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SECOND ORDER ASYMPTOTIC ANALYSIS IN OPTIMIZATION

ANÁLISIS ASINTÓTICO DE SEGUNDO ORDEN EN OPTIMIZACIÓN

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Felipe Ignacio Lara Obreque

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Felipe Ignacio Lara Obreque

Advisor: Dr. Fabián Flores-Bazán.
Co-Advisor: Dr. Nicolas Hadjisavvas.

EVALUATION PANEL

Signature: ____________________
Dr.

Signature: ____________________
Dr.

Signature: ____________________
Dr.

Signature: ____________________
Dr.

Dissertation date: ____________________

Score: ____________________

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Dedicado a mi hijo Valentín.

“Es, señores, política tradicional que honra a la República chilena la cariñosa atención que siempre se prestó por los gobiernos de todos los partidos a las instituciones de la educación popular; i esto, no con el banal intento de formar doctores, gramáticos i académicos, sino como lo expresaron los Senadores de 1818, con el nobilísimo intento de formar buenos ciudadanos, esto es, ciudadanos capaces de cooperar a los fines sociales del Estado i de la política.

Bajo de este respecto, creo yo, señores, que sin renunciar a la tarea mas noble i al medio más eficaz de gobierno, un Estado no puede ceder a ningún otro poder social la dirección superior de la enseñanza pública. Para el sociólogo y para el filósofo, señores , bajo el respecto indicado, bajo el respecto moral, Gobernar es Educar, i todo buen sistema de política es un verdadero sistema de educación, así como todo sistema jeneral de educación es un verdadero sistema político.”

Extracto del discurso “El Estado y la Educación Nacional”, 17 de Septiembre de 1888.

Valentín Leteleir Madariaga

Linares 16 del Diciembre de 1852 - Santiago 19-20 de Junio de 1919.

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Contents

ABSTRACT

Recently, the concepts of second order asymptotic directions and functions have been introduced and applied to global and vector optimization problems. In this work, we establish some new properties for these two concepts. In particular, in case of a convex set, a complete characterization of the second order asymptotic cone is given. Also, formulas that permit the easy computation of the second order asymptotic function of a convex function are established. It is shown that the second order asymptotic function provides a finer description of the behavior of functions at infinity, than the first order asymptotic function. Furthermore, we show the second order asymptotic function of a given convex one can be seen as the first order asymptotic function of another convex function.

We use second order asymptotic analysis to develop necessary and sufficient conditions for the scalar minimization problem in the noncoercive convex case. We give a new existence result for a point to be a proper efficient solution in the multiobjective optimization problem and a sufficient condition for the Domination Property. Finer estimates for the efficient and weak efficient solution sets (and for their second order asymptotic cones) to a convex/quasiconvex vector optimization problems are also provided.

We use asymptotic analysis to describe in a more systematic way the behaviour at the infinity of functions in the convex and quasiconvex case. Starting from the formulae for the first and second order asymptotic function in the convex case, we introduce similar notions suitable for dealing with quasiconvex functions. We characterize the nonemptiness and boundedness of the set of minimizers of any lsc quasiconvex function; finally, we also characterize boundedness from below, along lines, of any proper and lsc function.

Chapter 1

Introduction

En teoría de optimización existen tres aspectos que son de gran interés estudiar; primero, estudiar la existencia de soluciones óptimas; segundo, establecer condiciones necesarias para la existencia de estas soluciones; y tercero, entregar condiciones suficientes para la existencia de soluciones óptimas. Usualmente, se reconoce la entrega de condiciones suficientes para la existencia de solución como un problema de gran complejidad.

En general, estamos acostumbrados a buscar los puntos de mínimo y/o máximo partiendo de nuestros conocimientos del cálculo diferencial, tanto en una como en n dimensiones, esto es, derivando y encontrando los puntos críticos de las funciones objetivos, para después, por medio de las segundas derivadas, establecer la optimalidad de estos puntos en la función original. En este sentido, las funciones convexas juegan un rol fundamental, debido a este importante resultado: "En una función convexa, todo mínimo local es mínimo global".

Lamentablemente, no siempre podemos derivar las funciones, ya sea por la existencia de "manifolds" o por problemas de continuidad en la función. Es por esta situación que existen importantes conceptos de derivadas generalizadas como el clásico concepto de subdiferencial introducido por R. T. Rockafellar en su tesis doctoral. También a su generalización de epsilon-subdiferencial, y muchos otros conceptos desarrollados largamente en la literatura como puede verse, por ejemplo, en [19, 28, 58] y sus referencias.

En esta tesis, trabajaremos con el concepto de análisis asintótico (o de recesión), el cual intenta describir el comportamiento de la función en el infinito, es decir, describe su comportamiento muy lejos del lugar de en donde el mínimo y/o máximo podría estar. En este sentido, la función asintótica (de primer orden) geométricamente, nos entrega la pendiente de la recta tangente a la función original en el infinito.

Este concepto fue introducido por Steinitz en [66] hace más de un siglo. Luego fue redescubierto por Debreu en su libro Teoría del Valor en [20] y por Dieudonné en dimensión infinita en [27]. Otros usos fueron entregados en [17, 29] durante la década del sesenta. Pero quizás los avances más importantes fueron dados en la década de los setentas por Didieu en [25, 26] para el caso no convexo y por R. T. Rockafellar en el caso convexo, donde sus trabajos están resumidos en sus libros [62, 63]. De ahí en más, esta herramienta ha sido muy utilizada para la obtención de condiciones necesarias y suficientes para el problema de Minimización Escalar, Optimización Multiobjetivo, Desigualdades Variacionales, Teoría de Control Óptimo, Cálculo de Variaciones y Problemas de Ecuaciones en Derivadas Parciales, entre muchas otras áreas de la matemática aplicada.

A modo de ejemplo, propiedades básicas del análisis asintótico pueden encontrarse en los libros [9, 54, 62] para espacios de dimensión finita, mientras que en [5, 10, 63] para espacios de dimensión infinita. Aplicaciones al problema de Minimización Escalar pueden encontrarse en [6, 8, 60, 68]. Aplicaciones en Economía pueden verse en [20, 54, 57]. Aplicaciones a Optimización Multiobjetivo en [21, 22, 24, 32, 38, 39, 40]. Aplicaciones a Desigualdades Variacionales y Problemas de Equilibrio en [1, 2, 5, 31, 33, 43, 63].

Hace muy poco, los Profesores Dinh The Luc y Nicolás Hadjisavvas, han introducido el concepto de "dirección asintótica de segundo orden" en [45], concepto que busca controlar la distancia entre las direcciones asintóticas de primer orden y los puntos del conjunto que la generan. Para esto, han definido el conjunto asintótico de segundo orden y las funciones asintóticas de segundo orden superior e inferior. Además, han aplicado esta nueva herramienta al problema de minimización escalar (caso coercivo), al problema de optimización multiobjetivo, y también han entregado condiciones suficientes para que la suma de dos conjuntos sea cerrada. Todo esto en un contexto muy general, sin utilizar hipótesis de convexidad o convexidad generalizada, lo cual hace que las fórmulas para el cálculo de las funciones asintóticas superior e inferior sean complejas de manejar.

En esta Tesis Doctoral, tomamos esta idea de análisis asintótico de segundo orden y la desarrollamos detalladamente para conjuntos y funciones convexas. Aplicaciones a los problemas de optimización escalar y multiobjetivo son también desarrollados. Finalmente, entregamos atisbos para algunos casos de convexidad generalizada, como funciones casiconvexas y semiestrictamente casiconvexas. Todo esto organizado de la siguiente forma.

En Chapter 2, introducimos los conceptos básicos necesarios del Análisis Convexo y de la Teoría de Optimización que nos permitirán comprender las demostraciones y extensiones de los resultados en los capítulos posteriores. Nociones de convexidad y convexidad generalizada, propiedades de conos convexos cerrrados, el Teorema Bipolar, la transformada de Legendre-Fenchel y su biconjugada, son algunos de los conceptos revisados en Section 2.1. En Section 2.2 revisamos muchos de los conceptos y propiedades básicas del análisis asintótico (o de recesión) tanto para el caso convexo como no convexo, para conjuntos y funciones (ver [9, 33, 54, 62]). Dichas propiedades serán posteriormente extendidas (la gran mayoría de ellas), al caso del segundo orden.

En Chapter 3 desarrollamos profundamente la teoría básica del Análisis Asintótico de Segundo Orden para conjuntos y funciones convexas. Dos importantes caracterizaciones para el conjunto asintótico de segundo orden se entregan cuando el conjunto original es convexo, Proposition 3.4 y Proposition 3.5. A partir de aquí, interesantes propiedades para el caso convexo son desarrolladas en la Proposition 3.6. Además, fórmulas que permiten un fácil cálculo para la función asintótica de segundo orden son también desarrolladas en este capítulo, las cuales pueden verse en la Proposition 3.17 ecuaciones (3.17) y (3.18), como también una tercera fórmula en Proposition 3.18. Estos resultados están resumidos en nuestro primer artículo científico, ver [35].

En Chapter 4, entregamos aplicaciones del capítulo anterior a problemas de minimización escalar en dimensión finita para funciones convexas no coercivas. Una primera aproximación en la búsqueda de condiciones suficientes para el acotamiento por abajo de una función semicontinua inferiormente en términos de las funciones asintóticas de primer y segundo orden se muestra en Proposition 4.2. Relacionamos varias familias de funciones convexas no coercivas presentadas en [6, 7, 8, 51] con una nueva clase introducida por nosotros. Finalmente entregamos un resultado que caracteriza la solución óptima en Theorem 4.2.

Aplicaciones al problema de Optimización Multiobjetivo no convexo se entregan en este capítulo, condiciones suficientes para la existencia de una solución Eficiente Propia y de la Propiedad de Dominación pueden verse en Theorem 4.3, para la existencia de una solución Débil Eficiente en el caso convexo en Theorem 4.4. Finalmente, caracterizamos los conos asintóticos de segundo orden para los conjuntos eficiente y débil eficiente en Theorem 4.5.

En Chapter 5, intentamos dar los primeros pasos del desarrollo de esta teoría de análisis asintótico de segundo orden en dimensión infinita. En particular, trabajamos en espacios de Banach reflexivos en donde definimos las direcciones asintóticas de segundo orden y la función asintótica de segundo orden en el caso convexo utilizando los resultados obtenidos por nosotros en Chapter 3. Una aplicación al problema de minimización escalar no coercivo se entrega en este capítulo en Theorem 5.2, "generalizando" en algún sentido, un clásico resultado de [10], que también puede encontrarse en el libro de Attouch-Buttazzo-Michaille [5], capítulo 15, que aquí lo recordamos como Theorem 5.1.

Desgradaciadamente, estos avances en dimensión infinita tienen una fuerte limitante, pues sólo funcionan para funciones convexas con interior topológico del dominio efectivo no vacío, esto es, cuando intdom $F \neq \emptyset$, una condición demasiado restrictiva en espacios de dimensión infinita. Es por eso que queda abierto el problema de avanzar en una definición más adecuada de función asintótica de segundo orden en dimensión infinita, en la cual se utilice una noción de interior más débil, como por ejemplo, el concepto de "quasi interior relativo" introducido por Borwein y Lewis en [15], y desarrollado también en [14, 69, 70].

Finalmente, el concepto de convexidad generalizada ha sido incorporado hace mucho tiempo, y cuenta con un desarrollo teórico bastante avanzado como puede verse en [3, 16, 19, 46, 50, 58]. Es un área tan desarrollada que cuenta con vastas aplicaciones en economía, mecánica, ingeniería, finanzas y otras áreas de la matemática aplicada. Numerosos investigadores han aportados interesantes trabajos como puede verse en [30, 31, 33, 39, 40, 48, 53, 60, 67] para optimización escalar y multiobjetivo.

Un problema abierto hasta hoy, es el de entregar una definición correcta de función asintótica en los casos de convexidad generalizada, como por ejemplo, cuando la función es casiconvexa, semiestrictamente casiconvexa o pseudoconvexa. Algunos atisbos de análisis asintótico para el caso casiconvexo pueden encontrarse en [4, 25, 26, 34, 56, 59, 61]. En todo caso, en ninguno de los trabajos previos se ha logrado cerrar la discusión o entregar razones suficientes para terminar con la búsqueda de una función asintótica adecuada en el caso casiconvexo.

En Chapter 6 entregamos nuevos aportes y fundamentos para ayudar, en algún sentido, con la búsqueda en la vía correcta de la función asintótica en el caso casiconvexo. Desarrollamos en este capítulo propiedades de la función asintótica incidente, propuesta por J. P. Penot en [59], como puede verse en Proposition 6.2 and Proposition 6.3. Además, entregamos por primera vez una fórmula que ayuda a calcular esta función sin utilizar su definición a través del epígrafo de la función en Proposition 6.6, ecuación (6.5). Junto con esto, también comparamos esta definición con otros intentos que han aparecido en la literatura como los introducidos en [34]. Desigualdades, ejemplos y contraejemplos que muestran sus caracteristicas y propiedades, como también sus diferencias, son analizados en este capítulo. Aplicaciones al problema de optimización escalar y multiobjetivo pueden verse en Section 6.2. Caracterizaciones para el acotamiento y no vacuidad del conjunto de minimizadores en el caso escalar se presentan en Theorem 6.1 y Theorem 6.2. Finalmente, condiciones necesarias para la existencia de soluciones eficientes y débil eficiente pueden encontrarse en Lemma 6.1.

Por todo lo anterior, este trabajo puede ser visto como continuación de los estudios realizados por los autores antes mencionados y como la apertura de una nueva línea de investigación en el Análisis Asintótico de Convexidad Generalizada y en Teoría de Optimización.

Chapter 2

Convex Analysis

In this chapter we present the basic algebraic and topological notions that we will use for the rest of this thesis work. Such notions will be fundamental to understand the nature of the problem to study and to establish, correctly, the results of interest.

2.1 Basic notions and cones

We start this section with the basic properties of convex analysis for sets and functions.

Definition 2.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set. We say K is convex if for all $x, y \in K$*

$$\lambda x + (1-\lambda)y \in K, \ \forall \ \lambda \in [0,1].$$

A hyperplane $H = \{x \in \mathbb{R}^n : \ \langle x, a\rangle = \alpha\}$ where $a \in \mathbb{R}^n, a \neq 0$ and $\alpha \in \mathbb{R}$ is a convex set. A hyperplane divides the space $\mathbb{R}^n$ in two closed half space denoted by $H^+ = \{x \in \mathbb{R}^n : \ \langle x, a\rangle \geq \alpha\}$ and $H^- = \{x \in \mathbb{R}^n : \ \langle x, a\rangle \leq \alpha\}$. Clearly, the half spaces are convex sets.

We also consider the following sets;

Definition 2.2 *Let $K \subseteq \mathbb{R}^n$ be any set, then*

(a) *The affine hull of K, denoted by* aff K*,*

$$\alpha x + (1-\alpha)y \in K, \ \forall \ x, y \in K, \ \forall \ \alpha \in \mathbb{R}.$$

(a) *The convex hull of K, denoted by* $\mathrm{co}(K)$*, is the set*

$$\mathrm{co}(K) = \left\{ x \in \mathbb{R}^n : \ x = \sum_{i=1}^{n+1} \lambda_i x_i, \sum_{i=1}^{n+1} \lambda_i = 1, \lambda_i \geq 0, x_i \in K, \ \forall \ i \in \{1, 2, ..., n+1\} \right\}.$$

(b) *A point $\overline{x} \in K$ is an extreme point of K, if for all $x, y \in K$ we have that*

$$\overline{x} \neq \lambda x + (1-\lambda)y, \ \forall\ \lambda \in]0,1[.$$

(c) *The cone generated by K, is the smallest cone containing the set K, that is,*

$$\text{cone }(K) = \bigcup_{t \geq 0} tK.$$

Obviously, $\text{co}(K) \subseteq \text{aff } K$ and $\text{aff } K = \text{aff } (\text{co}(K))$.

Definition 2.3 *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set. We define the relative interior of K, and we denote it by* ri K, *to be the topological interior of k relative to* aff K. *That is,*

$$\text{ri } K = \{x \in \text{aff } K : \ \exists\ \varepsilon > 0, (x + \varepsilon B(0,1)) \cap \text{aff } K \subseteq K\}. \tag{2.1}$$

The relative interior is equal to the topological interior when the affine hull is the whole $\mathbb{R}^n$. Furthermore, if K is convex and $K \neq \emptyset$, then the relative interior ri $K \neq \emptyset$. That is, the relative interior of a nonempty convex set is always nonempty.

Proposition 2.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set. Then we have the following assertions:*

(a) *If $x \in$ ri K and $y \in$ cl K, then $\lambda x + (1-\lambda)y \in$ ri K for all $\lambda \in]0,1]$.*

(b) *$x \in$ ri K if and only if for every $y \in K$ there exists $\varepsilon > 0$ such that $x + \varepsilon(y - x) \in K$.*

(c) *$x \in$ int K if and only if for each $u \in \mathbb{R}^n$ there exists $\varepsilon > 0$ such that $x + \varepsilon u \in K$.*

Proof. See Theorem 6.1, 6.4 and Corollary 6.4.1 en [62]. ■

More properties and details on relative interior can be found in [9, 62].

Definition 2.4 *Let K be a nonempty convex set. We say that a function $f : K \subseteq \mathbb{R}^n \to \mathbb{R}$ is;*

(a) *convex, if for all $x, y \in K$ we have*

$$f(\lambda x + (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y), \ \forall\ \lambda \in [0,1] \tag{2.2}$$

(b) *quasiconvex, if for all $x, y \in K$ we have*

$$f(\lambda x + (1-\lambda)y) \leq \max\{f(y), f(x)\}, \ \forall\ \lambda \in [0,1]; \tag{2.3}$$

(*c*) *semistrictly quasiconvex if for all* $x, y \in K$ *with* $f(x) \neq f(y)$*, we have*

$$f(\lambda x + (1-\lambda)y) < \max\{f(y), f(x)\}, \ \forall \ \lambda \in \ ]0,1[. \tag{2.4}$$

Remark 2.1 *Every convex function is quasiconvex and semistrictly quasiconvex. But, in general, there is no relationship between quasiconvexity and semistrictly quasiconvexity.*

For functions defined on the extended real line, we define the following important sets.

Definition 2.5 *Given a function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$*, we define*

(*a*) *the effective domain of* f *by* $\mathrm{dom}\ f = \{x \in \mathbb{R}^n : \ f(x) < +\infty\}$.

(*b*) *The epigraph of* f *by* $\mathrm{epi}\ f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} : \ f(x) \leq t\}$.

(*c*) *The level set of* f *by* $S_\lambda(f) = \{x \in \mathbb{R}^n : f(x) \leq \lambda\}$*, with* $\lambda \in \mathbb{R}$.

And the following two concepts;

Definition 2.6 *We say that a function* $f : K \subseteq \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *is lower semicontinuous (lsc for short) at* $\overline{x} \in K$*, if for all* $\{x_n\}_{n\in\mathbb{N}} \subseteq K$ *such that* $x_n \to \overline{x}$*, we have*

$$f(\overline{x}) \leq \liminf_{n\to\infty} f(x_n).$$

f *is lsc on* K *if it is in every point of* K.

Definition 2.7 *A function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ *is said to be proper if* $\mathrm{dom}\ f \neq \emptyset$ *and* $f(x) > -\infty$ *for all* $x \in \mathrm{dom}\ f$.

Proposition 2.2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a function. Then the following assertions are equivalent;*

(*a*) f *is lsc on* $\mathbb{R}^n$.

(*b*) $\mathrm{epi}\ f$ *is closed in* $\mathbb{R}^n \times \mathbb{R}$.

(*c*) $S_\lambda(f)$ *is closed for all* $\lambda \in \mathbb{R}$.

Definition 2.8 *Let* K *be a nonempty convex set. We say that a function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{\pm + \infty\}$ *is;*

(*a*) f *is convex if* $\mathrm{epi}\ f$ *is convex.*

(*b*) *f is quasiconvex, if for all $\lambda \in \mathbb{R}$ $S_\lambda(f)$ is convex.*

It is easy to see that in case of f is a real-valued fucntion then (2.2) is equivalent to Definition 2.8 part (a) and (2.3) is equivalent with part (b).

There exists a relationship between quasiconvexity and semistrictly quasiconvexity under an additional assumption of lsc for f.

Proposition 2.3 *Suppose $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is semistrictly quasiconvex and lsc, then f is quasiconvex.*

More details on generalized convexity can be found in [3, 16, 46, 50] and references therein.

Duality plays an important role in optimization problems, under convexity or not. A key player in any duality framework is the Legendre-Fenchel transform, also called the conjugate function of a given function.

Definition 2.9 *For any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the function $f^* : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f^*(y) = \sup_{x \in \mathrm{dom}\ f} \{\langle x, y\rangle - f(x)\} \tag{2.5}$$

is called the conjugate function of f. The biconjugate function of f is defined by

$$f^{**}(x) = \sup_{y \in \mathrm{dom}\ f^*} \{\langle x, y\rangle - f^*(y)\} \tag{2.6}$$

If f is proper and lsc, then we always have $f^{**} \leq f$. Furthermore, if f is convex, proper and lsc function, then by Theorem 1.2.5 in [9] we know that $f^{**} = f$.

We also present basic notions and properties for cones and the Bipolar Theorem.

Definition 2.10 *Given a nonempty set $P \subseteq \mathbb{R}^m$, we define its positive polar cone P^* by*

$$P^* = \{q \in \mathbb{R}^m : \ \langle q, p\rangle \geq 0, \ \forall\ p \in P\}.$$

It is easy to see that P^* is always a closed convex cone. We denote $P^{**} = (P^*)^*$.

Proposition 2.4 *Let $P \subseteq \mathbb{R}^m$ be a closed convex cone with int $P^* \neq \emptyset$, then*

$$\mathrm{int}\ P^* = \{q \in \mathbb{R}^m : \ \langle q, p\rangle > 0, \ \forall\ p \in P\backslash\{0\}\}.$$

Remark 2.2 *If $P \subseteq \mathbb{R}^m$ is a polyhedral cone, we suppose there exists $\{q_1, q_2, ..., q_k\} \subseteq \mathbb{R}^m$ such that the positive polar cone P^* of P is given by*

$$P^* = \bigcup_{t \geq 0} t(\text{co}\{q_1, q_2, ..., q_k\}).$$

We now present the Bipolar Theorem.

Theorem 2.1 *Let P be a closed convex cone, then $P = P^{**}$.*

Remark 2.3 *From the previous theorem we obtain the following characterization for a closed convex cone;*

$$p \in P \Leftrightarrow \langle q, p \rangle \geq 0, \ \forall \ q \in P^*. \tag{2.7}$$

If int $P \neq \emptyset$ then

$$p \in \text{int } P \Leftrightarrow \langle q, p \rangle > 0, \ \forall \ q \in P^*, q \neq 0. \tag{2.8}$$

Definition 2.11 *Let $P \subseteq \mathbb{R}^m$ be a convex cone, we say that P is;*

(a) *pointed, if $P \cap (-P) = \{0\}$;*

(b) *acute, if there exists an open half space $H^{++} = \{x \in \mathbb{R}^m : \ \langle q, x \rangle > 0\}$ with $q \neq 0$, such that*

$$\text{cl } P \subseteq H^{++} \cup \{0\}.$$

If P is a closed convex cone, then the previous two definitions coincide, see Proposition 2.1.4 in [64].

Theorem 2.2 *Let P be a closed convex cone, then*

$$\text{int } P^* \neq \emptyset \Leftrightarrow P \text{ is pointed.}$$

2.2 First order asymptotic analysis

In this section we introduce the concept of asymptotic cone and function.

Definition 2.12 *Let $K \subseteq \mathbb{R}^n$ be any set, then its asymptotic cone K^∞ is defined by*

$$K^\infty = \{v \in \mathbb{R}^n : \ \exists \ \{x_n\}_{n \in \mathbb{N}} \subseteq K, \ \exists \ t_n \downarrow 0, \ t_n x_n \to v\}.$$

If $K = \emptyset$, we set by convention $K^\infty = (\emptyset)^\infty = \{0\}$.

Definition 2.13 *For any proper function $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, its asymptotic function g^∞, given by $g^\infty : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is defined as the function for which*

$$\text{epi } g^\infty = (\text{epi } g)^\infty \tag{2.9}$$

The basic properties for asymptotic cones and functions are listed in the following Lemma. For more details refer to [9], Chapter 3.

Lemma 2.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then the following assertions hold;*

(a) *K^∞ is always a closed cone.*

(b) $K^\infty = (\text{cl } K)^\infty$.

(c) *K is bounded if and only if $K^\infty = \{0\}$.*

(d) *If $K_1 \subseteq K$ then $K_1^\infty \subseteq K^\infty$;*

(e) *If K is a closed convex set, then*

$$K^\infty = \{u \in \mathbb{R}^n : \; x_0 + \lambda u \in K, \; \forall \; \lambda \geq 0\} \text{ for any } x_0 \in K. \tag{2.10}$$

(f) *If K is convex, not necessarily closed, then we have a similar characterization for its asymptotic cone K^∞ using elements of the* ri K, *that is,*

$$K^\infty = \{u \in \mathbb{R}^n : \; x_0 + \lambda u \in \text{ri } K, \; \forall \; \lambda \geq 0\} \text{ for any } x_0 \in \text{ri } K. \tag{2.11}$$

(g) *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a family of sets, then*

$$(\bigcap_{i \in I} K_i)^\infty \subseteq \bigcap_{i \in I} K_i^\infty,$$

and equality holds when K_i is closed and convex for all $i \in I$ and $\bigcap_{i \in I} K_i \neq \emptyset$.

(h) *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a family of sets, then*

$$\bigcup_{i \in I} K_i^\infty \subseteq (\bigcup_{i \in I} K_i)^\infty,$$

and equality holds when $|I| < +\infty$, with I the index set.

(*i*) *Let* $g : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function and let* $\alpha \in \mathbb{R}$ *be such that* $S_\alpha(g) \neq \emptyset$*, then*

$$\{x \in \mathbb{R}^n : \ g(x) \leq \alpha\}^\infty \subseteq \{x \in \mathbb{R}^n : \ g^\infty(x) \leq 0\},$$

and equality holds when g *is lsc, proper and convex.*

We also have of interest formulas for the calculus of the asymptotic function.

Proposition 2.5 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function, then its asymptotic function is given by,*

(*a*)

$$f^\infty(u) = \liminf_{\substack{t \to 0^+ \\ y \to u}} t f(\frac{y}{t}) \tag{2.12}$$

(*b*) *If* f *is also convex and lsc, then*

$$f^\infty(u) = \sup_{\lambda > 0} \frac{f(\overline{x} + \lambda u) - f(\overline{x})}{\lambda}, \ \overline{x} \in \operatorname{dom} \ f \tag{2.13}$$

(*c*) *and*

$$f^\infty(u) = \sup_{x \in \operatorname{dom} \ f} \left(f(x+u) - f(x) \right). \tag{2.14}$$

Proposition 2.6 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper function, then* $f^\infty(\cdot)$ *is lsc and positively homogeneous.* $f^\infty(0) = 0$ *or* $-\infty$*, and if* $f^\infty(0) = 0$ *then* $f^\infty(\cdot)$ *is proper.*

The following result will be very useful in the next chapter, see Lemma 7.3 in [62].

Proposition 2.7 *For every proper convex function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *we have that*

$$\operatorname{riepi} \ f = \{(x, \mu) \in \mathbb{R}^n \times \mathbb{R} : \ x \in \operatorname{ridom} \ f, \mu > f(x)\}. \tag{2.15}$$

Chapter 3

Second order asymptotic analysis: basic theory

3.1 Introduction

The concept of asymptotic (or recession) directions of a set has been introduced almost 100 years ago [66], and then it was rediscovered by Debreu [20], where its use concerns the closedness of the sum of any two closed sets, which was also studied by Dieudonné [27] in an infinite dimensional setting, for convex sets. A further use is developed in [29], where a duality scheme for linear relations is discussed, see also [17, 68]. The notion of asymptotic direction may be conceived as a main tool to describe the asymptotic behaviour of the set at infinity along these particular directions, so it is of primary importance for dealing with unbounded sets, as it gives rise to the "asymptotic analysis" approach. A vector $u \neq 0$, say with $\|u\| = 1$, is by definition an asymptotic direction of a set K if its direction is the limit of directions of a sequence of vectors $\{x_k\}_{k\in\mathbb{N}}$ in K such that $\|x_k\| \to +\infty$; equivalently, if there exists a sequence $\{x_k\}$ in K such that $\|x_k\| \to +\infty$ and $\dfrac{x_k}{\|x_k\|} \to u$. In many interesting cases, the vectors x_k do not approach the line $\mathbb{R}u$ containing u; actually, their distance from it converges to $+\infty$. How exactly do the directions of x_k approach the direction of u? One way to answer this question, is to consider the projections $\langle x_k, u\rangle\, u$ of x_k onto u, and the differences $x_k - \langle x_k, u\rangle\, u$. Then, if $\|x_k - \langle x_k, u\rangle\, u\| \to +\infty$, one can find the limit, if any, of the directions of the latter vectors. These are exactly the canonic (second order) directions, which are orthogonal to u. From this line of reasoning, and in order to have a finer tool in the study of the behavior of sets and functions at infinity, very recently the authors in [45] introduced, more generally, the notion of second order direction for sets, as the limit of directions of a sequence of vectors $x_k - \alpha_k u$ with $\alpha_k \to +\infty$, $\frac{x_k}{\|x_k\|} \to \frac{u}{\|u\|}$ and

$\|x_k - \alpha_k u\| \to +\infty$, which show how the vector x_k is "seen" from a vector $\alpha_k u$. The precise definition for sets (and functions) will be given in the two following sections.

We point out that (first order) asymptotic cones in infinite dimensional spaces, for non-convex sets, were considered in [25, 26], see also [54]; whereas in spaces of finite dimension, we refer to [9, 63]. More recent applications may be found in [21, 31, 33, 60] and references therein.

A related concept which is motivated mainly by minimization problems, is the concept of asymptotic function. A careful analysis of the behaviour of the asymptotic function (associated to objective) along the asymptotic directions of the feasible set is crucial for a study of the existence of minima. Similarly to the first order case, a second order asymptotic function can be defined.

We believe that in the same way as the first-order asymptotic analysis proved to be a powerful in the study of sets and functions at infinity, the second-order approach will yield finer results in optimization, economics, engineering, etc. Indeed, it was shown in [45] that these second order notions may be used to establish necessary or sufficient conditions for optimality, characterize the efficient points in vector optimization, or provide criteria for the closedness of the sum of closed sets, in cases where the results using the first order asymptotic notions are not adequate. It is worthwhile mentioning that in [45], mainly the general non convex case was treated, and no formula was provided for the convex case. This latter situation will be discussed in detail in the present chapter, so we will see that there are very simple and attractive formulas that provide the corresponding asymptotic notions for sets and functions.

We also must mention that the meaning of our concept of asymptotic cone is different from that considered, for instance, in [49]. For us the term asymptotic means far away, in contrast to the mentioned in that paper, compare (3) and (4) in Definition 2.1 of [49] and our Proposition 3.4 part (c).

One our main results is a characterization of the second order asymptotic cone in the case of a convex set. Based on this characterization, we will show several properties of such cones, and give formulas for the second order asymptotic function, that permit an easy computation.

The structure of the chapter is as follows. The basic definitions and notations are given in the next section. Section 3 contains some preliminary results on the properties of second order asymptotic cones, and also their relation to the so-called canonic directions. In addition, the convex case is discussed in details. For example, it is known that for a closed convex set $K \subseteq \mathbb{R}^n$, its asymptotic cone K^∞ is the set of vectors $u \in \mathbb{R}^n$ such that

$x_0 + tu \in K$ for every $x_0 \in K$ and $t > 0$. As we shall see, given a convex (not necessarily closed) set K and an element x_0 of its relative interior $\operatorname{ri} K$, the second order asymptotic cone of K with respect to u, which will be denoted by $K^{\infty 2}[u]$, is the set of all $v \in \mathbb{R}^n$ such that for every $s > 0$, $x_0 + tu + sv \in K$ for all t sufficiently large (Proposition 3.4); actually, this property implies both $u \in K^\infty$ and $v \in K^{\infty 2}[u]$. In the special case of polyhedral sets, we may have $x_0 \in K$ instead of $x_0 \in \operatorname{ri} K$, and $K^{\infty 2}[u]$ has a simple expression (Remark 3.2 and Proposition 3.8).

The above characterization, together with some other similar ones, will be used to obtain information about the structure of the second order asymptotic cone, as well as several of its properties. Section 3.3 is devoted to the second order asymptotic function. Using the characterizations found in Section 3.2, we obtain formulas which permit an easy calculation of this function in case the original function is (not necessarily lower semicontinuous) convex, as we show with some examples.

It should be noted that the notation differs with respect to the ones introduced in [45]. Also, the definition of second order asymptotic function (which in [45] was called lower second order asymptotic function) is different, even if it is proven to be equivalent to the one in [45]. The changes were deemed useful in order to approach the more usual notation and definitions of the corresponding first order notions. Similar changes affect the canonic directions.

3.2 Second order asymptotic cones

We start by establishing various properties for the second order asymptotic cone of any set, including its link to canonic directions. Then we provide some characterizations of the second order asymptotic cone in case the set is convex.

3.2.1 Preliminary results

Definition 3.1 *Given a nonempty set $K \subseteq \mathbb{R}^n$ and $u \in \mathbb{R}^n$, we say that $v \in \mathbb{R}^n$ is a second order asymptotic direction of K at u if there are sequences $x_k \in K$, s_k and $t_k \in \mathbb{R}$, with $s_k, t_k \to +\infty$ such that,*

$$v = \lim_{k \to +\infty} \left(\frac{x_k}{s_k} - t_k u \right). \tag{3.1}$$

The set of all such elements v is denoted by $K^{\infty 2}[u]$.

Equivalently, $v \in K^{\infty 2}[u]$ if for each $k \in \mathbb{N}$ there exist $s_k > k$ and $t_k > k$ such that

$$\left| \frac{x_k}{s_k} - t_k u - v \right| < \frac{1}{k}. \tag{3.2}$$

Note that if (3.1) holds for s_k, t_k, x_k as in the definition, then

$$\lim_{k \to +\infty} \frac{x_k}{s_k t_k} = u.$$

Consequently, one has necessarily $u \in K^{\infty}$.

The set $K^{\infty 2}[u]$ is a cone, termed the second order asymptotic cone of K at u. It is nonempty exactly when $u \in K^{\infty}$ (see [45]). $K^{\infty 2}[u]$ is always closed, and if $u = 0$ then $K^{\infty 2}[0] = K^{\infty}$.

Next proposition collects some basic properties of the second order asymptotic cone for any set, although they partly already appear in [45].

Proposition 3.1 *Let $\emptyset \neq K \subseteq \mathbb{R}^n$. The following assertions hold.*

(*a*) *If $K_0 \subseteq K$, then $K_0^{\infty 2}[u] \subseteq K^{\infty 2}[u]$ for all $u \in \mathbb{R}^n$.*

(*b*) *$K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ for all $u \in K^{\infty}$.*

(*c*) *$K^{\infty 2}[u] = \overline{K}^{\infty 2}[u]$ for all $u \in K^{\infty}$.*

(*d*) *For all $u \in K^{\infty}$, $(K^{\infty 2}[u])^{\infty} = K^{\infty 2}[u]$ and $K^{\infty 2}[u] = (K^{\infty 2}[u])^{\infty 2}[u]$.*

(*e*) *If $u \in K^{\infty}$, then $K^{\infty 2}[u] \subseteq \overline{\mathbb{R}_+ K - \mathbb{R}_+ u}$.*

(*f*) *If $u \in \operatorname{ri} K^{\infty}$, then $\operatorname{aff} K^{\infty} \subseteq K^{\infty} - K^{\infty} \subseteq K^{\infty 2}[u]$.*

(*g*) *Let $A, B \subseteq \mathbb{R}^n$, with $u_1 \in A^{\infty}, u_2 \in B^{\infty}$, then*

$$(A \times B)^{\infty 2}[(u_1, u_2)] = A^{\infty 2}[u_1] \times B^{\infty 2}[u_2].$$

(*h*) *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a family of sets and $u \in \mathbb{R}^n$. Then*

$$\bigcup_{i \in I} K_i^{\infty 2}[u] \subseteq \Big(\bigcup_{i \in I} K_i\Big)^{\infty 2}[u].$$

Equality holds when $|I| < +\infty$.

(i) Let $\{K_i\}_{i\in I} \subseteq \mathbb{R}^n$ be a family of sets satisfying $\bigcap_{i\in I} K_i \neq \emptyset$ and $u \in \mathbb{R}^n$. Then

$$\Big(\bigcap_{i\in I} K_i\Big)^{\infty 2}[u] \subseteq \bigcap_{i\in I} K_i^{\infty 2}[u].$$

Proof. (a) and (b) were proved in Proposition 2.2 of [45]. (c), (e) and (g) are straightfoward.

(d): The first equality is obvious since $K^{\infty 2}[u]$ is a closed cone. To show inclusion $(\supseteq)$ in the second equality, let $w \in (K^{\infty 2}[u])^{\infty 2}[u]$. In view of (3.2), for each $k \in \mathbb{N}$ there exist $w_k \in K^{\infty 2}[u]$ and $s_k > k$, $t_k > k$ such that $\left|\frac{w_k}{s_k} - t_k u - w\right| < \frac{1}{k}$; moreover, there exist $w_k' \in K$ and $s_k' > k$, $t_k' > k$ such that $\left|\frac{w_k'}{s_k'} - t_k' u - w_k\right| < \frac{1}{k}$. Dividing the second inequality by s_k and adding to the first we deduce that

$$\left|\frac{w_k'}{s_k' s_k} - \left(t_k + \frac{t_k'}{s_k}\right) u - w\right| < \frac{1}{k} + \frac{1}{k s_k} \leq \frac{1}{2k}.$$

Since $s_k' s_k > k$ and $t_k + \frac{t_k'}{s_k} > k$, we infer that $w \in K^{\infty 2}[u]$.

$(\subseteq)$ Let $w \in K^{\infty 2}[u]$. For every sequence $t_k \to +\infty$, we have $w + t_k u \in K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ by property (b), so $t_k w + t_k^2 u \in K^{\infty 2}[u]$ since $K^{\infty 2}[u]$ is a cone. From

$$w = \lim_{k\to+\infty} \Big(\frac{t_k w + t_k^2 u}{t_k} - t_k u\Big),$$

it then follows that $w \in (K^{\infty 2}[u])^{\infty 2}[u]$.

(f) Given $v \in \operatorname{aff} K^{\infty}$, we have that for all $k \in \mathbb{N}$ large enough, $u + \frac{1}{k} v \in K^{\infty}$. Since K^{∞} is a cone, $ku + v \in K^{\infty}$. Consequently one can find $x_k \in K$ and $t_k > k$ such that $\left|\frac{x_k}{t_k} - ku - v\right| < \frac{1}{k}$. In view of (3.2), this implies that $v \in K^{\infty 2}[u]$.

(h): $(\subseteq)$ is an obvious consequence of property (a).

$(\supseteq)$ $(|I| < \infty)$. Let $w \in (\bigcup_{i=1}^m K_i)^{\infty 2}[u]$. Then there exist sequences $\{x_k\}_{k\in\mathbb{N}} \subseteq \bigcup_{i=1}^m K_i$ and $s_k, t_k \to +\infty$ with $w = \lim_{k\to\infty}\left(\frac{x_k}{s_k} - t_k u\right)$. Since I is finite, there exists i_0 and a subsequence $\{x_{k_l}\}_{l\in\mathbb{N}}$ such that $x_{k_l} \in K_{i_0}$ for all $l \in \mathbb{N}$. Hence, $w \in K_{i_0}^{\infty 2}[u] \subseteq \bigcup_{i=1}^m K_i^{\infty 2}[u]$.

(i) is again a trivial consequence of property (b). ∎

As we will see in the next section, the reverse inclusion does not hold in general, but it does hold in some cases.

Remark 3.1 *We do not have all the good properties of the first order asymptotic analysis in the second order. For example, consider $K \subseteq \mathbb{R}^2$, with $K = (\{0\} \times \mathbb{R}) \cup (\mathbb{R} \times \{0\})$, that is, K is the union of axis x and y. Here K is a closed cone, thus $K^{\infty} = K$.*

On the other hand, $K^{\infty 2}[u]$ change with respecto to the vector u. If we pick $u_1 = (1,0)$ then $K^{\infty 2}[(1,0)] = \mathbb{R} \times \{0\}$ while if $u_2 = (0,1)$ then $K^{\infty 2}[(0,1)] = \{0\} \times \mathbb{R}$.

Then, given a closed cone K, its not necessarily true that $K = K^{\infty 2}[u]$ for all $u \in K^{\infty}$.

In second order asymptotic analysis, changing the convergence rates of the sequences t_k, s_k, we can have finer estimates is some circumstances. To that end, the authors in [45] introduced the second order asymptotic directions called canonic directions, which are of especial importance in our work.

Definition 3.2 *Given $u \neq 0$, a second order asymptotic direction $v \in K^{\infty 2}[u]$ is canonic if $\langle u, v\rangle = 0$. The set of all canonic directions for u is denoted by $K^{\nu}[u]$*

Note that $K^{\nu}[u] = K^{\infty 2}[u] \cap u^{\perp}$. Furthermore, for each $u \neq 0$ and $\alpha > 0$, we have by Proposition 2.2 (vi) in [45] that $K^{\infty 2}[\alpha u] = K^{\infty 2}[u]$. Thus $K^{\infty 2}[u] = K^{\infty 2}[\frac{u}{\|u\|}]$ and $K^{\nu}[u] = K^{\nu}[\frac{u}{\|u\|}]$, then we always restrict ourselves to the case $\|u\| = 1$.

We note that in [45] canonic directions were defined differently; we use here another definition which is simpler. The equivalence of the two definitions is a consequence of the following Proposition (see Proposition 2.8 in [45] for details). We recall it here for convenience of the reader.

Proposition 3.2 *Let $K \subseteq \mathbb{R}^n$ be nonempty and $u \neq 0$. Then the following assertions holds,*

(*a*) $K^{\nu}[u] \neq \emptyset$ *if and only if* $u \in K^{\infty}$.

(*b*) $K^{\infty 2}[u] = \mathbb{R}u + K^{\nu}[u]$.

(*c*) $v \in K^{\nu}[u]$ *if and only if it can be written in the form*

$$v = \lim_{t_k \to +\infty} t_k \left(\frac{x_k}{\|x_k\|} - \frac{u}{\|u\|} \right) \text{ with } t_k \to +\infty, x_k \in K \text{ and } \frac{t_k}{\|x_k\|} \to 0$$

In what follows $P_{u^{\perp}}$ denotes the projection on $u^{\perp}$.

Corollary 3.1 *For every $u \in K^{\infty} \backslash \{0\}$, $K^{\infty 2}[u] = \mathbb{R}u + K^{\nu}[u]$. Then we have*

$$K^{\nu}[u] = P_{u^{\perp}}(K^{\infty 2}[u]).$$

Proof. We may assume that $\|u\| = 1$. To show inclusion $\subseteq$, write any $v \in K^{\infty 2}[u]$ as $v = \langle u, v\rangle u + (v - \langle v, u\rangle u)$. Then $\langle u, v\rangle u \in \mathbb{R}u$ and $v - \langle v, u\rangle u \in u^{\perp}$. In addition, $v - \langle v, u\rangle u \in K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$ by Proposition 3.1 part (b). Thus inclusion $\subseteq$ follows. The opposite inclusion follows from $\mathbb{R}u + K^{\nu}[u] \subseteq \mathbb{R}u + K^{\infty 2}[u] = K^{\infty 2}[u]$.

The second assertion is a consequence of the first one. ■

It is interesting that the canonic directions in $K^{\nu}[u]$ are first order directions of the projection of K onto $u^{\perp}$:

Proposition 3.3 *Let $u \in K^{\infty}\backslash\{0\}$. Then $K^{\nu}[u] \subseteq (P_{u^{\perp}}K)^{\infty}$.*

Proof. Take any $v \in K^{\nu}[u]$. Then there exist sequences $\{x_n\} \subseteq K$ and $t_n, s_n \to +\infty$ such that (3.1) holds. It follows that

$$v = P_{u^{\perp}}v = \lim \left(\frac{P_{u^{\perp}}x_n}{t_n} - s_n P_{u^{\perp}}u \right) = \lim \frac{P_{u^{\perp}}x_n}{t_n} \in (P_{u^{\perp}}K)^{\infty}.$$

■

Note that in general $K^{\nu}[u] \neq (P_{u^{\perp}}K)^{\infty}$. For instance, if $K \subseteq \mathbb{R}^2$ is the set $\{(x_1, 0) : x_1 \in \mathbb{R}\} \cup \{(0, x_2) : x_2 \in \mathbb{R}\}$ and $u = (0, 1)$, then $u^{\perp} = \{(x_1, 0) : x_1 \in \mathbb{R}\}$, $(P_{u^{\perp}}K)^{\infty} = u^{\perp}$ but $K^{\nu}[u] = \{0\}$.

3.2.2 The case of convex sets

In case K is a convex (not necessarily closed) subset of $\mathbb{R}^n$, we have a characterization of $K^{\infty 2}[u]$ which reminds the one for K^{∞} given by (2.10). This will permit us to show some properties of the second order asymptotic cone of convex sets.

Proposition 3.4 *Let $K \subseteq \mathbb{R}^n$ be a nonempty convex set with $x \in \operatorname{ri} K$. Consider the following assertions;*

(a) $u \in K^{\infty}$ *and* $v \in K^{\infty 2}[u]$.

(b) *for all* $s > 0$ *there exists* $\bar{t} > 0$ *such that for every* $t > \bar{t}$, $x + tu + sv \in K$.

(c) *there exist sequences* $s_n \to +\infty$, $t_n \to +\infty$ *such that* $x + s_n t_n u + s_n v \in K$.

Thus $b) \Rightarrow c) \Rightarrow a)$. If K is also convex, then $a) \Rightarrow b)$.

Proof. $(a) \Rightarrow (b)$. Suppose that K is convex. Let $x \in \operatorname{ri} K$ be arbitrary. Let P be the projection on $u^{\perp}$. Write $x = t_1 u + Px$ and $v = t_2 u + Pv$. By Theorem 6.6 in [62], $\operatorname{ri} PK = P(\operatorname{ri} K)$, hence $Px \in \operatorname{ri} PK$. By Propositions 3.1 and 3.3, $Pv \in K^{\nu}[u] \subseteq (PK)^{\infty}$.

Since PK is convex, from (2.11) we deduce that for every $s > 0$, $Px + sPv \in \operatorname{ri} PK$. Hence there exists $y \in \operatorname{ri} K$ such that $Py = Px + sPv$. Write $y = t_3 u + Py$. Substituting Py, Px and Pv we deduce $y = x + \bar{t}u + sv$ where $\bar{t} = t_3 - t_1 - st_2$. From $y \in \operatorname{ri} K$ and $u \in K^{\infty}$ we infer that for every $t > \bar{t}$, $x + tu + sv = y + (t - \bar{t})u \in \operatorname{ri} K$. Since this is true for all $t > \bar{t}$, it is clear that we can choose some $\bar{t}_1 > 0$ such that for all $t > \bar{t}_1$, $x + tu + sv \in \operatorname{ri} K$.

$(b) \Rightarrow (c)$. If (b) holds for some $x \in \operatorname{ri} K$, then set $s_k = k$ and choose λ_k large enough, say $\lambda_k > k^2$, such that $x + \lambda_k u + kv \in K$. Define $t_k = \lambda_k / k$. Then (c) holds for the same x.

$(c) \Rightarrow (a)$. If (c) holds for some x, then for the sequence $a_k \doteq x + s_k t_k u + s_k v$ we will have

$$\frac{a_k}{s_k t_k} = \frac{x}{s_k t_k} + \frac{v}{t_k} + u \to u,$$

so $u \in K^\infty$. In addition,

$$\frac{a_k}{s_k} - t_k u = \frac{x}{s_k} + v \to v,$$

hence $v \in K^{\infty 2}[u]$. ■

Note that if (b) is true for some $x \in \operatorname{ri} K$, then this implies (a), which in turn implies (b) and (c) for every $x \in \operatorname{ri} K$. Hence if (b) is true for some $x \in \operatorname{ri} K$, then it is true for all.

We also note, as is clear from the proof, that (b) in Proposition 3.4 may be replaced by

(b') for all $s > 0$ there exists $\bar{t} > 0$ such that for every $t > \bar{t}$, $x + tu + sv \in \operatorname{ri} K$.

We also have some equivalent characterizations, when we know that $u \in K^\infty$.

Proposition 3.5 *Let $K \subseteq \mathbb{R}^n$ be convex and $u \in K^\infty$. Given $v \in \mathbb{R}^n$, the following are equivalent.*

(a) $v \in K^{\infty 2}[u]$.

(b) *For every (equivalently, for some) $x \in \operatorname{ri} K$ and every $s > 0$, there exists $t \in \mathbb{R}$ such that $x + tu + sv \in K$.*

(c) *For every $x \in \operatorname{ri} K$, there exists $t \in \mathbb{R}$ such that $x + tu + v \in K$.*

Proof. If (a) holds, then (b) holds in view of Proposition 3.4. Conversely, if (b) holds for some $x \in \operatorname{ri} K$, given $s > 0$ choose $t \in \mathbb{R}$ so that $x + 2tu + 2sv \in K$. Then $x + tu + sv = \frac{1}{2}x + \frac{1}{2}(x + 2tu + 2sv) \in \operatorname{ri} K$. Thus for every $t' > t$, $x + t'u + sv = x + tu + sv + (t' - t)u \in K$. Hence (a) holds by Proposition 3.4.

So we have only to prove that (c) implies (b). Assume that (c) holds and let $s > 0$. Given $x \in \operatorname{ri} K$, there exists $t_1 \in \mathbb{R}$ such that $x + t_1 u + v \in K$. Then

$$x + \frac{t_1}{2}u + \frac{1}{2}v \in \,]x, x + t_1 u + v[\,,$$

so $x + \frac{t_1}{2}u + \frac{1}{2}v \in \operatorname{ri} K$. By using again (c) on $x + \frac{t_1}{2}u + \frac{1}{2}v$ we can find $t_2 \in \mathbb{R}$ such that $x + (\frac{t_1 + t_2}{2})u + 2\frac{v}{2} \in \operatorname{ri} K$. Using induction, we conclude that for every $k \in \mathbb{N}$ we can find $t'_k \in \mathbb{R}$ such that $x + t'_k u + \frac{k}{2}v \in \operatorname{ri} K$. Take k large enough so that $\frac{k}{2} > s$, and set

$\lambda = \frac{2s}{k} \in \,]0,1[$. Then $x + \lambda t'_k u + sv \in \left]x, x + t'_k u + \frac{k}{2} v\right[$. This implies $x + \lambda t'_k u + sv \in K$ and (b) holds. ■

As a first application of the characterization, we show:

Proposition 3.6 *Let $K \subseteq \mathbb{R}^n$ be convex.*

- (a) *$K^{\infty 2}[u]$ is convex for all $u \in K^\infty$.*
- (b) *$K^\infty \subseteq K^{\infty 2}[u]$ for all $u \in K^\infty$.*
- (c) *If $u \in K^\infty \cap (-K^\infty)$, then $K^{\infty 2}[u] = K^\infty$.*
- (d) *If $u \in K^\infty$ then $K^\infty - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$, and so if additionaly K is a cone, we get $K^{\infty 2}[u] = \overline{K - \mathbb{R}_+ u}$.*

Proof. (a) Fix $x \in \mathrm{ri}\, K$. If $v_1, v_2 \in K^{\infty 2}[u]$, then for every $s > 0$ we can find t_1, t_2 such that for all $t \geq t_i$, $x + tu + sv_i \in K$. Since K is convex, for every $t > \max\{t_1, t_2\}$ and $\lambda \in \,]0,1[$ we have $x + tu + s((1-\lambda)v_1 + \lambda v_2) \in K$. Thus $(1-\lambda)v_1 + \lambda v_2 \in K^{\infty 2}[u]$.

(b) Given $x \in \mathrm{ri}\, K$, we note that for every $u, v \in K^\infty$, and for every $s, t > 0$, $x + su \in \mathrm{ri}\, K$ so $x + tu + sv \in K$. Thus by the characterization of Proposition 3.4, $v \in K^{\infty 2}[u]$.

(c) Let $x \in \mathrm{ri}\, K$ and $v \in K^{\infty 2}[u]$. For every $s > 0$, we can find t such that $x + tu + sv \in K$. Since $-u \in K^\infty$, $(x + tu + sv) + t(-u) \in K$. Thus $x + sv \in K$ for all $s > 0$, so $v \in K^\infty$. This shows that $K^{\infty 2}[u] \subseteq K^\infty$. Using (b) we obtain the equality.

(d) Since K is convex, $K^\infty \subseteq K^{\infty 2}[u]$ by part (b). Proposition 3.1 part (b) implies that $\mathbb{R}u \subseteq K^{\infty 2}[u]$, and therefore $K^\infty - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$ due to the convexity of the cone $K^{\infty 2}[u]$. In case K is a cone, the equality follows from Proposition 3.1 part (e). ■

Another application is that the inclusion in Proposition 3.3 is an equality for convex sets:

Proposition 3.7 *Let $K \in \mathbb{R}^n$ be nonempty and convex, $u \in K^\infty \backslash \{0\}$ and P be the projection on $u^\perp$. Then $K^\nu[u] = (PK)^\infty$ and, consequently, $K^{\infty 2}[u] = \mathbb{R}u + (PK)^\infty$.*

Proof. We already know by Proposition 3.3 that for any nonempty set K, $K^\nu[u] \subseteq (PK)^\infty$. Take any $v \in (PK)^\infty$. If $x \in \mathrm{ri} K$ then $Px \in \mathrm{ri}\,(PK)$. As in the proof of implication $(a) \Rightarrow (b)$ in Proposition 3.4, we infer that for every $s > 0$ there exists $\bar{t} > 0$ such that for $t > \bar{t}$, $x + tu + sv \in K$. Hence $v \in K^{\infty 2}[u]$. Since $\langle v, u \rangle = 0$, v is canonic, so $K^\nu[u] = (PK)^\infty$. ■

In other words, the canonic directions of K at u are exactly the first order directions of the projection of K on $u^\perp$. The second order directions are the vectors whose projection on $u^\perp$, are first order directions of PK.

Remark 3.2 *The assumption $x \in \operatorname{ri} K$ is used in the proof of (a) $\Rightarrow$ (b) of Proposition 3.4, in order to ensure that $Px \in \operatorname{ri}PK$, so that $Px + sPv \in \operatorname{ri}PK$ since $Pv \in (PK)^{\infty}$. Whenever K is a closed set such that PK is also closed, this assumption is not needed and the same proof shows that one can take simply $x \in K$. Such a case occurs for example when K is polyhedral.*

The following counterexample shows that $x \in \operatorname{ri} K$ cannot be replaced by $x \in K$ in the general case.

Example 3.1 *Consider the cone*

$$K = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 \leq 2x_2x_3, \quad x_3 \geq 0\}.$$

This is the cone generated by the circle $x_1^2 + (x_2 - 1)^2 \leq 1$, $x_3 = 1$. Then $K^{\infty} = K$, and $u \doteq (0, 0, 1) \in K$. If we set $P = P_{u^{\perp}}$, then

$$\begin{aligned} PK &= \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_2 > 0\} \cup \{(0, 0, 0)\} \\ (PK)^{\infty} &= \{(x_1, x_2, 0) \in \mathbb{R}^3 : x_2 \geq 0\} \\ K^{\infty 2}[u] &= \mathbb{R}u + (PK)^{\infty} = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_2 \geq 0\}. \end{aligned}$$

By taking $v = (1, 0, 0)$ we see that $u \in K^{\infty}$, $v \in K^{\infty 2}[u]$, $0 \in K$ but $0 + tu + sv = (s, 0, t) \notin K$ for any $s > 0$, t arbitrary. Hence condition (b) in Proposition 3.4 does not hold.

Further, we show that the inclusion in Proposition 3.1 part (i) is an equality in case of finitely many convex sets satisfying a regularity condition.

Proposition 3.8 *Let $\{K_i\}_{i \in I} \subseteq \mathbb{R}^n$ be a finite family of convex sets with $\bigcap_{i \in I} \operatorname{ri} K_i \neq \emptyset$. If $u \in (\bigcap_{i \in I} K_i)^{\infty}$, then*

$$(\bigcap_{i \in I} K_i)^{\infty 2}[u] = \bigcap_{i \in I} K_i^{\infty 2}[u]. \tag{3.3}$$

Proof. We only need to prove $(\supseteq)$.
Let $v \in \bigcap_{i \in I} K_i^{\infty 2}[u]$. Choose $x_0 \in \bigcap_{i \in I} \operatorname{ri} K_i$. Then, given $i \in I$ and $s > 0$, there exist $\overline{t_i} \geq 0$ such that for all $t_i > \overline{t_i}$, $x_0 + t_i u + sv \in K_i^{\infty 2}[u]$. Let $\bar{t} = \max_{i \in I}\{\overline{t_i}\}$, then $x_0 + tu + sv \in \bigcap_{i \in I} K_i$, for all $t > \bar{t}$. By Proposition 3.4, $v \in (\bigcap_{i \in I} K_i)^{\infty 2}[u]$. ∎

Remark 3.3 *a) The previous proposition is not true for an infinite family, even of polyhedral sets. For example, take for every $m \in \mathbb{N}$, $K_m \doteq \{(x_1, x_2) : x_2 \geq m\,|x_1|\}$. Clearly $\bigcap_{m \in \mathbb{N}} K_m = \mathbb{R}_+(0, 1)$, and $(\bigcap_{m \in \mathbb{N}} K_m)^{\infty 2}[u] = \mathbb{R}u$, while $\bigcap_{m \in \mathbb{N}} K_m^{\infty 2}[u] = \mathbb{R}^2$.*

b) The proposition is also not true in general if $\bigcap_{i=1}^{k} \operatorname{ri} K_i = \emptyset$, *even for a finite family of closed convex sets. Consider for example the cones* K_1 *and* K_2 *generated, respectively, by the circles* $x_1^2 + (x_2 - 1)^2 \leq 1$, $x_3 = 1$ *and* $x_1^2 + (x_2 + 1)^2 \leq 1$, $x_3 = 1$ *(see example 3.1). The cones have the half axis* $K = \mathbb{R}_+ u$, $u = (0, 0, 1)$ *as common generatrix. It is easy to see that* $K_1 \cap K_2 = K$. *From Example 3.1 we see that* $K_1^{\infty 2}[u] = \{(x_1, x_2, x_3) : x_2 \geq 0\}$ *and, likewise,* $K_2^{\infty 2}[u] = \{(x_1, x_2, x_3) : x_2 \leq 0\}$. *Hence,* $\mathbb{R}[u] = K^{\infty 2}[u] \subsetneq K_1^{\infty 2}[u] \cap K_2^{\infty 2}[u]$.

When the set is polyhedral, the second order asymptotic cone has a simple expression. In what follows, set for $i = 1, 2, \ldots, m$,

$$H_i = \{x : \ \langle a_i, x \rangle \leq \alpha_i\},$$

for some $a_1, a_2, \ldots, a_m \in \mathbb{R}^n$ and $\alpha_1, \alpha_2, \cdots, \alpha_m \in \mathbb{R}$. Then $H_i^\infty = \{x : \ \langle a_i, x \rangle \leq 0\}$. Given $u \in (\bigcap_{i=1}^m H_i)^\infty$, set $I_1 = \{i \in \{1, 2, \ldots, m\} : \ u \in \operatorname{bd} H_i^\infty\}$ (set of active inequalities for u) and $I_2 = \{i \in \{1, 2, \ldots, m\} : \ u \in \operatorname{int} H_i^\infty\}$. The next proposition gives an expression for $K^{\infty 2}[u]$ when K is polyhedral. As usual, the intersection of an empty family of subsets of $\mathbb{R}^n$ is considered to be the whole space $\mathbb{R}^n$.

Proposition 3.9 *Assume for* $i = 1, 2, \ldots, m$, *that* H_i *is a halfspace as above, and that* $\bigcap_{i=1}^m H_i \neq \emptyset$. *If* $u \in (\bigcap_{i=1}^m H_i)^\infty$, *then*

$$(\bigcap_{i=1}^{m} H_i)^{\infty 2}[u] = \bigcap_{i \in I_1} H_i^\infty = \bigcap_{i=1}^{m} H_i^{\infty 2}[u],$$

where $I_1 = \{i \in \{1, 2, \ldots, m\} : \ u \in \operatorname{bd} H_i^\infty\}$.

Proof. By Proposition 3.1 part (f), $H_i^{\infty 2}[u] = \mathbb{R}^n$ for all $i \in I_2$. For $i \in I_1$, $u \in \operatorname{bd} H_i^\infty$ means that $u \in H_i^\infty \cap (-H_i^\infty)$. By Proposition 3.6 part (c), $H_i^\infty = H_i^{\infty 2}[u]$ from which the second equality follows.

To show the first equality, we use Proposition 3.5 part (b): Fix any $x \in \operatorname{ri}(\bigcap_{i=1}^m H_i)$. Note that $\langle a_i, x \rangle \leq \alpha_i$ for all $i \in I_1 \cup I_2$, $\langle a_i, u \rangle = 0$ for all $i \in I_1$ and $\langle a_i, u \rangle < 0$ for all $i \in I_2$. Given $v \in \mathbb{R}^n$, we have that $v \in (\bigcap_{i=1}^m H_i)^{\infty 2}[u]$ if and only if for all $s > 0$ there exists $t > 0$ such that $x + tu + sv \in \bigcap_{i=1}^m H_i$, i.e., $\langle a_i, x + tu + sv \rangle \leq \alpha_i$ for all i. For $i \in I_2$ it always holds that $\langle a_i, x + tu + sv \rangle \leq \alpha_i$ if t is sufficiently large. For $i \in I_1$, $\langle a_i, x + tu + sv \rangle = \langle a_i, x + sv \rangle \leq \alpha_i$ holds for large $s > 0$ if and only if $\langle a_i, v \rangle \leq 0$, i.e., $v \in \bigcap_{i \in I_1} H_i^\infty$. This shows the first equality. ■

It follows immediately that in case of polyhedral sets, equality (3.3) holds without any assumption on the relative interiors:

Corollary 3.2 *Let $\{K_i\}_{i\in I}$ be a finite family of polyhedral sets. If $\bigcap_{i\in I} K_i \neq \emptyset$, then*

$$(\bigcap_{i\in I} K_i)^{\infty 2}[u] = \bigcap_{i\in I} K_i^{\infty 2}[u].$$

3.3 Second order asymptotic functions

We first give a formula for $f^{\infty 2}(u;\cdot)$ for any proper function f, followed by some basic properties linking second and first order asymptotic notions. Afterwards, the case of a convex function is considered, and we provide various formulas for $f^{\infty 2}(u;\cdot)$.

3.3.1 Some preliminaries

Let us fix a direction $u \in \mathbb{R}^n$ for which $f^\infty(u)$ is finite; then $u \in (\operatorname{dom} f)^\infty$ and $(u, f^\infty(u)) \in (\operatorname{epi} f)^\infty$. Set $A = (\operatorname{epi} f)^{\infty 2}[(u, f^\infty(u))]$. Then $(v,\alpha) \in A$ iff there exist sequences $(x_k, \alpha_k) \in \operatorname{epi} f$ and $s_k, t_k \to +\infty$ such that

$$\frac{(x_k,\alpha_k)}{s_k} - t_k\left(u, f^\infty(u)\right) \to (v,\alpha). \tag{3.4}$$

In this case, for every $h > 0$, $(x_k, \alpha_k + s_k h) \in \operatorname{epi} f$ and

$$\frac{(x_k,\alpha_k + s_k h)}{s_k} - t_k\left(u, f^\infty(u)\right) \to (v,\alpha + h).$$

Thus, $(v, \alpha + h) \in A$ for every $h > 0$. Since A is a closed cone, this means that A is the epigraph of some lsc, positively homogeneous function. We call this function second order asymptotic function of f at u and we denote its value at v by $f^{\infty 2}(u; v)$, that is,

$$\operatorname{epi}\ f^{\infty 2}(u;\cdot) = (\operatorname{epi}\ f)^{\infty 2}[(u, f^\infty(u))]. \tag{3.5}$$

This yields the following straightforward result.

Proposition 3.10 *For $u \in \mathbb{R}^n$ satisfying $f^\infty(u) \in \mathbb{R}$, the function $f^{\infty 2}(u;\cdot) : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$, defined as in (3.5), is lsc and positively homogeneous; it satisfies $f^{\infty 2}(u; 0) = 0$ or $-\infty$, while $f^{\infty 2}(u; 0) = 0$ if and only if $f^{\infty 2}(u;\cdot)$ is proper.*

From (3.5) we derive the next formula.

Proposition 3.11 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and $u \in \mathbb{R}^n$ be such that $f^\infty(u)$ is* *finite. Then for every* $v \in \mathbb{R}^n$,

$$f^{\infty 2}(u;v) = \inf\left\{ \liminf_{k\to\infty} \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) : \ x_k \in \operatorname{dom} f, s_k, t_k \to +\infty, \ \frac{x_k}{s_k} - t_k u \to v \right\} \tag{3.6}$$

Proof. Let $g(v)$ be the expression at the right-hand side of (3.6). We will show that $f^{\infty 2}(u;\cdot)$ and g have the same epigraph.

If $(v,\alpha) \in \operatorname{epi} f^{\infty 2}(u;\cdot)$, i.e., $(v,\alpha) \in (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right]$, then by Definition 3.1 there exist sequences $(x_k,\alpha_k) \in \operatorname{epi} f$, $s_k, t_k \to +\infty$ such that $\frac{x_k}{s_k} - t_k u \to v$ and $\frac{\alpha_k}{s_k} - t_k f^{\infty}(u) \to \alpha$.

Since $f(x_k) \leq \alpha_k$, it follows that

$$\liminf \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) \leq \lim \left(\frac{\alpha_k}{s_k} - t_k f^{\infty}(u) \right) = \alpha.$$

Hence $g(v) \leq \alpha$, i.e., $(v,\alpha) \in \operatorname{epi} g$.

Conversely, assume that $(v,\alpha) \in \operatorname{epi} g$. Then for every $\varepsilon > 0$, $g(v) < \alpha + \varepsilon$. It follows that there exist sequences $x_k \in \operatorname{dom} f$, $s_k, t_k \to +\infty$ such that

$$\frac{x_k}{s_k} - t_k u \to v, \tag{3.7}$$

$$\liminf \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) < \alpha + \varepsilon. \tag{3.8}$$

By taking a subsequence if necessary, we may assume that the lim inf is actually lim. Let $\gamma_k \in \mathbb{R}$ be such that

$$\frac{f(x_k) + \gamma_k}{s_k} - t_k f^{\infty}(u) = \alpha + \varepsilon, \tag{3.9}$$

that is,

$$\gamma_k = s_k \left(\alpha + \varepsilon - \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) \right).$$

Relation (3.8) implies that $\gamma_k > 0$ for large k. Hence $(x_k, f(x_k) + \gamma_k) \in \operatorname{epi} f$. From (3.7) and (3.9) we deduce that

$$\frac{(x_k, f(x_k) + \gamma_k)}{s_k} - t_k(u, f^{\infty}(u)) \to (v, \alpha + \varepsilon),$$

hence $(v, \alpha + \varepsilon) \in (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right] = \operatorname{epi} f^{\infty 2}(u;\cdot)$ for every $\varepsilon > 0$. Since the second order cone is closed, we deduce that $(v,\alpha) \in \operatorname{epi} f^{\infty 2}(u;\cdot)$. ∎

Note that in [45] the second order asymptotic function was defined directly through formula (3.6), was called "lower second order asymptotic function", and was denoted by

$R''_{-}f(u;\cdot)$. Furthermore, in [45] there is also defined the upper second order asymptotic function as follows;

Definition 3.3 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function and let $u \in \mathbb{R}^n$ be such that $f^{\infty}(u) \in \mathbb{R}$. We define the upper second order asymptotic function of f at u and we denote its value at v by*

$$f_{+}^{\infty 2}(u;v) = \inf \left\{ \limsup_{k\to\infty} \left(\frac{f(x_k)}{s_k} - t_k f^{\infty}(u) \right) : \ x_k \in \operatorname{dom} f, s_k, t_k \to +\infty, \ \frac{x_k}{s_k} - t_k u \to v \right\} \tag{3.10}$$

Its clear from the definition that $f^{\infty 2} \leq f_{+}^{\infty 2}$. Along to this thesis we work preferently with $f^{\infty 2}$ over $f_{+}^{\infty 2}$. We recall the following result presented in Proposition 3.2 of [45].

Proposition 3.12 *Let f be as before and $u \in \mathbb{R}^n$ such that $f^{\infty}(u) \in \mathbb{R}$. Then the following assertions holds;*

(*a*) *For every $\alpha, \beta > 0$ we have*

$$f^{\infty 2}(\alpha u; \beta v) = \beta f^{\infty 2}(u;v) \text{ and } f_{+}^{\infty 2}(\alpha u; \beta v) = \beta f_{+}^{\infty 2}(u;v)$$

(*b*) *For every $\beta > 0$ we have*

$$(\beta f)^{\infty 2}(u;v) = \beta f^{\infty 2}(u;v) \text{ and } (\beta f)_{+}^{\infty 2}(u;v) = \beta f_{+}^{\infty 2}(u;v)$$

In the special case $u = 0$ formula (3.6) implies that $f^{\infty 2}(0;v) = f^{\infty}(v)$ for all $v \in \mathbb{R}$. This is also a consequence of the fact that the functions $f^{\infty 2}(0;\cdot)$ and f^{∞} have the same epigraph, in view of the equality $K^{\infty 2}[0] = K^{\infty}$ for $K = \operatorname{epi} f$.

We have a simple inequality between $f^{\infty}(u)$ and $f^{\infty 2}(u;u)$:

Proposition 3.13 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then $f^{\infty 2}(u;u) \leq f^{\infty}(u)$, for every $u \in \mathbb{R}^n$ such that $f^{\infty}(u) \in \mathbb{R}$.*

Proof. By Proposition 3.1, $u \in K^{\infty 2}[u]$ whenever $u \in K^{\infty}$. Consequently, we have that $(u, f^{\infty}(u)) \in (\operatorname{epi} f)^{\infty 2}[(u, f^{\infty}(u))] = \operatorname{epi} f^{\infty 2}(u;\cdot)$. This implies immediately that $f^{\infty 2}(u;u) \leq f^{\infty}(u)$. ■

In the study of minimization problems, one must analyze the behaviour of the objective function along unbounded minimizing sequences $\{x_k\}$. Given an objective function f, a control of the growth rate of the quotient $\dfrac{f(x_k)}{\|x_k\|}$ is provided by $f^\infty(u)$ whenever $\dfrac{x_k}{\|x_k\|} \to u$. We will show that still the second order asymptotic function $f^{\infty 2}$ provides a finer description of the growth of f at infinity. To see this, let $\{x_k\}$ be a sequence in $\operatorname{dom} f$ with $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u$, then

$$\liminf \frac{f(x_k)}{\|x_k\|} \geq f^\infty(u). \tag{3.11}$$

In other words, if $f^\infty(u) \in \mathbb{R}$, then the rate of growth of $f(x_k)$ is at least the rate of growth of $\|x_k\| f^\infty(u)$. However, this does not mean that $f(x_k) - \|x_k\| f^\infty(u)$ is bounded from below, i.e., that

$$\liminf \left(f(x_k) - \|x_k\| f^\infty(u)\right) > -\infty. \tag{3.12}$$

It can be easily seen that (3.12) is a stronger statement than (3.11). In fact, the second order asymptotic function gives a necessary condition for (3.12) to hold.

Proposition 3.14 *Let f be arbitrary and $u \in \mathbb{R}^n$ be such that $\|u\| = 1$ and $f^\infty(u) \in \mathbb{R}$. If (3.12) holds for every sequence $x_k \in \operatorname{dom} f$ such that $\|x_k\| \to +\infty$, $\frac{x_k}{\|x_k\|} \to u$, then $f^{\infty 2}(u; u) = f^\infty(u)$.*

Proof. By Proposition 3.13, $f^{\infty 2}(u;u) \leq f^\infty(u)$. Assume that $f^{\infty 2}(u;u) < f^\infty(u)$. Then there exist sequences $x_k \in \operatorname{dom} f$, $s_k, t_k \to +\infty$, such that $\frac{x_k}{s_k} - t_k u \to u$ and $\frac{f(x_k)}{s_k} - t_k f^\infty(u) - f^\infty(u) \to -\alpha$ with $\alpha \in \left]0, +\infty\right]$. From $\frac{x_k}{s_k} - (t_k+1)u \to 0$ follows that $\|x_k\| \to +\infty$, $\frac{\|x_k\|}{s_k} - (t_k+1)\|u\| \to 0$ and

$$\frac{x_k}{\|x_k\|} = \frac{x_k}{s_k(t_k+1)} \frac{s_k(t_k+1)}{\|x_k\|} \to u.$$

In addition,

$$-\alpha = \lim\left(\frac{f(x_k)}{s_k} - (t_k+1) f^\infty(u)\right) = \lim\left(\frac{f(x_k)}{s_k} - \frac{\|x_k\|}{s_k\|u\|} f^\infty(u)\right). \tag{3.13}$$

Using $s_k \to +\infty$ and $\|u\| = 1$ we obtain $\lim\left(f(x_k) - \|x_k\| f^\infty(u)\right) = -\infty$. This contradicts our assumption, so $f^{\infty 2}(u;u) = f^\infty(u)$. ■

The converse is not true, even for a convex function.

Example 3.2 *Define f on $\mathbb{R}^2$ by*

$$f(\alpha,\beta) = \begin{cases} -\sqrt{\beta}, & \beta \geq 0 \\ +\infty, & \beta < 0 \end{cases}.$$

Then f is convex and lsc. For every $x \in \mathrm{ridom} f$ the function $f(x + t(1,0))$ does not depend on t, so one can easily see that $f^\infty(1,0) = 0 = f^{\infty 2}((1,0);(1,0))$. However, if we take $x_k = (k, \sqrt{k})$ then we can check that $\|x_k\| \to +\infty$, $\frac{x_k}{\|x_k\|} \to (1,0)$ but

$$\lim \left(f(x_k) - \|x_k\| f^\infty(1,0)\right) = -\infty.$$

As we will see in the next section (cf. Remark 3.4), whenever f is a convex function and $f^\infty(u) = f^{\infty 2}(u;u)$, then given a sequence $\{x_k\}$ such that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u$, we are sure that (3.12) holds for sequences that belong to a line of the form $x + tu$, $t > 0$ for some $x \in \mathrm{ridom} f$. For more general sequences, (3.12) might not hold.

Given a proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ let $S_\lambda \doteq \{x \in \mathbb{R}^n : f(x) \le \lambda\}$ be its sublevel set. Next proposition shows the relation between the zero-level set of $f^{\infty 2}(u;\cdot)$ and the second order cone of the level set of f.

Proposition 3.15 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function and $\lambda \ge \inf f$. If $u \in \mathbb{R}^n$ with $f^\infty(u) = 0$, then*

$$(S_\lambda)^{\infty 2}[u] \subseteq \{w : f^{\infty 2}(u;w) \le 0\}.$$

Proof. Let $v \in (S_\lambda)^{\infty 2}[u]$. Then there exist $x_k \in S_\lambda, t_k, s_k \to +\infty$ such that $\frac{x_k}{s_k} - t_k u \to v$. Thus

$$\frac{f(x_k)}{s_k} - t_k f^\infty(u) = \frac{f(x_k)}{s_k} \le \frac{\lambda}{s_k} \to 0,$$

and $f^{\infty 2}(u;v) \le 0$. ■

3.3.2 The case of convex functions

Whenever f is convex, $f^{\infty 2}(u;\cdot)$ is convex too since $(\mathrm{epi}\, f)^{\infty 2}\left[(u, f^\infty(u))\right]$ is convex by Proposition 3.6. In this case, $f^{\infty 2}(u;\cdot)$ has a simpler form, as we will see. In preparation of what follows, we first show a formula for f^∞ which is analogous to (2.13), but does not assume that f is lower semicontinuous.

Proposition 3.16 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex function. Then*

(*a*) *Given $x_0 \in \mathrm{ridom} f$ and $(u,\mu) \in \mathbb{R}^n \times \mathbb{R}$, one has*

$$(u,\mu) \in \mathrm{epi}\, f^\infty \Longleftrightarrow (x_0, f(x_0)) + t(u,\mu) \in \mathrm{epi}\, f, \ \forall t > 0.$$

(*b*) *For every $x_0 \in \mathrm{ridom} f$, $u \in \mathbb{R}^n$,*

$$f^\infty(u) = \lim_{t \to +\infty} \frac{f(x_0 + tu) - f(x_0)}{t} = \sup_{t>0} \frac{f(x_0 + tu) - f(x_0)}{t}$$

Proof. (a) For every $\beta > f(x_0)$, one has $(x_0,\beta) \in \text{riepi}f$. Hence, if $(u,\mu) \in \text{epi}\, f^\infty = (\text{epi}f)^\infty$ then $(x_0,\beta) + t(u,\mu) \in \text{epi}\, f$, $\forall t > 0$. The last inclusion means that $f(x_0 + tu) \le \beta + t\mu$. Since this is true for all $\beta > f(x_0)$, we deduce that $f(x_0+tu) \le f(x_0)+t\mu$, i.e., $(x_0, f(x_0))+t(u,\mu) \in \text{epi}\, f$. The converse is similar.
(b) is proved by using (a), exactly as the analogous equalities when f is lsc. ∎

It follows from the above proposition that whenever f is a proper convex function, $f^\infty(0) = 0$ so f^∞ is also proper.

Corollary 3.3 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1,\dots,k$, be convex functions such that $\bigcap_{i=1}^{k} \text{ridom}\ f_i \neq \emptyset$. Then*

$$\left(\sum_{i=1}^k f_i\right)^\infty = \sum_{i=1}^k f_i^\infty \quad , \quad \left(\max_{1\le i\le k} f_i\right)^\infty = \max_{1\le i\le k} f_i^\infty.$$

Proof. Since $\text{dom} \sum_{i=1}^k f_i = \bigcap_{i=1}^k \text{dom}\, f_i = \text{dom}\max_i f_i$, from the assumption we get that $\bigcap_{i=1}^k \text{ri}\,\text{dom}\, f_i = \text{ri}\,\text{dom}\sum_{i=1}^k f_i = \text{ridom}\max_i f_i$. We then apply Proposition 3.16 part (b). ∎

We now establish some useful monotonicity properties.

Lemma 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be proper and convex.*

(a) *For every $x \in \text{ridom}f$ and u such that $f^\infty(u)$ is finite, the function $g(t) := f(x+tu) - tf^\infty(u)$ is decreasing on $\mathbb{R}$.*

(b) *If $(y,\delta) \in \text{epi}\, f$, then for every $v \in \mathbb{R}^n$ the function $s \to \frac{f(y+sv)-\delta}{s}$ is increasing on $]0,+\infty[$.*

(c) *Let $x \in \text{ridom}f$, $\beta \ge f(x)$, $f^\infty(u)$ be finite and $v \in (\text{dom}\, f)^{\infty 2}[u]$. If we set*

$$k_\beta(s,t) = \frac{f(x+tu+sv) - tf^\infty(u) - \beta}{s}, \qquad s>0, t>0 \tag{3.14}$$

then the function $s \to \lim_{t\to+\infty} k_\beta(s,t)$ is increasing. Consequently for all $\beta \ge f(x)$,

$$\lim_{s\to+\infty}\lim_{t\to+\infty} k_{f(x)}(s,t) = \lim_{s\to+\infty}\lim_{t\to+\infty} k_\beta(s,t) = \sup_{s>0}\inf_{t>0} k_\beta(s,t) = \sup_{s>0}\inf_{t>0} k_{f(x)}(s,t). \tag{3.15}$$

Proof. (a) Let $t' > t > 0$. Since $x \in \text{ridom}f$, we have $x+tu \in \text{ridom}f$ and $x+t'u \in \text{ridom}f$. Setting $x_1 = x+tu$, we know by Proposition 3.16 that

$$\frac{f(x+t'u) - f(x+tu)}{t'-t} = \frac{f(x_1 + (t'-t)u) - f(x_1)}{t'-t} \le f^\infty(u).$$

From this we obtain $f(x+t'u)-t'f^\infty(u) \le f(x+tu)-tf^\infty(u)$, then g is decreasing on $]0,+\infty[$, and since g is convex, it is decreasing on $\mathbb{R}$.
(b) The function is the sum of two increasing functions:

$$\frac{f(y+sv)-\delta}{s} = \frac{f(y+sv)-f(y)}{s} + \frac{f(y)-\delta}{s}.$$

(c) Using (a) we deduce that for every $s>0$, $\lim_{t\to+\infty} k_\beta(s,t)$ exists and is equal to $\inf_{t>0} k_\beta(s,t)$. Let $s'>s>0$. By using Proposition 3.4 on $(\operatorname{dom} f)^{\infty 2}[u]$ we deduce that there exists $\bar{t}>0$ such that for all $t \ge \bar{t}$, $x+tu+sv \in \operatorname{dom} f$ and $x+tu+s'v \in \operatorname{dom} f$. Then by Proposition 3.16, $(x+tu, \beta+tf^\infty(u)) = (x,\beta)+t(u,f^\infty(u)) \in \operatorname{epi} f$ since $(u,f^\infty(u)) \in (\operatorname{epi} f)^\infty$. Using (b) for $y=x+tu$, $\delta=\beta+tf^\infty(u)$ we obtain

$$\frac{f(x+tu+sv)-tf^\infty(u)-\beta}{s} \le \frac{f(x+tu+s'v)-tf^\infty(u)-\beta}{s'}, \qquad \forall t \ge \bar{t}.$$

Taking the limit as $t\to+\infty$ we find that $\lim_{t\to+\infty} k_\beta(s,t) \le \lim_{t\to+\infty} k_\beta(s',t)$, i.e., the function $s \to \lim_{t\to+\infty} k_\beta(s,t)$ is increasing. Thus,

$$\lim_{s\to+\infty}\lim_{t\to+\infty} k_\beta(s,t) = \sup_{s>0}\inf_{t>0} k_\beta(s,t), \qquad \forall \beta \ge f(x). \tag{3.16}$$

On the other hand, it is clear that

$$\lim_{s\to+\infty}\lim_{t\to+\infty} \frac{f(x+tu+sv)-tf^\infty(u)-\beta}{s} = \lim_{s\to+\infty}\lim_{t\to+\infty} \frac{f(x+tu+sv)-tf^\infty(u)}{s},$$

hence $\lim_{s\to+\infty}\lim_{t\to+\infty} k_\beta(s,t) = \lim_{s\to+\infty}\lim_{t\to+\infty} k_{f(x)}(s,t)$. From this and (3.16) we deduce the equalities (3.15). ■

Proposition 3.17 *Let $f:\mathbb{R}^n \to \mathbb{R}\cup\{+\infty\}$ be proper and convex and $x \in \operatorname{ridom} f$. For every u such that $f^\infty(u)$ is finite and $v \in (\operatorname{dom} f)^{\infty 2}[u]$,*

$$f^{\infty 2}(u;v) = \sup_{s>0}\inf_{t>0} \frac{f(x+tu+sv)-tf^\infty(u)-f(x)}{s} \tag{3.17}$$

$$f^{\infty 2}(u;v) = \lim_{s\to+\infty}\lim_{t\to+\infty} \frac{f(x+tu+sv)-tf^\infty(u)-f(x)}{s} \tag{3.18}$$

Proof. Take any $\beta > f(x)$. Then we have $(x,\beta) \in \operatorname{riepi} f$. Define $k_\beta(s,t)$ as in (3.14). We show that $\sup_{s>0}\inf_{t>0} k_\beta(s,t) \le f^{\infty 2}(u;v)$ by showing that for every $\alpha\in\mathbb{R}$, $f^{\infty 2}(u;v) \le \alpha$ implies the following $\sup_{s>0}\inf_{t>0} k_\beta(s,t) \le \alpha$.

Since $\operatorname{epi} f$ is convex, using Proposition 3.5 we have the following implications:

$$\begin{aligned} f^{\infty 2}(u;v) \leq \alpha &\Rightarrow (v,\alpha) \in (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right] \\ &\Rightarrow \forall s>0, \exists t>0, (x,\beta)+t(u,f^{\infty}(u))+s(v,\alpha) \in \operatorname{epi} f \\ &\Rightarrow \forall s>0, \exists t>0, (x+tu+sv, \beta+tf^{\infty}(u)+s\alpha) \in \operatorname{epi} f \\ &\Rightarrow \forall s>0, \exists t>0, f(x+tu+sv) \leq \beta+tf^{\infty}(u)+s\alpha \\ &\Rightarrow \forall s>0, \exists t>0, k_\beta(s,t) \leq \alpha \\ &\Rightarrow \sup_{s>0}\inf_{t>0} k_\beta(s,t) \leq \alpha. \end{aligned}$$

We now show $f^{\infty 2}(u;v) \leq \sup_{s>0}\inf_{t>0} k_\beta(s,t)$ by showing that for every $\alpha \in \mathbb{R}$, $\sup_{s>0}\inf_{t>0} k_\beta(s,t) < \alpha$ implies $f^{\infty 2}(u;v) \leq \alpha$. Following the previous implications in reverse order, we obtain

$$\begin{aligned} \sup_{s>0}\inf_{t>0} k_\beta(s,t) < \alpha &\Rightarrow \forall s>0, \exists t>0, k_\beta(s,t) < \alpha \\ &\Rightarrow \forall s>0, \exists t>0, (x+tu+sv, \beta+tf^{\infty}(u)+s\alpha) \in \operatorname{epi} f \\ &\Rightarrow \forall s>0, \exists t>0, (x,\beta)+t(u,f^{\infty}(u))+s(v,\alpha) \in \operatorname{epi} f \\ &\Rightarrow (v,\alpha) \in (\operatorname{epi} f)^{\infty 2}[(u,f^{\infty}(u))] \Rightarrow f^{\infty 2}(u;v) \leq \alpha. \end{aligned}$$

It follows that $f^{\infty 2}(u;v) = \sup_{s>0}\inf_{t>0} k_\beta(s,t)$ for every $\beta > f(x)$. Using (3.15) we deduce equalities (3.18) and (3.17). ■

A formula analogous to (2.14) holding for f^{∞} also holds.

Proposition 3.18 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex. For every u such that $f^{\infty}(u)$ is finite and $v \in (\operatorname{dom} f)^{\infty 2}[u]$,*

$$f^{\infty 2}(u;v) = \sup_{x \in \operatorname{ri}\operatorname{dom} f}\inf_{t>0}\left(f(x+tu+v) - tf^{\infty}(u) - f(x)\right).$$

Proof. As in Proposition 3.17, we use a representation of $(\operatorname{epi} f)^{\infty 2}[(u, f^{\infty}(u))]$. Let us show first that

$$f^{\infty 2}(u;v) = \sup_{(x,\beta)\in \operatorname{ri}\operatorname{epi} f}\inf_{t>0}\left(f(x+tu+v) - tf^{\infty}(u) - \beta\right). \tag{3.19}$$

Assume that $\alpha \geq f^{\infty 2}(u;v)$, i.e., $(v,\alpha) \in \operatorname{epi} f^{\infty 2}(u;\cdot) = (\operatorname{epi} f)^{\infty 2}[(u, f^{\infty}(u))]$. By the equivalence $(a) \iff (c)$ of Proposition 3.5, for every $(x,\beta) \in \operatorname{ri}\operatorname{epi} f$ there exists $t > 0$ such that $(x,\beta)+t(u,f^{\infty}(u))+(v,\alpha)\in \operatorname{epi} f$. This amounts to

$$\forall\,(x,\beta)\in \operatorname{riepi} f, \exists t>0: f(x+tu+v)-tf^{\infty}(u)-\beta\leq \alpha$$

or

$$\sup_{(x,\beta)\in \operatorname{riepi} f}\inf_{t>0}\left(f(x+tu+v)-tf^{\infty}(u)-\beta\right)\leq \alpha.$$

Since this is true for every $\alpha\geq f^{\infty 2}(u;v)$ we deduce inequality $\geq$ in (3.19). The reverse inequality is deduced similarly, by taking any α such that

$$\sup_{(x,\beta)\in \operatorname{riepi} f}\inf_{t>0}\left(f(x+tu+v)-tf^{\infty}(u)-\beta\right)< \alpha$$

and deducing, using again Proposition 3.5, that $f^{\infty 2}(u;v)\leq\alpha$. Thus, equation (3.19) holds. Note that $(x,\beta)\in \operatorname{riepi} f$ if and only if $x\in \operatorname{ridom} f$ and $\beta>f(x)$. Hence,

$$\begin{aligned} f^{\infty 2}(u;v) &= \sup_{x\in \operatorname{ridom} f}\ \sup_{\beta>f(x)}\ \inf_{t>0}\left(f(x+tu+v)-tf^{\infty}(u)-\beta\right)\\ &= \sup_{x\in \operatorname{ridom} f}\ \sup_{\beta>f(x)}\left(\inf_{t>0}\left(f(x+tu+v)-tf^{\infty}(u)\right)-\beta\right)\\ &= \sup_{x\in \operatorname{ridom} f}\ \inf_{t>0}\left(f(x+tu+v)-tf^{\infty}(u)-f(x)\right)\end{aligned}$$

which proves the proposition. ■

Formula (3.18) comes in handy for calculating $f^{\infty 2}$ for convex functions.

Example 3.3 *(a) Take $f(x)=\|x\|$. Then $f^{\infty}=f$. To calculate $f^{\infty 2}(u;v)$ we may consider that $u\neq 0$ since as we remarked, $f^{\infty 2}(0;v)=f^{\infty}(v)$. We use (3.18) with $x=0$. We first calculate:*

$$\lim_{t\to+\infty}\left(f(tu+sv)-tf^{\infty}(u)\right)=\lim_{t\to+\infty}\left(\|tu+sv\|-t\,\|u\|\right)$$

$$=\lim_{t\to+\infty}\frac{2ts\,\langle u,v\rangle+s^{2}\,\|v\|^{2}}{\|tu+sv\|+t\,\|u\|}=\lim_{t\to+\infty}\frac{2s\,\langle u,v\rangle+\frac{1}{t}s^{2}\,\|v\|^{2}}{\left\|u+\frac{s}{t}v\right\|+\|u\|}=\frac{s\,\langle u,v\rangle}{\|v\|}.$$

Hence, $f^{\infty 2}(u;v)=\frac{\langle u,v\rangle}{\|v\|}$.
(b) Let f be the quadratic convex function $f(x)=\frac{1}{2}\langle Ax,x\rangle+\langle c,x\rangle+k$ where A is a symmetric positive semidefinite matrix, $c\in\mathbb{R}^{n}$ and $k\in\mathbb{R}$. It is known that $f^{\infty}(u)=\langle c,u\rangle$ if $u\in\ker A$, while $f^{\infty}(u)=+\infty$ if $u\notin\ker A$. An application of (3.18) yields immediately that $f^{\infty 2}(u;v)=f^{\infty}(v)$ for every $u\in\ker A$ and $v\in\mathbb{R}^{n}$.
(c) Consider $f(x)=(1+\langle Ax,x\rangle)^{\frac{1}{2}}$ where A is a symmetric positive semidefinite matrix.

Then $f^{\infty}(u) = \langle Au, u\rangle^{\frac{1}{2}}$. *Since* $\langle u, Au\rangle = 0$ *if and only if* $u \in \ker A$, *one can easily compute from (3.18) and obtain*

$$f^{\infty 2}(u;v) = \langle Av, v\rangle^{\frac{1}{2}},\ \text{if}\ u \in \ker A;\ f^{\infty 2}(u;v) = \frac{\langle Au, v\rangle}{\langle Au, u\rangle^{\frac{1}{2}}},\ \text{if}\ u \notin \ker A.$$

(*d*) *Let* $g : \mathbb{R}^m \to \mathbb{R} \cup \{+\infty\}$ *be proper and, and* $A : \mathbb{R}^n \to \mathbb{R}^m$ *be linear and such that* $A(\mathbb{R}^n) \cap \operatorname{ri}\operatorname{dom} g \neq \emptyset$. *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be defined by* $f(x) = g(Ax)$.

It is known that $f^{\infty}(u) \geq g^{\infty}(Au)$, $\forall u \in \mathbb{R}^n$, *see Proposition 2.6.3 in [9]. For every* u *such that* $f^{\infty}(u) = g^{\infty}(Au)$ *and are finite, and every* v,

$$f^{\infty 2}(u;v) = \inf\{\liminf\left(\frac{g(Ax_k)}{s_k} - t_k f^{\infty}(u)\right) : \frac{x_k}{s_k} - t_k u \to v, t_k, s_k \to +\infty\}$$

$$\geq \inf\{\liminf\left(\frac{g(y_k)}{s_k} - t_k g^{\infty}(Au)\right) : \frac{y_k}{s_k} - t_k Au \to Av, t_k, s_k \to +\infty\} = g^{\infty 2}(Au; Av).$$

If in addition g *is convex, in which case also* f *is convex, then* $f^{\infty}(u) = g^{\infty}(Au)$, *for all* $u \in \mathbb{R}^n$. *Under our assumption, there exists* $x_0 \in \mathbb{R}^n$ *such that* $Ax_0 \in \operatorname{ridom} g$. *In this case,* $x_0 \in \operatorname{ridom} f$. *Then for every* u *such that* $f^{\infty}(u)$ *is finite and every* v,

$$f^{\infty 2}(u;v) = \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{f(x_0 + tu + sv) - tf^{\infty}(u) - f(x_0)}{s}$$

$$= \lim_{s\to+\infty}\lim_{t\to+\infty}\frac{g(A(x_0 + tu + sv)) - tg^{\infty}(Au) - g(Ax_0)}{s} = g^{\infty 2}(Au; Av).$$

Note that in the examples (a), (b) and (c), $f^{\infty 2}(u,u) = f^{\infty}(u)$. This is not a coincidence, as shown by the next proposition.

Proposition 3.19 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function and* $u \in \mathbb{R}^n$ *be such that* $f^{\infty}(u) \in \mathbb{R}$. *The following hold:*

(*a*) $f^{\infty 2}(u; v - ru) \leq f^{\infty}(v) - rf^{\infty}(u)$ *for every* $r \geq 0$ *and* $v \in \operatorname{dom} f^{\infty}$. *In particular,* $f^{\infty 2}(u;v) \leq f^{\infty}(v)$.

(*b*) *If* $f^{\infty 2}(u;0) = 0$, *then* $f^{\infty 2}(u;u) = f^{\infty}(u) = -f^{\infty 2}(u;-u)$.

(*c*) *If* $f^{\infty 2}(u;0) = -\infty$, *then* $f^{\infty 2}(u;u) = -\infty$.

Proof. (a) We apply the inclusion $K^{\infty} - \mathbb{R}_+ u \subseteq K^{\infty 2}[u]$ (Proposition 3.6 part (d)) to the set $K = \operatorname{epi} f$. It follows that

$$\operatorname{epi} f^{\infty} - \mathbb{R}_+(u, f^{\infty}(u)) \subseteq (\operatorname{epi} f)^{\infty 2}\left[(u, f^{\infty}(u))\right] = \operatorname{epi} f^{\infty 2}(u; \cdot).$$

Thus,

$$(v - ru, t - rf^{\infty}(u)) \in \operatorname{epi} f^{\infty 2}(u; \cdot), \quad \forall r \geq 0, \forall (v, t) \in \operatorname{epi} f^{\infty},$$

which means

$$f^{\infty 2}(u; v - ru) \leq t - rf^{\infty}(u), \quad \forall r \geq 0, \forall (v, t) \in \operatorname{epi} f^{\infty},$$

proving (a).

(b) From (a) we obtain $f^{\infty 2}(u;u) \leq f^{\infty}(u)$ and $f^{\infty 2}(u;-u) \leq f^{\infty}(0) - f^{\infty}(u) = -f^{\infty}(u)$ (note that f^{∞}, being convex, lsc, and such that $f^{\infty}(u)$ is finite, never takes the value $-\infty$). The convexity of $f^{\infty 2}(u;\cdot)$ yields

$$0 = f^{\infty 2}(u; 0) \leq \frac{1}{2} f^{\infty 2}(u; u) + \frac{1}{2} f^{\infty 2}(u; -u) \leq 0.$$

Hence

$$f^{\infty 2}(u; u) = -f^{\infty 2}(u; -u) = f^{\infty}(u),$$

the desired result.

(c) If $f^{\infty 2}(u; 0) = -\infty$, then $f^{\infty 2}(u; u)$ cannot be finite. As it is bounded above by $f^{\infty}(u)$, necessarily $f^{\infty 2}(u; u) = -\infty$. ■

Remark 3.4 *By Lemma 3.1, for every $x \in \operatorname{ridom} f$ and $u \in (f^{\infty})^{-1}(\mathbb{R})$, the function $t \to f(x + tu) - tf^{\infty}(u)$ is decreasing, hence $\lim_{t \to +\infty} (f(x + tu) - tf^{\infty}(u))$ exists. By inspecting formula (3.18) we see that $f^{\infty 2}(u; 0) = 0$ holds if and only if $\lim_{t \to +\infty} (f(x + tu) - tf^{\infty}(u)) \in \mathbb{R}$, while $f^{\infty 2}(u; 0) = -\infty$ holds if and only if $\lim_{t \to +\infty} (f(x + tu) - tf^{\infty}(u)) = -\infty$.*

Consequently, if $f^{\infty 2}(u; u) \in \mathbb{R}$, then $\lim_{t \to +\infty} (f(x + tu) - tf^{\infty}(u)) \in \mathbb{R}$.

We also provide some more calculus rules.

Proposition 3.20 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, $i = 1, \ldots, k$, be convex functions such that $\bigcap_{i=1}^{k} \operatorname{ri} \operatorname{dom} f_i \neq \emptyset$. For every $u \in \mathbb{R}^n$ such that $f_i^{\infty}(u) \in \mathbb{R}$ for all i, and every $v \in \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$ the following equality holds:*

$$\left(\max_{1 \leq i \leq k} f_i \right)^{\infty 2} (u; v) = \max_{1 \leq i \leq k} f_i^{\infty 2}(u; v)$$

Also, the equality

$$(f_1 + f_2 + \cdots + f_k)^{\infty 2}(u; v) = \sum_{i=1}^{k} f_i^{\infty 2}(u; v)$$

holds, provided that the right-hand side is defined, i.e., if $f_i^{\infty 2}(u;v) = +\infty$ *for some* i*, then* $f_j^{\infty 2}(u;v) > -\infty$ *for all* $j \neq i$*.*

Proof. Set $f = f_1 + f_2 + \cdots + f_k$. Then $\operatorname{dom} f = \bigcap_{i=1}^{k} \operatorname{dom} f_i$ and $\operatorname{ri}\operatorname{dom} f = \bigcap_{i=1}^{k} \operatorname{ri}\operatorname{dom} f_i$. By Proposition 3.8, $(\operatorname{dom} f)^{\infty 2}[u] = \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$. Take any $x \in \bigcap_{i=1}^{k} \operatorname{ri}\operatorname{dom} f_i$. For every $v \in \bigcap_{i=1}^{k} (\operatorname{dom} f_i)^{\infty 2}[u]$, using Corollary 3.3 we find

$$\sum_{i=1}^{k} f_i^{\infty 2}(u;v) = \sum_{i=1}^{k} \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{f_i(x+tu+sv) - tf_i^{\infty}(u) - f_i(x)}{s}$$

$$= \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{f(x+tu+sv) - tf^{\infty}(u) - f(x)}{s} = f^{\infty 2}(u;v).$$

The proof of the other equality is similar. ■

In case of convex functions and for $\lambda > \inf f$, the inclusion of Proposition 3.15 becomes an equality, as we now show.

Proposition 3.21 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper convex function and* $\lambda > \inf f$*. If* $u \in \mathbb{R}^n$ *with* $f^{\infty}(u) = 0$*, then*

$$(S_\lambda)^{\infty 2}[u] = \{w : f^{\infty 2}(u;w) \leq 0\}.$$

Proof. Inclusion ($\subseteq$) follows from Proposition 3.15, so we have only to show ($\supseteq$).

Let $v \in \mathbb{R}^n$ with $f^{\infty 2}(u;v) \leq 0$. Since $\lambda > \inf f$ and f is convex, by Corollary 7.3.2 in [62] there exists $y \in \operatorname{ri}\operatorname{dom} f$ such that $f(y) < \lambda$. Then

$$\sup_{s>0} \inf_{t>0} \frac{f(y+tu+sv) - tf^{\infty}(u) - f(y)}{s} = f^{\infty 2}(u;v) \leq 0.$$

Thus for every $s > 0$,

$$\inf_{t>0} f(y+tu+sv) \leq f(y) < \lambda.$$

Since by Lemma 3.1 the function $t \to f(y+tu+sv)$ is nonincreasing, we deduce that for every $s > 0$ there exists $\bar{t} \geq 0$ such that $f(y+tu+sv) < \lambda$ for all $t \geq \bar{t}$, that is $y+tu+sv \in S_\lambda$ for all $t \geq \bar{t}$. One can easily see that we have also $y \in \operatorname{ri} S_\lambda$. Hence, $v \in (S_\lambda)^{\infty 2}[u]$. ■

We deduce from Proposition 3.7 that second order asymptotic directions of a convex set, K, can be seen as first order asymptotic directions of another convex set. To be more precise, it was proved that $K^{\infty 2}[u] = \mathbb{R}u + (PK)^{\infty} = (\mathbb{R}u + PK)^{\infty}$, where P is the projection on

$u^\perp$ (for the second equality apply Corollary 9.1.1 in [62]). Thus, a natural question arises: is it true that the second order asymptotic function of a given convex one can be seen as the first order asymptotic function of another convex function? the answer is yes, as we will see now.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex. For every $x \in \operatorname{ri} \operatorname{dom} f$ and $u \in \mathbb{R}^n$ such that $f^\infty(u) \in \mathbb{R}$, define the function $g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R} \cup \{+\infty\}$ by

$$g(v,t) \doteq f(x + tu + v) - tf^\infty(u).$$

This function is clearly convex (note that $(v,t) \to x + tu + v$ and $(v,t) \to tf^\infty(u)$ are linear functions). Consequently the function $f_u : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ defined by

$$f_u(v) \doteq \inf_{t>0} \left(f(x + tu + v) - tf^\infty(u) \right)$$

is convex but not necessarily proper or lsc. By setting $f_u^\infty = (f_u)^\infty$, we have the following expected proposition.

Proposition 3.22 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be proper and convex and $x \in \operatorname{ri} \operatorname{dom} f$. For every $v \in \mathbb{R}^n$,*

$$f^{\infty 2}(u; v) = \lim_{s \to +\infty} \frac{f_u^\infty(sv)}{s} \tag{3.20}$$

$$f_u^\infty(v) = \lim_{s \to +\infty} \frac{f_u(sv)}{s} \tag{3.21}$$

and thus $f^{\infty 2}(u; v) = f_u^\infty(v)$.

Proof. We first show (3.20). If $v \in (\operatorname{dom} f)^{\infty 2}[u]$, then this is already known (see formula (3.18)). If $v \notin (\operatorname{dom} f)^{\infty 2}[u]$, then we know that $f^{\infty 2}(u; v) = +\infty$. Also, by Proposition 3.5 part (b) there exists $s_0 > 0$ such that for every $t \geq 0$, $x + tu + s_0 v \notin \operatorname{dom} f$. This implies $f_u(s_0 v) = +\infty$. By the definition of f_u, $f_u(0) < +\infty$. Since f_u is convex, this implies $f_u(sv) = +\infty$ for all $s \geq s_0$, so (3.20) holds.

Now we show (3.21). Let $v \in \operatorname{aff}(\operatorname{dom} f)$ be such that $\|v\|$ is sufficiently small so that $x + v \in \operatorname{ri}(\operatorname{dom} f)$. From the definition of f_u it is clear that $f_u(v) < +\infty$. Hence, $0 \in \operatorname{ri} \operatorname{dom} f_u$.

We already know that $f^{\infty 2}(u; 0)$ equals 0 or $-\infty$. Assume first that $f^{\infty 2}(u; 0) = 0$ (that is, $f^{\infty 2}(u; \cdot)$ is proper). Also from (3.20) we obtain $f_u(0) = 0$. It follows that f_u is proper, since otherwise 0 should be at the relative boundary of $\operatorname{ri}(\operatorname{dom} f_u)$ by Theorem 7.2 in [62]. By using Proposition 3.16 we obtain that for all $v \in \mathbb{R}^n$, (3.21) holds.

Now assume that $f^{\infty 2}(u;0) = -\infty$. Since $f^{\infty 2}(u;\cdot)$ is convex and lsc, it can only take the values $+\infty$ and $-\infty$. We consider two cases.

(*i*) If $f^{\infty 2}(u;v) = -\infty$. By (3.20), $f_u(sv) < +\infty$ for s sufficiently large and $f_u(0) = -\infty$. Using convexity, we deduce that $f_u(sv) = -\infty$ for all $s \geq 0$. It follows that $f_u^\infty(v) = -\infty$. Indeed, for every $s > 0$ and $\mu \in \mathbb{R}$, $(sv, \mu) \in \operatorname{epi} f_u$; thus, $\forall s > 0$ and $\mu \in \mathbb{R}$, $s\,(v, \mu) \in \operatorname{epi} f_u$. It follows that $\forall \mu \in \mathbb{R}$, $(v, \mu) \in \operatorname{epi} f_u^\infty$, i.e., $f_u^\infty(v) = -\infty$, proving the claim.

(*ii*) Suppose $f^{\infty 2}(u;v) = +\infty$, we want to show that $f_u^\infty(v) = +\infty$. Assume to the contrary that $f_u^\infty(v) < +\infty$. In this case, we know that $v \in (\operatorname{dom} f_u)^\infty$. Since $0 \in \operatorname{ri} \operatorname{dom} f_u$, we deduce that $f_u(sv) < +\infty$ for all $s > 0$. Using convexity of f_u and $f_u(0) = -\infty$, we deduce as before that $f_u(sv) = -\infty$, for all $s > 0$. By (3.20), $f^{\infty 2}(u;v) = -\infty$, a contradiction.

Thus $f^{\infty 2}(u;\cdot) = f_u^\infty(\cdot)$ in all cases. ■

Remark 3.5 *Note that for a general convex function g, is not true that* $g^\infty(v) = \lim_{s \to +\infty} \dfrac{g(sv)}{s}$ *so* (3.21) *is not trivial. For instance, take* $g : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ *defined by* $g(x_1, x_2) = +\infty$ *if* $x_2 \geq 0$ *and* $g(x_1, x_2) = -\infty$ *if* $x_2 < 0$. *Then* $g(s(1,0)) = +\infty$ *for all* $s \in \mathbb{R}$, *but* $g^\infty(1,0) = -\infty$ *by definition of* g^∞ *(remember also that* g^∞ *is lsc).*

Chapter 4

Applications in optimization

In this Chapter we develop necessary and sufficient conditions for the existence of the optimal solution in the scalar minimization problem under convexity assumptions. We also obtain two existence results for the Multiobjective Optimization Problem in the convex and nonconvex case. Finally, finer estimates for the second order asymptotic cones of the efficient and weakly efficient solution sets are established.

4.1 The convex scalar problem

In this section we will treat non coercive convex minimization problems. We will obtain sufficient and necessary conditions to the existence of an optimal point, provided some compatibility conditions hold.

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper convex and lsc function and consider the minimization problem

$$\min_{x\in\mathbb{R}^n} \; f(x), \tag{4.1}$$

and the level set minimization problem, given $p \in \mathbb{R}^n$ and $K_p = \{x \in \mathbb{R}^n : \; f(x) \leq f(p)\}$, defined by

$$\min_{x\in K_p} f(x). \tag{4.2}$$

4.1.1 Necessary conditions

The following result shows two necessary conditions for the function f to be bounded from below.

Proposition 4.1 *(Theorem 3.3 in [45]) Let f be as before. Suppose that $m = \inf\limits_{v\in\mathbb{R}^n} f(x) > -\infty$, then*

($a1$) *$f^{\infty}(u) \geq 0$ for all $u \in (\text{dom } f)^{\infty}$.*

($a2$) *If $f^{\infty}(u) = 0$, then $f^{\infty 2}(u; v) \geq 0$ for every $v \in (\text{dom } f)^{\infty 2}[u]$.*

From asymptotical analysis we know that a necessary condition for the problem (4.2) to have a solution is that $f^{\infty}(u) \geq 0$, for all $u \in (K_p)^{\infty}$.

Then, given $p \in K$ such that K_p is unbounded, if $f^{\infty}(u) \geq 0$ for all $u \in (K_p)^{\infty} \backslash \{0\}$, then $-t_n f^{\infty}(u) \leq 0$ for all $t_n > 0$ with $t_n \to +\infty$, thus for every $\{x_n\} \subseteq K_p$ such that $\frac{x_n}{\|x_n\|} \to u$ we have

$$\frac{f(x_n)}{s_n} - t f^{\infty}(u) \leq \frac{f(x_n)}{s_n} \leq \frac{f(p)}{s_n}, \ \forall \ n \in \mathbb{N},$$

then $f^{\infty 2}(u; v) \leq 0$. We actually proved the next result.

Corollary 4.1 *Let f be as before. Given $p \in K$, if the problem* (4.2) *is bounded from below, then*

$$f^{\infty 2}(u; v) = 0, \ \forall \ u \in (K_p)^{\infty} \backslash \{0\} \text{ with } f^{\infty}(u) = 0 \text{ and } v \in (K_p)^{\infty 2}[u]. \tag{4.3}$$

By an easy application of Proposition 3.19, the necessary condition (4.3) is true for the global optimization problem (4.1) when the second order asymptotic direction is the same as the first order direction. We do not have the same property for other second-order asymptotic directions. Furthermore, both necessary conditions do not imply that f is bounded from below.

Example 4.1 *Define $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ by*

$$f(\alpha, \beta) = \begin{cases} -\sqrt{\beta}, & \text{if } \alpha \geq 0 \text{ and } \beta \in [0, \sqrt{\alpha}] \\ +\infty, & \text{otherwise.} \end{cases}$$

One has $\lim_{x \to +\infty} f(x, \sqrt{x}) = -\infty$, so $\inf f = -\infty$. Let us check that f satisfies conditions ($a1$) *and* ($a2$) *of Proposition 4.1. It is evident that $f^{\infty}(1, 0) = 0$. Also, for every $(\alpha, \beta) \notin \mathbb{R}_+(1, 0)$ and every $(x_1, x_2) \in \text{dom } f$, $(x_1, x_2) + t(\alpha, \beta) \notin \text{dom } f$ for all $t > 0$ sufficiently large; consequently, $f^{\infty}(\alpha, \beta) = +\infty$. Hence condition* ($a1$) *is satisfied.*

Let $v = (\alpha, \beta) \in \mathbb{R}^2$ and choose $(x_1, x_2) \in \text{ridom } f$. Suppose first that $\beta > 0$. Given $s > 0$, for t sufficiently large, $(x_1, x_2) + t(1, 0) + s(\alpha, \beta) \in \text{dom } f$. It is then clear that

$$f^{\infty 2}((1, 0); (\alpha, \beta)) = \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{f((x_1, x_2) + t(1, 0) + s(\alpha, \beta)) - t f^{\infty}(1, 0)}{s}$$

$$= \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{-\sqrt{x_2 + s\beta}}{s} = 0.$$

If $\beta < 0$, then for s sufficiently large one has $x_2 + s\beta < 0$, so $(x_1, x_2) + t(1,0) + s(\alpha, \beta) \notin \operatorname{dom} f$ for all $t > 0$. Hence in this case $f^{\infty 2}((1,0);(\alpha,\beta)) = +\infty$. Thus, f satisfies also condition $(a2)$.

We use a slight modification of condition $(a2)$ into the following.

$(a2')$: If $f^{\infty}(u) = 0$ then $f^{\infty 2}(u;u) = 0$.

Note that in view of Proposition 3.13, condition $(a2')$ is equivalento to: If $f^{\infty}(u) = 0$ then $f^{\infty 2}(u;u) \geq 0$.

The calculus of the second order asymptotic function is very important to determine easily if a lsc function is bounded from below or not.

If f is proper convex and $n = 1$, then the converse is also true. More generally, if f is proper convex and satisfies $(a1)$ and $(a2')$ then the restriction of f on every straight line $l = \{x + tu : t \in \mathbb{R}\}$ with $x \in \operatorname{ridom} f$, is bounded from below. Indeed, if $f^{\infty}(u) > 0$ then $\lim_{t\to+\infty} f(x+tu) = +\infty$. If $f^{\infty}(u) = 0$ then by Proposition 3.19 and Remark 3.4, $\lim_{t\to+\infty} f(x+tu)$ is finite.

The same is true for $\lim_{t\to-\infty} f(x+tu) = \lim_{t\to+\infty} f(x+t(-u))$, thus f has a lower bound on l. The characterization reveals the connection between the dimension of the epigraph of the function and the order of the asymptotics functions involved in the result.

With a little more effort, one can show a similar result for every lsc (not necessarily convex) proper function.

Proposition 4.2 *For every lsc (not necessarily convex) proper function f, condition $(a1)$ and $(a2')$ imply that f is bounded from below on every straight line $l = \{x + tu : t \in \mathbb{R}\}$ with $x \in \operatorname{ridom} f$.*

Proof. Assume to the contrary that there exists a sequence $x_n = x + \alpha_n u$ in l such that $f(x_n) \to -\infty$. If the sequence is bounded, we can assume that it converges to some $y \in l$ and obtain that $f(y) = -\infty$, a contradiction. If it is unbounded, by selecting a subsequence and using $-u$ instead of u if necessary, we may assume that $\alpha_n \to +\infty$. Note that $\frac{x_n}{\alpha_n} \to u$, so

$$0 \leq f^{\infty}(u) \leq \liminf_{n\to+\infty} \frac{f(x_n)}{\alpha_n} \leq 0.$$

Thus $f^{\infty}(u) = 0$ and by selecting again a subsequence we may assume that $\lim \frac{f(x_n)}{\alpha_n} = 0$. Set $s_n = \sqrt{-f(x_n)}$ and $t_n = \frac{\alpha_n}{s_n} - 1$. Then $\lim s_n = +\infty$, $\lim t_n \geq \lim(\frac{\alpha_n}{-f(x_n)} - 1) = +\infty$, and $\frac{x_n}{s_n} - t_n u \to u$. Also,

$$f^{\infty 2}(u;u) \leq \lim_{n\to+\infty} \left(\frac{f(x_n)}{s_n} - t_n f^{\infty}(u) \right) = -\lim_{n\to+\infty} \sqrt{-f(x_n)} = -\infty.$$

This contradicts $f^{\infty 2}(u;u) = 0$. ■

Proposition 4.2 does not imply that a lsc proper function that satisfies conditions $(a1)$ and $(a2')$ is bounded from below on higher dimension spaces. See the counterexample Example 4.1, where the function f satisfies stronger conditions like convexity, $(a1)$ and $(a2)$.

4.1.2 Sufficient conditions

Until now, we only know one sufficient condition for the existence of solutions for the minimization problem using second order asymptotic functions. This result was presented in [45]. We recall the result for convenience of the reader.

Theorem 4.1 *(Theorem 3.6 in [45]) Let $K \subseteq \operatorname{dom} f$ be a closed set and f be a proper and lsc (not necessarily convex) function. If the following conditions hold,*

(a) *$f^{\infty}(u) \geq 0$ for all $u \in K^{\infty}$.*

(b) *$f_{+}^{\infty 2}(u;u) > 0$ for all $u \in K^{\infty}, u \neq 0$ with $f^{\infty}(u) = 0$.*

then the problem (4.1) *has an optimal solution.*

Remark 4.1 *Let now consider the next condition;*

(b'): *For every $u \in K^{\infty}, u \neq 0$ with $f^{\infty}(u) = 0$, there exists $p \in K^{\infty 2}[u]$ such that $f_{+}^{\infty 2}(u;p) > 0$.*

The previous Theorem cannot be strengthened in the sense that we replace (b) by (b') as the next example shows.

Let $K = \{(x^2, x) : x \in \mathbb{R}\} = \{(x, \sqrt{x}, x \geq 0\} \cup \{x, -\sqrt{x}), x \geq 0\}$, and define $f(x, \sqrt{x}) = \sqrt{x}$ and $f(x, -\sqrt{x}) = -\sqrt{x}$. Again $K^{\infty} = \mathbb{R}_{+}u$ with $u := (1, 0)$, and $f^{\infty}(1, 0) = 0$. Note again that $p := (0, 1) \in K^{\infty 2}[u]$ and by using formula (3.6) *we can easily calculate $f_{+}^{\infty 2}(u;p) = 1$. Thus (b') holds but f has no minimum.*

It is clear that for the constant function $f(x) = 0$ the problem (4.1) has optimal solutions, while condition (b) in Theorem 4.1 does not hold. Thus the converse is not true. We also show the next example, which will be used in the next chapter.

Example 4.2 *Let $f : \mathbb{R} \to \mathbb{R}$ be the convex and continuous function given by $f(x) = 0$ if $x \leq 0$, and $f(x) = x^2$ if $x \geq 0$. Here the solution set is $S =]-\infty, 0]$ and the asymptotic function is given by $f^{\infty}(u) = 0$ if $u \leq 0$, and $f^{\infty}(u) = +\infty$ if $u > 0$.*

Let $u < 0$ and $\{x_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be any sequence with $|x_n| \to +\infty$ and $\dfrac{x_n}{|x_n|} \to \dfrac{u}{|u|}$, let $s_n, t_n \uparrow +\infty$ be such that $\left(\dfrac{x_n}{s_n} - t_n u\right) \to u$, then

$$f_+^{\infty 2}(u;u) = \inf_{\substack{\{x_n\}\in K(u)\\ L(\{x_n\},w)\neq\emptyset}} \sup_{\{(s_n,t_n)\}\in L(\{x_n\},w)} \lim_{n\to\infty} \left(\frac{f(x_n)}{s_n} - t_n f^\infty(u)\right) = 0,$$

because $f(x_n) = 0$ for all $n \in \mathbb{N}$ and $f^\infty(u) = 0$ for $u < 0$. Then $f_+^{\infty 2}(u;u) = 0$ and the second order asymptotic sufficient condition cannot be satisfied.

The main reason is that, the previous function is non coercive. We recall the Following definitions.

Definition 4.1 *The function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is called*

(a) *level bounded if for each $\lambda > \inf_{\mathbb{R}^n} f$, the level set $S_\lambda(f)$ is bounded.*

(b) *coercive if $f^\infty(u) > 0$ for all $u \neq 0$.*

If f is level bounded, then $\lim_{|x|\to+\infty} f(x) = +\infty$ and vice-versa. If f is also proper lsc and convex, then by Proposition 3.1.3 in [9] we have that f is coercive if and only if f is level bounded if and only if $0 \in \operatorname{intdom} f^*$.

The next proposition shows, in one dimension, that whenever f is level bounded, condition (b) in Theorem 4.1 holds.

Proposition 4.3 *If $f : \mathbb{R} \to \mathbb{R}$ is level bounded with $f^\infty(1) = 0$, then $f_+^{\infty 2}(1;1) = +\infty$.*

Proof. Indeed, let $x_n \to +\infty$. Set $s_n = \min(\sqrt{x_n}, \sqrt{f(x_n)})$ and $t_n = \frac{x_n}{s_n} - 1$, then $\frac{x_n}{|x_n|} = 1$, $s_n \to +\infty$, $t_n \geq \frac{x_n}{\sqrt{x_n}} - 1 \to +\infty$, $\frac{x_n}{s_n} - t_n = 1 \to 1$, thus

$$\frac{f(x_n)}{s_n} - t_n f^\infty(1) = \frac{f(x_n)}{s_n} \geq \sqrt{f(x_n)} \to +\infty.$$

Hence, $f_+^{\infty 2}(1;1) = +\infty$. ∎

This is not true in higher dimension as we can see in the next example.

Example 4.3 *Consider the set $K = \{(\alpha,\beta) \in \mathbb{R}_+^2 : \beta \leq \sqrt{\alpha}\}$ and the function $f(\alpha,\beta) = \sqrt{\alpha}$ on K. The function f is level bounded, because whenever $\|(\alpha_n,\beta_n)\| \to +\infty$ for $(\alpha_n,\beta_n) \in K$, one has necessarily $\alpha_n \to +\infty$. It is easy to see that $K^\infty = \mathbb{R}_+ u$ where $u = (1,0)$ and $f^\infty(u) = 0$. Now consider the sequence $x_n = (n, \sqrt{n})$. This sequence belongs to $K(u)$ and there exists at least one sequence (s_n,t_n) satisfying $s_n, t_n \to +\infty$ and*

$\frac{x_n}{s_n} - t_n u \to u$ *by Lemma 3.5 in [45], or as can be seen directly. Let* (s_n, t_n) *be any such sequence, then* $\frac{\sqrt{n}}{s_n} \to 0$ *and*

$$\lim \frac{f(x_n)}{s_n} = \lim \frac{\sqrt{n}}{s_n} = 0.$$

Hence the supremum in the definition of $f_+^{\infty 2}(u;u)$ *is zero for the sequence* $\{x_n\}$*. Since* $f_+^{\infty 2}(u;u)$ *is defined by taking the infimum over all sequences in* $K(u)$*, we get* $f_+^{\infty 2}(u;u) = 0$*.*

The goal of this section is to provide an existence result for the minimization problem for non coercive convex functions using second order asymptotic analysis.

To this end, we recall weaker definitions of coercivity presented in the literature. The asymptotically directionally constant (adc for short) and weakly coercive functions introduced by Auslender in [6] and developed in [6, 7, 8, 9] and references therein.

Definition 4.2 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, convex and lsc function. Then* f *is;*

(a) *adc if* $f^\infty(u) = 0 \Rightarrow f^\infty(-u) = 0$.

(b) *weakly coercive if* f *is adc and* $f^\infty(u) \geq 0$ *for all* $u \neq 0$.

By Theorem 3.2.1 in [9] we know that f is weakly coercive if and only if $0 \in \operatorname{ridom} f^*$.

Note that a proper, lsc and convex function is adc if and only if it is constant on every line $x + \mathbb{R}u$ where $x \in \operatorname{dom} f$ and $f^\infty(u) = 0$.

We introduce a new class of convex function in terms of the first and second order asymptotic analysis.

Definition 4.3 *We say a proper convex and lsc function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *is in the class* $\mathcal{F}$ *if* f *satiesfies conditions* $(a1)$ *and* $(a2')$*.*

Our class $\mathcal{F}$ contains the weakly coercive functions.

Proposition 4.4 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be a proper, convex and lsc function. If* f *is weakly coercive, then* $f \in \mathcal{F}$*.*

Proof. If f is weakly coercive, then $f^\infty(u) \geq 0$ for all $u \neq 0$ and $(a1)$ holds. On the other hand, since f is adc, take $u \in (\operatorname{dom} f)^\infty$ such that $f^\infty(u) = 0$. Then

$$f(x + \rho u) = f(x), \ \forall \ x \in \operatorname{dom} f, \ \forall \ \rho \in \mathbb{R},$$

that is,

$$f(x + (s+t)u) = f(x), \ \forall \ x \in \operatorname{dom} f, \ \forall \ s, t > 0,$$

since $f^\infty(u) = 0$ then we have,

$$f(x + (s + t)u) - tf^\infty(u) - f(x) = 0, \ \forall\, x \in \text{ridom}\, f, \ \forall\, s, t > 0,$$

$$\Rightarrow f^{\infty 2}(u; u) = \sup_{s>0} \inf_{t>0} \frac{f(x + (s + t)u) - tf^\infty(u) - f(x)}{s} = 0,$$

thus $(a2')$ holds. ■

The reverse implication does not hold as the following example shows.

Example 4.4 *Consider the convex, proper and lsc function $f : \mathbb{R} \to \mathbb{R}$ given by*

$$f(x) = \begin{cases} -x & x < 0 \\ 0 & x \geq 0. \end{cases}, \ f^\infty(u) = \begin{cases} -u & u < 0 \\ 0 & u \geq 0. \end{cases}$$

For $u = 1$ then $f^\infty(1) = 0$, thus $f^{\infty 2}(1; 1) = 0$ and $f \in \mathcal{F}$. But f is not weakly coercive (because is not adc).

Remark 4.2 *(i) The class of asymtotically level stable functions (als for short) introduced by Auslender in [8] (see also [9] chapter 3, section 3 for details) also includes the class of weakly convex functions. Anyway, our family $\mathcal{F}$ is not contained in the als class as the following example shows.*

Consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = e^x$. This function is not als (see Remark 3 in [8]), but $f \in \mathcal{F}$. In fact,

$$f^\infty(u) = \begin{cases} 0 & u \leq 0 \\ +\infty & u > 0. \end{cases}$$

Furthermore, for $u = -1$ we have $f^{\infty 2}(-1; -1) = 0$.

(ii) The class of Weakly analytic functions (waf for short) introduced by Kummer in [51] (see also [9] chapter 5, section 4 for details) has no relationship with our family $\mathcal{F}$. In fact, consider $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = \max\{0, x\}$, here

$$f^\infty(u) = \begin{cases} 0 & u \leq 0 \\ u & u > 0. \end{cases}$$

Then f is not waf, but for $u = -1$ we have $f^{\infty 2}(-1; -1) = 0$, then $f \in \mathcal{F}$.

On the other hand, consider $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x_1, x_2) = -\sqrt{x_2}$ if $x_2 \geq 0$, and $f(x_1, x_2) = +\infty$ elsewhere. Here f is waf, but for $u = (0, 1)$ we have $f^\infty(u) = 0$ and $f^{\infty 2}(u; u) = -\infty$, then $f \notin \mathcal{F}$.

In order to obtain a sufficient condition for the non coercive optimization problem (4.1) we introduce the following condition related to the one introduced in [6], which is a slight modification of that used in [10].

Define the asymptotic cone $R = \{u \in \mathbb{R}^n : \ f^\infty(u) = 0, \ f^{\infty 2}(u; u) = 0\}$ and consider the condition:

$(a3)$: If the sequence $x_k \in \text{dom } f, \|x_k\| \to +\infty$ is such that $\frac{x_k}{\|x_k\|} \to u \in R$ and for all $z \in \text{dom } f$, there exists $k_z \in \mathbb{N}$ such that

$$f(x_k) \leq f(z), \text{ when } k \geq k_z,$$

then there exists $\overline{x} \in \text{dom } f$ and $\overline{k}$ such that $\|\overline{x}\| < \|x_{\overline{k}}\|$ and $f(\overline{x}) \leq f(x_k), \ \forall \ k \geq \overline{k}$.

We present the main result of this section.

Theorem 4.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, convex and lsc function, then conditions $(a1), (a2')$ and $(a3)$ are satisfied if and only if the problem* (4.1) *has a solution.*

Proof. $(\Rightarrow)$: Suppose condition $(a1), (a2')$ and $(a3)$ are satisfied, For every $k \in \mathbb{N}$, set $B_k = \{x \in \mathbb{R}^n : \ \|x\| \leq k\}$. We may suppose, without loss of generality, that $B_k \cap \text{dom } f \neq \emptyset$ for all $n \in \mathbb{N}$.
Take any solution $x_k \in B_k \cap \text{dom } f$ to the problem

$$\min_{x \in B_k} \ f(x). \tag{4.4}$$

If for some $k_0 \in \mathbb{N}, \|x_{k_0}\| < k_0$, then x_k is solution of the problem (4.1) by the convexity of f.

Suppose $\|x_k\| = k$ for all $k \in \mathbb{N}$ and define $w_k = \frac{x_k}{k}$ for all $k \in \mathbb{N}$, then we may extract a subsequence (which we still denote by $\{w_k\}_{k \in \mathbb{N}}$) converging to some $\overline{w} \in \mathbb{R}^n$. Thus

$$f(kw_k) = f(x_k) < +\infty,$$

for every $k \in \mathbb{N}$ sufficiently large. Take $x_0 \in \text{dom } f$ any fixed point. By the lsc and convexity of f, we have for every $\lambda > 0$,

$$f(x_0 + \lambda \overline{w}) \leq \liminf_{k \to +\infty} f\left(\lambda w_k + \left(1 - \frac{\lambda}{k}\right) x_0\right) \leq \liminf_{k \to +\infty} \left(\frac{\lambda}{k} f(kw_k) + \left(1 - \frac{\lambda}{k}\right) f(x_0)\right)$$

$$= \liminf_{k \to +\infty} \left(\frac{\lambda}{k} f(x_k) + \left(1 - \frac{\lambda}{k}\right) f(x_0)\right) = f(x_0),$$

thus

$$f^{\infty}(\overline{w}) = \lim_{\lambda\to+\infty} \frac{f(x_0+\lambda\overline{w})}{\lambda} \leq \lim_{\lambda\to+\infty} \frac{f(x_0)}{\lambda} = 0,$$

which together with the assumption $(a1)$ implies that $f^{\infty}(\overline{w}) = 0$. By assumption $(a2')$ we have $f^{\infty 2}(\overline{w};\overline{w}) = 0$ and then $\overline{w} \in R$. Thus $\{x_k\}_{k\in\mathbb{N}}$ satisfies the premises of $(a3)$ and therefore there exists $z \in \mathbb{R}^n$ and $\overline{k} \in \mathbb{N}$ such that

$$\|z\| < \|x_{\overline{k}}\| \text{ and } f(z) \leq f(x_k),\ \forall\ k \geq \overline{k},$$

giving a contradiction with the fact that $x_{\overline{k}}$ is a solution of the problem for $\overline{k} \in \mathbb{N}$.

($\Leftarrow$) If the problem (4.1) has a solution then $(a1)$ and $(a2')$ are consequences of Proposition 4.1. Let $z \in \mathbb{R}^n$ be the solution of the problem, taken any sequence $\{x_k\}_{k\in\mathbb{N}}$ such that $\|x_k\| \to +\infty$, for $\overline{x} = u$ the condition $(a3)$ holds and the proof is complete. ■

4.2 Multiobjective optimization

In this section, we develop some new existence results for the multiobjective optimization problems under second order asymptotic analysis. A sufficient condition for the Domination Property and for a point to be proper efficient and weakly efficient are given in the convex and nonconvex case.

Let $P \subsetneq \mathbb{R}^m$ be a closed convex cone and $F = (f_1, f_2, \ldots, f_m) : \mathbb{R}^n \to \mathbb{R}^m$. Let $K \subseteq \cap_{I=1}^{m} \text{dom } f_i$ be a closed convex set. We say that $\bar{x} \in K$ is a:

• "weakly efficient" point of F (on K) if $\text{int } P \neq \emptyset$ and

$$F(x) - F(\bar{x}) \notin -\text{int } P, \quad \forall\ x \in K,$$

or equivalently, $(F(K) - F(\bar{x})) \cap (-\text{int } P) = \emptyset$, or equivalently, $\overline{\text{cone}}(F(K) - F(\bar{x}) + P) \cap (-\text{int } P) = \emptyset$;

• "efficient" point of F (on K) if

$$F(x) - F(\bar{x}) \notin -P \setminus l(P), \quad \forall\ x \in K,$$

or equivalently, $(F(K) - F(\bar{x})) \cap (-P \setminus l(P)) = \emptyset$, or equivalently, $\text{cone}(F(K) - F(\bar{x}) + P) \cap (-P \setminus l(P)) = \emptyset$.

• "proper efficient" point of F (on K) if

$$\overline{\text{cone}}(F(K) - F(\bar{x}) + P) \cap -P = \{0\}.$$

The concept of Proper Efficient solution was discussed long time ago by many important mathematicians. See for example [12, 13, 42]. We work here with the Benson definition who ended the discussion in [12].

Notice that every proper efficient point is efficient and every efficient point is weakly efficient. The set of weakly efficient points is denoted by E_W, that of efficient by E, and the set of proper efficient by E_{Pr}. It is easy to see that $E_{\mathrm{Pr}} \neq \emptyset$ implies the pointedness of P, that is, $l(P) = \{0\}$.

We start this section with vector-valued functions definitions.

Definition 4.4 *Let P, K be as before. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector-valued function. We say that F is,*

(a) *P-convex* [30]*, if for all $x, y \in K$,*

$$\alpha F(x) + (1-\alpha)F(y) \in F(\alpha x + (1-\alpha)y) + P, \ \forall \ \alpha \in]0,1[. \tag{4.5}$$

In particular, F is $\mathbb{R}^m_+$-convex if and only if each component f_i is convex.

(b) *Naturally P-quasiconvex* [67] *if,*

$$\forall \ x, y \in K, \ F([x,y]) \subseteq [F(x), F(y)] - P. \tag{4.6}$$

(c) *$*$-quasiconvex* [48] *if,*

$$\langle p^*, F(\cdot)\rangle \text{ is quasiconvex}, \ \ \forall \ \ p^* \in P^*. \tag{4.7}$$

(d) *P-quasiconvex* [30] *if the set,*

$$\{u \in K : \ F(u) \in \lambda - P\} \text{ is convex for all } \lambda \in \mathbb{R}^m. \tag{4.8}$$

In particular, F is $\mathbb{R}^m_+$-quasiconvex if and only if each component f_i is quasiconvex.

Without any further assumption on P, Naturally P-quasiconvex functions are strictly larger than P-convex, see [67]. The class of P-quasiconvex functions is strictly larger than $*$-quasiconvex functions by [48], and the class of $*$-quasiconvex functions are equivalent to Naturally P-quasiconvex functions by Theorem 2.3 of [40]. For several applications in vector optimization see [40, 48, 67] and reference therein.

The following result give us a general sufficient condition for a set to has the Domination Property and a proper efficient solution. Recall that $F(K)$ has the domination property, which means that for every $y \in F(K)$ there exists an efficient point $z \in F(K)$ such that z dominates y in the sense that $z \in y + P$. For more details on Domination Property see [47, 54].

Theorem 4.3 *Let P be a closed convex and pointed cone. Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a vector valued function and K as before. Assume that $F(K)$ is closed and $F(K) + P$ is a convex set. If there exists $u_0 \in (F(K) + P)^\infty$ such that*

$$(F(K) + P)^{\infty 2}[u_0] \cap (-P \backslash \{0\}) = \emptyset, \tag{4.9}$$

then $F(K)$ has the domination property and $E_{\mathrm{Pr}} \neq \emptyset$.

Proof. Let $u_0 \in (F(K) + P)^\infty$ be such that $(F(K) + P)^{\infty 2}[u_0] \cap (-P \backslash \{0\}) = \emptyset$, since $F(K) + P$ is a convex set, by Proposition 3.6 we have that

$$(F(K) + P)^\infty \cap (-P \backslash \{0\}) = \emptyset.$$

Since $F(K) \subseteq F(K) + P$, thus $(F(K))^\infty \cap (-P \backslash \{0\}) = \emptyset$, then by Corollary 4.6, Chapter 2 in [54], $F(K)$ has the domination property and $E \neq \emptyset$.

Furthermore, from $(F(K))^\infty \cap (-P \backslash \{0\}) = \emptyset$, we have $(F(K))^\infty \cap -P = \{0\}$, thus $F(K) + P$ is closed. Finally, Corollary 4.5, Chapter 2 in [54] implies that $E_{\mathrm{Pr}} \neq \emptyset$. ∎

Under the assumption of the previous theorem, vector u has to be in $-\mathrm{bd}\,(F(K)+P)^\infty$, therefore there is no contradiction with the fact that, $(F(K) + P)^{\infty 2}[u] = \mathbb{R}^n$ when $u \in \mathrm{int}\,(F(K) + P)^\infty$, Proposition 3.1 part (f).

We need to check the condition (4.9) for only one first order asymptotic direction, while several results in the literature show us when $F(K) + P$ is convex. For example, if F is Naturally P-quasiconvex and for every $q \in P^*$ the scalarization function $h_q = \langle q, F(\cdot) \rangle$ is lsc on any line segment of K, then Corollary 3.11 in [37] implies that $F(K) + P$ is a convex set.

The difficult part is that $F(K)$ has to be a closed set. The closedness of images of sets under linear or nonlinear operations has been extensively studied in optimization. For a linear transformation see Theorem 9.1 in [62], for the image of a closed set under linear or nonlinear mappings see Exercise 3.16 in [63].

We recall Theorem 4.2 in [23] for convex functions under coercivity conditions. In this result we consider $P = \mathbb{R}^m_+$.

Proposition 4.5 *(Theorem 4.2 in [23]) Let $F : \mathbb{R}^n \to \mathbb{R}^m$ be a $\mathbb{R}^m_+$-convex function such* *that each f_i is finite. If $R_0 = \bigcap_{i=1}^{m} \{u \in K^{\infty} : \ f_i^{\infty}(u) \leq 0\}$ is a linear subspace, then $F(K)$ is closed.*

The previous result will be extended to semistrictly quasiconvex functions in Chapter 6.

For another existence result under second order asymptotic analysis, we will consider $P \subsetneq \mathbb{R}^m$ be a closed convex cone such that $P^* \setminus \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ with extrd P^* being a finite set, that is, P is a polyhedral cone. By extrd P^* we mean the set of extreme directions of P^*.

Given $q \in P^*$, we consider the scalarization function $h_q : K \to \mathbb{R}$ defined as $h_q(x) = \langle q, F(x) \rangle$.

Motivated by [32] and Theorem 5.2 we define,

$$R_1 = \bigcup_{q \in \mathrm{extrd}\ P^*} \{u \in K^{\infty} : \ h_q^{\infty}(u) = 0, h_q^{\infty 2}(u; u) = 0\},$$

Consider the followings conditions,

$(f1)$ There exists $z \in \mathbb{R}^m$ such that $\langle q, z \rangle \leq h_q(x)$, for all $q \in \mathrm{extrd}\ P^*$ and for all $x \in K$.

$(f2)$ For every $q \in \mathrm{extrd}\ P^*$, if $h_q^{\infty}(u) = 0$ then $h_q^{\infty 2}(u; u) = 0$.

$(f3)$ For any sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq K$ satisfying

(i) $\|x_n\| \to +\infty, \dfrac{x_n}{\|x_n\|} \to u \in R_1$, and

(ii) $\forall\ y \in K,\ \exists\ n_y \in \mathbb{N}$ such that $F(y) - F(x_n) \notin -\mathrm{int}\ P,\ \forall\ n \geq n_y$,

there exists $w \in K, \overline{n} \in \mathbb{N}$ such that $\|w\| < \|x_{\overline{n}}\|$ and $F(w) - F(x_{\overline{n}}) \in -P$.

Theorem 4.4 *Let $K \subseteq \mathbb{R}^n$ be a closed convex set and P as before. Let h_q be proper, convex and lsc for all $q \in \mathrm{extrd}\ P^*$. If assumptions $(f1), (f2)$ and $(f3)$ hold then $E_W \neq \emptyset$.*

Proof. For every $n \in \mathbb{N}$, set $K_n = \{x \in K : \ \|x\| \leq n\}$. We may suppose $K_n \neq \emptyset$ for all $n \in \mathbb{N}$. Consider the problem

$$\text{find } \overline{x} \in K_n : \ F(y) - F(\overline{x}) \notin -\mathrm{int}\ P,\ \forall\ y \in K_n. \tag{4.10}$$

Since the solution exists for every $n \in \mathbb{N}$ by Lemma 3.2 in [32], say $x_n \in K$ for all $n \in \mathbb{N}$.

(i) If $\|x_n\| < n$ for some n, we claim $x_n \in E_W$. In fact, if there is $y \in K$ with $\|y\| > n$ such that $F(y) - F(x_n) \in -\mathrm{int}\ P$, we take $z \in K$ with $z \in \ ]x_n, y[$ and $\|z\| < n$. Writing

$z = \lambda x_n + (1-\lambda)y$ for some $\lambda \in]0,1[$, we have

$$\lambda F(x_n) + (1-\lambda)F(y) - F(z) \in P.$$

This implies

$$-(F(z) - F(x_n)) \in -(1-\lambda)(F(y) - F(x_n)) + P \subseteq (1-\lambda)\text{int } P + P = \text{int } P + P = \text{int } P,$$

which contradicts the choice of x_n. Then $x_n \in E_W$.

(ii) If $\|x_n\| = n$ for all $n \in \mathbb{N}$, then $\frac{x_n}{\|x_n\|} \to u \neq 0, u \in K^\infty$. For any fixed $y \in K$, $F(y) - F(x_n) \notin -\text{int } P$, for all n sufficiently large. This shows that the sequence satisfies (ii) of condition $(f3)$.

On the other hand, for every $\lambda > 0$ and for all n sufficiently large, the convexity of each component implies

$$\left(1 - \frac{\lambda}{\|x_n\|}\right) F(y) + \frac{\lambda}{\|x_n\|} F(x_n) \in F\left(\left(1 - \frac{\lambda}{\|x_n\|}\right) y + \frac{\lambda}{\|x_n\|} x_n\right) + P.$$

Hence

$$-\left[F\left(\left(1 - \frac{\lambda}{\|x_n\|}\right) y + \frac{\lambda}{\|x_n\|} x_n\right) - F(y)\right] \in \mathbb{R}^m \backslash -\text{int } P \;+\; P \subseteq \mathbb{R}^m \backslash -\text{int } P,$$

Thus, there exists $q \in \text{extrd } P^*$ such that

$$h_q\left(\left(1 - \frac{\lambda}{\|x_n\|}\right) y + \frac{\lambda}{\|x_n\|} x_n\right) \leq h_q(y),$$

and since h_q is lsc, we have

$$h_q(y + \lambda u) \leq \liminf_{n \to +\infty} h_q\left(\left(1 - \frac{\lambda}{\|x_n\|}\right) y + \frac{\lambda}{\|x_n\|} x_n\right) \leq h_q(y),$$

where q depends of y and λ. If we take $\lambda = n$, then for all $y \in K$ and $n \in \mathbb{N}$ there exists $q_{y,n} \in \text{extrd } P^*$ such that

$$h_{q_{y,n}}(y + nu) \leq h_{q_{y,n}}(y),$$

since $q_{y,n} \in \text{extrd } P^*$ for all n and the index set is finite, there exists q_y and a subsequence $\{y + lu\}_{l \in \mathbb{N}}$ such that

$$\sup_{l \in \mathbb{N}} \frac{h_{q_y}(y + lu) - h_{q_y}(y)}{l} = h_{q_y}^\infty(u) \leq 0,$$

by condition $(f1)$ we have $h_{q_y}^\infty(u) = 0$. Since $(f2)$ holds then $h_{q_y}^{\infty 2}(u;u) = 0$ and $u \in R_1$, then the sequence x_n satisfies condition $(f3)$ and there exists $w \in K$ and $\overline{n} \in \mathbb{N}$ such that $F(w) - F(x_{\overline{n}}) \in -P$.

We claim $x_{\overline{n}} \in E_W$. If not, there exists $y \in K, \|y\| > \overline{n}$ such that $F(y) - F(x_{\overline{n}}) \in -\text{int } P$. Since $\|w\| < \|x_{\overline{n}}\| = \overline{n}$, we can find $z \in \ ]w, y[$ such that $\|z\| < \overline{n}$, we proceed as in the first part to get a contradiction with the choice of $x_{\overline{n}}$, so $E_W \neq \emptyset$. ■

If $m = 1$, the previous result coincides with Theorem 4.1.

The following example shows the importance of the previous theorem compared with Theorem 4.1.

Example 4.5 *(Example 4.2) Let $f : \mathbb{R} \to \mathbb{R}$ with $f(x) = 0, x \leq 0$ and $f(x) = x^2, x \geq 0$. The solution set is $S =]-\infty, 0]$ and the asymptotic function is given by $f^\infty(u) = 0, u \leq 0, f^\infty(u) = +\infty, u > 0$. As we seem in Example 4.2, condition (b) of the Theorem 4.1 can not be satisfied.*

On the other hand, $(f1)$ and $(f2)$ holds. We only need to check $(f3)$ for $m = 1$. Let $x_k < 0$ as before and take any $y \in \mathbb{R}$, since $f(x_k) = 0$ there exists k_y sufficiently large such that

$$f(x_k) \leq f(y), \ \forall \ k \geq k_y,$$

then we can find $z \in \mathbb{R}$ and $\overline{k} \in \mathbb{N}$ with $f(z) \leq f(x_{\overline{k}})$ and $\|z\| < \|x_{\overline{k}}\|$, proving that $\text{argmin}_{\mathbb{R}} f \neq \emptyset$.

4.3 Second order asymptotic estimates

We study the behaviour of the efficient and weakly efficient solution sets at the infinity, we provide finer estimates for their asymptotic cones using the second order asymptotic analysis.

We deal with the following scalarization problem, given $q \in P^*$

$$\min_{x \in S(z)} h_q(x). \tag{4.11}$$

Let $K \subseteq \cap_{q \in P^*} \text{dom } h_q$ be a closed convex set. We say that a vector $u \in K^\infty, u \neq 0$ satisfies the compatibility condition on $\emptyset \neq J \subseteq P^* \backslash \{0\}$, if

$$h_q^\infty(u) = 0, \ \forall \ q \in J.$$

Lemma 4.1 *Let P be such that $P^* \backslash \{0\} = \text{co}(\text{extrd } P^*)$ with $\text{extrd } P^*$ being a finite set, that is, P is a polyhedral cone. If there exists $z \in K, \ q_0 \in \text{int } P^*$ such that $\text{argmin}_{S(z)} h_{q_0} \neq \emptyset$*

and $u \in (\mathrm{argmin}_{S(z)} h_{q_0})^{\infty}, u \neq 0$, *satisfies the compatibility condition on* extrd P^*, *then* $E \neq \emptyset$, *unbounded and*

$$\left(\underset{S(z)}{\mathrm{argmin}} h_{q_0}\right)^{\infty 2}[u] \subseteq E^{\infty 2}[u] \subseteq (E_W)^{\infty 2}[u] \subseteq \bigcup_{q \in \mathrm{extrd}\ P^*} \{w \in K^{\infty 2}[u]:\ h_q^{\infty 2}(u,w) \leq 0\}.$$

Proof. Lemma 1 in [38] implies that $\emptyset \neq (\mathrm{argmin}_{S(z)} h_{q_0}) \subseteq E$. Since $u \neq 0$ then E is unbounded, and Proposition 3.1 (a) implies the first inclusion (and the second is obvious).

For the third inclusion, assume that extrd $P^* = \{q_i:\ i = 1, \ldots, p\}$. Let $w \in (E_W)^{\infty 2}[u]$, then there exists $\{x_k\}_{k \in \mathbb{N}} \subseteq E_W, s_k, t_k \to +\infty$ with $w = \lim_{k \to +\infty} \left(\frac{x_k}{s_k} - t_k u\right)$.

(i): If $\liminf_{k \to +\infty} h_{q_{i_0}}(x_k) \leq \alpha < +\infty$ for some $i_0 \in \{1, 2, \ldots, p\}$, then there exists a subsequence $\{x_k^1\}$ such that $\lim_{k \to +\infty} \left(\frac{h_{q_{i_0}}(x_k^1)}{s_k^1} - t_k^1 h_{q_{i_0}}^{\infty}(u)\right) = 0$, then

$$\{x \in K:\ h_{q_{i_0}}(x) \leq \alpha\}^{\infty 2}[u] \subseteq \{w \in K^{\infty 2}[u]:\ h_{q_{i_0}}^{\infty 2}(u; w) \leq 0\}$$

$$\subseteq \bigcup_{i=1}^{p} \{w \in K^{\infty 2}[u]:\ h_{q_i}^{\infty 2}(u; w) \leq 0\}.$$

(ii): If on the contrary, for all $i = 1, 2, \ldots, p$, we have $\liminf\limits_{k \to +\infty} h_{q_i}(x_k) = +\infty$, then $\lim\limits_{k \to +\infty} h_{q_i}(x_k^1) = +\infty$, for all $i = 1, 2, \ldots, p$.

For $\bar{k} \in \mathbb{N}$ fixed and any $i \in \{1, 2, \ldots, p\}$ there exists $k_i \in \mathbb{N}$ such that $h_{q_i}(x_k^1) > h_{q_i}(x_{\bar{k}}^1)$ for all $k \geq k_i$. That is, $\langle q_i, F(x_{\bar{k}}^1) - F(x_k^1)\rangle < 0$ for all $k \geq k_i$ and for all $i = 1, 2, \ldots, p$. Let $k_0 = \max_{1 \leq i \leq p} k_i$, then for all $i = 1, 2, \ldots, p$, we have

$$\langle q_i, F(x_{\bar{k}}^1) - F(x_k^1)\rangle < 0, \quad \forall \ \ k \geq k_0.$$

This implies that $F(x_{\bar{k}}^1) - F(x_k^1) \in -\mathrm{int}\ P$ which cannot happen if $x_k^1 \in E_W$, proving that case (i) is only possible. ∎

We recall the following (first order) asymptotic cone introduced in [32] to define their extension to second order asymptotic analysis,

$$R_P \doteq \bigcap_{y \in K} \{u \in K^{\infty}: \quad F(y + \lambda u) - F(y) \in -P, \ \ \forall\ \lambda > 0\}. \tag{4.12}$$

In the following, we consider the closed convex cone P such that $P^* \backslash \{0\} = \mathrm{co}(\mathrm{extrd}\ P^*)$ is not necessarily polyhedral.

Definition 4.5 *Given* $u \in K^{\infty}$ *with* $h_q^{\infty}(u) \in \mathbb{R}$ *for all* $q \in J$, *we say that a vector* v *belongs* *to the P-second order asymptotic cone at u, denoted by $v \in R_P^2[u]$, if*

$$\forall\ y \in K,\ \forall\ s > 0,\ \exists\ \bar{t} > 0 : h_q(y + tu + sv) - th_q^{\infty}(u) \le h_q(y),\ \forall\ q \in J,\ \forall\ t \ge \bar{t}. \quad (4.13)$$

If u satisfies the compatibility condition, then (4.13) can be expressed by

$$\forall\ y \in K,\ \forall\ s > 0,\ \exists\ \bar{t} > 0 :\ h_q(y + tu + sv) \le h_q(y),\ \forall\ q \in J,\ \forall\ t \ge \bar{t}. \quad (4.14)$$

We present the following inner estimates for the second order asymptotic cone of the efficient solution set.

Lemma 4.2 *Let P be a closed convex cone. If $u \in K^{\infty}, u \neq 0$ satisfies the compatibility condition on* extrd P^* *and* ri $E \neq \emptyset$*, then $u \in E^{\infty}$ (i.e. E is unbounded) and*

$$R_P^2[u] \subseteq E^{\infty 2}[u].$$

Proof. Let $v \in R_P^2[u]$ and $\overline{x} \in \text{ri } E$. Take any $y \in K$ and $s > 0$, then there exists $\bar{t} > 0$ such that

$$F(y) - F(\overline{x} + tu + sv) = F(y) - F(\overline{x}) + F(\overline{x}) - F(\overline{x} + tu + sv).$$

Thus, $F(y) - F(\overline{x}) + F(\overline{x}) - F(\overline{x} + tu + sv) \in \mathbb{R}^m \setminus (-P \setminus l(P)) + P \subseteq \mathbb{R}^m \setminus (-P \setminus l(P))$, for all $t \ge \bar{t}$, which proves that $u \in E^{\infty}$ (i.e. E is unbounded) and $v \in E^{\infty 2}[u]$. ■

We only work with the case $u \neq 0$, since if $u = 0$ satisfies the compatibility condition on $J = P^* \setminus \{0\}$, then $R_P^2[0] = R_P$ and Lemma 4.2 coincides with Lemma 7 in [38].

Remark 4.3 *If h_q is convex for all $q \in J \subseteq P^* \setminus \{0\}$ and the conditions of the Lemma 4.2 hold, then*

$$\bigcap_{q \in J} \{v \in K^{\infty 2}[u] :\ h_q^{\infty 2}(u; v) \le 0\} \subseteq E^{\infty 2}[u]. \quad (4.15)$$

In addition, if int $P \neq \emptyset$*, then*

$$\bigcap_{q \in J} \{v \in K^{\infty 2}[u] :\ h_q^{\infty 2}(u; v) \le 0\} \subseteq E_W^{\infty 2}[u]. \quad (4.16)$$

By Proposition 3.19 we know that for every proper convex function h_q we have that $h_q^{\infty 2}(u; v) \le h_q^{\infty}(v)$ for all $v \in \text{dom } h_q^{\infty}$. Then (4.15) and (4.16) are finer inner estimates than Remark 5 in [38].

The following example shows that, even in the case when $u \in K^{\infty}, u \neq 0$ and the compatibility condition is not satisfied, we still have a better inner estimate for the efficient solution set with second order asymptotic cones than the estimates given by the first order.

Example 4.6 *Let $K = [0, +\infty[$, $P = \mathbb{R}^2_+$ and $f_1(x) = x^2, f_2(x) = -\sqrt{x}$. Here $f_1, f_2 : K \to \mathbb{R} \cup \{+\infty\}$ are convex, proper and continuous functions. $E = [0, +\infty[$ and*

$$f_1^\infty(u) = \begin{cases} 0 & \text{if } u = 0, \\ +\infty & \text{if } u \neq 0, \end{cases} \quad \textit{and } f_2^\infty(u) = \begin{cases} 0 & \text{if } u \geq 0, \\ +\infty & \text{if } u < 0, \end{cases}$$

Then,

$$\bigcap_{q \in P^* \backslash \{0\}} \{u \in K^\infty : \; h_q^\infty(u) \leq 0\} = \{0\}.$$

On the other hand, for $u = 1$ we have that $f_1^{\infty 2}(1; v) = f_2^{\infty 2}(1; v) = -\infty$, for all $v > 0$, then

$$\bigcap_{q \in P^* \backslash \{0\}} \{v \in K^{\infty 2}[1] : \; h_q^{\infty 2}(1; v) \leq 0\} = [0, +\infty[= E^\infty = E^{\infty 2}[1].$$

Next lemma gives us an outer second order estimate.

Lemma 4.3 *Assume that P is a closed convex cone. Suppose $A \neq \emptyset$ and $u \in A^\infty$ satisfies the compatibility condition on J. If*

$$\sup_{x \in A} \; \sup_{q \in J} \; h_q(x) \; < \; +\infty \tag{4.17}$$

then

$$A^{\infty 2}[u] \subseteq \bigcap_{q \in J} \{w \in K^{\infty 2}[u] : \; h_q^{\infty 2}(u; w) \leq 0\}.$$

Proof. Let $w \in A^{\infty 2}[u]$, then there exist $x_k \in A$ and sequences $t_k, s_k \to +\infty$ such that $w = \lim_{k \to +\infty} \left(\frac{x_k}{s_k} - t_k u \right)$. By (4.17) there exists $M > 0$ such that $h_q(x_k) \leq M$ for all $k \in \mathbb{N}$ and $q \in J \subseteq P^* \backslash \{0\}$. If $q_0 \in J \subseteq P^* \backslash \{0\}$ is any index, since u satisfies the compatibility condition, thus

$$\lim_{k \to \infty} \left(\frac{h_{q_0}(x_k)}{s_k} - t_k h_{q_0}^\infty(u) \right) \leq \lim_{k \to \infty} \frac{M}{s_k} = 0.$$

Hence $h_q^{\infty 2}(u; w) \leq 0$ for all $q \in J \subseteq P^* \backslash \{0\}$. ■

Remark 4.4 *A condition like* (4.17) *was first introduced in [21] by Deng. Others authors used later in [38, 44].*

We are in position to give a complete characterization of the second order asymptotic cone of the efficient and weakly efficient solution sets.

Theorem 4.5 *Consider* $J = \text{extrd } P^*$ *and* h_q *be a proper, convex and lsc function for all* $q \in \text{extrd } P^*$*. Then,*

(*a*) *If* $E \neq \emptyset$ *and* (4.17) *holds for* $A = E$*, then*

$$\bigcap_{q\in\text{extrd } P^*} \{u \in K^\infty : \ h_q^\infty(u) \leq 0\} = E^\infty.$$

In addition, if ri $E \neq \emptyset$ *and* $u \in K^\infty, u \neq 0$, *satisfies the compatibility condition on* extrd P^**, then* $u \in E^\infty, E$ *is unbounded and*

$$\bigcap_{q\in\text{extrd } P^*} \{w \in K^{\infty 2}[u] : \ h_q^{\infty 2}(u; w) \leq 0\} = E^{\infty 2}[u].$$

(*b*) *Suppose* int $P \neq \emptyset$*. If* $E_W \neq \emptyset$ *and* (4.17) *holds for* $A = E_W$*, then*

$$\bigcap_{q\in\text{extrd } P^*} \{u \in K^\infty : \ h_q^\infty(u) \leq 0\} = (E_W)^\infty.$$

In addition, if ri $E_W \neq \emptyset$ *and* $u \in K^\infty, u \neq 0$, *satisfies the compatibility condition on* extrd P^**, then* $u \in E_W^\infty, E_W$ *is unbounded and*

$$\bigcap_{q\in\text{extrd } P^*} \{w \in K^{\infty 2}[u] : \ h_q^{\infty 2}(u; w) \leq 0\} = (E_W)^{\infty 2}[u].$$

Proof. For E^∞ and $(E_W)^\infty$ see Theorem 1 in [38]. Characterizations for the second order are consequences from Lemma 4.2 and Lemma 4.3. ■

Suppose (4.17) holds for $J = \text{extrd } P^*$ and $\emptyset \neq \text{ri } E \subseteq E$. If h_q is convex, proper and lsc for all $q \in J = \text{extrd } P^*$, part (*a*) of Theorem 4.5 implies that

$$\bigcap_{q\in\text{extrd } P^*} \{w \in K^\infty : \ h_q^\infty(w) \leq 0\} \subseteq \bigcap_{q\in\text{extrd } P^*} \{w \in K^{\infty 2}[u] : \ h_q^{\infty 2}(u; w) \leq 0\},$$

that is $E^\infty \subseteq E^{\infty 2}[u]$ for all nonzero vector $u \in K^\infty$ satisfying the compatibility condition on $J = \text{extrd } P^*$. In addition, if int $P \neq \emptyset$ then $E_W^\infty \subseteq E_W^{\infty 2}[u]$ for all nonzero vector $u \in K^\infty$ satisfying the compatibility condition on $J = \text{extrd } P^*$.

We actually proved the next corollary.

Corollary 4.2 *Assume that* h_q *is convex, proper and lsc for all* $q \in \text{extrd } P^*$*. Then the following assertions hold;*

(a) *If* ri $E \neq \emptyset$ *and condition* (4.17) *holds for* $A = E$*, then*

$$E^{\infty} \subseteq E^{\infty 2}[u],$$

for all $u \in K^{\infty}$ *satisfying the compatibility condition on* $J = \text{extrd } P^*$.

(b) *If* int $P \neq \emptyset$, ri $E_W \neq \emptyset$ *and condition* (4.17) *holds for* $A = E_W$*, then*

$$E_W^{\infty} \subseteq E_W^{\infty 2}[u],$$

for all $u \in K^{\infty}$ *satisfying the compatibility condition on* $J = \text{extrd } P^*$.

Notice that even if E or E_W are not necessarily convex, we still have the inclusion given by Proposition 3.6 part (b).

Chapter 5

Second order asymptotic analysis in Banach spaces

In this chapter we try to extend the definition of second order asymptotic cone and function to infinite dimensional spaces. In particular, we work in a reflexive Banach space. We develop the basic properties of Chapter 3 for cones and functions under an additional assumption. We also extend, in some sense, a classical existence result for non coercive convex functions.

5.1 Preliminary results

Let V be a reflexive Banach space and $K \subseteq V$ be a nonempty set. Its strong closure is denoted by $\overline{K}$, its boundary by bd K, its topological interior by int K, its relative interior by ri K, and its convex hull by co(K). We set cone$(K) = \bigcup_{t \geq 0} tK$ being the smallest cone containing K and $\overline{\text{cone}}(K) = \overline{\bigcup_{t \geq 0} tK}$.

For any given weakly closed set K in V (actually, the asymptotic notion to be considered is blind to weak closure), we define the asymptotic cone of K as the weakly closed set

$$K^{\infty} = \{u \in V : \exists\, t_n \downarrow 0, \exists\, x_n \in K : t_n x_n \rightharpoonup u\}.$$

Here " $\rightharpoonup$" stands for the weak convergence. We set $\emptyset^{\infty} = \emptyset$. For any given function $F : V \to \mathbb{R} \cup \{+\infty\}$, the asymptotic function of F is defined as the function F^{∞} such that

$$\text{epi } F^{\infty} = (\text{epi } F)^{\infty}.$$

Consequently, by Remark 2.17 in [10] we have that

$$F^\infty(u) = \inf \left[\liminf_{n\to+\infty} \ t_n F\left(\frac{x_n}{t_n}\right) : \ t_n \downarrow 0, \ x_n \rightharpoonup u \right].$$

In the case when K is convex and closed it is well-known, by Proposition 15.1.5 in [5], that for all $x_0 \in K$

$$K^\infty = \{u \in V : \ x_0 + \lambda u \in K, \ \forall\, \lambda > 0\},$$

when F is a convex and lsc function, by Proposition 2.5 in [10], for all $x_0 \in \mathrm{dom}\, F$, we have

$$F^\infty(u) = \lim_{\lambda\to+\infty} \frac{F(x_0+\lambda u) - F(x_0)}{\lambda} = \sup_{\lambda>0} \frac{F(x_0+\lambda u) - F(x_0)}{\lambda} \tag{5.1}$$

where, as usual, $\mathrm{dom}\, F = \{u \in V : \ F(u) < +\infty\}$ and the epigraph of F is the set $\mathrm{epi}\, F = \{(u,t) \in V \times \mathbb{R} : \ F(u) \leq t\}$.

Recall that for a convex set, sequentially weakly closed is equivalent to closed, and for a convex function F, sequentially weakly lsc is equivalent to lsc.

We list some basic results on asymptotic cones that will be useful in what follows to understand properties of the second order asymptotic cones. Their proofs can be found in Proposition 15.1.5 from [5].

Proposition 5.1 *Let $K \subseteq V$ be a nonempty set. Then the following assertions hold:*

(*a*) *If $K_1 \subseteq K_2$ then $K_1^\infty \subseteq K_2^\infty$.*

(*b*) *$(K+x)^\infty = K^\infty$ for all $x \in V$.*

(*c*) *Let $\{K_i\}_{i\in I} \subseteq V$ be any family of nonempty sets, then*

$$\left(\bigcap_{i\in I} K_i\right)^\infty \subseteq \bigcap_{i\in I} K_i^\infty.$$

In addition, if $\cap_{i\in I} K_i \neq \emptyset$ and each set K_i is closed and convex, then equality holds.

Some important properties of the asymptotic function F^∞ of F are described in the next proposition, their proofs can be found in Proposition 15.1.1. from [5].

Proposition 5.2 *Let $F : V \to]-\infty, +\infty]$ be a proper, convex and lsc function, then:*

(*a*) *F^∞ is proper, convex, lsc, and positively homogeneous.*

(b) *For every $F_1, F_2, \ldots, F_k$ proper, convex and lsc functions such that $\cap_{i=1}^k \mathrm{dom}\ F_i \neq \emptyset$, one has*

$$\left(\sum_{i=1}^k F_i\right)^\infty = \sum_{i=1}^k F_i^\infty.$$

(c) $F^\infty(u) + F^\infty(-u) \geq 0$, *for every $u \in V$.*

We use the Proposition 3.4 of Chapter 3 to define directly the second order directions in reflexive Banach spaces.

Definition 5.1 *Given a nonempty, closed convex set $K \subseteq V$ with $\mathrm{int}\ K \neq \emptyset$ and $u \in K^\infty$. We say v is a second order asymptotic direction of K at u if*

$$\forall\ s > 0,\ \exists\ \bar{t} > 0 :\ x_0 + tu + sv \in K,\ \forall\ t > \bar{t},\ \forall\ x_0 \in \mathrm{int}\ K.$$

The set of all such elements v is denoted by $K^{\infty 2}[u]$. Furthermore, only a single element $x_0 \in \mathrm{int}\ K$ is necessary in the definition. Indeed, for every $x_0 \in K$ set

$$L(x_0) = \{v \in V :\ \forall\ s > 0,\ \exists\ \bar{t} > 0,\ x_0 + tu + sv \in K,\ \forall\ t > \bar{t}\}.$$

Let $x_0, x_1 \in \mathrm{int}\ K$ and $v \in L(x_0)$. We will show that $v \in L(x_1)$. Since $x_1 \in \mathrm{int}\ K$, the line through x_0 and x_1 contains an element $x_2 \in K$ such that $x_1 \in]x_0, x_2[$, so there exists $\lambda \in]0, 1[$ such that $x_1 = \lambda x_0 + (1-\lambda)x_2$. From $v \in L(x_0)$ we know that for all $s > 0$ (and in particular for $\frac{s}{\lambda}$) there exists $\bar{t} > 0$ such that $x_0 + tu + \frac{s}{\lambda}v \in K$, for all $t > \bar{t}$. By convexity, $\lambda\left(x_0 + tu + \frac{s}{\lambda}v\right) + (1-\lambda)\, x_2 \in K$, then $x_1 + \lambda tu + sv \in K$, for all $t > \bar{t}$, this implies that $x_1 + t'u + sv \in K$, for all $t' > \lambda\bar{t}$, so $L(x_0) \subseteq L(x_1)$. By the same argument, we can prove the other inclusion.

The set $K^{\infty 2}[u]$ is a cone, termed the second order asymptotic cone of K at u. It is nonempty exactly when $u \in K^\infty$, if $u = 0$ then $K^{\infty 2}[0] = K^\infty$.

Remark 5.1 *For every nonempty convex set $K \subseteq V$ with $\mathrm{int}\ K \neq \emptyset$ and $u \in K^\infty$, the cone $K^{\infty 2}[u]$ is convex. In fact, let $x \in \mathrm{int}\ K$ and $v_1, v_2 \in K^{\infty 2}[u]$, then for every $s > 0$ there exists $t_1, t_2 > 0$ such that $x + tu + sv_i \in K$ for all $t \geq t_i$. Since K is convex, for every $t > \max\{t_1, t_2\}$ and $\lambda \in]0, 1[$ we have that $x + tu + s((1-\lambda)v_1 + \lambda v_2) \in K$. Thus $(1-\lambda)v_1 + \lambda v_2 \in K^{\infty 2}[u]$ and $K^{\infty 2}[u]$ is convex.*

Proposition 5.3 *Let $K \subseteq V$ be a closed convex set with $\mathrm{int}\ K \neq \emptyset$, then*

(a) *If $K_0 \subseteq K$ is a closed convex set with $\mathrm{int}\ K_0 \neq \emptyset$, then $(K_0)^{\infty 2}[u] \subseteq K^{\infty 2}[u]$ for all $u \in (K_0)^\infty$.*

(*b*) $(K + x)^{\infty 2}[u] = K^{\infty 2}[u]$, *for all* $u \in K^\infty$ *and* $x \in V$.

(*c*) $K^{\infty 2}[u] + \mathbb{R}u = K^{\infty 2}[u]$, *for all* $u \in K^\infty$.

(*d*) *Let* $\{K_i\}_{i\in I} \subseteq V$ *be a family of closed convex sets with* $\text{int } K_i \neq \emptyset$ *for all* $i \in I$. *If* $u \in \bigcup_{i\in I} K_i^\infty$ *then*

$$\bigcup_{i\in I}(K_i)^{\infty 2}[u] \subseteq (\bigcup_{i\in I} K_i)^{\infty 2}[u],$$

and the equality hold when $|I| < +\infty$.

(*e*) *Let* $\{K_i\}_{i\in I} \subseteq V$ *be a finite family of closed convex sets such that* $\bigcap_{i\in I} \text{int } K_i \neq \emptyset$. *If* $u \in (\bigcap_{i\in I} K_i)^\infty$ *then*

$$(\bigcap_{i\in I} K_i)^{\infty 2}[u] \subseteq \bigcap_{i\in I}(K_i)^{\infty 2}[u],$$

and the equality hold when I *is finite.*

Proof. Their proofs are similar to those of the finite dimensional case, see Chapter 3 for the details. ■

We present the definition of the second order asymptotic function for a proper, convex and lsc function, based on results presented in Chapter 3.

Definition 5.2 *Let* $F : V \to]-\infty, +\infty]$ *be a proper, convex, lsc function with* $\text{intdom } F \neq \emptyset$. *Let* $u \in V$ *with* $F^\infty(u) \in \mathbb{R}$, *the second order asymptotic function* $F^{\infty 2}$ *of* F *at* u *is defined by the function which satisfies that*

$$\text{epi } F^{\infty 2}(u; \cdot) = (\text{epi F})^{\infty 2}[(u, F^\infty(u))]. \tag{5.2}$$

Since $(\text{epi } F)^{\infty 2}$ is always a cone, then the function $F^{\infty 2}(u; \cdot)$ is positively homogeneous. The convexity follows immediately from Remark 5.1 because epi $F^{\infty 2}(u; \cdot)$ is a convex set. Finally, in the special case when $u = 0$, (5.2) implies that $F^{\infty 2}(0; v) = F^\infty(v)$ for all $v \in V$.

We would like to obtain an easy formula for the second order asymptotic function analogous to (5.1). To this end, we will need the next result.

Lemma 5.1 *Let* $F : V \to]-\infty, +\infty[$ *be a proper, convex and lsc function. If* $x_0 \in \text{intdom } F$ *and* $\overline{\mu} \in \mathbb{R}$ *is such that* $F(x_0) < \overline{\mu}$, *then* $(x_0, \overline{\mu}) \in \text{intepi } F$.

Proof. Let $x_0 \in \text{intdom } F$ and $\overline{\mu} \in \mathbb{R}$ such that $\overline{\mu} > F(x_0)$. By Corollary 2.5 in [28] we have that F is continuous at x_0.

Take α such that $F(x_0) < \alpha < \overline{\mu}$, then there exists $\varepsilon > 0$ such that $F(v) < \alpha$ for all $v \in B(x_0, \varepsilon)$. This means that $B(x_0, \varepsilon) \times]\alpha, +\infty[\subseteq \text{epi } F$ is an open neighborhood of $(x_0, \overline{\mu})$, so $(x_0, \overline{\mu}) \in \text{intepi } F$. ■

We now establish some useful monotonicity properties.

Lemma 5.2 *Let $F : V \to]-\infty, +\infty[$ be a proper, convex and lsc function.*

(a) *For every $x_0 \in \text{intdom } F$ and u such that $F^\infty(u)$ is finite, the function $G(t) := F(x_0 + tu) - tF^\infty(u)$ is decreasing on $\mathbb{R}$.*

(b) *If $(y, \delta) \in \text{epi } F$, then for every $v \in \mathbb{R}^n$ the function $s \to \frac{F(y+sv)-\delta}{s}$ is increasing on $]0, +\infty[$.*

(c) *Let $x_0 \in \text{intdom } F$, $\beta \geq F(x_0)$. Let u such that $F^\infty(u)$ be finite and $v \in (\text{dom } F)^{\infty 2}[u]$. If we set*

$$k_\beta(s,t) = \frac{F(x_0 + tu + sv) - tF^\infty(u) - \beta}{s}, \quad s > 0, t > 0 \tag{5.3}$$

then the function $s \to \lim_{t \to +\infty} k_\beta(s,t)$ is increasing. Consequently for all $\beta \geq F(x_0)$,

$$\lim_{s \to +\infty} \lim_{t \to +\infty} k_{F(x_0)}(s,t) = \lim_{s \to +\infty} \lim_{t \to +\infty} k_\beta(s,t) = \sup_{s>0} \inf_{t>0} k_\beta(s,t) = \sup_{s>0} \inf_{t>0} k_{F(x_0)}(s,t). \tag{5.4}$$

Proof. (a) Let $t' > t > 0$. Since $x_0 \in \text{intdom } F$, we have $x_0 + tu \in \text{intdom } F$ and $x_0 + t'u \in \text{intdom } F$. Setting $x_1 = x_0 + tu$, we know that

$$\frac{F(x_0 + t'u) - F(x_0 + tu)}{t' - t} = \frac{F(x_1 + (t' - t)u) - F(x_1)}{t' - t} \leq F^\infty(u).$$

From this we obtain $F(x_0 + t'u) - t'F^\infty(u) \leq F(x_0 + tu) - tF^\infty(u)$, then G is decreasing on $]0, +\infty[$, and since G is convex, it is decreasing on $\mathbb{R}$.

(b) The function is the sum of two increasing functions:

$$\frac{F(y + sv) - \delta}{s} = \frac{F(y + sv) - F(y)}{s} + \frac{F(y) - \delta}{s}.$$

(c) Using (a) we deduce that for every $s > 0$, $\lim_{t \to +\infty} k_\beta(s,t)$ exists and is equal to $\inf_{t>0} k_\beta(s,t)$.

Let $s' > s > 0$. By using Definition 5.1 on $(\text{dom } F)^{\infty 2}[u]$ we deduce that there exists $\bar{t} > 0$ such that for all $t \geq \bar{t}$, $x_0 + tu + sv \in \text{dom } F$ and $x_0 + tu + s'v \in \text{dom } F$. Since $(u, F^\infty(u)) \in (\text{epi } F)^\infty$ then we have $(x_0 + tu, \beta + tF^\infty(u)) = (x_0, \beta) + t(u, F^\infty(u)) \in \text{epi } F$.

Using (b) for $y = x_0 + tu$, $\delta = \beta + tF^\infty(u)$ we obtain

$$\frac{F(x_0 + tu + sv) - tF^\infty(u) - \beta}{s} \leq \frac{F(x_0 + tu + s'v) - tF^\infty(u) - \beta}{s'}, \ \forall\, t \geq \bar{t}.$$

Taking the limit as $t \to +\infty$ then $\lim\limits_{t\to+\infty} k_\beta(s,t) \leq \lim\limits_{t\to+\infty} k_\beta(s',t)$, that is, the function $s \to \lim_{t\to+\infty} k_\beta(s,t)$ is increasing. Thus,

$$\lim_{s\to+\infty} \lim_{t\to+\infty} k_\beta(s,t) = \sup_{s>0} \inf_{t>0} k_\beta(s,t), \ \forall\, \beta \geq F(x_0). \tag{5.5}$$

On the other hand, it is clear that

$$\lim_{s\to+\infty} \lim_{t\to+\infty} \frac{F(x_0 + tu + sv) - tF^\infty(u) - \beta}{s} = \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{F(x_0 + tu + sv) - tF^\infty(u)}{s} \tag{5.6}$$

hence $\lim_{s\to+\infty} \lim_{t\to+\infty} k_\beta(s,t) = \lim_{s\to+\infty} \lim_{t\to+\infty} k_{F(x_0)}(s,t)$. From this and (5.5) we deduce the equalities (5.4). ■

The following proposition gives us two interesting formulas for an easy calculus of the second order asymptotic function.

Proposition 5.4 *Let $F : V \to]-\infty, +\infty]$ be a proper, convex and lsc function such that* $\operatorname{intdom} F \neq \emptyset$. *For every u such that $F^\infty(u)$ is finite and $v \in (\operatorname{dom} F)^{\infty 2}[u]$,*

$$F^{\infty 2}(u;v) = \sup_{s>0} \inf_{t>0} \frac{F(x_0 + tu + sv) - tF^\infty(u) - F(x_0)}{s} \tag{5.7}$$

$$F^{\infty 2}(u;v) = \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{F(x_0 + tu + sv) - tF^\infty(u) - F(x_0)}{s}, \tag{5.8}$$

for any $x_0 \in \operatorname{intdom} F$.

Proof. Take $x_0 \in \operatorname{intdom} F$ and $\beta > F(x_0)$, then by Lemma 5.1 we have that $(x_0, \beta) \in \operatorname{intepi} F$. Define

$$k_\beta(s,t) = \frac{F(x_0 + tu + sv) - tF^\infty(u) - \beta}{s}, \ s > 0, t > 0. \tag{5.9}$$

We show that $\sup_{s>0} \inf_{t>0} k_\beta(s,t) \leq F^{\infty 2}(u;v)$ by showing that for every $\alpha \in \mathbb{R}$, $F^{\infty 2}(u;v) \leq \alpha$ implies $\sup_{s>0} \inf_{t>0} k_\beta(s,t) \leq \alpha$.

Since epi F is convex, by using Definition 5.1 we have the following implications:

$$\begin{aligned}
F^{\infty 2}(u;v) \le \alpha &\Rightarrow (v,\alpha) \in (\text{epi } F)^{\infty 2}\left[(u, F^{\infty}(u))\right] \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, (x_0,\beta) + t(u,F^{\infty}(u)) + s(v,\alpha) \in \text{epi } F \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, (x_0+tu+sv, \beta + tF^{\infty}(u) + s\alpha) \in \text{epi } F \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, F(x_0+tu+sv) \le \beta + tF^{\infty}(u) + s\alpha \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, k_\beta(s,t) \le \alpha \\
&\Rightarrow \sup_{s>0}\inf_{t>0} k_\beta(s,t) \le \alpha.
\end{aligned}$$

We now show that $F^{\infty 2}(u;v) \le \sup_{s>0}\inf_{t>0} k_\beta(s,t)$ by showing that for every $\alpha \in \mathbb{R}$, we have that $\sup_{s>0}\inf_{t>0} k_\beta(s,t) < \alpha$ implies $F^{\infty 2}(u;v) \le \alpha$. Following the previous implications in reverse order, we obtain

$$\begin{aligned}
\sup_{s>0}\inf_{t>0} k_\beta(s,t) < \alpha &\Rightarrow \forall\ s>0,\ \exists\ t>0, k_\beta(s,t) < \alpha \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, (x_0+tu+sv, \beta + tF^{\infty}(u) + s\alpha) \in \text{epi } F \\
&\Rightarrow \forall\ s>0,\ \exists\ t>0, (x_0,\beta) + t(u,F^{\infty}(u)) + s(v,\alpha) \in \text{epi } F \\
&\Rightarrow (v,\alpha) \in (\text{epi } F)^{\infty 2}[(u,F^{\infty}(u))] \Rightarrow F^{\infty 2}(u;v) \le \alpha.
\end{aligned}$$

It follows that $F^{\infty 2}(u;v) = \sup_{s>0}\inf_{t>0} k_\beta(s,t)$ for every $\beta > F(x_0)$. Using (5.4) we deduce equalities (5.7) and (5.8). ∎

An easy application of the previous formula is the following result.

Proposition 5.5 *Consider a finite number of proper, convex, lsc functions $F_1, F_2, \dots, F_k$ with $\cap_{i=1}^k \text{intdom } F_i \neq \emptyset$, one has*

$$\left(\sum_{i=1}^k F_i\right)^{\infty 2} = \sum_{i=1}^k F_i^{\infty 2}.$$

Proof. Let $x_0 \in \cap_{i=1}^k \text{intdom } F_i$, and $u, v \in V$ then

$$\sum_{i=1}^k F_i^{\infty 2}(u;v) = \sum_{i=1}^k \lim_{s\to+\infty}\lim_{t\to+\infty} \frac{F_i(x_0+tu+sv) - tF_i^{\infty}(u) - F_i(x_0)}{s}$$

$$= \lim_{s\to+\infty}\lim_{t\to+\infty} \sum_{i=1}^k \frac{F_i(x_0+tu+sv) - tF_i^{\infty}(u) - F_i(x_0)}{s} = (\sum_{i=1}^k F_i)^{\infty 2}(u;v),$$

by Proposition 5.2 part (b). ∎

5.2 The convex minimization problem

In this section, we will treat non coercive convex minimizations problems. We will obtain sufficient and necessary conditions for the existence of an optimal point provided some compatibility conditions hold.

Consider the problem

$$\min_{v\in V} F(v), \tag{5.10}$$

and the level set minimization problem, given $p \in V$ and $V_p = \{x \in V : \ F(x) \leq F(p)\}$, defined by

$$\min_{x\in V_p} F(x). \tag{5.11}$$

The following conditions appear frequently in the literature,

$(F1)$ $F : V \to]-\infty, +\infty]$ is a proper convex and lsc function.

$(F2)$ If $t_k \to +\infty, v_k \rightharpoonup v$ and $\{F(t_k v_k)\}_{k\in\mathbb{N}}$ is bounded from above, then $\|v_k - v\| \to 0$.

Condition $(F2)$ is satisfied vacuously when dim $V < +\infty$. It is used very often for elasticity problems in mechanics, see [2, 5, 6, 10, 41].

The following result shows two necessary conditions for the function F to be bounded from below.

Proposition 5.6 *(Theorem 3.3 in [45]) Suppose F satisfies* $(F1)$. *If $m = \inf_{v\in V} F(v) > -\infty$, then*

$(A1)$ $F^{\infty}(u) \geq 0$ *for all* $u \in V$.

$(A2)$ *In case that* intdom $F \neq \emptyset$. *If* $F^{\infty}(u) = 0$, *then* $F^{\infty 2}(u; v) \geq 0$ *for all* $v \in V$.

Proof. We only prove $(A2)$, the proof of $(A1)$ can be found in [5]. Let $u \in V, u \neq 0$ be such that $F^{\infty}(u) = 0$, take $v \in V$ and $x_0 \in$ intdom F, then

$$F^{\infty 2}(u; v) = \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{F(x_0 + tu + sv) - F(x_0)}{s} \geq \lim_{s\to+\infty} \lim_{t\to+\infty} \frac{m - F(x_0)}{s} = 0,$$

and the proof is complete. ∎

Analogously to Corolario 4.1 we can prove the next result for the infinite dimensional case.

Corollary 5.1 *Let F be as before. Given $p \in V$, if the problem* (5.11) *is bounded from below, then*

$$F^{\infty 2}(u; v) = 0, \ \forall \ u \in (V_p)^{\infty} \backslash \{0\} \text{ with } F^{\infty}(u) = 0 \text{ and } v \in (V_p)^{\infty 2}[u]. \tag{5.12}$$

By an easy application of Proposition 3.19, the necessary condition (5.12) is true for the global optimization problem (5.10) when the second order asymptotic direction is the same as the first order direction. We do not have the same property for other second-order asymptotic directions even in finite dimensional spaces. Furthermore, both necessary conditions do not imply that F is bounded from below as we seem in Ejemplo 4.1.

Recall Theorem 15.1.1 from [5].

Theorem 5.1 *Let V be a reflexive and separable Banach space with norm $\|\cdot\|$, and let $F : V \to]-\infty, +\infty]$ be a function satisfying* $(F1)$ *and* $(F2)$. *If the following conditions are satisfied:*

$(A1)$ *$F^{\infty}(v) \geq 0$ for all $v \in V$,*

$(A3)$ *the set $R_0 = \{v \in V : \ F^{\infty}(v) = 0\}$ is a linear subspace of V,*

then the problem (5.10) *admits at least a solution.*

$(A1)$ and $(A3)$ imply that F is weakly coercive. But, a condition like $(A3)$ is not natural in many situations, because physical and mechanicals problems are interpreted frequently by time-dependend convex functions, and therefore, they are only defined for the positive axis. These functions can be extended to the whole space through the indicator function, but a condition like $(A3)$ cannot be satisfied in those cases.

To obtain an existence result, we introduce the following condition related to the one presented in [6], which are a slight modification of that used in [10].

$(F3)$: If the sequence $v_k \in \text{dom } F$, with $\|v_k\| \to +\infty$ is such that $\frac{v_k}{\|v_k\|} \rightharpoonup v \in R_0$ and for all $z \in \text{dom } F$, there exists $k_z \in \mathbb{N}$ such that

$$F(v_k) \leq F(z), \text{ when } k \geq k_z,$$

then there exists $\overline{v} \in \text{dom } F$ and $\overline{k} \in \mathbb{N}$ such that $\|\overline{v}\| < \|v_{\overline{k}}\|$ and $F(\overline{v}) \leq F(v_k), \ \forall \ k \geq \overline{k}$.

Recall that, the fact that a unit norm sequence $w_k = \frac{v_k}{\|v_k\|}$ converges weakly to a vector v does not imply that the vector v is not the null one.

We present the main result of this section.

Theorem 5.2 *Let V be a reflexive Banach space with norm $\|\cdot\|$ and let $F : V \to]-\infty, +\infty]$ be a function satisfying $(F1)$, then the following assertions are equivalent*

(a) *$(A1)$ and $(F3)$ hold.*

(b) *The problem (5.10) has a solution.*

Proof. ($\Rightarrow$): Suppose condition $(A1)$ and $(F3)$ are satisfied, then for every $k \in \mathbb{N}$ set $B_k = \{v \in V : \|v\| \leq k\}$. We may suppose, without loss of generality, that $B_k \cap \text{dom } F \neq \emptyset$ for all $k \in \mathbb{N}$.
Take any solution $v_k \in B_k \cap \text{dom } F$ to the problem

$$\min_{v \in B_k} F(v). \tag{5.13}$$

Such v_k exists applying the calculus of variations methods since B_k is sequentially weak compact for all $k \in \mathbb{N}$. If for some $k_0 \in \mathbb{N}, \|v_{k_0}\| < k_0$, then v_k is solution of the problem (5.10) by the convexity of F.

Suppose $\|v_k\| = k$ for all $k \in \mathbb{N}$ and define $w_k = \frac{v_k}{k}$ for all $k \in \mathbb{N}$, the unit sequence $\{w_k\}_{k\in\mathbb{N}}$ is weakly compact in V. Then we may extract a subsequence (which we still denoted by $\{w_k\}_{k\in\mathbb{N}}$) weakly converging to some $\overline{w} \in V$. Then

$$F(kw_k) = F(v_k) \leq F(x_0) < +\infty,$$

for every $k \in \mathbb{N}$ sufficiently large, where x_0 is any point in dom F. By the lsc and convexity of F, we have for every $\lambda > 0$,

$$F(x_0 + \lambda\overline{w}) \leq \liminf_{k\to+\infty} F\left(\lambda w_k + \left(1 - \frac{\lambda}{k}\right) x_0\right)$$

$$\leq \liminf_{k\to+\infty} \left(\frac{\lambda}{k} F(kw_k) + \left(1 - \frac{\lambda}{k}\right) F(x_0)\right) \leq F(x_0),$$

thus,

$$F^\infty(\overline{w}) = \lim_{\lambda\to+\infty} \frac{F(x_0 + \lambda\overline{w})}{\lambda} \leq \lim_{\lambda\to+\infty} \frac{F(x_0)}{\lambda} = 0,$$

which together with the assumption $(A1)$ imply $F^\infty(\overline{w}) = 0$. Then $\overline{w} \in R_0$. Thus $\{v_k\}_{k\in\mathbb{N}}$ satisfies the premises of $(F3)$ and therefore there exists $z \in V$ and $\overline{k} \in \mathbb{N}$ such that

$$\|z\| < \|v_{\overline{k}}\| \text{ and } F(z) \leq F(v_k), \ \forall\ k \geq \overline{k},$$

given a contradiction with the fact that $v_{\overline{k}}$ is a solution of the problem for $\overline{k} \in \mathbb{N}$.

($\Leftarrow$) If problem (5.10) has a solution, then $(A1)$ is consequence of Proposition 5.6. Let $z \in V$ be the solution of the problem, for any sequence $\{v_k\}_{k\in\mathbb{N}}$ such that $\|v_k\| \to +\infty$ and $\overline{v} = u$, condition $(F3)$ holds and the proof is complete. ■

Theorem 5.2 can be applied to a family of convex functions that Theorem 5.1 cannot be applied.

Example 5.1 *Let $\Omega \subseteq \mathbb{R}^n$ be a nonempty, bounded and open set with a Lipschitz boundary. Let $V = L^2(\Omega, \mu)$ with μ being the Lebesgue measure. Consider the function,*

$$F(v) = \int_{\Omega} \max\{0, v\} d\mu. \tag{5.14}$$

Since the Lebesgue measure is finite, F is everywhere finite and $\operatorname{dom} F = V$. *Clearly, F is convex, proper and lsc. Actually, following general results from calculus of variations, see for instance Corollary 1.2, Chapter VIII in [28], we can prove that F is continuous. But we will check it directly due to the structure of the problem.*

Assume that $\{v_n\}$ is a sequence in $L^2(\Omega, \mu)$, norm converging to v. Since for every $x \in \Omega$, we have that $||v_n(x)| - |v(x)|| \leq |v_n(x) - v(x)|$, we deduce from the definition of the norm ($\|v\| = \left(\int |v(x)|\, d\mu\right)^{1/2}$) that $\||v_n| - |v|\| \leq \|v_n - v\|$. Hence, $|v_n| \underset{L^2(\Omega)}{\longrightarrow} |v|$. Since $\max\{0, v\} = (|v| + v)/2$, *then*

$$\max\{0, v_n\} \underset{L^2(\Omega)}{\longrightarrow} \max\{0, v\}.$$

$F(v)$ is the scalar product of $\max\{0, v\}$ *with the constant function* 1, *then $F(v_n) \to F(v)$ and F is continuous.*

Furthermore, $F^\infty(v) = F(v) \geq 0$ for all $v \in L^2(\Omega)$ and condition $(A1)$ is satisfied. But $F^\infty(v) = 0$ if and only if $v \leq 0$, so $R_0 = \{v \in V : \ v \leq 0\}$ is not a linear subspace and Theorem 5.1 cannot be applied.

We clain that condition $(F3)$ is satisfied. In fact, If $v_k \in L^2(\Omega)$ is a sequence such that $\|v_k\| \to +\infty$ with $\frac{v_k}{\|v_k\|} \rightharpoonup v \in R_0$ and for $z \in L^2(\Omega)$, there exists $k_z \in \mathbb{N}$ such that

$$F(v_k) \leq F(z), \text{ when } k \geq k_z,$$

then if we take $\overline{v} = 0 \in L^2(\Omega)$ condition $(F3)$ is satisfied. Finally, Theorem 5.2 implies that (5.14) *has an optimal solution.*

Remark 5.2 *Our Theorem 5.2 extends and generalizes Theorem 5.1 for convex functions by the following two aspects;*

(*i*) *It was proved in Lemma 3.1 of [60] that, under condition* $(A1)$, *if* R_0 *is a linear subspace then condition* $(F3)$ *holds, that is,* $(A3) \Rightarrow (F3)$.

(*ii*) *We do not need the compactness condition* $(F2)$ *like in Theorem 5.1 or Theorem 2.1 in [6]. The requirement that* V *has to be Separable is also deleted.*

We define the cone

$$R_0^2 = \{v \in V : \ F^\infty(v) = 0, F^{\infty 2}(v; v) = 0\}.$$

Consider the following slight modification of condition $(F3)$ given by;

$(F3')$: If the sequence $v_k \in \text{dom } F, \|v_k\| \to +\infty$ is such that $\frac{v_k}{\|v_k\|} \rightharpoonup v \in R_0^2$ and for all $z \in \text{dom } F$, there exists $k_z \in \mathbb{N}$ such that

$$F(v_k) \leq F(z), \text{ when } k \geq k_z,$$

then there exists $\overline{v} \in \text{dom } F$ and $\overline{k} \in \mathbb{N}$ such that $\|\overline{v}\| < \|v_{\overline{k}}\|$ and $F(\overline{v}) \leq F(v_k), \ \forall \ k \geq \overline{k}$.

Lemma 5.3 *Let* V *be a reflexive Banach space with norm* $\|\cdot\|$ *and let* $F : V \to]-\infty, +\infty]$ *be a function satisfying* $(F1)$ *with* intdom $F \neq \emptyset$, *then conditions*

$(A1)$ $F^\infty(v) \geq 0$ *for all* $v \in V$,

$(A2')$ *If* $F^\infty(v) = 0$, *then* $F^{\infty 2}(v; v) = 0$,

and condition $(F3')$ *is satisfied if and only if problem* (5.10) *has a solution.*

It is easy to prove that, under condition $(A2')$, the sets R_0 and R_0^2 are equal.

The next example was given in [5], Example 15.1.8, and shows that condition $(F2)$ cannot be removed, but the anomaly is also detected by our Theorem 5.2 throughout condition $(F3)$.

Example 5.2 *Let* V *be an infinite dimensional separable Hilbert space, and let* $\{e_n\}_{n\in\mathbb{N}}$ *be a complete orthonormal system in* V. *We denote by* $\langle \cdot, \cdot \rangle$ *the scalar product in* V *and we define a function* $F : V \to \mathbb{R}$ *by setting*

$$F(v) = \sum_{n \in \mathbb{N}} 2^{-n} |\langle v, e_n \rangle - 1|^2, \ \forall \ v \in V.$$

It is clear that F *is convex, lsc and finite valued. Moreover, by definition* $F(v) \geq 0$ *for every* $v \in V$. *Taking* $x_0 = 0$, *one obtains for all* $v \in V$ *that*

$$F^\infty(v) = \lim_{\lambda \to +\infty} \frac{F(\lambda v) - F(0)}{\lambda} = \lim_{\lambda \to +\infty} \sum_{n \in \mathbb{N}} 2^{-n} \left[\lambda |\langle v, e_n \rangle|^2 - 2\langle v, e_n \rangle \right].$$

That is,

$$F^{\infty}(v) = \begin{cases} 0, & v = 0 \\ +\infty, & v \neq 0. \end{cases}$$

Hence $(A1)$ *holds and* $(A2')$ *is satisfied vacuously. Here* $(F3)$ *is equivalent to* $(F3')$. *Finally, we prove that* $(F3)$ *is violated. In fact, taking the vector* $v_k = \Sigma_{i=1}^{k} e_i$ *for every* $k \in \mathbb{N}$, *we get*

$$F(v_{k+1}) = \sum_{n=k+2}^{+\infty} 2^{-n} < F(v_k) = \sum_{n=k+1}^{+\infty} 2^{-n}, \ \forall \ k \in \mathbb{N}. \tag{5.15}$$

Since $R_0 = R_0^2 = \{0\}$ *then* $\frac{v_k}{\|v_k\|} \rightharpoonup v = 0$. *If not, then there exists* $w \in V$ *such that* $\langle \frac{v_k}{\|v_k\|}, w \rangle \nrightarrow 0$, *taking a subsequence, we may suppose that* $\langle \frac{v_k}{\|v_k\|}, w \rangle \to \alpha \neq 0$ *and that* $\frac{v_k}{\|v_k\|}$ *converges weakly to some* v. *This implies that, for all* i

$$\langle \frac{v_k}{\|v_k\|}, e_i \rangle \to \langle v, e_i \rangle.$$

Since $\langle \frac{v_k}{\|v_k\|}, e_i \rangle = \frac{1}{\sqrt{k}}$ *for* k *sufficiently large, we have* $\langle v, e_i \rangle = 0$ *for all* i, *so* $v = 0$, *which cannot happen if* $\alpha \neq 0$, *then* $\frac{v_k}{\|v_k\|} \rightharpoonup 0$. *By* (5.15) *there is no* $\overline{v} \in V$ *and* $\overline{k} \in \mathbb{N}$ *such that* $\|\overline{v}\| < \|v_{\overline{k}}\|$ *and* $F(\overline{v}) \leq F(v_k)$ *for all* $k \geq \overline{k}$, *then* $(F3)$ *is violated.*

The following example, Example 15.1.9 in [5], shows that condition $(A3)$ cannot be removed in Theorem 5.1. The anomaly is detected by condition $(A2')$ in Lemma 5.3 and Theorem 5.2 throughout $(F3)$.

Example 5.3 *Let* $V = \mathbb{R}$ *and* $F : \mathbb{R} \to]-\infty, +\infty]$ *be given by*

$$F(x) = \begin{cases} -\log(x), & x > 0, \\ +\infty, & x \leq 0. \end{cases}$$

The function F *is proper, convex, lsc and* $\operatorname{intdom} F =]0, +\infty[$. *Condition* $(F2)$ *holds vacuously by the dimension of the space,* $F^{\infty}(x) = 0$ *if* $x \geq 0$ *and* $F^{\infty}(x) = +\infty$ *in other case, then* $R_0 = \{x \in \mathbb{R};\ F^{\infty}(x) = 0\}$ *is not a linear subspace and Theorem 5.1 cannot be applied.*

On the other hand, for every $x > 0$

$$F^{\infty 2}(1;1) = \lim_{s \to +\infty} \lim_{t \to +\infty} \frac{-\log(x+t+s) + \log(x)}{s} = -\infty,$$

then the necessary condition $(A2')$ *is violated and Lemma 5.3 cannot be applied. Theorem 5.2 also detects the anomaly throughout condition* $(F3)$, *which is more difficult to check than condition* $(A2')$.

Remark 5.3 *(a) Note that if dom F is bounded in $(V, \|\cdot\|)$ then conditions $(A1), (A2')$ and $(F2)$ are automatically fulfilled and conditions $(F3) - (F3')$ hold vacuously.*

(b) Our condition $(A2')$ is very useful when exists a nonzero first order asymptotic directions $v \in V$ such that $F^{\infty}(v) = 0$, since the second order asymptotic function provides a finer description of the behavior of the function at the infinity than the first order asymptotic function. If $F^{\infty}(v) = 0$ only for $v = 0$, then condition $(A3')$ holds vacuously.

(iii) Example 5.3 shows that $R_0^2 \subsetneqq R_0$ even for convex functions.

Chapter 6

Asymptotic functions under generalized convexity assumptions

Asymptotic analysis involves a description of the behaviour of a mathematical object at infinity. Usually it concerns a set, or a function via its epigraph. When a minimization problem is considered, convexity is the desired condition since any local property has a global character: for example, any local minimizer is global, and first order necessary optimality conditions become also sufficient. Under lack of convexity an analysis of the behaviour of unbounded minimizing sequences is necessary, and then once we normalize them, their limit directions need to be compared with those stemming from the epigraph of the objective function.

In general, the existence issue in nonconvex minimization problems requires a global knowledge of the objects. However, quasiconvex objective functions still provide a good instance where we may apply the same tools, slightly modified, coming from convex situations. This Chapter goes in that direction.

Section 6.1 collects some basic definitions, some of them well known like (first order) asymptotic cones and functions; their second order counterparts are also recalled, in the general case. Then, first and second order asymptotic cones and functions, which seems to be suitable for dealing with quasiconvex functions, are introduced.

Section 6.2 shows some applications of the notions introduced in Section 6.1: we identify a new class of quasiconvex vector mappings and provide a sufficient condition under which the image, via a mapping belonging to that class, of a closed convex set is closed; when minimizing a quasiconvex function, we characterize the nonemptiness and boundedness of the optimal solution set; a new necessary condition for a point to be efficient and weakly efficient in the multiobjective optimization problem are also provide.

6.1 Quasiconvex asymptotic functions

We start this section with the definition of another (first order) asymptotic cone, very closed to the usual concept with several applications to generalized differentiability, as shown in [9, 59, 61], called the incident asymptotic cone.

Definition 6.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, the incident asymptotic cone of K is defined by*

$$K^{i\infty} = \{u \in \mathbb{R}^n : \ \forall\ t_k \to +\infty,\ \exists\ x_k \in K,\ \frac{x_k}{t_k} \to u\}. \tag{6.1}$$

It is well-known that the usual K^{∞} can also be expressed as outer limit in the sense of Painlevé-Kuratowski (from theory of set convergence) by,

$$K^{\infty} = \limsup_{t \to +\infty} \frac{K}{t} \tag{6.2}$$

and the incident asymptotic cone $K^{i\infty}$ as the inner limit,

$$K^{i\infty} = \liminf_{t \to +\infty} \frac{K}{t}. \tag{6.3}$$

By definition $K^{i\infty} \subseteq K^{\infty}$. When these two cones coincide, the set K is called asymptotically regular in [9] or asymptotable in [55]. A sufficient condition for a set K to be asymptotically regular is the (CD) condition given in [54], that is, for all $u \in K^{\infty}$ there exists a bounded set K_0 such that $(tu + K_0) \cap K \neq \emptyset$ for all $t > 0$.

In the next example we show that the asymptotic cone and the incident asymptotic cone are not equal in general.

Example 6.1 *(Chapter 1, Section 2, page 13 in [54]). Consider the set $K = \{(2^{2n}, 0) \in \mathbb{R}^2 : \ n = 0, 1, 2, \ldots\}$. Here K^{∞} is the ray generated by $u = (1, 0)$. Take now $t_n = 2^{n^2}$ for all $n \in \mathbb{N}$, then for any $x_n \in K$ there exists $n_0 \in \mathbb{N}$ such that $2^{n^2} > 2^{2n}$ for all $n \geq n_0$, then $\frac{x_n}{t_n} \to (0, 0)$ and $(1, 0) \notin K^{i\infty}$. Thus $K^{i\infty} = \{(0, 0)\}$ and the inclusion is strict.*

Furthermore, the previous example illustrates to us that the incident asymptotic cone could be $\{0\}$ even for unbounded sets.

For more details on theory of set convergence see [11, 52, 54, 55, 63] and references therein.

An important part of the analysis is knows when the asymptotic and the incident asymptotic cones are equals. We present two examples of asymptotically regular sets in the next proposition.

Proposition 6.1 *Let $K \subseteq \mathbb{R}^n$ be a nonempty set, then the following assertions holds;*

(a) *If K is convex then K is asymptotically regular.*

(b) *If K is a closed cone then K is asymptotically regular. Furthermore, $K^{i\infty} = K^{\infty} = K$.*

Proof. (a): Since K is convex, then by Proposition 2.1.3 in [9], we have that K is asymptotically regular and $K^{i\infty} = K^{\infty}$.

(b): Suppose K is a closed cone. We only need to prove $K^{\infty} \subseteq K^{i\infty}$. Take $u \in K^{\infty} = K$, then for all $t_n \to +\infty$ we pick $x_n = t_n u \in K$, thus $\frac{x_n}{t_n} = \frac{t_n u}{t_n} = u \in K^{i\infty}$. ∎

Could be natural to define the incident asymptotic function from the incident asymptotic cone of the epigraph of a function, as was given by Penot in [59].

Definition 6.2 *(Section 11 in [59]) For every proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define the incident asymptotic function of f by the function $f^{i\infty} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ for which*

$$\text{epi } f^{i\infty} = (\text{epi } f)^{i\infty} \tag{6.4}$$

Following standar arguments, see Lemma 2.1 part (g) and Proposition 2.7, we can prove the basic properties of $f^{i\infty}$, that is,

Proposition 6.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function, then*

(i) *$f^{i\infty}(\cdot)$ is lsc and positively homogeneous. $f^{i\infty}(0) = 0$ or $-\infty$, and if $f^{i\infty}(0) = 0$ then $f^{i\infty}(\cdot)$ is proper.*

(ii) *Let $\lambda \in \mathbb{R}$ be such that $S_\lambda(f) \neq \emptyset$, then $(S_\lambda(f))^{i\infty} \subseteq \{u \in \mathbb{R}^n : \ f^{i\infty}(v) \leq 0\}$.*

The inclusion in (ii) could be strict. In fact, take $f(x) = \frac{x}{1+x}$ if $x \geq 0$, and $f(x) = +\infty$ elsewhere. For $\lambda = \frac{1}{2}$ we have

$$(S_{\frac{1}{2}}(f))^{i\infty} = ([0,1])^{i\infty} = \{0\}$$

while $\{u \in \mathbb{R} : f^{i\infty}(u) \leq 0\} = [0, +\infty[$.

When f is a quasiconvex function was proved in [59] that $f^{i\infty}$ is also quasiconvex. We present the proof here for convenience or the reader.

Proposition 6.3 *(Proposition 11.1 in [59]) If $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex then $f^{i\infty} : \mathbb{R}^n \to \mathbb{R} \cup \{\pm\infty\}$ is quasiconvex.*

Proof. $(u,\alpha) \in \text{epi } f^{i\infty} = (\text{epi } f)^{i\infty}$ if and only if for all $t_k \to +\infty$, there exists $(x_k, \alpha_k) \in$ epi f such that $\frac{(x_k,\alpha_k)}{t_k} \to (v,\alpha)$, if and only if for all $t_k \to +\infty$, there exists $(x_k, \alpha_k) \in$ epi f with $f(x_k) \leq \alpha_k$ for all $k \in \mathbb{N}$, such that

$$\frac{x_k}{t_k} \to u, \ \frac{\alpha_k}{t_k} \to \alpha.$$

Let $u_1, u_2 \in (\text{dom } f)^{i\infty}$ and $u = \lambda u_1 + (1-\lambda)u_2$, with $\lambda \in]0,1[$. We claim that

$$f^{i\infty}(u) \leq \max\{f^{i\infty}(u_1), f^{i\infty}(u_2)\}.$$

Suppose that $f^{i\infty}(u_j) < +\infty$ for $j = 1, 2$. Since $(u_1, f^{i\infty}(u_1)) \in$ epi $f^{i\infty}$ then for all $t_k \to +\infty$ there exists $(x_k, \alpha_k) \in$ epi f such that,

$$f(x_k) \leq \alpha_k, \ \frac{x_k}{t_k} \to u_1, \ \frac{\alpha_k}{t_k} \to \alpha.$$

Since $(u_2, f^{i\infty}(u_2)) \in$ epi $f^{i\infty}$ then for all $t_k \to +\infty$ there exists $(y_k, \beta_k) \in$ epi f such that,

$$f(y_k) \leq \beta_k, \ \frac{y_k}{t_k} \to u_2, \ \frac{\beta_k}{t_k} \to \beta.$$

Let $z_k = \lambda x_k + (1-\lambda)y_k$, thus

$$f(z_k) \leq \max\{f(x_k), f(y_k)\} \leq \max\{\alpha_k, \beta_k\} = \gamma_k,$$

then $(z_k, \gamma_k) \in$ epi f.

In both cases we have convergence for an arbitrary $t_k \to +\infty$, then

$$\frac{(z_k, \gamma_k)}{t_k} \to (u, \max\{f^{i\infty}(u_1), f^{i\infty}(u_2)\}) \in \text{epi } f^{i\infty}.$$

Thus,

$$f^{i\infty}(u) \leq \max\{f^{i\infty}(u_1), f^{i\infty}(u_2)\}.$$

■

For the usual asymptotic function, we give the proof in one dimension. To that end, we will use the following result.

Proposition 6.4 *(Proposition 1.9 in [3].) A function $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex if and only if there exists an interval I of the form $]-\infty, b[$ or $]-\infty, b]$, where $t \in]-\infty, +\infty]$, such that f is nonincreasing on I and nondecreasing on its complement.*

Proposition 6.5 *If $f : \mathbb{R} \to \mathbb{R} \cup \{+\infty\}$ is quasiconvex, then $f^\infty : \mathbb{R} \to \mathbb{R} \cup \{\pm\infty\}$ is quasiconvex.*

Proof. Note first that for a quasiconvex function f, one has either $f(x) \geq f(0)$ for all $x > 0$, or $f(x) \geq f(0)$ for all $x \leq 0$ (or both); indeed, if we assume that this is not the case, then we get some $x_1 > 0$ and some $x_2 < 0$ such that $f(x_1) < f(0)$ and $f(x_2) < f(0)$, contradicting quasiconvexity.

Hence we may assume, for instance, that $f(x) \geq f(0)$ for all $x > 0$. Then we get $f^\infty(1) \geq 0$, so f^∞ is nondecreasing on $(0, \infty)$.

If $f^\infty(0) = 0$, then f^∞ is nondecreasing on $[0, \infty)$. Since it is either decreasing or nondecreasing on $(-\infty, 0]$, we deduce by Proposition 6.4 that f^∞ is quasiconvex.

Now assume that $f^\infty(0) = -\infty$. If $f^\infty(x) \geq 0$ for all $x < 0$ then by Proposition 6.4, f^∞ is quasiconvex. So we may assume that $f^\infty(-1) < 0$. In this case, we will show that $f^\infty(-1) = -\infty$, so f^∞ is nonincreasing on $\mathbb{R}$ and thus it is quasiconvex.

Since $f^\infty(0) = -\infty$, there exists a sequence $\{x_k\} \subseteq \operatorname{dom} f$ and $t_k \to +\infty$ such that $\frac{x_k}{t_k} \to 0$ and $\frac{f(x_k)}{t_k} \to -\infty$. It follows that $f(x_k) \to -\infty$ so we may assume that $f(x_k) < f(0)$ and $x_k < 0$.

Since $f^\infty(-1) < 0$, there exist sequences $\{y_k\} \subseteq \operatorname{dom} f$ and $t_k \to +\infty$ such that $\frac{y_k}{t_k} \to -1$ and $\lim_{k\to+\infty} \frac{f(y_k)}{t_k} = f^\infty(-1) < 0$. This implies that $y_k \to -\infty$ and $f(y_k) \to -\infty$. For each k and each $x < x_k$ we may choose k' such that $y_{k'} < x < x_k$ and $f(y_{k'}) < f(x_k)$. By quasiconvexity, $f(x) \leq f(x_k)$ for all $x < x_k$. Set $z_k = x_k - t_k$. Then $\frac{z_k}{t_k} \to -1$ and $\frac{f(z_k)}{t_k} \leq \frac{f(x_k)}{t_k} \to -\infty$. Hence $f^\infty(-1) = -\infty$ as was to be proved.

Then f^∞ is quasiconvex. ■

Remark 6.1 *We conjecture the assertion of J. P. Penot in section 11 of [59]; "f^∞ is not necessarily quasiconvex when f it is" is not true in finite dimensional spaces.*

The incident asymptotic function can be calculated by the following new formula. According to our best knowledge no formula appears in the literature.

Proposition 6.6 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function. Then for all $u \in \mathbb{R}^n$*

$$f^{i\infty}(u) = \sup_{t_k\to+\infty} \inf_{\frac{x_k}{t_k}\to u} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \tag{6.5}$$

Proof. ($\geq$) We denote by $h(u)$ the right hand of (6.5). Let $(u,\alpha) \in \text{epi } f^{i\infty}$ then for all $t_k \to +\infty$ there exists $(x_k,\alpha_k) \in \text{epi } f$ such that $\frac{(x_k,\alpha_k)}{t_k} \to (u,\alpha)$. Since $\frac{f(x_k)}{t_k} \leq \frac{\alpha_k}{t_k}$ then

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \leq \limsup_{k\to+\infty} \frac{\alpha_k}{t_k} = \alpha,$$

thus,

$$\inf_{\frac{x_k}{t_k}\to u} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \leq \alpha, \ \forall\ t_k \to +\infty,$$

then

$$\sup_{t_k\to+\infty} \inf_{\frac{x_k}{t_k}\to u} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \leq \alpha,$$

thus $h(u) \leq \alpha$ for all $(u,\alpha) \in \text{epi } f^{i\infty}$, so $h(u) \leq f^{i\infty}(u)$.

($\leq$) If $h(u) = +\infty$ then the inequality is obvious. If $h(u) \in \mathbb{R}$, let $\varepsilon > 0$, then $h(u) < h(u) + \varepsilon$, thus for all $t_k \to +\infty$

$$\inf_{\frac{x_k}{t_k}\to u} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} < h(u) + \varepsilon,$$

then for all $t_k \to +\infty$, there exists $\frac{x_k}{t_k} \to u$ such that

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} < h(u) + \varepsilon.$$

Let we consider,

$$\alpha_k = \begin{cases} t_k(h(u)+\varepsilon) & \text{if } \frac{f(x_k)}{t_k} \leq h(u)+\varepsilon, \\ f(x_k) & \text{if } \frac{f(x_k)}{t_k} > h(u)+\varepsilon. \end{cases}$$

Then, in any case, we have that $f(x_k) \leq \alpha_k$, then $(x_k,\alpha_k) \in \text{epi } f$. Since

$$\frac{(x_k,\alpha_k)}{t_k} \to (u, h(u)+\varepsilon), \ \forall\ \varepsilon > 0,$$

then $(u, h(u)+\varepsilon) \in \text{epi } f^{i\infty}$, thus $f^{i\infty}(u) \leq h(u)+\varepsilon$ for all $\varepsilon > 0$. Proving that $f^{i\infty}(u) \leq h(u)$.
■

Another attempts to define appropriate asymptotic functions in the quasiconvex case are listed in the following definition. We refer to [34] section 5 for a thorough study, here we give a short account of that work and we compare those asymptotic functions with the incident asymptotic function.

Definition 6.3 *For every proper function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ we consider the following* *definitions. Let $\lambda \in \mathbb{R}$ be such that $S_\lambda(f) \neq \emptyset$, then*

$$f^\infty(u;\lambda) = \sup_{x \in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-\lambda}{t} \tag{6.6}$$

$$f^{q\infty}(u) = \sup_{\substack{x \in \mathrm{dom}\ f \\ t>0}} \frac{f(x+tu)-f(x)}{t} \tag{6.7}$$

In case when $\lambda = f(\overline{x})$ for some $\overline{x} \in \mathrm{dom}\ f$, we symple write (as in [34])

$$f^\infty(u;\overline{x}) = f^\infty(u; f(\overline{x})) = \sup_{x \in S_{f(\overline{x})}(f)} \sup_{t>0} \frac{f(x+tu)-f(\overline{x})}{t}.$$

We can describe the previous functions with the following characterizations for their epigraphs, see Proposition 5.1 in [34]. For any proper function f and for all $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$, we have

$$\mathrm{epi}\ f^\infty(\cdot;\lambda) = \bigcap_{x \in S_\lambda(f)} \bigcap_{t>0} t(\mathrm{epi}\ f - (x,\lambda)) \tag{6.8}$$

$$\mathrm{epi}\ f^{q\infty}(\cdot) = \bigcap_{x \in \mathrm{dom}\ f} \bigcap_{t>0} t(\mathrm{epi}\ f - (x, f(x))) \tag{6.9}$$

On the other hand. $f^{q\infty}$ and $f^\infty(\cdot;\lambda)$ are also quasiconvex when f is quasiconvex by Proposition 3.28 in [34]. Furthermore, if f is, proper, convex and lsc, then all the definitions coincide with the usual asymptotic function, that is, for all $u \in \mathbb{R}^n$ and all $x \in \mathrm{dom}\ f$, we have

$$f^\infty(u) = f^{q\infty}(u) = f^\infty(u;x) = f^{i\infty}(u). \tag{6.10}$$

We have the following relationship between some of the previos attempts.

Proposition 6.7 *For a proper function f and $\lambda \in \mathbb{R}$ with $S_\lambda(f) \neq \emptyset$, we have*

(a) $f^{q\infty}(u) \geq \sup_{x \in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \geq f^\infty(u;\lambda)$.

(b) $f^{q\infty}(u) \geq \inf_{x \in \mathrm{dom}\ f} \limsup_{t \to +\infty} \frac{f(x+tu)}{t} \geq \inf_{x \in \mathrm{dom}\ f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \geq f^{i\infty}(u), \ \forall\ u \in \mathbb{R}^n$.

(c) $f^{i\infty}(u) \geq f^\infty(u)$ *for all* $u \in \mathbb{R}^n$.

Proof. (c): By definition of incident asymptotic cone.

(a): Let $\lambda \in \mathbb{R}$ be such that $S_\lambda(f) \neq \emptyset$, and let $x \in \mathrm{dom}\, f$. Since $f(x) \leq \lambda$, then for every $u \in \mathbb{R}^n$ we have that

$$\frac{f(x+tu)-\lambda}{t} \leq \frac{f(x+tu)-f(x)}{t}, \ \forall\, t>0,$$

thus taken $\sup_{t>0}$ and later $\sup_{x \in S_\lambda(f)}$ we have

$$f^{\infty}(u;\lambda) \leq \sup_{x\in S_\lambda(f)} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \leq f^{q\infty}(u).$$

(b): Let $x \in \mathrm{dom}\, f$ be given. For every $t_k \to +\infty$, set $y_k = x + t_k u$. Then $\frac{y_k}{t_k} \to u$. One obviously has

$$\limsup_{k\to+\infty} \frac{f(y_k)}{t_k} \leq \limsup_{t\to+\infty} \frac{f(x+tu)}{t}$$

hence for this sequence $\{t_k\}$,

$$\inf_{\frac{x_k}{t_t}\to u} \left\{ \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \right\} \leq \limsup_{t\to+\infty} \frac{f(x+tu)}{t}.$$

This is true for all $t_k \to +\infty$, so $f^{i\infty}(u) \leq \limsup_{t\to+\infty} \frac{f(x+tu)}{t}$. Since this is true for all x, we deduce the first inequality in (b). For the second, we remark that for every $x \in \mathrm{dom}\, f$,

$$\limsup_{t\to+\infty} \frac{f(x+tu)}{t} = \limsup_{t\to+\infty} \frac{f(x+tu)-f(x)}{t} \leq \sup_{t>0} \frac{f(x+tu)-f(x)}{t}.$$

Thus,

$$\inf_{x\in \mathrm{dom}\, f} \limsup_{t\to+\infty} \frac{f(x+tu)}{t} \leq \inf_{x\in \mathrm{dom}\, f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} \leq f^{q\infty}(u).$$

■

The reverse implications are not true in general as the following example shows.

Example 6.2 *Consider the quasiconvex and lsc function $f(x) = \frac{x}{1+x}$ if $x \geq 0$, and $f(x) = +\infty$ if $x < 0$. Computing,*

$$f^{\infty}(u) = f^{i\infty}(u) = \begin{cases} 0 & \text{if } u \geq 0, \\ +\infty & \text{if } u < 0. \end{cases}, \ f^{\infty}(u;\overline{x}) = \begin{cases} \frac{u}{(1+\overline{x})^2} & \text{if } u \geq 0, \\ +\infty & \text{if } u < 0. \end{cases}$$

$$f^{q\infty}(u) = \begin{cases} u & \text{if } u \geq 0, \\ +\infty & \text{if } u < 0. \end{cases}$$

Then $f^{q\infty}(1) > f^{\infty}(1;\overline{x}) > f^{i\infty}(1)$ *for all* $\overline{x} > 0$ *and* $f^{q\infty}(1) > f^{\infty}(1)$.

The equality beetwen $f^{i\infty}$ and f^{∞} does not hold when f is quasiconvex.

Example 6.3 *Consider the quasiconvex, proper and lsc function given by*

$$f(x) = \begin{cases} +\infty & \text{if } x < 0, \\ 0 & \text{if } 0 \leq x \leq 1, \\ 2^k & \text{if } 2^k < x \leq 2^{k+1},\ k = 0, 1, 2, \ldots. \end{cases}$$

It is easy to see that,

$$f^{\infty}(u) = \begin{cases} +\infty & \text{if } u < 0, \\ \frac{u}{2} & \text{if } u \geq 0. \end{cases}$$

We calculate $f^{i\infty}(1)$. *For every* $t_k \to +\infty$, *let* x_k *be such* $\frac{x_k}{t_k} \to 1$. *Then*

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} = \limsup_{k\to+\infty} \frac{f(x_k)}{x_k}\frac{x_k}{t_k} = \limsup_{k\to+\infty} \frac{f(x_k)}{x_k}.$$

Since by construction of f *we have* $f(x) \leq x$ *for all* $x \geq 0$, *using* (6.5) *we infer that* $f(1) \leq 1$. *Now, given* $\varepsilon > 0$ *take* $t_k = (1+\varepsilon)2^k$. *For every* x_k *such that* $\frac{x_k}{t_k} \to 1$, *we have that* $\frac{x_k}{t_k} \geq \frac{1}{1+\varepsilon}$ *for large* k, *thus* $x_k > \frac{t_k}{1+\varepsilon} = 2^k$. *Thus* $f(x_k) \geq 2^k$ *and using* (6.5) *we have*

$$f^{i\infty}(1) \geq \inf_{\frac{x_k}{t_k}\to 1} \limsup_{k\to+\infty} \frac{f(x_k)}{t_k} \geq \inf_{\frac{x_k}{t_k}\to 1} \limsup_{k\to+\infty} \frac{2^k}{(1+\varepsilon)2^k} = \frac{1}{1+\varepsilon}.$$

Thus $f^{i\infty}(u) > f^{\infty}(u)$ *for all* $u > 0$.

The equality beetwen $f^{q\infty}$ and $f^{i\infty}$ does not hold, even when f is continuous and quasiconvex.

Example 6.4 *Let* $f(x) = \sqrt{x}$ *if* $x > 0$, *and* $f(x) = +\infty$ *in other case. Here* f *is quasiconvex, proper and continuous function. It is well know that* $f^{\infty}(u) = 0$ *if* $u \geq 0$, *and* $f^{\infty}(u) = +\infty$ *if* $u < 0$. *Take* $u = 1$, *then*

$$f^{q\infty}(1) = \sup_{\substack{x\in \mathrm{dom}\ f \\ t>0}} \frac{f(x+t) - f(x)}{t} = \sup_{\substack{x\geq 0 \\ t>0}} \frac{\sqrt{x+t} - \sqrt{x}}{t} = +\infty,$$

that is,

$$f^{q\infty}(u) = \begin{cases} +\infty & \text{if } u \neq 0, \\ 0 & \text{if } u = 0. \end{cases}$$

On the other hand, for all $t_k \to +\infty$ there exists $x_k \geq 0$ such that $\frac{x_k}{t_k} \to 1$, then

$$\limsup_{k\to+\infty} \frac{f(x_k)}{t_k} = \limsup_{k\to+\infty} \frac{1}{\sqrt{t_k}}\sqrt{\frac{x_k}{t_k}} = 0,\ \forall\ t_k \to +\infty,\ \forall\ \frac{x_k}{t_k} \to 1.$$

Thus $f^{i\infty}(u) = f^{\infty}(u)$ for all $u \in \mathbb{R}$ and $f^{q\infty}(1) > f^{i\infty}(1)$.

The interesting reader could be thinking in the reason to define $f^{q\infty}$ or $f^{\infty}(\cdot;\lambda)$, for any $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$, because there are not come from some kind of epigraph of a function as the usual asymptotic function or the incident asymptotic function. We present the reason here.

Remark 6.2 (*a*) *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function. Let $\lambda \in \mathbb{R}$ be such that $S_\lambda(f) \neq \emptyset$. Then we have*

(*i*) $(\mathrm{argmin}_{\mathbb{R}^n} f)^{\infty} = \{u \in \mathbb{R}^n :\ f^{q\infty}(u) \leq 0\}$.

(*ii*) $(S_\lambda(f))^{\infty} = \{u \in \mathbb{R}^n :\ f^{\infty}(u;\lambda) \leq 0\}$.

(*b*) *Since for a proper, lsc and quasiconvex function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ and $\lambda \in \mathbb{R}$ such that $S_\lambda(f) \neq \emptyset$, the sets* $\mathrm{argmin}_{\mathbb{R}^n} f$ *and $S_\lambda(f)$ are convex, then there is no new information computing their incident asymptotic cones.*

We will now establish another formula for $f^{q\infty}$ in the general case under lsc. To that purpose, some notions are needed. Recall that for any function $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, the upper and lower Dini directional derivatives of f at $x \in \mathrm{dom}\ f$ in the direction $u \in \mathbb{R}^n$ are defined by

$$f^D(x;u) = \limsup_{t\to 0^+} \frac{f(x+tu) - f(x)}{t}.$$

$$f_D(x;u) = \liminf_{t\to 0^+} \frac{f(x+tu) - f(x)}{t}.$$

If f is lsc, then we know from the Diewert mean value theorem (see for example Theorem 10.1 in [46]): for each $a, b \in \mathrm{dom}\ f$, there exists $z \in [a, b[$ such that

$$f_D(z; b-a) \geq f(b) - f(a).$$

Proposition 6.8 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc with* $\mathrm{dom}\ f$ *nonempty and convex. Then*

$$f^{q\infty}(u) = \sup_{x \in \mathrm{dom}\ f} f^D(x;u). \tag{6.11}$$

Proof. Set $\alpha = \sup_{x \in \text{dom } f} f^D(x;u)$. Note that for every $y \in \text{dom } f$, $f^D(y;u) \leq \sup_{t>0} \dfrac{f(y+tu)-f(y)}{t}$. Hence

$$f^D(y;u) \leq \sup_{x \in \text{dom } f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t} = f^{q\infty}(u), \ \forall\, y \in \text{dom } f,$$

from which follows that $\alpha \leq f^{q\infty}(u)$.

To show the reverse inequality, take $u \in \mathbb{R}^n$. Assume first that for every $y \in \text{dom } f$ and $t > 0$, $y + tu \in \text{dom } f$. By Diewert's mean value theorem there exists $z \in [y, y+tu[$ such that

$$f(y+tu) - f(y) \leq f_D(z;tu).$$

Since $f_D(z;tu) \leq f^D(z;tu) = tf^D(z;u) \leq t \sup_{x \in \text{dom } f} f^D(x;u)$ it follows that

$$\frac{f(y+tu)-f(y)}{t} \leq \alpha, \ \forall\, y \in \text{dom } f, \ t > 0.$$

Hence $f^{q\infty}(u) \leq \alpha$.

Now assume that for some $y \in \text{dom } f$ and $t > 0$, $y + tu \notin \text{dom } f$. Then $f^{q\infty}(u) = +\infty$. Let $\bar{t} = \sup\{t : y + tu \in \text{dom } f\}$. If $y + \bar{t}u \in \text{dom } f$, then clearly $f^D(y+\bar{t}u;u) = +\infty$ and $\alpha = +\infty$. If not, then $f(y+\bar{t}u) = +\infty$ and by lsc we have that $\lim_{t \to \bar{t}_-} f(y+tu) = +\infty$. Using again Diewert's mean value theorem, it is easy to see that

$$\sup_{z \in [y, y+\bar{t}u[} f^D(z;u) = +\infty,$$

so we find again $\alpha = +\infty$. Thus in all cases, $f^{q\infty}(u) \leq \alpha$. ∎

For example, if f is the increasing, thus quasiconvex function $f(x) = x + \sin(x), x \in \mathbb{R}$, we get $f^{q\infty}(1) = \sup_{x \in \mathbb{R}} f'(x) = 2$.

Remark 6.3 (i): *Note that in spite of the fact that for a given $x \in \text{dom } f$, in general*

$$f^D(x;u) = \limsup_{t \to 0^+} \frac{f(x+tu)-f(x)}{t} \neq \sup_{t>0} \frac{f(x+tu)-f(x)}{t},$$

we still have $\sup_{x \in \text{dom } f} f^D(x;u) = \sup_{x \in \text{dom } f} \sup_{t>0} \frac{f(x+tu)-f(x)}{t}$. *In fact, the proof shows that for every $x_0 \in \text{dom } f$, if $l_{x_0} = \{x_0 + tu; t \geq 0\}$ then*

$$\sup_{x \in l_{x_0}} f^D(x;u) = \sup_{x \in l_{x_0}} \sup_{t>0} \frac{f(x+tu)-f(x)}{t}.$$

(ii): *When f is convex and lsc function, then $f^{q\infty} = f^{\infty}$ so we have still another formula* *for* f^∞, *that is,*

$$f^\infty(u) = \sup_{x \in \mathrm{dom}\ f} f^D(x;u).$$

We now introduce some notions on second order asymptotic functions suitable for dealing with quasiconvex functions.

Definition 6.4 *Given any proper function* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, *and* $u \in \mathbb{R}^n, u \neq 0$ *such that* $f^\infty(u)$ *is finite, define*

$$f_{qi}^{\infty 2}(u;v) = \sup_{\substack{x \in \mathrm{ridom}\ f \\ s>0}} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}, \ \forall\ v \in \mathbb{R}^n, \tag{6.12}$$

and the set

$$R_{qi} = \{u \in \mathbb{R}^n :\ f^\infty(u) = 0,\ f_{qi}^{\infty 2}(u;u) = 0\}.$$

Observe immediately that under convexity of f, it holds

$$f^{\infty 2}(u;v) = f_{qi}^{\infty 2}(u;v),\ v \in \mathbb{R}^n.$$

We start by establishing a simple but important fact.

Proposition 6.9 *Let* $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ *be any proper function. If* $\mu = \inf\limits_{\mathbb{R}^n} f$ *is finite, then*

$$[u \in (\mathrm{dom}\ f)^\infty,\ f^\infty(u) = 0] \Rightarrow f_{qi}^{\infty 2}(u;u) \geq 0$$

Proof. Take any $u \in (\mathrm{dom}\ f)^\infty$ such that $f^\infty(u) = 0$. Then

$$\frac{f(x+(s+t)u) - f(x)}{s} \geq \frac{\mu - f(x)}{s},\ \forall\ x \in \mathrm{ridom}\ f,\ \forall\ s,t > 0.$$

This implies that

$$f_{qi}^{\infty 2}(u;u) = \sup_{\substack{x \in \mathrm{ridom}\ f \\ s>0}} \inf_{t>0} \frac{f(x+(s+t)u) - f(x)}{s} \geq 0.$$

Thus $f_{qi}^{\infty 2}(u;u) \geq 0$. ∎

6.2 Applications in optimization

This section is devoted to show some potential applications of the notions introduced in the previous sections. The first characterizes the boundedness and nonemptiness of the set of minimizers of a quasiconvex function; whereas the second application deals with a new necessary condition for a point to be efficient or weakly efficient in the quasiconvex multiobjective optimization problem.

6.2.1 Characterizing boundedness and nonemptiness of the optimal solution set

Next two theorems go beyond coerciveness. Many authors have been worked with the non coercive minimization problem under asymptotic analysis, see for instance [10, 60] and very recently in [18]. Our results are different in the sense we do not require an asymptotic regularity at the infinity.

The continuity of f on $\text{dom}\, f$ and lsc (on $\mathbb{R}^n$) serve to ensure that whenever $x \in \text{dom}\, f$ and $u \in (\text{dom}\, f)^\infty$, we have $x + tu \in \text{dom}\, f$ for all $t > 0$, as we can see in the next proof.

Theorem 6.1 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper function, continuous on* $\text{dom}\, f$*, lsc on $\mathbb{R}^n$ and quasiconvex. Then $\underset{\mathbb{R}^n}{\text{argmin}} f \neq \emptyset$ and compact, if and only if the following assertions hold:*

(*a*) $f^\infty(u) \geq 0$ *for all* $u \in \mathbb{R}^n \backslash \{0\}$;

(*b*) $[u \in (\text{dom}\, f)^\infty,\ f^\infty(u) = 0] \Rightarrow f^{\infty 2}_{qi}(u; u) \geq 0$;

(*c*) $R_{qi} = \{0\}$.

Proof. Suppose first that $\underset{\mathbb{R}^n}{\text{argmin}} f \neq \emptyset$ and compact. Obviously (a) holds and (b) follows from Proposition 6.9. Let $u \in R_{qi}$, that is, $f^{\infty 2}_{qi}(u; u) = 0$ and $f^\infty(u) = 0$. Then

$$\sup_{\substack{x \in \text{ridom}\, f \\ s>0}} \inf_{t>0} \frac{f(x + (s + t)u) - f(x)}{s} = 0,$$

which implies that

$$\inf_{t>0} f(x + (s + t)u) \leq f(x),\ \forall\ x \in \text{ridom}\, f,\ \forall\ s > 0.$$

Let us prove that for every $x \in \text{ridom}\, f$ and $s > 0$, $f(x + su) \leq f(x)$. Assume that for some $s > 0$ we have $f(x + su) > f(x)$. Since $\inf_{t>0} f(x + (s + t)u) \leq f(x) < f(x + su)$, there exists $t > 0$ such that $f(x + su + tu) < f(x + su)$. This contradicts the quasiconvexity of f since $x + su$ belongs to the segment $]x, x + su + tu[$.

Take any $x_0 \in \underset{\mathbb{R}^n}{\operatorname{argmin}} f$. Then, there exists a sequence $x_k \in \operatorname{ridom} f$ such that $x_k \to x_0$. By continuity of f on $\operatorname{dom} f$, $f(x_k) \to f(x_0)$. By lower semicontinuity of f,

$$f(x_0 + su) \leq \liminf_{k \to +\infty} f(x_k + su) \leq \lim_{k \to +\infty} f(x_k) = f(x_0).$$

Thus, $x_0 + su \in \underset{\mathbb{R}^n}{\operatorname{argmin}} f$ for all $s > 0$. This contradicts the boundedness of $\underset{\mathbb{R}^n}{\operatorname{argmin}} f$ if $u \neq 0$.

Let us check the other implication. Take any minimizing sequence $\{x_k\}$, we will check that it is bounded. Thus, suppose that $\|x_k\| \to +\infty$ and $\frac{x_k}{\|x_k\|} \to u \neq 0$. Since $f(x_k)$ is a bounded sequence, $f^\infty(u) \leq 0$, and so $f^\infty(u) = 0$. Let $x \in \operatorname{ridom} f$. By quasiconvexity, given any $t > 0, s > 0$ we have

$$f\left((1 - \frac{s+t}{\|x_k\|})x + \frac{s+t}{\|x_k\|} x_k\right) \leq \max\{f(x), f(x_k)\}.$$

Given that $\lim_{k \to +\infty} \max\{f(x), f(x_k)\} = \max\{f(x), \inf\ f\} = f(x)$, the lower semicontinuity of f gives

$$f(x + (s+t)u) \leq \liminf_{k \to +\infty} f\left((1 - \frac{s+t}{\|x_k\|})x + \frac{s+t}{\|x_k\|} x_k\right) \leq f(x),$$

wich implies that $f(x + (s+t)u) - f(x) \leq 0$. Thus $f_{qi}^{\infty 2}(u; u) \leq 0$, which together with assumption (b) give $u \in R_{qi}$, yielding a contradiction. Hence $\{x_k\}$ is bounded, and so standard arguments show that any limit point is a minimizer for f. The same reasoning also proves that $\underset{\mathbb{R}^n}{\operatorname{argmin}} f$ is bounded, so compact. ■

The same condition that f is lsc (on $\mathbb{R}^n$) is necessary as can be seen by the following example.

Example 6.5 *Take $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(x_1, x_2) = \begin{cases} x_1^2, & x_1 > 0, x_2 > 0, \\ 0, & x_1 = x_2 = 0, \\ +\infty, & \text{otherwise.} \end{cases}$$

The function f is actually convex, continuous on $\operatorname{dom} f$ *and* $\underset{\mathbb{R}^n}{\operatorname{argmin}} f = \{(0,0)\}$. *It can be easily seen that f is not lsc on $\mathbb{R}^2$ (for instance at $(0,1)$) and (c) does not hold since for $u = (0,1), f^\infty(u) = f_{qi}^{\infty 2}(u; u) = 0$.*

The continuity assumption can be deleted in the preceding theorem at the cost of strengthening the definition of $f_{qi}^{\infty 2}$ and so the set R_{qi}. As before, given $u \in \mathbb{R}^n, u \neq 0$

with $f^\infty(u)$ being finite, define

$$f_q^{\infty 2}(u;v) = \sup_{\substack{x\in \text{dom } f\\ s>0}} \inf_{t>0} \frac{f(x+tu+sv) - tf^\infty(u) - f(x)}{s}, \ \forall\, v \in \mathbb{R}^n, \tag{6.13}$$

and the set

$$R_q = \{u \in \mathbb{R}^n : \ f^\infty(u) = 0, \ f_q^{\infty 2}(u;u) = 0\}.$$

Theorem 6.2 *Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be lsc continuous and quasiconvex function. Then $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$ and compact, if and only if the following assertions hold:*

(a) $f^\infty(u) \geq 0$ *for all* $u \in \mathbb{R}^n \backslash \{0\}$*;*

(b) $[u \in (\text{dom } f)^\infty, \ f^\infty(u) = 0] \Rightarrow f_q^{\infty 2}(u;u) \geq 0$*;*

(c) $R_q = \{0\}$.

Proof. Suppose first that $\operatorname*{argmin}_{\mathbb{R}^n} f \neq \emptyset$ and compact. Obviously (a) holds and (b) follows from previous theorem and the fact that $f_q^{\infty 2} \geq f_{qi}^{\infty 2}$. Let $u \in R_q$, that is, $f_q^{\infty 2}(u;u) = 0$ and $f^\infty(u) = 0$. Then

$$\sup_{\substack{x\in \text{dom } f\\ s>0}} \inf_{t>0} \frac{f(x+(s+t)u) - f(x)}{s} = 0,$$

which implies that

$$\inf_{t>0} f(x+(s+t)u) \leq f(x), \ \forall\, x \in \text{dom } f, \ \forall\, s > 0.$$

In particular, if $\overline{x} \in \operatorname*{argmin}_{\mathbb{R}^n} f$, we get

$$\inf_{t>0} f(\overline{x}+(s+t)u) = f(\overline{x}), \ \forall\, s > 0. \tag{6.14}$$

We claim that

$$\overline{x} + (s+t)u \in \operatorname*{argmin}_{\mathbb{R}^n} f, \ \forall\, t > 0, \ \forall\, s > 0.$$

In fact, suppose to the contrary that there exists $r_0 > 0$, such that $f(\overline{x} + r_0 u) > f(\overline{x})$. By quasiconvexity

$$f(\overline{x} + r_0 u) \leq \max\{f(\overline{x}), f(\overline{x} + r_0 u + tu)\} \leq f(\overline{x} + r_0 u + tu), \ \forall\, t > 0,$$

which implies that $f(\overline{x}+r_0u) \le \inf_{t>0} f(\overline{x}+r_0u+tu) = f(\overline{x})$ by (6.14), and so $\overline{x}+(s+t)u \in \underset{\mathbb{R}^n}{\operatorname{argmin}} f$ for all $t > 0$ and all $s > 0$, yielding a contradiction. Hence $u = 0$.

For the other implication, we proceed as the previous proof with obvious changes.

■

The next example shows that in fact the last two previous theorems cover situations where the function may be non coercive.

Example 6.6 *Let us consider the non coercive function*

$$f(x) = \begin{cases} -x, & \text{if } x < 0, \\ \frac{x}{1+x}, & \text{if } x \ge 0. \end{cases}$$

We immediately obtain

$$f^{\infty}(u) = \begin{cases} -u, & \text{if } u < 0, \\ 0, & \text{if } u \ge 0. \end{cases}$$

Moreover, if $u > 0$ we get

$$\sup_{\substack{x>0 \\ s>0}} \inf_{t>0} \frac{f(x+tu+su) - tf^{\infty}(u) - f(x)}{s} = u.$$

Thus, $f_{qi}^{\infty 2}(u;u) = f_q^{\infty 2}(u;u) \ge u$ for all $u > 0$. Hence $R_{qi} = R_q = \{0\}$.

Example 6.7 *Take the quasiconvex function $f(x) = \sqrt{|x|}, x \in \mathbb{R}$. We see that $\underset{\mathbb{R}^n}{\operatorname{argmin}} f = \{0\}$ and $f^{\infty}(u) = 0$ for all $u \in \mathbb{R}$. If $u > 0$ then*

$$\sup_{\substack{x\ge 0 \\ s>0}} \inf_{t>0} \frac{\sqrt{|x+tu+su|} - \sqrt{|x|}}{s} = \sup_{\substack{x\ge 0 \\ s>0}} \frac{\sqrt{x+su} - \sqrt{x}}{s} = +\infty.$$

If $u < 0$ then

$$\sup_{\substack{x<0 \\ s>0}} \inf_{t>0} \frac{\sqrt{|x+tu+su|} - \sqrt{|x|}}{s} = \sup_{\substack{x<0 \\ s>0}} \frac{\sqrt{-x-su} - \sqrt{-x}}{s} = +\infty.$$

Hence,

$$f_q^{\infty 2}(u;u) = \begin{cases} 0, & \text{if } u = 0, \\ +\infty, & \text{if } u \ne 0. \end{cases}$$

As a consequence, $R_q = \{0\}$.

One may wonder whether

$$f_{qi}^{\infty 2}(u;u) = f_q^{\infty 2}(u;u).$$

First of all, we note that

$$f_{qi}^{\infty 2}(u;v) \leq f_{q}^{\infty 2}(u;v),\ \forall\ v \in \mathbb{R}^n,$$

and that if dom $f = \mathbb{R}^n$ then the equality is trivially satisfied. The following instance shows that a strict inequality may hold in general.

Example 6.8 *Take $f : \mathbb{R}^2 \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$f(x_1, x_2) = \begin{cases} \frac{\pi}{2}, & \text{if } x_2 > 0, \\ \arctan x_1, & \text{if } x_2 = 0. \\ +\infty, & \text{otherwise.} \end{cases}$$

Then f is lsc and quasiconvex. Take $u = (1, 0)$. For every $x = (x_1, x_2) \in \operatorname{intdom} f$, $s, t > 0$ we have that

$$\frac{f(x + tu + su) - f(x)}{s} = 0,$$

so $f_{qi}^{\infty 2}(u;u) = 0$. But, for $x = (0,0), s > 0$

$$\inf_{t>0} \frac{f(x + tu + su) - f(x)}{s} > 0,$$

then $f_{q}^{\infty 2}(u;u) > 0$.

6.2.2 Necessary condition in quasiconvex multiobjective optimization

For the Multiobjective Optimization Problem defined in Chapter 4, section 4.2, next lemma give us a necessary conditions for existence of efficient and weak efficient solutions. It is a generalization of Proposition 8.3 in [33], Proposition 4.1 (b) in [21] and Lemma 9 in [38]. We work with the usual cone $P = \mathbb{R}^m_+$ and we denote $f_i^{q\infty} = (f_i)^{q\infty}$.

Lemma 6.1 *Let $f_i : \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, lsc and quasiconvex function for all $i \in \{1, 2, \ldots, m\}$.*

(a) If $E \neq \emptyset$ then,

$$u \in \mathbb{R}^n : f_i^{q\infty}(u) \leq 0,\ \forall\ i \in \{1, 2, \ldots, m\} \Rightarrow f_i^{q\infty}(u) = 0,\ \forall\ i \in \{1, 2, \ldots, m\}.$$

(b) If $E_W \neq \emptyset$ then,

$$u \in \mathbb{R}^n : f_i^{q\infty}(u) \leq 0,\ \forall\ i \in \{1, 2, \ldots, m\} \Rightarrow \exists\ i_0 \in \{1, 2, \ldots, m\}, f_{i_0}^{q\infty}(u) = 0.$$

Proof. (a): If on the contrary, there exists $i_0 \in \{1, 2, \ldots, m\}$ such that $f_{i_0}^{q\infty}(u) < 0$, then

$$\frac{f_{i_0}(y + \lambda u) - f_{i_0}(y)}{\lambda} \leq f_{i_0}^{q\infty}(u) < 0, \ \forall \ y \in \operatorname{dom} f, \ \forall \ \lambda > 0.$$

Thus $f_i(y + \lambda u) \leq f_i(y)$ for all $i \in \{1, 2, \ldots, m\}$ and $f_{i_0}(y + \lambda u) < f_{i_0}(y)$, that is $F(y + \lambda u) - F(y) \in -\mathbb{R}^m_+ \setminus \{0\}$, for all $y \in \operatorname{dom} f$ and all $\lambda > 0$, contradicting the fact that E is nonempty. Hence, $f_i^{q\infty}(u) = 0$ for all $i \in \{1, 2, \ldots, m\}$.
(b): Suppose, on the contrary, that for all $i \in \{1, 2, \ldots, m\}$, $f_i^{q\infty}(u) < 0$. Then for all $i \in \{1, 2, \ldots, m\}$ we have

$$\frac{f_i(y + \lambda u) - f_i(y)}{\lambda} \leq f_{i_0}^{q\infty}(u) < 0, \ \forall \ y \in \operatorname{dom} f, \ \forall \ \lambda > 0,$$

from which $f_i(y + \lambda u) < f_i(y)$, for all $i \in \{1, 2, \ldots, m\}$, for all $y \in \operatorname{dom} f$ and all $\lambda > 0$. That is, $F(y+\lambda u) - F(y) \in -\operatorname{int} \mathbb{R}^m_+$ for all $y \in \operatorname{dom} f$ and all $\lambda > 0$. Which cannot happen if $E_W \neq \emptyset$, proving the result. ■

Chapter 7

Conclusion and future work

We develop new properties for the second order asymptotic cone and function, in particular, in the convex case. We give a complete characterization for the second order asymptotic cone in the convex case for two cases; first when we know that the vector u belongs to the (first order) asymptotic cone in Proposition 3.4, and second, when we do not know it in Proposition 3.5.

We also develop new easy formulas for computing the second order asymptotic function in the convex case, similar to the first order, in Proposition 3.17 equation (3.17) and (3.18). Another formula using only elements of the relative interior of the domain of the function is given in Proposition 3.18. We obtain clearly the relationship between the first and second order asymptotic functions in the convex and nonconvex case. Furthermore, calculus rules are also provided in Chapter 3.

We obtain applications in Chapter 4 for the scalar and multiobjective optimization problems in the convex and nonconvex case. A new existence result is given in Theorem 4.2 for non coercive convex functions using second order asymptotic analysis and a first approach to obtain a characterization for the boundedness from below of a lsc function is given in Proposition 4.2. A sufficient condition for the Domination Property and the existence of a Proper efficient solution in the convex and nonconvex case are given in Theorem 4.3. Finally, we give finer estimates for the second order asymptotic cone of the efficient and weakly efficient solution sets in the convex case, and a complete characterization in Theorem 4.5.

The second order asymptotic function gives more information about when a function is bounded from below or not, than the (first order) asymptotic function, this could means that we can obtain more applications in optimization theory using this new tool.

A first step to obtain an adequate definition in Banach spaces was presented in Chapter 5, with a very important limitation, that is, the condition of intdom $F \neq \emptyset$ is very restrictive in these spaces. We would need to develop and employ a weaker notion of interior as the "quasi relative interior", introduced by Borwein and Lewis in [15] and studied further in [14, 69, 70]. This could be a very interesting work in the future.

Finally, a new step searching the correct definition of the asymptotic function under generalized convexity assumptions are given in Chapter 6. We develop an idea of definition given by J. P. Penot more than ten years ago. We give a formula for computing the incident asymptotic function in Proposition 6.6 under no convexity assumption and we compare it with other two attempts presented recently in [34]. Applications in the scalar and multiobjective optimization problem to this definitions are also provide.

The discussion of the correct asymptotic function in the quasiconvex case is not closed at all. We believe our step could help towards the solution to this problem. We will be very glad to work on this problem in the future.

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