}_h \rangle_{\partial \mathcal{T}_h} &= (\underline{\boldsymbol{z}}_h, \underline{\boldsymbol{s}}_h)_{\mathcal{T}_h}\,, && \text{(5.1d)}\\
\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \beta_h \rangle_{\partial \mathcal{T}_h} &= 0\,, && \text{(5.1e)}\\
\langle \hat{\boldsymbol{q}}_h, \underline{\boldsymbol{\alpha}}_h \boldsymbol{n} \rangle_{\partial \mathcal{T}_h} &= 0\,, && \text{(5.1f)}
\end{aligned}$$

for all $(\underline{\boldsymbol{s}}, \boldsymbol{v}_h, \boldsymbol{m}_h, w_h, \beta_h, \underline{\boldsymbol{\alpha}}_h) \in Z_h \times \Sigma_h \times Q_h \times V_h \times W_h^0 \times \Phi_h$, where

$$\begin{aligned} \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} &= \boldsymbol{\sigma}_h \cdot \boldsymbol{n} + \tau_1(u_h - \hat{u}_h) \quad \text{on} \quad \partial\mathcal{T}_h\,, \\ \hat{\boldsymbol{q}}_h &= \boldsymbol{q}_h + \tau_2(\underline{\boldsymbol{z}}_h - \hat{\underline{\boldsymbol{z}}}_h)\boldsymbol{n} \quad \text{on} \quad \partial\mathcal{T}_h\,, \end{aligned} \tag{5.2}$$

are the numerical traces.

In addition, we propose to obtain the a priori error estimate with the next projector:

$$(\underline{\Pi}, \overline{\Pi}, \mathbb{P}) : [H(\text{div}; \mathcal{T}_h)]^{d\times d} \times [H(\text{div}, \mathcal{T}_h)]^d \cup [H^1(\mathcal{T}_h)]^d \times H^1(\mathcal{T}_h) \longrightarrow Z_h \times \Sigma_h \cup Q_h \times V_h$$

is defined as follow; given the functions $(\underline{\tilde{\rho}}, \rho) \in [H(\text{div}; \mathcal{T}_h)]^{d\times d} \times [H(\text{div}; \mathcal{T}_h)]^d \cup [H^1(\mathcal{T}_h)]^d$ and a simplex $T \in \mathcal{T}_h$. The restriction of $(\underline{\Pi}\underline{\tilde{\rho}}, \overline{\Pi}\rho)$ to T is defined as the element of $[\mathbb{P}_k(T)]^{d\times d} \times [\mathbb{P}_k(T)]^d$ that satisfies:

$$\begin{aligned} (\overline{\Pi}\rho - \rho, \boldsymbol{v})_T &= 0 \quad \forall\, \boldsymbol{v} \in [\mathbb{P}_{k-1}(T)]^d\,,\ \text{if } k \geq 1\,, \\ \left\langle(\overline{\Pi}\rho - \rho)\cdot\boldsymbol{n}, w\right\rangle_e &= 0 \quad \forall\, w \in \mathbb{P}_k(e)\,, \\ (\underline{\Pi}\underline{\tilde{\rho}} - \underline{\tilde{\rho}}, \underline{\boldsymbol{v}})_T &= 0 \quad \forall\,, \underline{\boldsymbol{v}} \in [\mathbb{P}_{k-1}(T)]^{d\times d}\,,\ \text{if } k \geq 1\,, \\ \left\langle(\underline{\Pi}\underline{\tilde{\rho}} - \underline{\tilde{\rho}})\boldsymbol{n}, \boldsymbol{w}\right\rangle_e &= 0 \quad \forall\, \boldsymbol{w} \in \mathbb{P}_k(e)\,. \end{aligned}$$

Given $\eta \in \mathrm{H}^1(\mathcal{T}_h)$ and a simplex $T \in \mathcal{T}_h$. The restriction of $\mathbb{P}\eta$ to T is defined as the element of $\mathbb{P}_k(T)$ that satisfies

$$\begin{aligned} (\mathbb{P}\eta - \eta, w)_T &= 0 \quad \forall\, w \in \mathbb{P}_{k-1}(T)\,,\ \text{if } k \geq 1\,, \\ \langle \mathbb{P}\eta - \eta, w\rangle_e &= 0 \quad \forall\, w \in \mathbb{P}_k(e)\,. \end{aligned}$$

4. We would like to study the next biharmonic problem

$$\begin{aligned} \triangle^2 u &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega \\ \triangle u &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{5.3}$$

For doing that, we propose to define the variables $z = \triangle u$ and $\triangle z = f$, later $\boldsymbol{q} = \nabla u$ and $\boldsymbol{\sigma} = \nabla z$. this is,

$$\begin{aligned} \boldsymbol{\sigma} &= \nabla z \quad \text{in} \quad \Omega\,, \\ \nabla\cdot\boldsymbol{\sigma} &= f \quad \text{in} \quad \Omega\,, \\ z &= 0 \quad \text{on} \quad \partial\Omega\,, \\ \boldsymbol{q} &= \nabla u \quad \text{in} \quad \Omega\,, \\ \nabla\cdot\boldsymbol{q} &= z \quad \text{in} \quad \Omega\,, \\ u &= 0 \quad \text{on} \quad \partial\Omega\,. \end{aligned} \tag{5.4}$$

In addition, we propose to study the next HDG scheme. Let the space Σ_h, V_h and W_h as in (2.13), then we can to find $((\boldsymbol{\sigma}_h, \boldsymbol{q}_h), (z_h, u_h), (\hat{z}_h, \hat{u}_h)) \in \Sigma_h^2 \times V_h^2 \times W_h^2$ such that

$$\begin{aligned}
(\boldsymbol{\sigma}_h, \boldsymbol{v}_h)_{\mathcal{T}_h} + (z_h, \nabla \cdot \boldsymbol{v}_h)_{\mathcal{T}_h} - \langle \boldsymbol{v}_h, \hat{z}_h \rangle_{\partial \mathcal{T}_h} &= 0 && \text{(5.5a)}\\
-(\boldsymbol{\sigma}_h, \nabla_h w_h)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, w_h \rangle_{\partial \mathcal{T}_h} &= (f, w_h)_{\mathcal{T}_h} && \text{(5.5b)}\\
(\boldsymbol{q}_h, \boldsymbol{\tau}_h)_{\mathcal{T}_h} + (u_h, \nabla \cdot \boldsymbol{\tau}_h)_{\mathcal{T}_h} - \langle \boldsymbol{\tau}_h \cdot \boldsymbol{n}, \hat{u}_h \rangle_{\partial \mathcal{T}_h} &= 0 && \text{(5.5c)}\\
-(\boldsymbol{q}_h, \nabla \mu_h)_{\mathcal{T}_h} + \langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \mu_h \rangle_{\partial \mathcal{T}_h} &= (z_h, \mu_h)_{\mathcal{T}_h} && \text{(5.5d)}\\
\langle \hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n}, \beta_h \rangle_{\partial \mathcal{T}_h} &= 0 && \text{(5.5e)}\\
\langle \hat{\boldsymbol{q}}_h \cdot \boldsymbol{n}, \alpha_h \rangle_{\partial \mathcal{T}_h} &= 0 && \text{(5.5f)}\\
\langle \hat{z}_h, \beta_h \rangle_{\partial \Omega} &= 0 && \text{(5.5g)}\\
\langle \hat{u}_h, \alpha_h \rangle_{\partial \Omega} &= 0 && \text{(5.5h)}
\end{aligned}$$

for all $((\boldsymbol{v}_h, \boldsymbol{\tau}_h), (w_h, \mu_h), (\beta_h, \alpha_h)) \in \Sigma_h^2 \times V_h^2 \times W_h^2$. Where

$$\begin{aligned}
\hat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{n} &:= \boldsymbol{\sigma}_h \cdot \boldsymbol{n} + \tau_1 (z_h - \hat{z}_h) \boldsymbol{n} \quad \text{on} \quad \partial \mathcal{T}_h\\
\hat{\boldsymbol{q}}_h \cdot \boldsymbol{n} &:= \boldsymbol{q}_h \cdot \boldsymbol{n} + \tau_2 (u_h - \hat{u}_h) \boldsymbol{n} \quad \text{on} \quad \partial \mathcal{T}_h
\end{aligned}$$

are the numerical traces. By last, we propose to analyze the priori error with the HDG-projectors studied in [6].

Bibliography

[1] W. Rend, T. Hill, Triangular mesh methods for the neutron transport equation, Tech. Report LA-UR-73-479, Los Alamos, NM (1973) .

[2] F. Bassi, S. Rebay, A high-order accurate discontinuous finite element method for the numerical solution of the compressible Navier-Stokes equations, J. Comput. Phys 131 (1997) 267–279.

[3] D. Di Prieto, A. Ern, Mathematical aspects of discontinuous Galerkin methods, Vol. 69, Springer, Berlin, 2011.

[4] N. A. Douglas, F. Brezzi, B. Cockburn, L. Marini, Unified analysis of discontinuous Galerkin methods for elliptic problem, SIAM J. Numer. Anal. 39 (2001) 1749–1779.

[5] B. Cockburn, J. Gopalakrishnan, R. Lazarov, Unified hybridization of discontinuous Galerkin, mixed and continuous Galerkin methods for second order elliptic problems, SIAM J. Numer. Anal. 47 (2009) 1319–1365.

[6] B. Cockburn, J. Gopalakrishnan, F. J. Sayas, A projection-based error analysis of HDG methods, Mathematical of Computation 79 (2010) 1351–1367.

[7] B. Cockburn, B. Dong, J. Guzmán, A superconvergent LDG–hybridizable Galerkin method for second-order elliptic problems, Math. Comp. 77 (2008) 1887–1916.

[8] B. Cockburn, J. Guzmán, H. Wang, Superconvergent discontinuous Galerkin methods for second-order elliptic problems, Math. Comp. 78 (2009) 1–24.

[9] M. Kirby, S. Sherwin, B. Cockburn, To CG or to HDG: A comparative study, J. Sci. Comput. 51 (2012) 183–212.

[10] B. Cockburn, Q. Weifeng, K. Shi, Superconvergent HDG methods on isoparametric elements for second-order elliptic problems, SIAM J. Numer. Anal. 50 (2012) 1417–1432.

[11] C. Yanlai, B. Cockburn, Analysis of variable-degree HDG methods for Convection-Diffusion equations. Part II: Semimatching nonconforming meshes, Math. Comput. 83 (2014) 87–111.

[12] B. Cockburn, M. Solano, Solving dirichlet boundary-value problems on curved domains by extensions from subdomains, SIAM J. Sci. Comput. 34 (2012) A497–A519.

[13] B. Cockburn, S. F.J., M. Solano, Coupling at a distance HDG and BEM, SIAM J. Sci. Comput. 34 (2012) A28–A47.

[14] B. Cockburn, Q. Weifeng, M. Solano, A priori error analysis for HDG methods using extensions from subdomains to achieve boundary conformity, Math. Comput. 83 (2014) 665–699.

[15] B. Chabaud, B. Cockburn, Uniform-in-time superconvergence of HDG methods for the heat equation, Math. Comput. 81 (2012) 107–129.

[16] B. Cockburn, B. Dong, J. Guzmán, M. Restelli, R. Sacco, A hybridizable discontinuous Galerkin method for steady-state convection-diffusion-reaction problems, SIAM J. Sci. Comput. 31 (2009) 3827–3846.

[17] N. Cuong, P. J., B. Cockburn, A class of embedded discontinuous Galerkin methods for computational fluid dynamics, J. Comput. Physics 302 (2015) 674–692.

[18] S. Rhebergen, B. Cockburn, J. J. W. van der vegt, A space-time discontinuous Galerkin method for the incompressible Navier-Stokes equations, J. Comput. Physics 233 (2013) 339–358.

[19] B. Cockburn, M. Solano, Solving dirichlet boundary-value problems on curved domains by extensions from subdomains, SIAM J. Sci. Comput. 34 (2012) A497–A519.

[20] F. Gousheng, Q. Weifeng, Z. Wujun, An analysis of HDG method for convection-dominated diffusion problems, ESAIM Math. Mod. and Numer. Anal. 49 (2015) 225–256.

[21] C. Nguyen, J. Peraire, B. Cockburn, An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations, J. Comput. Physics 230 (2011) 1147–1170.

[22] C. Nguyen, J. Peraire, B. Cockburn, Hybridizable discontinuous Galerkin methods for the time-harmonic Maxwell's equations, J. Comput. Physics 230 (2011) 7151–7175.

[23] B. Cockburn, V. Quenneville-Belair, Uniform-in-time superconvergence of the HDG methods for the acoustic wave equation, Math. Comput. 83 (2014) 65–85.

[24] G. Gatica, F. Sequeira, A priori and a posteriori error analysis of an augmented HDG method, J. sci. Comput. 66 (2016) 1192–1250.

[25] J. Carrero, B. Cockburn, D. Schotzau, Hybridized globally divergence-free LDG methods. part I: The Stokes problem, Math. Comput. 75 (2006) 533–563.

[26] B. Cockburn, J. Gopalakrishnan, The derivation of hybridizable discontinuous Galerkin methods for Stokes flow, SIAM J. Numer. Anal. 47 (2009) 1092–1125.

[27] B. Cockburn, C. Hguyen, J. Peraire, A comparison of HDG methods for Stokes flow, J. Sci. Comput 45 (2010) 215–237.

[28] B. Cockburn, J. Gopalakrishnan, C. Nguyen, J. Peraire, F. Sayas, Analysis of HDG methods for Stokes flow, Math. Comput. 80 (2011) 723–760.

[29] B. Cockburn, K. Shi, Conditions for superconvergence of HDG methods for Stokes flow, Math. Comput. 82 (2013) 651–671.

[30] B. Cockburn, J. Cui, Divergence-free HDG methods for the vorticity-velocity formulation of the Stokes problem, J. Sci. Comput. 52 (2012) 256–270.

[31] A. Cesmelioglu, B. Cockburn, C. Nguyen, J. Peraire, Analysis of HDG methods for Oseen equations, J. Sci. Comput. 55 (2013) 392–431.

[32] L. Gatica, F. Sequeira, A priori and aposteriori error analyses od an HDG method for the Brinkman problem, Comput. Math. Appl. 75 (2018) 1191–1212.

[33] A. Cesmelioglu, B. Cockburn, Q. Weifeng, Analysis of a hybridizable discontinuous Galerkin method for the steady-state incompressible Navier-Stokes equations, Math. Comput. 86 (2017) 1643–1670.

[34] C. Nguyen, J. Peraire, B. Cockburn, An implicit high-order hybridizable discontinuous Galerkin method for the incompressible Navier-Stokes equations, J. Comput. 230 (2011) 1147–1170.

[35] S. Rhebergen, B. Cockburn, J. J. W. var der Vergt, A space-time discontinuous Galerkin method for the incompressible Navier-Stokes equations, J. Comput. Physics 233 (2013) 339–358.

[36] C. Nguyen, J. Peraire, B. Cockburn, High-order implicit hybridizable discontinuous Galerkin methods for acoustics and elastodynamics, J. Comput. Physics 230 (2011) 3695–3718.

[37] J. Peraire, C. Nguyen, B. Cockburn, A hybridizable discontinuous Galerkin method for the compressible Euler and Navier-Stokes equations, Proceedings of the 48th Aerospace Sciences Meeting and Exihibit (2010) 1–12.

[38] C. Nguyen, J. Peraire, An Adaptive shock-capturing HDG method for compressible flows, 20th AIAA Computational fluid Dynamics Conference (2011) 1–20.

[39] D. Moro, C. Nguyen, J. Peraire, A hybridized discontinuous Petrov-Galerkin scheme for scalar conservation laws, Internat. J. Numer. Methods Engrg. 68 (2011) 1530–1549.

[40] C. Soon, B. Cockburn, H. Stolarski, A hybridizable discontonuous Galerkin method for linear elasticity, Internat, J. Numer. Methods Engrg. 80 (2009) 1058–1092.

[41] B. Cockburn, K. Shi, Superconvergent HDG methods for linear elasticity with weakly symmetric stresses., SIAM J. Numer. Anal. 33 (2013) 747–770.

[42] C. Nguyen, J. Peraire, Hybridizable discontinuous Galerkin methods for partial differential equations in continuum mechanics, J. Comput. Physics 231 (2012) 5955–5988.

[43] B. Cockburn, B. Dong, J. Guzmán, A hybridizable and superconvergent discontinuous Galerkin method for biharmonic problems, J. Sci. Comput 40 (2009) 141–187.

[44] F. Celiker, B. Cockburn, K. Shi, A projection-based error analysis of HDG methods for Timoshenki Beams, Math. Comput. 81 (2012) 131–151.

[45] M. Edwin, Behrens, J. Guzmán, A new family of mixed methods for the Reisser-Mindlin plate model based on a system of first-order equations, J. Sci. Comput. 49 (2011) 137–166.

[46] R. Kupferman, A central-difference scheme for a pure stream function formulation of incompressible viscous flow, SIAM J. Sci. Comput. 23 (2001) 1–18.

[47] P. Ciarlet, P. Raviart, A mixed finite element for the biharmonic equation, Proc. Sympos., Math. Res. Center (1974) 125–145.

[48] Babuška, J. Osborn, J. Pitkaranta, Analysis of mixed methods using mesh dependent norms, Math. Comput. 35 (1980) 1039–1062.

[49] E. Behrens, J. Guzmán, A mixed method for the biharmonic problem based on a system of first-order equations, SIAM J. Numer. Anal. 49 (2011) 789–817.

[50] H. Blum, R. Rannacher, On the boundary value problem of the biharmonic operator on domains with angular corners, Math. Methods Appl. sci. 2 (1980) 556–581.

[51] F. Brezzi, J. Douglas, L. Marini, Two families of mixed finite elements for second order elliptic problems, Numer. Math. 47 (1985) 217–235.