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CONTROL Y "ADAPTIVIDAD" EN UN MODELO DE CIRCULACIÓN OCEÁNICA

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RESUMEN

Esta tesis considera modelos simplificados que describen la circulación oceánica, en particular el modelo cuasi-geostrófico lineal en sus formulaciones velocidad - presión y función de corriente - vorticidad.

Para abordar los problemas de datos iniciales incompletos, se utilizó la teoría de control insensibilizante. El principal objetivo fue estudiar la existencia de controles que permitan obtener mediciones en un observatorio, de manera independiente de las pequeñas variaciones en las condiciones iniciales. Este análisis se reduce a estudiar problemas de tipo controlabilidad aproximada o exacta. Se obtuvo una propiedad de continuación única y una desigualdad inversa para un sistema en cascada de tipo Stokes, donde el término de Coriolis jugó un papel determinante.

Se utilizó una técnica de asimilación de datos para determinar condiciones iniciales desconocidas con el objetivo de realizar predicciones. Se implemetó numéricamente este método. Se logró determinar una aproximación de la condición inicial a partir de mediciones del estado en instantes de tiempo anteriores y conociendo el término fuente, lo que permitió realizar buenas predicciones del sistema. En el aspecto teórico, la recuperación exacta de la condición inicial se reduce a un problema de controlabilidad nula, donde se probó una desigualdad de observabilidad.

Finalmente, se introduce una estrategia de refinamiento adaptivo de mallado para reducir las oscilaciones y pobre resolución que aparece cuando el término convectivo es dominante. Se propuso un indicador anisotrópico del error *a posteriori* para localizar la capa límite sin información *a priori* de la solución y para crear mallas bien adaptadas a la solución. Esta técnica se basó en la recuperación de la hessiana de la solución. Finalmente, se determinó la eficiencia de dicha estrategia por medio de varios experimentos numéricos.

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Motivación

El océano se puede considerar como un fluido débilmente compresible sujeto a la fuerza de Coriolis y la fuerza de gravedad; su movimiento y estado están gobernados por las ecuaciones generales de la hidrodinámica y las ecuaciones de difusión de temperatura y salinidad. Debido a la complejidad de la dinámica del océano se estudian modelos simplificados tanto desde el punto de vista físico como matemático.

Las ecuaciones generales de la hidrodinámica pueden ser aproximadas por las ecuaciones de Boussinesq (EB) o las ecuaciones primitivas (EP). Las (EB) se obtienen a partir de las ecuaciones generales para un fluido compresible donde se asume la aproximación de Boussinesq, es decir, las variaciones de la densidad son despreciables en las ecuaciones excepto en los términos de flotabilidad y en la ecuación de estado. Las (EP) se derivan de las (EB) bajo la aproximación hidrostática para la ecuación de momento vertical [LTW92].

Diferentes modelos pueden ser derivados de EB y/o de EP para el estudio de la circulación a gran escala. Uno de los más usados por los oceanógrafos para estudiar la circulación oceánica en latitudes medias, es el cuasi-geostrófico.

La escala del fenómeno esta caracterizada por la dimensión del número de Rossby (R_o). El caso de gran escala corresponde a un número de Rossby muy pequeño, es decir, $R_o \leq 0.007$, que corresponde a longitudes características de 10^2m en la vertical y 10^5m en la horizontal.

En este trabajo consideraremos el modelo lineal cuasi-geostrófico en su formulación velocidad-presión y en la formulación función de corriente-vorticidad.

Obtendremos primero las ecuaciones del modelo cuasi-geostrófico a partir de las ecuaciones de Navier-Stokes para un fluido gravitacional y el principio de conservación de la masa.

Ecuaciones Primitivas

Sea $(x, z) = ((x_1, x_2), z) \in \Omega$ un dominio abierto, acotado y conexo de $\mathbb{R}^3$ y $t \in (0, T)$, con $T > 0$ fijo. Consideremos las ecuaciones de Navier-Stokes para un fluido gravitacional [Lew97]

$$\rho\frac{D\vec{U}}{Dt} - \mu\Delta_c\vec{U} = -\nabla_c p + \vec{G} + \vec{C}, \tag{1}$$

donde

$$\frac{D\phi}{Dt} := \partial_t\phi + \left(\vec{U}\cdot\nabla_c\right)\phi, \qquad \Delta_c\phi := \Delta\phi + \partial^2_{zz}\phi, \qquad \nabla_c\phi := (\partial_{x_1}\phi, \partial_{x_2}\phi, \partial_z\phi).$$

En (1), $\vec{U}$ representa la velocidad del fluido en $((x,z),t)$

$$\vec{U}((x,z),t) = \vec{U} = \vec{u} + w\vec{k} = u_1\vec{\imath} + u_2\vec{\jmath} + w\vec{k},$$

donde $\vec{u} = (u_1, u_2)$ son las velocidades horizontales y w la velocidad vertical. Además de la presión, sobre el fluido actúan la fuerza de Coriolis $\vec{C}$ y la fuerza de gravedad $\vec{G}$:

$$\vec{C} := -2\rho\vec{\omega}_0 \times \vec{U}, \qquad \vec{G} := \rho g\vec{k},$$

donde $\vec{\omega}_0$ es la velocidad angular de la Tierra ($7.24\times 10^{-5}\,\mathrm{s}^{-1}$) y g es el coeficiente de gravedad ($\approx 9.81\,\mathrm{m\,s^{-1}}$). En un punto dado, la fuerza de Coriolis se escribe como

$$\vec{C} = -2\rho\omega_0\left(\cos(\theta_0)\vec{k}\times\vec{U} + \sin(\theta_0)\vec{\jmath}\times\vec{U}\right),$$

donde θ_0 es la latitud del punto considerado y $\omega_0 = |\vec{\omega}_0|$.

Por otro lado, a partir del principio de conservación de la masa, se tiene

$$\frac{1}{\rho}\frac{D\rho}{Dt} = -\mathrm{div}_c\vec{U},$$

donde $\mathrm{div}_c\,\vec{U} = \mathrm{div}\,\vec{u} + \partial_z w$.

Con esto, las ecuaciones dinámicas del fluido son

$$\left\{\begin{array}{l} \rho\dfrac{Du}{Dt} - \mu\Delta_c\vec{u} + 2\rho\omega_0\left(\cos(\theta_0)\vec{k}\times\vec{u} + \sin(\theta_0)w\vec{\imath}\right) + \nabla p = 0 \\ \rho\dfrac{Dw}{Dt} - \mu\Delta_c w - 2\rho\omega_0 u_1\sin(\theta_0) + \rho g + \partial_z p = 0 \\ \partial_t\rho + \mathrm{div}_c(\rho\vec{U}) = 0. \end{array}\right. \tag{2}$$

Principales Aproximaciones

1. Aproximación de Boussinesq:

La variación de la densidad en el océano es de aproximadamente 2% con respecto al valor característico ρ_0; entonces se sustituye ρ por el valor promedio ρ_0 en todas las ecuaciones excepto en los términos de flotabilidad, $-\rho g$. Por consecuencia, la ecuación de conservación de la masa se transforma en

$$\mathrm{div}_c \vec{U} = 0.$$

2. Órdenes para la velocidad y la fuerza de Coriolis:

Las magnitudes de las velocidades horizontales son varios órdenes más grandes que la magnitud de la velocidad vertical en el océano como se puede observar en los siguientes resultados

$$\mathcal{O}(\vec{u}) := U = 0.1\, m\, s^{-1}, \qquad \mathcal{O}(w\vec{k}) := W = 10^{-5} m\, s^{-1}.$$

Se puede deducir del análisis anterior que $\mathcal{O}(\sin(\theta_0) w\vec{k}) << \mathcal{O}(\cos(\theta_0)\vec{u})$, entonces el término de Coriolis se simplifica a

$$\vec{C} \approx f(x_2)\vec{k} \times \vec{U} = f(x_2)\vec{k} \times \vec{u},$$

donde $f(x_2) = 2\omega_0 \cos(\theta_0)$.

3. Aproximación hidrostática:

Para fluidos geostróficos de gran escala, la relación entre la escala vertical y la horizontal es muy pequeña. Del análisis anterior de las magnitudes de las velocidades, se tiene para la segunda ecuación en (2)

$$\frac{Dw}{Dt} - (\mu/\rho)\Delta_c w - 2\omega_0 u_1 \sin\theta_0 << g;$$

luego se tiene la llamada aproximación hidrostática

$$\partial_z p = -\rho g.$$

Teniendo en cuenta las aproximaciones anteriores se obtiene el siguiente sistema de ecuaciones

$$\begin{cases} \dfrac{Du}{Dt} - A\Delta_c \vec{u} + f\vec{k} \times \vec{u} + \dfrac{\nabla p}{\rho_0} = 0, \\ \partial_z p = -\rho g, \\ \mathrm{div}_c(\vec{U}) = 0, \end{cases} \tag{3}$$

donde $A = (\mu/\rho_0)$ denota la viscosidad cinemática. A este sistema se le agregan condiciones iniciales sobre la velocidad $\vec{u}$ y condiciones de borde. Denotemos por Γ_s la interfaz aire-mar y por $\Gamma_l \cup \Gamma_f$ los bordes laterales y fondo respectivamente. Se supone superficie rígida en Γ_s, es decir, $w|_{\Gamma_s} = 0$ y no adherencia sobre $\Gamma_l \cup \Gamma_f$, es decir, $\vec{U}|_{\Gamma_l \cup \Gamma_f} \cdot \vec{n} = 0$. También es usual suponer velocidad nula en las costas y en el fondo, $\vec{U}|_{\Gamma_l \cup \Gamma_f} = 0$.

Problema de Reynolds y la turbulencia

El sistema (3) no puede resolverse exactamente, salvo en algunos casos simples. Aún en el caso que se obtenga una solución, es difícil para un fenómeno en particular, determinar qué término es el responsable de dicho fenómeno. Una manera de resolver esto y obtener las ecuaciones que controlan un fenómeno particular es separar las ecuaciones en dos conjuntos: uno que expresa el movimiento promedio y el otro que describe las fluctuaciones en torno al valor promedio. Cambiando el período de tiempo promediado, se puede separar un fenómeno particular [KM93, Lew97].

Consideremos el problema abstracto de un fluido de viscosidad ν sobre un abierto $\mathcal{O} \in \mathbb{R}^n$ y consideremos una fuerza $\vec{F}$ dependiente de la velocidad W. Escribamos la ecuación de Navier-Stokes para este fluido

$$\partial_t \vec{U} + (\vec{U} \cdot \nabla_c)\vec{U} - A\Delta_c \vec{U} + \frac{\nabla P}{\rho} = \vec{F}(\vec{U}). \tag{4}$$

El promedio en el tiempo lo denotamos por

$$\phi_m := \frac{1}{T}\int_0^T \phi \, dt$$

y escribamos cada variable como

$$\phi := \phi_m + \phi',$$

donde ϕ' representa las fluctuaciones en torno a ϕ_m. Notar que, si $(\xi_1, \ldots, \xi_n)$ son las coordenadas de un punto en $\mathbb{R}^n$, se cumple que

$$(\phi')_m = 0, \qquad (\partial_t \phi)_m = \partial_t(\phi_m), \qquad (\partial_{\xi_k}\phi)_m = \partial_{\xi_k}(\phi_m) \quad \forall k = 1, \ldots, n.$$

Por otro lado, si $(u_1, \ldots, u_n)$ son las componentes de $\vec{U}$, promediando la ecuación (4) se obtiene

$$\partial_t \vec{U}_m + (\vec{U}_m \cdot \nabla_c)\vec{U}_m - A\Delta_c \vec{U}_m + \frac{\nabla P_m}{\rho} = \vec{F}(\vec{U}_m) - (\vec{U}' \cdot \nabla)\vec{U}'. \tag{5}$$

Para un fluido incompresible, el último término se reduce a

$$((\vec{U}' \cdot \nabla)\vec{U}')_m = \mathrm{div}_c(\vec{U}'\vec{U}'^t) - (\mathrm{div}_c \vec{U}')\vec{U}' = \mathrm{div}_c(\vec{U}'\vec{U}'^t),$$

así, (5) se transforma en

$$\partial_t \vec{U}_m + (\vec{U}_m \cdot \nabla_c)\vec{U}_m - A\Delta_c \vec{U}_m + \frac{\nabla P_m}{\rho} = \vec{F}(\vec{U}_m) - \mathrm{div}_c(\vec{U}'\vec{U}'^t). \tag{6}$$

Esta ecuación muestra que los flujos de momento asociados a las fluctuaciones actúan como una fuente a los movimientos de grandes períodos de tiempo.

Definamos el tensor de Reynolds por

$$\mathcal{R}_e := (\vec{U}'\vec{U}'^t) = (u_i' u_j')_{1\leq i,\, j\leq n}.$$

Así,

$$\mathrm{div}_c(\mathcal{R}_e) = \mathrm{div}_c(\vec{U}'\vec{U}'^t) = \sum_{i=1}^{n} \partial_{\xi_i}(u_i' u_j').$$

La determinación rigurosa del tensor de Reynolds es un problema inabordable [KM93, Lew97]. Una suposición admisible es asumir que el tensor de Reynolds es proporcional al gradiente de la velocidad promedio, es decir,

$$\mathcal{R}_e = -A_t \nabla_c \vec{U}_m,$$

donde A_t es la viscosidad turbulenta. El fenómeno de difusión turbulenta es predominante con respecto a los de viscosidad molecular ($\nu_t \gg \nu$), por lo que este último se puede despreciar, es decir,

$$\partial_t \vec{U}_m + (\vec{U}_m \cdot \nabla_c)\vec{U}_m - A_t \Delta_c \vec{U}_m + \frac{\nabla P_m}{\rho} = \vec{F}(\vec{U}_m).$$

En las ecuaciones que describen los movimientos de gran escala, la influencia de los movimientos de mesoescala vienen dados por los coeficientes turbulentos.

Definamos el número de Reynolds como $R_e := LU/A_t$, donde L y U representan longitudes y velocidades características. Este número nos permite identificar la naturaleza del fluido: para el régimen newtoniano ($R_e \to \infty$) y para el régimen de Stokes ($R_e \to 0$).

De ahora en adelante, trabajaremos con las variables promediadas en dependencia de la escala de tiempo que nos interesa, despreciando la viscosidad molecular. Para el caso $n = 3$, el tensor de Reynolds toma la forma, ([HB92])

$$(\mathcal{R}_e)_{x,z} = A_V \frac{\partial u_1}{\partial z} + A_H \frac{\partial w}{\partial x_1}, \qquad (\mathcal{R}_e)_{x,y} = A_H \frac{\partial u_1}{\partial x_2} + A_H \frac{\partial u_2}{\partial x_1}, \quad etc.$$

donde A_H, A_V son los coeficientes de viscosidad turbulenta horizontal y vertical, respectivamente. Aplicando el análisis anterior a (3), se tienen las EP del océano

$$\left\{ \begin{array}{l} \dfrac{Du}{Dt} - A_H \Delta \vec{u} - A_V \partial_{zz}^2 \vec{u} + f\vec{k} \times \vec{u} + \dfrac{1}{\rho_0} \nabla p = 0, \\ \partial_z p = -\rho g, \\ \mathrm{div}_c(\vec{U}) = 0, \\ w|_{\Gamma_s} = 0, \quad \vec{U}|_{\Gamma_l \cup \Gamma_f} = 0, \\ \vec{u}|_{t=0} = \vec{u}_0. \end{array} \right. \tag{7}$$

Modelo cuasi-geostrófico lineal

El balance entre la fuerza de Coriolis y el gradiente de presión es el llamado equilibrio geostrófico que gobierna las ecuaciones de un fluido geofísico de gran escala. El modelo cuasi-geostrófico describe la evolución temporal del equilibrio geostrófico.

El modelo cuasi-geostrófico o también conocido como modelo barotrópico, desprecia la circulación termohalina, pero es capaz de reproducir propiedades estadísticas de la circulación oceánica en latitudes medias ($20^o \leq \theta_0 \leq 50^o$) forzadas por la fuerza del viento.

La principal característica de la circulación forzada por el viento es la presencia de un doble giro con una corriente fuerte en la costa oeste produciendo un chorro energético fuerte hacia el interior del océano. Estos giros son persistentes y dominantes, tienen una escala típica horizontal de aproximadamente mil kilómetros [BGdS01]. Ejemplos de estas corrientes son: la Corriente del Golfo, en el Atlántico Norte (Figura 1) y la Corriente de Kuro Shio en el Pacífico Norte (ver por ejemplo [BVM94, Med99, Med00, MW95, Ped87]).

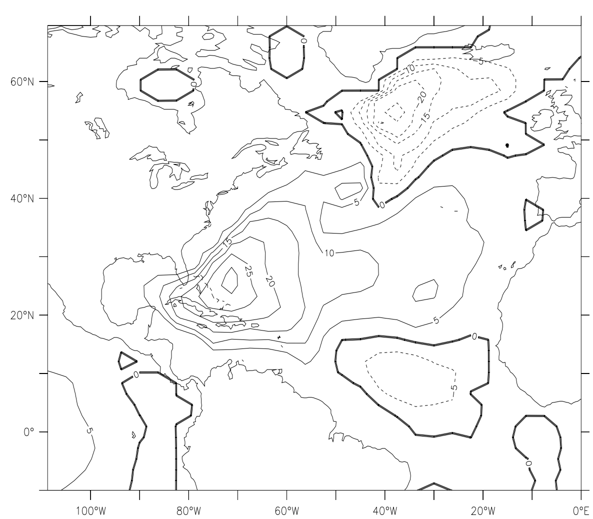

Figure 1: Líneas de corriente en el Atlántico Norte.

Para la obtención de las ecuaciones del modelo cuasi-geostrófico de una sola capa, despreciaremos la estratificación de la densidad e integraremos en la vertical considerando una profundidad fija $z = -D_0$.

Definamos el valor medio de una magnitud h cualquiera como sigue:

$$\underline{h} := \frac{1}{D_0} \int_{-D_0}^{0} h \, dz.$$

Se define también [Lew97],

$$\rho_0 \int_{-D_0}^{0} \frac{\partial}{\partial z} \left(A_V \frac{\partial \vec{u}}{\partial z} \right) dz := \tau_s - \tau_f,$$

donde $\tau_s := \rho_0 A_V \frac{\partial \vec{u}}{\partial z}$ es el estrés en la superficie, que corresponde al estrés del viento $\mathcal{T}$, es decir $\tau_s := \mathcal{T}$. El estrés en el fondo τ_f, se asume proporcional a la velocidad, es decir, $\tau_f := \gamma \vec{u}$, donde γ es el coeficiente de fricción en el fondo. Utilizando esta definición, si promediamos la primera ecuación en (7) se obtiene,

$$\frac{\partial \underline{\vec{u}}}{\partial t} - A_H \Delta \underline{\vec{u}} + \gamma \underline{\vec{u}} + f \vec{k} \times \underline{\vec{u}} + \frac{1}{\rho_0} \nabla \underline{p} = \frac{1}{\rho_0 D_0} \mathcal{T}. \tag{8}$$

Por otra parte, si integramos la tercera ecuación en (7), utilizando las condiciones de borde para w, se deduce

$$\operatorname{div} \underline{\vec{u}} + \frac{1}{D_0} w|_{-D_0}^{0} = \operatorname{div} \underline{\vec{u}} = 0.$$

De esta forma, hemos obtenido las ecuaciones del modelo cuasi-geostrófico lineal. Sea Ω un dominio de $\mathbb{R}^2$ y Γ su frontera. Definamos $Q := \Omega \times (0, T)$ y $\Sigma := \Gamma \times (0, T)$, omitiendo las barras, el sistema es el siguiente:

$$\begin{cases} \dfrac{\partial \vec{u}}{\partial t} - A_H \Delta \vec{u} + \gamma \vec{u} + f \vec{k} \times \vec{u} + \dfrac{1}{\rho_0} \nabla p = \dfrac{1}{\rho_0 D_0} \mathcal{T} \qquad \text{en } Q, \\ \operatorname{div} \vec{u} = 0 \qquad \text{en } Q, \\ \vec{u} = 0 \qquad \text{en } \Sigma, \\ \vec{u}|_{t=0} = \vec{u}_0 \qquad \text{en } \Omega. \end{cases} \tag{9}$$

Usualmente se asume la aproximación β-plano (ver [Ped87] por detalles) que consiste en proyectar la superficie esférica de la Tierra sobre un plano tangente a un punto con longitud φ_0 y latitud θ_0. Se define en este plano un sistema de coordenadas Cartesianas local (x_1, x_2) con origen en (φ_0, θ_0). Además, se sustituye el parámetro de Coriolis f por su aproximación lineal:

$$f = f_0 + \beta x_2, \qquad f_0 = 2\omega_0 \sin \theta_0, \qquad \beta = \frac{2\omega_0}{R} \cos \theta_0,$$

donde R es el radio de la Tierra (6.371×10^6 m).

Por otra parte, es sabido que la formulación en función de corriente - vorticidad del modelo cuasi-geostrófico es muy utilizada, principalmente por sus ventajas numéricas [Med99, Med00, MW95, Ver90, Ver92].

Consideremos Ω como al inicio. Denotemos por Γ_0 la frontera exterior y Γ_i, $1 \leq i \leq p$, las otras componentes conexas de Γ, en caso de que existan. Es conocido que la condición de divergencia nula de la velocidad se puede expresar introduciendo una función de corriente ψ de $\vec{u}$ como sigue:

$$\vec{u} = \vec{\mathrm{curl}}\,\psi := \left(\frac{\partial \psi}{\partial x_2}, -\frac{\partial \psi}{\partial x_1} \right).$$

Como $\vec{u}$ se anula en Σ, ψ debe ser constante en cada componente conexa $\Gamma_i \times (0,T)$. Además, puede ser unívocamente determinada si tomamos $\psi = 0$ en Γ_0.

Aplicando el operador rotacional a la primera ecuación en (9) e introduciendo la función de corriente ψ, se deduce

$$\left\{ \begin{array}{l} \dfrac{\partial}{\partial t}(-\Delta\psi) + A_H \Delta^2 \psi - \gamma \Delta \psi - \beta \dfrac{\partial \psi}{\partial x_1} = \dfrac{1}{\rho_0 D_0} \,\mathrm{curl}\, \mathcal{T} \quad \text{en } Q, \\ \psi = \dfrac{\partial \psi}{\partial n} = 0 \quad \text{en } \Sigma. \end{array} \right. \tag{10}$$

Para caracterizar la dinámica del modelo es conveniente introducir números adimensionales que aparecen cuando se adimensionalizan las ecuaciones. Sea $\mathcal{T}_0$ la amplitud del estrés del viento climatológico entonces, de la fórmula de Svedrup para la circulación horizontal de gran escala en el interior del océano [MW95, BGdS01], se puede obtener la escala típica para la función de corriente:

$$\psi_0 = \frac{\mathcal{T}_0}{\rho_0 D_0 \beta},$$

donde D_0 es la profundidad típica del océano. Consideremos las siguientes variables adimensionales:

$$(x_1', x_2') = L^{-1}(x_1, x_2), \quad t' = U^{-1} L t, \quad \psi' = \psi_0^{-1} \psi, \quad \mathcal{T}' = \mathcal{T}_0^{-1} \mathcal{T},$$

donde L representa la longitud horizontal característica de la circulación oceánica y U es la velocidad horizontal característica.

Escalamos la ecuación (10); es decir, sustituimos cada variable por las relaciones anteriores y, en el caso de los operadores, se tiene, por ejemplo,

$$\frac{\partial \psi}{\partial t} = \psi_0 U L^{-1} \frac{\partial \psi'}{\partial t}, \qquad \Delta \psi = \psi_0 L^{-2} \Delta \psi'.$$

Así, se obtiene la siguiente ecuación adimensional:

$$\begin{cases} R_o \dfrac{\partial}{\partial t}(-\Delta\psi) + \epsilon_m \Delta^2 \psi - \epsilon_s \Delta\psi - \dfrac{\partial \psi}{\partial x_1} = \operatorname{curl} \mathcal{T} \quad \text{en } Q, \\ \psi = \dfrac{\partial \psi}{\partial n} = 0 \quad \text{en } \Sigma, \end{cases} \tag{11}$$

donde hemos omitido el símbolo $'$ de las variables e introducido

$$R_o = \frac{U}{\beta L^2}, \quad \epsilon_m = \frac{A_H}{\beta L^3}, \quad \epsilon_s = \frac{\gamma}{\beta L}.$$

R_o se conoce como el número de Rossby, ϵ_m como el número de Munk y ϵ_s como el número de Stommel o Ekman. Por último, introducimos la vorticidad ω de $\vec{u}$ como:

$$\omega := \operatorname{curl} \vec{u} := \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = -\Delta\psi;$$

luego se tiene,

$$\begin{cases} R_o \dfrac{\partial \omega}{\partial t} - \epsilon_m \Delta\omega - \epsilon_s \omega - \dfrac{\partial \psi}{\partial x_1} = \operatorname{curl} \mathcal{T} \quad \text{en } Q, \\ \omega + \Delta\psi = 0 \quad \text{en } Q, \\ \psi = \dfrac{\partial \psi}{\partial n} = 0 \quad \text{en } \Sigma, \\ \omega(0) = \omega_0 = \operatorname{curl} \vec{u}_0 \quad \text{en } \Omega. \end{cases} \tag{12}$$

En este trabajo utilizaremos dos formulaciones para el modelo cuasi-geostrófico: en velocidad-presión, que viene dado por el sistema (9), y en función de corriente-vorticidad, que viene dado por el sistema (12).

Dos aplicaciones de la Teoría de Control

Para estudiar un fenómeno físico, en particular en el océano, contamos con un modelo del sistema dinámico que consiste en un conjunto de ecuaciones para cada variable de estado. A estas ecuaciones se le suman los términos fuentes, las condiciones iniciales y de frontera y por último los parámetros físicos como, por ejemplo, el coeficiente de viscosidad, el coeficiente de fricción en el fondo, etc. En principio, estos valores pueden obtenerse directamente de las mediciones, pero en la práctica esto es difícil y costoso. Por ello muchas veces estamos en presencia de problemas de datos iniciales desconocidos o incompletos.

Se han desarrollado varias técnicas para resolver los problemas de datos desconocidos. Las más utilizados son las técnicas de asimilación de datos. Estos métodos se basan en incorporar al modelo de sistema dinámico observaciones medidas con el objetivo de describir un fenómeno particular, así como de realizar predicciones.

Entre los métodos más usados de asimilación de datos se encuentran los métodos variacionales que se basan en la teoría de control óptimo [Lio71, Mar75, LD85, LDT86]. Recientemente en [Pue02] se presenta un método no clásico de asimilación de datos basado en la teoría de control. La aplicación de esta teoría al modelo cuasi-geotrófico se estudiará en el Capítulo 2.

También se tienen los modelos con datos iniciales incompletos, es decir, modelos donde los datos iniciales presentan pequeñas incertidumbres, por ejemplo, consideremos la condición inicial como sigue:

$$u(0) = u_0 + \tau \hat{u}_0,$$

donde u_0 es una valor aproximdo conocido de $u(0)$ y $\tau \hat{u}_0$ es el error (desconocido) que afecta a esta aproximación. En esta expresión, $\hat{u}_0$ representa una función de norma uno (por ejemplo, en el espacio de funciones de cuadrado integrable) y el parámetro τ un factor de escala que se supone pequeño.

Podemos plantearnos dos problemáticas:

- Si además de las condiciones iniciales, el término fuente también presenta una perturbación, nos interesaría, si es posible, identificar el término desconocido de la fuente independiente del ruido aportado por la incertidumbre sobre las condiciones iniciales.

- Por otra parte, nos planteamos si sería posible actúar sobre el sistema (acción humana) para obtener mediciones en cierta región del espacio independientes a las incertidumbres de las condiciones iniciales.

Para resolver la primera problemática se utiliza la *Teoría de los Centinelas* introducida en [Lio88] (ver más detalles en [Lio92b]). La segunda problemática se basa en la *Teoría de Control Insensibilizante* introducida en [Lio90] y que estudiaremos en el Capítulo 1. Ambas técnicas se reducen a estudiar problemas de tipo controlabilidad exacta o aproximada.

Capítulo 1: Controles insensibilizantes para el modelo cuasi-geostrófico del océano

En este capítulo nos centraremos en estudiar la segunda problemática aplicada al modelo cuasi-geostrófico del océano. Es necesario señalar que la teoría de control insensibilizante si bien resulta de menor interés práctico, constituye un problema interesante y complejo desde el punto de vista matemático.

Se puede demostrar que la existencia de controles (actuando en un "pequeño" abierto ω) tales que las mediciones en el observatorio $\mathcal{O} \subset\subset \Omega$ sean insensibles o casi insensibles a las pequeñas variaciones de las condiciones iniciales, es equivalente a un problema de controlabilidad nula o aproximada, respectivamente.

De forma general, la controlabilidad de una ecuación en derivadas parciales (EDP) o de un sistema de EDP se consigue cuando se sabe conducir la o las EDP de un estado inicial dado u_0 a un estado final deseado u_1, mediante un control h que actúa sobre el sistema desde el exterior.

Estaremos en presencia de un control distribuido, si el control actúa a través del segundo miembro de la ecuación. Por el contrario, si el control se ejerce a través de las condiciones de frontera, hablaremos de control frontera.

Fijando el control h en cierto conjunto $\mathcal{U}_{ad}$ (conjunto de controles admisibles), diremos que nuestro sistema es *exactamente controlable* en Y (un espacio de Banach adecuado en el que evolucionan las ecuaciones) en el tiempo T si, cualesquiera que sean u_0 e u_1 en Y, podemos encontrar un control $h \in \mathcal{U}_{ad}$ de tal forma que el estado asociado u_h, con dato inicial u_0, verifica

$$u_h(T) = u_1.$$

Esta condición puede relajarse. En este sentido, dados u_0 y u_1 arbitrarios, diremos que el sistema es *aproximadamente controlable* en Y en el instante T si, para cada $\varepsilon > 0$, existe un control h tal que el correspondiente estado u_h verifica la condición

$$\|u_h(T) - u_1\|_Y \leq \varepsilon.$$

Por último, diremos que hay *controlabilidad exacta a cero* o *controlabilidad nula* en el tiempo T, cuando $u_1 = 0$, es decir, para cada u_0, existe un control h tal que el estado u_h asociado a h verifica $u_h(T) = 0$.

Varios han sido los autores que en los últimos años han trabajado en la existencia de controles insensibilizantes. En [BF95], los autores estudian la existencia de controles $\varepsilon-$insensibilizantes para la ecuación del calor lineal y semilineal con condiciones iniciales y de frontera incompletas. Posteriormente en [dT00], se demuestra que existen los controles insensibilizantes para las mismas ecuaciones pero bajo ciertas suposiciones sobre el término fuente y condición inicial nula. Estos resultados fueron recientemente extendidos a no linealidades más generales en [BGBPG02].

En el Capítulo 1 estudiaremos la existencia de controles insensibilizantes y ε-insensibilizantes para ecuaciones de tipo Stokes. Hasta donde sabemos, éste es el primer resultado de existencia de controles insensibilizantes para este tipo de ecuaciones.

En los problemas de insensibilización, estamos en presencia de un sistema en cascada donde el control no actúa directamente en el sistema de la función que queremos conducir a cero después de un intervalo de tiempo T. Esto hace que el problema que estamos considerando sea más difícil que los problemas de controlabilidad nula para las ecuaciones de tipo Stokes.

El primer resultado del Capítulo 1 es relativo a la controlabilidad aproximada. Gracias a la presencia del término de Coriolis y utilizando los resultados de [Fab96, FL02], se deduce una propiedad de continuación única para un sistema en cascada.

El resultado más importante de este capítulo consiste en deducir una desigualdad de observabilidad. La demostración de esta desigualdad se basa en una estimación global de Carleman para un sistema en cascada. Los puntos claves consisten en seguir los pasos de la demostración del resultado de la continuación única y utilizar la forma de las ecuaciones, en particular el término de Coriolis. Este resultado se obtiene bajo ciertas suposiciones sobre el término fuente, la condición inicial y la intersección del espacio de control y el observatorio.

El contenido del Capítulo 1 corresponde a los artículos [FCGO03, FCGO04]:

- E. Fernández-Cara, G. C. García and A. Osses, Insensitizing controls for a large-scale ocean circulation model. *C. R. Math. Acad. Sci. Paris* 337 (2003), no. 4, 265–270.

- E. Fernández-Cara, G. C. García and A. Osses, Controls insensitizing the observation of a quasi-geostrophic ocean model, aceptado en SIAM Journal on Control and Optimization (2004).

Capítulo 2: Un problema de asimilación de datos

Como hemos mencionado antes, en el Capítulo 2 aplicaremos un método de asimilación de datos para el modelo cuasi-geostrófico del océano en su formulación función de corriente-vorticidad. Utilizaremos un método introducido recientemente en [Pue02], donde se presentan algunos resultados teóricos aplicados a la ecuación de difusión-convección y a las ecuaciones linealizadas de Navier-Stokes.

El método se puede formular como sigue: se quiere predecir la circulación del océano durante un intervalo de tiempo (T_0, T_0+T) pero no se conoce la condición inicial en T_0. Usando mediciones en un intervalo de tiempo anterior $(0, T_0)$ y distribuidas en cierta región del espacio que llamaremos observatorio $\mathcal{O}$, se trata de recuperar el valor de la condición inicial en T_0.

En las aproximaciones clásicas, usando el método variacional de asimilación de datos, se buscan las condiciones iniciales en $t = 0$ con el objetivo de calcular el estado en el intevalo $(0, T_0+T)$. Este método usa las técnicas de control óptimo para minimizar un funcional de costo conveniente junto con un método de regularización (por ejemplo la regularización de Tikhonov) [BLV98, LBV98].

La aproximación planteada en [Pue02] ya no busca la condición inicial en $t = 0$ para predecir la evolución del sistema en el intervalo de tiempo $(0, T_0 + T)$, sino que busca una aproximación en T_0, independiente del valor en $t = 0$. Para ello sólo será necesario conocer el término fuente y mediciones en el observatorio $\mathcal{O} \times (0, T_0)$.

En este capítulo, primeramente analizaremos la existencia y unicidad del problema adjunto. Este resultado se prueba usando el método de transposición introducido en [LM68].

El método que usaremos para la reconstrucción "exacta" del estado en T_0 se reducirá a estudiar un problema de controlabilidad nula. El punto esencial consiste en probar una desigualdad de observabilidad. La ausencia de condiciones de frontera sobre la vorticidad en el problema considerado hace que no se pueda obtener directamente una desigualdad de Carleman global. La estrategia consistirá en pasar a la formulación velocidad-presión, donde si se conoce una estimación de Carleman global (ver [FCGO04] o Anexo B).

Además, introduciremos un problema de control óptimo clásico para obtener una aproximación de la vorticidad en T_0 a partir de mediciones de la función de corriente en el observatorio $\mathcal{O} \times (0, T_0)$ y la tensión del viento en la superficie. Realizaremos la implementación numéricamente de dicho método donde estudiaremos la dependencia del tamaño de la región de medición, el tiempo de asimilación y los coeficientes del modelo para obtener una correcta predicción de la circulación.

Una estrategia de refinamiento adaptivo

Capítulo 3: Análisis del error a priori y a posteriori de un método de elementos finitos para un modelo de circulación oceánica de gran escala

Como se comentó al comienzo de esta introdución, un fenómeno típico de los modelos de gran escala en el océano, es la formación de una corriente fuerte en la costa occidental. Debido a esta corriente, las soluciones de estos modelos presentan capas límites que generan la aparición de oscilaciones y una mala resolución en esta región cuando se resuelve numéricamente. Una forma de reducir este efecto es el refinamiento de la malla numérica donde la solución presenta esa capa límite.

Las estrategias de refinamiento se basan principalmente en indicadores del error *a posteriori*. Muchos han sido los trabajos dedicados a este tema; ver por ejemplo

[BR78, Ver96, AO00, BS01].

En particular, en la región de la capa límite, la solución tiene gradientes fuertes en una dirección y casi ninguna variación en la dirección ortogonal. En este caso, resulta conveniente utilizar elementos alineados con esta capa.

Se han propuesto varios alternativas para crear " mallas anisotrópicas". Algunas de ellas se basan en la recuperación de la hessiana de la solución (ver, por ejemplo, [CD97, CDHMP97, AAYHT$^+$96, PPK92]). En particular, los autores en [AFG$^+$00] introdujeron una estrategia de mallado anisotrópico adaptivo guiada por un estimador del error direccional que se basa en la recuperación de la segunda derivada de la solución a partir de una aproximación por elementos finitos.

En este capítulo, presentaremos un análisis semejante para el caso estacionario del modelo cuasi-geostrófico formulado en función de corriente - vorticidad. Primeramente estudiaremos la existencia y unicidad de la solución continua y discreta del modelo. Bajo suposiciones apropiadas de regularidad se prueba que la función de corriente puede calcularse con un orden de error óptimo en $H^1(\Omega)$.

El segundo resultado consiste en analizar una estrategia de refinamiento adaptivo de mallado para reducir las oscilaciones y la pobre resolución que aparece cuando el término convectivo es dominante. Proponemos un indicador anisotrópico del error *a posteriori* para localizar la capa límite, sin utilizar información *a priori* de la misma y para crear mallas bien adaptadas a la solución. Esta técnica se basa en la recuperación de la hessiana de la solución. Finalmente, evaluamos la eficiencia de nuestra estrategia por medio de varios experimentos numéricos.

El contenido de este capítulo corresponde al artículo [CGR03]:

- J. M. Cascón, G. C. García and R. Rodríguez, A priori and a posteriori error analysis for a large-scale ocean circulation model. *Comput. Methods Appl. Mech. Engrg.*, 192 (2003) 5305–5327.

Part I

Two applications of Control Theory

Chapter 1

Insensitizing controls for a quasi-geostrophic ocean model

Abstract
In this chapter, we consider a linear quasi-geostrophic ocean model with partially known initial conditions. We search for controls that make the observation locally insensitive to the perturbations of the initial data. Their existence is equivalent to the null controllability property for an associated cascade Stokes-like system. Thanks to the presence of the Coriolis term, we are able to prove the existence of such controls. Our strategy is the following. First, we prove a unique continuation property for the adjoint of the state system that leads to approximate controllability. Then, under certain assumptions, an observability inequality is established for the adjoint. The proof is inspired by the arguments leading to unique continuation. This inequality leads to the desired null controllability result.

1.1 Introduction and main results

1.1.1 Incomplete initial data ocean model

Let Ω be a non-empty open bounded and connected subset of $\mathbb{R}^2$, with boundary Γ of class $\mathcal{C}^2$ and outwards unit normal vector $\nu = \nu(x)$. Let ω be a non-empty open subset of Ω, $T > 0$, $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. In this paper, we will consider a linear quasi-geostrophic ocean model [BGdS01, MAS96, MW95] described by the following equations:

$$\begin{cases} u_t - A\Delta u + \gamma u + (f_0 + \beta x_2)\,\mathrm{k}\wedge u + \dfrac{1}{\rho_0}\nabla p = \mathcal{T} + h1_\omega \quad \text{in } Q, \\ \operatorname{div} u = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(0) = u_0 + \tau\widehat{u}_0 \quad \text{in } \Omega, \end{cases} \tag{1.1}$$

where $u(x,t)$ and $p(x,t)$ respectively denote the velocity and the pressure of the fluid at $(x,t) = (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}_+$. In this model, A represents the horizontal *eddy viscosity* coefficient, γ is the bottom *friction* coefficient, ρ_0 is the fluid density and $(f_0 + \beta x_2)\mathrm{k}\wedge u$ is the Coriolis term, with $\mathrm{k}\wedge u = (-u_2, u_1)$. In the right hand side, 1_ω denotes the characteristic function of ω and $\mathcal{T}$ is a given source. The term $\tau\widehat{u}_0$, where $\tau \in \mathbb{R}$, represents a small unknown perturbation of the initial velocity field u_0 and $h = h(x,t)$ is a control function to be determined.

Notice that the Coriolis force is represented by a zero order coupling term in the equations. It introduces a different behavior of the system depending on the direction in space. In order to simplify the presentation of the results, we will assume that $A = 1$, $\gamma = 1$, $f_0 = 1$, $\beta = 1$ and $\rho_0 = 1$.

Let us introduce the following spaces, which are usual in the analysis of Stokes systems

$$\begin{aligned} H &= \{v \in L^2(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega,\ v\cdot\nu = 0 \text{ on } \Gamma\}, \\ V &= \{v \in H_0^1(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega\}, \qquad W = H^2(\Omega)^2 \cap V. \end{aligned}$$

Recall that,

$$W \hookrightarrow V \hookrightarrow H \equiv H' \hookrightarrow V' \hookrightarrow W',$$

where the embeddings are dense and compact. To simplify notation we denote $L^2(0,T;L^2(\Omega)^2)$ by $L^2(Q)$ and $L^2(0,T;L^2(\omega)^2)$ by $L^2(\omega\times(0,T))$.

For any given $u_0, \tau\widehat{u}_0 \in H$ with $\|\widehat{u}_0\|_{0,\Omega} = 1$, any $\mathcal{T} \in L^2(Q)$ and any $h \in L^2(\omega\times(0,T))$, the linear system (1.1) possesses a unique solution (u,p), with $u \in L^2(0,T;V)\cap H^1(0,T;V')$ and $p \in W^{-1,\infty}(0,T;L^2(\Omega))$ (p is unique up to an additive distribution only depending on t). This is easily proved by adapting the arguments of [Tem84] to the presence of a skew-symmetric Coriolis term in the equations (see Appendix A). Notice that, if we had $u_0 + \tau\widehat{u}_0 \in V$, then the couple (u,p) would satisfy $u \in L^2(0,T;W)\cap H^1(0,T;H)$ and $p \in L^2(0,T;H^1(\Omega))$.

We will be concerned with the search of controls such that the velocity measurements over an observation set are insensible or almost insensible to small variations of the initial conditions. To do this, we will use *insensitizing control* theory.

1.1.2 Insensitizing controls and controllability

Let $\mathcal{O}$ be a non-empty subset of Ω and let us introduce the following functional, defined on the family of solutions to (1.1):

$$\Phi(u) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |u(x,t)|^2 \, dx \, dt. \tag{1.2}$$

The notion of *insensitizing controls* was introduced by J. L. Lions [Lio90]. In the context of (1.1)–(1.2), it reads as follows:

Definition 1.1.1 *We say that the control $h \in L^2(\omega \times (0,T))$ is Φ insensitizing if*

$$\frac{d}{d\tau}\Phi(u)\Big|_{\tau=0} = 0 \qquad \forall \widehat{u}_0 \in H, \quad \textit{with} \quad \|\widehat{u}_0\|_{0,\Omega} = 1. \tag{1.3}$$

On the other hand, we say that $h \in L^2(\omega \times (0,T))$ is Φ $\varepsilon-$insensitizing if

$$\left|\frac{d}{d\tau}\Phi(u)\Big|_{\tau=0}\right| \le \varepsilon \qquad \forall \widehat{u}_0 \in H, \quad \textit{with} \quad \|\widehat{u}_0\|_{0,\Omega} = 1. \tag{1.4}$$

Of course, in (1.3) and (1.4) u is, together with p, the solution to (1.1).

The Φ insensitizing (resp. Φ ε-insensitizing) controls h must be interpreted as those leading to an observation $\Phi(u)$ that is locally independent (resp. almost independent) at the initial perturbation $\tau\widehat{u}_0$. The existence of such controls is a pertinent question, since it is realistic to assume that the true initial conditions for (1.1) are unknown. In fact, as noticed in [Lio90], it would be more convenient to search for Ψ insensitizing (or Ψ $\varepsilon-$insensitizing) controls, where

$$\Psi(u) = \frac{1}{2}\int_0^T \int_{\mathcal{O}} |\operatorname{curl} u(x,t)|^2 \, dx \, dt.$$

But this is much more complicate and will be the subject of future work.

It is easy to characterize the insensitivity (resp. ε-insensitivity) property in terms of exact null controllability (resp. approximate controllability) of a related cascade system. Indeed, let $(\bar{u}, \bar{p})$ and (q, r) be the solutions of the following systems:

$$\begin{cases} \bar{u}_t - \Delta\bar{u} + \bar{u} + (1+x_2)\, k \wedge \bar{u} + \nabla\bar{p} = \mathcal{T} + h1_\omega & \text{in } Q, \\ \operatorname{div} \bar{u} = 0 & \text{in } Q, \\ \bar{u} = 0 & \text{on } \Sigma, \\ \bar{u}(0) = u_0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

$$\begin{cases} -q_t - \Delta q + q - (1+x_2)\, k \wedge q + \nabla\pi = \bar{u}1_{\mathcal{O}} & \text{in } Q, \\ \operatorname{div} q = 0 & \text{in } Q, \\ q = 0 & \text{on } \Sigma, \\ q(T) = 0 & \text{in } \Omega. \end{cases} \tag{1.6}$$

Then the control h is Φ insensitizing (resp. Φ ε-insensitizing) if and only if

$$q(0) = 0 \qquad (\text{ resp. } \|q(0)\|_{0,\Omega} \leq \varepsilon). \tag{1.7}$$

Indeed, in view of (1.2), condition (1.3) is equivalent to

$$\int_0^T \int_{\mathcal{O}} \bar{u} \cdot u_\tau \, dx \, dt = 0 \quad (\text{ resp. (1.4) is equivalent to } \left| \int_0^T \int_{\mathcal{O}} \bar{u} \cdot u_\tau \, dx \, dt \right| \leq \varepsilon),$$

where $\bar{u}$ is the solution of (1.5) and u_τ is the solution of (1.1) differentiated with respect to τ. Using the definition of (q, π) and integrating by parts, we obtain

$$\int_\Omega q(0) \cdot \widehat{u}_0 \; dx = 0 \quad (\text{ resp. } \left| \int_\Omega q(0) \cdot \widehat{u}_0 \; dx \right| \leq \varepsilon), \quad \forall \widehat{u}_0 \in H, \quad \text{with } \|\widehat{u}_0\|_{0,\Omega} = 1.$$

This is equivalent to (1.7) (see [dT00] for more details).

Notice that, since $\bar{u} \in L^2(0,T;V)$, we also have $q \in L^2(0,T;W) \cap H^1(0,T;H)$ and $\pi \in L^2(0,T;H^1(\Omega))$.

We are thus in the presence of a null controllability problem (resp. an approximate controllability problem) for a cascade system, where the control h is not acting directly in the system satisfied by q (the function we want to drive to zero after a time interval of length T) but indirectly, through $\bar{u}1_{\mathcal{O}}$. In this sense, the problem under consideration is more difficult than the null controllability problem for Stokes type equations.

1.1.3 Main results

There have been several recent results concerning the existence of insensitizing and $\varepsilon-$insensitizing controls for parabolic problems.

Thus, in [BF95] the existence of $\varepsilon-$insensitizing controls for linear heat equations with partially known initial and boundary conditions was established. The same was also obtained for semilinear heat equations with globally Lipschitz-continuous nonlinearities. After that, it has been proved in [dT00] that insensitizing controls exist for the same equations completed with zero initial data, under suitable assumptions on the source term. In [BGBPG02], the authors have extended these results to other more general (slightly superlinear) nonlinearities.

In this work, we deal with the insensitizing and $\varepsilon-$insensitizing problems in the case of the Stokes type equations (1.1). Our results were sketched in [FCGO03]. To our knowledge, these are the first insensitivity results in the literature for equations of this kind.

As in the previous references, we will assume that the following geometrical hypothesis is satisfied:

$$\omega \cap \mathcal{O} \neq \emptyset. \tag{1.8}$$

Our main results are the following:

Theorem 1.1.1 *Let $T > 0$ and assume that (1.8) is satisfied. Then, for each $\varepsilon > 0$ there exists a control $h \in L^2(\omega \times (0,T))$ which is Φ $\varepsilon-$insensitizing.*

Theorem 1.1.2 *Under the assumptions of theorem 1.1.1, if we also have $u_0 = 0$ and*

$$\int_0^T \int_\Omega \exp\left(Mt^{-4}\right) \mathcal{T}^2 \, dx \, dt < +\infty, \tag{1.9}$$

for an appropriate constant M depending on Ω, ω, $\mathcal{O}$ and T, then there exists a control $h \in L^2(\omega \times (0,T))$ which is Φ insensitizing.

In [dT00], it was proved for the linear heat equation that, in general, we cannot expect the existence of insensitizing controls for nonvanishing initial data in $L^2(\Omega)$ when $\Omega \setminus \overline{\omega} \neq \emptyset$. The proof of this result is based on a counter-example for which the appropriate *observability inequality* fails when the initial data belong to $L^2(\Omega)$. Similar arguments could be used for Stokes systems. In view of this, it is reasonable to impose in Theorem 1.1.2 that $u_0 = 0$.

This chapter is organized as follows. In Section 1.2, we prove Theorem 1.1.1. Thanks to the presence of the Coriolis term, we can prove a unique continuation result for a cascade system similar to (1.5)–(1.6). In Section 1.3, we prove Theorem 1.1.2. We first check that if the observability inequality (1.13) is satisfied then insensitizing controls do exist. The proof of the observability inequality relies on an appropriate global Carleman inequality for the cascade adjoint system. We prove this inequality following a chain of estimates based on the same steps of the unique continuation proof. Finally, in order to be self-contained, we give in Appendix B a sketch of the proof of a Carleman estimate for Stokes-like systems that is needed in Section 1.3.

1.2 The existence of ε-insensitizing controls

In this section, we prove the approximate controllability of the cascade system (1.5)–(1.6). Since the considered system is linear, without loss of generality, we can assume that $\mathcal{T} = 0$ and $u_0 = 0$.

Theorem 1.2.1 *Suppose that $\omega \cap \mathcal{O} \neq \emptyset$. Then the linear space*

$$\{q(0) \,:\, h \in L^2(\omega \times (0,T))\}$$

is dense in H.

Proof: Let $\phi_0 \in H$ be such that

$$\int_\Omega q(0) \cdot \phi_0 \; dx = 0 \qquad \forall h \in L^2(\omega \times (0,T)).$$

Let us see that $\phi_0 = 0$. Now, let (ϕ, θ), (z, r) be the solution of

$$\left\{ \begin{array}{l} \phi_t - \Delta\phi + \phi + (1+x_2)\, k \wedge \phi + \nabla\theta = 0 \quad \text{in } Q, \\ \operatorname{div}\phi = 0 \quad \text{in } Q, \\ \phi = 0 \quad \text{on } \Sigma, \\ \phi(0) = \phi_0 \quad \text{in } \Omega, \end{array} \right. \tag{1.10}$$

$$\left\{ \begin{array}{l} -z_t - \Delta z + z - (1+x_2)\, k \wedge z + \nabla r = \phi 1_{\mathcal{O}} \quad \text{in } Q, \\ \operatorname{div} z = 0 \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(T) = 0 \quad \text{in } \Omega. \end{array} \right. \tag{1.11}$$

For each $\phi_0 \in H$, this coupled system possesses a unique solution (ϕ, θ), (z, r), with at least $\phi, z \in L^2(0,T;V) \cap H^1(0,T;V')$ and $\theta, r \in W^{-1,\infty}(0,T;L^2(\Omega))$ (again, θ and r are unique up to a distribution only depending on t).

Using (1.5) and (1.6), we deduce at once that

$$\int_0^T \int_\omega h \cdot z \, dx\, dt = \int_\Omega q(0) \cdot \phi_0 \, dx = 0 \qquad \forall h \in L^2(\omega \times (0,T)),$$

so z vanishes in $\omega \times (0,T)$. In order to conclude that $\phi_0 = 0$ in Ω, we will use the following unique continuation result:

Lemma 1.2.1 *Assume* (1.8)*. Let (ϕ, θ), (z, r) be a solution to* (1.10)-(1.11) *with $\phi_0 \in H$. Then, if $z = 0$ in $\omega \times (0,T)$, we necessarily have $z \equiv \phi \equiv 0$ and $\nabla r \equiv \nabla\theta \equiv 0$ in Q.*

Proof: Let us set $\widetilde{\omega} = \omega \cap \mathcal{O}$. By assumption this is a nonempty open set. Let us introduce the horizontal component of $\widetilde{\omega}$

$$C_1 = \{(x_1, x_2) \in \overline{\Omega} \,:\, \exists x_1^0 \;\text{ such that }\; (x_1^0, x_2) \in \widetilde{\omega}\}$$

and let us set $\Sigma_1 = (\Gamma \cap C_1) \times (0,T)$. We will prove a more general result saying that if we have

$$\begin{cases} \phi_t - \Delta\phi + \phi + (1+x_2)\, k \wedge \phi + \nabla\theta = 0 \quad \text{in } Q, \\ \operatorname{div} \phi = 0 \quad \text{in } Q, \\ \phi_1 = 0 \quad \text{on } \Sigma_1, \end{cases} \tag{1.12}$$

$$\begin{cases} -z_t - \Delta z + z - (1+x_2)\, k \wedge z + \nabla r = \phi 1_{\mathcal{O}} \quad \text{in } Q, \\ \operatorname{div} z = 0 \quad \text{in } Q \end{cases}$$

and $z = 0$ in $\omega \times (0,T)$, then $\phi \equiv 0$.

We first notice that $\operatorname{curl} \phi = 0$ in $\widetilde{\omega} \times (0,T)$. Then, applying the curl operator to (1.12), in view of the presence of the Coriolis term and the fact that $\operatorname{div} \phi = 0$, we deduce that $\phi_2 = 0$ in $\widetilde{\omega} \times (0,T)$ and ϕ_1 is a constant in $\widetilde{\omega} \times (0,T)$.

Let us introduce $\psi = \partial\phi/\partial x_1$ and $\pi = \partial\theta/\partial x_1$. We have

$$\begin{cases} \psi_t - \Delta\psi + \psi + (1+x_2)\, k \wedge \psi + \nabla\pi = 0 \quad \text{in } Q, \\ \operatorname{div} \psi = 0 \quad \text{in } Q, \\ \psi = 0 \quad \text{in } \ \widetilde{\omega} \times (0,T). \end{cases}$$

From the uniqueness property in [FL02], one has $\psi \equiv 0$ in Q. Now, since $\partial\phi_i/\partial x_1 \equiv 0$ for $i = 1, 2$ and $\operatorname{div} \phi = 0$ in Q, we also have $\nabla\phi_2 = 0$ in Q and, from the fact that $\phi_2 = 0$ in $\widetilde{\omega} \times (0,T)$, we deduce that $\phi_2 \equiv 0$.

On the other hand, since $\partial\phi_1/\partial x_1 = 0$ in Q and $\phi_1 = 0$ on Σ_1 , we see that $\phi_1 = 0$ in $C_1 \times (0,T)$. In view of the uniqueness properties in [Fab96], we must have $\phi_1 \equiv 0$ in Q, as desired. □

Remark 1.2.1 *Similar unique continuation properties for the Stokes system have been deduced in [Fab96] if all components of the velocity up to one vanish in an open nonempty subset of Q. This result requires additional conditions on the coefficients which are not satisfied when the Coriolis term appears in the equations.*

□

1.3 The existence of insensitizing controls

The proof of Theorem 1.1.2 relies on the following observability result for the cascade adjoint system (1.10)-(1.11):

Proposition 1.3.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$. There exist positive constants M and K, depending only on Ω, ω, $\mathcal{O}$ and T, such that the inequality*

$$\int_0^T \int_\Omega \exp\left(-Mt^{-4}\right)|z|^2 \, dx \, dt \leq K \int_0^T \int_\omega |z|^2 \, dx \, dt \tag{1.13}$$

holds for every solution of (1.10)-(1.11) *with $\phi_0 \in H$.*

The proof of Proposition 1.3.1 is based on the global Carleman inequality (1.18) for the cascade adjoint system (1.10)-(1.11). This Carleman inequality will be proved later, in Section 1.3.1. The fact that (1.18) implies (1.13) will be proved in Section 1.3.2.

Let us now give the proof of Theorem 1.1.2 assuming that Proposition 1.3.1 holds. Thus, let us assume that (1.8) is satisfied, $u_0 = 0$ and (1.9) holds with M being the constant furnished by Proposition 1.3.1.

The approximate control h of minimal norm in $L^2(\omega \times (0,T))$ corresponding to $u_0 = 0$, a source term $\mathcal{T}$ satisfying (1.9) and tolerance $\varepsilon > 0$ can be obtained by minimizing in $L^2(\Omega)$ the following convex functional [FPZ95, Lio92a]:

$$J_\varepsilon(\phi_0) = \frac{1}{2}\int_0^T \int_\omega |z|^2 \, dx \, dt + \int_0^T \int_\Omega \mathcal{T} \cdot z \, dx \, dt + \varepsilon \|\phi_0\|_{0,\Omega}. \tag{1.14}$$

Thus, if the minimum of J_ε in $L^2(\Omega)$ is attained at $\widehat{\phi}_{0\varepsilon}$ and we denote by $(\widehat{\phi}_\varepsilon, \widehat{\theta}_\varepsilon)$, $(\widehat{z}_\varepsilon, \widehat{r}_\varepsilon)$ the solution to (1.10)–(1.11) with $\phi_0 = \widehat{\phi}_{0\varepsilon}$, then the control

$$h_\varepsilon = \widehat{z}_\varepsilon 1_\omega \tag{1.15}$$

is such that the associated solution $(\bar{u}_\varepsilon, \bar{p}_\varepsilon)$, $(q_\varepsilon, \pi_\varepsilon)$ to (1.5)–(1.6) with $u_0 = 0$ satisfies $\|q_\varepsilon(0)\|_{0,\Omega} \leq \varepsilon$.

It is not difficult to see that

$$\liminf_{\|\phi_0\|_{0,\Omega} \to \infty} \frac{J_\varepsilon(\phi_0)}{\|\phi_0\|_{0,\Omega}} \geq \varepsilon.$$

The proof of this inequality is classical, see [FPZ95]. It is implied by the unique continuation property for the cascade adjoint system that we have presented above (see Lemma 1.2.1).

Furthermore, the following optimality condition must be satisfied at $\widehat{\phi}_{0\varepsilon}$:

$$\int_0^T \int_\omega |\widehat{z}_\varepsilon|^2 \, dx \, dt + \int_0^T \int_\Omega \mathcal{T} \cdot \widehat{z}_\varepsilon \, dx \, dt + \varepsilon \|\widehat{\phi}_{0\varepsilon}\|_{0,\Omega} = 0. \tag{1.16}$$

By replacing (1.15) in (1.16), introducing the weight $e^{Mt^{-4}}$ and using (1.13) and Young's inequality, we easily deduce that

$$\int_0^T \int_\omega |h_\varepsilon|^2 \, dx \, dt \le K^2 \int_0^T \int_\Omega \exp(Mt^{-4}) |\mathcal{T}|^2 \, dx \, dt.$$

Since h_ε is uniformly bounded in $L^2(\omega \times (0,T))$, we can extract a subsequence $\{h_{\varepsilon_n}\}$ with $\varepsilon_n \to 0$ and satisfying

$$\begin{array}{lll} h_{\varepsilon_n} \rightharpoonup h & \text{weakly in} & L^2(\omega \times (0,T)), \\ \bar{u}_{\varepsilon_n} \to \bar{u} & \text{strongly in} & L^2(Q) \quad \text{and} \\ q_{\varepsilon_n} \to q & \text{strongly in} & L^2(Q) \end{array}$$

as $n \to +\infty$. Of course, we have denoted here by $(\bar{u}_{\varepsilon_n}, \bar{p}_{\varepsilon_n})$, $(q_{\varepsilon_n}, \pi_{\varepsilon_n})$ and $(\bar{u}, \bar{p})$, (q, π) the solutions to (1.5)–(1.6) associated to h_{ε_n} and h, respectively. Notice that $\|q_{\varepsilon_n}(0)\|_{0,\Omega} \le \varepsilon_n$ for all $n \ge 1$. Consequently, we have $q(0) = 0$.

This ends the proof of Theorem 1.1.2.

1.3.1 A global Carleman estimate

The goal of this section is to present an estimate of the Carleman kind for the solutions to the adjoint cascade system (1.10)–(1.11). As mentioned above, this estimate will be crucial for the proof of Proposition 1.3.1.

Let us first introduce an open ball B_0 such that $B_0 \subset\subset \omega \cap \mathcal{O}$ and an auxiliary function $\eta_0 \in \mathcal{C}^2(\overline{\Omega})$ satisfying

$$\eta_0(x) > 0 \ \ \forall x \in \Omega, \quad \eta_0 = 0 \ \text{ on } \partial\Omega, \quad |\nabla \eta_0(x)| > 0 \ \ \forall x \in \overline{\Omega \setminus B_0}\,. \tag{1.17}$$

The existence of such a function is proved in [FI96].

Let us also introduce the weight functions

$$\begin{array}{lllllll} \alpha(x,t) & = & \dfrac{e^{2\lambda\|\eta_0\|_\infty} - e^{\lambda\eta_0}}{t^4(T-t)^4}, & \widehat{\alpha}(t) & = & \min_{\overline{\Omega}} \alpha(x,t), \quad \alpha^*(t) & = \max_{\overline{\Omega}} \alpha(x,t), \\ \varphi(x,t) & = & \dfrac{e^{\lambda\eta_0}}{t^4(T-t)^4}, & \widehat{\varphi}(t) & = & \max_{\overline{\Omega}} \varphi(x,t), \quad \varphi^*(t) & = \min_{\overline{\Omega}} \varphi(x,t). \end{array}$$

The following property of the functions α^* and $\widehat{\alpha}$ will be needed below:

Lemma 1.3.1 *For any $a > 1$ there exists $\lambda_a > 0$ such that*

$$a\,\widehat{\alpha}(t) > \alpha^*(t) \quad \forall \lambda > \lambda_a\,, \quad \forall t \in (0,T).$$

Proof: The proof is elementary. Indeed, it suffices to notice that we have $a(e^{2x} - e^x) > e^{2x} - 1$ if $a > 1$ and x is sufficiently large. □

The main result in this Section is the following:

Theorem 1.3.1 *Assume that $\omega \cap \mathcal{O} \neq \emptyset$ and let the functions α, φ, $\widehat{\alpha}$ and $\widehat{\varphi}$ be as above. For each $\widehat{\gamma} \in (0,1)$, there exist constants $\widehat{s}$, $\widehat{\lambda}$ and $\widehat{C}$ depending on Ω, ω, $\mathcal{O}$ and T such that one has*

$$\int_0^T\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}(|z_t|^2+|\Delta z|^2)+s\lambda^2\varphi|\nabla z|^2+s^3\lambda^4\varphi^3|z|^2\right)dx\,dt$$
$$+\int_0^T\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}(|\phi_t|^2+|\Delta\phi|^2)+s\lambda^2\varphi|\nabla\phi|^2+s^3\lambda^4\varphi^3|\phi|^2\right)dx\,dt$$
$$\leq \widehat{C}\int_0^T\int_\omega e^{-(1+\widehat{\gamma})s\widehat{\alpha}}s^{63}\lambda^{32}\widehat{\varphi}^{67}|z|^2\,dx\,dt, \tag{1.18}$$

for any $s > \widehat{s}$ and $\lambda > \widehat{\lambda}$ and for every solution (ϕ,θ), (z,r) to (1.10)-(1.11) *associated to an initial data $\phi_0 \in H$.*

The proof will be divided in several steps and will be given in the following subsections. First, we will apply a global Carleman estimate for the Stokes system subsequently to (1.10) and (1.11). This will lead to the estimate (1.22). Then, in order to deduce (1.18), we will have to estimate the integral in the right hand side of (1.22) containing ϕ in terms of z. To this end, we will follow the steps of the proof of Lemma 1.2.1 in converse order.

Step 1: A first direct Carleman estimate

Let $I(s,\lambda;v)$ stand for the quantity

$$I(s,\lambda;v)=\int_0^T\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}(|v_t|^2+|\Delta v|^2)+s\lambda^2\varphi|\nabla v|^2+s^3\lambda^4\varphi^3|v|^2\right)dx\,dt \tag{1.19}$$

for any positive s and λ and any sufficiently regular function $v = v(x,t)$. We then have:

Lemma 1.3.2 *For each $\gamma_1 \in (0,1)$ there exist positive constants s_1, λ_1 and C_1,* *depending on* Ω*,* ω*,* $\mathcal{O}$ *and* T*, with the following properties:*

$$\begin{aligned} I(s,\lambda;z) \le C_1 \bigg\{ &\int_0^T\!\!\int_{B_0} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}|z|^2\,dx\,dt \\ &+ \int_0^T\!\!\int_{B_0} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\phi|^2\,dx\,dt \\ &+ \int_0^T\!\!\int_{\mathcal{O}} e^{-2s\alpha}\left((s\varphi)^{1/2}|\phi|^2 + \frac{1}{s^3\varphi^{7/2}}|\phi_t|^2\right)\,dx\,dt \bigg\} \end{aligned} \tag{1.20}$$

and

$$I(s,\lambda;\phi) \le C_1 \int_0^T\!\!\int_{B_0} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}|\phi|^2\,dx\,dt \tag{1.21}$$

for any $s > s_1$ *and* $\lambda > \lambda_1$ *and for every solution of* (1.10)-(1.11) *with* $\phi_0 \in H$*.*

The proof of Lemma 1.3.2 is similar to the proof of other recent global Carleman inequalities for the Stokes system. The main ideas are due to O.Yu. Imanuvilov, see [Ima98, Ima01]; see also [Bar01] and [FCGIP04] for other related results. Due to the regularity properties of ϕ in (1.35), we have used a more straight argument than in [FCGIP04], where the right hand side only belongs to $L^2(Q)^2$. This argument allows us to obtain the last two integrals in (1.20). Combining the estimations (1.20) and (1.21), we will be able to absorb them by the left hand side of $I(s,\lambda;\phi)$. For the reader's convenience, we present the proof in Appendix B.

Let us fix $\widehat{\gamma}$, with $0 < \widehat{\gamma} < 1$. We are now going to deduce several estimates that hold for "sufficiently large s and λ". By this we mean that they are satisfied for any $s > \bar{s}$ and any $\lambda > \bar{\lambda}$, where $\bar{s}$ and $\bar{\lambda}$ are (large) positive constants depending only on Ω, ω, $\mathcal{O}$, T and $\widehat{\gamma}$.

In the sequel, C denotes a generic constant, not necessarily the same at each occurrence, depending on Ω, ω, $\mathcal{O}$, T and (possibly) $\widehat{\gamma}$.

Let γ_1 be given in $(\widehat{\gamma}, 1)$. In view of Lemma 1.3.2 applied to γ_1 , we get

$$I(s,\lambda;z) + I(s,\lambda;\phi) \le C \int_0^T\!\!\int_{B_0} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}(|z|^2+|\phi|^2)\,dx\,dt \tag{1.22}$$

for s and λ large enough.

Indeed, the last two integrals in (1.20) can be absorbed by the left hand side of $I(s,\lambda;\phi)$, since

$$Cs^{-3}\varphi^{-7/2} \le \frac{1}{2}(s\varphi)^{-1} \quad \text{and} \quad C(s\varphi)^{1/2} \le \frac{1}{2}s^3\varphi^3$$

for sufficiently large s.

Step 2: An estimate of ϕ in terms of $\operatorname{curl}\phi$

In order to simplify the notation, let us set $a=7$ and $b=15/2$. Then

$$I(s,\lambda;z)+I(s,\lambda;\phi)\le C\int_0^T\int_{B_0} e^{-(1+\gamma_1)s\widehat{\alpha}}s^a\lambda^4\widehat{\varphi}^b(|z|^2+|\phi|^2)\,dx\,dt. \tag{1.23}$$

We will denote by $B_1\,,B_2\,,\ldots$ a sequence of balls centered at the same point than B_0 and satisfying

$$B_0\subset\subset B_1\subset\subset\cdots\subset\subset\omega\cap\mathcal{O}.$$

It is not a restriction to assume that their common center is the origin. This will be supposed in the sequel for simplicity. We will consider some functions $\xi_i\in\mathcal{C}_0^\infty(B_i)$ satisfying

$$\begin{gathered}0\le\xi_i\le 1,\quad \xi_i(x)=1\text{ in }B_{i-1}\,,\\ \xi_i^{-1/2}\nabla\xi_i\in L^\infty(\Omega),\quad \xi_i^{-1/2}\Delta\xi_i\in L^\infty(\Omega)\end{gathered} \tag{1.24}$$

(see [dT00] for a justification of the existence of these ξ_i).

Since $\operatorname{div}\phi=0$, $\phi=0$ on Σ and Ω is connected, we can introduce the stream function ψ satisfying

$$\phi=\operatorname{curl}\psi\equiv\left(\frac{\partial\psi}{\partial x_2},-\frac{\partial\psi}{\partial x_1}\right),$$

with $\psi=0$ on one connected component of Σ and $\dfrac{\partial\psi}{\partial n}=0$ on Σ.

Let us set $\rho_1(t)=e^{-(1+\gamma_1)s\widehat{\alpha}}s^a\lambda^4\widehat{\varphi}^b$. Then we have

$$\int_0^T\int_{B_0}\rho_1|\phi|^2\,dx\,dt\le\int_0^T\int_{B_1}\rho_1\xi_1|\nabla\psi|^2\,dx\,dt.$$

We will now give an estimate of the last integral in terms of $|\operatorname{curl}\phi|^2$. To this end, let us introduce the vorticity w, given by

$$w=\operatorname{curl}\phi=\frac{\partial\phi_2}{\partial x_1}-\frac{\partial\phi_1}{\partial x_2}.$$

Applying the curl operator to (1.10), we obtain

$$\begin{cases} w_t-\Delta w+w-\dfrac{\partial\psi}{\partial x_1}=0 & \text{in } Q,\\ \Delta\psi+w=0 \quad \text{in } Q. \end{cases} \tag{1.25}$$

In order to estimate $|\nabla\psi|^2$, we multiply by $\rho_1\xi_1\psi$ the second equation of (1.25). Then, we integrate by parts with respect to the space variable x and we get

$$\int_0^T\int_{B_1}\rho_1\xi_1|\nabla\psi|^2\,dx\,dt=\int_0^T\int_{B_1}\rho_1\xi_1\psi w\,dx\,dt+\frac{1}{2}\int_0^T\int_{B_1}\rho_1(\Delta\xi_1)|\psi|^2\,dx\,dt. \tag{1.26}$$

Notice that, using $I(s,\lambda;\phi)$, we can get upper bounds for $|\psi|^2$, $|\nabla\psi|^2$ and $|\psi_t|^2$. Indeed, from the definition of α^*, φ^* and $\widehat{\varphi}$, we have

$$\begin{aligned} I(s,\lambda;\phi) &\geq \int_0^T\!\!\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}|\nabla\psi_t|^2 + s^3\lambda^4\varphi^3|\nabla\psi|^2\right)dx\,dt \\ &\geq \int_0^T\!\!\int_\Omega e^{-2s\alpha^*}\left(\frac{1}{s\widehat{\varphi}}|\nabla\psi_t|^2 + s^3\lambda^4(\varphi^*)^3|\nabla\psi|^2\right)dx\,dt \\ &\geq C\int_0^T\!\!\int_\Omega e^{-2s\alpha^*}\left(\frac{1}{s\widehat{\varphi}}|\psi_t|^2 + s^3\lambda^4(\varphi^*)^3\left(|\psi|^2+|\nabla\psi|^2\right)\right)dx\,dt. \quad (1.27) \end{aligned}$$

Here we have used the fact that $\psi=0$ on one of the connected components of Σ to apply Poincaré's inequality.

With this information, we will be able to absorb the first integral in (1.26). Indeed, after using Young's inequality, we can estimate this term as follows:

$$\begin{aligned} \int_0^T\!\!\int_{B_1}\rho_1\xi_1\psi w\,dx\,dt &\leq \delta\int_0^T\!\!\int_\Omega e^{-2s\alpha^*}s^3\lambda^4(\varphi^*)^3|\psi|^2\,dx\,dt \\ &+C_\delta\int_0^T\!\!\int_{B_1} e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}s^{2a-3}\lambda^4\widehat{\varphi}^{2b-3}|w|^2\,dx\,dt. \quad (1.28) \end{aligned}$$

Now, if we introduce γ_2 with $0<\gamma_2<2\gamma_1-1$, then $(1+2\gamma_1-\gamma_2)/2>1$ and, from Lemma 1.3.1, we see that $(1+2\gamma_1-\gamma_2)\widehat{\alpha}/2>\alpha^*$ for λ sufficiently large. Consequently, it can be assumed that

$$-2(1+\gamma_1)\widehat{\alpha}+2\alpha^* < -(1+\gamma_2)\widehat{\alpha}$$

and we can replace $e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}$ by $e^{-(1+\gamma_2)s\widehat{\alpha}}$ in the last integral in (1.28):

$$\begin{aligned} \int_0^T\!\!\int_{B_1}\rho_1\xi_1\psi w\,dx\,dt &\leq \delta\int_0^T\!\!\int_\Omega e^{-2s\alpha^*}s^3\lambda^4(\varphi^*)^3|\psi|^2\,dx\,dt \\ &+C_\delta\int_0^T\!\!\int_{B_1} e^{-(1+\gamma_2)s\widehat{\alpha}}s^{2a-3}\lambda^4\widehat{\varphi}^{2b-3}|w|^2\,dx\,dt. \quad (1.29) \end{aligned}$$

Notice that, if we had chosen γ_1 sufficiently close to 1 before, then we would still have the possibility to choose γ_2 satisfying $\widehat{\gamma}<\gamma_2<2\gamma_1-1$.

On the other hand, by choosing δ sufficiently small, we can absorb the first term in the right hand side of (1.29) with $I(s,\lambda;\phi)$.

It remains in this step to estimate the last integral in (1.26). Assume that ξ_1 has been constructed as before, but also satisfying

$$\xi_1(x) = \begin{cases} 1 & \text{in } |x|<r_0, \\ \widehat{\Psi}\left(\frac{|x|-r_0}{r_1-a-r_0}\right) & \text{in } r_0\leq|x|\leq r_1-a, \\ 0 & \text{in } |x|>r_1-a, \end{cases}$$

where r_i denotes the radio of B_i, a is small enough and $\widehat{\Psi}$ is a function satisfying $\widehat{\Psi} \in \mathcal{C}^\infty([0,1])$,

$$\widehat{\Psi}(0) = 1, \quad \widehat{\Psi}(1) = 0, \quad \text{and} \quad \widehat{\Psi}^{(n)}(0) = \widehat{\Psi}^{(n)}(1) = 0 \quad \forall n \geq 1.$$

Let us set

$$\eta(x) = \int_{\bar{x}_1}^{x_1} \Delta\xi_1(y_1, x_2)\, dy_1$$

for each $x = (x_1, x_2) \in \bar{B}_1$, we have $\bar{x}_1 < x_1$ and $(\bar{x}_1, x_2) \in \partial B_1$. Notice that $\dfrac{\partial \eta}{\partial x_1} = \Delta \xi_1$. It is also easy to see that Supp $\eta \subset \bar{B}_1(0; r_1 - a)$. And now, using the first equation in (1.25), we observe that

$$\begin{aligned} \frac{1}{2}\int_0^T\!\!\int_{B_1} \rho_1(\Delta\xi_1)|\psi|^2\, dx\, dt &= \frac{1}{2}\int_0^T\!\!\int_{B_1} \rho_1 \frac{\partial \eta}{\partial x_1}|\psi|^2\, dx\, dt \\ &= -\int_0^T\!\!\int_{B_1} \rho_1 \eta \psi \frac{\partial \psi}{\partial x_1}\, dx\, dt \\ &= -\int_0^T\!\!\int_{B_1} \rho_1 \eta \psi (w_t - \Delta w + w)\, dx\, dt. \end{aligned} \quad (1.30)$$

Remark 1.3.1 *Notice that we have used the term* $\dfrac{\partial \psi}{\partial x_1}$ *in equation* (1.25) *in order to estimate* $|\psi|^2$ *over* B_1*. The terms comes from Coriolis force and it is absent in Stokes system.*

We will now estimate this last integral in the right hand side of (1.30). Concerning the product $\rho_1 \eta \psi w_t$, we can integrate by parts with respect to time in $B_1 \times (0,T)$ and then apply Young's inequality to deduce that

$$\begin{aligned} \int_0^T\!\!\int_{B_1} \rho_1 \eta \psi w_t\, dx\, dt &= -\int_0^T\!\!\int_{B_1} \left(\rho_1 \eta \psi_t w + \rho_1' \eta \psi w\right)\, dx\, dt \\ &\leq \delta \int_0^T\!\!\int_\Omega e^{-2s\alpha^*}\left(\frac{1}{s\widehat{\varphi}}|\psi_t|^2 + s^3\lambda^4(\varphi^*)^3|\psi|^2\right)\, dx\, dt \\ &+ C_\delta \int_0^T\!\!\int_{B_1} e^{-(1+\gamma_2)s\widehat{\alpha}}(s^{2a+1}\lambda^8\widehat{\varphi}^{2b+1} + s^{2a-1}\lambda^4\widehat{\varphi}^{2b-1/2})|w|^2\, dx\, dt \end{aligned} \quad (1.31)$$

for sufficiently large s and λ.

To obtain this inequality, we have first used that

$$|\rho_1'| = \left|(e^{-(1+\gamma_1)s\widehat{\alpha}} s^a \lambda^4 \widehat{\varphi}^b)_t\right| \leq C e^{-(1+\gamma_1)s\widehat{\alpha}} s^{a+1}\lambda^4 \widehat{\varphi}^{b+5/4}.$$

Then, we have noticed that

$$\begin{aligned}
&\int_0^T\int_{B_1} \rho_1' \eta\psi w \, dx\, dt \\
&\quad \le \delta \int_0^T\int_{\Omega} e^{-2s\alpha^*} s^3\lambda^4(\varphi^*)^3|\psi|^2\, dx\, dt \\
&\quad + C_\delta \int_0^T\int_{B_1} e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*} s^{2a-1}\lambda^4\widehat{\varphi}^{2b+5/2}(\varphi^*)^{-3}|w|^2\, dx\, dt
\end{aligned}$$

and, finally, we have taken λ large enough to have

$$e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}\widehat{\varphi}^{2b+5/2}(\varphi^*)^{-3} \le e^{-(1+\gamma_2)s\widehat{\alpha}}\widehat{\varphi}^{2b-1/2}.$$

We can simplify the estimate (1.31) by using the inequality

$$s^{2a-1}\widehat{\varphi}^{2b-1/2} \le Cs^{2a+1}\widehat{\varphi}^{2b+1},$$

that must hold for large s. Thus, we obtain

$$\begin{aligned}
&\int_0^T\int_{B_1} \rho_1 \eta\psi w_t \, dx\, dt \\
&\quad \le \delta \int_0^T\int_{\Omega} e^{-2s\alpha^*} \left(\frac{1}{s\widehat{\varphi}}|\psi_t|^2 + s^3\lambda^4(\varphi^*)^3|\psi|^2\right) dx\, dt \\
&\quad + C_\delta \int_0^T\int_{B_1} e^{-(1+\gamma_2)s\widehat{\alpha}} s^{2a+1}\lambda^8\widehat{\varphi}^{2b+1}|w|^2\, dx\, dt.
\end{aligned} \tag{1.32}$$

Notice that the first integral in the right hand side of (1.32) also appears in (1.27) and can be absorbed later by choosing δ small enough.

Let us now consider the term $\rho_1\eta\psi(\Delta w)$ in the last integral of (1.30). Let us integrate by parts with respect to the space variable x, let us use the identity -$\Delta\psi = w$ and let us apply Young's inequality. Arguing as before, we obtain

$$\begin{aligned}
\int_0^T\int_{B_1} \rho_1\eta\psi(\Delta w)\, dx\, dt &= \int_0^T\int_{B_1} \rho_1\left((\Delta\eta)\psi w + 2\nabla\eta\cdot\nabla\psi\, w + \eta|w|^2\right) dx\, dt \\
&\le \delta \int_0^T\int_{\Omega} e^{-2s\alpha^*} s^3\lambda^4(\varphi^*)^3(|\psi|^2 + |\nabla\psi|^2)\, dx\, dt \\
&+ C_\delta \int_0^T\int_{B_1} e^{-(1+\gamma_2)s\widehat{\alpha}} s^{2a-3}\lambda^4\widehat{\varphi}^{2b-3}|w|^2\, dx\, dt
\end{aligned} \tag{1.33}$$

for any sufficiently large s and λ.

Finally, arguing in a similar way, we can also estimate the last term $\rho_1\eta\psi w$ in (1.30):

$$\begin{aligned}\int_0^T\int_{B_1}\rho_1\eta\psi w\,dx\,dt &\leq \delta\int_0^T\int_{B_1}e^{-2s\alpha^*}s^3\lambda^4(\varphi^*)^3|\psi|^2\,dx\,dt\\ &+ C_\delta\int_0^T\int_{B_1}e^{-(1+\gamma_2)s\widehat{\alpha}}s^{2a-3}\lambda^4\widehat{\varphi}^{2b-3}|w|^2\,dx\,dt \qquad (1.34)\end{aligned}$$

From (1.30) and (1.32)-(1.34), we find that

$$\begin{aligned}&\frac{1}{2}\int_0^T\int_{B_1}\rho_1(\Delta\xi_1)|\psi|^2\,dx\,dt\\ &\qquad\leq 3\delta\int_0^T\int_\Omega e^{-2s\alpha^*}\left(\frac{1}{s\widehat{\varphi}}|\psi_t|^2+s^3\lambda^4(\varphi^*)^3(|\psi|^2+|\nabla\psi|^2)\right)dx\,dt\\ &\qquad+C_\delta\int_0^T\int_{B_1}\rho_2|w|^2\,dx\,dt, \qquad (1.35)\end{aligned}$$

where

$$\rho_2(t)=e^{-(1+\gamma_2)s\widehat{\alpha}}s^{2a+1}\lambda^8\widehat{\varphi}^{2b+1}.$$

Replacing the estimates (1.28) and (1.35) in (1.22), with $\delta>0$ sufficiently small, we obtain

$$I(s,\lambda;z)+I(s,\lambda;\phi)\leq C\left\{\int_0^T\int_{B_0}\rho_1|z|^2\,dx\,dt+\int_0^T\int_{B_1}\rho_2|\operatorname{curl}\phi|^2\,dx\,dt\right\}. \quad (1.36)$$

Step 3: An estimate of $\operatorname{curl}\phi$ in terms of z

Let us apply the curl operator to (1.11). For $\zeta=\operatorname{curl}z$, we obtain the following:

$$-\zeta_t-\Delta\zeta+\zeta-z_2=w1_{\mathcal{O}} \quad \text{in } \mathcal{O}\times(0,T).$$

Recall that $\xi_2\in\mathcal{C}_0^\infty(B_2)$ satisfies (1.24) and $B_1\subset\subset B_2\subset\subset\omega\cap\mathcal{O}$. After multiplying the above equation by $\rho_2\xi_2 w$, integrating by parts in Q and using (1.25), it follows that

$$\begin{aligned}\int_0^T\int_{B_2}\rho_2\xi_2|w|^2\,dx\,dt &= -\int_0^T\int_{B_2}\rho_2\xi_2\phi_2\zeta\,dx\,dt+\int_0^T\int_{B_2}\rho_2'\xi_2 w\zeta\,dx\,dt\\ &-\int_0^T\int_{B_2}\rho_2((\Delta\xi_2)w\zeta+2(\nabla\xi_2\cdot\nabla w)\zeta+\xi_2 wz_2)\,dx\,dt. \qquad (1.37)\end{aligned}$$

As before, we choose γ_3 satisfying $0<\gamma_3<2\gamma_2-1$. Then, for sufficiently large λ we have $(1+2\gamma_2-\gamma_3)\widehat{\alpha}/2>\alpha^*$ and, consequently,

$$-2(1+\gamma_2)\widehat{\alpha}+2s\alpha<-2(1+\gamma_2)\widehat{\alpha}+2s\alpha^*<-(1+\gamma_3)\widehat{\alpha}.$$

Notice once more that, if γ_1 is sufficiently close to 1, then we can choose γ_3 in $(\widehat{\gamma}, 1)$.

Now, proceeding as in the previous step, we see that

$$\left|\int_0^T\int_{B_2} \rho_2\xi_2\phi_2\zeta\,dx\,dt\right| \le \delta\int_0^T\int_\Omega e^{-2s\alpha}s^3\lambda^4\varphi^3|\phi_2|^2\,dx\,dt$$
$$+C_\delta\int_0^T\int_{B_2}\rho_3\frac{1}{s^4\lambda^4\widehat{\varphi}^4}\xi_2^2|\zeta|^2\,dx\,dt$$

for any small $\delta > 0$ (to be fixed later). Here, ρ_3 stands for the function

$$\rho_3(t) = e^{-(1+\gamma_3)s\widehat{\alpha}(t)}s^{4a+3}\lambda^{16}\widehat{\varphi}^{4b+3}(t).$$

We also have

$$\int_0^T\int_{B_2}\rho_2'\xi_2 w\zeta\,dx\,dt \le \delta\int_0^T\int_{B_2}\rho_2\xi_2|w|^2\,dx\,dt$$
$$+C_\delta\int_0^T\int_{B_2}\rho_2 s^2\widehat{\varphi}^{5/2}\xi_2|\zeta|^2\,dx\,dt.$$

Furthermore, after separating the terms in the last integral in (1.37), we find that

$$\left|\int_0^T\int_{B_2}\rho_2(\Delta\xi_2)w\zeta\,dx\,dt\right| \le \delta\int_0^T\int_{B_2}\rho_2\xi_2|w|^2\,dx\,dt$$
$$+C_\delta\int_0^T\int_{B_2}\rho_2\frac{|\Delta\xi_2|^2}{\xi_2}|\zeta|^2\,dx\,dt$$

and

$$\left|\int_0^T\int_{B_2}\rho_2(\nabla\xi_2\cdot\nabla w)\,\zeta\,dx\,dt\right| \le \delta\int_0^T\int_\Omega e^{-2s\alpha}\frac{1}{s\varphi}|\Delta\phi|^2\,dx\,dt$$
$$+C_\delta\int_0^T\int_{B_2}\rho_3|\nabla\xi_2|^2|\zeta|^2\,dx\,dt.$$

In this last estimate we have used that $|\nabla w|^2 = |\Delta\phi|^2$. Finally,

$$\left|\int_0^T\int_{B_2}\rho_2\xi_2 w z_2\,dx\,dt\right| \le \delta\int_0^T\int_{B_2}\rho_2\xi_2|w|^2\,dx\,dt + C_\delta\int_0^T\int_{B_2}\rho_2\xi_2|z_2|^2\,dx\,dt.$$

In view of (1.37) and all these inequalities, we obtain

$$\int_0^T\int_{B_2}\rho_2\xi_2|\operatorname{curl}\phi|^2\,dx\,dt$$
$$\le \frac{\delta}{1-3\delta}\int_0^T\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}|\Delta\phi|^2 + s^3\lambda^4\varphi^3|\phi_2|^2\right)dx\,dt$$
$$+C_\delta\int_0^T\int_{B_2}\left(\rho_2|z_2|^2 + \rho_3\bar{\xi}_2|\operatorname{curl} z|^2\right)dx\,dt, \tag{1.38}$$

for some $\bar{\xi}_2 \in \mathcal{C}_0^\infty(B_2)$.

It remains to estimate the previous integral of $\rho_3\bar{\xi}_2|\operatorname{curl} z|^2$. Arguing as above, we see that

$$\begin{aligned}\int_0^T\int_{B_2}\rho_3\bar{\xi}_2|\operatorname{curl} z|^2\,dx\,dt &\le 2\int_0^T\int_{B_2}\rho_3\bar{\xi}_2|\nabla z|^2\,dx\,dt\\ &= -2\int_0^T\int_{B_2}\rho_3(\nabla\bar{\xi}_2\cdot\nabla z+\bar{\xi}_2(\Delta z))z\,dx\,dt\\ &\le \delta\int_0^T\int_\Omega e^{-2s\alpha}\left(s\lambda^2\varphi|\nabla z|^2+\frac{1}{s\varphi}|\Delta z|^2\right)dx\,dt\\ &+ C_\delta\int_0^T\int_{B_2}\rho_4|z|^2\,dx\,dt,\end{aligned} \tag{1.39}$$

where

$$\rho_4(t) = e^{-(1+\gamma_4)s\widehat{\alpha}}s^{8a+7}\lambda^{32}\widehat{\varphi}^{8b+7}$$

for some γ_4 satisfying $0 < \gamma_4 < 2\gamma_3 - 1$. For the reasons stated above, it is clear that γ_4 can be assumed to satisfy $\widehat{\gamma} < \gamma_4 < 1$.

Choosing $\delta > 0$ small enough and replacing the estimates (1.38) and (1.39) in (1.36), we obtain:

$$I(s,\lambda;z) + I(s,\lambda;\phi) \le C\int_0^T\int_\omega \rho_4|z|^2\,dx\,dt, \tag{1.40}$$

for all large s and λ. Taking into account the definition of ρ_4, that $\gamma_4 > \widehat{\gamma}$, $a = 7$ and $b = 15/2$, we see that (1.18) holds.

This ends the proof of Theorem 1.3.1.

1.3.2 Proof of the observability inequality

Let us now give the proof of the observability inequality in Proposition 1.3.1, which relies on the above result. First, we observe that, by classical estimates for Stokes system, the following energy inequalities hold:

$$\int_{T/2}^T\int_\Omega |z|^2\,dx\,dt \le \widehat{C}\int_{T/2}^T\int_{\mathcal{O}} |\phi|^2\,dx\,dt \tag{1.41}$$

and

$$\|\phi(t+T/4)\|_{0,\Omega}^2 \le C\|\phi(t)\|_{0,\Omega}^2 \qquad \forall t \in (T/4, 3T/4)\,.$$

From the latter, we can easily deduce

$$\int_{T/2}^T \|\phi(t)\|_{0,\Omega}^2 dt \le C\int_{T/4}^{3T/4} \|\phi(t)\|_{0,\Omega}^2 dt \tag{1.42}$$

where $\widehat{C}$ and C are independent of ϕ.

Let us introduce the notation $\alpha_0 = e^{2\lambda\|\eta_0\|_\infty} - e^{\lambda\eta_0}$ and $\beta = 8b + 7$, where $b = 15/2$. We will now establish two technical results very useful to estimate the weights appearing in (1.18).

Lemma 1.3.3 *Set* $\widehat{\alpha}_0 = \min_{\overline{\Omega}} \alpha_0(x)$. *One has*

$$e^{-(1+\widehat{\gamma})s\widehat{\alpha}} t^{-4\beta}(T-t)^{-4\beta} \leq 2^{8\beta} T^{-8\beta} \exp(-C_0(1+\widehat{\gamma})sT^{-8}), \tag{1.43}$$

for all $s \geq s_3 = \max\{\widehat{s}, \beta T^8 (2^8(1+\widehat{\gamma})\widehat{\alpha}_0)^{-1}\}$, *where* C_0 *is a positive constant.*

Proof: Let us put

$$e^{-(1+\widehat{\gamma})s\widehat{\alpha}} t^{-4\beta}(T-t)^{-4\beta} = 1/f(t) \tag{1.44}$$

for all $t \in (0,T)$, with

$$f(t) = \exp\left(\frac{(1+\widehat{\gamma})s\widehat{\alpha}_0}{t^4(T-t)^4}\right) t^{4\beta}(T-t)^{4\beta} = \exp\left(\frac{(1+\widehat{\gamma})s\widehat{\alpha}_0}{\tau}\right)\tau^\beta = g(\tau), \tag{1.45}$$

where $\tau = t^4(T-t)^4 \in [0, T^8/2^8]$. We can verify that the minimum of $g(t)$ is achieved at $\widehat{\tau} = (1+\widehat{\gamma})s\widehat{\alpha}_0\beta^{-1}$. Notice that, $g(0) = \infty$ and g is decreasing for $\tau \in (0,\widehat{\tau})$ and increasing for $\tau > \widehat{\tau}$. Thus,

$$\min_{0\leq t\leq T} f(t) = \min_{0\leq\tau\leq T^8/2^8} g(t)$$
$$= \begin{cases} g(\widehat{\tau}) = e^\beta((1+\widehat{\gamma})s\widehat{\alpha}_0\beta^{-1})^\beta & \text{if } T^8/2^8 \geq (1+\widehat{\gamma})s\widehat{\alpha}_0\beta^{-1}, \\ g(T^8/2^8) = 2^{-8\beta}T^{8\beta}\exp(2^8(1+\widehat{\gamma})s\widehat{\alpha}_0 T^{-8}) & \text{if } T^8/2^8 \leq (1+\widehat{\gamma})s\widehat{\alpha}_0\beta^{-1}. \end{cases}$$

Hence, if $s \geq s_3 = \max\{s_2, \beta T^8(2^8(1+\widehat{\gamma})\widehat{\alpha}_0)^{-1}\}$, we have

$$\min_{0\leq t\leq T} f(t) \geq 2^{-8\beta}T^{8\beta}\exp(C_0(1+\widehat{\gamma})sT^{-8}) \quad \text{with} \quad C_0 = 2^8\widehat{\alpha}_0.$$

From (1.44), we deduce (1.43). □

Lemma 1.3.4 *Set* $\alpha_0^* = \max_{\overline{\Omega}} \alpha_0(x)$. *For every* $s \geq s_4 = \max(s_3, 3T^8/28\alpha_0^*)$, *one has*

$$e^{-2s\alpha}t^{-12}(T-t)^{-12} \geq A_s \exp(-Mt^{-4}) \qquad \forall x \in \Omega, \quad \forall t \in (0, T/2), \tag{1.46}$$

where $A_s = 2^{24}T^{-24}\exp(-256s\alpha_0^*T^{-8})$ *and* $M = 16s\alpha_0^*T^{-4}$.

Proof: Let us consider the following decomposition of $t^{-4}(T-t)^{-4}$:

$$\frac{1}{t^4(T-t)^4} = \frac{20}{T^7 t} + \frac{10}{T^6 t^2} + \frac{4}{T^5 t^3} + \frac{1}{T^4 t^4} + \frac{20}{T^7(T-t)} + \frac{10}{T^6(T-t)^2} + \frac{4}{T^5(T-t)^3} + \frac{1}{T^4(T-t)^4}.$$

Notice that we can estimate $e^{-2s\alpha}t^{-12}(T-t)^{-12}$ from below as follows:

$$\begin{aligned} e^{-2s\alpha}t^{-12}(T-t)^{-12} &\geq \exp\left(-2s\alpha_0^*\left(\frac{20}{T^7 t} + \frac{10}{T^6 t^2} + \frac{4}{T^5 t^3} + \frac{1}{T^4 t^4}\right)\right)t^{-12}H(t) \\ &\geq \exp(-16s\alpha_0^* T^{-4}t^{-4})t^{-12}H(t), \end{aligned}$$

where

$$H(t) = \exp\left(-2s\alpha_0^*\left(\frac{20}{T^7(T-t)} + \frac{10}{T^6(T-t)^2} + \frac{4}{T^5(T-t)^3} + \frac{1}{T^4(T-t)^4}\right)\right)(T-t)^{-12}.$$

Computing the first derivative of $H(t)$, we deduce that $H(t)' \leq 0$ if

$$s \geq \frac{3T^7(T-t)^4}{2\alpha_0^*(5(T-t)^3 + 5T(T-t)^2 + 3T^2(T-t) + T^3)}.$$

In particular, for $s \geq 3T^8/28\alpha_0^*$, the function H is decreasing in $(0, T/2)$. Therefore, we have

$$\begin{aligned} e^{-2s\alpha}t^{-12}(T-t)^{-12} &\geq \exp(-16s\alpha_0^* T^{-4}t^{-4})(T/2)^{-12}H(T/2) \\ &= A_s \exp(-Mt^{-4}), \end{aligned}$$

for all $t \in (0, T/2)$, with $A_s = 2^{24}T^{-24}\exp(-256s\alpha_0^* T^{-8})$ and $M = 16s\alpha_0^* T^{-4}$. This proves the lemma. □

It is easy to see from Lemma 1.3.3, Lemma 1.3.4 and the Carleman estimate (1.18) that

$$\int_0^{T/2}\int_\Omega \exp(-Mt^{-4})|z|^2\,dx\,dt \leq C_1\int_0^T\int_\omega |z|^2\,dx\,dt, \tag{1.47}$$

with $C_1 = CA_s^{-1}2^{8\beta}T^{-8\beta}\exp(-C_0(1+\widehat{\gamma})sT^{-8})$, for all $s \geq s_4$.

We also have the following estimate for ϕ in terms of z:

Proposition 1.3.2 *There exists a positive constant C_2 such that*

$$\int_{T/2}^T\int_\Omega |\phi|^2\,dx\,dt \leq C_2\int_0^T\int_\omega |z|^2\,dx\,dt.$$

Proof: Let us first recall the estimation (1.40). Using the fact that $e^{-2s\alpha}t^{-12}(T-t)^{-12}$ is bounded far from $t=0$ and $t=T$, in view of (1.42), we can estimate from below the following term of the left hand side of (1.40):

$$\int_0^T\int_\Omega e^{-2s\alpha}t^{-12}(T-t)^{-12}|\phi|^2\,dx\,dt \geq \int_{T/4}^{3T/4}\int_\Omega e^{-2s\alpha}t^{-12}(T-t)^{-12}|\phi|^2\,dx\,dt$$
$$\geq 2^{24}T^{-24}e^{-sKT^{-8}}\int_{T/4}^{3T/4}\int_\Omega |\phi|^2\,dx\,dt \geq C2^{24}T^{-24}e^{-sKT^{-8}}\int_{T/2}^{T}\int_\Omega |\phi|^2\,dx\,dt,$$

where $K=2^{17}3^{-4}\alpha_0^*(x)$.

On the other hand, using Lemma 1.3.3 for bounding upperly the right hand side of (1.40), we conclude

$$\int_{T/2}^{T}\int_\Omega |\phi|^2\,dx\,dt \leq C_2\int_{T/2}^{T}\int_\omega |z|^2\,dx\,dt,$$

where $C_2=2^{8\beta-24}T^{-8\beta+24}\exp((K-C_0(1+\widehat{\gamma}))sT^{-8})$ and for all $s\geq s_4$. □

Finally, we obtain the desired observability inequality (1.13) using (1.47), the energy estimate (1.41) and Proposition 1.3.2:

$$\begin{aligned}\int_0^T\int_\Omega \exp(-Mt^{-4})|z|^2\,dx\,dt &\leq \int_0^{T/2}\int_\Omega \exp(-Mt^{-4})|z|^2\,dx\,dt+\int_{T/2}^{T}\int_\Omega |z|^2\,dx\,dt\\ &\leq (C_1+\widehat{C}C_2)\int_0^T\int_\omega |z|^2\,dx\,dt,\end{aligned}$$

for all $s\geq s_4$.

1.4 Open problems and some comments

- Intersection of the sets $\mathcal{O}$ and ω.
  The geometrical hypothesis $\omega\cap\mathcal{O}\neq\emptyset$ is required in order to prove the existence of both, $\varepsilon-$insensitizing and insensitizing controls. In the first case, the hypothesis is used to proved a unique continuation property (Lemma 1.2.1). In the case of insensitizing controls, this hypothesis is used to prove an observability inequality. The problem is completely open when $\omega\cap\mathcal{O}=\emptyset$ (see [dT00]).

- Extension to other problems.
  As we mention before, in the insensitizing problem ((1.5)–(1.6)) the control h acts indirectly in the system with unknown (q,π) through the variable $\bar{u}$. This

problem is more difficult than the classical null controllability for Stokes-type equations. The main difficult to prove the null controllability of the cascade system (1.11)–(1.10) is to estimate $|\phi|$ in terms of $|z|$ in the right hand side of (1.22). Since the weight function multiplying ϕ is larger than that on the left hand side ($I(s, \lambda; \phi)$), the right hand side can not be absorbed as in the case of the heat equation (see [dT00]). In our case we have proved the observability inequality thanks to the presence of the Coriolis term. For Stokes equations the existence of insensitizing controls is still an open problem.

Chapter 2

A data assimilation problem

Abstract

Data assimilation is a useful methodology for estimating oceanic parameters [RL00]. The estimate of a quantity of interest via data assimilation involves the combination of observational data with a dynamical system model. This is a powerful methodology, which makes possible the estimation of values that sometimes are difficult to measure directly.

In this work we consider a non classical approach to data assimilation introduced in [Pue02]. We are interested in assimilating the satellite altimeter data into a quasi-geostrophic ocean model in order to recover the unknown initial condition which allows us to make predictions of ocean circulation. We present a numerical implementation of this method. We study the role that the size of the observation region plays for the recovery of the initial value as well as the time of data assimilation for a good prediction of future times.

Theoretically, to obtain an exact reconstruction of the initial value, we solve a null controllability problem. In particular, we deduce an observability result from a global Carleman inequality for the associated velocity - pressure formulation introduced in [FCGO03].

2.1 Introduction

A dynamical system model to approximate a physical system consists of a set of equations for each state variable of interest. In addition, we need the values of physical parameters (for example, coefficients of viscosity, diffusivity, etc.), forcing terms and initial and boundary conditions. In principle these values could be estimated directly from measurements. In practice, directly measuring the parameters of an ocean system is difficult because of sampling, technical and resource requirements.

The aim of data assimilation is to incorporate measured observations into a dynamical system model in order to derive accurate estimates of the current and future states of the system.

Data assimilation has been extensively used in meteorology for operational weather forecasting. On the other hand, the application of data assimilation into ocean models is just a few years old. For a review of the status of the subject we refer to [GMR91, Ben92].

One of the methodologies in use today is the variational method, based on optimal control theory. This method relies on the work by Lions [Lio71] and Marchuk [Mar75]. The idea of variational data assimilation is the following: We know "measurements" of the state on a time interval $(0, T_0)$, $T_0 > 0$, and we look for the initial value at $t = 0$, in order to compute the state on the time interval $(0, T_0 + T)$. Variational data assimilation method uses optimal control theory to minimize a suitable cost function (usually least square methods). This problem is known to be ill-posed but this can be partially avoided by adding a regularization term (for example Tikhonov regularization) (see also [CT87, LDT86, BLV98, LBV98, **?**, **?**]).

Recently in [Pue02], the author introduced an approach where we do not look for the value at $t = 0$ in order to predict the evolution of the system in $(0, T_0 + T)$. Instead, we look for the value at T_0, without need of the initial data at $t = 0$. The idea is to compute an approximation of the state during a period of time $(T_0, T_0 + T)$ using "measurements" of the state in some space region during a time interval $(0, T_0)$.

The purpose of this work is to apply this method to a quasi-geostrophic model. Theoretically, we obtain an exact reconstruction of the initial value at T_0 solving a null controllability problem. In particular, we deduce an observability result from a global Carleman inequality for the corresponding velocity - pressure formulation introduced in [FCGO03].

This chapter is organized as follows. In Section 2.2, we introduce the quasi-geostrophic ocean model used and the data assimilation problem. We first prove an observability inequality which is based on a global Carleman inequality for the corresponding velocity-pressure formulation of the system (2.13). In Section 2.3, we prove Theorem 2.2.1 which allows us to obtain an exact reconstruction of the initial value $\omega(T_0)$. In Section 2.4, we give an approximation algorithm which uses classical optimal control auxiliary problems and, finally, in Section 2.5 we implement this method and present several numerical experiments. We end with some comments and conclusions.

2.2 The ocean model and the data assimilation problem

Let Ω be a non-empty open bounded and simply-connected subset of $\mathbb{R}^2$, with boundary Γ of class $\mathcal{C}^2$. In this chapter, we will consider the linear quasi-geostrophic ocean model formulated in terms of the stream function $\psi(x,t)$ and the vorticity $\omega(x,t)$ at $(x,t) = (x_1, x_2, t) \in \mathbb{R}^2 \times \mathbb{R}_+$. Given $T_1 > 0$, the model is described by the following equations:

$$\begin{cases} R_o \dfrac{\partial \omega}{\partial t} - \epsilon_m \Delta\omega + \epsilon_s \omega - \dfrac{\partial \psi}{\partial x_1} = \operatorname{curl} \mathcal{T} \quad \text{in } \Omega \times (0, T_1), \\ \omega + \Delta\psi = 0 \quad \text{in } \Omega \times (0, T_1), \\ \psi = \dfrac{\partial \psi}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_1), \\ \omega(0) = \omega_0 \quad \text{in } \Omega, \end{cases} \tag{2.1}$$

where the forcing term corresponds to the wind stress $\mathcal{T}$ over the ocean surface. The coefficients R_o, ϵ_s and ϵ_m are the non-dimensional Rossby, Stommel and Munk numbers respectively:

$$R_o = \frac{U}{\beta L^2}, \quad \epsilon_m = \frac{A_H}{\beta L^3}, \quad \epsilon_s = \frac{\gamma}{\beta L}.$$

Here, U denotes a typical horizontal velocity, L is a representative horizontal length scale of ocean circulation, A_H is a constant horizontal eddy viscosity coefficient and γ is a bottom friction coefficient (see [MW95, BGdS01] for typical values). For a given $\omega_0 \in L^2(\Omega)$ and $\mathcal{T} \in L^2(0,T_1; L^2(\Omega)^2)$, the problem (2.1) possesses a unique solution (ψ, ω), with $\psi \in L^2(0,T_1; H_0^1(\Omega))$ and $\omega \in L^2(0,T_1; H^1(\Omega)) \cap H^1(0,T_1; H^{-1}(\Omega))$. This result is proved in Appendix A.

An application of data assimilation in oceanography is the insertion of the altimetry satellite data into the ocean models in order to recover streamlines. In the framework of quasi-geostrophy, the sea-surface height or dynamical topography is proportional to the stream function (ψ_{obs}) [BLV98, LBV98]:

$$\text{sea} - \text{surface height} = \frac{f_0}{g} \psi_{obs},$$

where f_0 is the Coriolis parameter evaluated at reference latitude $\tilde{\theta}_0$ and g is the gravity.

Let us consider the ocean model (2.1) where we do not impose any initial condition on the vorticity ω. We suppose that we know (for example, from altimetric satellite measurements) the stream function ψ_{obs} on a time interval $(0, T_0)$, with

$0 < T_0 < T_1$, and distributed in the observation region $\mathcal{O}$, which is a non-empty open subset of Ω, i.e.,

$$\psi_{obs} = \psi|_{\mathcal{O}\times(0,T_0)}.$$

Our aim is to reconstruct the value of the state at time T_0, i.e., $\omega(T_0)$. This value will be the initial condition for the interval (T_0, T_1), where we want to predict the circulation of the ocean. From [Pue02], the data assimilation problem consists of determining an approximation of the initial data at T_0 from the known "measurements" of the state in $\mathcal{O} \times (0, T_0)$.

For the reconstruction of $\omega(T_0)$ we will introduce a control problem for the following adjoint problem: For φ_0 in $L^2(\Omega)$ and h in $L^2(\mathcal{O} \times (0, T_0))$, let us consider the following problem:

$$\begin{cases} -R_o\dfrac{\partial z}{\partial t} - \epsilon_m \Delta z + \epsilon_s z + s = 0 \quad \text{in } \Omega \times (0, T_0), \\ \Delta s + \dfrac{\partial z}{\partial x_1} = h 1_{\mathcal{O}} \quad \text{in } \Omega \times (0, T_0), \\ z = \dfrac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ z(T_0) = \varphi_0 \quad \text{in } \Omega. \end{cases} \tag{2.2}$$

To simplify notation we denote $L^2(0, T_0; L^2(\Omega))$ by $L^2(\Omega\times(0, T_0))$ and $L^2(0, T_0; L^2(\mathcal{O}))$ by $L^2(\mathcal{O} \times (0, T_0))$.

For the existence and uniqueness of the solution of system (2.2), we have used the transposition method introduced in [LM68].

Definition 2.2.1 *For each $\varphi_0 \in L^2(\Omega)$ and $h \in L^2(\mathcal{O}\times(0, T_0))$ we say that (z, s, z_0) is a weak solution to Problem* (2.2) *if (z, s, z_0) belongs to $L^2(0, T_0; H_0^1(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$, and*

$$\langle f, z\rangle + \int_0^{T_0}\int_\Omega gs\,dx\,dt + \int_\Omega \theta_0 z_0\,dx = \int_0^{T_0}\int_{\mathcal{O}} h\phi\,dx\,dt + \int_\Omega \theta(T_0)\varphi_0\,dx, \tag{2.3}$$

for each $f \in L^2(0, T_0; H^{-1}(\Omega))$, $g \in L^2(\Omega \times (0, T_0))$ and $\theta_0 \in L^2(\Omega)$, where (ϕ, θ) is the solution of

$$\begin{cases} R_o\dfrac{\partial \theta}{\partial t} - \epsilon_m\Delta\theta + \epsilon_s\theta - \dfrac{\partial \phi}{\partial x_1} = f \qquad \text{in } \Omega \times (0, T_0), \\ \theta + \Delta\phi = g \qquad \text{in } \Omega \times (0, T_0), \\ \phi = \dfrac{\partial \phi}{\partial n} = 0 \qquad \text{on } \Gamma \times (0, T_0), \\ \theta(0) = \theta_0 \qquad \text{in } \Omega. \end{cases} \tag{2.4}$$

In (2.3), $\langle \cdot, \cdot \rangle$ denotes the duality pairing between $L^2(0, T_0; H^{-1}(\Omega))$ and $L^2(0, T_0; H_0^1(\Omega))$.

We can prove (using Appendix A in combination with the abstract formulation for the stationary case in Section III.1.1 of [GR86]) that Problem (2.4) has a unique solution (ϕ, θ) satisfying

$$\phi \in L^2(0, T_0; H_0^1(\Omega)), \qquad \theta \in L^2(0, T_0; H^1(\Omega)) \cap H^1(0, T_0; H^{-1}(\Omega)). \tag{2.5}$$

Using the interpolation theorem in [LM68], we obtain that $\theta \in \mathcal{C}([0, T_0]; L^2(\Omega))$. Moreover,

$$\|\phi\|_{L^2(0,T_0;H_0^1(\Omega))} + \|\theta\|_{\mathcal{C}([0,T_0];L^2(\Omega))} \leq C\{\|f\|_{L^2(0,T_0;H^{-1}(\Omega))} + \|g\|_{L^2(\Omega\times(0,T_0))} + \|\theta_0\|_{0,\Omega}\}. \tag{2.6}$$

Lemma 2.2.1 *For each $\varphi_0 \in L^2(\Omega)$ and $h \in L^2(\mathcal{O} \times (0, T_0))$, there exists a unique solution (z, s, z_0) in $L^2(0, T_0; H_0^1(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$.*

Proof: Given $\varphi_0 \in L^2(\Omega)$ and $h \in L^2(\mathcal{O} \times (0, T_0))$, let us define the functional on $l : L^2(0, T_0; H^{-1}(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$ by

$$l(f, g, \theta_0) := \int_0^{T_0} \int_{\mathcal{O}} h\phi \, dx \, dt + \int_\Omega \varphi_0 \theta(T_0) \, dx, \tag{2.7}$$

where (ϕ, θ) is the solution of Problem (2.4).

Since (ϕ, θ) satisfies (2.5), it follows that l is well-defined, and using (2.6) it is easy to prove that it defines a linear continuous functional on $L^2(0, T_0; H^{-1}(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$.

On the other hand, from (2.3),

$$l(f, g, \theta_0) = \langle f, z \rangle + \int_0^{T_0} \int_\Omega gs \, dx \, dt + \int_\Omega \theta_0 z_0 \, dx.$$

Since l defines a linear continuous functional on $L^2(0, T_0; H^{-1}(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$, there exists a unique (z, s, z_0) in $L^2(0, T_0; H_0^1(\Omega)) \times L^2(\Omega \times (0, T_0)) \times L^2(\Omega)$ such that

$$l(f, g, \theta_0) = \langle f, z \rangle + \int_0^{T_0} \int_\Omega gs \, dx \, dt + \int_\Omega \theta_0 z_0 \, dx \quad \forall f \in L^2(0, T_0; H^{-1}(\Omega)), \forall g \in L^2(\Omega \times (0, T_0)), \forall \theta_0 \in L^2(\Omega).$$

Thus, (z, s, z_0) is the solution of (2.2) in the very weak sense of (2.3) and (2.4), and satisfies

$$\|z\|_{L^2(0,T_0;H_0^1(\Omega))} + \|s\|_{L^2(\Omega\times(0,T_0))} + \|z_0\|_{0,\Omega} \leq \|h\|_{L^2(\Omega\times(0,T_0))} + \|\varphi_0\|_{L^2(\Omega)}. \tag{2.8}$$

□

Remark 2.2.1 *In this very weaker case, it is not clear the relationship between z_0 and $z(0)$, and therefore (2.4) and (2.2) are no longer completely equivalent. Then, each time we talk about $z(0)$, we really mean z_0.*

Now, we present the main result which gives us an exact reconstruction of $\omega(T_0)$.

Theorem 2.2.1 *For any $\mathcal{O} \subset \Omega$, $T_0 > 0$ and $\varphi_0 \in L^2(\Omega)$, there exists $h = h(\varphi_0)$ in $L^2(\mathcal{O} \times (0, T_0))$ such that the solution (z, s) of system (2.2), in the very weak sense of (2.3) and (2.4), satisfies*

$$z(0) = 0 \qquad \text{in } \ \Omega. \tag{2.9}$$

We then have, $\forall \varphi_0 \in L^2(\Omega)$,

$$(\omega(T_0), \varphi_0) = \frac{1}{R_o} \left\{ \int_0^{T_0} \int_\Omega \operatorname{curl} \mathcal{T}\, z(\varphi_0)\, dx\, dt - \int_0^{T_0} \int_{\mathcal{O}} \psi_{obs} h(\varphi_0)\, dx\, dt \right\}. \tag{2.10}$$

Moreover, there exists a positive constant C depending on Ω, $\mathcal{O}$ and T_0 such that

$$\|\omega(T_0)\|^2_{0,\Omega} \leq C \left\{ \int_0^{T_0} \int_\Omega |\operatorname{curl} \mathcal{T}|^2\, dx\, dt + \int_0^{T_0} \int_{\mathcal{O}} |\psi_{obs}|^2\, dx\, dt \right\}. \tag{2.11}$$

Equation (2.10) allows us to calculate the component of $\omega(T_0)$ on φ_0 for any $\varphi_0 \in L^2(\Omega)$ from the wind stress at $(0, T_0)$, the measurement ψ_{obs}, and the control $h(\varphi_0)$, which has to be computed. Taking φ_{0j} as the elements of a Hilbert basis of $L^2(\Omega)$, we can reconstruct $\omega(T_0)$ as

$$\omega(T_0) = \sum_j (\omega(T_0), \varphi_{0j}) \varphi_{0j}. \tag{2.12}$$

Hereafter $(\cdot, \cdot)$ denotes the inner product of $L^2(\Omega)$ or $L^2(\Omega)^2$, as appropriate.

The proof of Theorem 2.2.1 reduces to study the null controllability of the system (2.2). The inequality (2.11) will be obtained from an observability inequality that will be proved in the next section.

2.2.1 Proof of the observability inequality

In this section we will prove an observability inequality. This inequality will be the main tool in the proof of Theorem 2.2.1.

Let (ϕ, θ) be the solution of the following problem:

$$\left\{ \begin{array}{l} R_o \dfrac{\partial \theta}{\partial t} - \epsilon_m \Delta \theta + \epsilon_s \theta - \dfrac{\partial \phi}{\partial x_1} = 0 \qquad \text{in } \Omega \times (0, T_0), \\ \theta + \Delta \phi = 0 \qquad \text{in } \Omega \times (0, T_0), \\ \phi = \dfrac{\partial \phi}{\partial n} = 0 \qquad \text{on } \Gamma \times (0, T_0), \\ \theta(0) = \theta_0 \qquad \text{in } \Omega. \end{array} \right. \tag{2.13}$$

Proposition 2.2.1 *For any solution* (ϕ, θ) *of* (2.13)*, with* $\theta_0 \in L^2(\Omega)$*, there exists a positive constant* C*, depending only on* Ω*,* $\mathcal{O}$ *and* T*, such that*

$$\|\theta(T_0)\|^2_{0,\Omega} \leq C \int_0^{T_0} \int_{\mathcal{O}} |\phi|^2 \, dx \, dt. \tag{2.14}$$

The proof of Proposition 2.2.1 is based on the global Carleman inequality for an equivalent problem of (2.13) in terms of the original variables: the velocity $v(x,t)$ and the pressure $p(x,t)$ (see Theorem 2.2.2 below). For this problem, we will apply a global Carleman estimate given in [FCGO04] and, after some computation, we arrive at the observability inequality. In order to simplify the notation, we will assume that $R_o = 1$, $\epsilon_m = 1$ and $\epsilon_s = 1$.

Remark 2.2.2 *We can directly obtain a global Carleman estimate for* (2.13) *only in the case that we have homogeneous Dirichlet boundary conditions for both variables, i.e.,* $\phi = 0$ *and* $\theta = 0$ *on* $\Gamma \times (0, T_0)$ *(see [FÈ96]).*

Theorem 2.2.2 *The stream function - vorticity problem* (2.13) *with initial condition* $\theta(0) = \operatorname{curl} v_0$*, where* $v_0|_\Gamma = 0$ *and* $\operatorname{div} v_0 = 0$*, is equivalent to the following problem:*

$$\begin{cases} \dfrac{\partial v}{\partial t} - \Delta v + v + x_2 k \wedge v + \nabla p = 0 \quad \text{in } \Omega \times (0, T_0), \\ \operatorname{div} v = 0 \quad \text{in } \Omega \times (0, T_0), \\ v = 0 \quad \text{on } \Gamma \times (0, T_0), \\ v(0) = v_0 \quad \text{in } \Omega. \end{cases} \tag{2.15}$$

Proof: First, let us remark that problem 2.15 actually has a unique solution. Indeed, using the spaces H and V defined in Section 1.1.1 of Chapter 1, we have the following result: For given $v_0 \in H$, the problem (2.15) has a unique solution (v, p), with $v \in L^2(0, T; V) \cap H^1(0, T; V')$ and $p \in W^{-1,\infty}(0, T; L^2_0(\Omega))$ (see Appendix A).

Now, the proof of the equivalence is classical. Following [Qua93], let us assume that (ϕ, θ) is the solution of (2.13) with $\theta(0) = \operatorname{curl} v_0$, where v_0 satisfies $\operatorname{div} v_0 = 0$ and $v_0|_\Gamma = 0$. Let us consider, for $t > 0$, the vector field $u = \overrightarrow{\operatorname{curl}}\, \phi$. Clearly $\operatorname{div} u = \operatorname{div}(\overrightarrow{\operatorname{curl}}\, \phi) = 0$. Its curl satisfies

$$\operatorname{curl} u = \operatorname{curl}(\overrightarrow{\operatorname{curl}}\, \phi) = -\Delta\phi = \theta.$$

Replacing $\theta = \operatorname{curl} u$ in (2.13) and using the fact that $-\dfrac{\partial \phi}{\partial x_1} = u_2$, we have

$$\frac{\partial}{\partial t}(\operatorname{curl} u) - \Delta(\operatorname{curl} u) + \operatorname{curl} u + u_2 = 0.$$

Using that $\operatorname{curl}(x_2(k \wedge u)) = x_2 \operatorname{div} u + u_2 = u_2$, we obtain

$$\operatorname{curl}\left(\frac{\partial u}{\partial t} - \Delta u + u + x_2 k \wedge u\right) = 0.$$

Therefore, since Ω is simply-connected,

$$\frac{\partial u}{\partial t} - \Delta u + v + x_2 k \wedge u = \nabla q,$$

for some function q which is unique up to an additive constant. In order that (u, q) can be identified with the solution $(v, -p)$ of problem (2.15), it remains to show that u satisfies the same initial and boundary conditions of v.

For the boundary values, it happens that $u = \vec{\operatorname{curl}}\, \phi$ on $\Sigma_0 = \Gamma \times (0, T_0)$. After separating normal and tangential components, this gives

$$u|_{\Sigma_0} \cdot n = \vec{\operatorname{curl}}\, \phi|_{\Sigma_0} \cdot n = \left.\frac{\partial \phi}{\partial \tau}\right|_{\Sigma_0} = 0,$$
$$u|_{\Sigma_0} \cdot \tau = \vec{\operatorname{curl}}\, \phi|_{\Sigma_0} \cdot \tau = -\left.\frac{\partial \phi}{\partial n}\right|_{\Sigma_0} = 0,$$

where n and τ represent the outward unit normal and unit tangential vectors, respectively. Hence, $u|_{\Sigma_0} = 0$.

For the initial values, we have to determine the value attained by u as $t \to 0^+$. We have, using the assumption $\theta(0) = \operatorname{curl} v_0$,

$$\operatorname{curl} u(0) = \operatorname{curl}\left(\lim_{t \to 0^+} u(t)\right) = \lim_{t \to 0^+} \operatorname{curl} u(t) = \lim_{t \to 0^+} \theta(t) = \theta(0) = \operatorname{curl} v_0.$$

Then, $\operatorname{curl}(u(0) - v_0) = 0$. Using the same argument as before, it follows that

$$u(0) - v_0 = \nabla \beta.$$

Now, because of $\operatorname{div} u(0) = \operatorname{div} v_0 = 0$, we have that β is harmonic in Ω. Furthermore, taking the normal component in the above equation, we obtain

$$\frac{\partial \beta}{\partial n} = (u(0) - v_0)|_{\Gamma} \cdot n = 0.$$

Thus, $\beta = \textit{constant}$ and we conclude $u(x, 0) = v_0$.

The converse follows from the condition of incompressibility $\operatorname{div} v = 0$ and the boundary condition $v|_{\Sigma_0} = 0$ in a bounded, connected domain (see details in Appendix A).

□

Now, let us recall the Carleman estimate for (2.15) given in Appendix B (see also [FCGO03]).

Theorem 2.2.3 *Let $\mathcal{O}_1$ be a subset of Ω and let the functions α, φ, $\widehat{\alpha}$ and $\widehat{\varphi}$ be as in Chapter 1. For each $\gamma_1 \in (0,1)$, there exist positive constants s_1, λ_1 and C_1, depending on Ω, $\mathcal{O}_1$ and T such that one has*

$$I(s,\lambda;v) := \int_0^{T_0}\int_\Omega e^{-2s\alpha}\left(\frac{1}{s\varphi}(|v_t|^2+|\Delta v|^2)+s\lambda^2\varphi|\nabla v|^2+s^3\lambda^4\varphi^3|v|^2\right)dx\,dt$$
$$\leq C_1\int_0^{T_0}\int_{\mathcal{O}_1} e^{-(1+\gamma_1)s\widehat{\alpha}}s^7\lambda^4\widehat{\varphi}^{15/2}|v|^2\,dx\,dt, \qquad (2.16)$$

for any $s > s_1$ and $\lambda > \lambda_1$ and for every solution (v,p) of (2.15) *associated with an initial data $v_0 \in H$.*

Proof of Proposition 2.2.1 We will divide the proof into two steps.
Step 1: Let us first obtain the left hand side of the observability inequality (2.14). To this end, we introduce $\tilde{v}(x,t) = \gamma(t)v(x,t)$ and $\tilde{p}(x,t) = \gamma(t)p(x,t)$, where γ is a regular function satisfying:

$$\gamma(t) = 1 \text{ in } (3T_0/4, T_0), \quad 1 \geq \gamma(t) \geq 0 \text{ in } (T_0/2, 3T_0/4), \quad \gamma(t) = 0 \text{ in } (0, T_0/2).$$

It is easy to check that $(\tilde{v},\tilde{p})$ verifies

$$\begin{cases} \tilde{v}_t - \Delta\tilde{v} + \tilde{v} + x_2 k\wedge\tilde{v} + \nabla\tilde{p} = -\gamma_t v & \text{in } \Omega\times(0,T_0), \\ \operatorname{div}\tilde{v} = 0 & \text{in } \Omega\times(0,T_0), \\ \tilde{v} = 0 & \text{on } \Gamma\times(0,T_0), \\ \tilde{v}(0) = 0 & \text{in } \Omega, \end{cases} \qquad (2.17)$$

and there exists a unique solution $(\tilde{v},\tilde{p})$, with $\tilde{v} \in L^2(0,T;V)\cap H^1(0,T;V')$ and $\tilde{p} \in W^{-1,\infty}(0,T;L^2_0(\Omega))$ (see Appendix A). Moreover, one has

$$\|\nabla\tilde{v}\|^2_{L^2(0,T_0;L^2(\Omega))} \leq C\int_{T_0/2}^{3T_0/4}\int_\Omega |v|^2\,dx\,dt. \qquad (2.18)$$

Now, we derive (2.17) with respect to the time variable t and introduce $\tilde{u} = \tilde{v}_t$ and $\tilde{\pi} = \tilde{p}_t$ such that

$$\begin{cases} \tilde{u}_t - \Delta\tilde{u} + \tilde{u} + x_2 k\wedge\tilde{u} + \nabla\tilde{\pi} = -\gamma_t v_t - \gamma_{tt} v & \text{in } \Omega\times(0,T_0), \\ \operatorname{div}\tilde{u} = 0 & \text{in } \Omega\times(0,T_0), \\ \tilde{u} = 0 & \text{on } \Gamma\times(0,T_0), \\ \tilde{u}(0) = 0 & \text{in } \Omega. \end{cases} \qquad (2.19)$$

This problem also has unique solution with $\tilde{u} = \tilde{v}_t \in L^2(0,T;V)\cap H^1(0,T;V')$. Notice that, the functions γ_t and γ_{tt} are defined over $(T_0/2, 3T_0/4)$, then we can use the Carleman estimate (2.16) to deduce that the right hand side of the first equation in (2.19) belongs to $L^2(0,T_0;L^2(\Omega))$. Thus,

$$\begin{aligned}\|\nabla\tilde{u}\|^2_{L^2(0,T_0;L^2(\Omega))} = \|\nabla\tilde{v}_t\|^2_{L^2(0,T_0;L^2(\Omega))} &\le C\int_0^{T_0}\int_\Omega \left(|\gamma_t v_t|^2 + |\gamma_{tt}v|^2\right)\,dx\,dt\\ &\le C\int_{T_0/2}^{3T_0/4}\int_\Omega \left(|v_t|^2+|v|^2\right)\,dx\,dt. \qquad (2.20)\end{aligned}$$

We have obtained that $\nabla\tilde{v}\in L^2(0,T_0;L^2(\Omega))$ and $\nabla\tilde{v}_t\in L^2(0,T_0;L^2(\Omega))$, in particular $\nabla\tilde{v}_t\in L^2(0,T_0;H^{-1}(\Omega))$, then $\nabla\tilde{v}\in\mathcal{C}([0,T_0];L^2(\Omega))$. It follows from (2.18) and (2.20) that,

$$\begin{aligned}\|\nabla v(T_0)\|^2_{0,\Omega} = \|\nabla\tilde{v}(T_0)\|^2_{0,\Omega} = \|\nabla\tilde{v}(T_0/2)\|^2_{0,\Omega} + 2\int_{T_0/2}^{T_0}\int_\Omega |\nabla\tilde{v}||\nabla\tilde{v}_t|\,dx\,dt\\ \le C\int_{T_0/2}^{3T_0/4}\int_\Omega\left(|v_t|^2+|v|^2\right)\,dx\,dt.\end{aligned}$$

Following the same steps as in the proof of Proposition 1.3.2 in Chapter 1, we obtain

$$\begin{aligned}\|\nabla v(T_0)\|^2_{0,\Omega} &\le C\int_{T_0/2}^{3T_0/4} e^{-2s\alpha^*}\left(\frac{1}{\widehat{\varphi}}\|v_t\|^2_{0,\Omega} + \varphi^{*3}\|v\|^2_{0,\Omega}\right)dt\\ &\le C\int_0^{T_0}\int_\Omega e^{-2s\alpha}\left(\frac{1}{\varphi}|v_t|^2+\varphi^3|v|^2\right)\,dx\,dt,\end{aligned}$$

where α^*, φ^*, $\widehat{\varphi}$, α and φ were defined in Section 1.3.1. Observe that the last integrals are bounded with $I(s,\lambda;v)$, hence

$$\|\nabla v(T_0)\|^2_{0,\Omega}\le CI(s,\lambda;v)$$

On the other hand, since $\operatorname{div} v = 0$ and $v=0$ on $\Gamma\times(0,T_0)$,

$$\begin{aligned}\int_\Omega |\nabla v(T_0)|^2\,dx &= -\int_\Omega \Delta v(T_0)\cdot v(T_0)\,dx = \int_\Omega \vec{\operatorname{curl}}(\operatorname{curl} v(T_0))\cdot v(T_0)\,dx\\ &= \int_\Omega |\operatorname{curl} v(T_0)|^2\,dx = \int_\Omega |\theta(T_0)|^2\,dx,\end{aligned}$$

here we have used the fact that $\operatorname{curl} v(t)=\theta(t)$. Then, we deduce the left hand side of (2.14)

$$\|\theta(T_0)\|^2_{0,\Omega}\le CI(s,\lambda;v), \qquad (2.21)$$

where C depends on Ω, $\mathcal{O}$ and T_0.

Step 2: In order to obtain the upper bound in (2.14), we will use the Carleman estimate (2.16), with $v = \vec{\text{curl}}\,\phi$,

$$\begin{aligned} I(s,\lambda;v) &\le C\int_0^{T_0}\int_{\mathcal{O}_1} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}|v|^2\,dx\,dt \\ &= C\int_0^{T_0}\int_{\mathcal{O}_1} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}|\vec{\text{curl}}\,\phi|^2\,dx\,dt \\ &\le C\int_0^{T_0}\int_{\mathcal{O}} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}\xi|\vec{\text{curl}}\,\phi|^2\,dx\,dt, \end{aligned}$$

where in the later, we consider a function $\xi \in \mathcal{C}_0^\infty(\mathcal{O})$, with $\mathcal{O}_1 \subset\subset \mathcal{O}$, satisfying

$$0 \le \xi \le 1, \quad \xi(x) = 1 \text{ in } \mathcal{O}_1.$$

Integrating by parts over the space variable, applying Young's inequality and taking into account that $v = \vec{\text{curl}}\,\phi$, we obtain

$$\begin{aligned} I(s,\lambda;v) &\le C\int_0^{T_0}\int_{\mathcal{O}} e^{-(1+\gamma_1)s\widehat{\alpha}} s^7\lambda^4\widehat{\varphi}^{15/2}(\vec{\text{curl}}\,\xi\cdot\vec{\text{curl}}\,\phi\,\phi + \xi\,\text{curl}(\vec{\text{curl}}\,\phi)\,\phi)\,dx\,dt \\ &\le \frac{\delta}{2}\int_0^{T_0}\int_\Omega e^{-2s\alpha}s^3\lambda^4\varphi^3|v|^2\,dx\,dt + \frac{\delta}{2}\int_0^{T_0}\int_\Omega e^{-2s\alpha^*}s\lambda^2\varphi^*|\,\text{curl}\,v|^2\,dx\,dt \\ &\quad + C_\delta\int_0^{T_0}\int_{\mathcal{O}} e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}s^{13}\lambda^6\widehat{\varphi}^{14}|\phi|^2\,dx\,dt \\ &\le \frac{\delta}{2}\int_0^{T_0}\int_\Omega e^{-2s\alpha}(s^3\lambda^4\varphi^3|v|^2 + s\lambda^2\varphi|\nabla v|^2)\,dx\,dt \\ &\quad + C_\delta\int_0^{T_0}\int_{\mathcal{O}} e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}s^{13}\lambda^6\widehat{\varphi}^{14}|\phi|^2\,dx\,dt. \end{aligned} \tag{2.22}$$

Arguing as in Section 1.3.1, we introduce γ_2 with $0 < \gamma_2 < 2\gamma_1 - 1$. Then $(1+2\gamma_1-\gamma_2)/2 > 1$ and, from Lemma 1.3.1 of Section 1.3.1, we see that $(1+2\gamma_1-\gamma_2)\widehat{\alpha}/2 > \alpha^*$ for λ sufficiently large. Consequently, it can be assumed that

$$-2(1+\gamma_1)\widehat{\alpha} + 2\alpha^* < -(1+\gamma_2)\widehat{\alpha}$$

and we can replace $e^{-2(1+\gamma_1)s\widehat{\alpha}+2s\alpha^*}$ by $e^{-(1+\gamma_2)s\widehat{\alpha}}$ in the last integral in (2.22).

By choosing δ sufficiently small, we can absorb the first two terms in the right hand side of (2.22) with $I(s,\lambda;v)$, hence

$$I(s,\lambda;v) \le C\int_0^{T_0}\int_{\mathcal{O}} e^{-(1+\gamma_2)s\widehat{\alpha}}s^{13}\lambda^6\widehat{\varphi}^{14}|\phi|^2\,dx\,dt.$$

Since the weight $e^{-(1+\gamma_2)s\widehat{\alpha}}\widehat{\varphi}^{14}$ is bounded (see Lemma 1.3.3 of Section 1.3.2), we have

$$I(s,\lambda;v) \leq C\int_0^{T_0}\int_{\mathcal{O}} |\phi|^2\,dx\,dt, \tag{2.23}$$

where C depends on Ω, $\mathcal{O}$ and T_0. Combining (2.23) and (2.21), we finally obtain the observability estimate:

$$\|\theta(T_0)\|_{0,\Omega}^2 \leq C\int_0^{T_0}\int_{\Omega} |\phi|^2\,dx\,dt.$$

□

Remark 2.2.3 *The uniqueness continuation property for the system* (2.13) *can be deduced from* (2.23)*. Indeed, using similar arguments as before, we have*

$$\begin{aligned}\int_0^{T_0}\int_{\mathcal{O}} |\phi|^2\,dx\,dt &\geq \tilde{C}I(s,\lambda;v) \geq \tilde{C}\int_0^{T_0}\int_{\Omega} e^{-2s\alpha^*}\varphi^*|\operatorname{curl} v|^2\,dx\,dt\\ &= \tilde{C}\int_0^{T_0}\int_{\Omega} e^{-2s\alpha^*}\varphi^*|\theta|^2\,dx\,dt.\end{aligned}$$

If $\phi = 0$ in $\mathcal{O}\times(0,T_0)$ we necessarily have $\theta \equiv 0$ in $\Omega\times(0,T_0)$. On the other hand, from the second equation in (2.13)*, ϕ satisfies*

$$\left\{\begin{array}{l}\Delta\phi(t) = 0 \quad \text{in } \Omega,\ t\in(0,T_0),\\ \phi = \dfrac{\partial\phi}{\partial n} = 0 \quad \text{on } \Gamma,\ t\in(0,T_0).\end{array}\right.$$

It follows also that $\phi \equiv 0$ in $\Omega\times(0,T_0)$.

2.3 Proof of Theorem 2.2.1

This section is devoted to proving Theorem 2.2.1. This proof is based on the observability result proved in Proposition 2.2.1 and the uniqueness continuation property given in Remark 2.2.3. We argue as follows.

The approximate control h of minimal norm in $L^2(\mathcal{O}\times(0,T_0))$ corresponding to $\varphi_0 \in L_0^2(\Omega)$ and $\varepsilon > 0$ can be obtained by minimizing the following convex functional in $L^2(\Omega)$ [FPZ95, Lio92a]:

$$J_\varepsilon(\theta_0) = \frac{1}{2}\int_0^{T_0}\int_{\mathcal{O}} |\phi|^2\,dx\,dt + \varepsilon\|\theta_0\|_{0,\Omega} + \int_{\Omega}\theta(T_0)\varphi_0\,dx\,dt. \tag{2.24}$$

Thus, if the minimum of J_ε in $L^2(\Omega)$ is attained at $\widehat{\theta}_{0\varepsilon}$ and we denote by $(\widehat{\phi}_\varepsilon, \widehat{\theta}_\varepsilon)$ the solution to (2.13) with $\theta_0 = \widehat{\theta}_{0\varepsilon}$, then the control

$$h_\varepsilon = \widehat{\phi}_\varepsilon 1_{\mathcal{O}} \tag{2.25}$$

is such that the associated solution $(z_\varepsilon, s_\varepsilon)$ to (2.2) satisfies $\|z_\varepsilon(0)\|_{0,\Omega} \leq \varepsilon$.

From (2.2) and (2.13) we deduce that

$$\int_\Omega \theta(T_0)\varphi_0 \, dx = -\int_0^{T_0}\int_{\mathcal{O}} h(\varphi_0)\phi \, dx \, dt,$$

hence, the functional (2.24) may be rewritten as

$$J_\varepsilon(\theta_0) = \frac{1}{2}\int_0^{T_0}\int_{\mathcal{O}} |\phi|^2 \, dx \, dt + \varepsilon\|\theta_0\|_{0,\Omega} - \int_0^{T_0}\int_{\mathcal{O}} h(\varphi_0)\phi \, dx \, dt.$$

In what follows we verify the uniqueness of the minimum of J_ε.

Lemma 2.3.1 *For $\varepsilon > 0$, the functional J_ε defined in* (2.24) *is continuous, strictly convex and satisfies*

$$\liminf_{\|\theta_0\|_{0,\Omega}\to\infty} \frac{J_\varepsilon(\theta_0)}{\|\theta_0\|_{0,\Omega}} \geq \varepsilon. \tag{2.26}$$

Proof: The proof of this inequality is classical (see [FPZ95]). For completeness we include it here. J_ε is strictly convex since it is the sum of convex and strictly convex terms. To see that J_ε is continuous, we only need to recall the continuity property of system (2.13) (see Appendix A),

$$\|\phi\|_{L^2(0,T_0;H_0^1(\Omega))} + \|\theta\|_{\mathcal{C}([0,T_0];H^1(\Omega))} \leq C\|\theta_0\|_{0,\Omega}. \tag{2.27}$$

In order to prove (2.26), we take a sequence $\{\theta_0^n\}_{n\leq 1}$, with $\|\theta_0^n\|_{0,\Omega} \to +\infty$ and we denote the solution of (2.13) associated with θ_0^n by (ϕ, θ). If we define

$$\tilde{\phi}^n = \frac{\phi^n}{\|\theta_0^n\|_{0,\Omega}}, \quad \tilde{\theta}^n = \frac{\theta^n}{\|\theta_0^n\|_{0,\Omega}} \quad \text{and} \quad \tilde{\theta}_0^n = \frac{\theta_0^n}{\|\theta_0^n\|_{0,\Omega}},$$

then from the continuity result (2.27), we deduce that $(\tilde{\phi}^n, \tilde{\theta}^n)$ is bounded in

$$L^2(0,T_0;H_0^1(\Omega)) \times L^2(0,T_0;H^1(\Omega)).$$

Since $\|\tilde{\theta}_0^n\|_{0,\Omega} = 1$, $\tilde{\theta}_0^n$ is also bounded in $L^2(\Omega)$. Then we can extract subsequences (still denoted by $\tilde{\phi}^n$, $\tilde{\theta}^n$ and $\tilde{\theta}_0^n$), such that

$$\begin{aligned}
\tilde{\phi}^n &\rightharpoonup \tilde{\phi} \quad \text{weakly in} \quad L^2(0,T_0;H_0^1(\Omega)),\\
\tilde{\theta}^n &\rightharpoonup \tilde{\theta} \quad \text{weakly in} \quad L^2(0,T_0;H^1(\Omega)),\\
\tilde{\theta}_0^n &\rightharpoonup \tilde{\theta}_0 \quad \text{weakly in} \quad L^2(\Omega).
\end{aligned}$$

On the other hand,

$$\frac{J_\varepsilon(\theta_0^n)}{\|\theta_0^n\|_{0,\Omega}} = \frac{1}{2}\|\theta_0^n\|_{0,\Omega}\int_0^{T_0}\int_{\mathcal{O}} |\tilde{\phi}^n|^2\,dx\,dt + \varepsilon - \int_0^{T_0}\int_{\mathcal{O}} h\tilde{\phi}^n\,dx\,dt. \tag{2.28}$$

We have two possibilities:

1. If $\int_0^{T_0}\int_\Omega |\tilde{\phi}^n|^2\,dx\,dt > 0$. Since $\|\theta_0^n\|_{0,\Omega} \to \infty$, the inequality (2.28) implies that
$$\liminf_n \frac{J_\varepsilon(\theta_0^n)}{\|\theta_0^n\|_{0,\Omega}} \to +\infty \geq \varepsilon.$$

2. If $\int_0^{T_0}\int_{\mathcal{O}} |\tilde{\phi}^n|^2\,dx\,dt = 0$. By using the continuity and convexity property of $\|\cdot\|^2_{L^2(\mathcal{O}\times(0,T_0))}$ and the weak-convergence of $\tilde{\phi}^n$, we can deduce that
$$\int_0^{T_0}\int_{\mathcal{O}} |\tilde{\phi}|^2\,dx\,dt \leq \liminf_n \int_0^{T_0}\int_{\mathcal{O}} |\tilde{\phi}^n|^2\,dx\,dt = 0.$$
Then $\tilde{\phi} = 0$ in $\mathcal{O}\times(0,T_0)$. We now use the uniqueness continuation result that was proved in Remark 2.2.3 to imply that $\tilde{\phi} = 0$ in $\Omega\times(0,T_0)$. Therefore,
$$\tilde{\phi}^n \rightharpoonup 0 \quad \text{weakly in } L^2(0,T_0;H_0^1(\Omega)).$$
Using this fact, from (2.28) we obtain
$$\liminf_n \frac{J_\varepsilon(\theta_0^n)}{\|\theta_0^n\|_{0,\Omega}} = \frac{1}{2}\liminf_n \|\theta_0^n\|_{0,\Omega}\int_0^{T_0}\int_{\mathcal{O}} |\tilde{\phi}^n|^2\,dx\,dt + \varepsilon - 0 \geq \varepsilon$$
and we conclude that (2.26) holds.

□

An immediate consequence of this lemma, we know that $\forall\varepsilon > 0$, the functional J_ε has a unique minimum at $\widehat{\theta}_{0\varepsilon}$ and the following optimality condition must be satisfied:

$$\int_0^{T_0}\int_{\mathcal{O}} |\widehat{\phi}_\varepsilon|^2\,dx\,dt + \varepsilon\|\widehat{\theta}_{0\varepsilon}\|_{0,\Omega} + \int_\Omega \widehat{\theta}_\varepsilon(T_0)\varphi_0\,dx = 0. \tag{2.29}$$

Next, applying Young's inequality to (2.29) and using (2.14), we obtain (for $a > 0$)

$$\begin{aligned}\int_0^{T_0}\int_{\mathcal{O}} |\widehat{\phi}_\varepsilon|^2\,dx\,dt + \varepsilon\|\widehat{\theta}_{0\varepsilon}\|_{0,\Omega} &\leq \frac{a^2}{2}\|\widehat{\theta}_\varepsilon(T_0)\|^2_{0,\Omega} + \frac{1}{2a^2}\|\varphi_0\|^2_{0,\Omega}\\ &\leq \frac{Ca^2}{2}\int_0^{T_0}\int_{\mathcal{O}} |\widehat{\phi}_\varepsilon|^2\,dx\,dt + \frac{1}{2a^2}\|\varphi_0\|^2_{0,\Omega}.\end{aligned}$$

Choosing $a^2 = 1/C$ and taking into account that $h_\varepsilon = \widehat{\phi}_\varepsilon 1_{\mathcal{O}}$, we deduce that

$$\int_0^{T_0}\int_{\mathcal{O}} |h_\varepsilon|^2 \, dx\, dt \leq C\|\varphi_0\|^2_{0,\Omega}. \tag{2.30}$$

Since h_ε is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$, and from (2.8), $(z_\varepsilon, s_\varepsilon, z_\varepsilon(0))$ is uniformly bounded in $L^2(0,T_0;H_0^1(\Omega))\times L^2(\Omega\times(0,T_0))\times L^2(\Omega)$, we can extract a subsequence $\{h_{\varepsilon_n}\}$ and $(z_{\varepsilon_n}, s_{\varepsilon_n}, z_{\varepsilon_n}(0))$, with $\varepsilon_n \to 0$, such that with

$$\begin{aligned}
h_{\varepsilon_n} &\rightharpoonup h \quad \text{weakly in} \quad L^2(\mathcal{O}\times(0,T_0)),\\
z_{\varepsilon_n} &\rightharpoonup z \quad \text{weakly in} \quad L^2(0,T_0;H_0^1(\Omega)),\\
s_{\varepsilon_n} &\rightharpoonup s \quad \text{weakly in} \quad L^2(\Omega\times(0,T_0)),\\
z_{\varepsilon_n}(0) &\rightharpoonup z(0) \quad \text{weakly in} \quad L^2(\Omega),
\end{aligned}$$

as $n\to+\infty$. Here we have denoted by $(z_{\varepsilon_n}, s_{\varepsilon_n})$ and (z,s) the solutions to (2.2) associated with h_{ε_n} and h, respectively. Notice that $\|z_{\varepsilon_n}(0)\|_{0,\Omega}\leq \varepsilon_n$ for all $n\geq 1$. Consequently, we have $z(0)=0$.

In order to obtain (2.10), we use (2.2) and (2.13) and the fact that $z(0)=0$ in Ω. To deduce (2.11), we use (2.8), (2.10) and (2.30) as follows:

$$\begin{aligned}
(\omega(T_0),\varphi_0) &= \left\{\int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z(\varphi_0)\,dx\,dt - \int_0^{T_0}\int_{\mathcal{O}} \psi_{obs} h(\varphi_0)\,dx\,dt\right\}\\
&\leq C\|\varphi_0\|_{0,\Omega}\left(\int_0^{T_0}\int_\Omega |\operatorname{curl}\mathcal{T}|^2\,dx\,dt + \int_0^{T_0}\int_{\mathcal{O}} |\psi_{obs}|^2\,dx\,dt\right)^{1/2}.
\end{aligned}$$

Then, it follows that

$$\|\omega(T_0)\|^2_{0,\Omega} \leq C\left\{\int_0^{T_0}\int_\Omega |\operatorname{curl}\mathcal{T}|^2\,dx\,dt + \int_0^{T_0}\int_{\mathcal{O}} |\psi_{obs}|^2\,dx\,dt\right\}.$$

This concludes the proof of Theorem 2.2.1.

2.4 Optimal control problem

In this section we give another method to prove approximate controllability which is useful for numerical purposes. This method uses an optimal control problem. We will be able to characterize the control of minimal norm in $L^2(Q)$ by an optimality system and then we will present the time-space discretization of this system.

Let us consider the following optimal control problem for fixed $\varphi_0\in L^2(\Omega)$. Let (z,s) be the solution of (2.2) and, for $\alpha>0$, let us define

$$J_\alpha(h) = \frac{1}{2\alpha}\int_\Omega |z(0)|^2\;dx + \int_0^{T_0}\int_\Omega |h|^2\;dxdt,$$

where we have penalized the exact final condition (2.9). We look for $h_\alpha \in L^2(\Omega \times (0,T_0))$ such that

$$J_\alpha(h_\alpha) = \min_{h\in L^2(\Omega\times(0,T_0))} J_\alpha(h). \tag{2.31}$$

Theorem 2.4.1 1. *For every $\alpha > 0$, there exists a unique solution h_α to* (2.31) *and h_α is characterized by the following optimality system:*

$$\begin{cases} -\dfrac{\partial z_\alpha}{\partial t} - \Delta z_\alpha + z_\alpha + s_\alpha = 0 \quad \text{in } \Omega\times(0,T_0), \\ \Delta s_\alpha + \dfrac{\partial z_\alpha}{\partial x_1} = h_\alpha 1_{\mathcal{O}} \quad \text{in } \Omega\times(0,T_0), \\ z_\alpha = \dfrac{\partial z_\alpha}{\partial n} = 0 \quad \text{on } \Gamma\times(0,T_0), \\ z_\alpha(T_0) = \varphi_0 \quad \text{in } \Omega, \end{cases} \tag{2.32}$$

$$\begin{cases} \dfrac{\partial \theta_\alpha}{\partial t} - \Delta\theta_\alpha + \theta_\alpha - \dfrac{\partial \phi_\alpha}{\partial x_1} = 0 \quad \text{in } \Omega\times(0,T_0), \\ \theta_\alpha + \Delta\phi_\alpha = 0 \quad \text{in } \Omega\times(0,T_0), \\ \phi_\alpha = \dfrac{\partial \phi_\alpha}{\partial n} = 0 \quad \text{on } \Gamma\times(0,T_0) \\ \theta_\alpha(0) = -\dfrac{1}{\alpha} z_\alpha(0) \quad \text{in } \Omega, \end{cases} \tag{2.33}$$

$$h_\alpha = \phi_\alpha 1_{\mathcal{O}} \quad \text{in } \Omega\times(0,T_0). \tag{2.34}$$

2. *When α tends to zero, we have*

$$\left\{ \int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z_\alpha(\varphi_0)\,dx\,dt - \int_0^{T_0}\int_\Omega \psi_{obs} h_\alpha(\varphi_0)\,dx\,dt \right\} \to (\omega(T_0),\varphi_0). \tag{2.35}$$

Proof:

1. It follows from [Lio71] that problem (2.31) has a unique solution h_α which is characterized by the optimality system (2.32)–(2.33).

2. Combining (2.32) and (2.33) and taking into account that $h_\alpha = \phi_\alpha 1_{\mathcal{O}}$, we obtain the following optimality condition

$$\int_0^{T_0}\int_{\mathcal{O}} |h_\alpha|^2\,dx\,dt + \frac{1}{\alpha}\|z_\alpha(0)\|^2_{0,\Omega} = -\int_\Omega \theta_\alpha(T_0)\varphi_0\,dx\,dt. \tag{2.36}$$

Applying Young's inequality in (2.36) and using the observability result (2.14) we are lead to

$$\begin{aligned} \int_0^{T_0}\int_{\mathcal{O}} |h_\alpha|^2\,dx\,dt + \frac{1}{\alpha}\|z_\alpha(0)\|^2_{0,\Omega} &\le \frac{a^2}{2}\|\theta_\alpha(T_0)\|^2_{0,\Omega} + \frac{1}{2a^2}\|\varphi_0\|^2_{0,\Omega} \\ &\le \frac{Ca^2}{2}\int_0^{T_0}\int_{\mathcal{O}} |\phi_\alpha|^2\,dx\,dt + \frac{1}{2a^2}\|\varphi_0\|^2_{0,\Omega}, \end{aligned}$$

for $a > 0$. Choosing $a^2 = 1/C$, we can deduce that

$$\int_0^{T_0}\int_{\mathcal{O}} |h_\alpha|^2\,dx\,dt + \frac{2}{\alpha}\|z_\alpha(0)\|^2_{0,\Omega} \le C\|\varphi_0\|^2_{0,\Omega}. \tag{2.37}$$

Since h_α is uniformly bounded in $L^2(\mathcal{O}\times(0,T))$, and from (2.8), $(z_\alpha, s_\alpha, z_\alpha(0))$ is uniformly bounded in $L^2(0,T_0;H_0^1(\Omega)) \times L^2(\Omega\times(0,T_0)) \times L^2(\Omega)$, we can extract subsequences $\{h_{\alpha_n}\}$ and $(z_{\alpha_n}, s_{\alpha_n}, z_{\alpha_n}(0))$, with $\alpha_n \to 0$, such that

$$\begin{aligned}
h_{\alpha_n} &\rightharpoonup h \quad \text{weakly in} \quad L^2(\mathcal{O}\times(0,T_0)),\\
z_{\alpha_n} &\rightharpoonup z \quad \text{weakly in} \quad L^2(0,T_0;H_0^1(\Omega)),\\
s_{\alpha_n} &\rightharpoonup s \quad \text{weakly in} \quad L^2(\Omega\times(0,T_0)),
\end{aligned}$$

and

$$z_{\alpha_n}(0) \rightharpoonup z(0) \quad \text{weakly in} \quad L^2(\Omega), \tag{2.38}$$

as $n \to +\infty$. Here we have denoted by $(z_{\alpha_n}, s_{\alpha_n})$ and (z, s) the solutions of (2.2) associated with h_{α_n} and h respectively. From (2.37) and (2.38), we deduce that $z(0) = 0$ in $L^2(\Omega)$. Hence, we conclude, for fixed φ_0,

$$\begin{aligned}
&\int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z_{\alpha_n}\,dx\,dt - \int_0^{T_0}\int_{\mathcal{O}} h_{\alpha_n}\psi_{obs}\,dx\,dt \to\\
&\qquad \int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z\,dx\,dt - \int_0^{T_0}\int_{\mathcal{O}} h\psi_{obs}\,dx\,dt = (\omega(T_0),\varphi_0).
\end{aligned}$$

Hence, the standard arguments allow us to conclude (2.35) as $\alpha \to 0$.

□

Remark 2.4.1 *For fixed $\varphi_0 \in L^2(\Omega)$, combing (2.1) and (2.32), with $\alpha > 0$, we deduce that*

$$(\omega(T_0),\varphi_0) = \int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z_\alpha(\varphi_0)\,dx\,dt - \int_0^{T_0}\int_{\mathcal{O}} \psi_{obs}h_\alpha(\varphi_0)\,dx\,dt + (\omega(0), z_\alpha(0)).$$

From (2.37) we obtain an estimate for $\|z_\alpha(0)\|_{0,\Omega}$. Then, we have an approximation of the projection $(\omega(T_0),\varphi_0)$ with an order of convergence $O(\sqrt{\alpha})$:

$$|(\omega(T_0),\varphi_0)| \le \left\{\int_0^{T_0}\int_\Omega \operatorname{curl}\mathcal{T}\, z_\alpha\,dx\,dt - \int_0^{T_0}\int_{\mathcal{O}} \psi_{obs}h_\alpha\,dx\,dt\right\} + C\sqrt{\alpha}\|\omega(0)\|_{0,\Omega}.$$

In the following analysis, we will split the solution of (2.32) in two problems. For this, let us introduce $(\tilde{z}, \tilde{s})$, the solution of

$$\left\{\begin{array}{l} -\dfrac{\partial \tilde{z}}{\partial t} - \Delta \tilde{z} + \tilde{z} + \tilde{s} = 0 \quad \text{in } \Omega \times (0, T_0), \\ \Delta \tilde{s} + \dfrac{\partial \tilde{z}}{\partial x_1} = 0 \quad \text{in } \Omega \times (0, T_0), \\ \tilde{z} = \dfrac{\partial \tilde{z}}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ \tilde{z}(T_0) = \varphi_0 \quad \text{in } \Omega, \end{array}\right. \tag{2.39}$$

and then

$$(z_\alpha, s_\alpha) = (z + \tilde{z}, s + \tilde{s}),$$

where (z, s) is the solution of

$$\left\{\begin{array}{l} -\dfrac{\partial z}{\partial t} - \Delta z + z + s = 0 \quad \text{in } \Omega \times (0, T_0), \\ \Delta s + \dfrac{\partial z}{\partial x_1} = \phi_\alpha 1_{\mathcal{O}} \quad \text{in } \Omega \times (0, T_0), \\ z = \dfrac{\partial z}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ z(T_0) = 0 \quad \text{in } \Omega. \end{array}\right. \tag{2.40}$$

To simplify notation we denote the function $\theta_\alpha(0)$ by g and consider the linear operator Λ defined by

$$\Lambda g = z(0),$$

where z is obtained from g as follows: First we solve (2.33) with $\theta_\alpha(0) = g$ and then the backward system (2.40).

Using the fact that $g = -\frac{1}{\alpha} z_\alpha(0) = -\frac{1}{\alpha}(z(0) + \tilde{z}(0))$, the optimality system (2.32)–(2.34) reduces to the following:

Find $g \in L^2(\Omega)$ such that

$$(\alpha I + \Lambda) g = -\tilde{z}(0), \tag{2.41}$$

where I is the identity.

The operator $\Lambda \in \mathcal{L}(L^2(\Omega), L^2(\Omega))$ satisfies for all g and $\tilde{g}$ in $L^2(\Omega)$:

$$(\Lambda g, \tilde{g}) = (z(0), \eta(0)) = \int_0^{T_0} \int_{\mathcal{O}} \phi_\alpha \xi \, dx \, dt, \tag{2.42}$$

where η is the solution of the following problem with initial condition $\tilde{g}$:

$$\left\{\begin{array}{l} \dfrac{\partial \eta}{\partial t} - \Delta \eta + \eta - \dfrac{\partial \xi}{\partial x_1} = 0 \quad \text{in } \Omega \times (0, T_0), \\ \eta + \Delta \xi = 0 \quad \text{in } \Omega \times (0, T_0), \\ \xi = \dfrac{\partial \xi}{\partial n} = 0 \quad \text{on } \Gamma \times (0, T_0), \\ \eta(0) = \tilde{g} \quad \text{in } \Omega. \end{array}\right. \tag{2.43}$$

Indeed, the last equation in (2.42) can be obtained from (2.40) and (2.43).

On the other hand, it follows from (2.42) that, for all g and $\tilde{g}$ in $L^2(\Omega)$,

$$(\Lambda g, \tilde{g}) = (\Lambda \tilde{g}, g) \quad \text{and} \quad (\Lambda g, g) \geq 0,$$

i.e., Λ is self-adjoint and positive semi-definite. We also have, from the uniqueness continuation property (see Remark 2.2.3), that Λ is positive definite. Indeed, from (2.42),

$$(\Lambda g, g) = \int_0^{T_0} \int_{\mathcal{O}} |\phi_\alpha|^2 \, dx \, dt = 0 \quad \Rightarrow \quad \phi_\alpha = 0 \ \text{ in } \mathcal{O} \times (0, T_0),$$

but we know that necessarily $\theta_\alpha \equiv 0$ in $\Omega \times (0, T_0)$ and $\theta_\alpha(0) = g = 0$.

On the other hand, if α is strictly positive, the operator $(\alpha I + \Lambda)$ is strongly elliptic, i.e.,

$$((\alpha I + \Lambda)g, g) = \alpha \|g\|_{0,\Omega}^2 + (\Lambda g, g) \geq \alpha \|g\|_{0,\Omega}^2.$$

Then, the equation (2.41) allows us to calculate the optimal initial condition and using this value in (2.33), we obtain the optimal control associated with φ_0.

In summary, given $\varphi_0 \in L^2(\Omega)$, in order to find an approximation to $(\omega(T_0), \varphi_0)$, we fist have to solve an optimal control problem for each φ_0, which reduces to solving (2.41). The operator Λ in the left-hand side of (2.41) represents the coupled optimality system (2.33) and (2.40) which does not depend on φ_0. The right-hand side of (2.41) depends on φ_0 and corresponds to solving (2.39). This is very important for the numerical approximation because, after discretization, the linear systems corresponding to different φ_0 all have the same matrices [Pue02]. Once the optimal control and states solutions of (2.41) are known, they are used with source terms and observations in formula (2.10) in order to compute the projection $(\omega(T_0), \varphi_0)$.

2.4.1 Approximation of Problems (2.32)–(2.33)

In this section we present the time-space discretization of Problem (2.32)–(2.33). Here we use a combination of time discretization by finite differences and space discretization by finite elements. The methodology used in this work to implement the data assimilation method is simple. Similar studies of the numerical solution of the approximate controllability problems are discussed in [Lio94] for diffusion equations.

Time discretization

Assuming that T_0 is finite, we introduce a discretization time step Δt, defined by $\Delta t = T_0/N$, where N is a positive integer. Using an implicit Euler time discretization, we approximate (2.33) by

$$\theta^0 = g.$$

Then assuming that $(\phi^{n-1}, \theta^{n-1})$ is known, we solve for $n = 1, \dots, N$:

$$\begin{cases} \dfrac{(\theta^n - \theta^{n-1})}{\Delta t} - \Delta\theta^n + \theta^n - \dfrac{\partial \phi^n}{\partial x_1} = 0 \quad \text{in } \Omega, \\ \theta^n + \Delta\phi^n = 0 \quad \text{in } \Omega, \\ \phi^n = \dfrac{\partial \phi^n}{\partial n} = 0 \quad \text{on } \Gamma, \end{cases}$$

where $\theta^n = \theta(n\Delta t)$. For simplicity we have dropped the subscripts α in (2.33). Let us recall that instead (2.32) we solve separately (2.39) and (2.40). We approximate (2.40) by

$$z^{N+1} = 0,$$

and assuming that (z^{n+1}, s^{n+1}) is known, we solve the following problem, for $n = N, N-1, \dots, 1$:

$$\begin{cases} \dfrac{(z^n - z^{n+1})}{\Delta t} - \Delta z^n + z^n + s^n = 0 \quad \text{in } \Omega, \\ \Delta s^n + \dfrac{\partial z^n}{\partial x_1} = \phi^n 1_{\mathcal{O}} \quad \text{in } \Omega, \\ z^n = \dfrac{\partial z^n}{\partial n} = 0 \quad \text{on } \Gamma. \end{cases} \tag{2.44}$$

We approximate Λ by $\Lambda^{\Delta t}$, which is defined as

$$\Lambda^{\Delta t} g = z^0.$$

We can prove that the operator $\Lambda^{\Delta t}$ is symmetric and positive semi-definite. Indeed, for g and $\tilde{g}$ in $L^2(\Omega)$, we have

$$(\Lambda^{\Delta t} g, \tilde{g}) = \int_\Omega z^1 \eta^0 \, dx,$$

where, from (2.43), $\eta^0 = \tilde{g}$.

Since $z^{N+1} = 0$, it follows that

$$z^1 \eta^0 = \Delta t \sum_{n=1}^{N} \left[\frac{z^n - z^{n+1}}{\Delta t} \eta^n - \frac{\eta^n - \eta^{n-1}}{\Delta t} z^n \right]. \tag{2.45}$$

Integrating (2.45) over Ω, taking into account (2.44) and the approximation of system (2.43), we obtain after some integrations by parts

$$
\begin{aligned}
(\Lambda^{\Delta t} g, \tilde{g}) &= \int_\Omega z^1 \eta^0 \, dx \\
&= \Delta t \sum_{n=1}^{N} \int_\Omega \left[(\Delta z^n - z^n - s^n)\eta^n - (\Delta \eta^n - \eta^n + \frac{\partial \xi^n}{\partial x_1}) z^n \right] dx \\
&= \Delta t \sum_{n=1}^{N} \int_\Omega (s^n \Delta \xi^n + \frac{\partial z^n}{\partial x_1} \xi^n) \, dx = \Delta t \sum_{n=1}^{N} \int_\Omega (\Delta s^n + \frac{\partial z^n}{\partial x_1}) \xi^n \, dx \\
&= \Delta t \sum_{n=1}^{N} \int_{\mathcal{O}} \phi^n \xi^n \, dx \geq 0, \qquad (2.46)
\end{aligned}
$$

which completes the proof.

For (2.39), we compute the approximate solution $(\tilde{z}, \tilde{s})$ by

$$\tilde{z}^{N+1} = \varphi_0$$

and for $n = 1, \ldots, N$, assuming that $(\tilde{z}^{n+1}, \tilde{s}^{n+1})$ is known, we solve the following system

$$
\begin{cases}
\dfrac{(\tilde{z}^n - \tilde{z}^{n+1})}{\Delta t} - \Delta \tilde{z}^n + \tilde{z}^n + \tilde{s}^n = 0 \quad \text{in } \Omega, \\
\Delta \tilde{s}^n + \dfrac{\partial \tilde{z}^n}{\partial x_1} = 0 \quad \text{in } \Omega, \\
\tilde{z}^n = \dfrac{\partial \tilde{z}^n}{\partial n} = 0 \quad \text{on } \Gamma.
\end{cases}
$$

Finally, we approximate problem (2.41) by:
Find $g^{\Delta t} \in L^2(\Omega)$ *such that*

$$(\alpha g^{\Delta t} + \Lambda^{\Delta t} g^{\Delta t}, \tilde{g}) = -(\tilde{z}^0, \tilde{g}) \qquad \forall \tilde{g} \in L^2(\Omega).$$

Remark 2.4.2 *The Euler schemes which have been used to discretize problem* (2.41) *in time are first order accurate. We can improve this by employing the Leap-frog scheme or the semi-Lagrangian scheme which have been used in [BGdS01].*

Space discretization

From now on , we suppose that Ω and $\mathcal{O}$ are convex and polygonal domains in $\mathbb{R}^2$. We introduce $\{\mathcal{T}_h\}$, a regular family of triangulations of $\overline{\Omega}$, where $h = \max_{T \in \mathcal{T}_h} h_T$, with $h_T = \operatorname{diam}(T)\, \forall T \in \mathcal{T}_h$. Next, we approximate $H^1(\Omega)$ and $H_0^1(\Omega)$ by the following finite dimensional spaces, with $\mathcal{P}_1(T)$ being the space of polynomial functions of degree ≤ 1,

$$\mathcal{L}_h := \left\{ \mu_h \in H^1(\Omega) : \; \mu_h|_T \in \mathcal{P}_1(T) \;\; \forall T \in \mathcal{T}_h \right\}$$

and define

$$\Phi_h := \mathcal{L}_h \cap H_0^1(\Omega).$$

We approximate $L^2(\Omega)$ by $\mathcal{L}_h$; this is reasonable since the closure of $H^1(\Omega)$ in $L^2(\Omega)$ is $L^2(\Omega)$.

Problem (2.41) will be approximated by:
Find $g_h^{\Delta t} \in \mathcal{L}_h$ *such that*

$$\int_\Omega (\alpha g_h^{\Delta t} + \Lambda_h^{\Delta t} g_h^{\Delta t}) \mu_h \, dx = -\int_\Omega \tilde{z}_h^0 \mu_h \, dx \qquad \forall \mu_h \in \mathcal{L}_h, \tag{2.47}$$

where $\Lambda_h^{\Delta t}$ and $\tilde{z}_h^0$ are obtained as described below. The term $\tilde{z}_h^0$ from the following full discretization of (2.39):

$$\tilde{z}_h^{N+1} = \varphi_{0h} \quad \text{with } \varphi_{0h} \in \Phi_h \text{ being an approximation of } \varphi_0,$$

and for $n = N, N-1, \ldots, 1$, compute $(\tilde{z}_h^n, \tilde{s}_h^n) \in \Phi_h \times \mathcal{L}_h$ from $(\tilde{z}_h^{n+1}, \tilde{s}_h^{n+1})$ by

$$\left\{ \begin{aligned} & \int_\Omega \frac{(\tilde{z}_h^n - \tilde{z}_h^{n+1})}{\Delta t} \mu_h \, dx + \int_\Omega \vec{\operatorname{curl}}\, \tilde{z}_h^n \cdot \vec{\operatorname{curl}}\, \mu_h \, dx + \int_\Omega \tilde{z}_h^n \mu_h \, dx + \int_\Omega \tilde{s}_h^n \mu_h \, dx = 0 \\ & \hspace{22em} \forall \mu_h \in \mathcal{L}_h, \\ & -\int_\Omega \vec{\operatorname{curl}}\, \tilde{s}_h^n \cdot \vec{\operatorname{curl}}\, v_h \, dx + \int_\Omega \frac{\partial \tilde{z}_h^n}{\partial x_1} v_h \, dx = 0 \quad \forall v_h \in \Phi_h. \end{aligned} \right. \tag{2.48}$$

The operator $\Lambda_h^{\Delta t}$ is defined by

$$\Lambda_h^{\Delta t} g_h = z_h^0 \qquad \forall g_h \in \mathcal{L}_h,$$

where we solve the following discrete cascade systems to obtain z_h^0 from g_h:

First Problem: Given $\theta_h^0 = g_h$, then for $n = 1, \ldots, N$, we compute $(\phi_h^n, \theta_h^n) \in \Phi_h \times \mathcal{L}_h$ from $(\phi_h^{n-1}, \theta_h^{n-1})$ via the solution of

$$\left\{ \begin{aligned} & \int_\Omega \frac{(\theta_h^n - \theta_h^{n-1})}{\Delta t} v_h \, dx + \int_\Omega \vec{\operatorname{curl}}\, \theta_h^n \cdot \vec{\operatorname{curl}}\, v_h \, dx + \int_\Omega \theta_h^n v_h \, dx - \int_\Omega \frac{\partial \phi_h^n}{\partial x_1} v_h \, dx = 0 \\ & \hspace{22em} \forall v_h \in \Phi_h, \\ & \int_\Omega \theta_h^n \mu_h \, dx - \int_\Omega \vec{\operatorname{curl}}\, \phi_h^n \cdot \vec{\operatorname{curl}}\, \mu_h \, dx = 0 \quad \forall \mu_h \in \mathcal{L}_h. \end{aligned} \right. \tag{2.49}$$

Second Problem: Given $z_h^{N+1} = 0$, then for $n = N, N-1, \dots, 0$, we compute $(z_h^n, s_h^n) \in \Phi_h \times \mathcal{L}_h$ from (z_h^{n+1}, s_h^{n+1}) via the solution of

$$\left\{ \begin{array}{l} \displaystyle\int_\Omega \frac{(z_h^n - z_h^{n+1})}{\Delta t} \mu_h \, dx + \int_\Omega \vec{\text{curl}}\, z_h^n \cdot \vec{\text{curl}}\, \mu_h \, dx + \int_\Omega z_h^n \mu_h \, dx + \int_\Omega s_h^n \mu_h \, dx = 0 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \mu_h \in \mathcal{L}_h, \\ \displaystyle -\int_\Omega \vec{\text{curl}}\, s_h^n \cdot \vec{\text{curl}}\, v_h \, dx + \int_\Omega \frac{\partial z_h^n}{\partial x_1} v_h \, dx = \int_{\mathcal{O}} \phi_h^n v_h \, dx \quad \forall v_h \in \Phi_h, \end{array} \right. \tag{2.50}$$

with ϕ_h^n being the output of the first problem.

We can prove an analogous relation to (2.46) for the operator $\Lambda_h^{\Delta t}$ which shows that the operator is symmetric and positive semi-definite, implying that (2.47) has a unique solution.

To solve problem (2.47) we can use either direct methods or iterative methods (such as conjugate gradient). The conjugate gradient method has been employed in [CGL94] to solve exact and approximate boundary controllability problems for the heat equations. In this work, we will use direct methods. When the dimension of the discrete domain is not too large, we can compute an explicit representation of $\Lambda_h^{\Delta t}$ in (2.47).

Let us denote by $\bar{x}_1, \dots, \bar{x}_k$ the nodes of the triangulation, the first ones, $\bar{x}_1, \dots, \bar{x}_m$ corresponding to the interior nodes and the last ones $\bar{x}_{m+1}, \dots, \bar{x}_k$ to the nodes on the boundary. Then, any element of $\mathcal{L}_h$ may be expressed as

$$\mu_h = \sum_{j=1}^{k} \mu_j \varphi_j(x) \qquad \text{with } \mu_j = \mu_h(\bar{x}_j)$$

and any element of Φ_h may be written as

$$v_h = \sum_{j=1}^{m} v_j \varphi_j(x) \qquad \text{with } v_j = v_h(\bar{x}_j),$$

where $\varphi_1, \dots, \varphi_k$ are the standard basis functions for these elements, satisfying $\varphi_i(\bar{x}_l) = \delta_{il}$. Thus, problem (2.47) reduces to:
Find $g_1^{\Delta t}, \dots, g_k^{\Delta t}$ such that

$$\sum_{i=1}^{k} \left(\alpha\, g_i^{\Delta t} + \sum_{j=1}^{k} a_{ij}\, g_j^{\Delta t} \right) \varphi_i = -\sum_{i=1}^{m} \tilde{z}_i^0 \varphi_i. \tag{2.51}$$

This can be written in matrix form as follows:

$$(\alpha I + A) G^{\Delta t} = -Z^0,$$

where $G^{\Delta t} = (g_1^{\Delta t}, \ldots, g_k^{\Delta t})$ is the unknown vector and $Z^0 = (\tilde{z}_1^0, \ldots, \tilde{z}_m^0)$ is the vector solution of (2.48) associated with φ_{0h}. To obtain each column of the matrix $A = (a_{ij})$, we solve (2.49)–(2.50) for each basis function φ_j, $j = 1, \ldots, k$, as the initial condition g_h for (2.49). Then, the vector of coefficient of the final result z_h^0 in the basis $\{\varphi_j\}_{j=1}^m$ gives the j^{th} column of the matrix A.

If the solution of (2.47) for each $\varphi_{0h} = \varphi_i$ is attained at $g_h^{\Delta t}$, we compute (2.49) with the initial condition $\theta_h^0 = g_h^{\Delta t}$. Then, we obtain the optimal control as $\hat{h}_h^n = \hat{\phi}_h^n 1_{\mathcal{O}}$ and the associated solution of (2.50) by $(\hat{z}_h^n, \hat{s}_h^n)$. These values will be used to recover $\omega(T_0)$.

To compute an approximation of $\omega(T_0)$ we must choose a discrete set of functions $\varphi_0 = \varphi_{0j}$ to be used in (2.12) and their corresponding finite element approximation φ_{0h}. In fact, we take directly the standard basis function for $\mathcal{L}_h$, i.e. $\{\varphi_i\}_{i=1,\ldots,k}$. Indeed, from (2.35) we have the following approximation (when $\alpha \to 0$)

$$\int_\Omega \omega_h^N \varphi_i \, dx = \Delta t \sum_{n=1}^N \int_\Omega \operatorname{curl} \mathcal{T}_h^n (\hat{z}_h^n(\varphi_i) + \tilde{z}_h^n(\varphi_i)) \, dx - \Delta t \sum_{n=1}^N \int_{\mathcal{O}} \psi_{obs,h}^n \hat{\phi}_h^n(\varphi_i) \, dx$$

$$i = 1, \ldots, k, \qquad (2.52)$$

where $\mathcal{T}_h^n$ and $\psi_{obs,h}^n$ are the wind stress and the measurements of the stream function in $\mathcal{O} \times (0, T_0)$, respectively.

To compute the vorticity we proceed as follows. Since

$$\omega_h^N = \sum_{i=1}^k W_i^N \varphi_i(x) \qquad \text{with } W_i^N = \omega_h^N(\bar{x}_i),$$

we multiply by φ_j and we integrate over Ω to obtain

$$\int_\Omega \omega_h^N \varphi_j \, dx = \sum_{i=1}^k \int_\Omega W_i^N \varphi_i \varphi_j \, dx \qquad j = 1, \ldots, k.$$

If we introduce $M_{ij} = \int_\Omega \varphi_i \varphi_j$, with $i = 1, \ldots, k$ and $j = 1, \ldots, k$, it follows that

$$\hat{W}^N = M^{-1} \sum_{j=1}^k \int_\Omega \omega_h^N \varphi_j \, dx,$$

where $\hat{W}^N = (W_1^N, \ldots, W_k^N)$ is the vector that we are looking for.

Table 2.1 summarize the implementation of the discrete method and shows the dependency of each stage on the principal parameters of the problem.

Table 2.1: The summarize of the numerical implementation. NV denotes the number of nodes and NT the number of time intervals.

	Stage	Applied method	Dependency on
1	Mesh and elementary matrices	$\mathcal{P}_1$-finite elements	Ω
2	Right − hand side (RHS) of (2.47) and $\int_0^{T_0}\int_\Omega \tilde{z}_i\varphi_j dxdt$	Evolution linear system (2.48) of NT(2NV × 2NV) for each φ_i, $i = 1, \ldots, \mathrm{NV}$, + NT matrix vector products	φ_i
3	Computation of Λ and $\int_0^{T_0}\int_\Omega z_i\varphi_j dxdt$, $\int_0^{T_0}\int_\mathcal{O} h_i\varphi_j dxdt$	Evolution linear system (2.49)(2.50) of NT(2NV × 2NV) for each φ_i, $i = 1, \ldots, \mathrm{NV}$, + NT matrix vector products	φ_i, $\mathcal{O}$
4	Optimality system (2.47)	Simultaneous linear system of NV × NV for each RHS $i = 1, \ldots, \mathrm{NV}$	φ_i, $\mathcal{O}$, α
5	Projection $(\omega(T_0), \varphi_i)$ (2.52)	Linear combination of steps 2, 3 and 4, and matrix vector products	$\mathcal{T}$, ψ_{obs}
6	Prediction	Evolution linear system (2.1) NT(2NV × 2NV)	$\mathcal{T}$, recovered $\omega(T_0)$

2.5 Numerical experiments

In this section, we present several numerical experiments. First, we consider different tests with Rossby, Munk and Stommel numbers equal to 1. Then, we consider more realistic examples with typical values for these numbers.

Let $\Omega = [0, L] \times [0, 2\,L]$ and $T_0 = 0.05\,T$ where $L = 1000\,\mathrm{km}$ is the typical horizontal length and $T = 1\,\mathrm{year}$. As in reference [MW95], we use for the wind stress,

$$(\tau_1, \tau_2) = \left(-\frac{1}{\pi}\cos \pi x_2, 0\right). \tag{2.53}$$

2.5.1 Tests with plain theoretical coefficients

For this series of test problems we have taken $R_o = \epsilon_m = \epsilon_s = 1$ and $\Delta t = T_0/50$. As we have no real measurements for testing our method, we will compare the results of our experiments with the results of the original model (2.1), i.e., we compute the ocean circulation using (2.1) over the time interval $(0, T_0)$, for some initial given value $\omega(0)$ and surface wind stress (2.53). Then, we save ψ in the observation region $\mathcal{O} \times (0, T_0)$ and $\omega(T_0)$, which will be our exact target values.

We compute the recovered final vorticity, denoted here by $\omega_{rec}(T_0)$, following the algorithm presented in Table 2.1, and the relative error as follows

$$\frac{\|\omega(T_0) - \omega_{rec}(T_0)\|_{0,\Omega}}{\|\omega(T_0)\|_{0,\Omega}}. \tag{2.54}$$

First, let us consider the observation region as the whole domain. Figure 2.1 shows that, for this case, we obtain very good recovery of $\omega(T_0)$. The relative error is 0.0408.

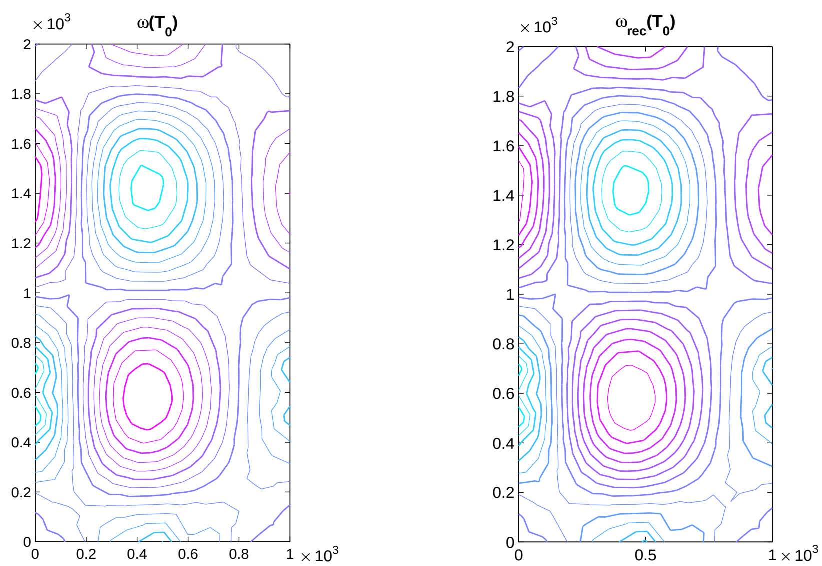

Figure 2.1: Test 1. Contour lines of final vorticity. Exact (left) and recovered (right). The observation region $\mathcal{O}$ for this test is the whole domain. We obtain good recovery of $\omega(T_0)$.

Now, let us choose the penalty parameter α. To do this, we present, in Figure 2.2, the graph of the dependence of the parameter α together with the relative error of the recovered final vorticity calculated using (2.54). Here, we consider the observation set $\mathcal{O}$ as the whole domain. We have chosen for our tests $\alpha = 10^{-7}$.

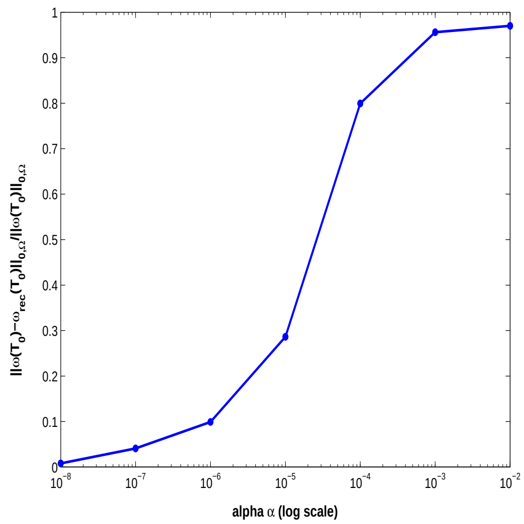

Figure 2.2: Test 1. Relative error of recovered final vorticity versus the penalty parameter α. The observation region $\mathcal{O}$ for this test is the whole domain $[0, 1000\text{km}] \times [0, 2000\text{km}]$. We see that the relative error decreases as the penalization parameter α decreases as expected.

Next, we consider different sizes of the observation set. In Figure 2.3, we show the exact and recovered final vorticity when $\mathcal{O} = [0, L] \times [0.5\, L, 1.5\, L]$. In this case, the relative error is 0.1856.

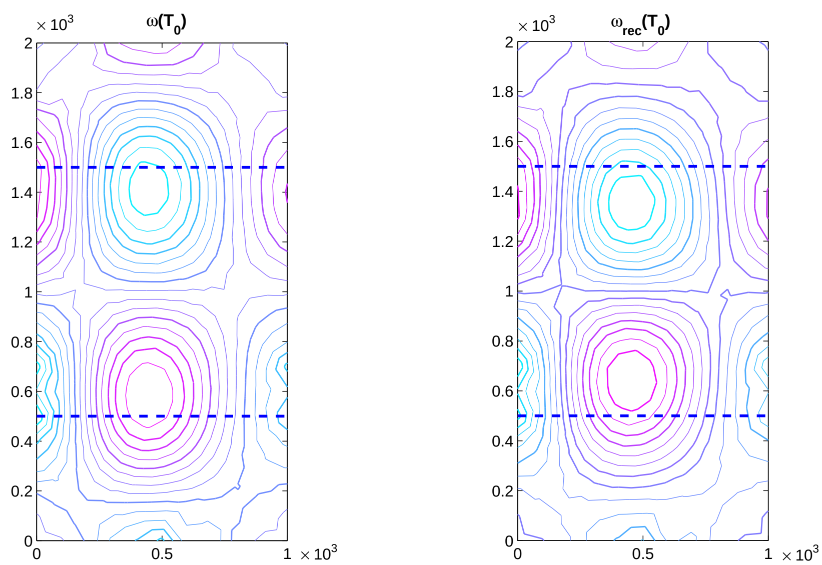

Figure 2.3: Test 1. Contour lines of final vorticity. Exact (left) and recovered (right). The area between the dotted lines corresponds to the observation region $\mathcal{O}$. The relative error is 0.1856.

Figure 2.4 shows that for the observation set $\mathcal{O} = [0, L] \times [0.8\,L, 1.2\,L]$, the recovery of $\omega(T_0)$ is not good, the relative error being 0.5267. In Figure 2.5, we

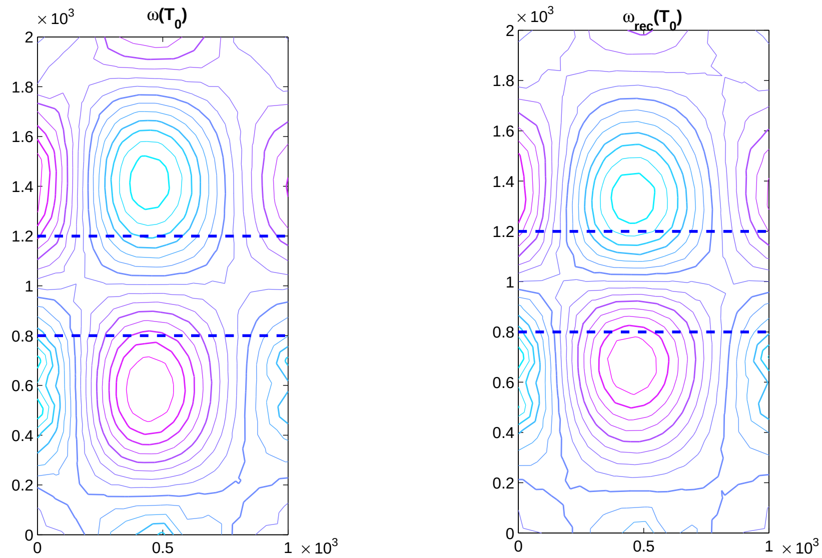

Figure 2.4: Test 1. Contour lines of final vorticity. Exact (left) and recovered (right). The area between the dotted lines corresponds to the observation region $\mathcal{O}$. The relative error is 0.5267. In this case the recovery is not acceptable.

observe that we can not expect to obtain good recovery of $\omega(T_0)$ when the size of the observation set is less than half that of the domain Ω.

For the following experiments, we consider $\mathcal{O} = [0, L] \times [0.5L, 1.5L]$. Figure 2.6 presents the graph of the norm of the optimal control $\|\hat{h}(t)\|_{0,\mathcal{O}}$ versus the time t. The optimal control $\hat{h}$ has been obtained using the following linear combination:

$$\hat{h}_h^n = \sum_{i=1}^{k} (\omega_h^n(T_0), \varphi_i) h_i^n(\varphi_i), \tag{2.55}$$

where h_i^n is the optimal control associated with φ_i.

Now, we want to predict the circulation at $T_{11} = T_{0i} + 0.01$ and $T_{12} = T_{0i} + 0.02$ using different data assimilation times T_{0i}, i.e., we compute the recovered $\omega(T_{0i})$ for each T_{0i} and then compute the solution of (2.1) for the time intervals (T_{0i}, T_{11}) and (T_{0i}, T_{12}) with initial condition $\omega(T_{0i})$. In Table 2.2, we present the relative error in $H^1(\Omega)$ and $L^2(\Omega)$ norms for the predicted stream function and vorticity respectively, at T_{11} and T_{12} using different data assimilation times T_{0i}. Notice that we obtain a better prediction at T_{11} and T_{12} if the data assimilation time (T_{0i}) is large.

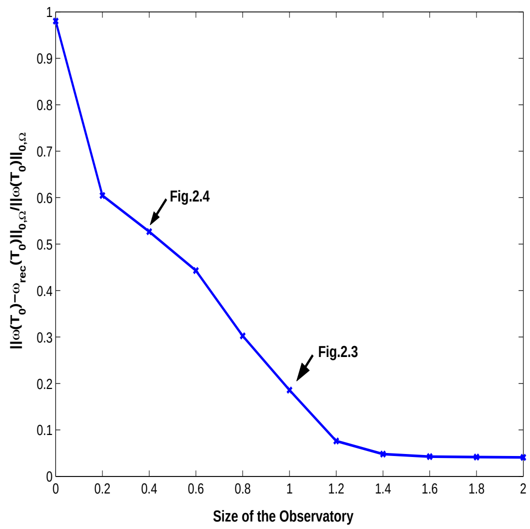

Figure 2.5: Test 1. Relative error for the recovered final vorticity versus the size of $\mathcal{O}$. We obtain good recovered values when the size of the observation region is more than half the size of Ω.

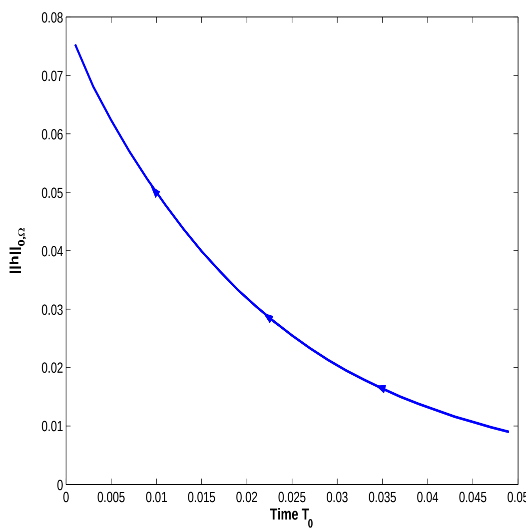

Figure 2.6: Test 1. Graph of $\|\hat{h}\|_{0,\mathcal{O}}$ versus t. The optimal control $\hat{h}$ has been obtained using (2.55). The observation region considered is [0,1000km]×[500km,1500km]. The norm of the control increases in value from T_0 to $t = 0$.

2.5.2 Tests with realistic coefficients

In the next experiments we consider the same domain but more realistic values for the Rossby, Munk and Stommel numbers, taken from [BGdS01]:

$$R_o = 1.5 \times 10^{-3}, \qquad \epsilon_m = 5 \times 10^{-5}, \qquad \epsilon_s = 5 \times 10^{-3},$$

Table 2.2: Test 1. Relative error for the predicted stream function (ψ_{pr}) and vorticity (ω_{pr}) at $T_{11} = T_{0i} + 0.01$ and $T_{12} = T_{0i} + 0.02$ using different T_{0i}. For a large data assimilation time, we obtain a better prediction at T_{11} and T_{12}.

	$\frac{\|(\psi-\psi_{pr})(T_{11})\|_{1,\Omega}}{\|\psi(T_{11})\|_{1,\Omega}}$	$\frac{\|(\omega-\omega_{pr})(T_{11})\|_{0,\Omega}}{\|\omega(T_{11})\|_{0,\Omega}}$	$\frac{\|(\psi-\psi_{pr})(T_{12})\|_{1,\Omega}}{\|\psi(T_{12})\|_{1,\Omega}}$	$\frac{\|(\omega-\omega_{pr})(T_{12})\|_{0,\Omega}}{\|\omega(T_{12})\|_{0,\Omega}}$
$T_{01} = 0.03$	0.1799	0.2240	0.1407	0.1722
$T_{02} = 0.04$	0.1422	0.1738	0.1154	0.1366
$T_{03} = 0.05$	0.1158	0.1372	0.0979	0.1114
$T_{04} = 0.06$	0.0981	0.1116	0.0858	0.0939

which correspond to

$$\gamma = 1 \times 10^{-7}\,\mathrm{s}^{-1}, \qquad A_H = 1 \times 10^3 \mathrm{m}^2\mathrm{s}^{-1}, \qquad L = 10^6\,\mathrm{m},$$
$$U = 0.03\mathrm{m\,s}^{-1}, \quad \beta = 2 \times 10^{-11}\mathrm{m}^{-1}\mathrm{s}^{-1}, \qquad D_0 = 800\,\mathrm{m}.$$

The following series of test problems have been done with $\Delta t = T_0/50$, $T_0 = 0.05$ and $\alpha = 5 \times 10^{-5}$ (α: penalty parameter). First, let us show in Figure 2.7 the exact and recovered final vorticity when the observation region is the whole domain. For this case, we obtain good recovery of $\omega(T_0)$, the relative error being 0.0258.

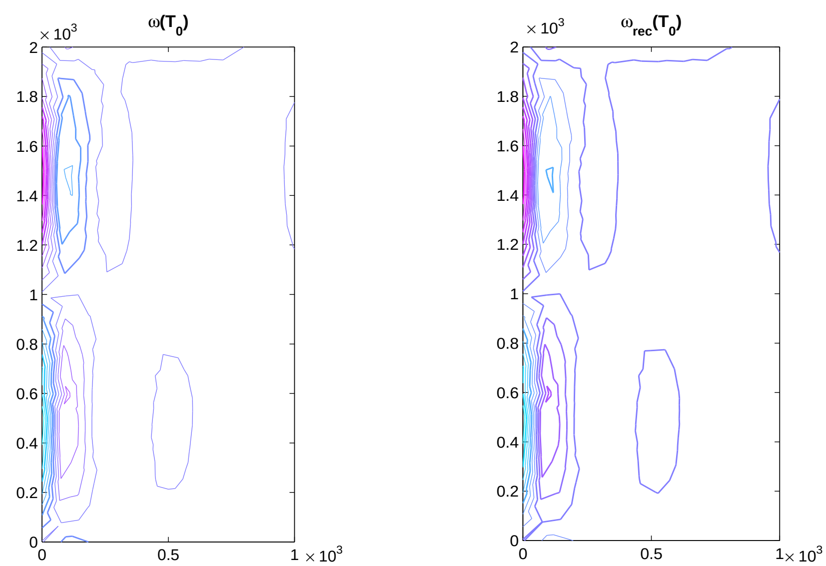

Figure 2.7: Test 2. Contour lines of final vorticity using real physical parameters. Exact (left) and recovered (right). The observation region $\mathcal{O}$ is the whole domain. We obtain good recovery of $\omega(T_0)$.

Since R_o, ϵ_m and ϵ_s are small, the convective term in (2.1) is dominant and introduce a boundary layer on the left-hand side of the domain. This can be seen in the stream lines shown in Figure 2.8. This figure shows the exact and recovered stream function at T_0 when the observation region is $\mathcal{O} = \Omega$. We have used the following problem to compute $\psi_{rec}(T_0)$:

$$\begin{cases} -\Delta\psi_{rec}(T_0) = \omega_{rec}(T_0) & \text{in } \Omega, \\ \psi_{rec}(T_0) = 0 & \text{on } \Gamma, \end{cases}$$

where $\omega_{rec}(T_0)$ is the recovered final vorticity.

Now, we consider, as in the previous section, different sizes of the observation set. In Figure 2.9, we show the exact and recovered final vorticity when $\mathcal{O} = [0, L] \times [0.5\,L, 1.5\,L]$. The relative error, in this case, is 0.2648.

In this convection dominated case, the recovery of $\psi(T_0)$ out of the observation region is not so good, as can be seen in Figure 2.10. This could be improved by using meshes refined on the boundary layer zone. Let us remark that, for the direct system, like (2.33), the boundary layer appears on the left side of the domain, but for the adjoint system, like (2.32), the boundary layer is on the right side.

Figure 2.11 shows that the values of the relative error for the recovered final vorticity $\omega(T_0)$ is acceptable for large sizes of the observatory region.

Figure 2.12 presents the graph of the norm of the optimal control $\|\hat{h}(t)\|_{0,\mathcal{O}}$ versus

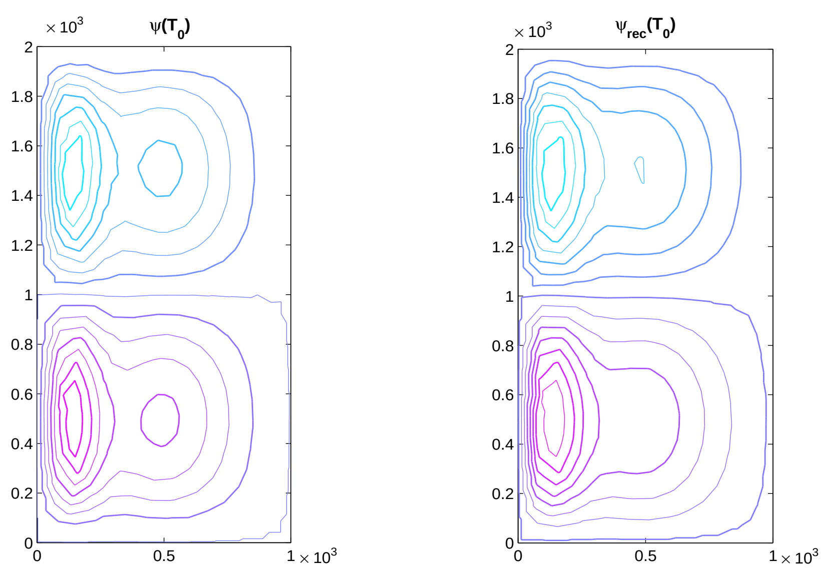

Figure 2.8: Test 2. Contour lines of stream function at T_0 using real physical parameters. Exact (left) and recovered (right). The observation region $\mathcal{O}$ is the whole domain. The recovery is good.

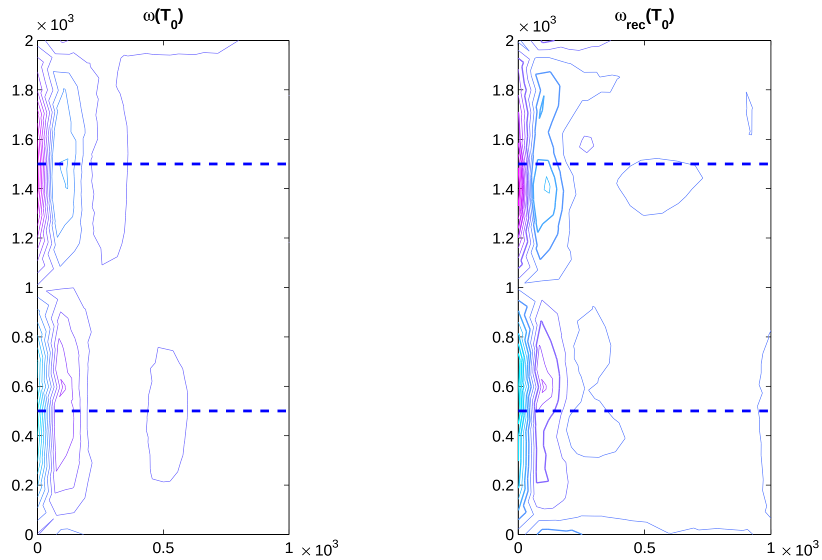

Figure 2.9: Test 2. Contour lines of final vorticity using real physical parameters. Exact (left) and recovered (right). The area between the dotted lines corresponds to the observation region $\mathcal{O}$. The relative error is 0.2648.

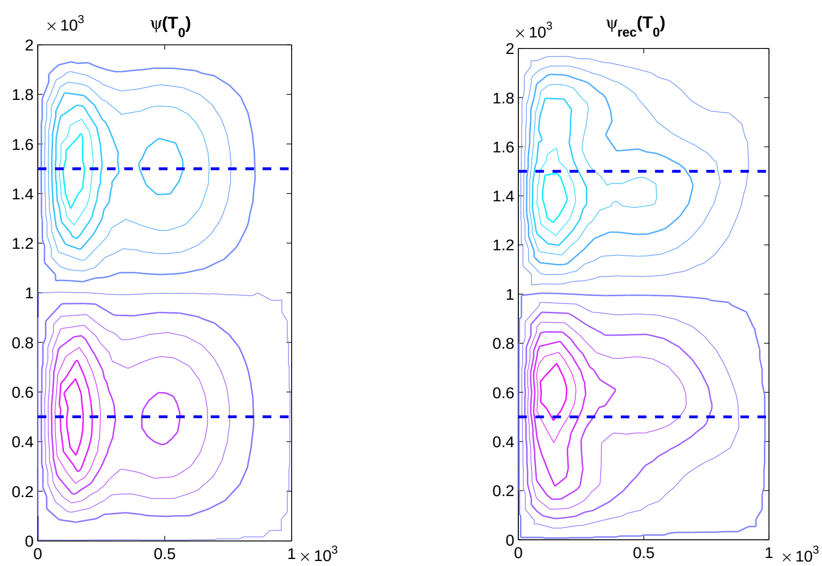

Figure 2.10: Test 2. Contour lines of stream function at T_0 using real physical parameters. Exact (left) and recovered (right). The area between the dotted lines corresponds to the observation region $\mathcal{O}$. Out of the observation region the recovery is not so good.

the time t when the observation region is $[0, L] \times [0.2L, 1.8L]$.

Finally, in Table 2.3, we present the relative error in $H^1(\Omega)$ and $L^2(\Omega)$ norms for

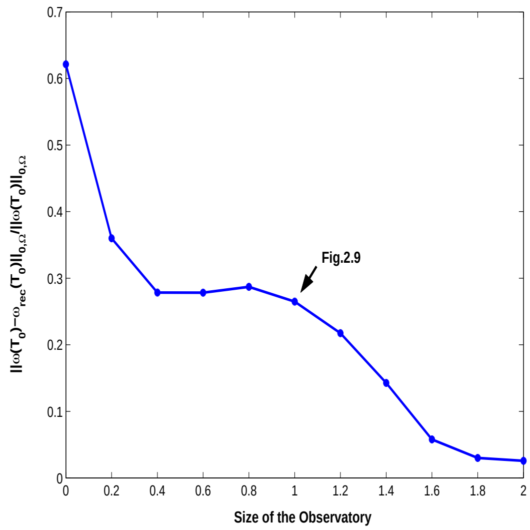

Figure 2.11: Test 2. Relative error for the recovered final vorticity versus the size of $\mathcal{O}$.

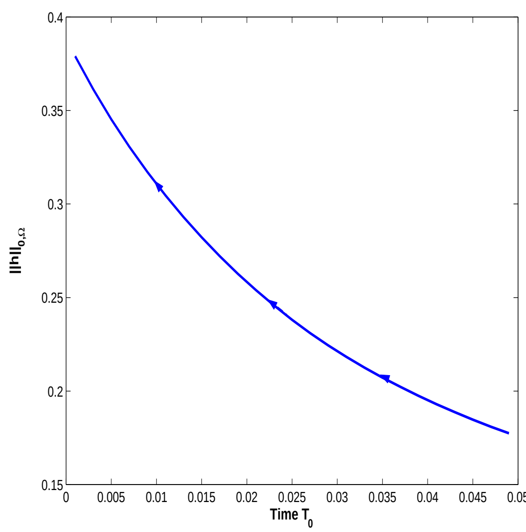

Figure 2.12: Test 2. Graph of $\|\hat{h}\|_{0,\mathcal{O}}$ versus t. The optimal control $\hat{h}$ has been obtained using (2.55). The observation region considered is [0,1000km]×[200km,1800km]. The norm of the control increases in value from T_0 to $t = 0$.

the predicted stream function and vorticity respectively, at T_{11} and T_{12} using different data assimilation times T_{0i}, when the observation region is $[0, L] \times [0.2L, 1.8L]$. Once more, we obtain a better prediction at T_{11} and T_{12} if the data assimilation time (T_{0i}) is larger. From the numerical results given in Table 2.3, we can also deduce that the model with real physical parameters is not very sensitive to the initial condition $\omega(T_0)$. In particular, the prediction of the variable of interest (ψ) is very good.

Table 2.3: Test 2. Relative error for the predicted stream function (ψ_{pr}) and vorticity (ω_{pr}) at $T_{11} = T_{0i} + 0.01$ and $T_{12} = T_{0i} + 0.02$ using different T_{0i}. These numerical results show that the model is not very sensitive to the initial condition $\omega(T_0)$.

	$\frac{\|(\psi-\psi_{pr})(T_{11})\|_{1,\Omega}}{\|\psi(T_{11})\|_{1,\Omega}}$	$\frac{\|(\omega-\omega_{pr})(T_{11})\|_{0,\Omega}}{\|\omega(T_{11})\|_{0,\Omega}}$	$\frac{\|(\psi-\psi_{pr})(T_{12})\|_{1,\Omega}}{\|\psi(T_{12})\|_{1,\Omega}}$	$\frac{\|(\omega-\omega_{pr})(T_{12})\|_{0,\Omega}}{\|\omega(T_{12})\|_{0,\Omega}}$
$T_{01} = 0.03$	0.0686	0.0656	0.0568	0.0522
$T_{02} = 0.04$	0.0589	0.0550	0.0486	0.0442
$T_{03} = 0.05$	0.0509	0.0467	0.0421	0.0379
$T_{04} = 0.06$	0.0442	0.0399	0.0370	0.0326

2.6 Comments and conclusions

A non classical approach to data assimilation has been applied to recovering the initial value $\omega(T_0)$ in order to predict the future state on the time interval (T_0, T_1). This method is based on null controllability theory. An observability inequality was proved using a global Carleman estimate for the associated velocity-pressure formulation.

In Section 2.4, we presented an approximate algorithm which makes use of a classical optimal control problem. In Theorem 2.4.1, we proved that, when $\alpha \to 0$, we can compute $(\omega(T_0), \varphi_0)$. We reported numerical evidence, in the form of plots of relative error versus α, that illustrates this behavior. Such plots are useful in choosing a suitable α. Since Λ is not strongly elliptic, the case $\alpha = 0$ is a difficult problem to solve numerically.

In the experiments presented in Section 2.5, we studied the role that the size of the observation region plays for the recovery of the final value at T_0, as well as the time of data assimilation for a good prediction of future times.

We could improve the numerical results of the tests with real coefficients using refined meshes where the boundary layer arises. It will be of interest in future work to use the mesh-refined strategy introduced in Chapter 3.

Part II

An adaptive mesh refinement strategy

Chapter 3

A priori and a posteriori error analysis for a large-scale ocean circulation finite element model

Abstract

We consider the finite element solution of the stream function-vorticity formulation for a large-scale ocean circulation model. First, we study existence and uniqueness of solution for the continuous and discrete problems. Under appropriate regularity assumptions we prove that the stream function can be computed with an error of order h in H^1–seminorm. Second, we introduce and analyze an h–adaptive mesh refinement strategy to reduce the spurious oscillations and poor resolution which arise when convective terms are dominant. We propose an a posteriori anisotropic error indicator based on the recovery of the Hessian from the finite element solution, which allows us to obtain well adapted meshes. The numerical experiments show an optimal order of convergence of the adaptive scheme. Furthermore, this strategy is efficient to eliminate the oscillations around the boundary layer.

3.1 Introduction

Different models have been proposed for large-scale horizontal ocean dynamics. Among them, the quasi-geostrophic model is one of the most widely used by oceanographers to predict wind-stress driven circulation at mid-latitudes; see [Med99, Med00, MW95, Ped87] and references therein. To study this model, it is usual to work with the stream function-vorticity mixed equations [BGdS01, Med99, Med00, MW95].

In the context of the Stokes equation, this formulation and an associated finite element scheme have been introduced by Ciarlet and Raviart in [CR74]. Since then, several authors have studied extensively this subject. A detailed mathematical analysis can be found in Girault and Raviart's book [GR86], which also includes further references. More recently, Amara *et. al.* [AB99, AD01] have introduced and analyzed improved finite element methods to deal with this formulation.

For a large-scale ocean model, a typical phenomenon is the formation of the Western boundary currents, as in the North Pacific and the North Atlantic. These currents have a typical horizontal scale of about one thousand kilometers and are persistent and dominant [BGdS01]. Because of them, the solutions of these models present boundary layers that, when numerically solved, lead to spurious oscillations and poor resolution. Nevertheless, the accuracy can be remarkably improved if correctly refined meshes are used where the boundary layer arises. Because of this, mesh-refinement strategies to create well-adapted meshes in an automatic manner are particularly useful for the numerical solution of the quasi-geostrophic model.

Mesh-refinement strategies are typically based on *a posteriori* error indicators. This subject has been introduced for finite element methods by Babuška and Rheinboldt [BR78] long time ago. Since then, many different approaches have been devised for many different problems. See for instance the monographs by Verfürth [Ver96], Ainsworth and J.T. Oden [AO00], and Babuška and Strouboulis [BS01]. In particular, for the stream function-vorticity formulation of Stokes problem, an error indicator have been analyzed in [ABYB98].

However, most of the existing adaptive techniques do not take care of the presence of boundary layers. In a boundary layer zone, the solution typically have strong gradients in one direction and almost no variation in the orthogonal one. Then, in this case, it turns out convenient to use non-shape regular stretched elements aligned with these layers.

Several alternatives have been proposed to create such "anisotropic" meshes. Some of them are based on appropriate anisotropic error indicators (see [AN98, ANS01, FP01, FP03, FPZ01, Kun00, Kun02, Pic03]). In practice, error indicators based on a post-processed Hessian of the computed solution are extensively used (see, for instance, [CD97, CDHMP97, AAYHT$^+$96, PPK92]). In particular, Almeida *et. al.* have introduced in [AFG$^+$00] an anisotropic mesh adaptation process guided by a directional error estimator which is based on the recovery of the second derivatives of the finite element solution.

The goal of this chapter is two-fold. On one hand, we analyze a finite element discretization of the stream function-vorticity formulation of a large-scale ocean circulation model. We show that under appropriate regularity assumptions this numerical method leads to an optimal order approximation of the velocity, which is the variable of interest.

However, in real problems, when the convective terms in the model are dominant, these estimates become meaningless. There is a vast literature dealing with methods for convection-dominated problems; see [RST96] for a survey. These methods invoke some form to boost the stability. The streamline diffusion finite element methods and the upwind finite element methods are two such. Frequently, these methods are used on Shishkin-type meshes; i.e., meshes very fine inside on the layer and coarse otherwise. These meshes yield much better results that uniform and adaptively-isotropic refined meshes containing a similar number of nodes [MS97], but their construction requires *a priori* knowledge about the behavior of the derivatives of the exact solution. Optimal (or quasi-optimal) convergence results are known for such methods in several norms, assuming that the solution can be decomposed in a regular part and layer terms [LS01, SRL01].

Our approach, instead, consists in using an *a posteriori* indicator to locate the boundary layer without any *a priori* information and to create meshes well adapted to the solution. We introduce a mesh refinement technique relying on an *a posteriori* anisotropic error indicator based on the recovery of the Hessian from the finite element solution, together with the anisotropic mesh generator `BL2D` [BLB95]. We assess the efficiency of our strategy by means of several numerical experiments.

The chapter is organized as follows: In Section 3.2, we recall the formulation of the steady-state linear quasi-geostrophic ocean model and its standard variational formulation. Then, in Section 3.3, we present its stream function-vorticity formulation and a finite element discretization. *A priori* error estimates are established in Section 3.4. We also present in this section some numerical experiments confirming the theoretical results for sufficiently smooth solutions and showing the need of correctly refined meshes on the wind-driven model subject to realistic values of the physical parameters. In Section 3.5, we introduce error indicators for the L^2 norms of the stream function and the velocity field. Then, we present a mesh adaptation technique. Finally, in Section3.6, we report several numerical experiments exhibiting the performance of the proposed strategy.

3.2 Simplified ocean model

Let Ω be an open bounded and connected, although not necessarily simply connected, subset of $\mathbb{R}^2$ with a Lipschitz continuous boundary Γ. We consider the steady-state linear quasi-geostrophic ocean model [BGdS01, MW95] described by the following equations

$$
\begin{cases}
-A_H \Delta u + \gamma u + f(x_2)\mathbf{k} \times u + \dfrac{1}{\rho_0}\nabla p = \dfrac{1}{\rho_0 D_0}\tau \quad \text{in } \Omega, \\
\operatorname{div} u = 0 \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \Gamma,
\end{cases}
\tag{3.1}
$$

where u and p are the velocity and the pressure of the fluid for each point $x = (x_1, x_2) \in \Omega$, $\tau = (\tau_1, \tau_2) \in L^2(\Omega)$ is the surface wind stress, A_H the constant horizontal coefficient of eddy viscosity, γ the bottom friction, D_0 and ρ_0 the depth and the density of the ocean, respectively. We denote by Γ_0 the exterior boundary and $\Gamma_i,\ 1 \le i \le p$, the other connected components of Γ if any. On the other hand, $\mathbf{k} \times u$ denotes the $90°$ rotation of the vector u; namely, $\mathbf{k} \times u = (-u_2, u_1)$. The Coriolis parameter f represents the planetary vorticity of the motion due to the rotation of the Earth.

The β-plane approximation is assumed (see, for instance, [Ped87]). It consists of substituting the Coriolis parameter f by a linear approximation

$$
f = f_0 + \beta x_2, \qquad f_0 = 2\omega_0 \sin\theta_0, \qquad \beta = \frac{2\omega_0}{R}\cos\theta_0,
$$

where ω_0 is the angular rotation rate of the Earth ($7.24 \times 10^{-5}\,\mathrm{s}^{-1}$), and R is the radius of the Earth ($6.371 \times 10^6\,\mathrm{m}$). This is a first order approximation to study the large-scale ocean dynamics valid at mid-latitudes ($20° \le \theta_0 \le 50°$).

We use standard notation for Sobolev spaces, norms and seminorms. Moreover, we introduce the space

$$
V := \left\{ v \in H_0^1(\Omega)^2 :\ \operatorname{div} v = 0 \right\}.
$$

A standard variational formulation of (3.1) is as follows:

Problem 1 *Find $u \in V$ such that*

$$
A_H(\operatorname{curl} u, \operatorname{curl} v) + \gamma(u, v) + ((f_0 + \beta x_2)\mathbf{k} \times u, v) = \frac{1}{\rho_0 D_0}(\tau, v) \quad \forall v \in V.
$$

Here and thereafter $(\cdot,\cdot)$ denotes the inner product of $L^2(\Omega)$ or $L^2(\Omega)^2$, as corresponds.

Theorem 3.2.1 *Problem* 1 *has a unique solution u in V and there exists a unique $p \in L_0^2(\Omega)$ such that (u, p) is the weak solution of (3.1).*

Proof: Let us introduce the bilinear form $a_1(\cdot,\cdot)$ by

$$a_1(u,v) = A_H(\operatorname{curl} u, \operatorname{curl} v) + \gamma(u,v) + ((f_0+\beta x_2)\mathbf{k}\times u, v)\,.$$

It is easy to check that the form $a_1(\cdot,\cdot)$ is V-elliptic. Indeed, for all $v \in V$

$$a_1(v,v) = A_H|v|^2_{1,\Omega} + \gamma\|v\|^2_{0,\Omega} + + ((f_0+\beta x_2)\mathbf{k}\times v, v) \geq C\|v\|_{1,\Omega},$$

since $(\mathbf{k}\times v, v)$ vanishes. Then, we apply Lax-Milgram's Theorem to deduce that Problem 1 has a unique solution u.

Let us check now that a solution of Problem 1 is a solution in some weak sense of (3.1), i.e., u is a solution of Problem 1 if and only if there exists a function $p \in L^2_0(\Omega)$ such that (u,p) is a weak solution of (3.1). Indeed, the equality $\operatorname{div} u = 0$ in Ω, is an easy consequence of $u \in V$ and $u = 0$ on Γ because of u is in $H^1_0(\Omega)^2$.

Integrating by parts in the first equation of Problem 1, we see that

$$(-A_H\Delta u + \gamma u + (f_0+\beta x_2)\mathbf{k}\times u - \frac{1}{\rho_0 D_0}\,\tau, v) = 0 \qquad \forall v \in V.$$

By an application of Theorem I.2.3 in [GR86], since $-A_H\Delta u + \gamma u + (f_0+\beta x_2)\mathbf{k}\times u - \frac{1}{\rho_0 D_0}\,\tau$ belong to $H^{-1}(\Omega)$, there exists a function $p \in L^2(\Omega)$, such that

$$\rho_0(-A_H\Delta u + \gamma u + (f_0+\beta x_2)\mathbf{k}\times u + \nabla p = \frac{1}{\rho_0 D_0}\,\tau)$$

where p is unique up to an additional constant. So, we conclude that $(u,p) \in H^1_0(\Omega)\times L^2_0(\Omega)$ is the weak solution of (3.1). □

Remark 3.2.1 *In the case that either Ω is a convex polygon or its boundary Γ is of class $\mathcal{C}^2$, the arguments of Theorem* I.5.4 *and Remark* I.5.6 *in* [*GR86*] *allow us to prove that the unique solution of* (3.1) *satisfies* $(u,p) \in H^2(\Omega)^2 \times H^1(\Omega)\cap L^2_0(\Omega)$ *and*

$$\|u\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C\|\tau\|_{0,\Omega},$$

where C is a constant independent of τ.

3.3 Stream function-vorticity formulation

3.3.1 The continuous problem

It is well-know that the divergence-free condition can be expressed by introducing a stream function ψ of u:

$$u = \vec{\operatorname{curl}}\,\psi := \left(\frac{\partial\psi}{\partial x_2}, -\frac{\partial\psi}{\partial x_1}\right).$$

Since u vanishes on Γ, ψ must be constant on each of its connected components Γ_i. Moreover, ψ is uniquely determined if we set $\psi = 0$ on Γ_0. Thus the stream function belongs to the space

$$\Phi := \left\{ \phi \in H^1(\Omega) : \ \phi|_{\Gamma_0} = 0, \ \phi|_{\Gamma_i} = c_i, \ 1 \le i \le p \right\},$$

where c_i denote arbitrary constants.

Let us introduce the vorticity ω of u:

$$\omega := \operatorname{curl} u := \left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) = -\Delta \psi. \tag{3.2}$$

If $\omega \in H^1(\Omega)$, by choosing $v = \vec{\operatorname{curl}}\, \phi$ in Problem 1 and integrating by parts, we obtain

$$A_H(\vec{\operatorname{curl}}\ \omega, \vec{\operatorname{curl}}\ \phi) + \gamma(\omega, \phi) - \beta \left(\frac{\partial \psi}{\partial x_1}, \phi \right) = \frac{1}{\rho_0 D_0} (\tau, \vec{\operatorname{curl}}\, \phi) \quad \forall \phi \in \Phi,$$

where we have also used that f_0 is constant.

After scaling the equations by introducing non-dimensional variables, we are led to the following problem:

Problem 2 *Find* $(\psi, \omega) \in \Phi \times H^1(\Omega)$ *such that*

$$\left\{ \begin{array}{l} \epsilon_m(\vec{\operatorname{curl}}\, \omega, \vec{\operatorname{curl}}\, \phi) + \epsilon_s(\omega, \phi) - \left(\dfrac{\partial \psi}{\partial x_1}, \phi \right) = (\tau, \vec{\operatorname{curl}}\, \phi) \qquad \forall \phi \in \Phi, \\ -(\omega, \mu) + (\vec{\operatorname{curl}}\, \psi, \vec{\operatorname{curl}}\, \mu) = 0 \qquad \forall \mu \in H^1(\Omega). \end{array} \right.$$

The last equation is a weak formulation of (3.2). The parameters ϵ_s and ϵ_m are the non-dimensional Stommel and Munk numbers, respectively, which are defined by

$$\epsilon_s := \frac{\gamma}{\beta L} \quad \text{and} \quad \epsilon_m := \frac{A_H}{\beta L^3},$$

with L being a typical horizontal length scale of the domain (see [MW95]).

Notice that a solution of Problem 1 provides a solution of Problem 2 only if $\omega = \operatorname{curl} u$ is smooth enough, for instance, if $u \in H^2(\Omega)$. The following theorem shows existence and uniqueness of solution of Problem 2 under regularity geometric constraints.

Theorem 3.3.1 *Let* Ω *be either a convex polygon or such that its boundary* Γ *is of class* $\mathcal{C}^2$*. Then, Problem 2 attains a unique solution* $(\psi, \omega) \in \Phi \times H^1(\Omega)$.

Proof: Let us consider the following Stokes-like problem: Find $(u,p) \in H_0^1(\Omega)^2 \times L_0^2(\Omega)$ such that

$$\begin{cases} a(u,v) + (p, \operatorname{div} v) = (\tau, v) & \forall v \in H_0^1(\Omega)^2, \\ (\operatorname{div} u, q) = 0 \quad \forall q \in L_0^2(\Omega), \end{cases} \tag{3.3}$$

with

$$a(u,v) = \epsilon_m(\operatorname{curl} u, \operatorname{curl} v) + \epsilon_s(u,v) + (x_2\mathbf{k} \times u, v).$$

This bilinear form is V-elliptic. In fact, $\forall v \in V$

$$a(v,v) = \epsilon_m|v|_{1,\Omega}^2 + \epsilon_s\|v\|_{0,\Omega}^2 + (x_2\mathbf{k} \times v, v) \geq \min\{\epsilon_m, \epsilon_s\}\|v\|_{1,\Omega}^2,$$

since $(x_2\mathbf{k} \times v, v)$ vanishes.

Then, the standard abstract theory (see for instance Theorem I.5.1 in [GR86]) applies to show that (3.3) attains a unique solution. Moreover, under the geometric assumptions on Ω, $(u,p) \in H^2(\Omega)^2 \times H^1(\Omega)$ and

$$\|u\|_{2,\Omega} + \|p\|_{1,\Omega} \leq C\|\tau\|_{0,\Omega},$$

with C independent of τ.

In fact, this is a consequence of Theorem I.5.4 and Remark I.5.6 in [GR86] applied to the equivalent formulation

$$\begin{cases} \epsilon_m(\operatorname{curl} u, \operatorname{curl} v) + (p, \operatorname{div} v) = (\tau - \epsilon_s(u,v) - x_2\mathbf{k} \times u, v) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in H_0^1(\Omega)^2, \\ (\operatorname{div} u, q) = 0 \quad \forall q \in L_0^2(\Omega). \end{cases}$$

Now, the abstract framework in Section III.1.1 and III.2.1 of [GR86] (in particular Theorem 2.1, 2.2, 2.3 and 2.4) applies to our problem allowing us to prove the equivalence between Problem 2 and (3.3) and, consequently, the theorem.

□

Remark 3.3.1 *The geometric assumptions on Ω have been used only to ensure that $(u,p) \in H^2(\Omega)^2 \times H^1(\Omega)$. Therefore, if (3.3) has such a smooth solution, we attain the same conclusion regarding existence and uniqueness for Problem 2.*

3.3.2 The discrete problem

Let $\{\mathcal{T}_h\}$ be a regular family of triangulations of $\bar{\Omega}$, where $h = \max_{T\in\mathcal{T}_h} h_T$, with $h_T = \operatorname{diam}(T)$ $\forall T \in \mathcal{T}_h$. Let $\mathcal{P}_l(T)$ be the space of polynomial functions of degree at most l defined on the triangle T; we set

$$\mathcal{L}_h := \left\{\phi_h \in H^1(\Omega) : \ \phi_h|_T \in \mathcal{P}_1(T) \ \ \forall T \in \mathcal{T}_h\right\}$$

and define

$$\Phi_h := \mathcal{L}_h \cap \Phi \qquad \text{and} \qquad \Theta_h := \mathcal{L}_h.$$

The corresponding Galerkin scheme is:

Problem 3 *Find* $(\psi_h, \omega_h) \in \Phi_h \times \Theta_h$ *such that*

$$\begin{cases} \epsilon_m(\vec{\operatorname{curl}}\,\omega_h, \vec{\operatorname{curl}}\,\phi_h) + \epsilon_s(\omega_h, \phi_h) - \left(\dfrac{\partial \psi_h}{\partial x_1}, \phi_h\right) = (\tau, \vec{\operatorname{curl}}\,\phi_h) \quad \forall \phi_h \in \Phi_h, \\ -(\omega_h, \mu_h) + (\vec{\operatorname{curl}}\,\psi_h, \vec{\operatorname{curl}}\,\mu_h) = 0 \quad \forall \mu_h \in \Theta_h. \end{cases}$$

The arguments of Section III.2.2 of [GR86] leading to Lemma III.2.4 and Theorem III.2.5 of this reference apply to our case allowing us to prove the following result:

Theorem 3.3.2 *Problem* 3 *attains a unique solution* $(\psi_h, \omega_h) \in \Phi_h \times \Theta_h$.

Proof: Firstly, we will study the existence and uniqueness of solution of an auxiliar problem and after that we obtain under certain assumption the equivalence to Problem 3 and the prove of Theorem 3.3.2.

Let Y_h be a finite-dimensional subspace of $L^2(\Omega)$, such that

$$\Phi_h \subset \Theta_h \subset Y_h.$$

Let us consider the following problem:
Find $(\psi_h, \omega_h) \in \Phi_h \times Y_h$ *and* $\lambda \in \Theta_h$ *such that*

$$\begin{cases} \epsilon_m(\omega_h, \theta_h) + \epsilon_s(\omega_h, \phi_h) - \left(\dfrac{\partial \psi_h}{\partial x_1}, \phi_h\right) - (\lambda_h, \theta_h) + (\vec{\operatorname{curl}}\,\phi_h, \vec{\operatorname{curl}}\,\lambda_h) \\ \qquad\qquad = (\tau, \vec{\operatorname{curl}}\,\phi_h) \quad \forall (\phi_h, \theta_h) \in \Phi_h \times Y_h, \\ -(\omega_h, \mu_h) + (\vec{\operatorname{curl}}\,\psi_h, \vec{\operatorname{curl}}\,\mu_h) = 0 \qquad \forall \mu_h \in Y_h. \end{cases} \tag{3.4}$$

We introduce the following bilinear forms:

$$\tilde{a}((\psi_h, \omega_h), (\phi_h, \theta_h)) = \epsilon_m(\omega_h, \theta_h) + \epsilon_s(\omega_h, \phi_h) - \left(\frac{\partial \psi_h}{\partial x_1}, \phi_h\right) \qquad \forall (\psi_h, \omega_h), (\phi_h, \theta_h) \in \Phi_h \times Y_h,$$

$$\tilde{b}((\phi_h, \theta_h), \mu_h) = -(\theta_h, \mu_h) + (\vec{\operatorname{curl}}\,\phi_h, \vec{\operatorname{curl}}\,\mu_h) \quad \forall (\phi_h, \theta_h) \in \Phi_h \times Y_h,\ \forall \mu \in \Theta_h,$$

and the space V_h

$$V_h = \{(\phi_h, \theta_h) \in (\Phi_h \times Y_h) : \ (\vec{\operatorname{curl}}\,\phi_h, \vec{\operatorname{curl}}\,\mu_h) = (\theta_h, \mu_h) \ \ \forall \mu_h \in \Theta_h\}.$$

Let us first check the V_h-ellipticity of $\tilde{a}(\cdot,\cdot)$. Indeed, for $(\phi_h, \theta_h) \in V_h$, we have

$$\begin{aligned}\tilde{a}((\phi_h,\theta_h),(\phi_h,\theta)) &= \epsilon_m \|\theta_h\|^2_{0,\Omega} + \epsilon_s(\theta_h,\phi_h) \\ &= \epsilon_m \|\theta_h\|^2_{0,\Omega} + \epsilon_s \|\vec{\operatorname{curl}}\,\phi_h\|^2_{0,\Omega} \\ &\geq \min\{\epsilon_m,\epsilon_s\} \|(\phi_h,\theta_h)\|^2_{H^1(\Omega)\times L^2(\Omega)},\end{aligned}$$

where we use the fact that $\Phi_h \subset \Theta_h$ to apply the definition of V_h. Observe also that the last term of $\tilde{a}(\cdot,\cdot)$ is zero.

Now, we check the inf-sup condition of $\tilde{b}(\cdot,\cdot)$. Since Θ_h is finite-dimensional space there exists a constant K depending of h such that

$$\|\mu_h\|_{1,\Omega} \leq K(h)\|\mu_h\|_{0,\Omega} \qquad \forall \mu_h \in \Theta_h,$$

then

$$\begin{aligned}\sup_{(\phi_h,\theta_h)\in\Phi_h\times Y_h} \frac{\tilde{b}((\phi_h,\theta_h),\mu_h)}{\|(\phi_h,\theta_h)\|_{H^1(\Omega)\times L^2(\Omega)}} &\geq \sup_{(0,\theta_h)\in\Phi_h\times Y_h} -\frac{(\theta_h,\mu_h)}{\|\theta_h)\|_{0,\Omega}} \\ &\geq \|\mu_h\|_{0,\Omega} \geq K(h)^{-1}\|\mu_h\|_{1,\Omega} \qquad \forall \mu_h \in \Theta_h,\end{aligned}$$

where we chose $\theta_h = -\mu_h$.

The V_h-ellipticity of $\tilde{a}(\cdot,\cdot)$ and the inf-sup condition of $\tilde{b}(\cdot,\cdot)$ allows us to conclude that (3.4) has exactly one solution $(\psi_h,\omega_h) \in \Phi_h \times Y_h$ and $\lambda \in \Theta_h$.

If in addition $\Theta_h = Y_h$ then $\lambda = \epsilon_m \omega_h$ and we obtain the equivalence between (3.4) and Problem 3. To prove these, let $\Theta_h = Y_h$ and let us show that $((\psi_h,\omega_h),\epsilon_m\omega_h)$ satisfies (3.4), i.e., that

$$\epsilon_m(\vec{\operatorname{curl}}\,\omega_h, \vec{\operatorname{curl}}\,\phi_h) + \epsilon_s(\omega_h,\phi_h) - \left(\frac{\partial \psi_h}{\partial x_1}, \phi_h\right) = (\tau, \vec{\operatorname{curl}}\,\phi_h) \quad \forall \phi_h \in \Phi_h,$$

notice that the above equation correspond to the first equation in Problem 3. Indeed, if $(\phi_h,\theta_h) \in V_h$ then the first equation in (3.4) becomes

$$\epsilon_m(\omega_h,\theta_h) + \epsilon_s(\omega_h,\phi_h) - \left(\frac{\partial \psi_h}{\partial x_1}, \phi_h\right) = (\tau, \vec{\operatorname{curl}}\,\phi_h). \tag{3.5}$$

Since $\Theta_h = Y_h$, taking $\mu_h = \omega_h$ in the definition of V_h, the functions ϕ_h and θ_h satisfy

$$(\vec{\operatorname{curl}}\,\phi_h, \vec{\operatorname{curl}}\,\omega_h) = (\theta_h,\omega_h).$$

Replacing the above relation in (3.5), we get

$$\epsilon_m(\vec{\operatorname{curl}}\,\omega_h, \vec{\operatorname{curl}}\,\phi_h) + \epsilon_s(\omega_h,\phi_h) - \left(\frac{\partial \psi_h}{\partial x_1}, \phi_h\right) = (\tau, \vec{\operatorname{curl}}\,\phi_h) \quad \forall (\phi_h,\theta_h) \in V_h,$$

but since $\Theta_h = Y_h$, each function $\phi_h \in \Phi_h$ has exactly one function $\theta_h \in \Theta_h$ such that (ϕ_h, θ_h) belongs to V_h. This proves that (3.4) is reduced to Problem 3.

On the other hand, as $\Theta_h = Y_h$ we have proved that $\lambda_h = \epsilon_m \omega_h$ in Θ_h, then

$$(\lambda_h, \theta_h) = \epsilon_m(\omega_h, \theta_h) \qquad \forall \theta_h \in Y_h. \tag{3.6}$$

Adding (3.6) to the first equation of Problem 3 and using that $\omega_h = \epsilon_m^{-1}\lambda_h$ we obtain (3.4). This prove the Theorem 3.3.2.

□

3.4 A priori error estimates

We introduce the following elliptic projection operators:

- $P_h : H^1(\Omega) \longrightarrow \Theta_h$ defined by

$$\epsilon_m(\vec{\operatorname{curl}}(P_h\mu - \mu), \vec{\operatorname{curl}}\,\theta_h) + \epsilon_s(P_h\mu - \mu, \theta_h) = 0 \qquad \forall \theta_h \in \Theta_h;$$

- $P_h^o : \Phi \longrightarrow \Phi_h$ defined by

$$\epsilon_m(\vec{\operatorname{curl}}(P_h^o\phi - \phi), \vec{\operatorname{curl}}\,\phi_h) = 0 \qquad \forall \phi_h \in \Phi_h.$$

It is proved in [Sco76] that, when Ω is a convex polygonal domain in $\mathbb{R}^2$ and $\{\mathcal{T}_h\}$ a family of quasi-uniform triangulations, given s and p in $\mathbb{R}$ such that $0 \le s \le 1$ and $2 \le p < \infty$, there exists a constant $C > 0$ such that the projections satisfy the following error estimates:

$$\|v - P_h v\|_{0,p,\Omega} + h|v - P_h v|_{1,p,\Omega} \le C h^{s+1}\|v\|_{s+1,p,\Omega} \quad \forall v \in W^{s+1,p}(\Omega), \tag{3.7}$$

$$\|v - P_h^o v\|_{0,p,\Omega} + h|v - P_h^o v|_{1,p,\Omega} \le C h^{s+1}\|v\|_{s+1,p,\Omega} \tag{3.8}$$
$$\forall v \in W^{s+1,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Here and thereafter C denotes a generic constant not necessarily the same at each occurrence but always independent of the mesh size h.

Let

$$W := \left\{(\phi, \theta) \in \Phi \times \Theta : \ (\vec{\operatorname{curl}}\,\phi, \vec{\operatorname{curl}}\,\mu) = (\theta, \mu) \ \forall \mu \in \Theta\right\}$$

and

$$W_h := \left\{(\phi_h, \theta_h) \in \Phi_h \times \Theta_h : \ (\vec{\operatorname{curl}}\,\phi_h, \vec{\operatorname{curl}}\,\mu_h) = (\theta_h, \mu_h) \ \forall \mu_h \in \Theta_h\right\}.$$

We prove the following theorem by adapting the arguments in Theorem III.2.6 of [GR86] to our case.

Theorem 3.4.1 *Let (ψ,ω) and (ψ_h,ω_h) be the respective solutions of Problems 2 and 3. If Ω is a convex polygon, then there exists a constant $C>0$ depending on ϵ_m and ϵ_s such that the following error estimate holds:*

$$\begin{aligned}|\psi-\psi_h|_{1,\Omega}+\|\omega-\omega_h\|_{0,\Omega} \;\le\; C\Bigg[&\inf_{(\phi_h,\theta_h)\in W_h}\left(|\psi-\phi_h|_{1,\Omega}+\|\omega-\theta_h\|_{0,\Omega}\right)\\ &+|\psi-P_h^o\psi|_{1,\Omega}+\|\omega-P_h\omega\|_{0,\Omega}\Bigg].\end{aligned} \tag{3.9}$$

Proof: Let $(\phi_h,\theta_h)\in W_h$. We have

$$\epsilon_m(\vec{\mathrm{curl}}(\omega-\omega_h),\vec{\mathrm{curl}}\,\phi_h)+\epsilon_s(\omega-\omega_h,\phi_h)-\left(\frac{\partial}{\partial x_1}(\psi-\psi_h),\phi_h\right)=0.$$

Then, by using the definition of P_h and the fact that $\phi_h\in\Theta_h$,

$$\begin{aligned}\epsilon_m(\vec{\mathrm{curl}}(P_h\omega &- \omega_h),\vec{\mathrm{curl}}\,\phi_h)+\epsilon_s(P_h\omega-\omega_h,\phi_h)\\ &- \left(\frac{\partial}{\partial x_1}(P_h^o\psi-\psi_h),\phi_h\right)-\left(\frac{\partial}{\partial x_1}(\psi-P_h^o\psi),\phi_h\right)=0.\end{aligned} \tag{3.10}$$

Now, the definition of W_h implies that

$$\begin{aligned}\epsilon_m(P_h\omega-\omega_h,\theta_h) &+ \epsilon_s(P_h\omega-\omega_h,\phi_h)\\ &- \left(\frac{\partial}{\partial x_1}(P_h^o\psi-\psi_h),\phi_h\right)-\left(\frac{\partial}{\partial x_1}(\psi-P_h^o\psi),\phi_h\right)=0.\end{aligned} \tag{3.11}$$

If we add and subtract ϕ_h and θ_h, we obtain

$$\begin{aligned}&\epsilon_m(P_h\omega-\theta_h,\theta_h)+\epsilon_s(P_h\omega-\theta_h,\phi_h)-\left(\frac{\partial}{\partial x_1}(P_h^o\psi-\phi_h),\phi_h\right)\\ &-\left(\frac{\partial}{\partial x_1}(\psi-P_h^o\psi),\phi_h\right)\\ &\qquad=\epsilon_m(\omega_h-\theta_h,\theta_h)+\epsilon_s(\omega_h-\theta_h,\phi_h)-\left(\frac{\partial}{\partial x_1}(\psi_h-\phi_h),\phi_h\right).\end{aligned}$$

This equation is valid in particular for $(\phi_h,\theta_h)=(\psi_h,\omega_h)$, since $(\psi_h,\omega_h)\in W_h$ because of the second equation of Problem 3. Thus, we obtain

$$\begin{aligned}&\epsilon_m(P_h\omega-\theta_h,\omega_h-\theta_h)+\epsilon_s(P_h\omega-\theta_h,\psi_h-\phi_h)\\ &-\left(\frac{\partial}{\partial x_1}(P_h^o\psi-\phi_h),\psi_h-\phi_h\right)-\left(\frac{\partial}{\partial x_1}(\psi-P_h^o\psi),\psi_h-\phi_h\right)\\ &\quad=\epsilon_m\|\omega_h-\theta_h\|^2+\epsilon_s(\omega_h-\theta_h,\psi_h-\phi_h)-\left(\frac{\partial}{\partial x_1}(\psi_h-\phi_h),\psi_h-\phi_h\right).\end{aligned}$$

It is easy to see that the last term vanishes. Then the definition of W_h implies

$$\begin{aligned}\epsilon_s\|\operatorname{curl} v(\psi_h-\phi_h)\|_{0,\Omega}^2+\epsilon_m\|\omega_h-\theta_h\|_{0,\Omega}^2 \\ \leq \epsilon_m\|P_h\omega-\theta_h\|_{0,\Omega}\|\omega_h-\theta_h\|_{0,\Omega} \\ +\left(\epsilon_s\|P_h\omega-\theta_h\|_{0,\Omega}+|P_h^o\psi-\phi_h|_{1,\Omega}+|\psi-P_h^o\psi|_{1,\Omega}\right)\|\psi_h-\phi_h\|_{0,\Omega}.\end{aligned}$$

On the other hand, since $\psi_h-\phi_h=0$ on Γ_0, Poincaré's Lemma yields

$$\|\psi_h-\phi_h\|_{0,\Omega}\leq C\|\vec{\operatorname{curl}}(\psi_h-\phi_h)\|_{0,\Omega}.$$

Then, the following inequality is a consequence of the two previous estimates and straightforward computations:

$$\begin{aligned}|\psi_h-\phi_h|_{1,\Omega}+\|\omega_h-\theta_h\|_{0,\Omega} \leq\ & C\left(\|P_h\omega-\theta_h\|_{0,\Omega}\right. \\ & \left.+|P_h^o\psi-\phi_h|_{1,\Omega}+|\psi-P_h^o\psi|_{1,\Omega}\right).\end{aligned}$$

Since this is valid $\forall(\phi_h,\theta_h)\in W_h$, the triangle inequality allows us to conclude the proof. □

Now, we want to estimate the right hand side of (3.9). To do this, let us first recall two useful estimates given in Lemma III.3.1 and Lemma III.3.2 of [GR86].

Lemma 3.4.1 *For all (ϕ,θ) in W one has*

$$\begin{aligned}&\inf_{(\tilde{\phi}_h,\tilde{\theta}_h)\in W_h}\|\theta-\tilde{\theta}_h\|_{0,\Omega} \\ &\leq \inf_{(\phi_h,\theta_h)\in\Phi_h\times\Theta_h}\left[2(|\phi-\phi_h|_{1,\Omega}+\|\theta-\theta_h\|_{0,\Omega})+\sup_{\mu_h\in\Theta_h}\frac{|(\vec{\operatorname{curl}}(\phi-\phi_h),\vec{\operatorname{curl}}\,\mu_h)|}{\|\mu_h\|_{0,\Omega}}\right].\end{aligned}$$

Lemma 3.4.2 *Let Ω be a bounded, convex polygon and let $\mathcal{T}_h$ be a quasi-uniform family of triangulations of $\bar{\Omega}$. Let p be a real number such that $2\leq p\leq\infty$. There exists a constant $C>0$ independent of h such that, for all $\phi\in W^{2,p}(\Omega)\cap H_0^1(\Omega)$*

$$\sup_{\mu_h\in\Theta_h}\frac{|(\vec{\operatorname{curl}}(\phi-\phi_h),\vec{\operatorname{curl}}\ \mu_h)|}{\|\mu_h\|_{0,\Omega}}\leq Ch^{1/2-1/p}\|\phi\|_{2,p,\Omega}.$$

The following theorem provides the error estimates for the stream function and vorticity.

Theorem 3.4.2 *Let Ω and $\{\mathcal{T}_h\}$ be like in Lemma 3.4.2. If $\psi\in H^3(\Omega)$, then for all $\varepsilon>0$,*

$$\begin{aligned}\|\omega-\omega_h\|_{0,\Omega}&\leq C(\varepsilon)h^{1/2-\varepsilon}\|\psi\|_{3,\Omega},\\ |\psi-\psi_h|_{1,\Omega}&\leq C(\varepsilon)h^{1-\varepsilon}\|\psi\|_{3,\Omega}.\end{aligned}$$

Proof: In the first part of the proof, we obtain the error estimate for the vorticity. We will proceed as in Section III.3.1 of [GR86] (see in particular Remark III.3.3).

We use Lemma 3.4.1 and Lemma 3.4.2 to estimate in (3.9) the error in $\mathcal{W}_h$

$$\|\omega-\omega_h\|_{0,\Omega} \leq C\left[\inf_{(\phi_h,\theta_h)\in\Phi_h\times\Theta_h}(|\psi-\psi_h|_{1,\Omega}+\|\omega-\omega_h\|_{0,\Omega})+h^{1/2-1/p}\|\psi\|_{2,p,\Omega}\right.$$
$$\left.+|\psi-P_h^o\psi|_{1,\Omega}+\|\omega-P_h\omega\|_{0,\Omega}\right].$$

Putting $\phi_h=\mathcal{P}_h^0\phi$ and $\omega_h=\mathcal{P}_h\omega$ in the above inequality and applying the error estimates (3.7) and (3.8) we obtain

$$\|\omega-\omega_h\|_{0,\Omega}\leq C\left(h\|\psi\|_{2,\Omega}+h\|\omega\|_{0,\Omega}+h^{1/2-1/p}\|\psi\|_{2,p,\Omega}\right).$$

Using Sobolev's imbedding Theorem: $H^3(\Omega)\overset{c}{\hookrightarrow}W^{2,p}(\Omega)$ for $1\leq p<\infty$, we deduce the error estimate for the vorticity by putting $\varepsilon=1/p>0$

$$\|\omega-\omega_h\|_{0,\Omega}\leq Ch^{1/2-\varepsilon}\|\psi\|_{3,\Omega}.$$

On the other hand, the second estimate follows from a duality argument similar to that in Theorem III.3.3 of [GR86] but based on the following auxiliary problem: Given $g\in L^2(\Omega)^2$, find $\phi_g\in H_0^1(\Omega)$ and $\lambda_g\in H^1(\Omega)$ such that

$$\left\{\begin{array}{ll}(\vec{\operatorname{curl}}\,\lambda_g,\vec{\operatorname{curl}}\,\chi)-\left(\dfrac{\partial\chi}{\partial x_1},\phi_g\right)=(g,\vec{\operatorname{curl}}\,\chi) & \forall\chi\in H_0^1(\Omega),\\ \epsilon_m(\vec{\operatorname{curl}}\,\phi_g,\vec{\operatorname{curl}}\,\mu)+\epsilon_s(\phi_g,\mu)-(\lambda_g,\mu)=0 & \forall\mu\in H^1(\Omega).\end{array}\right. \tag{3.12}$$

Because of convexity of Ω, the unique solution (ϕ_g,λ_g) belong to $H^3(\Omega)\cap H_0^1(\Omega)\times H^1(\Omega)$ and using the regularity of the corresponding velocity-pressure problem (see Remark 3.2.1) we deduce that

$$\|\phi_g\|_{3,\Omega}+\|\lambda_g\|_{1,\Omega}\leq C\|g\|_{0,\Omega}. \tag{3.13}$$

From Problem 2 and Problem 3 we have

$$\epsilon_m(\vec{\operatorname{curl}}(\omega-\omega_h),\vec{\operatorname{curl}}\,\phi_g)+\epsilon_s(\omega-\omega_h,\phi_h)-\left(\frac{\partial}{\partial x}(\psi-\psi_h),\phi_h\right)=0,$$
$$(\vec{\operatorname{curl}}\;(\psi-\psi_h),\vec{\operatorname{curl}}\,\lambda_h)=(\omega-\omega_h,\lambda_h).$$

In (3.12), we take $\chi = \psi - \psi_h$ and $\mu = \omega - \omega_h$ and we combine with the above equations, to obtain

$$\begin{aligned}(g, \vec{\mathrm{curl}}(\psi - \psi_h)) &= (\vec{\mathrm{curl}}(\lambda_g - \lambda_n), \vec{\mathrm{curl}}(\psi - \psi_h)) - \left(\frac{\partial}{\partial x}(\psi - \psi_h), \phi_g - \phi_n\right) \\ &\quad + (\omega - \omega_h, \lambda_h - \lambda_g) + \epsilon_m(\vec{\mathrm{curl}}(\omega - \omega_h), \vec{\mathrm{curl}}(\phi_g - \phi_h)) \\ &\quad + \epsilon_s(\omega - \omega_h, \phi_g - \phi_h) \qquad (3.14) \\ &\qquad\qquad \forall \lambda_h \in \Theta_h,\ \forall \phi_h \in \Phi_h.\end{aligned}$$

We separate the fourth term in the right hand side as follows

$$\begin{aligned}(\vec{\mathrm{curl}}(\omega - \omega_h), \vec{\mathrm{curl}}(\phi_g - \phi_h)) &= (\vec{\mathrm{curl}}(\omega - P_h\omega), \vec{\mathrm{curl}}(\phi_g - \phi_h)) \\ &\quad + (\vec{\mathrm{curl}}(P_h\omega - \omega_h), \vec{\mathrm{curl}}(\phi_g - \phi_h)). \qquad (3.15)\end{aligned}$$

The definition of W_h implies that

$$(\vec{\mathrm{curl}}\, \phi_g, \vec{\mathrm{curl}}(P_h\omega - \omega_h)) = (\lambda_g, P_h\omega - \omega_h). \qquad (3.16)$$

We also have from (3.11) and (3.10)

$$\begin{aligned}&\epsilon_m(\vec{\mathrm{curl}}(P_h\omega - \omega_h), \vec{\mathrm{curl}}\, \phi_h) + \epsilon_s(P_h\omega - \omega_h, \phi_h) \\ &\quad - \left(\frac{\partial}{\partial x_1}(P_h^o\psi - \psi_h), \phi_h\right) - \left(\frac{\partial}{\partial x_1}(\psi - P_h^o\psi), \phi_h\right) = 0, \qquad (3.17)\end{aligned}$$

$$\begin{aligned}&\epsilon_m(P_h\omega - \omega_h, \theta_h) + \epsilon_s(P_h\omega - \omega_h, \phi_h) \\ &\quad - \left(\frac{\partial}{\partial x_1}(P_h^o\psi - \psi_h), \phi_h\right) - \left(\frac{\partial}{\partial x_1}(\psi - P_h^o\psi), \phi_h\right) = 0. \qquad (3.18)\end{aligned}$$

Combining (3.16)-(3.18), we obtain for (3.15)

$$\begin{aligned}(\vec{\mathrm{curl}}(\omega - \omega_h), \vec{\mathrm{curl}}(\phi_g - \phi_h)) &= (\vec{\mathrm{curl}}\ (\omega - P_h\omega), \vec{\mathrm{curl}}(\phi_g - \phi_h)) \\ &\quad + (P_h\omega - \omega_h, \lambda_g - \theta_h).\end{aligned}$$

Replacing in (3.14), we have

$$\begin{aligned}&(g, \vec{\mathrm{curl}}(\psi - \psi_h)) = (\vec{\mathrm{curl}}(\lambda_g - P_n\lambda_g), \vec{\mathrm{curl}}(\psi - \psi_h)) - \left(\frac{\partial}{\partial x}(\psi - \psi_h), \phi_g - \phi_n\right) \\ &\quad + (\omega - \omega_h, P_h\lambda_g - \lambda_g) + \epsilon_m(\vec{\mathrm{curl}}(\omega - P_h\omega), \vec{\mathrm{curl}}(\phi_g - \phi_h)) \\ &\quad + \epsilon_m(P_h\omega - \omega_h, \lambda_g - \theta_h) + \epsilon_s(\omega - \omega_h, \phi_g - \phi_h) \quad \forall(\phi_h, \theta_h) \in W_h, \qquad (3.19)\end{aligned}$$

where we chose $\lambda_h = P_h\lambda_g$.

Now we will use a duality argument in order to estimate the error. Firstly, we know that

$$|\psi - \psi_h|_{1,\Omega} = \sup_{g \in L^2(\Omega)} \frac{(g, \vec{\operatorname{curl}}(\psi - \psi_h))}{\|g\|_{0,\Omega}}. \tag{3.20}$$

Secondly, we need to estimate each term of the right hand side in (3.19). Notice as ψ and ϕ_g belong at least to $H^2(\Omega)$, we take

$$\psi_h = I_h\psi, \quad \phi_h = I_h\phi_g,$$

where I_h is the interpolant operator $I_h \in \mathcal{L}(H^2(\Omega); \Theta_h)$ and satisfies the following error estimate for all integer m, with $0 \le m \le 2$ (see Lemma I.A.2 in [GR86])

$$|\psi - I_h\psi|_{m,\Omega} \le C_1 h^{2-m} |\psi|_{2,\Omega} \qquad \forall \psi \in H^2(\Omega).$$

Then, using (3.7) and (3.8), for $\varepsilon > 0$, we have

$$\begin{aligned}
|(\vec{\operatorname{curl}}(\lambda_g - P_n\lambda_g), \vec{\operatorname{curl}}(\psi - I_h\psi))| &\le C_2 h |\lambda_g|_{1,\Omega} |\psi|_{2,\Omega}, \\
\left| \left(\frac{\partial}{\partial x}(\psi - I_h\psi), \phi_g - I_h\phi_g \right) \right| &\le C_3 h^3 |\psi|_{2,\Omega} |\phi_g|_{2,\Omega}, \\
|(\omega - \omega_h, P_h\lambda_g - \lambda_g)| &\le C_4(\varepsilon) h^{3/2-\varepsilon} |\omega|_{1,\Omega} |\lambda_g|_{1,\Omega}, \\
|(\vec{\operatorname{curl}}(\omega - P_h\omega), \vec{\operatorname{curl}}(\phi_g - I_h\phi_g))| &\le C_5 h |\omega|_{1,\Omega} |\phi_g|_{2,\Omega}, \\
|(P_h\omega - \omega_h, \lambda_g - \theta_h)| &\le C_6(\varepsilon) h^{1-\varepsilon} |\omega|_{1,\Omega} \|\phi_g\|_{3,\Omega}, \\
|(\omega - \omega_h, \phi_g - I_h\phi_g)| &\le C_7(\varepsilon) h^{3/2-\varepsilon} |\omega|_{1,\Omega} |\phi_g|_{2,\Omega},
\end{aligned}$$

Replacing these estimates into (3.19) and according to (3.20) and (3.13) we deduce that

$$|\psi - \psi_h|_{1,\Omega} \le C(\varepsilon) h^{1-\varepsilon} \|\psi\|_{3,\Omega} \frac{\|\phi_g\|_{3,\Omega} + \|\lambda_g\|_{1,\Omega}}{\|\phi_g\|_{3,\Omega} + \|\lambda_g\|_{1,\Omega}} \le C(\varepsilon) h^{1-\varepsilon} \|\psi\|_{3,\Omega}.$$

□

3.4.1 Numerical experiments

The goal of our first experiment is to verify the error estimate above for the stream function. Let $\Omega = [0, 3] \times [0, 1]$, $\epsilon_m = \epsilon_s = 1$ and a right hand side such that the exact solution is the smooth function

$$\psi(x) = \sin^2 \frac{\pi x_1}{3} \, \sin^2 \pi x_2.$$

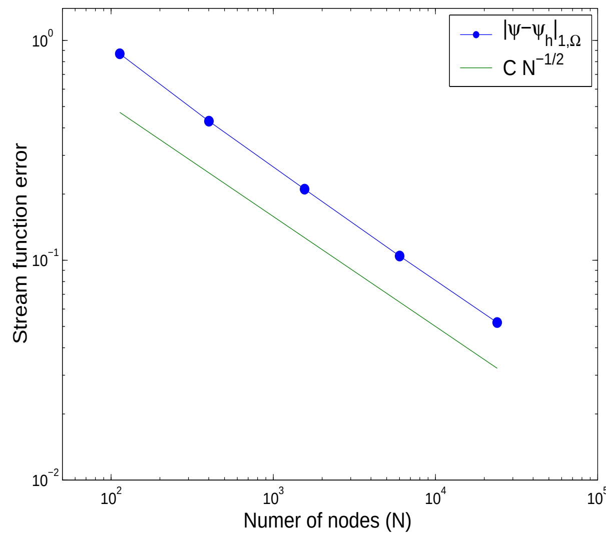

Figure 3.1: Error curve for $|\psi - \psi_h|_{1,\Omega}$ on a quasi-uniform mesh. Smooth solution.

We have used several quasi-uniform meshes with different degrees of refinement. Fig. 3.1 shows the error curve of the method which exhibits an $\mathcal{O}(h)$ of convergence, confirming the theoretical result of Theorem 3.4.2.

In the next experiment we consider the same domain but more realistic values for the Munk and Stommel numbers, taken from Myers and Weaver [MW95]:

$$\epsilon_m = 6 \times 10^{-5}, \qquad \epsilon_s = 0.05,$$

which correspond to

$$\gamma = 1 \times 10^{-6}\,\mathrm{s}^{-1}, \qquad A_H = 1.2 \times 10^3 \mathrm{m}^2\mathrm{s}^{-1}, \qquad L = 10^6\,\mathrm{m}, \qquad D_0 = 800\,\mathrm{m}.$$

For the wind stress we use, also as in reference [MW95],

$$(\tau_1, \tau_2) = \left(-\frac{1}{\pi}\cos \pi x_2, 0\right).$$

We have used the uniform mesh shown in Fig. 3.2. Since ϵ_m and ϵ_s are small, the convective term in Problem 2 is dominant and introduces a strong boundary layer on the left side of the domain. This can be seen in the computed stream lines shown in Fig. 3.3. This figure also shows a poor resolution of this layer and that spurious oscillations arise.

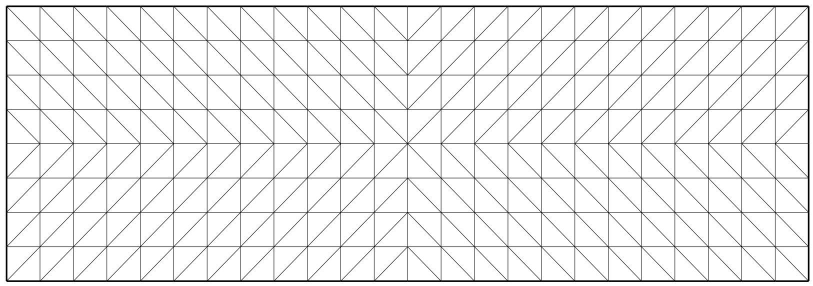

Figure 3.2: Uniform mesh.

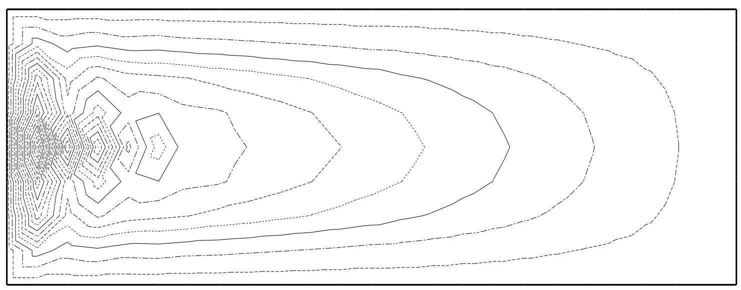

Figure 3.3: Stream lines obtained with a uniform mesh.

A typical strategy to avoid these defects is to use non shape-regular stretched elements aligned with the stream lines in the zone of the boundary layer (see [AFG+00]). We have done this with the mesh shown in Fig. 3.4. Thus, we have obtained a much better resolution of the boundary layer and avoided spurious oscillations, as shown in Fig. 3.5.

The remainder of this work is devoted to design an algorithm for creating well-adapted meshes in an automatic manner by using the computed solution on coarser triangulations.

3.5 A mesh-refinement strategy

We propose an h-adaptive mesh-refinement strategy, based on *a posteriori* error indicators providing information on the stretching direction and ratio where refinement and/or coarsening are needed. Our error indicator is based on a recovery of second derivatives (Hessian) from the finite element solution (see [AFG+00]). We use this information to devise an adaptive process leading to meshes well adapted to the solution, correctly refined, with stretched and oriented elements aligned with its singularities.

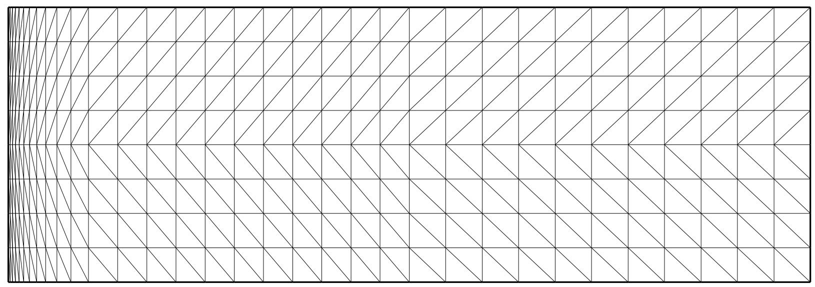

Figure 3.4: Graded mesh over-refined around the boundary layer.

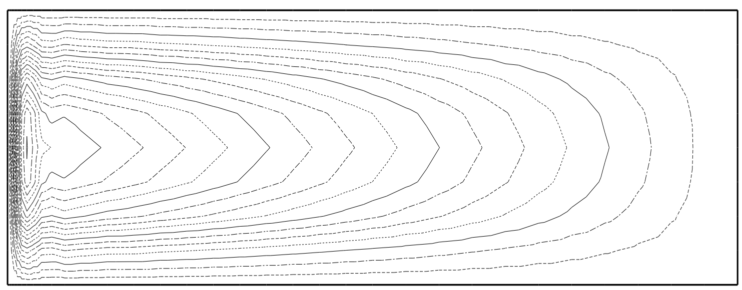

Figure 3.5: Stream lines obtained with the graded mesh.

3.5.1 An anisotropic error indicator

The heuristics behind our indicator is similar to that in [AFG+00]. It consists of assuming that the finite element solution error can be reasonably approximated by the linear Taylor expansion of the exact solution around the barycenter of each element. Then, given $T \in \mathcal{T}_h$, $\forall x \in T$ we have

$$\psi(x) - \psi_h(x) \approx \psi(x) - \psi_{\mathrm{L}}(x) \approx \frac{1}{2}\,(x - x_T)^{\mathrm{t}} H\psi(x_T)(x - x_T), \tag{3.21}$$

where ψ is the exact solution, ψ_h the finite element solution, ψ_{L} the linear Taylor expansion of ψ around the barycenter x_T of T, and $H\psi$ denotes the Hessian matrix of ψ. Therefore, straightforward computations lead to

$$\|\psi - \psi_h\|_{0,T} \approx \frac{1}{2}\left\{\int_T \left[(x - x_T)^{\mathrm{t}} H\psi(x_T)(x - x_T)\right]^2 dx\right\}^{1/2}. \tag{3.22}$$

Our error indicator consists of computing the last expression by using an approximate Hessian. This is obtained by means of a post-process of the computed solution ψ_h, based on applying twice a technique to recover the gradient.

First, we smooth $\nabla\psi_h$. To do this we use the Clément interpolation operator

$\Pi : L^2(\Omega) \to \mathcal{L}_h$, which is defined by

$$\Pi v := \sum_{i=1}^{N} P_i v(Q_i)\varphi_i,$$

where, for $i = 1, \dots, N$, Q_i are the nodes of the triangulation $\mathcal{T}_h$, φ_i are the standard nodal basis functions of $\mathcal{L}_h$, and $P_i : L^2(S_i) \to \mathcal{P}_0(S_i)$ denotes the orthogonal projection onto the subspace of constant functions defined in $S_i = \operatorname{supp}\varphi_i = \bigcup\{T \in \mathcal{T}_h :\ T \ni Q_i\}$.

Now, we introduce the gradient-recovery operator $\nabla_{\mathrm{R}} :\ \Phi_h \longrightarrow \mathcal{L}_h^2$ given by

$$\nabla_{\mathrm{R}}\psi_h := \begin{bmatrix} \Pi \dfrac{\partial \psi_h}{\partial x_1} \\ \Pi \dfrac{\partial \psi_h}{\partial x_2} \end{bmatrix}.$$

Notice that, since ψ_h is piecewise linear, we have

$$\nabla_{\mathrm{R}}\psi_h(Q_i) = \sum_{T \subset S_i} \frac{|T|}{|S_i|} \nabla(\psi_h|_T),$$

where $|T|$ and $|S_i|$ denote the corresponding areas.

Next, we define the symmetric recovered Hessian matrix H_{S} by

$$H_{\mathrm{S}}\psi_h := \frac{H_{\mathrm{R}}\psi_h + H_{\mathrm{R}}^{\mathrm{t}}\psi_h}{2} \in \mathcal{L}_h^{2\times 2},$$

where

$$H_{\mathrm{R}}\psi_h := \left[\ \nabla_{\mathrm{R}}\left(\Pi \frac{\partial \psi_h}{\partial x_1}\right) \ \middle|\ \nabla_{\mathrm{R}}\left(\Pi \frac{\partial \psi_h}{\partial x_2}\right)\ \right],$$

namely, the columns of $H_{\mathrm{R}}\psi_h$ are the recovered gradients of $\Pi \frac{\partial \psi_h}{\partial x_i}$.

The numerical evidence reported in [AFG⁺00] shows that H_{S} can be safely used as an approximation of H in adaptive processes for convection dominated problems. Then, from (3.22), we define the anisotropic error indicator

$$\eta_T := \left\{ \int_T \left[(x - x_T)^{\mathrm{t}} H_{\mathrm{S}}\psi_h(x_T)(x - x_T)\right]^2 dx \right\}^{1/2} \tag{3.23}$$

and the corresponding global error estimator

$$\eta := \left(\sum_{T \in \mathcal{T}_h} \eta_T^2 \right)^{1/2}.$$

The numerical experiments in Section 3.6 below give evidence that this estimator is equivalent to the error for both, regular and singular solutions.

3.5.2 The adaptive procedure

Our next goal is to design an adaptive procedure to solve Problem 3 on a sequence of meshes up to finally attain a solution with an error within a prescribed tolerance. To attain this purpose, the created meshes must be correctly refined and must contain stretched elements to take care of the anisotropy of the solution.

At each step, a new mesh $\mathcal{T}_{h'}$ better adapted to the solution of Problem 3 must be created. This will be done by using information of the solution computed on the "old" mesh $\mathcal{T}_h$. Two features of $\mathcal{T}_{h'}$ must be determined: the diameter $h_{T'}$ of the elements on different parts of the domain and the stretching and orientation of these elements to correctly solve the boundary layers of the solution.

To do this we use the recovered Hessian matrix on each element $T \in \mathcal{T}_h$. Let λ_1 and λ_2 be the eigenvalues of the symmetric matrix $H_{\mathrm{S}}\psi_h(x_T)$, with $|\lambda_1| \geq |\lambda_2|$. Let q_1 and q_2 be the corresponding orthonormal eigenvectors. Then

$$H_{\mathrm{S}}\psi_h(x_T) = Q\Lambda Q^{\mathrm{t}}, \tag{3.24}$$

with $\Lambda = \mathrm{diag}\{\lambda_1, \lambda_2\}$ and Q the orthogonal matrix of columns q_1 and q_2. Thus, by substituting (3.24) in (3.23), straightforward computations lead to

$$\eta_T^2 = \int_T \left\{ \sum_{i=1}^{2} |\lambda_i| \left[(x - x_T)^{\mathrm{t}} q_i \right]^2 \right\}^2 dx. \tag{3.25}$$

We define $h_i := \max_{x \in T} |(x - x_T)^{\mathrm{t}} q_i|$, $i = 1, 2$, which can be seen as a typical length of the element T in the direction q_i (actually, it is proportional to the length of the projection of T onto the span of q_i). We also call x^1, x^2 and x^3 the coordinates of the vertices of T. The following lemmas will be used to obtain a useful equivalence for the local indicator η_T.

Lemma 3.5.1 *For all $w \in \mathbb{R}^2$*

$$\int_T \left[(x - x_T)^{\mathrm{t}} w \right]^4 dx = \frac{|T|}{30} \sum_{k=1}^{3} \left[(x^k - x_T)^{\mathrm{t}} w \right]^4 .$$

Proof: Let $x \mapsto A\widehat{x} + b$ be a transformation applying the reference triangle $\widehat{T}$ of vertices $(0, 0)$, $(1, 0)$, and $(0, 1)$ onto T. Then we have

$$\int_T \left[(x - x_T)^{\mathrm{t}} w \right]^4 dx = \det(A) \int_{\widehat{T}} \left[(\widehat{x} - \widehat{x}_T)^{\mathrm{t}} A^{-t} w \right]^4 d\widehat{x}.$$

On the reference element the integration rule of the lemma can be verified by testing it with adequate fourth order polynomials. Then we conclude the proof by a new change of coordinates.

□

Lemma 3.5.2 *For all $T \in \mathcal{T}_h$, there exists $\alpha_T \in [1/30, 2]$ such that*

$$\eta_T^2 = \alpha_T |T| \sum_{i=1}^{2} \lambda_i^2 h_i^4.$$

Proof: It is easy to see from (3.25) that

$$\int_T \sum_{i=1}^{2} \lambda_i^2 \left[(x - x_T)^{\mathrm{t}} q_i\right]^4 dx \le \eta_T^2 \le 2 \int_T \sum_{i=1}^{2} \lambda_i^2 \left[(x - x_T)^{\mathrm{t}} q_i\right]^4 dx.$$

On one hand it is immediate that

$$\int_T \lambda_i^2 \left[(x - x_T)^{\mathrm{t}} q_i\right]^4 dx \le |T| \lambda_i^2 h_i^4.$$

On the other hand, from Lemma 3.5.1 we have

$$\int_T \lambda_i^2 \left[(x - x_T)^{\mathrm{t}}\ q_i\right]^4 dx = \frac{|T|}{30} \sum_{k=1}^{3} \lambda_i^2 \left[(x^k - x_T)^{\mathrm{t}} q_i\right]^4 \ge \frac{1}{30} |T| \lambda_i^2 h_i^4,$$

since $h_i = \max_{1 \le k \le 3} |(x^k - x_T)^{\mathrm{t}} q_i|$. Thus we conclude the lemma. □

In what follows we show that if the triangle is correctly oriented, then its area is equivalent to the product of its typical lengths $h_1 h_2$. In fact, assuming that the triangle T is stretched in the direction of q_2 (which corresponds to the direction of smallest variation of the solution) we prove that $|T| = c_T h_1 h_2$ with a constant c_T essentially independent of T.

First we prove this result for a conveniently scaled arbitrary triangle as that in Fig. 3.6.

Lemma 3.5.3 *Let T be a triangle as shown in Fig. 3.6. If $|\tan \beta| \le x_1/(2x_2)$, then*

$$\frac{8}{21} |\widehat{T}_A| \le \widehat{h}_1 \widehat{h}_2 \le |\widehat{T}_A|,$$

where $\widehat{h}_i := \max_{x \in \widehat{T}_A} |(x - x_{\widehat{T}_A})^{\mathrm{t}} q_i|$, $i = 1, 2$.

Proof: First notice that according to Fig. 3.6 we have

$$\frac{1}{2} \le x_2 \le 1, \qquad 0 \le x_1 \le \frac{\sqrt{3}}{2},$$

and the barycenter of the triangle is $x_{\widehat{T}_A} = (0, 0)$. Moreover, the vertical axis splits $\widehat{T}_A$ into two triangles of area $3x_1/4$. Hence, $|\widehat{T}_A| = 3x_1/2$.

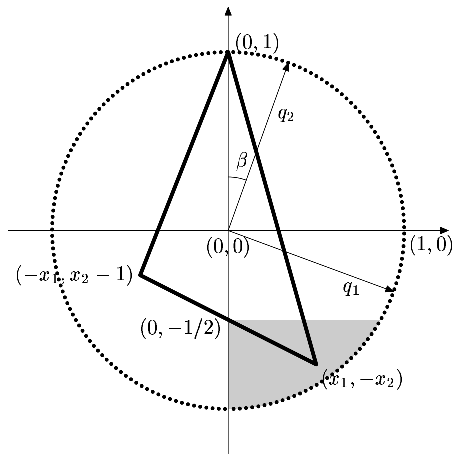

Figure 3.6: Anisotropic reference element $\widehat{T}_A$

Second, the explicit computation of the typical lengths yields

$$\begin{aligned}\widehat{h}_1 &= \max\{|\sin\beta|\,,\ |x_1\cos\beta + x_2\sin\beta|\,,\ |x_1\cos\beta-(1-x_2)\sin\beta|\}.\\ \widehat{h}_2 &= \max\{\cos\beta,\ |(1-x_2)\cos\beta + x_1\sin\beta|\,,\ |x_2\cos\beta - x_1\sin\beta|\},\end{aligned}$$

Then, we have

$$\widehat{h}_1 \le \max\{|\sin\beta|\,,\ x_1\cos\beta + x_2\,|\sin\beta|\,,\ x_1\cos\beta + (1-x_2)\,|\sin\beta|\}.$$

Now, since $x_2 \ge 1/2$, for $|\tan\beta| \le x_1/(2x_2)$ we obtain

$$\begin{aligned}&|\sin\beta| \le x_1\cos\beta,\\ &x_2\,|\sin\beta| \le \frac{x_1}{2}\cos\beta \quad\Longrightarrow\quad x_1\cos\beta + x_2\,|\sin\beta| \le \frac{3x_1}{2}\cos\beta,\\ &(1-x_2)\,|\sin\beta| \le x_2\,|\sin\beta| \le \frac{x_1}{2}\cos\beta\\ &\qquad\qquad\Longrightarrow\quad x_1\cos\beta + (1-x_2)\,|\sin\beta| \le \frac{3x_1}{2}\cos\beta.\end{aligned}$$

Hence

$$\widehat{h}_1 \le \frac{3x_1}{2}\cos\beta \le \frac{3x_1}{2} = |\widehat{T}_A|.$$

On the other hand

$$\widehat{h}_2 = \max_{x\in\widehat{T}_A} |x^{\mathrm{t}} q_2| \le \max_{x\in\widehat{T}_A} |x| \le 1,$$

and thus

$$\widehat{h}_2\widehat{h}_1 \le |\widehat{T}_A|.$$

For the other inequality notice that, independently of the sign of β, either $x_2 \sin\beta$ or $-(1-x_2)\sin\beta$ must be non negative. Then, the explicit expression of $\widehat{h}_1$ above yields

$$\widehat{h}_1 \geq x_1 \cos\beta.$$

On the other hand, clearly

$$\widehat{h}_2 \geq \cos\beta.$$

Hence, since $x_1 \leq \sqrt{3}/2$ and $x_2 \geq 1/2$, for $|\tan\beta| \leq x_1/(2x_2)$ we obtain

$$\widehat{h}_2\widehat{h}_1 \geq x_1 \cos^2\beta = \frac{x_1}{1+\tan^2\beta} \geq \frac{x_1}{1+x_1^2/(4x_2^2)} \geq \frac{4}{7}\,x_1 = \frac{8}{21}\,|\widehat{T}_A|.$$

Thus we conclude the proof. □

Notice that any triangle $T \in \mathcal{T}_h$ is the image by a scaling followed by a rigid motion of a triangle $\widehat{T}_A$ as in Fig. 3.6. Indeed by choosing orthogonal coordinates centered at the barycenter of T with the vertical axis coinciding with the longest median of the triangle, a scaling map applies T onto a triangle like $\widehat{T}_A$. Then, since the properties in Lemma 3.5.3 are rigid-motion and scale invariant, they are also valid for $T \in \mathcal{T}_h$. On the other hand, the constraint $\tan\beta \leq x_1/(2x_2)$ is fulfilled whenever the triangle is reasonably stretched in the direction q_2. Then, in this case, we have

$$|T| = c_T h_1 h_2, \qquad \text{with } c_T \in [8/21, 1]. \tag{3.26}$$

An adaptive algorithm to minimize the error indicator (3.25) should choose the typical lengths h_i' of each element T' in the new mesh $\mathcal{T}_{h'}$ in order to equilibrate the two terms in this sum; namely, such that

$$|\lambda_1| h_1'^2 \approx |\lambda_2| h_2'^2. \tag{3.27}$$

Actually, the values λ_1 and λ_2 correspond to the "old" mesh $\mathcal{T}_h$. However, they are approximations of the eigenvalues of the Hessian of the exact solution $H\psi(x_T)$. Then, it is reasonable to ask for $|\lambda_1| h_1'^2 \approx |\lambda_2| h_2'^2$.

Thus, it only remains to decide how to choose one of the diameters (for example h_1') to compute a solution with estimated error within a prescribed tolerance `tol`.

If we assume that the elements T of the mesh $\mathcal{T}_h$ are correctly stretched as to satisfy $|\lambda_1| h_1^2 \approx |\lambda_2| h_2^2$, then from Lemma 3.5.2 we have

$$\eta_T^2 \approx 2\alpha_T \lambda_1^2 |T| h_1^4.$$

Moreover, under the assumption of Lemma 3.5.3, from (3.26) there holds

$$|T| = c_T h_1 h_2 \approx c_T \left(\frac{|\lambda_1|}{|\lambda_2|}\right)^{1/2} h_1^2.$$

Then

$$\eta_T^2 \approx 2\alpha_T c_T \frac{|\lambda_1|^{5/2}}{|\lambda_2|^{1/2}} h_1^6 = C_T h_1^6. \tag{3.28}$$

Let us remark that the constant $C_T := 2\alpha_T c_T |\lambda_1|^{5/2}/|\lambda_2|^{1/2}$ depends on the Hessian of the solution on T. Thus it depends on the localization of the element T in Ω, but neither on h_1 nor on the triangle shape ratio.

Therefore, if the new mesh $\mathcal{T}_{h'}$ is created as to satisfy (3.27), then we also have

$$\eta_{T'}^2 \approx C_{T'} h_1'^6 \approx C_T h_1'^6, \tag{3.29}$$

for any element $T' \in \mathcal{T}_{h'}$ located in the same zone of Ω as T.

The new mesh will be created such that

$$\eta' = \left(\sum_{T' \in \mathcal{T}_{h'}} \eta_{T'}^2\right)^{1/2} \approx \texttt{tol},$$

with $\eta_{T'}$ being approximately the same for all elements $T' \in \mathcal{T}_{h'}$ (the latter is typically the alternative yielding the mesh with a least number of elements). Therefore

$$\eta_{T'}^2 \approx \frac{\texttt{tol}^2}{N'},$$

with N' being the number of elements $T' \in \mathcal{T}_{h'}$. Hence, by using (3.29) and (3.28), we have that h_1' must be chosen as follows:

$$h_1' \approx \left(\frac{1}{C_T}\frac{\texttt{tol}^2}{N'}\right)^{1/6} \approx \left(\frac{h_1^6}{\eta_T^2}\frac{\texttt{tol}^2}{N'}\right)^{1/6} \approx h_1 \left(\frac{\texttt{tol}}{\eta_T \sqrt{N'}}\right)^{1/3}. \tag{3.30}$$

This value of h_1' depends on the unknown number of elements N' of $\mathcal{T}_{h'}$. However this number appears as a scale factor affecting in the same way all the estimated h_1'. Therefore, different values of N' will produce different degrees of refinement but with the same refinement pattern. (i.e., the relative sizes of the elements remain equal). Because of this, it is not so important to know the value of N' in advance. In the experiment reported below, we have just considered $N' = N$.

In practice a quasi-uniform isotropic mesh is initially used. In such case it is not convenient to prescribe a very small tolerance from the beginning. A better strategy consists of using a set of diminishing tolerances up to attaining the prescribed one. Then, at each step, a new more stringent tolerance is used.

The adaptive strategy consists of obtaining an approximate solution from an initial isotropic mesh and to recover the corresponding second derivatives at each node. Then, we introduce this information to the mesh generator.

For our experiments we have used the mesh generator `BL2D` (see [BLB95]). The information that must be transmitted to this software to create a new mesh is a metric at each vertex. In our case the transmitted metrics are obtained by averaging element metrics on all the triangles sharing each vertex. These element metrics are given by

$$Q \begin{bmatrix} 1/h_1' & 0 \\ 0 & 1/h_2' \end{bmatrix} Q^{\mathrm{t}},$$

where, according to (3.30) and (3.27),

$$h_1' = h_1 \left(\frac{\mathtt{tol}}{\eta_T \sqrt{N'}} \right)^{1/3} \quad \text{and} \quad h_2' = \sqrt{\frac{|\lambda_2|}{|\lambda_1|}} h_1',$$

and λ_1, λ_2, and Q are given by (3.24).

Let us remark that we have chosen to generate a new mesh at each step instead of refining the previous one. This choice looks much simpler and avoids eventual constraints imposed by an initial coarse isotropic mesh. On the other hand, in all the numerical tests reported in Section 3.6, the elapsed time to generate each new mesh has been always negligible compared to the time needed to solve the corresponding linear systems.

3.5.3 An error estimator for the velocity field

Since the variable of interest is usually the velocity field $u = \vec{\mathrm{curl}}\, \psi$, in what follows we introduce an estimator for the computed velocities $u_h = \vec{\mathrm{curl}}\, \psi_h$.

Under the same assumptions of Section 3.5.2, we have from (3.21)

$$\|u - u_h\|_{0,T} = |\psi - \psi_h|_{1,T} \approx \left[\int_T |H\psi(x_T)(x - x_T)|^2 \, dx \right]^{1/2},$$

(see also [CD96]). Then, we obtain the following local estimator:

$$\widetilde{\eta}_T := \left\{ \int_T (x - x_T)^{\mathrm{t}} [H_{\mathrm{R}} \psi_h(x_T)]^{\mathrm{t}} [H_{\mathrm{R}} \psi_h(x_T)] (x - x_T) \, dx \right\}^{1/2}.$$

In the experiments below we show that the local effectivity indices defined by $\widetilde{\eta}_T/|\psi - \psi_h|_{1,T}$ are very close to one and the estimated global error

$$\widetilde{\eta} := \left(\sum_{T \in \mathcal{T}_h} \widetilde{\eta}_T^2 \right)^{1/2}$$

attains an optimal order of convergence, too.

Remark 3.5.1 *In spite of the fact that $\widetilde{\eta}_T$ seems to be an excellent estimator of the error in $H_0^1(\Omega)$ norm, we have chosen η_T to create the meshes. Indeed, we have tried also $\widetilde{\eta}_T$ for the adaptive procedure, but the obtained results were not so reliable, very likely because of limitations of the mesh generator.*

3.6 Numerical results

In this section we present several numerical experiments. First, to test our strategy, we consider a problem with a known smooth exact solution. In the second example we simulate the typical Western boundary currents by using an analytically known exponential boundary layer solution. In the third one, we consider a more realistic example with the wind stress taken from [MW95]. Finally, in the last one, we repeat this experiment in a non-convex domain.

In all the tests we have taken the following realistic physical parameters:

$$\epsilon_m = 6 \times 10^{-5}, \qquad \epsilon_s = 0.05.$$

We have initiated always the adaptivity process with a coarse isotropic mesh and a rough tolerance, which has been reduced to one half at each step.

3.6.1 Test 1

We take a problem with the same solution as in Section 3.4.1,

$$\psi(x) = \sin^2 \frac{\pi x_1}{3} \, \sin^2 \pi x_2,$$

but with the realistic physical parameters ϵ_m and ϵ_s indicated above. The geometry of the domain is shown in Fig. 3.7. This figure shows an intermediate obtained mesh and the corresponding stream lines solution. Notice that our strategy generate isotropic meshes as expected.

Fig. 3.8 shows that both, the exact and estimated global errors, attain optimal orders of convergence in $L^2(\Omega)$ and $H_0^1(\Omega)$ norms in terms of the number N of nodes:

$$\eta \approx \|\psi - \psi_h\|_{0,\Omega} = \mathcal{O}(N^{-1}), \qquad \widetilde{\eta} \approx |\psi - \psi_h|_{1,\Omega} = \mathcal{O}(N^{-1/2}).$$

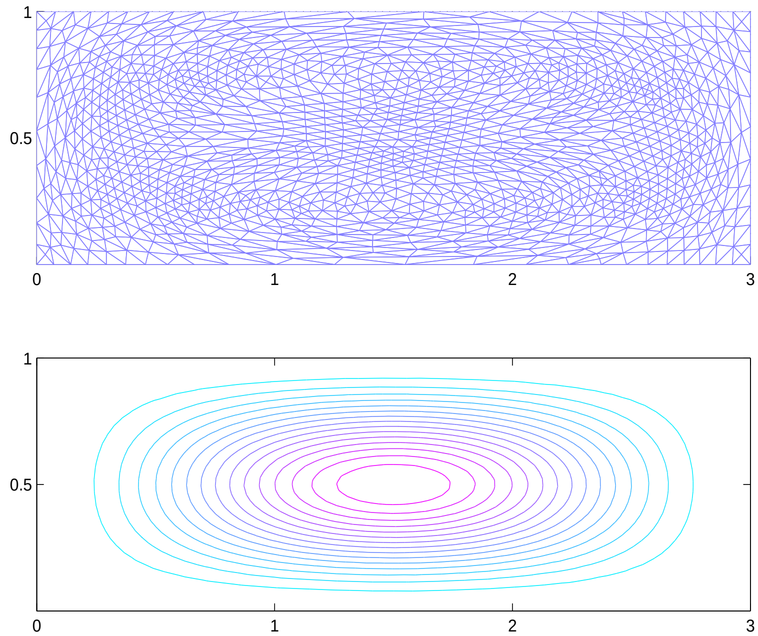

Figure 3.7: Test 1; iteration 2: 1298 nodes mesh. Computed stream lines.

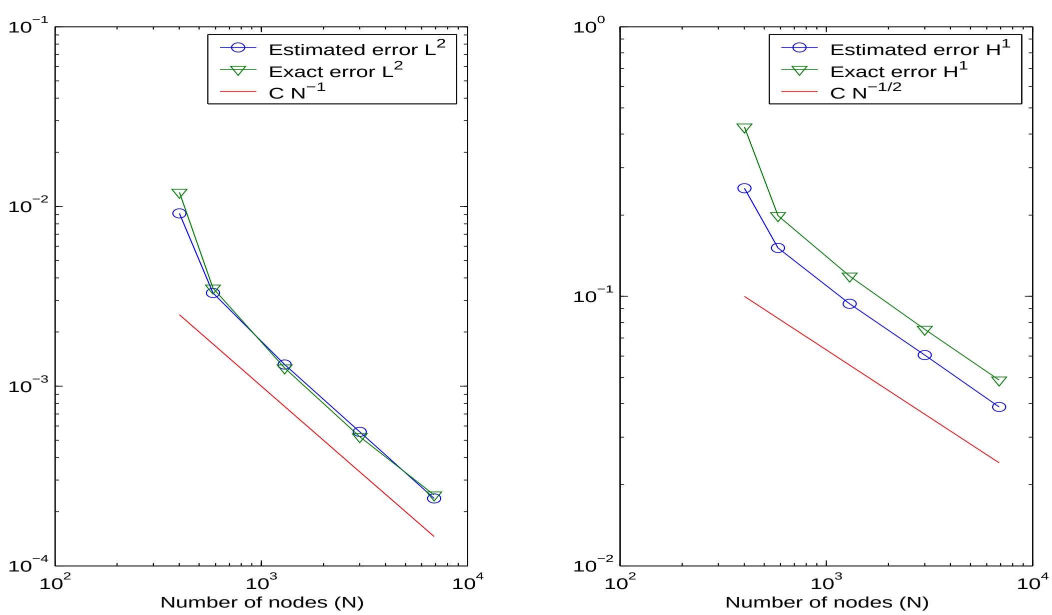

Figure 3.8: Test 1. Estimated and exact errors versus number of nodes (log-log scale).

Finally, Fig. 3.9 shows that the effectivity indices for $\widetilde{\eta}_T$ range between 0.4 and

1 for almost all the elements.

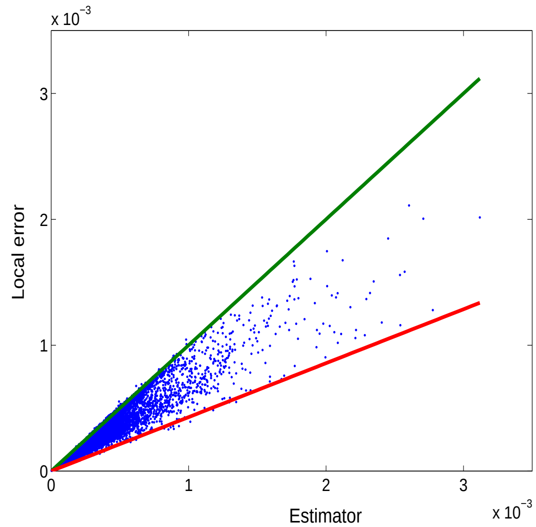

Figure 3.9: Test 1. Local estimator $\widetilde{\eta}_T$ versus local error $|\psi - \psi_h|_{1,T}$ $\forall T \in \mathcal{T}_h$ (6878 nodes mesh). The slope of the solid lines are 0.4 and 1.

3.6.2 Test 2

We consider a problem with exact solution

$$\psi(x) = \left[\left(1 - \frac{x_1}{d}\right)\left(1 - e^{-cx_1}\right)\sin \pi x_2\right]^2.$$

For our experiment we have taken $d = 3$ and $c = 20$. Then the function ψ exhibits a boundary layer on the left.

Fig. 3.10 shows an intermediate refined mesh and the corresponding stream lines solution. Let us remark that our refinement strategy recognizes the boundary layer location and allows avoiding numerical spurious oscillations.

Fig. 3.11 shows that optimal orders of convergence are attained for estimated and exact errors in both norms, again.

Finally, Fig. 3.12 shows that, in spite of the boundary layer, the effectivity indices are close to one, too.

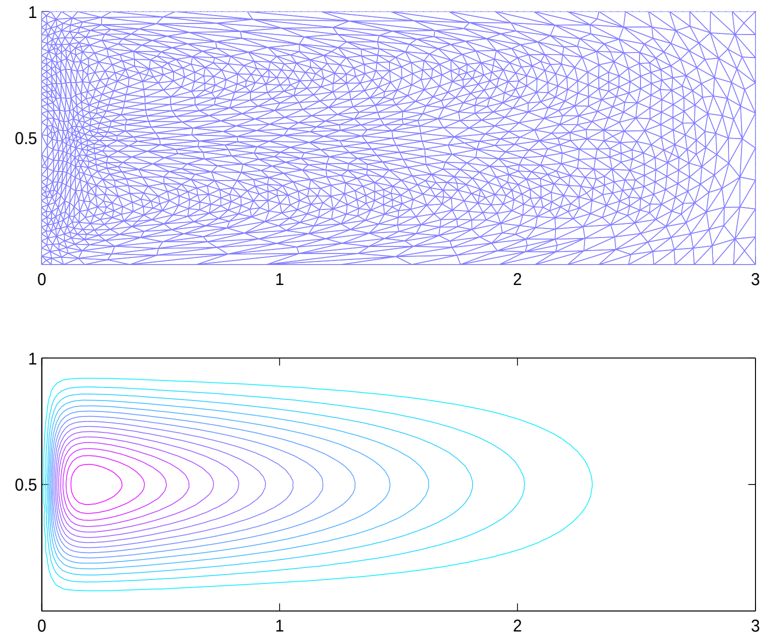

Figure 3.10: Test 2; iteration 4: 3267 nodes mesh. Computed stream lines.

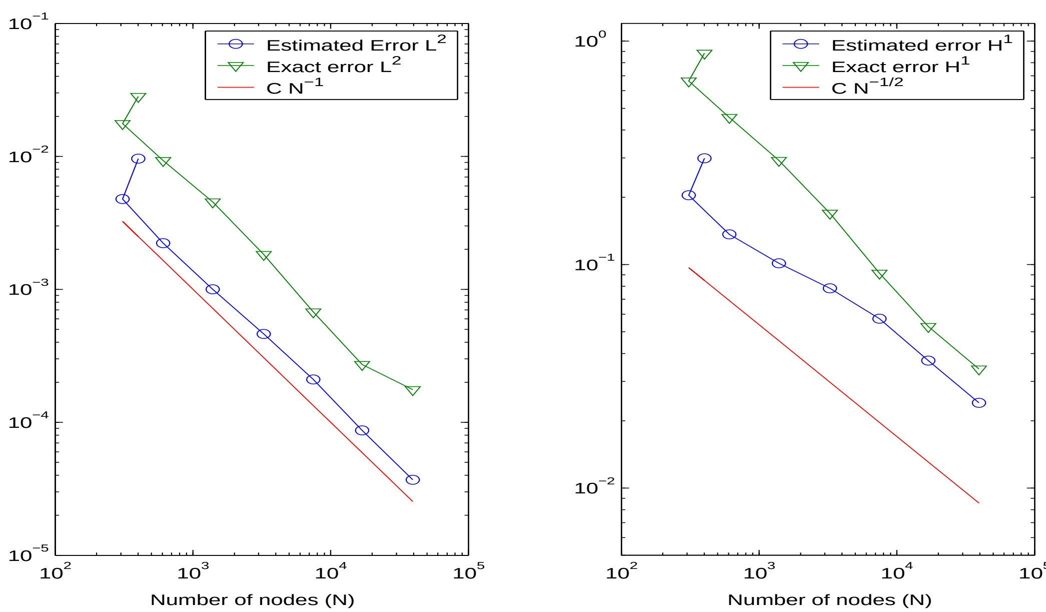

Figure 3.11: Test 2. Estimated and exact errors versus number of nodes (log-log scale).

3.6.3 Test 3

In this example we consider the wind stress used by Myers and Weaver (see [MW95]):

$$(\tau_1, \tau_2) = \left(-\frac{1}{\pi} \cos \pi x_2, 0 \right).$$

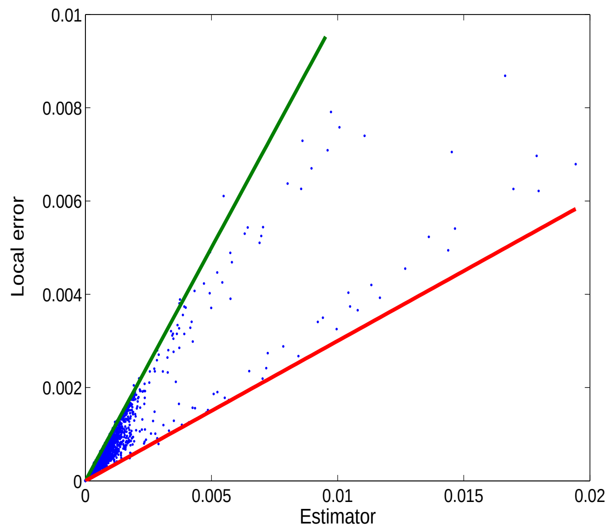

Figure 3.12: Test 2. Local estimator $\widetilde{\eta}_T$ versus local error $|\psi - \psi_h|_{1,T}$ $\forall T \in \mathcal{T}_h$ (7482 nodes mesh). The slope of the solid lines are 0.3 and 1.

Fig. 3.13 shows a zoom of the mesh at an intermediate iteration and the corresponding stream lines solution. The anisotropic nature of the adaptive mesh can be clearly observed from the zoom.

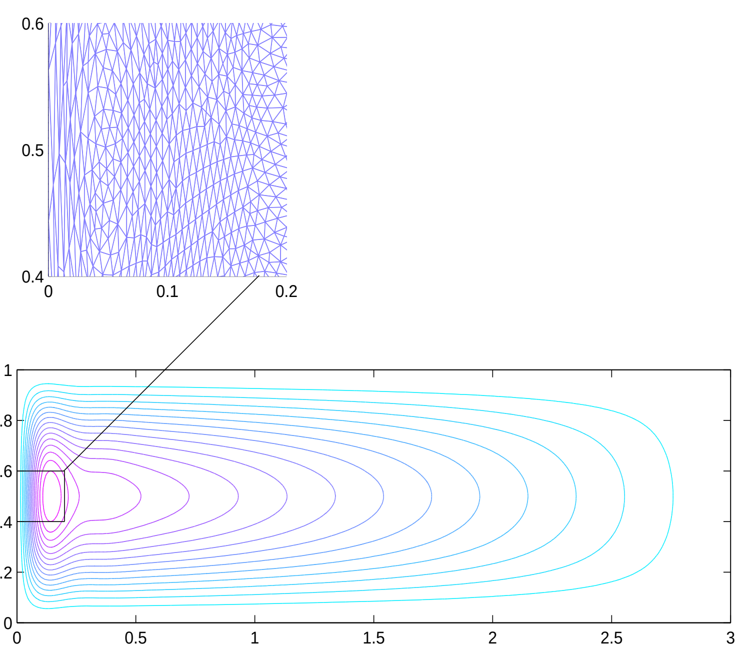

Figure 3.13: Test 3; iteration 6: zoom of a 9103 nodes mesh. Computed stream lines.

In this case there is no analytical solution available. However, Fig. 3.14 shows that the estimated errors attain optimal orders of convergence.

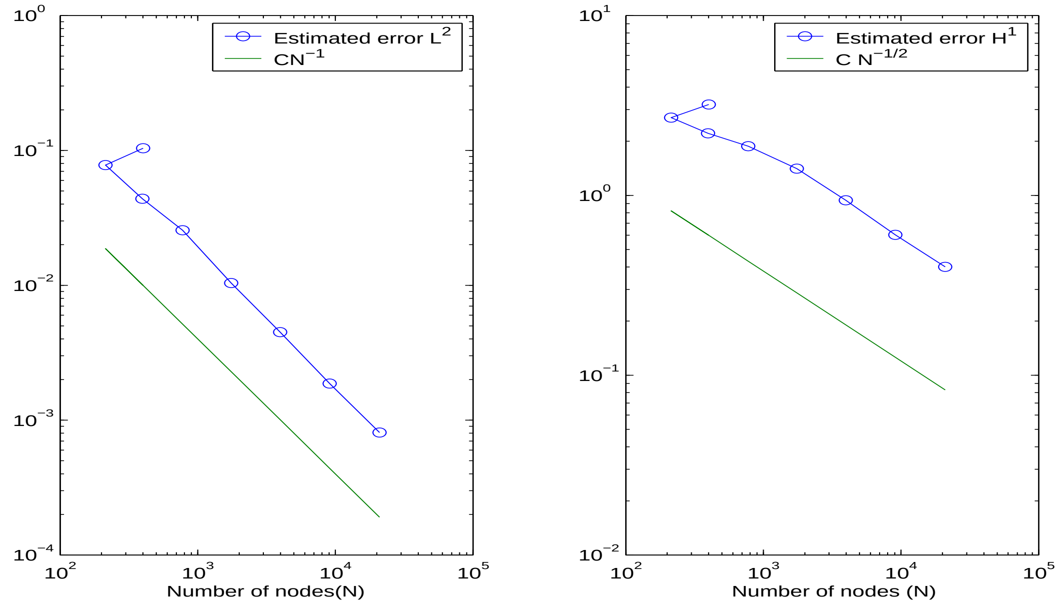

Figure 3.14: Test 3. Estimated error versus number of nodes (log-log scale).

3.6.4 Test 4

We consider the non convex domain shown in Fig. 3.15. This figure shows an intermediate refined mesh and the corresponding stream lines solution.

Fig. 3.16 shows that, for both norms, the estimated errors attain once more optimal orders of convergence.

Let us remark that although the theory in Section 3.4 does not cover this case, the experimental results show that the method combined with the proposed adaptive strategy behaves as well as for convex domains.

3.7 Conclusions

A finite element method to numerically solve the stream function-vorticity formulation of the quasi-geostrophic model has been analyzed. *A priori* error estimates with constants depending on the physical parameters have been proved under appropriate regularity assumptions. Thus, results already known for the two-dimensional Stokes problem have been extended to this model.

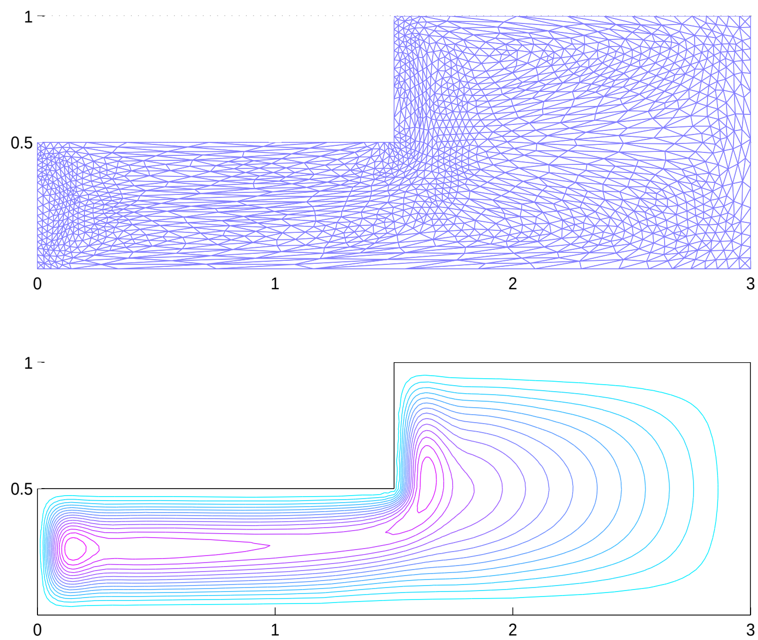

Figure 3.15: Test 4; iteration 4: 1522 nodes mesh. Computed stream lines.

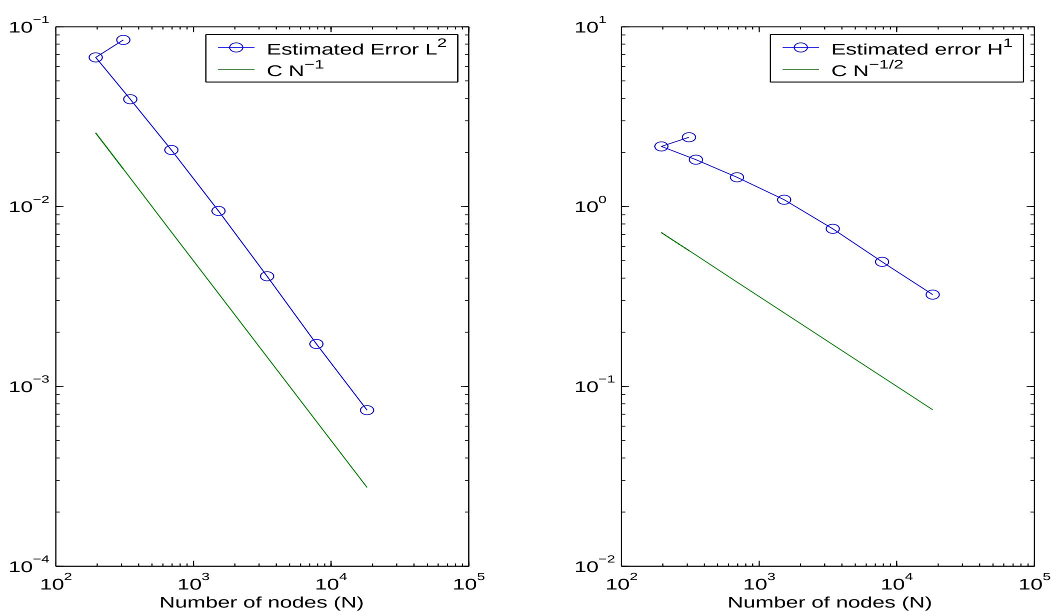

Figure 3.16: Test 4. Estimated error versus number of the nodes (log-log scale).

For large-scale ocean dynamics, these *a priori* estimates become meaningless.

Indeed, the Coriolis convective term dominates these equations and Western-current boundary layers appear. Then, well-adapted meshes become necessary for the method to work, avoiding spurious oscillations. An adaptive procedure to create such meshes in an automatic fashion is introduced. This strategy relies on an anisotropic error indicator based on a recovered Hessian. Several numerical experiments allow assessing the efficiency of this approach. In particular, optimal orders of convergence are attained in all the experiments.

Appendices

Appendix A

An existence and uniqueness result

In this appendix we will describe a few results concerning the existence and uniqueness of solutions of the linear evolution ocean model introduced in Section .

Let Ω be a non-empty open bounded subset of $\mathbb{R}^2$, with a Lipschitz-continuous boundary Γ and let $T > 0$ be fixed. Let $Q = \Omega \times (0, T)$ and $\Sigma = \Gamma \times (0, T)$. We consider the linear quasi-geostrophic ocean model described by the following equations:

$$\begin{cases} R_o \dfrac{\partial u}{\partial t} - A_H \Delta u + \gamma u + (f_0 + \beta x_2)\, \mathrm{k} \wedge u + \dfrac{1}{\rho_0} \nabla p = \dfrac{1}{\rho_0 D_0} \mathcal{T} \quad \text{in } Q, \\ \operatorname{div} u = 0 \quad \text{in } Q, \\ u = 0 \quad \text{on } \Sigma, \\ u(0) = u_0 \quad \text{in } \Omega. \end{cases} \tag{A.1}$$

where $u(x,t)$ and $p(x,t)$ denote the velocity and the pressure of the fluid and $\mathcal{T}$ is a given source. In order to simplify the notation, we take $R_o = 1$, $A_H = 1$, $\gamma = 1$, $f_0 = 1$, $\beta = 1$, $\rho_0 = 1$ and $D_0 = 1$.

Let us introduce the following spaces, which are usual in the analysis of Stokes systems

$$\begin{aligned} H &:= \{v \in L^2(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega, \ v \cdot \nu = 0 \text{ on } \Gamma\}, \\ V &:= \{v \in H_0^1(\Omega)^2 : \ \operatorname{div} v = 0 \text{ in } \Omega\}. \end{aligned}$$

A standard variational formulation of problem (A.1) is as follows. For a given $\mathcal{T}$ in $L^2(0, T; H^{-1}(\Omega)^2)$ and a given u_0 in H, we consider the problem:

Find $u \in L^2(0,T;V)$ with $du/dt \in L^2(0,T;V')$ such that

$$\begin{cases} \left(\dfrac{\partial u}{\partial t}, v\right) + (\nabla u, \nabla v) + (u,v) + (1+x_2)(\mathrm{k} \wedge u, v) = (\mathcal{T}, v) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall v \in V,\ a.e.\ \text{in } (0,T), \\ u(0) = u_0, \end{cases} \tag{A.2}$$

where we denote by $(\cdot,\cdot)$ the inner product of $L^2(\Omega)$ or $L^2(\Omega)^2$. Notice that, $\left(\frac{\partial u}{\partial t}, v\right) = \frac{d}{dt}(u,v)$.

We know that problems (A.1) and (A.2) are equivalent. Indeed, followig the same idea in Proposition III.1.1 of [Tem84], we have that if u is a solution of (A.2), there exists a distribution p on $(0,T)$ such that (u,p) is a solution of (A.1) in the sense of distribution.

For simplicity, we will consider the case of a simply connected domain Ω. Then, it is well known that the velocity can be expressed by introducing a stream function ψ of u:

$$u := \vec{\mathrm{curl}}\, \psi, \quad \psi \in H_0^1(\Omega).$$

Let us introduce the following mixed formulation of problem (A.1) where the stream function ψ and the vorticity $\omega := \mathrm{curl}\, u$ are the variables. For a given $\mathcal{T}$ in $L^2(0,T;L^2(\Omega)^2)$ and a given ω_0 in $L^2(\Omega)$, we consider the following problem:
Find (ψ, ω) *in* $L^2(0,T;H_0^1(\Omega)) \times L^2(0,T;H^1(\Omega))$ *such that*

$$\begin{cases} \dfrac{d}{dt}(\omega,\phi) + (\vec{\mathrm{curl}}\,\omega, \vec{\mathrm{curl}}\,\phi) + (\omega,\phi) - \left(\dfrac{\partial \psi}{\partial x_1}, \phi\right) = (\mathcal{T}, \vec{\mathrm{curl}}\,\phi) \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \forall \phi \in H_0^1(\Omega),\ a.e.\ \text{in } (0,T), \\ -(\omega,\mu) + (\vec{\mathrm{curl}}\,\psi, \vec{\mathrm{curl}}\,\mu) = 0 \qquad \forall \mu \in H^1(\Omega),\ a.e.\ \text{in } (0,T), \\ \omega(0) = \omega_0. \end{cases} \tag{A.3}$$

In Section A.1, we analyze an abstract variational problem. In Section A.2, we apply this framework to problem (A.2) and in Section A.3 to problem (A.3). We have based our analysis on [BR85, Tem84] where we have introduced some modifications coming from the introduction of a new bilinear form, named $d(\cdot,\cdot)$.

A.1 An abstract time dependent problem

Let X and M denote two real reflexive Banach spaces. Let Y be a real Hilbert space such that X is contained in Y with a continuous and dense imbedding. If X' and Y' are the dual spaces of X and Y respectively, we denote by $\langle\cdot,\cdot\rangle$ the duality pairing between Y' and Y, and also between X' and X.

Let us introduce four continuous bilinear forms $r(\cdot,\cdot)$, $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot)$ defined on $Y\times Y$, $X\times X$, $X\times X$ and $X\times M$, respectively. We also define the closed subset V of X by

$$V := \{v \in X;\ b(v,\mu) = 0,\ \forall \mu \in M\} \tag{A.4}$$

and H the closure of V in Y. We will consider the following identification

$$V \subset H \equiv H' \subset V', \tag{A.5}$$

where each space is dense in the following one and the injections are continuous. As a direct consequence of the previous inclusions, we get

$$\langle f, v\rangle_{V',V} = (f,v)_H \quad \forall f \in H, \forall v \in V. \tag{A.6}$$

We will denote the scalar product in H by $(\cdot,\cdot)$ and the duality pairing between V' and V by $\langle\cdot,\cdot\rangle$.

Given f in $L^2(0,T;X')$ and $u_0 \in H$, let us consider the Problem (Q):

Find $(u,\lambda) \in L^2(0,T;X) \times \mathcal{D}'(0,T;M)$ *such that*

$$\begin{cases} \dfrac{d}{dt} r(u,v) + a(u,v) + d(u,v) + b(v,\lambda) = \langle f, v\rangle \qquad \forall v \in X,\ \text{in}\, \mathcal{D}'(0,T), \\ b(u,\mu) = 0 \qquad \forall \mu \in M,\ a.e.\ \ \text{in}\,(0,T), \\ r(u(0) - u_0, v) = 0 \qquad \forall v \in Y, \end{cases} \tag{A.7}$$

and we associate the Problem (P):

Find $u \in L^2(0,T;V)$ *such that*

$$\begin{cases} \dfrac{d}{dt} r(u,v) + a(u,v) + d(u,v) = \langle f, v\rangle \quad \forall v \in V,\ a.e.\ \text{in}\,(0,T), \\ r(u(0) - u_0, v) = 0 \quad \forall v \in H. \end{cases} \tag{A.8}$$

Notice that, for any (u,λ) solution of (Q), u is solution of (P).

Let us consider the following hypotheses:

(h.1) the space V is separable;

(h.2) the form $a(\cdot,\cdot)$ is symmetric and V-elliptic, i.e., $\exists \alpha > 0$ such that

$$a(v,v) \geq \alpha \|v\|_X^2 \quad \forall v \in V;$$

(h.3) the form $b(\cdot,\cdot)$ satisfies the inf-sup condition, i.e., $\exists \beta > 0$ such that

$$\sup_{v\in X} \frac{b(v,\mu)}{\|v\|_X} \geq \beta \|\mu\|_M \quad \forall \mu \in M; \tag{A.9}$$

(h.4) the form $r(\cdot,\cdot)$ is symmetric in H and H-elliptic, i.e., $\exists\gamma > 0$ such that

$$r(v,v) \geq \gamma\|v\|_Y^2 \quad \forall v \in H;$$

(h.5) the form $d(\cdot,\cdot)$ is antisymmetric and satisfies

$$d(v,v) = 0 \quad \forall v \in V.$$

Let us denote by $\mathcal{W}_R$ the following space:

$$\mathcal{W}_R := \{v \in L^2(0,T;V);\ (Rv)_t \in L^2(0,T;V')\}, \tag{A.10}$$

where R is the linear operator associated with $r(\cdot,\cdot)$, defined from Y into Y' as

$$\langle Ru, v\rangle = r(u,v) \qquad \forall u \in Y,\ \ \forall v \in Y. \tag{A.11}$$

Let us first see that if $u \in L^2(0,T;V)$, the second equation in $(A.8)$ makes sense. We denote by A the linear and continuous operator from V into V', such that

$$\langle Au, v\rangle = a(u,v) \quad \forall v \in V. \tag{A.12}$$

From (h.2), the bilinear form $a(\cdot,\cdot)$ is a scalar product and $\|Au\|_{V'} = \|u\|_V$. It is easy to see that if $u \in L^2(O,T;V)$ then $Au \in L^2(0,T;V')$. We also define the linear and continuous operator $D : V \to V'$ as

$$\langle Du, v\rangle = d(u,v) \quad \forall v \in V. \tag{A.13}$$

We can deduce that $Du \in L^2(0,T;V')$. Indeed, because of the continuity of $d(\cdot,\cdot)$:

$$\|Du\|_{V'} \leq C\|u\|_V \quad \forall u \in V,\ C \geq 0.$$

Integrating with respect to t the above inequality, one has

$$\int_0^T \|Du\|_{V'}^2\, dt \leq C \int_0^T \|u\|_V^2\, dt < \infty.$$

From $(A.11)$, $(A.12)$ and $(A.13)$, we can write $(A.8)$ as follows

$$\frac{d}{dt}\langle Ru, v\rangle = \langle f - Au - Du, v\rangle \quad \forall v \in V. \tag{A.14}$$

From the above analysis $f - Au - Du$ belong to $L^2(0,T;V')$. Using Lemma III.1.1 of [Tem84] and (A.14), we show that $(Ru)_t \in L^2(0,T;V')$ (hence $u \in \mathcal{W}_R$) and that Ru is *a.e.* equal to a continuous function from $[0,T]$ into V'. Then, the second equation in (A.8) has sense.

Now, we will prove the main result of this section.

Theorem A.1.1 *Assume that the hypotheses* (h.1)–(h.5) *are satisfied. Given f and u_0 as in Problem* (Q)*. Then, Problem* (P) *has a unique solution u in $\mathcal{W}_R$, and*

$$u \in \mathcal{C}([0,T];H), \tag{A.15a}$$

$$\|u\|_{\mathcal{W}_R} \leq C\{\|f\|_{L^2(0,T;V')} + \|u_0\|_H\}. \tag{A.15b}$$

Moreover, Problem (Q) *has a unique solution (u, λ) in $\mathcal{W}_R \times \mathcal{D}'(0,T;M)$ where u is the solution of Problem* (P)*.*

We will divide the proof of Theorem A.1.1 into several steps. First, we will proof the existence of the solution of Problem (P). Next, we analyze the uniqueness of the solution of Problem (P) and finally we study Problem (Q).

Proof of the existence in the Theorem A.1.1.

We will use the Faedo-Galerkin method (Theorem III.1.1 in [Tem84]).

1. *Discrete model*

Since A is an adjoint, positive and compact linear operator in a Hilbert space, there exists an orthogonal basis of V, $\{w_i\}_{i\in\mathbb{N}}$ such that

$$Aw_j = \lambda_j w_j, \quad j = 1, 2, \cdots, \quad 0 < \lambda_1 \leq \cdots \lambda_j \leq \cdots \to \infty.$$

For each m, let us define an approximate solution u_m of (A.7) by

$$u_m = \sum_{i=1}^{m} u_{im}(t) w_i \qquad u_{im} = (u_m, w_i) \tag{A.16}$$

and

$$\left\{ \begin{array}{l} \dfrac{d}{dt} r(u_m, w_j) + a(u_m(t), w_j) + d(u_m(t), w_j) = \langle f, w_j \rangle, \quad j = 1, \cdots, m, \\ r(u_m(0) - u_{0m}, w_j) = 0, \end{array} \right. \tag{A.17}$$

where u_{0m} is, for example, the orthogonal projection in H of u_0 on the space spanned by $w_1, \cdots, w_m$ (such that $u_{0m} \to u_0$ in H when $m \to \infty$).

The functions u_{im}, $1 \leq i \leq m$, are scalar functions defined on $[0,T]$. The system $(A.17)$ can be written in matrical form as:

$$\begin{aligned} R\hat{u}' + A\hat{u} + D\hat{u} &= \hat{f}, \\ \hat{u}(0) &= u_{0m}, \end{aligned}$$

where the vector components of $\hat{u}$ are u_{im}, the vector components of $\hat{f}$ are $f_j = \langle f, w_j \rangle$ and the matrix components of R, A and D are

$$R_{ij} = r(w_i, w_j), \quad A_{ij} = a(w_i, w_j), \quad D_{ij} = d(w_i, w_j)$$

respectively. From (h.4) R is invertible then, from the theory of ordinary differential equations, we know that it attains a unique solution on the whole interval $[0, T]$.

Now, we will obtain *a priori* estimates independent of m for the functions u_m and then we pass to the limit.

2. *A priori estimates.*

We multiply equation (A.17) by $u_{im}(t)$ and add these equations for $j = 1, \cdots, m$. Using (h.5), we have

$$\frac{d}{dt} r(u_m, u_m) + a(u_m, u_m) = \langle f, u_m \rangle.$$

From (h.2), there exists a constant $\alpha > 0$, such that

$$\frac{d}{dt} r(u_m, u_m) + \alpha \|u_m\|_X^2 \leq \langle f, u_m \rangle. \tag{A.18}$$

Using Young's inequality, we can majorize the right-hand side of (A.18) by

$$\langle f, u_m \rangle \leq \frac{\alpha}{2} \|u_m\|_X^2 + \frac{1}{2\alpha} \|f\|_{V'}^2, \quad \forall \alpha > 0.$$

Therefore,

$$\frac{d}{dt} r(u_m, u_m) + \frac{\alpha}{2} \|u_m\|_X^2 \leq \frac{1}{2\alpha} \|f\|_{V'}^2. \tag{A.19}$$

Integrating (A.19) from 0 to s, with $s > 0$, we obtain in particular,

$$r(u_m(s), u_m(s)) \leq r(u_{0m}, u_{0m}) + \frac{1}{2\alpha} \int_0^s \|f(t)\|_{V'}^2 \, dt$$

from (h.4), the form $r(\cdot, \cdot)$ define a norm equivalent to the norm in Y, then

$$\gamma \sup_{s \in [0,T]} \|u_m(s)\|_Y^2 \leq \sup_{s \in [0,T]} r(u_m(s), u_m(s)) \leq r(u_0, u_0) + \frac{1}{2\alpha} \int_0^T \|f(t)\|_{V'}^2 \, dt, \tag{A.20}$$

where the right hand side of (A.20) is finite and independent of m; then the sequence u_m is bounded in $L^\infty(0, T; H)$.

On the other hand, integrating $(A.19)$ from 0 to T leads to

$$\begin{aligned} r(u_m(T), u_m(T)) + \frac{\alpha}{2}\int_0^T \|u_m(t)\|_X^2 \, dt &\leq r(u_{0m}, u_{0m}) + \frac{1}{2\alpha}\int_0^T \|f(t)\|_{V'}^2 \, dt \\ &\leq r(u_0, u_0) + \frac{1}{2\alpha}\int_0^T \|f(t)\|_{V'}^2 \, dt. \end{aligned}$$

This shows that the sequence u_m is bounded in $L^2(0,T;V)$.

3. *Passage to the limit.*

Since the sequence of discrete solution u_m is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V)$, then

i) $\exists \{u_{m'}\}_{m' \in \mathbb{N}} \subseteq \{u_m\}_{m \in \mathbb{N}}$ such that $u_{m'} \rightharpoonup u$ in $L^\infty(0,T;H)$, when $m' \to \infty$, i.e.,

$$\int_0^T (u_{m'}(t) - u(t), v(t)) \, dt \to 0 \qquad \forall v \in L^1(0,T;H). \tag{A.21}$$

ii) $\exists \{u_{m'}\}_{m' \in \mathbb{N}} \subseteq \{u_m\}_{m \in \mathbb{N}}$ such that $u_{m'} \rightharpoonup u^*$ in $L^2(0,T;V)$, when $m' \to \infty$, i.e.,

$$\int_0^T \langle u_{m'}(t) - u^*(t), v(t) \rangle \, dt \to 0 \qquad \forall v \in L^2(0,T;V'). \tag{A.22}$$

In particular, ii) is valid for each $v \in L^2(0,T;H)$. Using $(A.6)$ we have for (A.22)

$$\int_0^T (u_{m'}(t), v(t)) \, dt \to \int_0^T (u^*(t), v(t)) \, dt. \tag{A.23}$$

Moreover, since $L^2(0,T;H) \subset L^1(0,T;H)$, we compare $(A.23)$ with $(A.21)$, and using the uniqueness of the limit, we see that

$$\int_0^T (u(t) - u^*(t), v(t)) \, dt = 0 \qquad \forall v \in L^2(0,T;H).$$

Hence,

$$u = u^* \quad \text{in } L^\infty(0,T;H) \cap L^2(0,T;V). \tag{A.24}$$

In order to pass to the limit in $(A.17)$, let us first multiply $(A.17)$ by $\varphi \in \mathcal{C}^1([0,T])$, with $\varphi(T) = 0$, and integrate with respect to t

$$\int_0^T \frac{d}{dt} r(u_m, w_j)\varphi(t) \, dt + \int_0^T (a(u_m, w_j) + d(u_m, w_j))\varphi(t) \, dt = \int_0^T \langle f, w_j \rangle \varphi(t) \, dt,$$

$$j = 1, \cdots, m'.$$

Integrating by parts the first term, we obtain

$$\int_0^T \frac{d}{dt} r(u_m, w_j)\varphi(t)\ dt = -\int_0^T r(u_m, w_j)\varphi'(t)\ dt - r(u_{0m}, w_j)\varphi(0).$$

We pass to the limit for $m = m' \to \infty$ in the following integrals

$$\begin{aligned} -\int_0^T r(u_m, w_j)\varphi'(t)\ dt &\to -\int_0^T r(u, w_j)\varphi'(t)\ dt, \\ \int_0^T a(u_m, w_j)\varphi(t)\ dt &\to \int_0^T a(u, w_j)\varphi(t)\ dt, \\ (u_{0m}, w_j)\varphi(0) &\to r(u_0, w_j)\varphi(0). \end{aligned}$$

Here we have used the fact that $r(\cdot, \cdot)$ and $a(\cdot, \cdot)$ are scalar products in H and V respectively. It remains to prove that

$$\int_0^T d(u_m(t), w_j)\varphi(t)\ dt \to \int_0^T d(u(t), w_j)\varphi(t) dt.$$

To do this, we have

$$\begin{aligned} \int_0^T \{(Du_m(t), w_j)\varphi(t) - (Du(t), w_j)\varphi(t)\}\, dt &\le \int_0^T |(Du_m(t) - Du(t), w_j)\varphi(t)|\, dt \\ &\le \int_0^T \|Du_m(t) - Du(t)\|_{V'} |\varphi(t)|\ dt \\ &\le C \int_0^T \|u_m(t) - u(t)\|_V |\varphi(t)|\ dt \\ &\le \tilde{C} \left\{ \int_0^T \|u_m(t) - u(t)\|_V^2\ dt \right\}^{1/2} \to 0. \end{aligned}$$

Therefore, in the limit we find

$$\begin{aligned} -\int_0^T r(u, w_j)\varphi'(t) dt &+ \int_0^T (a(u, w_j) + d(u, w_j))\varphi(t)\ dt \\ &= r(u_0, w_j)\varphi(0) + \int_0^T \langle f, w_j \rangle \varphi(t)\ dt. \end{aligned}$$

The above equality holds for each j; this allows to write

$$\begin{aligned} -\int_0^T r(u, v)\varphi'(t)\ dt &+ \int_0^T (a(u, v) + d(u, v))\varphi(t)\ dt \\ &= r(u_0, v)\varphi(0) + \int_0^T \langle f, v \rangle \varphi(t)\ dt, \end{aligned} \tag{A.25}$$

where v is a finite linear combination of the w_j's. The equality (A.25) is still valid, by density, for each $v \in V$.

Now, let us integrate by part over $(0,T)$, with $\varphi \in \mathcal{D}((0,T))$, we obtain the following equation valid in the distribution sense on $(0,T)$

$$\frac{d}{dt}r(u,v) + a(u,v) + d(u,v) = \langle f, v\rangle \qquad \forall v \in V, \tag{A.26}$$

which is exactly the first equation in (A.8). From (A.26) and (A.24), this implies that $(Ru)_t \in L^2(0,T;V')$ and

$$(Ru)_t + Au + Du = f.$$

It remains to check the initial condition. To this aim, we multiply (A.26) by $\varphi(t)$ (the same function as before), integrate with respect to t and integrate by parts,

$$\begin{aligned} -\int_0^T r(u,v)\varphi'(t)\,dt \;&+\; \int_0^T (a(u,v)+d(u,v))\varphi(t)\,dt \\ &= \; r(u(0),v)\varphi(0) + \int_0^T \langle f, v\rangle \varphi(t)\,dt. \end{aligned}$$

By comparison with (A.25), we notice that

$$r(u_0 - u(0), v)\varphi(0) = 0 \qquad \forall v \in V, \;\; \forall \varphi \in \mathcal{D}((0,T)).$$

We can choose φ such that $\varphi(0) \neq 0$, then

$$r(u_0 - u(0), v) = 0 \quad \forall v \in V.$$

This equality implies that $Ru(0) = Ru_0$ in V' and ends the proof of the existence. □

Proof of the continuity and uniqueness of the solution of Problem (P).

The continuity result follows from Lemma III.1.2 in [Tem84] where, if the function u belongs to $L^2(0,T;V)$ and its derivative u_t belongs to $L^2(0,T;V')$, then u is almost everywhere equal to a function continuous from $[0,T]$ into H. In our problem we apply this lemma for R being the Riesz isomorphism from H onto H'.

Lemma A.1.1 *Assume that* $(h.4)$ *is satisfied. The space* $\mathcal{W}_R$ *is contained in* $\mathcal{C}^0([0,T];H)$ *with a continuous imbedding.*

Lemma A.1.2 *Problem* (P) *has a unique solution* u *in* $\mathcal{W}_R$.

Proof: Let u_1 and u_2 be two solutions of Problem (P),

$$\begin{aligned}\frac{d}{dt}r(u_2-u_1,v)+a(u_2-u_1,v)+d(u_2-u_1,v) &= 0 \quad \forall v\in V,\\ r(u_2(0)-u_1(0),v) &= 0 \quad \forall v\in H.\end{aligned}$$

Choosing $v=u_2-u_1$ and considering (h.2) and (h.5), we have

$$\begin{aligned}\frac{d}{dt}r(v,v)+\alpha\|v\|_X^2 &\leq 0,\\ \frac{d}{dt}r(v,v) &\leq 0,\\ r(v(t),v(t)) &\leq r(v(0),v(0))=0 \quad \forall t\in[0,T].\end{aligned}$$

From (h.4), the form $r(\cdot,\cdot)$ define a norm, hence we conclude the uniqueness of the solution of Problem (P), i.e., $u_1(t)=u_2(t)$ for each t.

Problem (Q).

Let us now analyze Problem (Q). The uniqueness is a direct consequence of (h.3) and the above result. To prove its existence, we follow the same steps as in Proposition III.1.1 of [Tem84] and Theorem 2.1 of [BR85]: If u is the solution of Problem (P) for *a.e.* t in $(0,T)$, we define $L(t)$ in X' by

$$\langle L(t),v\rangle=\int_0^t(\langle f(s),v\rangle-a(u(s),v)-d(u(s),v))\,ds+r(u_0,v)-r(u(t),v) \quad \forall v\in X.$$

Notice that $L\in\mathcal{C}^0([0,T];H)$ and satisfies $\langle L(t),v\rangle=0$, $\forall v\in V$. If B is the operator associated with the form $b(\cdot,\cdot)$, then, by (h.3), we know that the adjoint operator B' is an isomorphism from M onto $V^0=\{g\in X' \ : \ \langle g,v\rangle=0 \ \ \forall v\in V\}$ (see Lemma I.4.1 in [GR86]) and there exists a unique $\Lambda\in\mathcal{C}([0,T];M)$ such that, for every t in $[0,T]$,

$$b(v,\Lambda(t))=\langle B'\Lambda(t),v\rangle=\langle L(t),v\rangle \quad \forall v\in X.$$

If λ denotes the derivative of Λ in $\mathcal{D}'(0,T)$, from (A.7) yields

$$\frac{d}{dt}r(u,v)+a(u,v)+d(u,v)+b(v,\lambda)=\langle f,v\rangle \quad \forall v\in X \quad \text{in } \mathcal{D}'(0,T).$$

From the definition of V, we also have

$$b(u,\mu)=0 \quad a.e. \text{ in } (0,T) \quad \forall \mu\in M.$$

Finally, from (h.4), the form $r(\cdot,\cdot)$ defines a scalar product and yields: $u(0)=u_0$. Then, (u,λ) is the solution of Problem (Q). □

We can prove by the same way as in Proposition III.1.2 of [Tem84], further regularity of the solution.

Proposition A.1.1 *Assume that the hypotheses* (h.1)–(h.4) *are satisfied and that* Ω *is of class* $\mathcal{C}^2$ *or is a convex polygonal. The mapping* $(f, u_0) \to (u, \lambda)$ *is linear continuous from* $L^2(0, H; Y') \times V$ *into* $H^1(0, T; H) \times L^2(0, T; M)$*.*

A.1.1 Another related variational problem

Let us consider a weaker formulation of Problems (P) and (Q). Following [BGR87], we introduce two reflexive Banach spaces $\tilde{X}$ and $\tilde{M}$ such that:

$$X \subset \tilde{X} \subset Y, \qquad \tilde{M} \subset M,$$

where all the imbeddings are continuous and dense. We introduce three continuous bilinear forms $\tilde{a}(\cdot,\cdot)$, $\tilde{d}(\cdot,\cdot)$ and $\tilde{b}(\cdot,\cdot)$ on $\tilde{X} \times X$, $\tilde{X} \times X$ and $\tilde{X} \times \tilde{M}$ respectively, such that

$$\begin{aligned} \tilde{a}(u,v) = a(u,v), \quad \tilde{d}(u,v) &= d(u,v) \qquad \forall u \in X,\ \forall v \in X, \\ \tilde{b}(v,\mu) &= b(v,\mu) \qquad \forall v \in X,\ \forall \mu \in \tilde{M}. \end{aligned} \tag{A.27}$$

Next, we define the closed subset $\tilde{V}$ of $\tilde{X}$ by

$$\tilde{V} := \{ v \in \tilde{X} :\ \tilde{b}(v,\mu) = 0\ \forall \mu \in \tilde{M} \}.$$

We denote by $\langle \cdot,\cdot \rangle$ the duality pairing between $\tilde{X}$ and $\tilde{X}'$.

Here, we suppose that f belong to $L^2(0,T;\tilde{X}')$. We consider the Problem ($\tilde{\mathrm{Q}}$):
Find $(u,\lambda) \in L^2(0,T;\tilde{X}) \times L^2(0,T;\tilde{M})$ *such that*

$$\begin{cases} \dfrac{d}{dt} r(u,v) + \tilde{a}(u,v) + \tilde{d}(u,v) + \tilde{b}(v,\lambda) = \langle f, v \rangle \qquad \forall v \in \tilde{X},\ a.e. \text{ in } (0,T), \\ \tilde{b}(u,\mu) = 0 \qquad \forall \mu \in \tilde{M},\ a.e. \text{ in } (0,T), \\ r(u(0) - u_0, v) = 0 \qquad \forall v \in Y, \end{cases}$$

with ($\tilde{\mathrm{Q}}$), we associate Problem ($\tilde{\mathrm{P}}$):
Find $u \in L^2(0,T;\tilde{V})$ *such that*

$$\begin{cases} \dfrac{d}{dt} r(u,v) + \tilde{a}(u,v) + \tilde{d}(u,v) = \langle f, v \rangle \qquad \forall v \in \tilde{V},\ a.e. \text{ in } (0,T), \\ r(u(0) - u_0, v) = 0 \qquad \forall v \in H. \end{cases}$$

For any (u,λ) solution of ($\tilde{\mathrm{Q}}$), u is a solution of ($\tilde{\mathrm{P}}$).

We shall assume the following additional hypotheses:

(h.6) V is dense in $\tilde{V}$;

(h.7) the form $\tilde{a}(\cdot,\cdot)$ is $\tilde{V}$-elliptic, i.e., $\exists \tilde{\alpha} > 0$ such that

$$\tilde{a}(v,v) \geq \tilde{\alpha}\|v\|^2_{\tilde{X}} \qquad \forall v \in \tilde{V}.$$

(h.8) the form $\tilde{b}(\cdot,\cdot)$ satisfies the weak inf-sup condition, i.e., $\exists \tilde{\beta} > 0$ such that

$$\sup_{v\in\tilde{X}} \frac{\tilde{b}(v,\mu)}{\|v\|_{\tilde{X}}} \geq \tilde{\beta}\|\mu\|_M \quad \forall \mu \in \tilde{M}.$$

Theorem A.1.2 *Assume that the hypotheses* (h.1) − (h.7) *hold. Let* (u,λ) *be the solution of Problem* (Q)*.*

(1) *Problem* ($\tilde{\text{P}}$) *has a unique solution* $\tilde{u}$ *such that* $(R\tilde{u})_t$ *belongs to* $L^2(0,T;\tilde{V}')$*. If the solution of Problem* (P) *is such that* $(Ru)_t$ *belongs to* $L^2(0,T;\tilde{V}')$*, then* u *and* $\tilde{u}$ *coincide.*

(2) *In addition, if the solution* (u,λ) *of Problem* (Q) *is such that* λ *belongs to* $L^2(0,T;\tilde{M})$ *and* $(Ru)_t$ *belongs to* $L^2(0,T;\tilde{X}')$*, then it is the only solution of Problem* ($\tilde{\text{Q}}$)*.*

Proof:

(1) Proceeding in the same way as for Problem (P), we show that Problem ($\tilde{\text{P}}$) has a unique solution. On the other hand, if $(Ru)_t$ belong to $L^2(0,T;\tilde{V}')$ and since V is dense in $\tilde{V}$, we imply that u is a solution of Problem ($\tilde{\text{P}}$).

(2) In addition, if λ belongs to $L^2(0,T;\tilde{M})$ and $(Ru)_t$ belongs to $L^2(0,T;\tilde{X}')$, and since X is dense in $\tilde{X}$, we can show that any solution (u,λ) of Problem (Q) is a solution of Problem ($\tilde{\text{Q}}$). It remains to prove that it is the only solution of ($\tilde{\text{Q}}$). For the first component u it is obvious. For the second one, let us assume that

$$\tilde{b}(v,\lambda) = 0 \qquad \forall v \in \tilde{X}.$$

But then, from (h.7) we necessarily have $\lambda = 0$. This completes the proof.

□

A.2 Application to the velocity - pressure formulation

We will use the abstract framework given in Section A.1 to study Problem (A.1). To do this, we set

$$X = H_0^1(\Omega)^2, \quad Y = L^2(\Omega)^2 \quad \text{and} \quad M = L_0^2(\Omega) = \{q \in L^2(\Omega) \ : \ \int_\Omega q \ dx = 0\}.$$

We denote by $(\cdot,\cdot)$ the inner product on $L^2(\Omega)$ or $L^2(\Omega)^2$. The bilinear form $r(\cdot,\cdot)$ on $Y\times Y$ is the inner product $(\cdot,\cdot)$, so R is the identity operator on Y.

We define the continuous bilinear form $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot)$ by

$$\begin{aligned} a(u,v) &= (\nabla u,\nabla v)+(u,v) \qquad \forall u\in X,\ \forall v\in X,\\ d(u,v) &= (1+x_2)(k\wedge u,v) \qquad \forall u\in X,\ \forall v\in X,\\ b(v,q) &= -(q,\operatorname{div} v) \qquad \forall v\in X,\ \forall q\in M, \end{aligned}$$

and the given source $f=\mathcal{T}$.

One has

$$\begin{aligned} V &= \{v\in H_0^1(\Omega)^2:\ \operatorname{div} v=0 \text{ in } \Omega\},\\ H &= \{v\in L^2(\Omega)^2:\ \operatorname{div} v=0 \text{ in } \Omega,\ v\cdot n=0 \text{ on } \Gamma\}, \end{aligned}$$

where n is the unit outward normal to Ω and V satisfies (h.1). Then Problems (P) and (Q) are well defined and we have the following main result.

Theorem A.2.1 *Given $\mathcal{T}$ in $L^2(0,T;H^{-1}(\Omega)^2)$ and $u_0\in H$, Problem* (P) *has a unique solution u in $L^2(0,T;V)$ such that $du/dt\in L^2(0,T;V')$ and*

$$\|u\|_{L^2(0,T;H_0^1(\Omega)^2)}+\left\|\frac{du}{dt}\right\|_{L^2(0,T;V')}\le c\{\|\mathcal{T}\|_{L^2(0,T;V')}+\|u_0\|_{0,\Omega}\}. \qquad \text{(A.28)}$$

Moreover there exists $p\in\mathcal{D}'(0,T;L_0^2(\Omega))$ such that (u,p) is the unique solution of Problem (Q).

Proof:
This theorem is an immediate consequence of Theorem $A.1.1$. Let us verify the assumptions. The ellipticity property of form $a(\cdot,\cdot)$ is obvious since

$$a(v,v)=\|\nabla v\|_{0,\Omega}^2+\|v\|_{0,\Omega}^2=\|v\|_{1,\Omega}^2.$$

The inf-sup condition of form $b(\cdot,\cdot)$ follows from Corollary I.2.4 of [GR86]: For $q\in L_0^2(\Omega)$ there exists a unique function $v\in V^\perp$ such that

$$\operatorname{div} v=q, \qquad |v|_{1,\Omega}\le C\|q\|_{0,\Omega}.$$

Hence,

$$\frac{(q,\operatorname{div} v)}{|v|_{1,\Omega}}=\frac{\|q\|_{0,\Omega}^2}{|v|_{1,\Omega}}\ge (1/C)\|q\|_{0,\Omega},$$

from which the inf-sup condition (A.9) follows with $\beta = 1/C$. Finally, we also have (h.5). Indeed,

$$\begin{aligned} d(u,v) &= (1+x_2)\{(-u_2,v_1)+(u_1,v_2)\} \\ &= -(1+x_2)\{(u_1,-v_2)+(u_2,v_1)\} = -d(v,u) \end{aligned}$$

which completes the proof. □

Let us consider now the stationary problem (see Chapter 3):
For $g \in X'$, find (u,p) in $X \times M$ such that

$$\begin{cases} -\Delta u + u + (1+x_2)k \wedge u + \nabla p = g \quad \text{in } \Omega, \\ \operatorname{div} u = 0 \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \Gamma. \end{cases} \tag{A.29}$$

We know that if Γ is of class $\mathcal{C}^2$ or if Ω is a convex polygonal domain, the solution (u,p) belongs to $[H^2(\Omega)^2 \cap V] \times [H^1(\Omega) \cap L^2_0(\Omega)]$ and $-\Delta u + u + (1+x_2)k \wedge u + \nabla p$ belongs to $L^2(\Omega)$.

Proposition A.2.1 *The mapping: $(f, u_0) \to (u,p)$ is linear continuous from*

$$L^2(0,T;L^2(\Omega)^2) \times V \text{ into } [L^2(0,T;H^2(\Omega)^2) \cap H^1(0,T;H)] \times L^2(0,T;H^1(\Omega)).$$

Proof:
It is direct consequence of Proposition A.1.1 and of the regularity of stationary problem (A.29) (see Proposition III.1.2 of [Tem84]). □

A.3 Application to the stream function - vorticity formulation.

Let set $Y = H^1_0(\Omega) \times H^{-1}(\Omega)$, $M = L^2(\Omega)$ and $X = H^2_0(\Omega) \times L^2(\Omega)$. We define the continuous bilinear forms $a(\cdot,\cdot)$, $d(\cdot,\cdot)$ and $b(\cdot,\cdot)$ on $X \times X$, $X \times X$ and $X \times M$ respectively, by

$$\begin{aligned} &a(u,v) = (\omega,\theta) + (\omega,\phi), \quad d(u,v) = -\left(\frac{\partial \psi}{\partial x_1}, \phi\right), \\ &b(v,\mu) = -(\Delta\phi + \theta, \mu), \end{aligned}$$

where $u = (\psi,\omega)$ and $v = (\phi,\theta)$. We have

$$V := \{(\phi,\theta) \in X : \ \theta = -\Delta\phi\}, \qquad H := \{(\phi,\theta) \in Y : \ \theta = -\Delta\phi\},$$

where V satisfies (h.1).

We also introduce the bilinear form $r(\cdot,\cdot)$ on $Y \times Y$:

$$r(u,v) = (\omega,\phi),$$

where $u = (\psi,\omega)$ and $v = (\phi,\theta)$ and the corresponding operator $R(\psi,\omega) = (\omega,0)$.

Remark A.3.1 *Notice that, we could also work with $r(\cdot,\cdot)$ defined as $r(u,v) = (\vec{\text{curl}}\,\psi, \vec{\text{curl}}\,\phi)$ and the corresponding operator R by $Ru = (-\Delta\psi, 0)$, with $u = (\psi,\omega)$ and $v = (\phi,\theta)$.*

Finally, given $\mathcal{T}$ in $H^{-1}(\Omega)^2$, we set, for all $v = (\phi,\theta)$ in X,

$$\langle f, v\rangle = (\mathcal{T}, \vec{\text{curl}}\,\phi).$$

Let us consider the Problems (Q) and (P) as in Section A.1.

Theorem A.3.1 *For $(\mathcal{T}, u_0 = (\psi_0,\omega_0))$ given in $L^2(0,T;H^{-1}(\Omega)^2) \times H$, Problem* (P) *has a unique solution $u = (\psi,\omega)$ in $\mathcal{W}_R$, defined in* (A.10)*, and there exists λ in $\mathcal{D}'(0,T;L^2(\Omega))$ such that (u,λ) is the unique solution of Problem* (Q)*. Moreover, λ is equal to ω and belongs to $L^2(0,T;L^2(\Omega))$.*

Proof: Let us verify the assumptions (h.2)-(h.5). The ellipticity property of form $a(\cdot,\cdot)$ follows from the definition of V,

$$\begin{aligned} a(v,v) &= \|\theta\|_{0,\Omega}^2 + (\theta,\phi) = \|\theta\|_{0,\Omega}^2 - (\Delta\phi,\phi) \\ &= \frac{1}{2}\|\theta\|_{0,\Omega}^2 + \frac{1}{2}\|\Delta\phi\|_{0,\Omega}^2 + |\phi|_{1,\Omega}^2 \geq C_1\|(\phi,\theta)\|_X^2, \end{aligned}$$

where $v = (\phi,\theta)$. The inf-sup condition of form $b(\cdot,\cdot)$ follows from the choice $v = (0,-\mu)$, i.e.,

$$\sup_{(\phi,\theta)\in X} \frac{-(\Delta\phi+\theta,\mu)}{\|(\phi,\theta)\|_X} \geq \sup_{(0,\theta)\in X} \frac{-(\theta,\mu)}{\|\theta\|_{0,\Omega}} \geq \|\mu\|_{0,\Omega}.$$

The assumption (h.4) follows from

$$r(u,v) = (\omega,\phi), \qquad r(v,v) = \|\phi\|_{0,\Omega}^2$$

for all $u = (\psi,\omega)$ in H and for all $v = (\phi,\theta)$ in H. Now, we can apply Theorem A.1.1 to prove the first part of the theorem.

To prove that λ is equal to ω, we can follow the same ideas as in the stationary case (see Theorem III.2.1 in [GR86]). Since Problem (Q) admits a unique solution

$((\psi,\omega),\lambda)$, it suffices to check that $((\psi,\omega),\omega)$ is a solution of Problem (Q). Indeed, taking $\phi = 0$ in the first equation of Problem (Q), we obtain

$$(\omega - \lambda, \theta) = 0 \qquad \forall \theta \in L^2(\Omega),$$

then $\lambda = \omega$. □

Proposition A.3.1 *Let Ω be of class $\mathcal{C}^2$ or a convex polygonal. Let $\mathcal{T} \in L^2(0,T;L^2(\Omega)^2)$ and $(\psi_0,\omega_0) \in V$, then*

$$(\psi,\omega) \in L^2(0,T;H^3(\Omega) \times H^1(\Omega)) \cap H^1(0,T;H) \quad \textit{and } \lambda \in L^2(0,T;H^1(\Omega)).$$

Proof: It is a direct consequence of Proposition A.1.1 and of the regularity properties of the stationary problem. □

Let us introduce the weaker formulations of Problems (Q) and (P) as in Section A.1.1. We set $\tilde{X} = H_0^1(\Omega) \times L^2(\Omega)$ and $\tilde{M} = H^1(\Omega)$.

We introduce the bilinear forms

$$\tilde{a}(u,v) = (\omega,\theta) + (\omega,\phi), \qquad \tilde{d}(u,v) = -\left(\frac{\partial \psi}{\partial x_1}, \phi\right),$$
$$\tilde{b}(v,\mu) = (\vec{\operatorname{curl}}\,\phi, \vec{\operatorname{curl}}\,\mu) - (\theta,\mu),$$

where $u = (\psi,\omega)$ and $v = (\phi,\theta)$. We define the space $\tilde{V}$ by

$$\tilde{V} := \{(\phi,\theta) \in H_0^1(\Omega) \times L^2(\Omega) \;:\; (\vec{\operatorname{curl}}\,\phi, \vec{\operatorname{curl}}\,\mu) - (\theta,\mu) = 0 \;\forall \mu \in \tilde{M}\}.$$

We have the following result proved in Lemma III.2.1 of [GR86], which check the assumption (h.6).

Lemma A.3.1 *The spaces $\tilde{V}$ and V are the same. Moreover, Problem* ($\tilde{\mathrm{P}}$) *and* (P) *coincide.*

Now we can state the following theorem.

Theorem A.3.2 *Let us assume that Ω is of class $\mathcal{C}^2$ or is a convex polygonal. For $(\mathcal{T},(\psi_0,\omega_0))$ given in $L^2(0,T;L^2(\Omega)^2) \times V$, the solution of Problems* (Q) *and* ($\tilde{\mathrm{Q}}$) *are the same, i.e.,*

$$\psi \in L^2(0,T;H_0^1(\Omega)) \qquad \omega \in L^2(0,T;H^1(\Omega)) \cap H^1(0,T;H^{-1}(\Omega)).$$

Proof: The proof is an application of Theorem A.1.2 together with Proposition A.3.1 and Lemma A.3.1. Let us check the assumptions. We have already checked (h.2)-(h.5). The ellipticity condition (h.7) follows from the definition of $\tilde{V}$ and Poincaré's inequality,

$$a(v,v) = \|\theta\|^2_{0,\Omega} + (\theta,\phi) = \|\theta\|^2_{0,\Omega} + \|\nabla\phi\|^2_{0,\Omega} \geq C\|(\phi,\theta)\|^2_{\tilde{X}}.$$

On the other hand, the form $\tilde{b}(\cdot,\cdot)$ satisfies the weak inf-sup condition (h.8), which follows from (A.9) and (A.27):

$$\begin{aligned}\sup_{(\phi,\theta)\in\tilde{X}} \frac{\tilde{b}((\phi,\theta),\mu)}{\|(\phi,\theta)\|_{\tilde{X}}} &\geq \frac{1}{C}\sup_{(\phi,\theta)\in X} \frac{\tilde{b}((\phi,\theta),\mu)}{\|(\phi,\theta)\|_{X}} = \frac{1}{C}\sup_{(\phi,\theta)\in X} \frac{b((\phi,\theta),\mu)}{\|(\phi,\theta)\|_{X}} \\ &\geq \frac{\beta}{C}\|\mu\|_{0,\Omega} \qquad \forall\mu\in H^1(\Omega).\end{aligned}$$

In view of Proposition A.3.1 (λ belongs to $H^1(\Omega)$) and Lemma A.3.1, we have proved all the assumptions of Theorem A.1.2. Then we conclude the proof of Theorem A.3.2. □

Appendix B

Proof of a global Carleman inequality

In this appendix we will proof Lemma 1.3.2. We will adapt to this framework the arguments presented in [FCGIP04] and [Ima01]. Let us consider the system

$$\begin{cases} -z_t - \Delta z + z - (1+x_2)\,\mathrm{k}\wedge z + \nabla r = \phi 1_{\mathcal{O}} \quad \text{in } Q, \\ \operatorname{div} z = 0 \quad \text{in } Q, \\ z = 0 \quad \text{on } \Sigma, \\ z(T) = 0 \quad \text{in } \Omega, \end{cases} \tag{B.1}$$

where $\phi \in L^2(0,T;W) \cap H^1(0,T;H)$.

Recall that B_0 is an open ball satisfying $B_0 \subset\subset \omega \cap \mathcal{O}$ and the auxiliary function η_0 satisfies $\eta_0 \in \mathcal{C}^2(\overline{\Omega})$,

$$\eta_0(x) > 0 \ \ \forall x \in \Omega, \quad \eta_0 = 0 \ \text{ on } \partial\Omega, \quad |\nabla \eta_0(x)| > 0 \ \ \forall x \in \overline{\Omega \setminus B_0}\,.$$

We will need an additional open ball $B_{00} \subset\subset B_0$, such that we still have

$$|\nabla \eta_0(x)| > 0 \ \ \forall x \in \overline{\Omega \setminus B_{00}}\,.$$

We will divide the proof of Lemma 1.3.2 in several steps.

Step 1. Following [FCGIP04], we apply some well known Carleman estimates for the heat equation to (B.1). Thus, there exist constants s_0, λ_0 and $C > 0$ depending on Ω, ω and T such that, for every $\lambda > \lambda_0$ and $s > s_0$, the following estimate holds:

$$I(s,\lambda;z) \leq C \left\{ \int_0^T \int_{B_{00}} e^{-2s\alpha} s^3 \lambda^4 \varphi^3 |z|^2 \, dx\, dt \right.$$
$$\left. + \int_0^T \int_{\Omega} e^{-2s\alpha} \left(|\nabla r|^2 + |(1+x_2)k \wedge z|^2 + |\phi 1_{\mathcal{O}}|^2 \right) \, dx\, dt. \right\}, \tag{B.2}$$

Recall that the definitions of $I(s,\lambda;z)$ and the weights α and φ are given in Section 1.3.1.

Of course, we can choose s large enough in order to absorb the previous term $|(1+x_2)k\wedge z|^2$ with the left hand side (B.2). We then have:

$$I(s,\lambda;z) \leq C\left\{\int_0^T\int_{B_{00}} e^{-2s\alpha}s^3\lambda^4\varphi^3|z|^2\,dx\,dt + \int_0^T\int_\Omega e^{-2s\alpha}|\nabla r|^2\,dx\,dt + \int_0^T\int_{\mathcal{O}} e^{-2s\alpha}|\phi|^2\,dx\,dt\right\} \tag{B.3}$$

for any $\lambda > \lambda_0$ and any $s > s_{01}$.

Step 2. In order to estimate the pressure gradient ∇r in (B.3), we first apply the divergence operator to (B.1), i.e., we write

$$\Delta r(t) = \operatorname{div}((1+x_2)\mathrm{k}\wedge z)(t) \quad \text{in} \quad \Omega, \qquad t\in(0,T) \tag{B.4}$$

and then we use the following result by Imanuvilov and Puel [IP03], which is satisfied by weak solutions to second order elliptic equations:

Lemma B.1.1 *Let us set $\beta(x) = e^{\lambda\eta_0(x)}$ and let $v\in H^1(\Omega)$ be a solution of*

$$\Delta v = \operatorname{div} h \quad \text{in} \quad \Omega, \tag{B.5}$$

where $h\in L^2(\Omega)^2$. Then there exist positive constants τ_2, λ_{01} and C such that

$$\int_\Omega e^{2\tau\beta}|\nabla v|^2\,dx \leq C\left\{\tau\int_\Omega e^{2\tau\beta}\beta|h|^2\,dx + \tau^{1/2}e^{2\tau}\|g\|^2_{1/2,\partial\Omega} + \tau^2\lambda^2\int_{B_{00}} e^{2\tau\beta}\beta^2|v|^2\,dx + \int_{B_{00}} e^{2\tau\beta}|\nabla v|^2\,dx\right\}, \tag{B.6}$$

for any $\tau > \tau_2$ and any $\lambda > \lambda_{01}$, where $g = v|_{\partial\Omega}$. □

In particular, we have the following for $r(t)$ and $g(t) = r(t)|_{\partial\Omega}$:

$$\int_\Omega e^{2\tau\beta}|\nabla r(t)|^2\,dx \leq C\left\{\tau\int_\Omega e^{2\tau\beta}\beta|(1+x_2)k\wedge z(t)|^2\,dx + \tau^{1/2}e^{2\tau}\|g(t)\|^2_{1/2,\partial\Omega} + \tau^2\lambda^2\int_{B_{00}} e^{2\tau\beta}\beta^2|r(t)|^2\,dx + \int_{B_{00}} e^{2\tau\beta}|\nabla r(t)|^2\,dx\right\}. \tag{B.7}$$

In order to estimate the last integral in (B.7), let us introduce an open set B_{01} such that $B_{00} \subset\subset B_{01} \subset\subset B_0$ and a function $\xi_{01} \in \mathcal{C}_0^2(B_{01})$ such that

$$0 \leq \xi_{01} \leq 1 \quad \text{and} \quad \xi_{01} = 1 \text{ in } B_{00}\,.$$

Integrating by parts, it follows from (B.4) that

$$\begin{aligned}\int_{B_{00}} e^{2\tau\beta}|\nabla r(t)|^2\,dx &\leq \int_{B_{01}} e^{2\tau\beta}\xi_{01}|\nabla r(t)|^2\,dx \\ &= -\int_{B_{01}} e^{2\tau\beta}\xi_{01}\,\mathrm{div}((1+x_2)\mathrm{k}\wedge z)(t)r(t)\,dx \\ &\quad -\frac{1}{2}\int_{B_{01}} e^{2\tau\beta}\nabla\xi_{01}\cdot\nabla|r(t)|^2\,dx - \int_{B_{01}} \xi_{01}\nabla e^{2\tau\beta}\cdot\nabla|r(t)|^2\,dx.\end{aligned}$$

Integrating again by parts, applying Young's inequality, and taking into account that $|\Delta(e^{2\tau\beta}\xi_{01})| \leq C\tau^2\lambda^2\beta^2 e^{2\tau\beta}$ for some positive constant C, after some straightforward computations we deduce that

$$\int_{B_{00}} e^{2\tau\beta}|\nabla r(t)|^2\,dx \;\;\leq\;\; C\left\{\tau^2\lambda^2\int_{B_{01}} e^{2\tau\beta}\beta^2|r(t)|^2\,dx + \int_{B_{01}} e^{2\tau\beta}|z(t)|^2\,dx\right\}.$$

Replacing this inequality in (B.7), we obtain the following for each $t \in (0,T)$:

$$\begin{aligned}\int_\Omega e^{2\tau\beta}|\nabla r(t)|^2\,dx \leq C\Big\{&\tau\int_\Omega e^{2\tau\beta}\beta|z(t)|^2\,dx \\ &+\tau^{1/2}e^{2\tau}\|g(t)\|^2_{1/2,\partial\Omega} + \tau^2\lambda^2\int_{B_{01}} e^{2\tau\beta}\beta^2|r(t)|^2\,dx\Big\}.\end{aligned}$$

Now, let us put $\tau = s/(t^4(T-t)^4)$ and let us choose $s > s_{02} = \max(s_{01}\,, \tau_2(T/2)^8)$. Then $\tau > \tau_2$. Let us multiply by $\exp(-2s\exp(2\lambda\|\eta_0\|_\infty)/(t^4(T-t)^4))$ the previous inequality and let us integrate with respect to t in $(0,T)$. This leads to the estimate

$$\begin{aligned}\int_0^T\!\!\int_\Omega e^{-2s\alpha}|\nabla r|^2\,dx\,dt \;\;\leq\;\; C\Big\{&\int_0^T\!\!\int_\Omega e^{-2s\alpha}s\varphi|z|^2\,dx\,dt \\ &+\int_0^T e^{-2s\alpha^*}(s\varphi^*)^{1/2}\|g(t)\|^2_{1/2,\partial\Omega}\,dt \\ &+\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha}(s\lambda\varphi)^2|r|^2\,dx\,dt\Big\}, \qquad \text{(B.8)}\end{aligned}$$

where α^* and φ^* were also introduced in Section 1.3.1.

The first term in the right hand side of (B.8) can be absorbed by the left hand side $I(s,\lambda;z)$ in (B.2) for s large enough. Hence, we obtain:

$$\begin{aligned}I(s,\lambda;z) \;\;\leq\;\; C\Big\{&\int_0^T\!\!\int_{\omega_0} e^{-2s\alpha}s^3\lambda^4\varphi^3|z|^2\,dx\,dt + \int_0^T e^{-2s\alpha^*}(s\varphi^*)^{1/2}\|g(t)\|^2_{1/2,\partial\Omega}\,dt \\ &+\int_0^T\!\!\int_{\omega_1} e^{-2s\alpha}(s\lambda\varphi)^2|r|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s\alpha}|\phi|^2\,dx\,dt\Big\} \qquad \text{(B.9)}\end{aligned}$$

for any $\lambda > \lambda_{01}$ and any $s > s_{03}$.

Step 3. This step is devoted to estimate the norm of the trace of the pressure on the boundary. To this end, we introduce three new functions

$$\chi(t) = e^{-s\alpha^*(t)}(s\varphi^*(t))^{1/4}, \quad \tilde{z} = \chi(t)z, \qquad \tilde{r} = \chi(t)r.$$

From (B.1), we see that $(\tilde{z}, \tilde{r})$ satisfies

$$\begin{cases} -\tilde{z}_t - \Delta\tilde{z} + \tilde{z} + \nabla\tilde{r} = -\chi' z + \chi(1 + x_2)\,\mathrm{k} \wedge z + \chi\phi 1_{\mathcal{O}} \quad \text{in } Q, \\ \operatorname{div} \tilde{z} = 0 \quad \text{in } Q, \\ \tilde{z} = 0 \quad \text{on } \Sigma, \\ \tilde{z}(T) = 0 \quad \text{in } \Omega. \end{cases}$$

Using the continuity of the trace operator and standard *a priori* estimates for the pressure, we deduce that

$$\begin{aligned} \int_0^T \|\tilde{r}(t)\|^2_{1/2,\partial\Omega}\,dt &\le \int_0^T \|\tilde{r}(t)\|^2_{1,\Omega}\,dt \\ &\le C\left\{\int_0^T\!\!\int_\Omega e^{-2s\alpha^*} s^{5/2}(\varphi^*)^3|z|^2\,dx\,dt + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s\alpha^*}(s\varphi^*)^{1/2}|\phi|^2\,dx\,dt\right\}. \end{aligned}$$

We have used here that $|\chi'(t)|^2 \le Ce^{-2s\alpha^*} s^{5/2}(\varphi^*(t))^3$ for all $t \in (0,T)$. We thus obtain a new estimate from (B.9):

$$\begin{aligned} I(s,\lambda;z) \;\;\le\;\; & C\left\{\int_0^T\!\!\int_{B_{00}} e^{-2s\alpha} s^3\lambda^4\varphi^3|z|^2\,dx\,dt + \int_0^T\!\!\int_{B_{01}} e^{-2s\alpha}(s\lambda\varphi)^2|r|^2\,dx\,dt \right. \\ & \left. + \int_0^T\!\!\int_{\mathcal{O}} e^{-2s\alpha}(s\varphi)^{1/2}|\phi|^2\,dx\,dt\right\} \end{aligned} \tag{B.10}$$

for any $\lambda > \lambda_{01}$ and any $s > s_{04}$.

Step 4. It remains to estimate the “local” term in the right hand side of (B.10) containing $|r|^2$ in terms of z and ϕ.

Assume that the pressure r has been normalized in such a way that

$$\int_{B_{01}} r(t)\,dx = 0 \quad \forall t \in (0,T).$$

Then there exists $C > 0$ such that

$$\int_{B_{01}} |r(t)|^2\,dx \le C\int_{B_{01}} |\nabla r(t)|^2\,dx \quad \forall t \in (0,T)$$

and also

$$\int_0^T\int_{B_{01}} e^{-2s\alpha}(s\lambda\varphi)^2|r|^2\,dx\,dt \le C\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\nabla r|^2\,dx\,dt,$$

where the functions $\widehat{\alpha}=\widehat{\alpha}(t)$ and $\widehat{\varphi}=\widehat{\varphi}(t)$ were introduced in Section 1.3.1.

From (B.1), we see that

$$\begin{aligned}\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\nabla r|^2\,dx\,dt \;&\le\; C\left\{\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2(|z|^2+|\phi|^2)\,dx\,dt\right.\\ &+\left.\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2(|z_t|^2+|\Delta z|^2)\,dx\,dt\right\}.\end{aligned}$$

Therefore, in view of (B.10), we obtain

$$\begin{aligned}I(s,\lambda;z) \;&\le\; C\left\{\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}\left(s^3\lambda^4\widehat{\varphi}^3|z|^2+(s\lambda\widehat{\varphi})^2|\phi|^2\right)\,dx\,dt\right.\\ &+\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2(|z_t|^2+|\Delta z|^2)\,dx\,dt\\ &+\left.\int_0^T\int_{\mathcal{O}} e^{-2s\alpha}(s\varphi)^{1/2}|\phi|^2\,dx\,dt\right\}.\end{aligned} \tag{B.11}$$

Step 5. The rest of the proof deals with the estimates of the "local" integrals containing $|\Delta z|^2$ and $|z_t|^2$. First, we will be concerned with $|\Delta z|^2$.

Let us introduce a function $\xi_0\in\mathcal{C}_0^4(B_0)$ such that

$$0\le\xi_0\le 1 \quad\text{and}\quad \xi_0=1 \text{ in } B_{02},$$

where $B_{01}\subset\subset B_{02}\subset\subset B_0$ Let us set $\widehat{z}(x,t)=e^{-s\widehat{\alpha}}\widehat{\varphi}\,\xi_0\Delta z(T-t)$. We want to estimate the norm $\|\widehat{z}\|_{L^2(B_{01}\times(0,T))}$.

In order to simplify, we introduce the notations

$$\widehat{\eta}(t)=e^{-s\widehat{\alpha}}\widehat{\varphi} \qquad\text{and}\qquad (\tau_T f)(t)=f(T-t).$$

It is easy to check that $\widehat{z}$ verifies

$$\begin{cases}\widehat{z}_t-\Delta\widehat{z}=\widehat{\eta}'\rho_1\Delta\tau_T z-\widehat{\eta}\rho_1\Delta\tau_T z_t-\widehat{\eta}\rho_1\Delta^2\tau_T z\\ \qquad\qquad -\widehat{\eta}\Delta\rho_1\Delta\tau_T z-2\widehat{\eta}\nabla\rho_1\cdot\nabla\Delta\tau_T z \quad \text{in } \mathbb{R}^2\times(0,T),\\ \widehat{z}(0)=0 \quad \text{in } \mathbb{R}^2.\end{cases}$$

Using (B.1) and making some computations, we obtain

$$\left\{\begin{array}{l}\widehat{z}_t - \Delta\widehat{z} = \widehat{\eta}'\Delta(\rho_1\tau_T z) - 2\widehat{\eta}'\nabla\rho_1\nabla\tau_T z - \widehat{\eta}'\Delta\rho_1\tau_T z - \widehat{\eta}\Delta(\rho_1\tau_T z) + 2\widehat{\eta}\nabla\rho_1\nabla\tau_T z \\ \qquad + \widehat{\eta}\Delta\rho_1\tau_T z + \widehat{\eta}\Delta(\rho_1(1+x_2)\mathrm{k}\wedge\tau_T z) - \widehat{\eta}\nabla\rho_1\nabla((1+x_2)\mathrm{k}\wedge\tau_T z) \\ \qquad - \widehat{\eta}\Delta\rho_1((1+x_2)\mathrm{k}\wedge\tau_T z) - \widehat{\eta}\nabla\operatorname{div}(\rho_1(1+x_2)\mathrm{k}\wedge\tau_T z) \\ \qquad + \widehat{\eta}\nabla\rho_1\operatorname{div}((1+x_2)\mathrm{k}\wedge\tau_T z) + \widehat{\eta}H(\rho_1)(1+x_2)\mathrm{k}\wedge\tau_T z + \widehat{\eta}\Delta(\rho_1\tau_T\phi) \\ \qquad - 2\widehat{\eta}\nabla\rho_1\nabla\tau_T\phi - \widehat{\eta}\Delta\rho_1\tau_T\phi - \widehat{\eta}\Delta\rho_1\Delta\tau_T z - 2\widehat{\eta}\nabla\rho_1\nabla\Delta\tau_T z \quad \text{in } \mathbb{R}^2\times(0,T), \\ \widehat{z}(0) = 0 \qquad \text{in } \mathbb{R}^2,\end{array}\right.$$

where $H_{jk}(\rho_1) = \partial_{jk}\rho_1$ is the Hessian associated to ρ_1.

Let us introduce $\widehat{z}(x,t) = z_1(x,t) + z_2(x,t)$ such that

$$\left\{\begin{array}{l} z_{1t} - \Delta z_1 = \widehat{\eta}'\Delta(\rho_1\tau_T z) - \widehat{\eta}\Delta(\rho_1\tau_T z) + \widehat{\eta}\Delta(\rho_1(1+x_2)\mathrm{k}\wedge\tau_T z) \\ \qquad\qquad - \widehat{\eta}\nabla\operatorname{div}(\rho_1(1+x_2)\mathrm{k}\wedge\tau_T z) + \widehat{\eta}\Delta(\rho_1\tau_T\phi) \quad \text{in } \mathbb{R}^2\times(0,T), \\ z_1(0) = 0 \qquad \text{in } \mathbb{R}^2. \end{array}\right. \tag{B.12}$$

$$\left\{\begin{array}{l} z_{2t} - \Delta z_2 = -2\widehat{\eta}'\nabla\rho_1\nabla\tau_T z - \widehat{\eta}'\Delta\rho_1\tau_T z + 2\widehat{\eta}\nabla\rho_1\nabla\tau_T z + \widehat{\eta}\Delta\rho_1\tau_T z \\ \qquad\qquad - \widehat{\eta}\nabla\rho_1\nabla((1+x_2)\mathrm{k}\wedge\tau_T z) - \widehat{\eta}\Delta\rho_1((1+x_2)\mathrm{k}\wedge\tau_T z) \\ \qquad\qquad + \widehat{\eta}\nabla\rho_1\operatorname{div}((1+x_2)\mathrm{k}\wedge\tau_T z) + \widehat{\eta}H(\rho_1)(1+x_2)\mathrm{k}\wedge\tau_T z \\ \qquad\qquad - 2\widehat{\eta}\nabla\rho_1\nabla\tau_T\phi - \widehat{\eta}\Delta\rho_1\tau_T\phi - \widehat{\eta}\Delta\rho_1\Delta\tau_T z \\ \qquad\qquad - 2\widehat{\eta}\nabla\rho_1\nabla\Delta\tau_T z \qquad \text{in } \mathbb{R}^2\times(0,T), \\ z_2(0) = 0 \qquad \text{in } \mathbb{R}^2. \end{array}\right. \tag{B.13}$$

On one hand we have the following result

Lemma B.1.2 *Let z_1 be the solution of (B.12). Then there exists $C > 0$ such that*

$$\int_0^T\int_{\mathbb{R}^2} |z_1|^2\,dx\,dt \le C\left\{\int_0^T\int_\omega e^{-2s\widehat{\alpha}}s^2\widehat{\varphi}^{9/2}|z|^2\,dx\,dt + \int_0^T\int_\omega e^{-2s\widehat{\alpha}}\widehat{\varphi}^2|\phi|^2\,dx\,dt\right\}.$$

Proof: Let us consider $v(x,t)$ the solution of

$$\left\{\begin{array}{l} -v_t - \Delta v = F \quad \text{in } \mathbb{R}^2\times(0,T), \\ v(T) = 0 \quad \text{in } \mathbb{R}^2, \end{array}\right.$$

with $F(x,t) \in L^2(0,T;\mathbb{R}^2)$. By standard regularity results for the heat equation we have that $v \in L^2(0,T;H^2(\mathbb{R}^2)^2) \cap H^1(0,T;\mathbb{R}^2)$. We say that z_1 is a very weak solution of (B.12) if and only if the following condition holds:

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^2} z_1\cdot F\,dx\,dt &= \int_0^T\int_{\mathbb{R}^2} \rho_1[\widehat{\eta}'\tau_T z - \widehat{\eta}\tau_T z + \widehat{\eta}(1+x_2)\mathrm{k}\wedge\tau_T z]\cdot\Delta v\,dx\,dt \\ &+ \int_0^T\int_{\mathbb{R}^2} \widehat{\eta}\rho_1(\tau_T\phi\cdot\Delta v - \nabla\operatorname{div}(v)\cdot(1+x_2)\mathrm{k}\wedge\tau_T z)\,dx\,dt.\end{aligned}$$

Moreover, there exists a unique solution $z_1 \in L^2(0,T;\mathbb{R}^2)$ such that

$$\|z_1\|^2_{L^2(0,T;\mathbb{R}^2)} \leq C\left\{\|\widehat{\eta} z\|^2_{L^2(\omega\times(0,T))} + \|\widehat{\eta}' z\|^2_{L^2(\omega\times(0,T))} + \|\widehat{\eta}\phi\|^2_{L^2(\omega\times(0,T))}\right\}.$$

Taking into account that $\widehat{\eta}' \leq C(T)e^{-s\widehat{\alpha}}s\widehat{\varphi}^{9/4}$ we conclude the prove of the lemma. □

On the other hand for z_2, we have

Lemma B.1.3 *Let z_2 be the solution of* $(B.13)$*. then there exists $C > 0$ such that*

$$\int_0^T\int_{B_{01}} |z_2|^2\,dx\,dt \leq C\left\{\int_0^T\int_\omega e^{-2s\widehat{\alpha}}s^2\widehat{\varphi}^{9/2}|z|^2\,dx\,dt + \int_0^T\int_\omega e^{-2s\widehat{\alpha}}\widehat{\varphi}^2|\phi|^2\,dx\,dt\right\}.$$

Proof: Let us denote by $F_2 = F_2(x,t)$ (which is in $L^2(0,T;H^{-1}(\mathbb{R}^2)^2)$) the right hand side in (B.13). Since z_2 is the solution of heat equation in $\mathbb{R}^2\times(0,T)$ with homogeneous initial condition, we can express the solution in terms of the fundamental solution, i.e., in the 2-dimensional case

$$G(x,t) = \frac{1}{4\pi t}e^{-\frac{|x|^2}{2t}} \qquad \forall x \in \mathbb{R}^2,\ t > 0,$$

hence,

$$z_2(x,t) = \int_0^t\int_{\mathbb{R}^2} G(x-\theta,t-\tau)F_2(\theta,\tau)\,d\theta\,d\tau.$$

Notice that in F_2 appears only derivatives of ρ_1 then the support of F_2 is $B_0\backslash\bar{B}_{02}$ for each $t > 0$. Thus,

$$z_2(x,t) = \int_0^t\int_{B_0\backslash\bar{B}_{02}} G(x-\theta,t-\tau)F_2(\theta,\tau)\,d\theta\,d\tau.$$

Now, the main idea is to eliminate all the derivatives that appear in z and ϕ by integrating by parts. Let consider the first order derivatives on space appearing in F_2

$$\int_0^t\int_{B_0\backslash\bar{B}_{02}} G(x-\theta,t-\tau)\Big\{-2\widehat{\eta}'\nabla\rho_1\nabla\tau_T z(\tau) + 2\widehat{\eta}\nabla\rho_1\nabla\tau_T z(\tau) - 2\widehat{\eta}\nabla\rho_1\nabla\tau_T\phi(\tau)$$
$$- \widehat{\eta}\nabla\rho_1\nabla((1+x_2)\mathrm{k}\wedge\tau_T z(\tau)) + \widehat{\eta}\nabla\rho_1\,\mathrm{div}((1+x_2)\mathrm{k}\wedge\tau_T z(\tau))\Big\}\,d\theta\,d\tau$$
$$= \int_0^t\int_{B_0\backslash\bar{B}_{02}}\Big\{(\Delta\rho_1 G(x-\theta,t-\tau) + \nabla\rho_1\cdot\nabla G(x-\theta,t-\tau))(2\widehat{\eta}'\tau_T z(\tau) - 2\widehat{\eta}\tau_T z(\tau)$$
$$+ 2\widehat{\eta}\tau_T\phi(\tau) + \widehat{\eta}(1+x_2)\mathrm{k}\wedge\tau_T z(\tau)) - \widehat{\eta}G(x-\theta,t-\tau)H(\rho_1)(1+x_2)\mathrm{k}\wedge\tau_T z(\tau)$$
$$+ \widehat{\eta}\nabla\rho_1\nabla G(x-\theta,t-\tau)\cdot((1+x_2)\mathrm{k}\wedge\tau_T z(\tau))\Big\}\,d\theta\,d\tau.$$

For the higher order derivatives on space we consider first the following term

$$\int_0^t\int_{B_0\backslash\bar{B}_{02}} G(x-\theta,t-\tau)\widehat{\eta}\Delta\rho_1\Delta\tau_T z(\tau)\,d\theta\,d\tau$$
$$= \int_0^t\int_{B_0\backslash\bar{B}_{02}} \widehat{\eta}\tau_T z(\tau)\left\{\Delta^2\rho_1 G(x-\theta,t-\tau)\right.$$
$$\left. + 2\nabla\Delta\rho_1\cdot\nabla G(x-\theta,t-\tau) + \Delta\rho_1\Delta G(x-\theta,t-\tau)\right\}\,d\theta\,d\tau.$$

For the other term we have

$$\int_0^t\int_{B_0\backslash\bar{B}_{02}} G(x-\theta,t-\tau)\widehat{\eta}\nabla\rho_1\cdot\nabla\Delta\tau_T z(\tau)\,d\theta\,d\tau$$
$$= -\int_0^t\int_{B_0\backslash\bar{B}_{02}} \widehat{\eta}\tau_T z(\tau)\left\{\Delta^2\rho_1 G(x-\theta,t-\tau) + 3\nabla\Delta\rho_1\cdot\nabla G(x-\theta,t-\tau)\right.$$
$$\left. +\nabla\rho_1\cdot\nabla\Delta G(x-\theta,t-\tau) + \Delta\rho_1\Delta G(x-\theta,t-\tau)\right\}\,d\theta\,d\tau$$
$$-2\int_0^t\int_{B_0\backslash\bar{B}_{02}} \widehat{\eta}H(\rho_1)H(G(x-\theta,t-\tau))\tau_T z(\tau)\,d\theta\,d\tau.$$

We can deduce the following bounds

$$|\nabla G(y,t)| \le CG(y,t)t^{-1}|y|, \quad |\partial_{jk}G(y,t)| \le CG(y,t)t^{-2}(1+|y|^2),$$
$$|\nabla\Delta G(y,t)| \le CG(y,t)t^{-3}|y|(1+|y|^2).$$

Taking into account these bounds, after some computations we obtain the following estimate for z_2

$$|z_2(x,t)| \le C\left\{\int_0^t\int_{B_0\backslash\bar{B}_{02}} e^{-s\widehat{\alpha}}s\widehat{\varphi}^{9/4}G(x-\theta,t-\tau)(1+|t-\tau|^{-1}|x-\theta|\right.$$
$$+|t-\tau|^{-2}(1+|x-\theta|)+|t-\tau|^{-3}|x-\theta|(1+|x-\theta|^2))|\tau_T z(\tau)|\,d\theta\,d\tau$$
$$\left.+\int_0^t\int_{B_0\backslash\bar{B}_{02}} e^{-s\widehat{\alpha}}\widehat{\varphi}G(x-\theta,t-\tau)(1+|t-\tau|^{-1}|x-\theta|)|\tau_T\phi(\tau)|\,d\theta\,d\tau\right\}.$$

Let us now multiply the above inequality by the characteristic function of B_{01}, i.e., $1_{B_{01}}(x)$. Since $x\in B_{01}$ and $\theta\in B_0\backslash\bar{B}_{02}$, we have

$$0 < d = \mathrm{dist}(\bar{B}_{01},\bar{B}_{02}^c) \le |x-\theta| \le C(B_0,B_{01},B_{02}),$$

and we can write $1_{B_{01}}(x)1_{B_0\backslash\bar{B}_{02}}(\theta) \le 1_{B(0,d)^c}(x-\theta)$, where $B(0,d)$ represents a ball centered in 0 with radio d. Hence,

$$|z_2(x,t)|1_{B_{01}}(x) \le C\int_0^{+\infty}\int_{\mathbb{R}^2} e^{-s\widehat{\alpha}}G(x-\theta,t-\tau)1_{B(0,d)^c}(x-\theta)1_{B_0\times(0,T)}(\theta,\tau)$$
$$\left(s\widehat{\varphi}^{9/4}|t-\tau|^{-3}|\tau_T z(\tau)| + \widehat{\varphi}|t-\tau|^{-1}|\tau_T\phi(\tau)|\right)\,d\theta\,d\tau.$$

Let us recall the following Young's inequality. Let $f \in L^p((0,+\infty);\mathbb{R}^2)$ and $g \in L^q((0,+\infty);\mathbb{R}^2)$, then $f * g \in L^r((0,+\infty);\mathbb{R}^2)$ with $1+\frac{1}{r}=\frac{1}{p}+\frac{1}{q}$. Also, it holds

$$\|f * g\|_{L^r((0,+\infty);\mathbb{R}^2)} \leq \|f\|_{L^p((0,+\infty);\mathbb{R}^2)}\|g\|_{L^q((0,+\infty);\mathbb{R}^2)}.$$

We define

$$\begin{aligned} f_1(y,t) &= G(y,t)t^{-3}1_{B(0,d)^c}(y), \qquad & g_1(y,t) &= e^{-s\widehat{\alpha}}s\widehat{\varphi}^{9/4}1_{B_0\times(0,T)}(y,t)|\tau_T z|, \\ f_2(y,t) &= G(y,t)t^{-1}1_{B(0,d)^c}(y), \qquad & g_2(y,t) &= e^{-s\widehat{\alpha}}\widehat{\varphi}1_{B_0\times(0,T)}(y,t)|\tau_T \phi|. \end{aligned}$$

Since G contains a negative exponent in the exponential, it is easy to check that $f_1, f_2 \in L^1((0,T);\mathbb{R}^2)$ and moreover, because of the regularity of z and ϕ, we have $g_1, g_2 \in L^2((0,T);\mathbb{R}^2)$. Now we are in condition of apply the above Young's inequality (with $r=2$) to complete the proof. □

It follows, using the estimations of Lemma B.1.2 and Lemma B.1.3, that

$$\begin{aligned} &\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\Delta z|^2\,dx\,dt \leq \int_0^T\int_{B_0} s^2\lambda^2|\widehat{z}|^2\,dx\,dt \\ &\leq C\left(\int_0^T\int_{B_0} e^{-2s\widehat{\alpha}}s^4\lambda^2\widehat{\varphi}^{9/2}|z|^2\,dx\,dt + \int_0^T\int_{B_0} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\phi|^2\,dx\,dt\right). \end{aligned} \tag{B.14}$$

Thus, from (B.11) we have,

$$\begin{aligned} &I(s,\lambda;z) \\ &\quad\leq C\left(\int_0^T\int_{B_0} e^{-2s\widehat{\alpha}}s^4\lambda^4\widehat{\varphi}^{9/2}|z|^2\,dx\,dt + \int_0^T\int_{B_0} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\phi|^2\,dx\,dt\right. \\ &\quad\left.+\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|z_t|^2\,dx\,dt + \int_0^T\int_{\mathcal{O}} e^{-2s\alpha}(s\varphi)^{1/2}|\phi|^2\,dx\,dt\right). \end{aligned} \tag{B.15}$$

Step 6. Now, we want to estimate $|z_t|^2$. Due to the regularity properties of ϕ, we can use here a more straight argument than in [FCGIP04], where the right hand side only belongs to $L^2(Q)^2$.

First, notice that

$$\begin{aligned} \int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|z_t|^2\,dx\,dt &\leq \delta\int_0^T\int_{B_{01}} e^{-2s\alpha}\frac{1}{s\varphi}|z_t|^2\,dx\,dt \\ &+\delta\int_0^T\int_{B_{01}} e^{-2s\alpha^*}\frac{1}{s^3(\varphi^*)^{7/2}}|z_{tt}|^2\,dx\,dt \\ &+C_\delta\int_0^T\int_{B_{01}} e^{-4s\alpha^*+2s\alpha^*}s^7\lambda^4\widehat{\varphi}^{15/2}|z|^2\,dx\,dt. \end{aligned}$$

This is easily obtained by integrating by parts in time. We will later choose $\delta > 0$ small enough.

We have the following auxiliary result:

Lemma B.1.4 *Let* (z, r) *be the solution of* $(B.1)$. *Then the following estimate holds*

$$\int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s^3(\varphi^*)^{7/2}} |z_{tt}|^2 \, dx \, dt$$
$$\leq C \left(I(s, \lambda; z) + \int_0^T \int_{\mathcal{O}} e^{-2s\alpha^*} \left(\frac{1}{s\varphi^*} |\phi|^2 + \frac{1}{s^3(\varphi^*)^{7/2}} |\phi_t|^2 \right) dx \, dt \right). \tag{B.16}$$

Proof: Let us multiply (B.1) by $e^{-2s\alpha^*} s^{-2} (\varphi^*)^{-9/4} z_{tt}$ and let us integrate in Q. Noticing that

$$\left| (e^{-2s\alpha^*} (\varphi^*)^{-9/4})_t \right| \leq C e^{-2s\alpha^*} s (\varphi^*)^{-1},$$

after some computations we deduce that

$$\int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s^2(\varphi^*)^{9/4}} |\nabla z_t|^2 \, dx \, dt$$
$$\leq C \left\{ \int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s\varphi^*} (|z|^2 + |z_t|^2) \, dx \, dt \right.$$
$$\left. + \int_0^T \int_\Omega e^{-2s\alpha^*} s (\varphi^*)^{1/4} |\nabla z|^2 \, dx \, dt + \int_0^T \int_{\mathcal{O}} e^{-2s\alpha^*} \frac{1}{s\varphi^*} |\phi|^2 \, dx \, dt \right\}$$
$$+ \frac{1}{2} \int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s^3(\varphi^*)^{7/2}} |z_{tt}|^2 \, dx \, dt. \tag{B.17}$$

On the other hand, if we compute the time derivative of $(B.1)$ and then we multiply the result by $e^{-2s\alpha^*} s^{-3} (\varphi^*)^{-7/2} z_{tt}$, we find that

$$\int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s^3(\varphi^*)^{7/2}} |z_{tt}|^2 \, dx \, dt$$
$$\leq \int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s^2(\varphi^*)^{9/4}} |\nabla z_t|^2 \, dx \, dt \tag{B.18}$$
$$+ C \left(\int_0^T \int_\Omega e^{-2s\alpha^*} \frac{1}{s\varphi^*} |z_t|^2 \, dx \, dt + \int_0^T \int_{\mathcal{O}} e^{-2s\alpha^*} \frac{1}{s^3(\varphi^*)^{7/2}} |\phi_t|^2 \, dx \, dt \right).$$

From $(B.17)$ and $(B.18)$, we see that $(B.16)$ holds. □

In view of this lemma, we have

$$\begin{aligned}
&\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|z_t|^2\,dx\,dt\\
&\quad\le C\delta\left(I(s,\lambda;z)+\int_0^T\int_{\mathcal{O}} e^{-2s\alpha^*}\left(\frac{1}{s\varphi^*}|\phi|^2+\frac{1}{s^3(\varphi^*)^{7/2}}|\phi_t|^2\right)\,dx\,dt\right)\\
&\quad+C_\delta\int_0^T\int_{B_{01}} e^{-4s\alpha^*+2s\alpha^*}s^7\lambda^4\widehat{\varphi}^{15/2}|z|^2\,dx\,dt.
\end{aligned}$$

If we assume that $\gamma_1 < 1$ then $(3-\gamma_1)/2 > 1$ and from Lemma 1.3.1 we deduce that $(3-\gamma_1)\widehat{\alpha}/2 > \alpha^*$, for sufficiently large λ, say $\lambda > \lambda_{02}$. Consequently, $-4\widehat{\alpha} + 2\alpha^* < -(1+\gamma_1)\widehat{\alpha}$ and

$$\begin{aligned}
&\int_0^T\int_{B_{01}} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|z_t|^2\,dx\,dt\\
&\quad\le C\delta\left(I(s,\lambda;z)+\int_0^T\int_{\mathcal{O}} e^{-2s\alpha^*}\left(\frac{1}{s\varphi^*}|\phi|^2+\frac{1}{s^3(\varphi^*)^{7/2}}|\phi_t|^2\right)\,dx\,dt\right)\\
&\quad+C_\delta\int_0^T\int_{B_{01}} e^{-(1+\gamma_1)s\widehat{\alpha}}s^7\lambda^4\widehat{\varphi}^{15/2}|z|^2\,dx\,dt
\end{aligned}$$

for any $\lambda > \lambda_{02}$ and any $s > s_{04}$.

From $(B.15)$ and this estimate, choosing $\delta > 0$ small enough, we find:

$$\begin{aligned}
&I(s,\lambda;z)\\
&\quad\le C\left\{\int_0^T\int_{B_0} e^{-(1+\gamma_1)s\widehat{\alpha}}s^7\lambda^4\widehat{\varphi}^{15/2}|z|^2\,dx\,dt+\int_0^T\int_{B_0} e^{-2s\widehat{\alpha}}(s\lambda\widehat{\varphi})^2|\phi|^2\,dx\,dt\right.\\
&\quad\left.+\int_0^T\int_{\mathcal{O}} e^{-2s\alpha}\left((s\varphi)^{1/2}|\phi|^2+\frac{1}{s^3(\varphi^*)^{7/2}}|\phi_t|^2\right)\,dx\,dt\right\},
\end{aligned}$$

for all $\lambda > \lambda_{02}$ and $s > s_{04}$.

Obviously, this yields (1.20). The proof of (1.21) is very similar and in fact much simpler, since the left hand side of (1.10) is zero.

Thus, we have proved Lemma 1.3.2 for $\lambda_1 = \lambda_{02}$ and $s_1 = s_{04}$ (two parameters depending on Ω, ω, $\mathcal{O}$ and T).

Bibliography

[AAYHT+96] D. Ait-Ali-Yahia, W. G. Habashi, A. Tam, M.-G. Vallet, and M. Fortin. A directionally adaptive methodology using an edge-based error estimate on quadrilateral grids. *Internat. J. Numer. Methods Fluids*, 23(7):673–690, 1996.

[AB99] M. Amara and C. Bernardi. Convergence of a finite element discretization of the Navier-Stokes equations in vorticity and stream function formulation. *M2AN Math. Model. Numer. Anal.*, 33(5):1033–1056, 1999.

[ABYB98] M. Amara, M. Ben Younes, and C. Bernardi. Error indicators for the Navier-Stokes equations in stream function and vorticity formulation. *Numer. Math.*, 80(2):181–206, 1998.

[AD01] M. Amara and F. Dabaghi. An optimal $\mathbf{C}^0$ finite element algorithm for the 2D biharmonic problem: theoretical analysis and numerical results. *Numer. Math.*, 90(1):19–46, 2001.

[AFG+00] R. C. Almeida, R. A. Feijóo, A. C. Galeão, C. Padra, and R. S. Silva. Adaptive finite element computational fluid dynamics using an anisotropic error estimator. *Comput. Methods Appl. Mech. Engrg.*, 182(3-4):379–400, 2000. IV WCCM (Buenos Aires, 1998).

[AN98] T. Apel and S. Nicaise. The finite element method with anisotropic mesh grading for elliptic problems in domains with corners and edges. *Math. Methods Appl. Sci.*, 21(6):519–549, 1998.

[ANS01] T. Apel, S. Nicaise, and J. Schöberl. A non-conforming finite element method with anisotropic mesh grading for the Stokes problem in domains with edges. *IMA J. Numer. Anal.*, 21(4):843–856, 2001.

[AO00] M. Ainsworth and J. T. Oden. *A posteriori error estimation in finite element analysis*. Pure and Applied Mathematics. Wiley-Interscience [John Wiley & Sons], New York, 2000.

[Bar01] V. Barbu. Local internal controllability of the Navier-Stokes equations. *Adv. Differential Equations*, 6(12):1443–1462, 2001.

[Ben92] A. F. Bennett. *Inverse methods in physical oceanography*. Cambridge Monographs on Mechanics and Applied Mathematics. Cambridge University Press, Cambridge, 1992.

[BF95] O. Bodart and C. Fabre. Controls insensitizing the norm of the solution of a semilinear heat equation. *J. Math. Anal. Appl.*, 195(3):658–683, 1995.

[BGBPG02] O. Bodart, M. González-Burgos, and R. Pérez-García. Insensitizing controls for a semilinear heat equation with a superlinear nonlinearity. *C. R. Math. Acad. Sci. Paris*, 335(8):677–682, 2002.

[BGdS01] R. Bermejo and P. Galán del Sastre. Numerical studies of the long-term dynamics of the 2D Navier-Stokes equations applied to ocean circulation. In *XVII CEDYA: Congress on Differential Equations and Applications/VII CMA: Congress on Applied Mathematics (Spanish) (Salamanca, 2001)*, pages 15–34. Dep. Mat. Apl., Univ. Salamanca, Salamanca, 2001.

[BGR87] C. Bernardi, E. Godlewski, and G. Raugel. A mixed method for the time-dependent Navier-Stokes problem. *IMA J. Numer. Anal.*, 7(2):165–189, 1987.

[BLB95] H. Borouchaki and P. Laug Bernardi. Le mailleur adaptatif bidimensionnel bl2d: Manuel d‘utilisation et documentation. *Rapport Technique INRIA RT-0185*, 1995.

[BLV98] J. Blum, B. Luong, and J. Verron. Variational assimilation of altimeter data into a non-linear ocean model: temporal strategies. In *Control and partial differential equations (Marseille-Luminy, 1997)*, volume 4 of *ESAIM Proc.*, pages 21–57 (electronic). Soc. Math. Appl. Indust., Paris, 1998.

[BR78] I. Babuška and W. C. Rheinboldt. Error estimates for adaptive finite element computations. *SIAM J. Numer. Anal.*, 15(4):736–754, 1978.

[BR85] C. Bernardi and G. Raugel. A conforming finite element method for the time-dependent Navier-Stokes equations. *SIAM J. Numer. Anal.*, 22(3):455–473, 1985.

[BS01] I. Babuška and T. Strouboulis. *The finite element method and its reliability*. Numerical Mathematics and Scientific Computation. The Clarendon Press Oxford University Press, New York, 2001.

[BVM94] E. Blayo, J. Verron, and J. M. Molines. Assimilation of topex/poseidon altimeter data into a circulation model of the north atlantic. *J. Geophys. Res.*, 99:24691–24705, 1994.

[CD96] M. J. Castro-Díaz. *Generación y adaptación anisótropica de mallados de elementos finitos para la resolución numérica de E.D.P. Aplicaciones*. Ph.d. thesis, Universidad de Málaga, 1996.

[CD97] Buscaglia G. C. and E. A. Dari. Anisotropic mesh optimization and its applications in adaptivity. *Internat. J. Numer. Methods Engrg.*, 40:4119–4136, 1997.

[CDHMP97] M. J. Castro-Díaz, F. Hecht, B. Mohammadi, and O. Pironneau. Anisotropic unstructured mesh adaption for flow simulations. *Internat. J. Numer. Methods Fluids*, 25(4):475–491, 1997.

[CGL94] C. Carthel, R. Glowinski, and J.-L. Lions. On exact and approximate boundary controllabilities for the heat equation: a numerical approach. *J. Optim. Theory Appl.*, 82(3):429–484, 1994.

[CGR03] J. M. Cascón, G. C. García, and R. Rodríguez. A priori and a posteriori error analysis for a large-scale ocean circulation model. *Comput. Methods Appl. Mech. Engrg.*, 192:5305–5327, 2003.

[CR74] P. G. Ciarlet and P.-A. Raviart. A mixed finite element method for the biharmonic equation. In *Mathematical aspects of finite elements in partial differential equations (Proc. Sympos., Math. Res. Center, Univ. Wisconsin, Madison, Wis., 1974)*, pages 125–145. Publication No. 33. Math. Res. Center, Univ. of Wisconsin-Madison, Academic Press, New York, 1974.

[CT87] Ph. Courtier and O. Talagrand. Variational assimilation of meteorological observations with the adjoint vorticity equation. i: Theory. *Q. J. R. Meteorol. Soc.*, 113:1311–1328, 1987.

[dT00] Luz de Teresa. Insensitizing controls for a semilinear heat equation. *Comm. Partial Differential Equations*, 25(1-2):39–72, 2000.

[Fab96] C. Fabre. Uniqueness results for Stokes equations and their consequences in linear and nonlinear control problems. *ESAIM Contrôle Optim. Calc. Var.*, 1:267–302 (electronic), 1995/96.

[FCGIP04] E. Fernández-Cara, S. Guerrero, O. Yu. Imanuvilov, and J.-P. Puel. Local null controllability of the navier-stokes system. *to appear*, 2004.

[FCGO03] E. Fernández-Cara, G. C. García, and A. Osses. Insensitizing controls for a large-scale ocean circulation model. *C. R. Math. Acad. Sci. Paris*, 337(4):265–270, 2003.

[FCGO04] E. Fernández-Cara, G. C. García, and A. Osses. Controls insensitizing the observation of a quasi-geostrophic ocean model. *to appear*, 2004.

[FÈ96] A. V. Fursikov and O. Yu. Èmanuilov. Exact local controllability of two-dimensional Navier-Stokes equations. *Mat. Sb.*, 187(9):103–138, 1996.

[FI96] A. V. Fursikov and O. Yu. Imanuvilov. *Controllability of evolution equations*, volume 34 of *Lecture Notes Series*. Seoul National University Research Institute of Mathematics Global Analysis Research Center, Seoul, 1996.

[FL02] C. Fabre and G. Lebeau. Régularité et unicité pour le problème de Stokes. *Comm. Partial Differential Equations*, 27(3-4):437–475, 2002.

[FP01] L. Formaggia and S. Perotto. New anisotropic a priori error estimates. *Numer. Math.*, 89(4):641–667, 2001.

[FP03] L. Formaggia and S. Perotto. Anisotropic error estimates for elliptic problems. *Numer. Math.*, 94(1):67–92, 2003.

[FPZ95] C. Fabre, J.-P. Puel, and E. Zuazua. Approximate controllability of the semilinear heat equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 125(1):31–61, 1995.

[FPZ01] L. Formaggia, S. Perotto, and P. Zunino. An anisotropic a-posteriori error estimate for a convection-diffusion problem. *Comput. Vis. Sci.*, 4(2):99–104, 2001. Second AMIF International Conference (Il Ciocco, 2000).

[GMR91] M. Ghil and P. Malanotte-Rizzoli. Data assimilation in meteorology and oceanography. *Adv. Geophys.*, 33:141–266, 1991.

[GR86] V. Girault and P.-A. Raviart. *Finite element methods for Navier-Stokes equations*, volume 5 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1986. Theory and algorithms.

[HB92] D. B. Haidvogel and F. O. Bryan. *Ocean general circulation modeling*. In: Climate System Modeling. K. Trenberth, Cambridge University Press. Cambridge., 1992.

[Ima98] O. Yu. Imanuvilov. On exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 3:97–131 (electronic), 1998.

[Ima01] O. Yu. Imanuvilov. Remarks on exact controllability for the Navier-Stokes equations. *ESAIM Control Optim. Calc. Var.*, 6:39–72 (electronic), 2001.

[IP03] O. Yu. Imanuvilov and J.-P. Puel. Global Carleman estimates for weak solutions of elliptic nonhomogeneous Dirichlet problems. *Int. Math. Res. Not.*, (16):883–913, 2003.

[KM93] Z. Kowalik and T. S. Murty. *Numerical Modeling of Ocean Dynamics*, volume 5 of *Advanced Series on Ocean Engineering*. Word Scientific, 1993.

[Kun00] G. Kunert. An a posteriori residual error estimator for the finite element method on anisotropic tetrahedral meshes. *Numer. Math.*, 86(3):471–490, 2000.

[Kun02] G. Kunert. Toward anisotropic mesh construction and error estimation in the finite element method. *Numer. Methods Partial Differential Equations*, 18(5):625–648, 2002.

[LBV98] B. Luong, J. Blum, and J. Verron. A variational method for the resolution of a data assimilation problem in oceanography. *Inverse Problems*, 14(4):979–997, 1998.

[LD85] J. L. Lewis and J. C. Derber. The use of adjoint equations to solve a variational adjustment problem with advective constraints. *Tellus*, 37A:309–322, 1985.

[LDT86] F.-X. Le Dimet and O. Talagrand. Variational algorithms for analysis and assimilation of meteorological observations. *Tellus*, 38A:97 –110, 1986.

[Lew97] R. Lewandowski. *Analyse mathématique et océanographie*. Recherches en Mathématiques Appliquées [Research in Applied Mathematics]. Masson, Paris, 1997.

[Lio71] J.-L. Lions. *Optimal control of systems governed by partial differential equations.*. Translated from the French by S. K. Mitter. Die Grundlehren der mathematischen Wissenschaften, Band 170. Springer-Verlag, New York, 1971.

[Lio88] J.-L. Lions. Sur les sentinelles des systèmes distribués. Le cas des conditions initiales incomplètes. *C. R. Acad. Sci. Paris Sér. I Math.*, 307(16):819–823, 1988.

[Lio90] J.-L. Lions. Quelques notions dans l'analyse et le contrôle de systèmes à données incomplètes. In *Proceedings of the XIth Congress on Differential Equations and Applications/First Congress on Applied Mathematics (Spanish) (Málaga, 1989)*, pages 43–54. Univ. Málaga, Málaga, 1990.

[Lio92a] J.-L. Lions. Remarks on approximate controllability. *J. Anal. Math.*, 59:103–116, 1992. Festschrift on the occasion of the 70th birthday of Shmuel Agmon.

[Lio92b] J.-L. Lions. *Sentinelles pour les systèmes distribués à données incomplètes*, volume 21 of *Recherches en Mathématiques Appliquées [Research in Applied Mathematics]*. Masson, Paris, 1992.

[Lio94] J.-L. Lions. Exact and approximate controllability for distributed parameter system. In *VI Escuela de Otoño Hispano-Francesa sobre simulación numérica en física e ingeniería*, pages 1–238. Universidad de Sevilla, España, 1994.

[LM68] J.-L. Lions and E. Magenes. *Problèmes aux Limites non homogènes et Applications*, volume 1. Dunod, 1968.

[LS01] T. Linß and M. Stynes. The SDFEM on Shishkin meshes for linear convection-diffusion problems. *Numer. Math.*, 87(3):457–484, 2001.

[LTW92] J.-L. Lions, R. Temam, and S. H. Wang. On the equations of the large-scale ocean. *Nonlinearity*, 5(5):1007–1053, 1992.

[Mar75] G. I. Marchuk. Formulation of theory of perturbations for complicated models. *Appl. Math. Optimization*, 2:1–33, 1975.

[MAS96] G. I. Marchuk, V. I. Agoshkov, and V. P. Shutyaev. *Adjoint equations and perturbation algorithms in nonlinear problems*. CRC Press, Boca Raton, FL, 1996.

[Med99] T. T. Medjo. Mixed formulation of two-layer quasi-geostrophic equations of the ocean. *Numer. Methods PDEs.*, 15:489–502, 1999.

[Med00] T. T. Medjo. Numerical simulations of a two-layer quasi-geostrophic equation of the ocean. *SIAM J. Numer. Anal.*, 37(6):2005–2022 (electronic), 2000.

[MS97] N. Madden and M. Stynes. Efficient generation of oriented meshes for solving convection-diffusion problems. *Internat. J. Numer. Methods Engrg.*, 40:565–576, 1997.

[MW95] P. G. Myers and A. J. Weaver. A diagnostic barotropic finite-element ocean circulation model. *J. Atmos. Ocean Tech.*, 12:511–526, 1995.

[Ped87] J. Pedlosky. *Geophysical Fluid Dynamics*. Springer-Verlag, Berlin, 1987.

[Pic03] M. Picasso. Numerical study of the effectivity index for an anisotropic error indicator based on Zienkiewicz-Zhu error estimator. *Comm. Numer. Methods Engrg.*, 19(1):13–23, 2003.

[PPK92] J. Peraire, J. Peiró, and Morgan K. Adaptive remeshing for three-dimensional compressible flow computations. *J. Comput. Phys.*, 103:269–285, 1992.

[Pue02] J.-P. Puel. Une approche non classique d'un problème d'assimilation de données. *C. R. Math. Acad. Sci. Paris*, 335(2):161–166, 2002.

[Qua93] L. Quartapelle. *Numerical solution of the incompressible Navier-Stokes equations*, volume 113 of *International Series of Numerical Mathematics*. Birkhäuser Verlag, Basel, 1993.

[RL00] A. R. Robinson and P. F. J. Lermusiaux. Overview of data assimilation. In *Harvard Reports in Physical/Interdisciplinary Ocean Science Number 62*. August 2000.

[RST96] H.-G. Roos, M. Stynes, and L. Tobiska. *Numerical methods for singularly perturbed differential equations*, volume 24 of *Springer Series in Computational Mathematics*. Springer-Verlag, Berlin, 1996. Convection-diffusion and flow problems.

[Sco76] R. Scott. Optimal L^∞ estimates for the finite element method on irregular meshes. *Math. Comp.*, 30(136):681–697, 1976.

[SRL01] D. Schneider, H.-G. Roos, and T. Linß. Uniform convergence of an upwind finite element method on layer-adapted grids. *Comput. Methods Appl. Mech. Engrg.*, 190(35-36):4519–4530, 2001.

[Tem84] R. Temam. *Navier-Stokes equations*, volume 2 of *Studies in Mathematics and its Applications*. North-Holland Publishing Co., Amsterdam, third edition, 1984. Theory and numerical analysis, With an appendix by F. Thomasset.

[Ver90] J. Verron. Altimeter data assimilation into ocean model: Sensitivity to orbital parameters. *J. Geophys. Res.*, 95:11443–11459, 1990.

[Ver92] J. Verron. Nudging satellite altimeter data into quasi-geostrophic ocean models. *J. Geophys. Res.*, 97(C5):7479–7492, 1992.

[Ver96] R. Verfürth. *A Review of A Posteriori Error Estimation and Adaptive Mesh-refinement Techniques*. Wiley-Teubner, Chichester,, Stuttgart, 1996.