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METODOS DE ELEMENTOS FINITOS PARA PROBLEMAS DE ESTABILIDAD DE ESTRUCTURAS DELGADAS

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Resumen

El objetivo principal de esta tesis es análizar métodos numéricos para la aproximación de los coeficientes y modos de pandeo de estructuras delgadas. Específicamente, se estudia la aproximación por elementos finitos del problema de pandeo de placas y vigas.

En el primer trabajo, se estudia una formulación en términos de los momentos para los problemas de pandeo y de vibraciones de una placa poligonal elástica no necesariamente convexa modelada por las ecuaciones de Kirchhoff-Love. Para la discretización se consideran elementos finitos lineales a trozos y continuos para todas las variables. Usando la teoría espectral para operadores compactos, se obtienen resultados de convergencia óptimos para las autofunciones (desplazamiento transversal) y un doble orden para los autovalores (coeficientes de pandeo).

En el segundo trabajo, se estudia el problema de pandeo de una placa elástica modelada por las ecuaciones de Reissner-Mindlin. Este problema conduce al estudio espectral de un operador no compacto. Se demuestra que el espectro esencial del mismo está bien separado de los autovalores relevantes (coeficientes de pandeo) que se quieren calcular. Para la aproximación numérica se usan elementos finitos de bajo orden (DL3). Adaptando la teoría espectral para operadores no compactos, se demuestra convergencia óptima para las autofunciones y un doble orden para los autovalores, con estimaciones del error independientes del espesor de la placa, lo que demuestra que el método propuesto es libre de bloqueo ("locking-free").

En el tercer trabajo, se estudia un método de elementos finitos de bajo orden para el problema de pandeo de una viga no homogénea modelada por las ecuaciones de Timoshenko. Se da una caracterización espectral del problema continuo y usando la teoría

espectral para operadores no compactos, se demuestran órdenes óptimos de convergencia para las autofunciones (desplazamiento transversal, rotaciones y esfuerzos de corte) y un orden doble para los autovalores (coeficientes de pandeo), también con constantes independientes del espesor de la viga.

En todos los casos, se presentan ensayos numéricos que confirman los resultados teóricos obtenidos.

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Chapter 1

Introducción

Los principales objetivos en diseños de ingeniería son la seguridad y la durabilidad a lo largo del tiempo, aunque la importancia de los costos y los aspectos ambientales en el diseño ha crecido significativamente durante la última década. Autos, puentes y aviones, por ejemplo, tienen que cumplir cuidadosamente ciertos requerimientos mínimos prescritos de resistencia mecánica. Hoy en día, para cumplir este objetivo, existen herramientas eficientes tales como los métodos computacionales y el modelamiento matemático.

En las aplicaciones, al comenzar un proceso, se fijan el problema físico y los criterios de diseño. Luego, el problema se formula mediante un modelo matemático general, el cual es una idealización de la realidad (con posibles imperfecciones). En general, los problemas descritos por modelos matemáticos complejos no pueden ser resueltos de manera exacta y por lo tanto los métodos computacionales y las soluciones aproximadas son herramientas necesarias. Dependiendo de la necesidad y costo de los recursos computacionales, el modelo matemático general puede simplificarse por la experiencia de los ingenieros. Finalmente, el problema basado en el modelo matemático simplificado se resuelve aproximadamente por métodos numéricos y la solución obtenida se usa por los ingenieros en la toma de decisiones.

En el proceso de la resolución numérica debemos controlar el *error*, en particular, el llamado *error de discretización*, es decir, la diferencia entre la solución exacta del modelo matemático simplificado y su aproximación numérica. En todo el proceso también existen otro tipo de errores, por ejemplo, el error de modelamiento, el cual surge de la simplicación del modelo matemático general, el error de idealización que es la diferencia entre el modelo matemático general y el problema físico, etc. En lo que sigue, en este trabajo, solo nos preocuparemos del *error de discretización*.

1.1 Pandeo (Buckling) de estructuras delgadas

Un problema importante que ocurre en el diseño de estructuras delgadas en aplicaciones de ingeniería tales como carrocerías de automoviles, pilares de puentes, alas de un avion, etc., es el llamado *pandeo*. En estas aplicaciones, se pretende que una estructura resistente tenga un comportamiento estable, conservando sus características geométricas y de resistencia.

Cuando una estructura delgada se comprime mediante pequeñas cargas, ésta se deforma sin ningún cambio perceptible en la geometría y las cargas son soportadas. Cuando se alcanza el valor crítico de carga, inmediatamente la estructura experimenta una gran deformación y esta pierde las propiedades de resistencia. En este estado se dice que la estructura colapsó (pandeó). Por ejemplo, cuando una barra es sometida a una fuerza compresiva axial al principio ésta se comprime levemente, pero cuando alcanza la carga crítica la barra pandea. Un caso similar ocurre cuando tomamos un bastón de caminar y nos apoyamos sobre él dejando caer todo el peso del cuerpo que, si es considerable, hará que el bastón se curve produciéndose el pandeo. El pandeo también es conocido como inestabilidades estructurales.

Existen varios de tipos de inestabilidad en estructuras, pero en este trabajo nos centraremos en uno de los más importantes, el pandeo por flexión, el cual ya fue estudiado por Leonhard Euler (1707–1783). Este tipo de fenómeno inestable se produce al aplicar una carga axial de compresión, de cierta magnitud, a un elemento estructural.

El pandeo por flexión es la forma más elemental de pandeo y su estudio es un paso esencial para entender el comportamiento de pandeo de estructuras complejas, incluyendo estructuras con comportamiento inelástico, imperfecciones iniciales, etc. Es muy importante conocer la carga para la cual ocurre este tipo de pandeo, pues ésta rige el diseño de la estructura. Esta carga se denomina carga crítica de pandeo (*critical buckling load*, *critical load* o *limit of elastic stability*).

En la literatura, la carga crítica de pandeo para diferentes tipos de estructuras bajo distintos tipos de carga y condiciones de frontera se expresa usualmente mediante fórmulas simples de aproximación o tablas. Sin embargo, hoy en día los ingenieros requieren resultados más precisos para problemas en los cuales no existe soluciones analíticas disponibles. Cabe mencionar que salvo en unos pocos problemas (tal como el pandeo elástico de una barra ideal apoyada, bajo una fuerza axial), generalmente es muy trabajoso y en muchos casos imposible obtener soluciones analíticas exactas. Por lo tanto, es necesario utilizar métodos numéricos y en particular en este trabajo usaremos el *método de elementos finitos.*

El problema (que gobierna el fenómeno de pandeo) que surge de la modelación de este fenómeno es un *problema de autovalores* en el cual el autovalor representa la carga de pandeo y el autovector asociado el *modo de pandeo (buckling mode)*. El autovalor más pequeño corresponde a la carga crítica de pandeo.

Consideremos una placa elástica tridimensional de espesor $t > 0$ con configuración de referencia $\widetilde{\Omega} := \Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, donde Ω es un polígono $\mathbb{R}^2$ que describe la superficie media de la placa. Asumimos que la placa está empotrada en su frontera lateral $\partial\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$. En lo que sigue, resumiremos los argumentos dados en [17], para obtener las correspondientes ecuaciones del problema de pandeo (ver esta referencia y también [45] para más detalles).

Suponemos que $\widetilde{\boldsymbol{\sigma}}^0 := (\sigma_{ij}^0)_{1\leq i,j\leq 3}$, es un estado de tensiones pre-existente en la placa. Estas tensiones $\widetilde{\boldsymbol{\sigma}}^0$ que están ya presentes en la configuración de referencia, satisfacen las ecuaciones de equilibrio y se asume que son independientes de cualquier desplazamiento posterior que la configuración de referencia puede sufrir.

Sea $\widetilde{V} := \{\mathbf{v} \in H^1(\widetilde{\Omega})^3 : \mathbf{v} = \mathbf{0} \quad \text{on } \partial\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)\}$ el espacio de desplazamientos admisibles de la placa tridimensional. Si la configuración de referencia es perturbada por un pequeño cambio $\mathcal{F} \in V'$ (el cual podria ser una pequeña fuerza), entonces el trabajo hecho por $\widetilde{\boldsymbol{\sigma}}^0$ no puede ser despreciado. El desplazamiento correspondiente $\mathbf{u} = \{u_i\}_{1\leq i\leq 3}$, puede expresarse como la solución del siguiente problema (ver [17]):

Hallar $\mathbf{u} \in V$ *tal que*

$$\int_{\widetilde{\Omega}} C_{ijkl} u_{i,j} v_{k,l} + \int_{\widetilde{\Omega}} \widetilde{\boldsymbol{\sigma}}^0 u_{m,i} v_{m,j} = \langle \mathcal{F}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in V, \tag{1.1.1}$$

donde C_{ijkl} es el tensor de constantes elásticas del material, $u_{i,j} = \partial_j u_i$, y $\langle \cdot, \cdot \rangle$ denota la dualidad entre V' y V. El segundo término en el lado izquierdo es el trabajo hecho por $\widetilde{\boldsymbol{\sigma}}^0$. Restringimos nuestro análisis a múltiplos fijos de una *pre-tensión de pandeo* $\widetilde{\boldsymbol{\sigma}}$, es decir,

$$\widetilde{\boldsymbol{\sigma}}^0 = -\lambda_b \widetilde{\boldsymbol{\sigma}}, \tag{1.1.2}$$

donde λ_b representa la *carga de pandeo*. Luego, (1.1.1) queda:

Hallar $\mathbf{u} \in V$ *tal que*

$$\int_{\widetilde{\Omega}} C_{ijkl} u_{i,j} v_{k,l} - \lambda_b \int_{\widetilde{\Omega}} \widetilde{\boldsymbol{\sigma}} u_{m,i} v_{m,j} = \langle \mathcal{F}, \mathbf{v} \rangle \qquad \forall \mathbf{v} \in V. \tag{1.1.3}$$

Siguiendo [17], diremos que este problema es *establemente resoluble* si tiene una única solución para cada $\mathcal{F} \in V'$ y existe una constante C, independiente de $\mathcal{F}$, tal que

$$\|\mathbf{u}\|_V \leq C \|\mathcal{F}\|_{V'}.$$

Como antes mencionamos, nuestro objetivo será hallar el valor positivo más pequeño λ_b para el cual (1.1.3) no es establemente resoluble. Este λ_b es la carga crítica de pandeo que también se denomina el *límite de estabilidad elástico* de la estructura. Físicamente, representa al múltiplo más pequeño de la pre-tensión de pandeo $\widetilde{\boldsymbol{\sigma}}$, para el cual una pequeña perturbación en las condiciones externas sobre la placa puede causar pandeo. En [17] se mostró que este problema puede formularse como hallar el mínimo autovalor positivo λ_b del siguiente problema:

Hallar $\lambda_b \in \mathbb{R}$ *y* $\mathbf{0} \neq \mathbf{u} \in V$ *tal que*

$$\int_{\widetilde{\Omega}} C_{ijkl} u_{i,j} v_{k,l} = \lambda_b \int_{\widetilde{\Omega}} \widetilde{\boldsymbol{\sigma}} u_{m,i} v_{m,j} \qquad \forall \mathbf{v} \in V. \tag{1.1.4}$$

La aproximación por elementos finitos de la solución de problemas de autovalores tiene una larga historia. Referimos, por ejemplo, el libro de Babuška y Osborn [6]. La teoría de aproximación generalmente se desarrolla en términos del espectro de un operador $T : V \to V$ (donde V es un espacio de Sobolev apropiado) y de un operador discreto

$T_h : V_h \to V_h$ (donde V_h es el subespacio de elementos finitos de V). Dependiendo de la estructura delgada que consideremos y de las hipótesis cinemáticas de los desplazamientos, el operador T puede ser *compacto* [6, 35] o *no compacto* [18, 19]. Cuando T es compacto, usualmente el operador T_h converge a T en norma y se pueden derivar resultados de convergencia para los autovalores y autovectores (el espectro se reduce al $\{0\}$ y a una sucesión de autovalores aislados de multiplicidad finita cuyo único punto de acumulación es el 0). Por otra parte, cuando T es no compacto surgen varias complicaciones. Primero, el espectro esencial de T no se reduce al $\{0\}$ (como ocurre para operadores compactos). Esto significa que el espectro puede ahora contener, por ejemplo, autovalores de multiplicidad infinita, puntos de acumulación, espectro continuo, etc. Además, los resultados de convergencia no están garantizados y pueden existir autovalores espurios en la aproximación por elementos finitos.

1.2 Modelos de Placas

Los modelos que consideraremos serán simplificaciones de modelos basados en la teoría de elasticidad tridimensional. Mediante una reducción dimensional e hipótesis cinemáticas podemos obtener modelos para diferentes estructuras elásticas delgadas tales como: barras, vigas (una dimensión), membranas y placas (dos dimensiones).

En este trabajo consideraremos los problemas de pandeo de:

- Una placa modelada por las ecuaciones de Kirchhoff-Love.
- Una placa modelada por las ecuaciones de Reissner-Mindlin.
- Una viga no homogénea modelada por las ecuaciones de Timoshenko.

En el análisis de placas, los modelos más usados son el de Reissner-Mindlin (para placas delgadas y moderadamente gruesas) y el modelo de Kirchhoff-Love (placas delgadas) [10, 20].

En lo que sigue trataremos brevemente la reducción dimensional de los modelos de placa de Reissner-Mindlin y Kirchhoff-Love. Además mencionaremos los principios y suposiciones más importantes para estos modelos [10].

Sea como antes $\widetilde{\Omega} := \Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$ el dominio de una placa elástica tridimensional de espesor $t > 0$.

El campo de desplazamiento de la placa se denota por $\mathbf{u} = \{u_i(x, y, z)\}_{i=1}^3$ en coordenadas cartesianas globales x, y, z. En la teoría de placas de Reissner-Mindlin se asumen las siguientes suposiciones:

- Los puntos sobre la superficia media se deforman solamente en la dirección z.
- Todos los puntos contenidos en una normal al plano medio tienen el mismo desplazamiento vertical.
- Los puntos que antes de la deformación estaban sobre una recta normal al plano medio de la placa, permanecen al deformarse sobre una misma recta, sin que ésta tenga que ser necesariamente ortogonal a la deformada del plano medio.
- La tensión normal σ_{33} es despreciable.

Bajo estas condiciones, se tiene que el campo de desplazamientos admisibles tiene la forma:

$$\mathbf{u}_{\mathrm{RM}}(x, y, z) = \begin{pmatrix} -z\beta_1(x, y) \\ -z\beta_2(x, y) \\ w(x, y) \end{pmatrix}, \tag{1.2.1}$$

donde w es el desplazamiento transversal y $\beta = (\beta_1, \beta_2)$ son los ángulos que definen el giro de la normal.

En la teoría de Kirchhoff-Love, se mantienen las hipótesis anteriores pero la tercera se modifica como sigue:

- Los puntos sobre rectas normales al plano medio antes de la deformación, permanecen sobre rectas también ortogonales a la deformada del plano medio después de la deformación.

Bajo esta suposición, ahora el campo de desplazamientos toma la forma:

$$\mathbf{u}_{\mathrm{KL}}(x, y, z) = \begin{pmatrix} -z\frac{\partial w(x,y)}{\partial x} \\ -z\frac{\partial w(x,y)}{\partial y} \\ w(x, y) \end{pmatrix}, \tag{1.2.2}$$

donde w es el desplazamiento transversal.

Finalmente, para obtener los modelos matemáticos simplificados que describen el problema de pandeo de una placa (Reissner-Mindlin o Kirchhoff-Love) se consideran hipótesis adicionales sobre las relaciones entre deformaciones y tensiones (ley de Hooke), sobre la pre-tensión de pandeo $\widetilde{\boldsymbol{\sigma}}$, sobre el material (homogéneo, isotrópico), etc. Se sustituyen los desplazamientos admisibles (1.2.1) o (1.2.2) en (1.1.4) y se integra sobre el espesor t.

Se sabe que los métodos de elementos finitos conformes para placas de Kirchhoff-Love necesitan elementos $\mathcal{C}^1$, pues la formulación variacional natural para el problema del bilaplaciano es en H^2, lo cual implica usar aproximación de alto orden [14]. Otro tipo de técnica para aproximar este problema es usar métodos mixtos de elementos finitos [16, 2].

Por otra parte, éste no es el caso para placas de Reissner-Mindlin donde basta considerar elementos finitos $\mathcal{C}^0$. Sin embargo, debido al fenómeno de bloqueo ("locking") no se pueden utilizar elementos finitos estándar pues llevan a malos resultados cuando el espesor de la placa es muy pequeño. Para evitar este fenómeno se han considerado varias técnicas, entre las cuales podemos mencionar métodos basados en integración reducida (MITC) introducidos por Bathe y Dvorkin, o variaciones de éste propuestas por Durán y Liberman. Otra solución para este problema es escribir una formulación equivalente del problema en términos de dos problemas de Poisson y un problema tipo Stokes rotado, por medio de una descomposición de Helmholtz del esfuerzo de corte propuesta por Brezzi y Fortin y analizada por Arnold y Falk. Otras estrategias propuestas para evitar el bloqueo son los "Linked Interpolation Methods"analizados entre otros por Auricchio y Lovadina, y más recientemente Amara, Capatina-Papaghiuc y Chatti estudiaron una formulación en términos de momentos.

1.3 Organización de la tesis

En el Capítulo 2 de este trabajo consideramos la aproximación por elementos finitos de dos problemas espectrales para: (i) la determinación de los coeficientes y modos de pandeo y (ii) la aproximación de los primeros modos y frecuencias de vibración, de una placa empotrada no necesariamente convexa modelada por las ecuaciones de Kirchhoff-

Love. El método se basa en una discretización conforme de una formulación en términos de momentos [2]. El contenido de este capítulo corresponde al artículo [36]:

- D. Mora and R. Rodríguez, *A piecewise linear finite element method for the buckling and the vibration problems of thin plates*. Mathematics of Computation, **78** (2009), pp. 1891–1917.

Ya se indicó cual es el interés de conocer los coeficientes y modos de pandeo. Cabe mencionar que el conocimiento de las frecuencias y modos de vibración son necesarios para evitar efectos de resonancia. Cuando una fuerza externa periódica actúa sobre un sistema dinámico, la intensidad de la respuesta dependerá de la frecuencia de la fuerza externa y será máxima cuando ésta sea igual a una de las frecuencias naturales del sistema (es decir, la raíz cuadrada de alguno de los primeros valores propios del sistema). Si la fuerza periódica externa tiene un periodo cercano a los de resonancia se producirá un efecto importante sobre el sistema, lo cual podría corresponder a tensiones máximas o posibles rupturas.

En este artículo se ha probado convergencia y estimaciones de error óptimas para la aproximación del problema de pandeo y del problema de vibraciones usando la teoría abstracta de convergencia espectral presentada en [6] para operadores compactos. En ambos casos, todas las ecuaciones fueron discretizadas con elementos finitos lineales a trozos y continuos. Incluimos también resultados numéricos que muestran el buen comportamiento del método y comparamos con otros métodos clásicos.

En el Capítulo 3 de este trabajo consideramos la aproximación por elementos finitos de los coeficientes y modos de pandeo de una placa modelada por las ecuaciones de Reissner-Mindlin. Estos coeficientes son los recíprocos de los autovalores de un operador no compacto. Damos una caracterización espectral para este operador y mostramos que el espectro esencial del mismo está confinado a una bola centrada en el origen con radio proporcional al cuadrado del espesor de la placa. En cambio los coeficientes de pandeo relevantes corresponden a autovalores aislados de multiplicidad finita separados del espectro esencial, al menos si el espesor de la placa es suficientemente pequeño. El contenido de este capítulo corresponde al artículo [33], enviado para su publicación a *SIAM Journal on Numerical Analysis* y que se encuentra en la etapa de revisión:

- C. Lovadina, D. Mora, and R. Rodríguez, *Approximation of the buckling problem for Reissner-Mindlin plates*. Universidad de Concepción, Departamento de Ingeniería Matemática, Preprint **2009-01** (2009).

Para la aproximación numérica de los coeficientes y modos de pandeo, consideramos los elementos propuestos por Durán y Liberman en [22], los cuales se ha demostrado que son libres de bloqueo numérico en problemas de cargas y de vibraciónes. Luego se extiende la teoría clásica para operadores no compactos propuesta por Descloux, Nassif y Rappaz en [18, 19], para obtener estimaciones del error optimales uniformemente con respecto al espesor de la placa para las autofunciones y un orden doble para los autovalores, bajo la hipótesis de que las mallas son cuasi-uniformes. Las constantes de las estimaciones del error son independientes del espesor y dependen de normas de la solución que no degeneran cuando este espesor tiende a cero. Esto nos permite afirmar que en método propuesto es libre de bloqueo. Finalmente, incluimos resultados numéricos que muestran el buen comportamiento del método.

En el Capítulo 4 de este trabajo consideramos la aproximación de los coeficientes y modos de pandeo de una viga de Timoshenko no homogénea (la geometría y las propiedades físicas del material no se asumen constantes a lo largo de la viga). Al igual que en el capítulo anterior, los coeficientes y modos de pandeo se vinculan con los autovalores y autofunciones de un operador no compacto. Probamos que cuando el espesor de la viga es suficientemente pequeño los coeficientes de pandeo relevantes corresponden a autovalores aislados de multiplicidad finita. Para la aproximación por elementos finitos se considera el método mixto introducido por Arnold en [4] para el problema de flexión de vigas homogéneas de Timoshenko. Para la convergencia espectral y estimaciones del error adaptamos la teoría abstracta desarrollada en [18, 19] para operadores no compactos, pero de una manera alternativa a la del capítulo anterior. Así, se obtienen estimaciones del error óptimas para las autofunciones y un doble orden para los autovalores simples. Incluimos también resultados numéricos que muestran el buen comportamiento del método propuesto y confirman los resultados teóricos obtenidos. El contenido de este capítulo corresponde al artículo en preparación:

- C. Lovadina, D. Mora, and R. Rodríguez, *A locking-free finite element method for the buckling problem of a non-homogeneous Timoshenko beam.*

Finalmente, en el Capítulo 5 se presentan las conclusiones y las lineas de investigación abiertas de este trabajo.

Chapter 2

A piecewise linear finite element method for the buckling and the vibration problems of thin plates

2.1 Introduction

The analysis of finite element methods to solve plate eigenvalue problems has a long history. Let us mention among the oldest references the papers by Canuto [13], Ishihara [29, 30], Rannacher [37], and Mercier *et al.* [35, Section 7(b,d)]. While [37] deals with nonconforming methods for the biharmonic equation, all the other papers are based on different mixed formulations of the Kirchhoff model. These formulations turn out to be equivalent to the biharmonic equation when the solution is smooth enough (typically H^3). Therefore, in order to allow for such regularity to hold (see [27]), the plate is assumed to be convex in these references.

One of the most well-known mixed methods to deal with the biharmonic equation is the method introduced by Ciarlet and Raviart [16]. This was thoroughly studied by many authors (see, for instance, [12], [43], [24, Section 3(a)], [7, Section 4(a)], [26, Section III.3], [25], [3]). The method was applied to the plate vibration problem in [13] and [35, Section 7(b)], where it was proved to converge for finite elements of degree $k \geq 2$.

A formulation of the eigenvalue problem for the Stokes equation, which turns out to be equivalent to a plate buckling problem, is also analyzed in [35, Section 7(d)], where it is proved to converge for degree $k \geq 2$, as well. Although there is numerical evidence of optimal order convergence for piecewise linear elements applied to the vibration an the buckling plate problems (see in particular Section 2.5 below), to the best of our knowledge this has not been proved.

Other classical mixed method to deal with Kirchhoff plates was introduced by Miyoshi in [34] for load problems. This method is based on piecewise linear elements and was extended by Ishihara to the vibration problem in [29] and to the buckling problem in [30]. The method was proved to converge with a suboptimal order $\mathcal{O}(h^{1/2})$, but only for meshes uniform in the interior of the domain. This hypothesis cannot be avoided. In fact, we report in Section 2.5 numerical experiments which show that this method converges to wrong results when used on particular regular non-uniform meshes.

Another low-order method was introduced much more recently by Amara *et al.* in [2] to deal with the load problem for a Kirchhoff-Love plate subject to arbitrary boundary conditions. This method is based on a standard discretization by low-order conforming elements of a bending moment formulation. In the present paper we adapt this approach to the buckling and the vibration problems. We restrict our analysis to simply-connected polygonal clamped plates, not necessarily convex. In this case, all the equations are discretized by piecewise linear elements. We prove that the method leads to optimal orders of convergence for both, the vibration and the buckling problem. Since the analysis of the former is much straightforward, we describe in whole detail only the latter and summarize the results for the former.

The outline of the paper is as follows: We introduce in Section 2.2 both eigenvalue problems. We recall the mixed formulation in terms of bending moments and a third equivalent formulation considered in [2], which allows using standard finite elements for its discretization. In Section 2.3 we develop the numerical analysis of the buckling problem. With this aim, we introduce a linear operator whose spectrum is related with the solution of the buckling problem. A spectral characterization is given and additional regularity results are proved. Then, the finite element method is introduced and it is proved that it leads to optimal order approximation of the eigenfunctions. We end this section by proving that an improved order of convergence holds for the approximation of the eigenvalues. The same steps are briefly presented in Section 2.4 for the vibration problem, emphasizing the differences between both analyses. In Section 2.5 we report some numerical tests which confirm the theoretical results. We also include in this section numerical experiments with lowest-order Ciarlet-Raviart's and Ishihara's methods. These experiments show that Ciarlet-Raviart's method seems to converge with optimal order. The reported experiments also show that Ishihara's method fails when used on regular non-uniform meshes. We summarize some conclusions in Section 2.6. Finally, we give the matrix form of the discrete buckling problem in an appendix, which allows us to prove a spectral characterization of this generalized eigenvalue problem.

2.2 Problem statement

Let $\Omega \subset \mathbb{R}^2$ be a polygonal bounded simply-connected domain occupied by the mean surface of a plate, clamped on its whole boundary Γ. The plate is assumed to be homogeneous, isotropic, linearly elastic, and sufficiently thin as to be modeled by Kirchhoff-Love equations. We denote by u the transverse displacement of the mean surface of the plate.

The plate vibration problem reads as follows:

Find $(\lambda, u) \in \mathbb{R} \times H^2(\Omega)$, $u \neq 0$, such that

$$\begin{cases} \Delta^2 u = \lambda u & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \Gamma, \end{cases} \tag{2.2.1}$$

where $\lambda = \omega^2$, with $\omega > 0$ being the vibration frequency, and ∂_n denotes the normal derivative. To simplify the notation we have taken the Young modulus and the density of the plate, both equal to 1.

On the other hand, when the plate is subjected to a plane stress tensor field $\boldsymbol{\eta} : \Omega \to \mathbb{R}^{2\times 2}$, the corresponding linear buckling problem reads as follows:

Find $(\lambda, u) \in \mathbb{R} \times H^2(\Omega)$, $u \neq 0$, such that

$$\begin{cases} \Delta^2 u = -\lambda \left(\boldsymbol{\eta} : D^2 u\right) & \text{in } \Omega, \\ u = \partial_n u = 0 & \text{on } \Gamma, \end{cases} \tag{2.2.2}$$

where λ is in this case the critical load and $D^2u := (\partial_{ij}u)_{1\le i,j\le 2}$ denotes the Hessian matrix of u. The applied stress tensor field is assumed to satisfy the equilibrium equations:

$$\boldsymbol{\eta}^{\mathrm{T}} = \boldsymbol{\eta} \qquad \text{in } \Omega, \tag{2.2.3}$$

$$\mathbf{div}\,\boldsymbol{\eta} = 0 \qquad \text{in } \Omega. \tag{2.2.4}$$

Moreover, $\boldsymbol{\eta}$ is assumed to be essentially bounded, namely,

$$\boldsymbol{\eta} \in L^\infty(\Omega)^{2\times 2}. \tag{2.2.5}$$

However, we do not need to assume $\boldsymbol{\eta}$ to be positive definite. Let us remark that, in practice, $\boldsymbol{\eta}$ is the stress distribution on the plate subjected to in-plane loads, which does not need to be positive definite (see, for instance, Test 3 in Section 2.5.3 below).

Here and thereafter we use the following notation for any 2×2 tensor field $\boldsymbol{\tau}$, any 2D vector field $\boldsymbol{v}$, and any scalar field v:

$$\operatorname{div} \boldsymbol{v} := \partial_1 v_1 + \partial_2 v_2, \qquad \operatorname{rot} \boldsymbol{v} := \partial_1 v_2 - \partial_2 v_1, \qquad \operatorname{curl} v := \begin{pmatrix} \partial_2 v \\ -\partial_1 v \end{pmatrix},$$

$$\mathbf{div}\,\boldsymbol{\tau} := \begin{pmatrix} \partial_1 \tau_{11} + \partial_2 \tau_{12} \\ \partial_1 \tau_{21} + \partial_2 \tau_{22} \end{pmatrix}, \qquad \mathsf{Curl}\,\boldsymbol{v} := \begin{pmatrix} \partial_2 v_1 & -\partial_1 v_1 \\ \partial_2 v_2 & -\partial_1 v_2 \end{pmatrix}.$$

Moreover, we denote

$$\mathbf{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{J} := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

To obtain a weak formulation of each of the two spectral problems above, we multiply the corresponding equation by $v \in H_0^2(\Omega)$ and integrate twice by parts in Ω. Thus, for the vibration problem (2.2.1) we obtain:

Find $(\lambda, u) \in \mathbb{R} \times H_0^2(\Omega)$, $u \neq 0$, such that

$$\int_\Omega \Delta u\, \Delta v = \lambda \int_\Omega uv \qquad \forall v \in H_0^2(\Omega). \tag{2.2.6}$$

For the linear buckling problem we do the same and use the following lemma, which is easily proved by integrating by parts.

Lemma 2.2.1 *For all* $u \in H^2(\Omega)$, $v \in H_0^1(\Omega)$, *and* $\boldsymbol{\eta}$ *satisfying* (2.2.3)-(2.2.5),

$$\int_\Omega \left(\boldsymbol{\eta} : D^2 u\right) v = -\int_\Omega (\boldsymbol{\eta}\nabla u) \cdot \nabla v.$$

Thus, we obtain the following symmetric weak formulation of the buckling problem (2.2.2):

Find $(\lambda, u) \in \mathbb{R} \times H_0^2(\Omega)$, $u \neq 0$, such that

$$\int_\Omega \Delta u \, \Delta v = \lambda \int_\Omega (\boldsymbol{\eta}\nabla u) \cdot \nabla v \qquad \forall v \in H_0^2(\Omega). \tag{2.2.7}$$

It is well known that the eigenvalues of problem (2.2.6) are real and positive. Whenever $\boldsymbol{\eta}$ is positive definite, it is immediate to prove that those of problem (2.2.7) are real and positive, too. In any case these eigenvalues are real (see Lemma 2.3.1 below).

2.2.1 Formulation of the spectral problems in terms of bending moments

In what follows we adapt to the spectral problems of the previous section, an approach introduced and analyzed in [2] to deal with the load problem for Kirchhoff plates. Since the adaptation to the buckling problem presents several additional difficulties which must be tackled, we will describe this case in more detail and only summarize the analogous results for the vibration problem.

Let us denote

$$\mathcal{V} := H_0^1(\Omega) \qquad \text{and} \qquad \boldsymbol{\mathcal{X}} := \left\{ \boldsymbol{\tau} \in L^2(\Omega)^{2\times 2} : \ \operatorname{div}(\mathbf{div}\, \boldsymbol{\tau}) \in L^2(\Omega) \right\}.$$

It was proved in [2] that $\boldsymbol{\mathcal{X}}$ endowed with the norm

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\mathcal{X}}} := \left[\|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\operatorname{div}(\mathbf{div}\, \boldsymbol{\tau})\|_{0,\Omega}^2 \right]^{1/2}$$

is a Hilbert space and that $\mathcal{D}(\bar{\Omega})^{2\times 2}$ is a dense subspace of $\boldsymbol{\mathcal{X}}$. Moreover,

$$\int_\Omega \operatorname{div}(\mathbf{div}\, \boldsymbol{\tau}) v = \int_\Omega \boldsymbol{\tau} : D^2 v \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}, \quad \forall v \in H_0^2(\Omega). \tag{2.2.8}$$

Problem (2.2.7) can be rewritten as follows:

Find $(\lambda, \boldsymbol{\sigma}, u) \in \mathbb{R} \times \boldsymbol{\mathcal{X}} \times H_0^2(\Omega)$, $u \neq 0$, such that

$$\left\{\begin{array}{ll} \boldsymbol{\sigma} = \mathsf{C}(D^2 u) & \text{in } \Omega, \\ \operatorname{div}(\mathbf{div}\,\boldsymbol{\sigma}) = -\lambda \boldsymbol{\eta} : D^2 u & \text{in } \Omega. \end{array}\right. \tag{2.2.9}$$

In the expression above, $\boldsymbol{\sigma} = (\sigma_{ij})_{1 \leq i,j \leq 2}$ is the so called stress tensor and C is the linear operator arising from Hooke's law:

$$\mathsf{C}(\boldsymbol{\tau}) := (1-\nu)\,\boldsymbol{\tau} + \nu\,(\operatorname{tr}\boldsymbol{\tau})\,\mathbf{I}, \qquad \boldsymbol{\tau} \in \mathbb{R}^{2\times 2},$$

with $\nu \in (0, \frac{1}{2})$ being the Poisson coefficient. Let us remark that $\boldsymbol{\sigma}$ is a symmetric tensor as a consequence of the symmetry of $D^2 u$.

The equivalence between problems (2.2.7) and (2.2.9) is a straightforward consequence of (2.2.8) and the identity

$$\int_\Omega \mathsf{C}(D^2 u) : D^2 v = \int_\Omega \Delta u\,\Delta v \qquad \forall u, v \in H_0^2(\Omega),$$

which in its turn follows from the density of $\mathcal{D}(\Omega)$ in $H_0^2(\Omega)$ and integration by parts.

To obtain a weak formulation of problem (2.2.9) we proceed as in [2]. First note that the operator C is invertible, its inverse being given by

$$\mathsf{C}^{-1}(\boldsymbol{\tau}) = \frac{1}{1-\nu}\boldsymbol{\tau} - \frac{\nu}{1-\nu^2}\,(\operatorname{tr}\boldsymbol{\tau})\,\mathbf{I}, \qquad \boldsymbol{\tau} \in \mathbb{R}^{2\times 2}.$$

Next, consider the following closed subspace of $\boldsymbol{\mathcal{X}}$:

$$\boldsymbol{\mathcal{X}}^0 := \left\{\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}} :\ \operatorname{div}(\mathbf{div}\,\boldsymbol{\tau}) = 0\right\}.$$

The first equation of problem (2.2.9) can be equivalently written $\mathsf{C}^{-1}(\boldsymbol{\sigma}) = D^2 u$. By testing this equation with $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$ and using (2.2.8), we obtain

$$\int_\Omega \mathsf{C}^{-1}(\boldsymbol{\sigma}) : \boldsymbol{\tau} = \int_\Omega D^2 u : \boldsymbol{\tau} = \int_\Omega \operatorname{div}(\mathbf{div}\,\boldsymbol{\tau}) u = 0 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0. \tag{2.2.10}$$

On the other hand, taking traces in the first equation of (2.2.9), it follows that u is the unique solution of the problem

$$\left\{\begin{array}{ll} \Delta u - \dfrac{1}{1+\nu}\operatorname{tr}\boldsymbol{\sigma} = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma. \end{array}\right. \tag{2.2.11}$$

Moreover, let ϕ be the solution of the problem

$$\begin{cases} \Delta\phi = -\lambda\boldsymbol{\eta} : D^2 u & \text{in } \Omega, \\ \phi = 0 & \text{on } \Gamma, \end{cases} \tag{2.2.12}$$

and let

$$\boldsymbol{\sigma}^0 := \boldsymbol{\sigma} - \phi\mathbf{I}.$$

Since $\operatorname{div}(\mathbf{div}\,\phi\mathbf{I}) = \Delta\phi$, from the second equation in (2.2.9) and the first one in (2.2.12), we have that $\operatorname{div}(\mathbf{div}\,\boldsymbol{\sigma}^0) = 0$ and, hence, $\boldsymbol{\sigma}^0 \in \boldsymbol{\mathcal{X}}^0$.

Therefore, by testing problems (2.2.11) and (2.2.12) with functions in $\mathcal{V}$, substituting $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \phi\mathbf{I}$ in (2.2.10) and (2.2.11), and using Lemma 2.2.1, we arrive at the following weak formulation of problem (2.2.9):

Find $(\lambda, \phi, \boldsymbol{\sigma}^0, u) \in \mathbb{R} \times \mathcal{V} \times \boldsymbol{\mathcal{X}}^0 \times \mathcal{V}$, $u \neq 0$, such that

$$\begin{cases} \displaystyle\int_\Omega \nabla\phi \cdot \nabla v = -\lambda \int_\Omega (\boldsymbol{\eta}\nabla u) \cdot \nabla v & \forall v \in \mathcal{V}, \\ \displaystyle\int_\Omega \mathsf{C}^{-1}(\boldsymbol{\sigma}^0 + \phi\mathbf{I}) : \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla\gamma + \frac{1}{1+\nu}\int_\Omega \left(\operatorname{tr}\boldsymbol{\sigma}^0 + 2\phi\right)\gamma = 0 & \forall \gamma \in \mathcal{V}. \end{cases} \tag{2.2.13}$$

The following lemma will be used to prove that this problem is actually equivalent to problem (2.2.7).

Lemma 2.2.2 *Given $\boldsymbol{\chi} \in L^2(\Omega)^{2\times 2}$, there holds $\int_\Omega \boldsymbol{\chi} : \boldsymbol{\tau} = 0$ for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$ if and only if there exists $v \in H_0^2(\Omega)$ such that $\boldsymbol{\chi} = D^2 v$.*

Proof. Let $\boldsymbol{\chi} \in L^2(\Omega)^{2\times 2}$ be such that $\int_\Omega \boldsymbol{\chi} : \boldsymbol{\tau} = 0$ for all $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$. Let $v \in H_0^2(\Omega)$ be the solution of the following problem:

$$\int_\Omega D^2 v : D^2 w = \int_\Omega \boldsymbol{\chi} : D^2 w \qquad \forall w \in H_0^2(\Omega).$$

Hence, $\boldsymbol{\chi} - D^2 v \in \boldsymbol{\mathcal{X}}^0$ and, consequently,

$$\int_\Omega \boldsymbol{\chi} : \left(\boldsymbol{\chi} - D^2 v\right) = 0.$$

On the other hand, testing the problem above with $w = v$, we have that

$$\int_\Omega \big(\boldsymbol{\chi} - D^2 v\big) : D^2 v = 0.$$

Subtracting this equation from the previous one, we obtain

$$\int_\Omega \big(\boldsymbol{\chi} - D^2 v\big) : \big(\boldsymbol{\chi} - D^2 v\big) = 0$$

and, hence, $\boldsymbol{\chi} = D^2 v$. Since the converse is a direct consequence of the definition of $\boldsymbol{\mathcal{X}}^0$, we conclude the proof. $\square$

Now we are in a position to prove that problems (2.2.13) and (2.2.7) are equivalent.

Proposition 2.2.3 $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ *is a solution of problem* (2.2.13) *if and only if* (λ, u) *is a solution of problem* (2.2.7) *and* $\boldsymbol{\sigma} := \boldsymbol{\sigma}^0 + \phi \mathbf{I} = \mathsf{C}(D^2 u)$.

Proof. It has been already shown that problems (2.2.7) and (2.2.9) are equivalent. So, it is enough to prove the equivalence between problems (2.2.13) and (2.2.9).

Let $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ be a solution of problem (2.2.13). The first equation of this problem and Lemma 2.2.1 imply that ϕ satisfies (2.2.12). Therefore, since $\boldsymbol{\sigma}^0 \in \boldsymbol{\mathcal{X}}^0$, $\boldsymbol{\sigma} := \boldsymbol{\sigma}^0 + \phi \mathbf{I}$ satisfies the second equation of (2.2.9).

On the other hand, the second equation of (2.2.13) and Lemma 2.2.2 imply that there exists $v \in H_0^2(\Omega)$ such that $\mathsf{C}^{-1}(\boldsymbol{\sigma}^0 + \phi \mathbf{I}) = D^2 v$ or, equivalently, $\boldsymbol{\sigma}^0 + \phi \mathbf{I} = \mathsf{C}(D^2 v)$. By taking traces in this expression, we observe that v is the unique solution of the following problem,

$$\begin{cases} \Delta v = \dfrac{1}{1+\nu}\big(\operatorname{tr} \boldsymbol{\sigma}^0 + 2\phi\big) & \text{in } \Omega, \\ v = 0 & \text{on } \Gamma, \end{cases}$$

whose weak form coincides with the third equation of problem (2.2.13). Consequently, $v = u$. Therefore, $u \in H_0^2(\Omega)$ and $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \phi \mathbf{I} = \mathsf{C}(D^2 u)$, which allows us to conclude that $(\lambda, \boldsymbol{\sigma}, u)$ is a solution of problem (2.2.9).

The converse has been already proved when deducing (2.2.9), so we conclude the proof. $\square$

Remark 2.2.4 *Although no symmetry constraint is explicitly imposed in problem* (2.2.13) *on* $\boldsymbol{\sigma}^0$ *(and hence on* $\boldsymbol{\sigma} = \boldsymbol{\sigma}^0 + \phi\mathbf{I}$*), according to the theorem above* $\boldsymbol{\sigma} = \mathsf{C}(D^2u)$*. Consequently,* $\boldsymbol{\sigma}$ *and a fortiori the term* $\boldsymbol{\sigma}^0$ *in the solution of problem* (2.2.13) *turn out to be symmetric, anyway.*

Analogously, the vibration problem (2.2.1) can be rewritten as follows:

Find $(\lambda, \boldsymbol{\sigma}, u) \in \mathbb{R} \times \boldsymbol{\mathcal{X}} \times H_0^2(\Omega)$, $u \neq 0$, such that

$$\begin{cases} \boldsymbol{\sigma} = \mathsf{C}(D^2u) & \text{in } \Omega, \\ \operatorname{div}(\operatorname{\mathbf{div}} \boldsymbol{\sigma}) = \lambda u & \text{in } \Omega. \end{cases}$$

The same arguments used for the buckling problem lead to the following weak formulation of this problem:

Find $(\lambda, \phi, \boldsymbol{\sigma}^0, u) \in \mathbb{R} \times \mathcal{V} \times \boldsymbol{\mathcal{X}}^0 \times \mathcal{V}$, $u \neq 0$, such that

$$\begin{cases} \displaystyle\int_\Omega \nabla\phi \cdot \nabla v = -\lambda \int_\Omega uv & \forall v \in \mathcal{V}, \\ \displaystyle\int_\Omega \mathsf{C}^{-1}(\boldsymbol{\sigma}^0 + \phi\mathbf{I}) : \boldsymbol{\tau} = 0 & \forall \boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla\gamma + \frac{1}{1+\nu} \int_\Omega \left(\operatorname{tr} \boldsymbol{\sigma}^0 + 2\phi\right) \gamma = 0 & \forall \gamma \in \mathcal{V}. \end{cases} \tag{2.2.14}$$

Finally, the following equivalence result holds true:

Proposition 2.2.5 $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ *is a solution of problem* (2.2.14) *if and only if* (λ, u) *is a solution of problem* (2.2.6) *and* $\boldsymbol{\sigma} := \boldsymbol{\sigma}^0 + \phi\mathbf{I} = \mathsf{C}(D^2u)$*.*

2.2.2 Equivalent variational formulations

Our next step is to introduce new variational formulations of the buckling and the vibration spectral problems, which allow using standard finite elements for their discretization. With this purpose, we follow once more the arguments proposed in [2] to obtain a convenient decomposition of the space $\boldsymbol{\mathcal{X}}^0$.

Consider the following space:

$$\mathrm{H} := \left\{ \boldsymbol{\xi} \in H^1(\Omega)^2 : \int_\Omega \xi_1 = 0, \int_\Omega \xi_2 = 0 \text{ and } \int_\Omega \operatorname{div} \boldsymbol{\xi} = 0 \right\},$$

endowed with the norm

$$\|\boldsymbol{\xi}\|_{\mathrm{H}} := \left(\|\partial_2\xi_1\|_{0,\Omega}^2 + \tfrac{1}{2}\,\|\partial_2\xi_2 - \partial_1\xi_1\|_{0,\Omega}^2 + \|\partial_1\xi_2\|_{0,\Omega}^2\right)^{1/2}.$$

It is shown in [2] that $\|\cdot\|_{\mathrm{H}}$ and $\|\cdot\|_{1,\Omega}$ are equivalent norms in H, as a consequence of Korn's inequality.

In the same reference, it is also shown that, for each symmetric $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$, there exists a unique $\boldsymbol{\xi} \in \mathrm{H}$ such that

$$\boldsymbol{\tau} = \mathsf{Curl}\,\boldsymbol{\xi} + \tfrac{1}{2}\,(\operatorname{div}\boldsymbol{\xi})\,\mathbf{J}. \tag{2.2.15}$$

Since by virtue of Remark 2.2.4 the term $\boldsymbol{\sigma}^0$ in the solution of problem (2.2.13) turns out to be symmetric, then it can be accordingly written

$$\boldsymbol{\sigma}^0 = \mathsf{Curl}\,\boldsymbol{\psi} + \tfrac{1}{2}\,(\operatorname{div}\boldsymbol{\psi})\,\mathbf{J},$$

for a unique $\boldsymbol{\psi} \in \mathrm{H}$.

Remark 2.2.6 *The simple-connectedness assumption on Ω is necessary for the representation* (2.2.15) *to hold true for all symmetric $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$. This is tacitly assumed in the proofs of [2, Section 4.1].*

We introduce the following continuous bilinear form in H:

$$\begin{aligned} A(\boldsymbol{\psi},\boldsymbol{\xi}) :&= \int_\Omega \mathsf{C}^{-1}\big(\mathsf{Curl}\,\boldsymbol{\psi} + \tfrac{1}{2}\,(\operatorname{div}\boldsymbol{\psi})\,\mathbf{J}\big) : \big(\mathsf{Curl}\,\boldsymbol{\xi} + \tfrac{1}{2}\,(\operatorname{div}\boldsymbol{\xi})\,\mathbf{J}\big) \\ &= \frac{1}{1-\nu}\int_\Omega \left[\partial_2\psi_1\partial_2\xi_1 + \partial_1\psi_2\partial_1\xi_2 + \tfrac{1}{2}\,(\partial_2\psi_2 - \partial_1\psi_1)\,(\partial_2\xi_2 - \partial_1\xi_1)\right] \\ &\quad - \frac{\nu}{1-\nu^2}\int_\Omega \operatorname{rot}\boldsymbol{\psi}\,\operatorname{rot}\boldsymbol{\xi}. \end{aligned} \tag{2.2.16}$$

Straightforward calculus leads to

$$A(\boldsymbol{\xi},\boldsymbol{\xi}) = \frac{1}{1+\nu}\,\|\boldsymbol{\xi}\|_{\mathrm{H}}^2 + \frac{\nu}{1-\nu^2}\int_\Omega \left[(\partial_2\xi_1 + \partial_1\xi_2)^2 + (\partial_2\xi_2 - \partial_1\xi_1)^2\right],$$

which shows that $A(\boldsymbol{\xi},\boldsymbol{\xi}) \geq \frac{1}{1+\nu}\|\boldsymbol{\xi}\|_{\mathrm{H}}^2$ and, consequently, $A(\cdot,\cdot)$ is H-elliptic.

On the other hand, explicit computations lead to

$$\int_\Omega \mathsf{C}^{-1}(\phi\mathbf{I}) : \big(\mathsf{Curl}\,\boldsymbol{\xi} + \tfrac{1}{2}\,(\operatorname{div}\boldsymbol{\xi})\,\mathbf{J}\big) = -\frac{1}{1+\nu}\int_\Omega \phi\,\operatorname{rot}\boldsymbol{\xi} \tag{2.2.17}$$

and

$$\operatorname{tr}\boldsymbol{\sigma}^0 = \operatorname{tr}\left(\mathsf{Curl}\,\boldsymbol{\psi} + \tfrac{1}{2}\left(\operatorname{div}\boldsymbol{\psi}\right)\mathbf{J}\right) = -\operatorname{rot}\boldsymbol{\psi}. \tag{2.2.18}$$

Using all this in problem (2.2.13), we obtain the following new formulation of the buckling problem:

Find $(\lambda, \phi, \boldsymbol{\psi}, u) \in \mathbb{R} \times \mathcal{V} \times \mathrm{H} \times \mathcal{V}$, $u \neq 0$, such that

$$\left\{\begin{array}{ll}\displaystyle\int_\Omega \nabla\phi\cdot\nabla v = -\lambda\int_\Omega (\boldsymbol{\eta}\nabla u)\cdot\nabla v & \forall v \in \mathcal{V},\\ \displaystyle A(\boldsymbol{\psi},\boldsymbol{\xi}) - \frac{1}{1+\nu}\int_\Omega \phi\operatorname{rot}\boldsymbol{\xi} = 0 & \forall \boldsymbol{\xi} \in \mathrm{H},\\ \displaystyle\int_\Omega \nabla u\cdot\nabla\gamma + \frac{1}{1+\nu}\int_\Omega (-\operatorname{rot}\boldsymbol{\psi} + 2\phi)\,\gamma = 0 & \forall \gamma \in \mathcal{V}.\end{array}\right. \tag{2.2.19}$$

In what follows we show that problems (2.2.13) and (2.2.19) are equivalent.

Proposition 2.2.7 $(\lambda, \phi, \boldsymbol{\psi}, u)$ *is a solution of problem* (2.2.19) *if and only if* $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ *is a solution of problem* (2.2.13), *with* $\boldsymbol{\sigma}^0 = \mathsf{Curl}\,\boldsymbol{\psi} + \frac{1}{2}\left(\operatorname{div}\boldsymbol{\psi}\right)\mathbf{J}$.

Proof. Let $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ be a solution of problem (2.2.13). Let $\boldsymbol{\psi} \in \mathrm{H}$ be such that $\boldsymbol{\sigma}^0 = \mathsf{Curl}\,\boldsymbol{\psi} + \frac{1}{2}\left(\operatorname{div}\boldsymbol{\psi}\right)\mathbf{J}$. Given $\boldsymbol{\xi} \in \mathrm{H}$, let $\boldsymbol{\tau} := \mathsf{Curl}\,\boldsymbol{\xi} + \frac{1}{2}\left(\operatorname{div}\boldsymbol{\xi}\right)\mathbf{J} \in \boldsymbol{\mathcal{X}}^0$. Then, the last two equations in (2.2.19) follow from the corresponding ones in (2.2.13) by using (2.2.16)–(2.2.18).

Conversely, let $(\lambda, \phi, \boldsymbol{\psi}, u)$ be a solution of problem (2.2.19) and $\boldsymbol{\sigma}^0 := \mathsf{Curl}\,\boldsymbol{\psi} + \frac{1}{2}\left(\operatorname{div}\boldsymbol{\psi}\right)\mathbf{J} \in \boldsymbol{\mathcal{X}}^0$. The third equation in (2.2.19) and (2.2.18) yield the third equation in (2.2.13). On the other hand, the second equation in (2.2.19), (2.2.16), and (2.2.17) yield the second equation in (2.2.13), but only for symmetric test functions $\boldsymbol{\tau} = \mathsf{Curl}\,\boldsymbol{\xi} + \frac{1}{2}\left(\operatorname{div}\boldsymbol{\xi}\right)\mathbf{J} \in \boldsymbol{\mathcal{X}}^0$. To end the proof we will show that this equation also holds true for skew-symmetric test functions. In fact, since $\boldsymbol{\sigma}^0$ is symmetric, $\mathsf{C}^{-1}(\boldsymbol{\sigma}^0)$ is symmetric too and so is $\mathsf{C}^{-1}(\phi\mathbf{I})$ as well. Hence, for any skew-symmetric $\boldsymbol{\tau} \in \boldsymbol{\mathcal{X}}^0$, there holds $\int_\Omega \mathsf{C}^{-1}(\boldsymbol{\sigma}^0 + \phi\mathbf{I}) : \boldsymbol{\tau} = 0$ and we conclude the proof. $\square$

Analogously, the vibration problem (2.2.14) can be written as follows:

Find $(\lambda, \phi, \boldsymbol{\psi}, u) \in \mathbb{R} \times \mathcal{V} \times \mathrm{H} \times \mathcal{V}$, $u \neq 0$, such that

$$\left\{\begin{array}{ll} \displaystyle\int_\Omega \nabla\phi \cdot \nabla v = -\lambda \int_\Omega uv & \forall v \in \mathcal{V}, \\ \displaystyle A(\boldsymbol{\psi}, \boldsymbol{\xi}) - \frac{1}{1+\nu}\int_\Omega \phi \operatorname{rot} \boldsymbol{\xi} = 0 & \forall \boldsymbol{\xi} \in \mathrm{H}, \\ \displaystyle\int_\Omega \nabla u \cdot \nabla\gamma + \frac{1}{1+\nu}\int_\Omega (-\operatorname{rot}\boldsymbol{\psi} + 2\phi)\,\gamma = 0 & \forall \gamma \in \mathcal{V}. \end{array}\right. \tag{2.2.20}$$

The following equivalence result also holds true:

Proposition 2.2.8 *$(\lambda, \phi, \boldsymbol{\psi}, u)$ is a solution of problem* (2.2.20) *if and only if $(\lambda, \phi, \boldsymbol{\sigma}^0, u)$ is a solution of problem* (2.2.14)*, with $\boldsymbol{\sigma}^0 = \mathsf{Curl}\,\boldsymbol{\psi} + \frac{1}{2}(\operatorname{div}\boldsymbol{\psi})\,\mathbf{J}$.*

Remark 2.2.9 *In both problems,* (2.2.19) *and* (2.2.20)*, the eigenvalues cannot vanish. In fact, in both cases, if $\lambda = 0$, then the first equation yields $\phi = 0$, the second one and the* H*-ellipticity of A lead to $\boldsymbol{\psi} = 0$, and, from the third one, $u = 0$. Moreover, $\int_\Omega (\boldsymbol{\eta}\nabla u)\cdot\nabla u \neq 0$ in problem* (2.2.19)*, despite the fact that $\boldsymbol{\eta}$ is not necessarily positive definite. This is a consequence of the equivalence between problems* (2.2.19) *and* (2.2.7) *(cf. Propositions 2.2.7 and 2.2.3). Indeed, in problem* (2.2.7)*, $\int_\Omega (\boldsymbol{\eta}\nabla u)\cdot\nabla u = 0$ implies $\Delta u = 0$ and, hence, $u = 0$.*

Finally, to end this section, we introduce a more compact notation for the spectral problems (2.2.19) and (2.2.20). Let $\mathscr{A} : (\mathcal{V} \times \mathrm{H} \times \mathcal{V}) \times (\mathcal{V} \times \mathrm{H} \times \mathcal{V}) \to \mathbb{R}$, $\mathscr{B} : L^2(\Omega) \times L^2(\Omega) \to \mathbb{R}$, and $\mathscr{C} : H^1(\Omega) \times H^1(\Omega) \to \mathbb{R}$, be the continuous and symmetric bilinear forms respectively defined by

$$\begin{aligned} \mathscr{A}((\phi, \boldsymbol{\psi}, u), (\gamma, \boldsymbol{\xi}, v)) &:= A(\boldsymbol{\psi}, \boldsymbol{\xi}) + \int_\Omega \nabla\phi\cdot\nabla v + \int_\Omega \nabla u \cdot \nabla\gamma \\ &\qquad - \frac{1}{1+\nu}\left[\int_\Omega \phi \operatorname{rot}\boldsymbol{\xi} + \int_\Omega \gamma \operatorname{rot}\boldsymbol{\psi}\right] + \frac{2}{1+\nu}\int_\Omega \phi\gamma, \\ \mathscr{B}(u, v) &:= \int_\Omega uv, \\ \mathscr{C}(u, v) &:= \int_\Omega (\boldsymbol{\eta}\nabla u)\cdot\nabla v. \end{aligned}$$

Using this notation, problems (2.2.19) and (2.2.20) can be respectively written as follows:

Find $(\lambda, \phi, \boldsymbol{\psi}, u) \in \mathbb{R} \times \mathcal{V} \times \mathrm{H} \times \mathcal{V}$, $u \neq 0$, such that

$$\mathscr{A}((\phi, \boldsymbol{\psi}, u), (\gamma, \boldsymbol{\xi}, v)) = -\lambda \mathscr{C}(u, v) \qquad \forall (\gamma, \boldsymbol{\xi}, v) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}. \tag{2.2.21}$$

Find $(\lambda, \phi, \boldsymbol{\psi}, u) \in \mathbb{R} \times \mathcal{V} \times \mathrm{H} \times \mathcal{V}$, $u \neq 0$, such that

$$\mathscr{A}((\phi, \boldsymbol{\psi}, u), (\gamma, \boldsymbol{\xi}, v)) = -\lambda \mathscr{B}(u, v) \qquad \forall (\gamma, \boldsymbol{\xi}, v) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}. \tag{2.2.22}$$

2.3 Numerical analysis of the buckling problem

Before introducing the numerical method, we define the linear operator corresponding to the source problem associated with the buckling spectral problem (2.2.21) and prove some properties that will be used for the subsequent convergence analysis. Consider the following source problem:

Given $f \in \mathcal{V}$, find $(\phi, \boldsymbol{\psi}, u) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}$ such that

$$\mathscr{A}((\phi, \boldsymbol{\psi}, u), (\gamma, \boldsymbol{\xi}, v)) = -\mathscr{C}(f, v) \qquad \forall (\gamma, \boldsymbol{\xi}, v) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}. \tag{2.3.1}$$

This problem is well posed. In fact, it can be decomposed into the following sequence of three well posed problems:

1. Find $\phi \in \mathcal{V}$ such that

$$\int_\Omega \nabla \phi \cdot \nabla v = -\int_\Omega (\boldsymbol{\eta} \nabla f) \cdot \nabla v \qquad \forall v \in \mathcal{V}. \tag{2.3.2}$$

2. Find $\boldsymbol{\psi} \in \mathrm{H}$ such that

$$A(\boldsymbol{\psi}, \boldsymbol{\xi}) = G^{\phi}(\boldsymbol{\xi}) := \frac{1}{1+\nu} \int_\Omega \phi \operatorname{rot} \boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathrm{H}. \tag{2.3.3}$$

3. Find $u \in \mathcal{V}$ such that

$$\int_\Omega \nabla u \cdot \nabla \gamma = R^{\phi, \boldsymbol{\psi}}(\gamma) := \frac{1}{1+\nu} \int_\Omega (\operatorname{rot} \boldsymbol{\psi} - 2\phi)\, \gamma \qquad \forall \gamma \in \mathcal{V}. \tag{2.3.4}$$

Let T be the bounded linear operator defined by

$$\begin{aligned} T:\ \mathcal{V} &\to \mathcal{V}, \\ f &\mapsto u, \end{aligned}$$

with $(\phi, \boldsymbol{\psi}, u) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}$ being the solution of (2.3.1). Clearly λ is an eigenvalue of problem (2.2.21) if and only if $\mu := \frac{1}{\lambda}$ is a non-zero eigenvalue of T, with the same multiplicity and corresponding eigenfunctions u (recall $\lambda \neq 0$; cf. Remark 2.2.9).

The arguments used in the previous sections applied now to problem (2.3.1) allow us to show its equivalence with the following one:

Given $f \in \mathcal{V}$, find $u \in H_0^2(\Omega)$ such that

$$\int_\Omega \Delta u \, \Delta v = \int_\Omega (\boldsymbol{\eta} \nabla f) \cdot \nabla v \qquad \forall v \in H_0^2(\Omega). \tag{2.3.5}$$

More precisely, u coincides in both problems and

$$\boldsymbol{\sigma} := \mathsf{C}(D^2 u) = \mathsf{Curl}\, \boldsymbol{\psi} + \tfrac{1}{2} (\operatorname{div} \boldsymbol{\psi}) \, \mathbf{J} + \phi \mathbf{I}.$$

As a consequence, we can prove the following spectral characterization:

Lemma 2.3.1 *The spectrum of T satisfies $\mathrm{Sp}(T) = \{0\} \cup \{\mu_n :\ n \in \mathbb{N}\}$, where $\{\mu_n\}_{n \in \mathbb{N}}$ is a sequence of real eigenvalues which converges to 0. The multiplicity of each non-zero eigenvalue is finite and its ascent is 1.*

Proof. By virtue of the equivalence between problems (2.3.1) and (2.3.5), T is also a bounded linear operator from $\mathcal{V}$ into $H_0^2(\Omega)$. Hence, because of the compact inclusion $H_0^2(\Omega) \hookrightarrow \mathcal{V}$ and the spectral characterization of compact operators, we have that $\mathrm{Sp}(T) = \{0\} \cup \{\mu_n :\ n \in \mathbb{N}\}$, with $\{\mu_n\}_{n \in \mathbb{N}}$ a sequence of finite-multiplicity eigenvalues which converges to 0.

Moreover, it is simple to prove by using (2.2.3) that $T|_{H_0^2(\Omega)} : H_0^2(\Omega) \to H_0^2(\Omega)$ is self-adjoint with respect to the inner product $(u, v) \mapsto \int_\Omega \Delta u \, \Delta v$. Therefore, since $\mathrm{Sp}(T) = \{0\} \cup \mathrm{Sp}(T|_{H_0^2(\Omega)})$, we conclude that the non-zero eigenvalues of T are real and have ascent 1. Thus we end the proof. $\square$

Another conclusion of the equivalence between problems (2.3.1) and (2.3.5) is the following additional regularity result.

Lemma 2.3.2 *There exist $s \in (\frac{1}{2}, 1]$ and $C > 0$ such that, for all $f \in \mathcal{V}$, the solution $(\phi, \boldsymbol{\psi}, u)$ of problem* (2.3.1) *satisfies $u \in H^{2+s}(\Omega)$, $\boldsymbol{\psi} \in H^{1+s}(\Omega)^2$, and*

$$\|\phi\|_{1,\Omega} + \|u\|_{2+s,\Omega} + \|\boldsymbol{\psi}\|_{1+s,\Omega} \leq C \|f\|_{1,\Omega}.$$

Proof. The estimate for ϕ (which does not involve any additional regularity) follows directly from (2.3.2) and (2.2.5). The estimate for u follows from the equivalence between problems (2.3.1) and (2.3.5) and the classical regularity result for the biharmonic problem with right-hand side in $H^{-1}(\Omega)$ (cf. [27]).

To prove the estimate for $\boldsymbol{\psi}$, we use the explicit expression (2.2.16) for A to write

$$A(\boldsymbol{\psi}, \boldsymbol{\xi}) = \frac{1}{1-\nu} \int_\Omega \boldsymbol{\varepsilon}(\tilde{\boldsymbol{\psi}}) : \boldsymbol{\varepsilon}(\tilde{\boldsymbol{\xi}}) - \frac{\nu}{1-\nu^2} \int_\Omega \operatorname{div} \tilde{\boldsymbol{\psi}} \operatorname{div} \tilde{\boldsymbol{\xi}}, \tag{2.3.6}$$

with $\tilde{\boldsymbol{\psi}} = (\psi_2, -\psi_1)$, $\tilde{\boldsymbol{\xi}} = (\xi_2, -\xi_1)$, and $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1 \leq i,j \leq 2}$ being the standard strain tensor defined by $\varepsilon_{ij}(\boldsymbol{v}) := \frac{1}{2}(\partial_i v_j + \partial_j v_i)$, $1 \leq i, j \leq 2$. By substituting (2.3.6) into (2.3.3) and integrating by parts the right-hand side, we find that $\tilde{\boldsymbol{\psi}}$ is the solution of an elasticity-like problem with Lamé coefficients $\tilde{\mu} := \frac{1}{2(1-\nu)}$ and $\tilde{\lambda} := -\frac{\nu}{1-\nu^2}$, source term $-\frac{1}{1+\nu}\nabla\phi \in L^2(\Omega)^2$, and traction free boundary conditions. Notice that $\tilde{\mu} > 0$ and $\tilde{\lambda} + \tilde{\mu} = \frac{1}{2(1+\nu)} > 0$, too. Moreover, since the source term is orthogonal to the set of rigid motions, because of the constraints in the definition of H, the elasticity-like problem is well posed. Hence, from a classical regularity result for the elasticity equations (see, for instance, [41, Theorem 5.2]), there exists $s \in (\frac{1}{2}, 1]$ such that $\tilde{\boldsymbol{\psi}}$ and a fortiori $\boldsymbol{\psi}$ satisfy

$$\|\boldsymbol{\psi}\|_{1+s,\Omega} \leq C \|\nabla\phi\|_{0,\Omega} \leq C \|f\|_{1,\Omega}.$$

Thus we conclude the proof. □

Remark 2.3.3 *The constant s in the lemma above is the Sobolev regularity for the biharmonic equation with right-hand side in $H^{-1}(\Omega)$ and homogeneous Dirichlet boundary conditions. In fact, for the linear elasticity equations with right-hand side in $L^2(\Omega)^2$ and purely homogeneous Neumann conditions, the Sobolev regularity s is the same one. This constant only depends on the domain Ω. If Ω is convex, then $s = 1$. Otherwise, the lemma holds for all $s < s_0$, where $s_0 \in (\frac{1}{2}, 1)$ depends on the largest reentrant angle of Ω (see [27] for the precise equation determining s_0).*

Remark 2.3.4 *The lemma above does not fix any further regularity for ϕ. Indeed, no additional regularity can be expected for arbitrary $f \in \mathcal{V}$. For instance, from* (2.3.2)*, if $\boldsymbol{\eta} = \mathbf{I}$, then $\phi \equiv f$.*

2.3.1 Finite element approximation

For the numerical approximation, we consider a regular family $\{\mathcal{T}_h\}_{h>0}$ of triangular meshes in $\bar{\Omega}$ and the standard piecewise linear continuous finite element space

$$\mathcal{L}_h := \left\{ v_h \in \mathcal{C}(\bar{\Omega}) : \ v_h|_T \in \mathcal{P}_1(T) \quad \forall T \in \mathcal{T}_h \right\}.$$

Let $\mathcal{V}_h$ and H_h be the finite-dimensional subspaces of $\mathcal{V}$ and H, respectively defined by

$$\mathcal{V}_h := \mathcal{L}_h \cap \mathcal{V} = \{ v_h \in \mathcal{L}_h : \ v_h = 0 \text{ on } \Gamma \},$$
$$\mathrm{H}_h := \mathcal{L}_h^2 \cap \mathrm{H} = \left\{ \boldsymbol{\xi}_h \in \mathcal{L}_h^2 : \ \int_\Omega \xi_{h1} = 0, \ \int_\Omega \xi_{h2} = 0 \text{ and } \int_\Omega \operatorname{div} \boldsymbol{\xi}_h = 0 \right\}.$$

The discrete version of problem (2.2.21) reads as follows:

Find $(\lambda_h, \phi_h, \boldsymbol{\psi}_h, u_h) \in \mathbb{R} \times \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h$, $u_h \neq 0$, such that

$$\mathscr{A}((\phi_h, \boldsymbol{\psi}_h, u_h), (\gamma_h, \boldsymbol{\xi}_h, v_h)) = -\lambda_h \mathscr{C}(u_h, v_h) \qquad \forall (\gamma_h, \boldsymbol{\xi}_h, v_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h. \quad (2.3.7)$$

Let T_h be the bounded linear operator defined by

$$T_h : \ \mathcal{V} \to \mathcal{V},$$
$$f \mapsto u_h,$$

with $(\phi_h, \boldsymbol{\psi}_h, u_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h$ being the solution of the discrete analog of problem (2.3.1):

$$\mathscr{A}((\phi_h, \boldsymbol{\psi}_h, u_h), (\gamma_h, \boldsymbol{\xi}_h, v_h)) = -\mathscr{C}(f, v_h) \qquad \forall (\gamma_h, \boldsymbol{\xi}_h, v_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h. \quad (2.3.8)$$

As in the continuous case, this problem decomposes into a sequence of three well-posed problems, which are the respective discretizations of (2.3.2)–(2.3.4):

$$\phi_h \in \mathcal{V}_h : \qquad \int_\Omega \nabla \phi_h \cdot \nabla v_h = -\int_\Omega (\boldsymbol{\eta} \nabla f) \cdot \nabla v_h \qquad \forall v_h \in \mathcal{V}_h, \quad (2.3.9)$$

$$\boldsymbol{\psi}_h \in \mathrm{H}_h : \qquad A(\boldsymbol{\psi}_h, \boldsymbol{\xi}_h) = G^{\phi_h}(\boldsymbol{\xi}_h) \qquad \forall \boldsymbol{\xi}_h \in \mathrm{H}_h, \quad (2.3.10)$$

$$u_h \in \mathcal{V}_h : \qquad \int_\Omega \nabla u_h \cdot \nabla \gamma_h = R^{\phi_h, \boldsymbol{\psi}_h}(\gamma_h) \qquad \forall \gamma_h \in \mathcal{V}_h. \quad (2.3.11)$$

Also as in the continuous case, λ_h is an eigenvalue of problem (2.3.7) if and only if $\mu_h := \frac{1}{\lambda_h}$ is a non-zero eigenvalue of T_h, with the same multiplicity and corresponding eigenfunctions u_h.

Remark 2.3.5 *The same arguments leading to Remark 2.2.9 allow us to show that any solution of problem* (2.3.7) *satisfies* $\lambda_h \neq 0$. *Moreover,* $\int_\Omega (\boldsymbol{\eta}\nabla u_h) \cdot \nabla u_h \neq 0$ *also holds true, but the proof of this fact is postponed to the Appendix (cf. Remark 2.6.3, below).*

In what follows we will prove that $T_h \to T$ in norm as $h \to 0$. As a consequence, for all non-zero $\mu \in \mathrm{Sp}(T)$ and h small enough, there exists $\mu_h \in \mathrm{Sp}(T_h)$ such that $\mu_h \to \mu$. In particular, this implies that the discrete spectral problem (2.3.7) has solutions, at least for h sufficiently small, as long as $\mathrm{Sp}(T) \neq \{0\}$. A thorough spectral characterization is postponed to the Appendix (cf. Proposition 2.6.2, below), where the matrix form of problem (2.3.7) is introduced.

The following lemma yields the uniform convergence of T_h to T as $h \to 0$.

Lemma 2.3.6 *There exist* $C > 0$ *and* $r \in (\frac{1}{2}, 1]$ *such that, for all* $f \in \mathcal{V}$,

$$\|(T - T_h) f\|_{1,\Omega} \leq C h^r \|f\|_{1,\Omega} .$$

Proof. Given $f \in \mathcal{V}$, let $(\phi, \boldsymbol{\psi}, u)$ and $(\phi_h, \boldsymbol{\psi}_h, u_h)$ be the solutions of problems (2.3.1) and (2.3.8), respectively, so that $u = Tf$ and $u_h = T_h f$. From (2.3.4), (2.3.11), and the first Strang Lemma (cf. [15]), we have

$$\|u - u_h\|_{1,\Omega} \leq C \left[\inf_{\gamma_h \in \mathcal{V}_h} \|u - \gamma_h\|_{1,\Omega} + \sup_{\gamma_h \in \mathcal{V}_h} \frac{R^{\phi_h, \boldsymbol{\psi}_h}(\gamma_h) - R^{\phi, \boldsymbol{\psi}}(\gamma_h)}{\|\gamma_h\|_{1,\Omega}} \right] . \tag{2.3.12}$$

To estimate the first term in the right-hand side above, we use standard approximation results and the regularity of u proved in Lemma 2.3.2:

$$\inf_{\gamma_h \in \mathcal{V}_h} \|u - \gamma_h\|_{1,\Omega} \leq C h \|u\|_{2,\Omega} \leq C h \|f\|_{1,\Omega} . \tag{2.3.13}$$

For the second term, we use the definition of R (cf. (2.3.4)) and integration by parts to obtain

$$\sup_{\gamma_h \in \mathcal{V}_h} \frac{R^{\phi_h, \boldsymbol{\psi}_h}(\gamma_h) - R^{\phi, \boldsymbol{\psi}}(\gamma_h)}{\|\gamma_h\|_{1,\Omega}} \leq C \left(\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega} + \|\phi - \phi_h\|_{0,\Omega} \right) . \tag{2.3.14}$$

Now, we resort to a duality argument to estimate $\|\phi - \phi_h\|_{0,\Omega}$, since no additional regularity holds for ϕ (cf. Remark 2.3.4). Let

$$\chi \in H^1(\Omega) : \qquad \int_\Omega \nabla\chi \cdot \nabla\gamma = \int_\Omega (\phi - \phi_h)\,\gamma \qquad \forall \gamma \in H_0^1(\Omega). \tag{2.3.15}$$

By virtue of standard regularity results for the Laplace equation (see [27]), there exists $r \in (\frac{1}{2}, 1]$ such that $\chi \in H^{1+r}(\Omega)$ and

$$\|\chi\|_{1+r,\Omega} \leq C\,\|\phi - \phi_h\|_{0,\Omega}\,.$$

Let $\chi^{\mathrm{I}} \in \mathcal{V}_h$ be the Lagrange interpolant of χ. Taking $\gamma = \phi - \phi_h$ in (2.3.15) and using (2.3.2), (2.3.9), and standard approximation results, we have

$$\begin{aligned}
\|\phi - \phi_h\|_{0,\Omega}^2 &= \int_\Omega \nabla\chi \cdot \nabla(\phi - \phi_h) = \int_\Omega \nabla(\chi - \chi^{\mathrm{I}}) \cdot \nabla(\phi - \phi_h) \\
&\leq Ch^r\,\|\chi\|_{1+r,\Omega}\,\|\nabla(\phi - \phi_h)\|_{0,\Omega} \\
&\leq Ch^r\,\|\phi - \phi_h\|_{0,\Omega}\,\|\nabla(\phi - \phi_h)\|_{0,\Omega}\,.
\end{aligned}$$

Therefore, from (2.3.2) and (2.3.9), again, and (2.2.5),

$$\|\phi - \phi_h\|_{0,\Omega} \leq Ch^r\left(\|\nabla\phi\|_{0,\Omega} + \|\nabla\phi_h\|_{0,\Omega}\right) \leq Ch^r\,\|f\|_{1,\Omega}\,. \tag{2.3.16}$$

The next step is to estimate $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}$. With this aim, we consider first $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega}$. From (2.3.3), (2.3.10), the ellipticity of A, and the first Strang Lemma again, we have

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega} \leq C\left[\inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi} - \boldsymbol{\xi}_h\|_{1,\Omega} + \sup_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \frac{G^{\phi_h}(\boldsymbol{\xi}_h) - G^{\phi}(\boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{1,\Omega}}\right].$$

We use standard approximation results and Lemma 2.3.2 once more, to obtain

$$\inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi} - \boldsymbol{\xi}_h\|_{1,\Omega} \leq Ch^s\,\|\boldsymbol{\psi}\|_{1+s,\Omega} \leq Ch^s\,\|f\|_{1,\Omega}\,,$$

with $s \in (\frac{1}{2}, 1]$, whereas from the definition of G (cf. (2.3.3)),

$$\sup_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \frac{G^{\phi_h}(\boldsymbol{\xi}_h) - G^{\phi}(\boldsymbol{\xi}_h)}{\|\boldsymbol{\xi}_h\|_{1,\Omega}} \leq C\,\|\phi - \phi_h\|_{0,\Omega} \leq Ch^r\,\|f\|_{1,\Omega}\,.$$

Thus, defining $t := \min\{s, r\} \in (\frac{1}{2}, 1]$, we obtain that

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega} \leq Ch^t \|f\|_{1,\Omega}.$$

Next, we use another duality argument to estimate $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}$. Let

$$\boldsymbol{\rho} \in \mathrm{H}: \qquad A(\boldsymbol{\rho}, \boldsymbol{\xi}) = \int_\Omega (\boldsymbol{\psi} - \boldsymbol{\psi}_h) \cdot \boldsymbol{\xi} \qquad \forall \boldsymbol{\xi} \in \mathrm{H}. \tag{2.3.17}$$

The same arguments used in the proof of Lemma 2.3.2 allow us to show that

$$\|\boldsymbol{\rho}\|_{1+s,\Omega} \leq C \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}.$$

Hence, using again standard approximation results, we know that there exists $\boldsymbol{\rho}_h \in \mathrm{H}_h$ such that

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{1,\Omega} \leq Ch^s \|\boldsymbol{\rho}\|_{1+s,\Omega} \leq Ch^s \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}.$$

Thus, taking $\boldsymbol{\xi} = \boldsymbol{\psi} - \boldsymbol{\psi}_h$ in (2.3.17) and using (2.3.3) and (2.3.10), we obtain

$$\begin{aligned}
\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}^2 &= A(\boldsymbol{\rho}, \boldsymbol{\psi} - \boldsymbol{\psi}_h) = A(\boldsymbol{\rho} - \boldsymbol{\rho}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) + A(\boldsymbol{\rho}_h, \boldsymbol{\psi} - \boldsymbol{\psi}_h) \\
&\leq C \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{1,\Omega} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega} + \frac{1}{1+\nu} \left| \int_\Omega (\phi - \phi_h) \operatorname{rot} \boldsymbol{\rho}_h \right| \\
&\leq Ch^{s+t} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega} \|f\|_{1,\Omega} + Ch^r \|f\|_{1,\Omega} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega}.
\end{aligned}$$

Therefore, since $s + t > 1$,

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega} \leq Ch^r \|f\|_{1,\Omega}. \tag{2.3.18}$$

Thus, the lemma follows from (2.3.12), (2.3.13), (2.3.14), (2.3.16), and (2.3.18). □

Remark 2.3.7 *The order of convergence r depends on the maximum Sobolev regularity of the domain for the Laplace equations with right-hand side in $L^2(\Omega)$ and homogeneous Dirichlet boundary conditions. In particular, if Ω is convex, then $r = 1$. Otherwise, the lemma holds for all $r < r_0 := \frac{\pi}{\theta}$, with θ being the largest reentrant angle of Ω (cf. [27]).*

The following lemma shows that the error estimate for $\|(T - T_h)f\|_{1,\Omega}$ can be improved when f is smoother.

Lemma 2.3.8 *There exists $C > 0$ such that, for all $f \in \mathcal{V} \cap H^2(\Omega)$,*

$$\|(T - T_h)\, f\|_{1,\Omega} \leq Ch\, \|f\|_{2,\Omega} .$$

Proof. We follow exactly the same steps as in the proof of Lemma 2.3.6. However, now $\phi \in H^{1+r}(\Omega)$, with $r > \frac{1}{2}$ as in Remark 2.3.7, and $\|\phi\|_{1+r} \leq C\|f\|_{2,\Omega}$. In fact, ϕ is the solution of (2.3.2), which by virtue of Lemma 2.2.1 is a weak form of

$$\begin{cases} -\Delta\phi = \boldsymbol{\eta} : D^2 f \in L^2(\Omega), \\ \phi = 0 \qquad \text{on } \Gamma. \end{cases}$$

Hence, the estimate for $\|\phi - \phi_h\|_{0,\Omega}$ can be improved by using that

$$\|\nabla(\phi - \phi_h)\|_{0,\Omega} \leq Ch^r\, \|\phi\|_{1+r,\Omega} \leq Ch^r\, \|f\|_{2,\Omega} .$$

Consequently, we obtain instead of (2.3.16)

$$\|\phi - \phi_h\|_{0,\Omega} \leq Ch^{2r}\, \|f\|_{2,\Omega} . \tag{2.3.19}$$

This last inequality can be used to improve (2.3.18) as follows:

$$\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{0,\Omega} \leq Ch^{s+t}\, \|f\|_{1,\Omega} + Ch^{2r}\, \|f\|_{2,\Omega} . \tag{2.3.20}$$

Therefore, since $s + t > 1$ and $2r > 1$, too, the lemma follows from (2.3.12), (2.3.13), (2.3.14), (2.3.19), and (2.3.20). □

2.3.2 Spectral convergence and error estimates

As a direct consequence of Lemma 2.3.6, T_h converges in norm to T as h goes to zero. Hence, standard results of spectral approximation (see, for instance, [31]) show that isolated parts of $\mathrm{Sp}(T)$ are approximated by isolated parts of $\mathrm{Sp}(T_h)$. More precisely, let $\mu \neq 0$ be an eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. There exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces.

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces $\mathcal{M}$ and $\mathcal{N}$ of $H_0^1(\Omega)$:

$$\hat{\delta}(\mathcal{M},\mathcal{N}) := \max\left\{\delta(\mathcal{M},\mathcal{N}),\delta(\mathcal{N},\mathcal{M})\right\},$$

where

$$\delta(\mathcal{M},\mathcal{N}) := \sup_{\substack{x\in\mathcal{M}\\ \|x\|_{1,\Omega}=1}} \left(\inf_{y\in\mathcal{N}} \|x-y\|_{1,\Omega}\right).$$

The following theorem implies spectral convergence with an optimal order for the approximation of the eigenfunctions.

Theorem 2.3.9 *There exists a strictly positive constant C such that*

$$\hat{\delta}(\mathcal{E},\mathcal{E}_h) \leq Ch,$$
$$\left|\mu - \mu_h^{(i)}\right| \leq Ch, \qquad i = 1,\ldots,m.$$

Proof. As a consequence of Lemma 2.3.6, T_h converges in norm to T as h goes to zero. Then, the proof follows as a direct consequence of Lemma 2.3.8 and Theorems 7.1 and 7.3 from [6] and the fact that, for $f \in \mathcal{E}$, $\|f\|_{2,\Omega} \leq C\|f\|_{1,\Omega}$, because of Lemma 2.3.2. □

The error estimates for the eigenvalues $\mu \neq 0$ of T yield analogous estimates for the eigenvalues $\lambda = \frac{1}{\mu}$ of problem (2.2.21). However, the order of convergence in Theorem 2.3.9 is not optimal for μ. Our next goal is to improve this order.

With this purpose, let us denote $\lambda_h := 1/\mu_h^{(i)}$, with $\mu_h^{(i)}$ being any particular eigenvalue of T_h converging to μ. Let u_h, ϕ_h, and $\boldsymbol{\psi}_h$ be such that $(\lambda_h,\phi_h,\boldsymbol{\psi}_h,u_h)$ is a solution of problem (2.3.7) with $\|u_h\|_{1,\Omega} = 1$. According to Theorem 2.3.9, there exists a solution $(\lambda,\phi,\boldsymbol{\psi},u)$ of problem (2.2.21) with $\|u\|_{1,\Omega} = 1$ such that

$$\|u-u_h\|_{1,\Omega} \leq Ch. \tag{2.3.21}$$

The following lemma, which will be used to prove an improved order of convergence for the corresponding eigenvalues, shows estimates for $\phi - \phi_h$ and $\boldsymbol{\psi} - \boldsymbol{\psi}_h$.

Lemma 2.3.10 *There exists $C > 0$ such that*

$$\begin{aligned}\|\phi-\phi_h\|_{1,\Omega} + \|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,\Omega} &\leq C\left(h + \inf_{v_h\in\mathcal{V}_h}\|\phi - v_h\|_{1,\Omega} + \inf_{\boldsymbol{\xi}_h\in\mathrm{H}_h}\|\boldsymbol{\psi}-\boldsymbol{\xi}_h\|_{1,\Omega}\right)\\ &\leq Ch^t,\end{aligned}$$

where $t := \min\{s,r\} \in (\frac{1}{2},1]$, with s and r as in Lemma 2.3.2 and 2.3.6, respectively.

Proof. First note that $(\phi, \boldsymbol{\psi}, u)$ is the solution of problem (2.3.1) with $f = \lambda u$. Hence, from Lemma 2.3.2, $u \in H^2(\Omega)$ with $\|u\|_{2,\Omega} \leq C\lambda \|u\|_{1,\Omega}$. Hence, the same arguments used in the proof of Lemma 2.3.8 allows us to show that

$$\|\phi\|_{1+r,\Omega} \leq C \|u\|_{2,\Omega} \leq C\lambda \|u\|_{1,\Omega} .$$

On the other hand, $(\phi_h, \boldsymbol{\psi}_h, u_h)$ is the solution of problem (2.3.8) with $f = \lambda_h u_h$. Thus, from the equivalence between this problem and (2.3.9)–(2.3.11), ϕ_h is the solution of (2.3.9) with $f = \lambda_h u_h$. Hence, from the first Strang Lemma again,

$$\|\phi - \phi_h\|_{1,\Omega} \leq C \left[\inf_{v_h \in \mathcal{V}_h} \|\phi - v_h\|_{1,\Omega} + \sup_{v_h \in \mathcal{V}_h} \frac{\displaystyle\int_\Omega [\boldsymbol{\eta}\,(\lambda \nabla u - \lambda_h \nabla u_h)] \cdot \nabla v_h}{\|v_h\|_{1,\Omega}} \right].$$

To estimate the first term in the right-hand side above, we use standard approximation results:

$$\inf_{v_h \in \mathcal{V}_h} \|\phi - v_h\|_{1,\Omega} \leq C h^r \|\phi\|_{1+r,\Omega} \leq C h^r \|u\|_{1,\Omega} .$$

For the second term, we use the Cauchy-Schwarz inequality, (2.2.5), (2.3.21), and Theorem 2.3.9:

$$\begin{aligned} \sup_{v_h \in \mathcal{V}_h} \frac{\displaystyle\int_\Omega [\boldsymbol{\eta}\,(\lambda \nabla u - \lambda_h \nabla u_h)] \cdot \nabla v_h}{\|v_h\|_{1,\Omega}} &\leq C \|\lambda \nabla u - \lambda_h \nabla u_h\|_{0,\Omega} \\ &\leq C |\lambda| \|u - u_h\|_{1,\Omega} + |\lambda - \lambda_h| \|u_h\|_{1,\Omega} \\ &\leq C h. \end{aligned}$$

On the other hand, to estimate the term $\|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega}$, we repeat the arguments in the proof of Lemma 2.3.6 (with $f = \lambda u$) to obtain

$$\begin{aligned} \|\boldsymbol{\psi} - \boldsymbol{\psi}_h\|_{1,\Omega} &\leq C \left(\inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi} - \boldsymbol{\xi}_h\|_{1,\Omega} + \|\phi - \phi_h\|_{0,\Omega} \right) \\ &\leq C h^s \lambda \|u\|_{1,\Omega} + C \|\phi - \phi_h\|_{0,\Omega} . \end{aligned}$$

Next, repeating the arguments in the proof of Lemma 2.3.8, we have from (2.3.19) that

$$\|\phi - \phi_h\|_{0,\Omega} \leq C h^{2r} \lambda \|u\|_{1,\Omega} .$$

Thus, we conclude the proof. □

Now we are in a position to prove an improved order of convergence for the eigenvalues.

Theorem 2.3.11 *There exists a strictly positive constant C such that*

$$|\lambda - \lambda_h| \leq C\left(h^2 + \inf_{v_h \in \mathcal{V}_h} \|\phi - v_h\|_{1,\Omega}^2 + \inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi} - \boldsymbol{\xi}_h\|_{1,\Omega}^2\right) \leq Ch^{2t},$$

with $t \in (\frac{1}{2}, 1]$ as in Lemma 2.3.10.

Proof. We adapt to our case a standard argument (cf. [6, Lemma 9.1]). Let $U := (\phi, \boldsymbol{\psi}, u)$ and $U_h := (\phi_h, \boldsymbol{\psi}_h, u_h)$ be as in the proof of Lemma 2.3.10. Because of (2.2.21) and (2.3.7),

$$\begin{aligned}\mathscr{A}(U - U_h, U - U_h) &= \mathscr{A}(U, U) - 2\mathscr{A}(U, U_h) + \mathscr{A}(U_h, U_h) \\ &= -\lambda\mathscr{C}(u, u) + 2\lambda\mathscr{C}(u, u_h) - \lambda_h\mathscr{C}(u_h, u_h),\end{aligned}$$

whereas

$$\lambda\mathscr{C}(u - u_h, u - u_h) = \lambda\mathscr{C}(u, u) - 2\lambda\mathscr{C}(u, u_h) + \lambda\mathscr{C}(u_h, u_h).$$

Therefore, since $\mathscr{C}(u_h, u_h) \neq 0$ (cf. Remark 2.3.5),

$$\lambda - \lambda_h = \frac{\mathscr{A}(U - U_h, U - U_h) + \lambda\mathscr{C}(u - u_h, u - u_h)}{\mathscr{C}(u_h, u_h)}.$$

Moreover, from (2.3.21), $\mathscr{C}(u_h, u_h) \xrightarrow{h} \mathscr{C}(u, u) \neq 0$ (cf. Remark 2.2.9). Hence,

$$\begin{aligned}|\lambda - \lambda_h| &\leq C\left(|\mathscr{A}(U - U_h, U - U_h)| + |\lambda|\,|\mathscr{C}(u - u_h, u - u_h)|\right) \\ &\leq C\left(\|U - U_h\|_{\mathcal{V}\times\mathrm{H}\times\mathcal{V}}^2 + \|u - u_h\|_{1,\Omega}^2\right) \\ &\leq C\left(h^2 + \inf_{v_h \in \mathcal{V}_h} \|\phi - v_h\|_{1,\Omega}^2 + \inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi} - \boldsymbol{\xi}_h\|_{1,\Omega}^2\right) \\ &\leq Ch^{2t},\end{aligned}$$

the last two inequalities because of (2.3.21) and Lemma 2.3.10. Thus, we conclude the proof. □

Remark 2.3.12 *The order of convergence for the eigenvalues do not depend on the regularity of the eigenfunction u, which always belongs to $H^2(\Omega)$, but on the regularity of the auxiliary quantities ϕ and $\boldsymbol{\psi}$. In fact, the $\mathcal{O}(h^t)$ error estimate in Lemma 2.3.10 could be improved, provided ϕ and $\boldsymbol{\psi}$ were more regular.*

2.4 Numerical analysis of the vibration problem

In this section we summarize the results for the vibration problem. We do not include most of the proofs since they are either similar to the corresponding ones for the buckling problem or simpler. We only emphasize those aspects that differ from the buckling problem.

Consider the well-posed source problem associated with the vibration problem (2.2.22): Given $f \in L^2(\Omega)$, find $(\phi, \boldsymbol{\psi}, u) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}$ such that

$$\mathscr{A}((\phi, \boldsymbol{\psi}, u), (\gamma, \boldsymbol{\xi}, v)) = -\mathscr{B}(f, v) \qquad \forall (\gamma, \boldsymbol{\xi}, v) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}. \tag{2.4.1}$$

Let T be the bounded linear operator defined by

$$\begin{aligned} T: \ L^2(\Omega) &\to L^2(\Omega), \\ f &\mapsto u, \end{aligned}$$

with $(\phi, \boldsymbol{\psi}, u) \in \mathcal{V} \times \mathrm{H} \times \mathcal{V}$ being the solution of (2.4.1). Clearly λ is an eigenvalue of problem (2.2.22) if and only if $\mu := \frac{1}{\lambda}$ is a non-zero eigenvalue of T, with the same multiplicity and corresponding eigenfunctions u (recall $\lambda \neq 0$; cf. Remark 2.2.9).

For the vibration problem, the operator T is self-adjoint with respect to the $L^2(\Omega)$ inner product. Moreover T is compact, because of the compact inclusion $\mathcal{V} \hookrightarrow L^2(\Omega)$, and the following spectral characterization holds:

Lemma 2.4.1 *The spectrum of T satisfies* $\mathrm{Sp}(T) = \{0\} \cup \{\mu_n : \ n \in \mathbb{N}\}$*, where* $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of real positive eigenvalues which converges to* 0*. The multiplicity of each eigenvalue is finite and its ascent is* 1*.*

The following additional regularity result holds true in this case:

Lemma 2.4.2 *There exist* $r, s \in (\frac{1}{2}, 1]$ *and* $C > 0$ *such that, for all* $f \in L^2(\Omega)$*, the solution* $(\phi, \boldsymbol{\psi}, u)$ *of problem* (2.4.1) *satisfies* $\phi \in H^{1+r}(\Omega)$*,* $u \in H^{2+s}(\Omega)$*,* $\boldsymbol{\psi} \in H^{1+s}(\Omega)^2$*, and*

$$\|\phi\|_{1+r,\Omega} + \|u\|_{2+s,\Omega} + \|\boldsymbol{\psi}\|_{1+s,\Omega} \leq C \|f\|_{0,\Omega}.$$

Constants r and s above are the same as those in the proof of Lemma 2.3.2 and 2.3.6. By comparing this result with Lemma 2.3.2, we observe that ϕ is smoother in this case than for the buckling problem. This is the key-point which makes the analysis of the vibration problem a bit simpler.

2.4.1 Finite element approximation

The discrete version of problem (2.2.22) reads as follows:

Find $(\lambda_h, \phi_h, \boldsymbol{\psi}_h, u_h) \in \mathbb{R} \times \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h$, $u_h \neq 0$, such that

$$\mathscr{A}((\phi_h, \boldsymbol{\psi}_h, u_h), (\gamma_h, \boldsymbol{\xi}_h, v_h)) = -\lambda_h \mathscr{B}(u_h, v_h) \qquad \forall (\gamma_h, \boldsymbol{\xi}_h, v_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h. \quad (2.4.2)$$

Let T_h be the bounded linear operator defined by

$$T_h : \; L^2(\Omega) \to L^2(\Omega),$$
$$f \mapsto u_h,$$

with $(\phi_h, \boldsymbol{\psi}_h, u_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h$ being the solution of

$$\mathscr{A}((\phi_h, \boldsymbol{\psi}_h, u_h), (\gamma_h, \boldsymbol{\xi}_h, v_h)) = -\mathscr{B}(f, v_h) \qquad \forall (\gamma_h, \boldsymbol{\xi}_h, v_h) \in \mathcal{V}_h \times \mathrm{H}_h \times \mathcal{V}_h. \quad (2.4.3)$$

Once more, λ_h is an eigenvalue of problem (2.4.2) if and only if $\mu_h := \frac{1}{\lambda_h}$ is a non-zero eigenvalue of T_h, with the same multiplicity and corresponding eigenfunctions u_h. Also, as in the continuous case, $\lambda_h \neq 0$.

In this case, T_h is self-adjoint with respect to the $L^2(\Omega)$ inner product. Because of this, it is easy to prove the following spectral characterization:

Lemma 2.4.3 *Problem* (2.4.2) *has exactly* $\dim \mathcal{V}_h$ *eigenvalues, repeated accordingly to their respective multiplicities. All of them are real and positive.*

The following lemma yields the uniform convergence of T_h to T as $h \to 0$. Its proof follows the lines of the proof of Lemma 2.3.8, by taking advantage of the additional regularity of ϕ (cf. Lemma 2.4.2).

Lemma 2.4.4 *There exists* $C > 0$ *such that, for all* $f \in L^2(\Omega)$,

$$\|(T - T_h) f\|_{1,\Omega} \leq Ch \|f\|_{0,\Omega} .$$

2.4.2 Spectral convergence and error estimates

As a direct consequence of Lemma 2.4.4, T_h converges in $H^1(\Omega)$ norm to T as h goes to zero (as well as in $L^2(\Omega)$ norm). Hence, isolated parts of $\mathrm{Sp}(T)$ are approximated by isolated parts of $\mathrm{Sp}(T_h)$. Let $\mu \neq 0$ be an eigenvalue of T with multiplicity m and let $\mathcal{E}$ be its associated eigenspace. There exist m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ of T_h (repeated according to their respective multiplicities) which converge to μ. Let $\mathcal{E}_h$ be the direct sum of their corresponding associated eigenspaces. The following error estimate is again a direct consequence of standard spectral approximation results (cf. [6]):

Theorem 2.4.5 *There exists a strictly positive constant C such that*

$$\hat{\delta}(\mathcal{E}, \mathcal{E}_h) \leq Ch.$$

Finally an improved order of convergence also holds for the eigenvalues. To prove this, we do not need to resort to the analog of Lemma 2.3.10. We include in this case the simpler proof of the following theorem, where, for each $f \in \mathcal{E}$, we denote by $U^f := (\phi^f, \boldsymbol{\psi}^f, u^f)$ the solution of problem (2.4.1). (Notice that $u^f = Tf = \mu f$.)

Theorem 2.4.6 *There exists a strictly positive constant C such that*

$$\begin{aligned} \left|\mu - \mu_h^{(i)}\right| &\leq C \left[h + \sup_{f \in \mathcal{E}} \left(\frac{\inf_{v_h \in \mathcal{V}_h} \left\| \phi^f - v_h \right\|_{1,\Omega} + \inf_{\boldsymbol{\xi}_h \in \mathrm{H}_h} \|\boldsymbol{\psi}^f - \boldsymbol{\xi}_h\|_{1,\Omega}}{\|f\|_{0,\Omega}} \right) \right]^2 \\ &\leq Ch^{2t}, \qquad i = 1, \ldots, m, \end{aligned}$$

where $t = \min\{s, r\}$, with $r, s \in (\frac{1}{2}, 1]$ as in Lemma 2.4.2.

Proof. By applying Theorem 7.3 from [6] and taking into account that T and T_h are self-adjoint with respect to the $L^2(\Omega)$ inner product, we have

$$\left|\mu - \mu_h^{(i)}\right| \leq C \left[\sup_{f,g \in \mathcal{E}} \frac{\int_\Omega (Tf - T_h f)\, g}{\|f\|_{0,\Omega} \|g\|_{0,\Omega}} + \sup_{f \in \mathcal{E}} \frac{\|(T - T_h)\, f\|_{0,\Omega}^2}{\|f\|_{0,\Omega}^2} \right], \qquad i = 1, \ldots, m.$$

The second term in the right-hand side above is directly bounded by means of Lemma 2.4.4, so there only remains to estimate the first one. With this aim, let $f, g \in \mathcal{E}$. Let U^f and

U^g be defined as above. Let $U_h^f := (\phi_h^f, \boldsymbol{\psi}_h^f, u_h^f)$ and $U_h^g := (\phi_h^g, \boldsymbol{\psi}_h^g, u_h^g)$ be the solutions of problem (2.4.3) with data f and g, respectively. There holds

$$\begin{aligned}\int_\Omega (Tf - T_h f)\, g = \mathscr{B}(u^f - u_h^f, g) &= -\mathscr{A}(U^f - U_h^f, U^g) = -\mathscr{A}(U^f - U_h^f, U^g - U_h^g)\\ &\le C\, \|U^f - U_h^f\|_{\mathcal{V}\times\mathrm{H}\times\mathcal{V}}\, \|U^g - U_h^g\|_{\mathcal{V}\times\mathrm{H}\times\mathcal{V}}\,,\end{aligned}$$

because of the standard Galerkin orthogonality and the continuity of $\mathscr{A}$. Now,

$$\begin{aligned}\|U^f - U_h^f\|_{\mathcal{V}\times\mathrm{H}\times\mathcal{V}} &\le \|\phi^f - \phi_h^f\|_{1,\Omega} + \|\boldsymbol{\psi}^f - \boldsymbol{\psi}_h^f\|_{\mathrm{H}} + \|u^f - u_h^f\|_{1,\Omega}\\ &\le \inf_{v_h\in\mathcal{V}_h} \|\phi^f - v_h\|_{1,\Omega} + \inf_{\boldsymbol{\xi}_h\in\mathrm{H}_h} \|\boldsymbol{\psi}^f - \boldsymbol{\xi}_h\|_{1,\Omega} + Ch\, \|f\|_{0,\Omega}\\ &\le Ch^t\, \|f\|_{0,\Omega}\,,\end{aligned}$$

where we have used results from [2, Section 5] to estimate the terms $\|\phi^f - \phi_h^f\|_{1,\Omega}$ and $\|\boldsymbol{\psi}^f - \boldsymbol{\psi}_h^f\|_{\mathrm{H}}$ and Lemma 2.4.4 for $\|u^f - u_h^f\|_{1,\Omega}$. Since the same holds for $\|U^g - U_h^g\|_{\mathcal{V}\times\mathrm{H}\times\mathcal{V}}$, we conclude the proof. □

The error estimate from the previous lemma yields a similar one for the eigenvalues $\lambda = \frac{1}{\mu}$ of problem (2.2.22). Moreover, the analogous to Remark 2.3.12 also holds in this case.

2.5 Numerical results

We report in this section some numerical experiments which confirm the theoretical results proved above. Moreover, we compare in the first two tests the performance of the proposed method with those of Ciarlet-Raviart's [16, 13, 35] and Ishihara's [29, 30] methods.

Ciarlet-Raviart's method is based on a mixed form of the biharmonic equation, which is equivalent to this equation for convex domains. The method was proved to converge for the vibration and the buckling problems for finite elements of degree $k \ge 2$ (see [35, Section 7(b,d)]). Our experiments will give evidence of optimal order convergence for piecewise linear finite elements, although, to the best of our knowledge, this has not been proved.

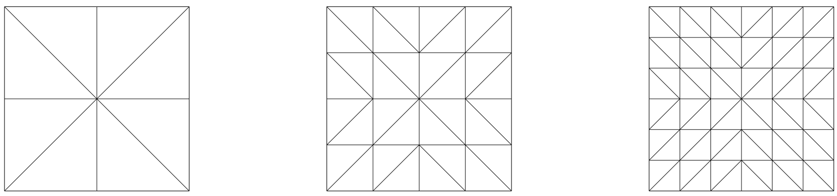
Ishihara's method is based on an alternative mixed formulation, also equivalent to the biharmonic equation for convex domains. Its piecewise linear discretization was analyzed in [29] for the vibration problem and in [30] for the buckling problem. It was proved that it converges in both cases, with a suboptimal order $\mathcal{O}(h^{1/2})$, only for meshes which are uniform in the interior of the domain. Our numerical experiments will show that this constraint is not a technicality, since the method converges to wrong results when is used on particular regular non-uniform meshes.

Since there is no significant difference in our experiments between the vibration and the buckling problems, we will only report the numerical results for the latter. We have taken in all our experiments a Poisson ratio $\nu = 0.25$.

2.5.1 Test 1: Uniformly compressed square plate; uniform meshes

We have taken as an example of a convex domain the unit square $\Omega := (0,1)\times(0,1)$. We have used the stress distribution corresponding to a uniformly compressed plate: $\boldsymbol{\eta} = \mathbf{I}$.

We have used uniform meshes as those shown in Figure 2.1. The refinement parameter N used to label each mesh is the number of elements on each edge of the plate.



$N = 2$ $N = 4$ $N = 6$

Figure 2.1: Square plate: uniform meshes.

We report in Table 2.1 the lowest buckling coefficients (i.e., the lowest eigenvalues of the buckling problem) computed with the method analyzed in this paper, with Ciarlet-Raviart's method, and with Ishihara's method. The table includes computed orders of convergence and extrapolated more accurate values of each eigenvalue obtained by means of a least-squares fitting.

Table 2.1: Lowest buckling coefficients of a uniformly compressed clamped square plate computed on uniform meshes with the method analyzed in this paper (A), Ciarlet-Raviart's method (CR), and Ishihara's method (I).

	Method	$N = 24$	$N = 36$	$N = 48$	$N = 60$	Order	Extrapolated
	A	5.3051	5.3042	5.3039	5.3038	2.61	5.3037
λ_1	CR	5.3830	5.3395	5.3239	5.3167	1.95	5.3033
	I	5.3529	5.3254	5.3159	5.3114	2.02	5.3037
	A	9.3578	9.3444	9.3398	9.3378	2.09	9.3343
$\lambda_2 = \lambda_3$	CR	9.5390	9.4261	9.3861	9.3675	1.97	9.3337
	I	9.4650	9.3912	9.3659	9.3544	2.06	9.3347
	A	13.0346	13.0091	13.0007	12.9969	2.14	12.9908
λ_4	CR	13.3977	13.1710	13.0919	13.0553	2.01	12.9909
	I	13.2128	13.0827	13.0407	13.0219	2.21	12.9930

It can be seen from Table 2.1 that the three methods converge in this case to the same values with optimal quadratic order, although this has been proved only for the method analyzed in this paper (cf. Remark 2.3.7). Notice that, for all the methods, the second computed eigenvalue is double, because the meshes preserve the symmetry of the domain leading to an eigenvalue of multiplicity 2 in the continuous problem.

Figure 2.2 shows the transverse displacements of the principal buckling mode (i.e., the eigenfunction corresponding to the lowest eigenvalue of the buckling problem) computed with the method analyzed in this paper.

2.5.2 Test 2: Uniformly compressed square plate; non-uniform meshes

We have tested the same three methods as above on non-uniform meshes, as well. We have solved the same problem as in the previous example with tiled meshes as those shown in Figure 2.3. The refinement parameter N used to label each mesh is now the number

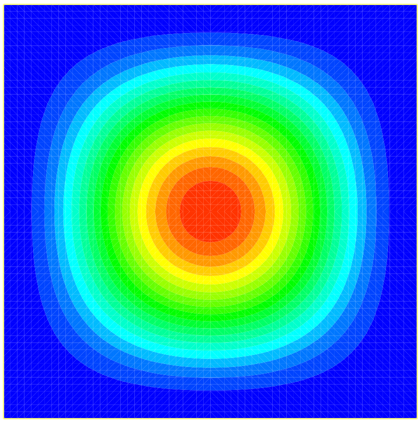
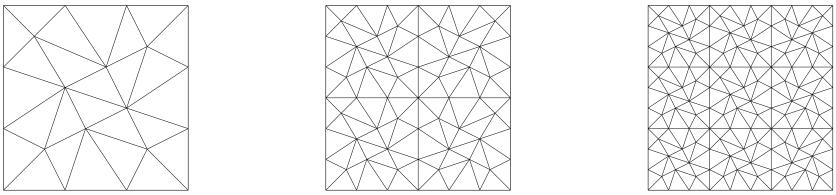


Figure 2.2: Uniformly compressed square plate; principal buckling mode.

of tiles on each edge of the plate. The reason for this choice is to avoid asymptotically uniform meshes.



$N = 1$ $N = 2$ $N = 3$

Figure 2.3: Square plate: tiled meshes.

We report in Table 2.2 the lowest buckling coefficients computed on these meshes with each of the three methods again. Notice that in this case, since the meshes do not preserve the symmetry of the domain, the second eigenvalue, which has multiplicity 2 in the continuous problem, will be in general approximated by two simple eigenvalues.

It can be seen from Table 2.2 that the method analyzed in this paper and Ciarlet-Raviart's method do not deteriorate on these meshes and converge to the same values with quadratic order again. Instead, this is not the case for Ishihara's method, which converges to wrong results.

Table 2.2: Lowest buckling coefficients of a uniformly compressed clamped square plate computed on non-uniform meshes with the method analyzed in this paper (A), Ciarlet-Raviart's method (CR), and Ishihara's method (I).

	Method	$N = 15$	$N = 25$	$N = 35$	$N = 45$	Order	Extrapolated
	A	5.3030	5.3034	5.3035	5.3036	1.91	5.3036
λ_1	CR	5.3134	5.3075	5.3058	5.3050	1.88	5.3038
	I	5.4491	5.4384	5.4348	5.4328	1.42	5.4285
	A	9.3323	9.3335	9.3338	9.3339	1.96	9.3342
λ_2	CR	9.3622	9.3449	9.3399	9.3379	1.92	9.3345
	I	9.5788	9.5520	9.5433	9.5388	1.56	9.5302
	A	9.3336	9.3340	9.3340	9.3341	1.79	9.3342
λ_3	CR	9.3641	9.3455	9.3402	9.3380	1.94	9.3345
	I	9.6337	9.6106	9.6030	9.5991	1.54	9.5914
	A	12.9830	12.9876	12.9890	12.9895	1.96	12.9904
λ_4	CR	13.0437	13.0103	13.0010	12.9970	1.95	12.9908
	I	13.3387	13.2949	13.2811	13.2743	1.64	13.2617

2.5.3 Test 3: Shear loaded square plate

For this test we have computed the buckling coefficients of the same plate as in the previous example, subjected to a uniform shear load. This corresponds to a plane stress field

$$\boldsymbol{\eta} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that $\boldsymbol{\eta}$ is not positive definite in this case.

We report in Table 2.3 the lowest buckling coefficients computed on the same uniform meshes used in Test 1 (cf. Figure 2.1) with the method analyzed in this paper.

Once more, the method converges with optimal quadratic order. Although we do not report the results obtained with the other two methods, both converge on uniform meshes to the same eigenvalues.

Table 2.3: Lowest buckling coefficients of a shear loaded clamped square plate computed on uniform meshes with the method analyzed in this paper.

	$N = 24$	$N = 36$	$N = 48$	$N = 60$	Order	Extrapolated
λ_1	14.8218	14.7215	14.6867	14.6706	2.01	14.6420
λ_2	17.3111	17.0922	17.0161	16.9810	2.02	16.9195
λ_3	36.0905	34.5656	34.0304	33.7825	1.99	33.3376

Figure 2.4 shows the transverse displacements of the principal buckling mode for the shear loaded square plate computed with the method analyzed in this paper.

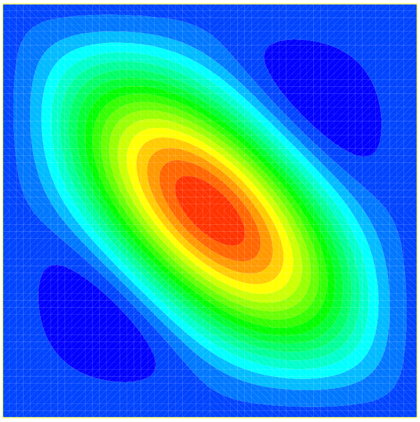


Figure 2.4: Shear loaded square plate; principal buckling mode.

2.5.4 Test 4: L-shaped plate

Finally, we have computed the buckling coefficients of an L-shaped plate: $\Omega := (0,1) \times (0,1) \setminus [0.5,1) \times [0.5,1)$. We have used $\boldsymbol{\eta} = \mathbf{I}$ (uniform compression) and uniform meshes as those shown in Figure 2.5. The meaning of the refinement parameter N is clear from this figure.

We report in Table 2.4 the lowest buckling coefficients computed with the method

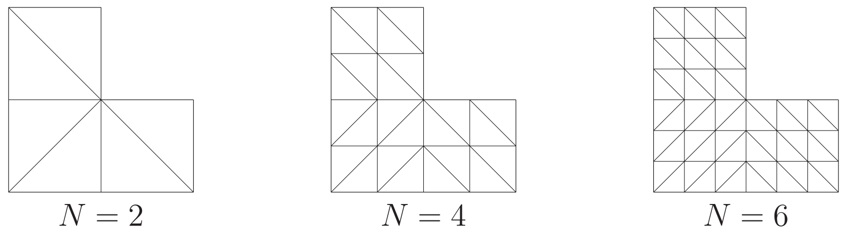


Figure 2.5: L-shaped plate: uniform meshes.

analyzed in this paper.

Table 2.4: Lowest buckling coefficients of an L-shaped clamped plate computed on uniform meshes with the method analyzed in this paper.

	$N=40$	$N=60$	$N=80$	$N=100$	Order	Extrapolated
λ_1	12.8379	12.9010	12.9328	12.9518	0.99	13.0290
λ_2	14.9175	14.9586	14.9752	14.9838	1.60	15.0036
λ_3	17.0083	16.9993	16.9968	16.9960	2.75	16.9949

In this case, for the first buckling coefficient, the method converges with order close to 1.089, which is the expected one because of the singularity of the solution (see [27]). Instead, the method converges with larger orders for the second and the third buckling coefficients.

Notice that, according to Theorem 2.3.11, the order of convergence for the buckling coefficients must double the worst among those of $\|\phi-\phi_h\|_{1,\Omega}$, $\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,\Omega}$, and $\|u-u_h\|_{1,\Omega}$. In the case of λ_1 and λ_2, the worst order should be that of $\|\boldsymbol{\psi}-\boldsymbol{\psi}_h\|_{1,\Omega}$. In fact, according to (2.3.21), the transverse displacement u satisfies $\|u-u_h\|_{1,\Omega}=\mathcal{O}(h)$ for any polygonal domain Ω. Moreover, since in this case $\boldsymbol{\eta}=\mathbf{I}$, we have $\phi=\lambda u$ and $\phi_h=\lambda_h u_h$, so that $\|\phi-\phi_h\|_{1,\Omega}=\mathcal{O}(h)$, too.

We include in Table 2.5 computed orders of convergence $\|u_h-u_{\mathrm{ex}}\|_{1,\Omega}$, where we have used as 'exact' transverse displacements u_{ex} the ones computed with a highly refined mesh corresponding to $N=200$.

Table 2.5: Errors of the transverse displacements $\|u_h - u_{\text{ex}}\|_{1,\Omega}$ for the lowest buckling coefficients of an L-shaped clamped plate computed on uniform meshes with the method analyzed in this paper.

	$N = 8$	$N = 16$	$N = 24$	$N = 32$	Order
λ_1	0.4514	0.2297	0.1477	0.1059	1.04
λ_2	0.4424	0.2218	0.1411	0.1005	1.07
λ_3	0.5028	0.2469	0.1570	0.1121	1.08

It can be seen from this table that the eigenfunctions u_h actually converge with order $\mathcal{O}(h)$ as Theorem 2.3.9 predicts, in spite of the non-convex angle of the domain.

Finally, Figure 2.6 shows the transverse displacement of the principal buckling mode.

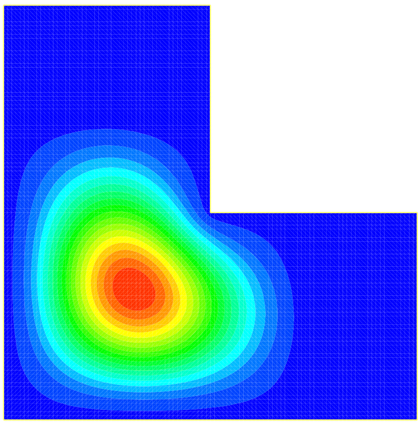


Figure 2.6: Uniformly compressed L-shaped plate; principal buckling mode.

2.6 Conclusions

We have introduced a finite element method for two eigenvalue problems: the computation of buckling and vibration modes of a clamped Kirchhoff polygonal plate. The method is based on discretizing a bending moment formulation by means of standard piecewise linear finite elements. This approach was proposed and analyzed by Amara *et al.* [2] to solve the corresponding load problem for a thin plate subject to arbitrary boundary conditions.

We have proved that the method yields an $\mathcal{O}(h)$ approximation to the transverse displacements of buckling and vibration modes. Moreover, it yields $\mathcal{O}(h^t)$ approximations to two auxiliary quantities, ϕ and $\boldsymbol{\psi}$, which allow us to compute to the same order of accuracy the bending moment $\boldsymbol{\sigma} = \mathsf{Curl}\,\boldsymbol{\psi} + \frac{1}{2}(\operatorname{div}\boldsymbol{\psi})\,\mathbf{J} + \phi\mathbf{I}$. The order t depends on the Sobolev regularity of the domain for the biharmonic and the Laplace equations. If Ω is convex, then $t = 1$; otherwise, $t \in (\frac{1}{2}, 1)$ depends on the largest reentrant angle of Ω. The method yields $\mathcal{O}(h^{2t})$ approximation to the buckling coefficients or the vibration frequencies, too.

Furthermore, Lemma 2.4.4 shows that the method leads to an $\mathcal{O}(h)$ approximation to the transverse displacement in the case of the source problem, too, even for non-convex polygonal clamped plates. Let us remark that such optimal order agrees with the fact that the transverse displacement always belongs to $H^2(\Omega)$. This improves in this particular case the estimate given in [2, Theorem 5.3] for this variable.

The numerical tests confirm the theoretical results, including the $\mathcal{O}(h)$ approximation to the transverse displacements even for plates with reentrant corners. The performance of the method analyzed in this paper is comparable to that of lowest-order Ciarlet-Raviart's method [16] (for which, to the best of the authors' knowledge, there is no proof of convergence for either of the eigenvalue problems). We have also tested numerically other well-known method for Kirchhoff plates, which was analyzed by Ishihara [29, 30] for both eigenvalue problems on meshes uniform in the interior of the domain. The numerical tests show that the uniformity constraint is not a technicality. In fact, it converges to wrong results when used on particular regular non-uniform meshes.

Appendix

The matrix form of the discrete spectral problem (2.3.7) reads as follows:

$$\begin{pmatrix} \mathbf{A} & \mathbf{B} & \mathbf{C} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} & \mathbf{0} \\ \mathbf{C} & \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Phi}_h \\ \boldsymbol{\Psi}_h \\ \mathbf{U}_h \end{pmatrix} = \lambda_h \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{E} \end{pmatrix} \begin{pmatrix} \boldsymbol{\Phi}_h \\ \boldsymbol{\Psi}_h \\ \mathbf{U}_h \end{pmatrix}, \tag{2.6.1}$$

where $\boldsymbol{\Phi}_h$, $\boldsymbol{\Psi}_h$, and $\mathbf{U}_h$ denote the vectors whose entries are the components of ϕ_h, $\boldsymbol{\psi}_h$, and u_h, respectively, in particular given bases of the discrete spaces. Let us remark that $\boldsymbol{\psi}_h \in \mathrm{H}_h$, whose definition involves three linear constraints. Actually, these constraints are imposed by means of three scalar Lagrange multipliers, which leads to an augmented spectral problem exactly equivalent with (2.6.1).

In this generalized eigenvalue problem, matrices $\mathbf{A}$, $\mathbf{C}$, $\mathbf{D}$, and $\mathbf{E}$ are symmetric, whereas $\mathbf{A}$, $\mathbf{C}$, and $\mathbf{D}$ are also positive definite. Let us define

$$\mathbf{F} := \begin{pmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^{\mathrm{T}} & \mathbf{D} \end{pmatrix}, \qquad \mathbf{G} := \begin{pmatrix} \mathbf{C} \\ \mathbf{0} \end{pmatrix}, \qquad \text{and} \qquad \mathbf{V}_h := \begin{pmatrix} \boldsymbol{\Phi}_h \\ \boldsymbol{\Psi}_h \end{pmatrix}.$$

Matrix $\mathbf{F}$ is non-singular. In fact, the following result holds true:

Lemma 2.6.1 $\mathbf{F}$ *is a positive definite matrix.*

Proof. Let $\phi_h \in \mathcal{V}_h$ and $\boldsymbol{\psi}_h = (\psi_{h1}, \psi_{h2}) \in \mathrm{H}_h$. Let $\boldsymbol{\Phi}_h$ and $\boldsymbol{\Psi}_h$ be the vectors whose entries are the components of ϕ_h and $\boldsymbol{\psi}_h$, respectively, and $\mathbf{V}_h$ as defined above. Straightforward computations lead to

$$\begin{aligned} \mathbf{V}_h^{\mathrm{T}} \mathbf{F} \mathbf{V}_h = {} & \frac{2}{1+\nu} \int_\Omega \phi_h^2 + \frac{2}{1+\nu} \int_\Omega (\partial_2 \psi_{h1} - \partial_1 \psi_{h2})\, \phi_h \\ & + \frac{1}{1-\nu} \int_\Omega \left[(\partial_2 \psi_{h1})^2 + (\partial_1 \psi_{h2})^2 + \tfrac{1}{2} (\partial_2 \psi_{h2} - \partial_1 \psi_{h1})^2 \right] \\ & - \frac{\nu}{1-\nu^2} \int_\Omega (\partial_2 \psi_{h1} - \partial_1 \psi_{h2})^2 \\ = {} & \frac{2}{1+\nu} \int_\Omega \left[\phi_h + \tfrac{1}{2} (\partial_2 \psi_{h1} - \partial_1 \psi_{h2}) \right]^2 \\ & + \frac{1}{2(1-\nu)} \int_\Omega \left[(\partial_2 \psi_{h1} + \partial_1 \psi_{h2})^2 + (\partial_2 \psi_{h2} - \partial_1 \psi_{h1})^2 \right] \geq 0. \end{aligned}$$

Hence $\mathbf{F}$ is non-negative definite. Moreover, the expression above vanishes if and only if $\phi_h = -\frac{1}{2}(\partial_2\psi_{h1} - \partial_1\psi_{h2})$, $\partial_2\psi_{h1} + \partial_1\psi_{h2} = 0$ and $\partial_2\psi_{h2} - \partial_1\psi_{h1} = 0$. Now, $\phi_h \in \mathcal{V}_h$ is piecewise linear and continuous, whereas for $\boldsymbol{\psi}_h \in \mathrm{H}_h$, $\partial_2\psi_{h1} - \partial_1\psi_{h2}$ is piecewise constant. Hence, if the expression above vanishes, then $\phi_h = -\frac{1}{2}(\partial_2\psi_{h1} - \partial_1\psi_{h2})$ has to be constant and, since it vanishes on Γ, it has to vanish in the whole Ω.

In such a case, $\partial_2\psi_{h1} - \partial_1\psi_{h2} = 0$ and $\partial_2\psi_{h1} + \partial_1\psi_{h2} = 0$, too, which leads to $\partial_2\psi_{h1} = \partial_1\psi_{h2} = 0$. Since $\partial_2\psi_{h2} - \partial_1\psi_{h1} = 0$, as well, there holds $\|\boldsymbol{\psi}_h\|_{\mathrm{H}} = 0$ and hence $\boldsymbol{\psi}_h = \mathbf{0}$. Thus $\mathbf{F}$ is positive definite and we conclude the proof. $\square$

Now we are in a position to prove the following characterization of the discrete spectral problem (2.3.7):

Proposition 2.6.2 *Let* $\mathcal{Z}_h := \{u_h \in \mathcal{V}_h : \ \mathscr{C}(u_h, v_h) = 0 \quad \forall v_h \in \mathcal{V}_h\}$*. Then, problem* (2.3.7) *has exactly* $\dim \mathcal{V}_h - \dim \mathcal{Z}_h$ *eigenvalues, repeated accordingly to their respective multiplicities. All of them are real and non-zero.*

Proof. Since according to the previous lemma $\mathbf{F}$ is positive definite and consequently non-singular, $\mathbf{\Phi}_h$ and $\mathbf{\Psi}_h$ can be eliminated in (2.6.1) as follows:

$$\mathbf{V}_h = -\mathbf{F}^{-1}\mathbf{G}\mathbf{U}_h \qquad \Longrightarrow \qquad \mathbf{E}\mathbf{U}_h = -\mu_h \left(\mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G}\right)\mathbf{U}_h,$$

with $\mu_h := \frac{1}{\lambda_h}$ (recall $\lambda_h \neq 0$; cf. Remark 2.3.5).

Now, since also $\mathbf{C}$ is non-singular, the columns of $\mathbf{G}$ are linearly independent. Hence, $\mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G}$ is symmetric and positive definite and, $\mathbf{E}$ being symmetric too, the generalized eigenvalue problem $\mathbf{E}\mathbf{U}_h = -\mu_h(\mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G})\mathbf{U}_h$ is well posed and all its eigenvalues are real. Therefore the number of eigenvalues of problem (2.6.1) (which is the matrix form of problem (2.3.7)) equals the number of non-zero eigenvalues of this problem, namely, $\dim \mathcal{V}_h - \dim(\mathrm{Ker}(\mathbf{E}))$. Thus, we conclude the lemma by noting that $\mathbf{E}\mathbf{U}_h = \mathbf{0}$ if and only if $u_h \in \mathcal{Z}_h$. $\square$

As an immediate consequence of the proof of this proposition, note that problem (2.3.7) always has real non-zero eigenvalues, as long as $\mathbf{E} \neq \mathbf{0}$.

Remark 2.6.3 *For all the solutions* $(\lambda_h, \phi_h, \boldsymbol{\psi}_h, u_h)$ *of problem* (2.3.7)*, there holds* $\int_\Omega (\boldsymbol{\eta}\nabla u_h)\cdot \nabla u_h \neq 0$*, despite the fact that* $\boldsymbol{\eta}$ *is not necessarily positive definite. In fact, as shown in the proof of Proposition 2.6.2,*

$$\int_\Omega (\boldsymbol{\eta}\nabla u_h) \cdot \nabla u_h = \mathscr{C}(u_h, u_h) = \mathbf{U}_h^{\mathrm{T}}\mathbf{E}\mathbf{U}_h = -\frac{1}{\lambda_h}\mathbf{U}_h^{\mathrm{T}}\left(\mathbf{G}^{\mathrm{T}}\mathbf{F}^{-1}\mathbf{G}\right)\mathbf{U}_h \neq 0. \tag{2.6.2}$$

Chapter 3

Approximation of the Buckling Problem for Reissner-Mindlin Plates.

3.1 Introduction

This paper deals with the analysis of the elastic stability of plates, in particular the so-called *buckling problem*. This problem has attracted much interest since it is frequently encountered in engineering applications such as bridge, ship, and aircraft design. It can be formulated as a spectral problem whose solution is related with the limit of elastic stability of the plate (i.e., eigenvalues-buckling coefficients and eigenfunctions-buckling modes).

The buckling problem has been studied for years by many researchers, being the Kirchhoff-Love and the Reissner-Mindlin plate theories the most used. For the Kirchhoff-Love theory, there exists a thorough mathematical analysis; let us mention, for instance, [13, 30, 35, 36, 37]. This is not the case for the Reissner-Mindlin theory, for which only numerical experiments (cf. [32, 44]) or analytical solutions in particular cases (cf. [46]) have been reported so far. Recently, Dauge and Suri introduced in [17] the mathematical spectral analysis of a problem of this kind based on three-dimensional elasticity. In the present paper, we will perform a similar analysis for Reissner-Mindlin plates.

The Reissner-Mindlin theory is the most used model to approximate the deformation of a thin or moderately thick elastic plate. It is very well understood that standard finite elements applied to this model lead to wrong results when the thickness is small with respect to the other dimensions of the plate due to the *locking* phenomenon. Several families of methods have been rigorously shown to be free of locking and optimally convergent. We mention the recent monograph by Falk [23] for a thorough description of the state of the art and further references.

The aim of this paper is to analyze one of these methods applied to compute the buckling coefficients and buckling modes of a clamped plate. We choose the low-order, nonconforming finite elements introduced by Durán and Liberman in [22] (see also [21] for the analysis of this method applied to the plate vibration problem). However, the developed framework could be useful to analyze other methods, as well.

One drawback of the Reissner-Mindlin formulation for plate buckling is the fact that the corresponding solution operator is non-compact. This is the reason why the essential spectrum no longer reduces to zero (as is the case for compact operators). This means that the spectrum may now contain nonzero eigenvalues of infinite multiplicity, accumulation points, continuous spectrum, etc. Thus, our first task is to prove that the eigenvalue corresponding to the limit of elastic stability (i.e., the smallest buckling coefficient) can be isolated from the essential spectrum, at least for sufficiently thin plates.

On the other hand, the abstract spectral theory for non-compact operators introduced by Descloux, Nassif, and Rappaz in [18, 19] cannot be directly applied to analyze the numerical method, because we look for error estimates valid uniformly in the plate thickness. However, using optimal order convergence results for the Durán-Liberman elements (cf. [21, 22]) and the theoretical framework used to prove additional regularity for Reissner-Mindlin equations (cf. [5]), under the assumption that the family of meshes is *quasi-uniform*, we can adapt the theory from [18, 19] to obtain optimal order error estimates for the approximation of the buckling modes, including a double order for the buckling coefficients. Moreover, these estimates are shown to be valid with constants independent of the plate thickness, which allows us to conclude that the proposed method is locking-free.

An outline of the paper is as follows. In the next section we derive the buckling problem and introduce a non-compact linear operator whose spectrum is related with the solution of this problem. In Section 3.3 we provide a thorough spectral characterization of this operator. In Section 3.4 we introduce a finite element discretization of the problem based on Duran-Liberman elements and prove some auxiliary results. In Section 3.5 we prove that the proposed numerical scheme is free of spurious modes and that optimal order error estimates hold true. In Section 3.6 we report some numerical tests which confirm the theoretical results. We include in this section a benchmark with a known analytical solution for a simply supported plate, which shows the efficiency of the method under other kind of boundary conditions, as well. Finally, in an appendix, we show that the results of Sections 3.3, 3.4, and 3.5 can be refined when considering the particular case of a uniformly compressed plate.

Throughout the paper we will use standard notations for Sobolev spaces, norms, and seminorms. Moreover, we will denote with C a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

3.2 The buckling problem

The first step will be to derive the equations for the Reissner-Mindlin plate buckling problem. With this aim, we will begin by considering the plate as a three-dimensional elastic solid and we will write the corresponding equations for the buckling in this case. Then, we will perform the dimensional reduction by means of the usual Reissner-Mindlin assumptions.

Consider a (three-dimensional) elastic plate of thickness $t > 0$ with reference configuration $\widetilde{\Omega} := \Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$, where Ω is a convex polygonal domain of $\mathbb{R}^2$ occupied by the midsection of the plate. We assume that the plate is clamped on its lateral boundary $\partial\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)$. In what follows, we summarize the arguments given in [17] to obtain the equations for the corresponding buckling problem (see this reference and also [45] for further details). We will use tildes on the quantities corresponding to the three-dimensional elastic model (as in $\widetilde{\Omega}$, for instance) to help distinguishing them form the corresponding ones in the Reissner-Mindlin model.

Suppose that $\widetilde{\boldsymbol{\sigma}}^0 := (\widetilde{\sigma}^0_{ij})_{1\le i,j\le 3}$ is a pre-existing stress state in the plate. This stress $\widetilde{\boldsymbol{\sigma}}^0$ is already present in the reference configuration. It satisfies the equations of equilibrium and it is assumed to be independent of any subsequent displacements that the reference configuration may undergo.

Let $\widetilde{V} := \left\{\widetilde{v} \in \mathrm{H}^1(\widetilde{\Omega})^3 : \widetilde{v} = 0 \text{ on } \partial\Omega \times \left(-\frac{t}{2}, \frac{t}{2}\right)\right\}$ be the space of admissible displacements of the three-dimensional plate. If the reference configuration is now perturbed by a small change $\widetilde{f} \in \widetilde{V}'$ (which could be a change in loading, for instance), then the work done by $\widetilde{\boldsymbol{\sigma}}^0$ cannot be neglected. The corresponding displacement $\widetilde{u} = (\widetilde{u}_i)_{1\le i\le 3}$ may be expressed as the solution of the following problem (see [17]):

Given $\widetilde{f} \in \widetilde{V}'$, find $\widetilde{u} \in \widetilde{V}$ such that

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k + \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}^0_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m = \langle \widetilde{f}, \widetilde{v}\rangle \qquad \forall \widetilde{v} \in \widetilde{V}.$$

Above, $(\widetilde{C}_{ijkl})_{1\le i,j,k,l\le 3}$ is the tensor of elastic constants of the material and $\langle\cdot,\cdot\rangle$ denotes the duality between $\widetilde{V}'$ and $\widetilde{V}$. The second term in the left hand side is the work done by $\widetilde{\boldsymbol{\sigma}}^0$.

We restrict our attention to multiples of a fixed *pre-buckling stress* $\widetilde{\boldsymbol{\sigma}}$, namely,

$$\widetilde{\boldsymbol{\sigma}}^0 = -\widetilde{\lambda}\widetilde{\boldsymbol{\sigma}}.$$

Then, the equation above reads

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k - \widetilde{\lambda} \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m = \langle \widetilde{f}, \widetilde{v}\rangle \qquad \forall \widetilde{v} \in \widetilde{V}.$$

According to [17], we will say that this problem is *stably solvable* if it has a unique solution for every $\widetilde{f} \in \widetilde{V}'$ and there exists a constant C, independent of $\widetilde{f}$, such that

$$\|\widetilde{u}\|_{\widetilde{V}} \le C\|\widetilde{f}\|_{\widetilde{V}'}.$$

Our goal will be to find the smallest value of $\widetilde{\lambda}$ for which this problem is not stably solvable. This value, which we will denote $\widetilde{\lambda}_{\mathrm{b}}$, is called the *limit of elastic stability*. Physically, it represents the smallest multiple of the pre-buckling stress $\widetilde{\boldsymbol{\sigma}}$ for which a small perturbation in external conditions on the plate may cause it to buckle. As shown in [17], this can be formulated as finding the minimum positive spectral value of the following problem:

Find $\widetilde{\lambda}_{\mathrm{b}} \in \mathbb{R}$ *and* $0 \neq \widetilde{u} \in \widetilde{V}$ *such that*

$$\int_{\widetilde{\Omega}} \sum_{i,j,k,l=1}^{3} \widetilde{C}_{ijkl}\, \partial_j \widetilde{u}_i\, \partial_l \widetilde{v}_k = \widetilde{\lambda}_{\mathrm{b}} \int_{\widetilde{\Omega}} \sum_{i,j,m=1}^{3} \widetilde{\sigma}_{ij}\, \partial_i \widetilde{u}_m\, \partial_j \widetilde{v}_m \qquad \forall \widetilde{v} \in \widetilde{V}. \tag{3.2.1}$$

The eigenvalues of this problem are called the *buckling coefficients* and the eigenfunctions the *buckling modes*.

The above analysis is valid for any three-dimensional solid. In what follows we use it to derive the equations for the corresponding Reissner-Mindlin plate model. In such a case, the deformation of the plate is described by means of the rotations $\beta = (\beta_1, \beta_2)$ of the fibers initially normal to the plate midsurface and the transverse displacement w, as follows:

$$\widetilde{u}(x, y, z) = \begin{bmatrix} -z\beta_1(x, y) \\ -z\beta_2(x, y) \\ w(x, y) \end{bmatrix}. \tag{3.2.2}$$

The pre-buckling stress $\widetilde{\boldsymbol{\sigma}}$ is assumed to arise from an elastic plane strain problem, so that

$$\widetilde{\boldsymbol{\sigma}} = \begin{bmatrix} \boldsymbol{\sigma} & \mathbf{0} \\ \mathbf{0} & 0 \end{bmatrix},$$

with $\boldsymbol{\sigma}(x, y) \in \mathbb{R}^{2\times 2}$ a symmetric tensor. For the remaining arguments of this section, it is enough to consider $\boldsymbol{\sigma} \in \mathrm{L}^{\infty}(\Omega)^{2\times 2}$. However, we will assume some additional regularity which will be used in the forthcoming sections, namely,

$$\boldsymbol{\sigma} \in \mathrm{W}^{1,\infty}(\Omega)^{2\times 2}. \tag{3.2.3}$$

Notice that we do not assume $\boldsymbol{\sigma}$ to be positive definite. Avoiding such assumption allows us to apply this approach, for instance, to shear loaded plates (cf. Section 3.6.3). Therefore, the buckling coefficients can be in principle positive or negative, the limit of elastic stability being that of smallest absolute value.

Next, we use Hooke's law with the plane stress assumption and the kinematically admissible displacements from Reissner-Mindlin model. Thus, by substituting $\widetilde{u}$ and $\widetilde{v}$ in (3.2.1) by means of (3.2.2), using the appropriate elastic constants $\widetilde{C}_{ijkl}$, and integrating over the thickness, we obtain the following variational spectral problem (see [44] for an alternative derivation):

Find $\lambda_{\mathrm{b}} \in \mathbb{R}$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that*

$$t^3 a(\beta,\eta) + \kappa t\,(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda_{\mathrm{b}}\Big[t\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} + t^3\,(\boldsymbol{\sigma}\nabla\beta_1, \nabla\eta_1)_{0,\Omega} + t^3\,(\boldsymbol{\sigma}\nabla\beta_2, \nabla\eta_2)_{0,\Omega}\Big] \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega). \tag{3.2.4}$$

Above, $\kappa := Ek/(2\,(1+\nu))$ is the shear modulus, with E being the Young modulus, ν the Poisson ratio, and k a correction factor (usually taken as 5/6 for clamped plates); $a(\cdot,\cdot)$ is the $\mathrm{H}_0^1(\Omega)^2$ elliptic bilinear form defined by

$$a(\beta,\eta) := \frac{E}{12\,(1-\nu^2)} \int_\Omega \left[(1-\nu)\,\boldsymbol{\varepsilon}(\beta) : \boldsymbol{\varepsilon}(\eta) + \nu\,\mathrm{div}\,\beta\,\mathrm{div}\,\eta\right],$$

where $\boldsymbol{\varepsilon} = (\varepsilon_{ij})_{1\le i,j\le 2}$ is the standard strain tensor with components $\varepsilon_{ij}(\beta) := \frac{1}{2}\,(\partial_i\beta_j + \partial_j\beta_i)$, $1 \le i, j \le 2$. Finally, $(\cdot,\cdot)_{0,\Omega}$ denotes the usual L^2 inner product.

Since the terms involving the rotations β in the right hand size of (3.2.4) are $O(t^3)$, they are typically negligible (see, for instance, [32, 46]). Thus, neglecting these terms, scaling the problem, and defining $\lambda := \lambda_{\mathrm{b}}/t^2$, we obtain

$$a(\beta,\eta) + \frac{\kappa}{t^2}\,(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega).$$

Finally, introducing the shear stress $\gamma := \dfrac{\kappa}{t^2}\,(\nabla w - \beta)$, we arrive at the following problem:

Problem 3.2.1 *Find* $\lambda \in \mathbb{R}$ *and* $0 \neq (\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}\,(\nabla w - \beta). \end{cases}$$

The goal of this paper is to propose and analyze a finite element method to solve Problem 3.2.1. In particular, our aim is to obtain accurate approximations of the smallest (in absolute value) eigenvalues λ, which correspond to the buckling coefficients $\lambda_{\mathrm{b}} = t^2\lambda$, and the associated eigenfunctions or buckling modes. For the analysis of this problem and its finite element approximation, we will rewrite it in several different forms and will consider other auxiliary problems. However Problem 3.2.1 is the only one to be discretized for the numerical computations.

The first step is to obtain a thorough spectral characterization of Problem 3.2.1, which will be the goal of the following section. With this end we introduce the so called *solution operator* whose spectrum is related with that of Problem 3.2.1. Let

$$\begin{aligned} T_t :\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega), \\ f &\mapsto w, \end{aligned} \tag{3.2.5}$$

where w is the second component of the solution to the following *source problem*:

Given $f \in \mathrm{H}_0^1(\Omega)$*, find* $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). \end{cases} \tag{3.2.6}$$

The operator T_t is linear and bounded and it is easy to see that (μ, w), with $\mu \neq 0$, is an eigenpair of T_t (i.e., $T_t w = \mu w$, $w \neq 0$) if and only if (λ, β, w) is a solution of Problem 3.2.1, with $\lambda = 1/\mu$ and a suitable $\beta \in \mathrm{H}_0^1(\Omega)^2$. Let us recall that our aim is to approximate the smallest eigenvalues of Problem 3.2.1, which correspond to the largest eigenvalues of the operator T_t.

To end this section we prove an additional regularity result for the solution to problem (3.2.6) which will be used in the sequel. To do this, first we rewrite problem (3.2.6) in a convenient way (see [5]). Using the following Helmholtz decomposition,

$$\gamma = \nabla\psi + \operatorname{curl} p, \qquad \psi \in \mathrm{H}_0^1(\Omega),\ p \in \mathrm{H}^1(\Omega)/\mathbb{R}, \tag{3.2.7}$$

we have that problem (3.2.6) is equivalent to the following one:

Given $f \in \mathrm{H}_0^1(\Omega)$*, find* $(\psi, \beta, p, w) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}^1(\Omega)/\mathbb{R} \times \mathrm{H}_0^1(\Omega)$ *such that*

$$\begin{cases} (\nabla\psi, \nabla v)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall v \in \mathrm{H}_0^1(\Omega), \\ a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla\psi, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2 (\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^2 (\nabla\psi, \nabla\xi)_{0,\Omega} & \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases} \tag{3.2.8}$$

We recall the following result for the solution of problem (3.2.8) (see [5]):

Theorem 3.2.1 *Let* Ω *be a convex polygon or a smoothly bounded domain in the plane. For any* $t > 0$, $\boldsymbol{\sigma} \in \mathrm{L}^\infty(\Omega)^{2\times 2}$*, and* $f \in \mathrm{H}_0^1(\Omega)$*, there exists a unique solution of problem* (3.2.8)*. Moreover,* $\beta \in \mathrm{H}^2(\Omega)^2$, $p \in \mathrm{H}^2(\Omega)$ *and there exists a constant* C*, independent of* t *and* f*, such that*

$$\|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|p\|_{1,\Omega} + t\,\|p\|_{2,\Omega} + \|w\|_{1,\Omega} \le C\,\|f\|_{1,\Omega}\,.$$

As a consequence of Theorem 3.2.1, by virtue of (3.2.7) and the equivalence between problems (3.2.6) and (3.2.8), we have that problem (3.2.6) is well-posed and there exists a constant C, independent of t and f, such that

$$\|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|\gamma\|_{0,\Omega} \le C\,\|f\|_{1,\Omega}\,. \tag{3.2.9}$$

3.3 Spectral properties

The aim of this section is threefold: (i) to prove a spectral characterization for the operator T_t defined above, (ii) to study the convergence of T_t and the behavior of its spectrum as t goes to zero, and (iii) to prove additional regularity for the eigenfunctions of T_t.

3.3.1 Spectral characterization

As stated above, we are only interested in approximating the largest eigenvalues of T_t. However, we will show that the spectrum of this operator does not reduce to eigenvalues. In fact, T_t is not compact and it has a non-trivial *essential spectrum*. Such essential spectrum is not relevant from the physical viewpoint, but its presence is a potential source of spectral pollution in the numerical methods (see, for instance [18]).

This will not be the case for the numerical method that we will propose, thanks to the results that will be proved in this subsection, which can be summarized as follows: Although T_t has a non-trivial essential spectrum, this is confined within a small ball around the origin, which is well separated from the largest eigenvalues of T_t (that are the goal of our numerical computation). To prove this, first we recall some basic definitions from spectral theory.

Given a generic linear bounded operator $T : X \to X$, defined on a Hilbert space X, the *spectrum* of T is the set $\mathrm{Sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and the *resolvent set* of T is its complement: $\rho(T) := \mathbb{C} \setminus \mathrm{Sp}(T)$. For any $z \in \rho(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ is the *resolvent operator* of T corresponding to z.

We recall the definitions of the following components of the spectrum.

- *Discrete spectrum*:
$$\mathrm{Sp_d}(T) := \{z \in \mathbb{C} : \mathrm{Ker}(zI - T) \neq \{0\} \text{ and } (zI - T) : X \to X \text{ is Fredholm}\}.$$

- *Essential spectrum*:
$$\mathrm{Sp_e}(T) := \{z \in \mathbb{C} : (zI - T) : X \to X \text{ is not Fredholm}\}.$$

The main result of this subsection is the following theorem which provides a suitable spectral characterization for the operator T_t defined in (3.2.5).

Theorem 3.3.1 *The spectrum of T_t decomposes as follows:* $\mathrm{Sp}(T_t) = \mathrm{Sp_d}(T_t) \cup \mathrm{Sp_e}(T_t)$*, with*

- $\mathrm{Sp_d}(T_t)$*, the discrete spectrum, which consists of real isolated eigenvalues of finite multiplicity and ascent one,*
- $\mathrm{Sp_e}(T_t)$*, the essential spectrum.*

Moreover, $\mathrm{Sp}_{\mathrm{e}}(T_t) \subset \left\{ z \in \mathbb{C} : \, |z| \leq \kappa^{-1} t^2 \, \|\boldsymbol{\sigma}\|_{\infty,\Omega} \right\}$.

The proof of this theorem will be given at the end of this subsection. Here and thereafter, we denote $\|\boldsymbol{\sigma}\|_{\infty,\Omega} := \max_{x\in\bar{\Omega}} |\boldsymbol{\sigma}(x)|$, with $|\cdot|$ being the matrix norm induced by the standard Euclidean norm in $\mathbb{R}^2$. Notice that the maximum above is well defined because of (3.2.3) and the fact that $\mathrm{W}^{1,\infty}(\Omega) \subset \mathcal{C}(\bar{\Omega})$.

As a consequence of this theorem we know that, although T_t may have essential spectrum, all the points of $\mathrm{Sp}(T_t)$ outside a ball centered at the origin of the complex plane are non-defective isolated eigenvalues. Moreover, the thinner the plate, the smaller the ball containing the essential spectrum.

The proof of Theorem 3.3.1 will be an immediate consequence of the results that follow. Consider the following continuous bilinear forms defined in $\mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$:

$$A((\beta, w), (\eta, v)) := a(\beta, \eta) + \frac{\kappa}{t^2} (\nabla w - \beta, \nabla v - \eta)_{0,\Omega} \,, \tag{3.3.1}$$

$$B((g, f), (\eta, v)) := (\boldsymbol{\sigma} \nabla f, \nabla v)_{0,\Omega} \,. \tag{3.3.2}$$

We notice that $A(\cdot,\cdot)$ is symmetric and elliptic (cf. [11]). Moreover, from the symmetry of $\boldsymbol{\sigma}$, it follows that $B(\cdot,\cdot)$ is symmetric too. Consider the bounded linear operator

$$\begin{aligned} \widetilde{T}_t : \, \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ (g, f) &\mapsto (\beta, w), \end{aligned} \tag{3.3.3}$$

where $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ is the solution of

$$A((\beta, w), (\eta, v)) = B((g, f), (\eta, v)) \qquad \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega).$$

We will prove in Lemma 3.3.4 below that the spectra of T_t and $\widetilde{T}_t$ coincide.

By virtue of the symmetry of $A(\cdot,\cdot)$ and $B(\cdot,\cdot)$, we have

$$A(\widetilde{T}_t(g, f), (\eta, v)) = B((g, f), (\eta, v)) = B((\eta, v), (g, f)) = A((g, f), \widetilde{T}_t(\eta, v)),$$

for every $(g, f), (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$. Therefore, $\widetilde{T}_t$ is self-adjoint with respect to the inner product $A(\cdot,\cdot)$. As a consequence, we have the following theorem (see, for instance, [17, Theorem 3.3]).

Theorem 3.3.2 *The spectrum of* $\widetilde{T}_t$ *is real (i.e.,* $\mathrm{Sp}(\widetilde{T}_t) \subset \mathbb{R}$*) and it decomposes as follows:* $\mathrm{Sp}(\widetilde{T}_t) = \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t) \cup \mathrm{Sp}_{\mathrm{e}}(\widetilde{T}_t)$*. Finally, if* $\mu \in \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t)$*, then* μ *is an isolated eigenvalue of finite multiplicity.*

The following result shows that the essential spectrum of $\widetilde{T}_t$ is confined in a neighborhood of the origin of diameter proportional to t^2.

Proposition 3.3.3 *Let* $\mu \in \mathrm{Sp}(\widetilde{T}_t)$ *be such that* $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$*. Then* $\mu \in \mathrm{Sp}_{\mathrm{d}}(\widetilde{T}_t)$*.*

Proof. Let $\mu \in \mathrm{Sp}(\widetilde{T}_t)$ be such that $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$. By virtue of Theorem 3.3.2, we only have to prove that $(\mu\widetilde{I} - \widetilde{T}_t)$ is a Fredholm operator. To this end, it is enough to show that there exists a compact operator $\widetilde{G}$ such that $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible. Let us introduce the operator S as follows:

$$\begin{aligned} S:\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega)^2, \\ f &\mapsto \beta, \end{aligned}$$

where β is the first component of the the unique solution (β, w) of problem (3.2.6). Notice that

$$\widetilde{T}_t(g, f) = (Sf, T_t f). \tag{3.3.4}$$

According to (3.2.9), we have that $\beta \in \mathrm{H}^2(\Omega)^2$ and hence S is compact. Let us now define the operator G as follows:

$$\begin{aligned} G:\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega), \\ f &\mapsto u, \end{aligned} \tag{3.3.5}$$

where $u \in \mathrm{H}_0^1(\Omega)$ is the unique solution of

$$(\nabla u, \nabla \xi)_{0,\Omega} = (Sf, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).$$

The operator G is compact as a consequence of the compactness of S. Next, we define $\widetilde{G}$ as follows:

$$\begin{aligned} \widetilde{G}:\ \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ (g, f) &\mapsto (Sf, Gf). \end{aligned}$$

Since S and G are compact, $\widetilde{G}$ is compact, too. In addition,

$$(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})(g, f) = ((\mu g - Sf + Sf), (\mu I - T_t + G) f) = (\mu g, (\mu I - T_t + G) f).$$

Therefore, $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible if and only if $(\mu I - T_t + G)$ is invertible.

From the fourth equation in (3.2.8), we notice that $v := (\mu I - T_t + G) f$ satisfies

$$\begin{aligned}(\nabla v, \nabla \xi)_{0,\Omega} &= \mu (\nabla f, \nabla \xi)_{0,\Omega} - (\nabla w, \nabla \xi)_{0,\Omega} + (\beta, \nabla \xi)_{0,\Omega} \\ &= \left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).\end{aligned}$$

Consequently, the operator $(\mu I - T_t + G)$ will be invertible if and only if, given $v \in \mathrm{H}_0^1(\Omega)$, there exists a unique $f \in \mathrm{H}_0^1(\Omega)$ solution of

$$\left(\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) \nabla f, \nabla \xi\right)_{0,\Omega} = (\nabla v, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega). \tag{3.3.6}$$

Now, because of the symmetry of $\boldsymbol{\sigma}(x)$, there exists an orthogonal matrix $\boldsymbol{P}(x)$ such that $\boldsymbol{\sigma}(x) = \boldsymbol{P}(x)\mathbf{D}(x)\boldsymbol{P}^{\mathrm{t}}(x)$, where

$$\mathbf{D}(x) := \begin{bmatrix} \overline{\omega}(x) & 0 \\ 0 & \underline{\omega}(x) \end{bmatrix},$$

with $\underline{\omega}(x) \leq \overline{\omega}(x)$ being the two real eigenvalues of $\boldsymbol{\sigma}(x)$. Hence, we write

$$\left(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma}\right) = \boldsymbol{P}(x) \begin{bmatrix} \mu - \kappa^{-1} t^2 \overline{\omega}(x) & 0 \\ 0 & \mu - \kappa^{-1} t^2 \underline{\omega}(x) \end{bmatrix} \boldsymbol{P}(x)^{\mathrm{t}}.$$

Let us denote $\omega_{\max} := \max_{x \in \bar{\Omega}} \overline{\omega}(x)$ and $\omega_{\min} := \min_{x \in \bar{\Omega}} \underline{\omega}(x)$. Since $\|\boldsymbol{\sigma}\|_{\infty,\Omega} = \max_{x \in \bar{\Omega}} |\boldsymbol{\sigma}(x)| = \max\{|\omega_{\max}|, |\omega_{\min}|\}$, for $|\mu| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$, there holds either $\mu > \kappa^{-1} t^2 \omega_{\max}$ or $\mu < \kappa^{-1} t^2 \omega_{\min}$. Hence, $(\mu \boldsymbol{I} - \kappa^{-1} t^2 \boldsymbol{\sigma})$ is uniformly positive definite in the first case or uniformly negative definite in the second one. Therefore, in both cases, there exists a unique solution $f \in \mathrm{H}_0^1(\Omega)$ of (3.3.6). Consequently, $(\mu I - T_t + G)$ is invertible and hence $(\mu\widetilde{I} - \widetilde{T}_t + \widetilde{G})$ is invertible, too. Thus, we have that $(\mu\widetilde{I} - \widetilde{T}_t)$ is Fredholm and we conclude the proof. $\square$

The following result shows that T_t and $\widetilde{T}_t$ have the same spectrum.

Lemma 3.3.4 *If T_t and $\widetilde{T}_t$ are the operators defined in* (3.2.5) *and* (3.3.3)*, respectively, then* $\mathrm{Sp}(\widetilde{T}_t) = \mathrm{Sp}(T_t)$.

Proof. We will prove that $\rho(\widetilde{T}_t) = \rho(T_t)$. Let z be such that $(z\widetilde{I} - \widetilde{T}_t)$ is invertible. We will prove that $(zI - T_t)$ is invertible, too. By hypothesis, for every $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ there exists a unique $(g, f) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w). \tag{3.3.7}$$

Recalling (3.3.4), we infer that there is a unique (g, f) such that $zg - Sf = \beta$ and $(zI - T_t) f = w$. Hence, we deduce that the operator $(zI - T_t) : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ is onto. Now, let us assume that there exists another $\hat{f}$ such that $(zI - T_t) \hat{f} = w$. Taking $\hat{g} = \frac{1}{z}(S\hat{f} + \beta)$, we have that $(z\widetilde{I} - \widetilde{T}_t)(\hat{g}, \hat{f}) = (\beta, w)$. Since by hypothesis $(z\widetilde{I} - \widetilde{T}_t)$ is invertible, from (3.3.7) it follows that $f = \hat{f}$. Therefore, $(zI - T_t)$ is also one-to-one and thus invertible.

Conversely, let z be such that $(zI - T_t)$ is invertible. We will prove that $(z\widetilde{I} - \widetilde{T}_t)$ is invertible, too. Recalling (3.3.4) again, we have to show that for every $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$, there exists a unique $(g, f) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} zg - Sf = \beta, \\ zf - T_t f = w. \end{cases}$$

Let $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ be given. There exists a unique $f \in \mathrm{H}_0^1(\Omega)$ such that $(zI - T_t) f = w$. Therefore, taking $g := \frac{1}{z}(Sf + \beta)$, we obtain $(z\widetilde{I} - \widetilde{T}_t)(g, f) = (\beta, w)$. The uniqueness of g follows immediately from the uniqueness of f and the first equation of the system above. The proof is complete. $\square$

The following result shows that the eigenvalues of T_t are non-defective.

Lemma 3.3.5 *Suppose that $\mu \neq 0$ is an isolated eigenvalue of T_t. Then its ascent is one.*

Proof. By contradiction. Let (μ, w) be an eigenpair of T_t, $\mu \neq 0$, and let us assume that T_t has a corresponding generalized eigenfunction, namely, $\exists \hat{w} \neq 0$ such that $T_t \hat{w} = \mu \hat{w} + w$.

Since (μ, w) is an eigenpair of T_t, there exists $\beta \in \mathrm{H}_0^1(\Omega)^2$ such that (cf. (3.2.5) and Problem 3.2.1)

$$a(\beta,\eta) + \frac{\kappa}{t^2}\left(\nabla w - \beta, \nabla v - \eta\right)_{0,\Omega} = \frac{1}{\mu}\left(\boldsymbol{\sigma}\nabla w, \nabla v\right)_{0,\Omega} \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega). \tag{3.3.8}$$

On the other hand, since $T_t\hat{w} = \mu\hat{w} + w$, the definition of T_t implies the existence of $\hat{\beta} \in \mathrm{H}_0^1(\Omega)^2$ such that

$$a(\hat{\beta},\eta) + \frac{\kappa}{t^2}(\nabla\left(w + \mu\hat{w}\right) - \hat{\beta}, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla\hat{w}, \nabla v)_{0,\Omega} \qquad \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega).$$

Defining $\bar{\beta} := (\hat{\beta} - \beta)/\mu$, the equation above can be written as follows:

$$\mu a(\bar{\beta},\eta) + a(\beta,\eta) + \frac{\kappa\mu}{t^2}(\nabla\hat{w} - \bar{\beta}, \nabla v - \eta)_{0,\Omega} + \frac{\kappa}{t^2}(\nabla w - \beta, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla\hat{w}, \nabla v)_{0,\Omega}.$$

We now take $(\eta, v) = \mu(\bar{\beta}, \hat{w})$ in (3.3.8) and $(\eta, v) = (\beta, w)$ in the equation above and subtract the resulting equations. Using also the symmetry of $a(\cdot,\cdot)$ and $\boldsymbol{\sigma}$, we obtain

$$a(\beta,\beta) + \frac{\kappa}{t^2}\left\|\nabla w - \beta\right\|_{0,\Omega}^2 = 0.$$

Thus, from the ellipticity of $a(\cdot,\cdot)$, we infer $\beta = 0$ and hence $w = 0$, which is a contradiction since w is an eigenfunction of T_t. The proof is complete. □

We are now in a position to prove Theorem 3.3.1.

Proof of Theorem 3.3.1. The proof follows easily by combining Lemma 3.3.4 with Theorem 3.3.2, Proposition 3.3.3, and Lemma 3.3.5. □

3.3.2 Limit problem

In this subsection we study the convergence properties of the operator T_t as t goes to zero. First, let us recall that it is well-known (see [11]) that, when t goes to zero, the solution (β, w, γ) of problem (3.2.6) converges to the solution of the following problem:

Given $f \in \mathrm{H}_0^1(\Omega)$, find $(\beta_0, w_0, \gamma_0) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0(\mathrm{rot};\Omega)'$ such that

$$\begin{cases} a(\beta_0,\eta) + \langle\gamma_0, \nabla v - \eta\rangle = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} & \forall(\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \nabla w_0 - \beta_0 = 0. \end{cases} \tag{3.3.9}$$

Above, $\langle \cdot, \cdot \rangle$ stands now for the duality pairing in $\mathrm{H}_0(\mathrm{rot};\Omega)$. Problem (3.3.9) is a mixed formulation for the following well-posed problem, which corresponds to the buckling of a Kirchhoff plate:

Given $f \in \mathrm{H}_0^1(\Omega)$, find $w_0 \in \mathrm{H}_0^2(\Omega)$ such that

$$\frac{E}{12\,(1-\nu^2)}\,(\Delta w_0, \Delta v)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f, \nabla v)_{0,\Omega} \qquad \forall v \in \mathrm{H}_0^2(\Omega). \tag{3.3.10}$$

Let T_0 be the bounded linear operator defined by

$$\begin{aligned} T_0 :\ \mathrm{H}_0^1(\Omega) &\to \mathrm{H}_0^1(\Omega), \\ f &\mapsto w_0, \end{aligned}$$

where w_0 is the second component of the solution of problem (3.3.9). Since $w_0 \in \mathrm{H}_0^2(\Omega)$, the operator T_0 is compact and hence its spectrum satisfies $\mathrm{Sp}(T_0) = \{0\} \cup \{\mu_n :\ n \in \mathbb{N}\}$, where $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of positive eigenvalues which converges to 0. The multiplicity of each non-zero eigenvalue is finite and its ascent is 1. The following lemma, which yields the convergence in norm of T_t to T_0 has been essentially proved in [21, Lemma 3.1].

Lemma 3.3.6 *There exists a constant C, independent of t, such that*

$$\|(T_t - T_0)\,f\|_{1,\Omega} \leq Ct\,\|f\|_{1,\Omega} \qquad \forall f \in \mathrm{H}_0^1(\Omega).$$

As a consequence of this lemma, standard properties about the separation of isolated parts of the spectrum (see [31], for instance) yield the following result.

Lemma 3.3.7 *Let $\mu_0 \neq 0$ be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then there exists $t_0 > 0$ such that, $\forall t < t_0$, D contains exactly m isolated eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each nonzero eigenvalue μ_0 of T_0 is a limit of isolated eigenvalues μ_t of T_t, as t goes to zero.*

Our next goal is to show that the largest eigenvalues of T_t converge to the largest eigenvalues of T_0 as t goes to zero. With this aim, we prove first the following lemma. Here and thereafter, we will use $\|\cdot\|$ to denote the operator norm induced by the $\mathrm{H}^1(\Omega)$ norm.

Lemma 3.3.8 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. Then there exist strictly positive constants t_0 and C such that, $\forall t < t_0$, $F \cap \mathrm{Sp}(T_t) = \emptyset$ and*

$$\|R_z(T_t)\| := \sup_{\substack{w \in \mathrm{H}_0^1(\Omega) \\ w \neq 0}} \frac{\|R_z(T_t)w\|_{1,\Omega}}{\|w\|_{1,\Omega}} \leq C \qquad \forall z \in F.$$

Proof. The mapping $z \mapsto \| (zI - T_0)^{-1} \|$ is continuous for all $z \in \rho(T_0)$ and goes to zero as $|z| \to \infty$. Consequently, it attains its maximum on any closed subset $F \subset \rho(T_0)$. Let $C_1 := 1/\max_{z \in F} \| (zI - T_0)^{-1} \|$; there holds

$$\|(zI - T_0)\, w\|_{1,\Omega} \geq \frac{1}{C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}_0^1(\Omega) \quad \forall z \in F.$$

Now, according to Lemma 3.3.6, there exists $t_1 > 0$ such that, for all $t < t_1$,

$$\|(T_t - T_0)\, w\|_{1,\Omega} \leq \frac{1}{2C_1} \|w\|_{1,\Omega} \qquad \forall w \in \mathrm{H}_0^1(\Omega).$$

Therefore, for all $w \in \mathrm{H}_0^1(\Omega)$, for all $z \in F$, and for all $t < t_1$,

$$\|(zI - T_t)\, w\|_{1,\Omega} \geq \|(zI - T_0)\, w\|_{1,\Omega} - \|(T_t - T_0)\, w\|_{1,\Omega} \geq \frac{1}{2C_1} \|w\|_{1,\Omega} \tag{3.3.11}$$

and, consequently, $z \notin \mathrm{Sp_d}(T_t)$.

On the other hand, $d := \min_{z \in F} |z|$ is strictly positive, because $\mathrm{Sp}(T_0) \ni 0$, $F \cap \mathrm{Sp}(T_0) = \emptyset$, and F is closed. Let $t_2 > 0$ be such $\kappa^{-1} t_2^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega} < d$. Hence, for all $z \in F$ and for all $t < t_2$, we have $|z| > \kappa^{-1} t^2 \|\boldsymbol{\sigma}\|_{\infty,\Omega}$ and, consequently, by virtue of Theorem 3.3.1, either $z \in \mathrm{Sp_d}(T_t)$ or $z \notin \mathrm{Sp}(T_t)$.

Altogether, if $t_0 := \min\{t_1, t_2\}$, then $(zI - T_t)$ is invertible for all $t < t_0$ and all $z \in F$. Moreover, because of (3.3.11),

$$\|R_z(T_t)\| = \| (zI - T_t)^{-1} \| \leq 2C_1$$

and we conclude the proof. □

It is easy to show that the spectrum of T_0 is real; in fact, this follows readily from the symmetric formulation (3.3.10). Since T_0 is compact, its nonzero eigenvalues are isolated and of finite multiplicity, so that we can order the positive ones as follows:

$$\mu_0^{(1)} \geq \mu_0^{(2)} \geq \cdots \geq \mu_0^{(k)} \geq \cdots,$$

where each eigenvalue is repeated as many times as its corresponding multiplicity. A similar ordering holds for the negative eigenvalues, too, if they exist.

According to Lemma 3.3.7, for t sufficiently small there exist eigenvalues of T_t close to each $\mu_0^{(k)}$. On the other hand, according to Theorem 3.3.1, the essential spectrum of T_t is confined within a ball centered at the origin of the complex plane with radius proportional to t^2. Therefore, at least for t sufficiently small, the points of the spectrum of T_t largest in modulus have to be isolated eigenvalues of finite multiplicity. Since the spectrum of T_t is also real, we order the positive eigenvalues as we did with those of T_0:

$$\mu_t^{(1)} \geq \mu_t^{(2)} \geq \cdots \geq \mu_t^{(k)} \geq \cdots .$$

Once more, a similar ordering holds for the negative eigenvalues of T_t, if they exist.

The following theorem shows that the k-th positive eigenvalue of T_t converges to the k-th positive eigenvalue of T_0 as t goes to zero. A similar result holds for the negative eigenvalues, as well.

Theorem 3.3.9 *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as defined above. For all $k \in \mathbb{N}$, $\mu_t^{(k)} \to \mu_0^{(k)}$ as $t \to 0$.*

Proof. We will prove the result for the largest eigenvalue $\mu_t^{(1)}$. The proof for the others is a straightforward modification of this one.

Let D be an open disk in the complex plane centered at $\mu_0^{(1)}$ with radius $r < [\mu_0^{(1)} - \mu_0^{(k)}]/2$, where $\mu_0^{(k)}$ is the largest eigenvalue of T_0 satisfying $\mu_0^{(k)} < \mu_0^{(1)}$. Therefore, $D \cap \mathrm{Sp}(T_0) = \{\mu_0^{(1)}\}$.

Let H be the half plane $\{z \in \mathbb{C} : \mathrm{Re}(z) < [\mu_0^{(k)} + \mu_0^{(1)}]/2\}$. Hence $\mathrm{Sp}(T_0) \subset D \cup H$. Let $F := \mathbb{C} \setminus (D \cup H)$. The set F is closed and $F \cap \mathrm{Sp}(T_0) = \emptyset$. Hence, according to Lemma 3.3.8, there exists $t_0 > 0$ such that, for all $t < t_0$, $F \cap \mathrm{Sp}(T_t) = \emptyset$, too, and hence $\mathrm{Sp}(T_t) \subset D \cup H$, as well.

On the other hand, because of Lemma 3.3.7, there exists $t_1 > 0$ such that, for all $t < t_1$, D contains as many eigenvalues of T_t as the multiplicity of $\mu_0^{(1)}$. Therefore, for all $t < \min\{t_0, t_1\}$, the largest eigenvalue of T_t, $\mu_t^{(1)}$, has to lie in D. Since D can be taken arbitrarily small, we conclude that $\mu_t^{(1)}$ converges to $\mu_0^{(1)}$ as t goes to zero. Thus, we conclude the proof. $\square$

3.3.3 Additional regularity of the eigenfunctions

The aim of this subsection is to prove a regularity result for the eigenfunctions of Problem 3.2.1. More precisely, we have the following proposition.

Proposition 3.3.10 *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as in Theorem 3.3.9. Let $(\lambda, \beta, w, \gamma)$ be a solution of Problem 3.2.1 with $\lambda = 1/\mu_t^{(k)}$. Then there exists $t_0 > 0$ such that, for all $t < t_0$, $\beta \in \mathrm{H}^2(\Omega)^2$, $w \in \mathrm{H}^2(\Omega)$, $\operatorname{div} \gamma \in \mathrm{L}^2(\Omega)$, and there holds*

$$\|\beta\|_{2,\Omega} \leq C\,|\lambda|\,\|w\|_{1,\Omega}\,, \tag{3.3.12}$$

$$\|w\|_{2,\Omega} \leq C\,|\lambda|\,\|w\|_{1,\Omega}\,, \tag{3.3.13}$$

$$\|\operatorname{div} \gamma\|_{0,\Omega} \leq C\,|\lambda|\,\|w\|_{2,\Omega}\,, \tag{3.3.14}$$

with C a positive constant independent of t.

Proof. Using the Helmholtz decomposition (3.2.7), Problem 3.2.1 is equivalent to finding $\lambda \in \mathbb{R}$ and $0 \neq (\psi, \beta, p, w) \in \mathrm{H}_0^1(\Omega) \times \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}^1(\Omega)/\mathbb{R} \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} (\nabla\psi, \nabla v)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v)_{0,\Omega} & \forall v \in \mathrm{H}_0^1(\Omega), \\ a(\beta, \eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla\psi, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -\,(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2\,(\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla\xi)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} + \kappa^{-1}t^2\,(\nabla\psi, \nabla\xi)_{0,\Omega} & \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases}$$

From Theorem 3.2.1 applied to the problem above, we immediately obtain that $\beta \in \mathrm{H}^2(\Omega)^2$ and the estimate (3.3.12).

On the other hand, the first and the last equations of the system above lead to

$$\left(\left(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma}\right)\nabla w, \nabla\xi\right)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).$$

Since $\mu_t^{(k)} \to \mu_0^{(k)} > 0$ as $t \to 0$, there exists $t_1 > 0$ such that $\mu_t^{(k)} > \mu_0^{(k)}/2\ \ \forall t < t_1$. Hence $\lambda = 1/\mu_t^{(k)} < 2/\mu_0^{(k)}$. We take $t_0 < t_1$ such that $\kappa^{-1}t_0^2\,\|\boldsymbol{\sigma}\|_{\infty,\Omega} < \mu_0^{(k)}/2$. Therefore, for all $t < t_0$, $(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma})$ is uniformly positive definite. Thus, since w is the solution of the problem

$$\begin{cases} \operatorname{div}\left[\left(\boldsymbol{I} - \lambda\kappa^{-1}t^2\boldsymbol{\sigma}\right)\nabla w\right] = \operatorname{div}\beta & \text{in } \Omega, \\ w = 0 & \text{on } \partial\Omega, \end{cases}$$

using a standard regularity result (see [42]), we have that $w \in \mathrm{H}^2(\Omega)$ and

$$\|w\|_{2,\Omega} \leq C \|\operatorname{div} \beta\|_{0,\Omega} \leq C \|\beta\|_{1,\Omega} \leq C |\lambda| \|w\|_{1,\Omega},$$

the last inequality because of (3.3.12).

Furthermore, taking $\eta = 0$ in Problem 3.2.1, using the estimate above and (3.2.3), it follows that

$$\operatorname{div} \gamma = \lambda \operatorname{div}(\boldsymbol{\sigma} \nabla w) \in \mathrm{L}^2(\Omega).$$

and

$$\|\operatorname{div} \gamma\|_{0,\Omega} \leq C |\lambda| \|w\|_{2,\Omega}.$$

The proof is complete. □

Once more a similar result holds for negative eigenvalues $\mu_t^{(k)} \to \mu_0^{(k)} < 0$.

3.4 Spectral approximation

For the numerical approximation, we focus on the finite element method proposed and studied in [22]. In what follows we introduce briefly this method (see this reference for further details). Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of triangular meshes of $\bar{\Omega}$. We will define finite element spaces H_h, W_h, and Γ_h for the rotations, the transverse displacements, and the shear stress, respectively.

For $K \in \mathcal{T}_h$, let α_1, α_2, α_3 be its barycentric coordinates. We denote by τ_i a unit vector tangent to the edge $\alpha_i = 0$ and define

$$p_1^K = \alpha_2 \alpha_3 \tau_1, \qquad p_2^K = \alpha_1 \alpha_3 \tau_2, \qquad p_3^K = \alpha_1 \alpha_2 \tau_3.$$

The finite element space for the rotations is defined by

$$H_h := \left\{ \eta_h \in \mathrm{H}_0^1(\Omega)^2 : \eta_h|_K \in \mathbb{P}_1^2 \oplus \langle p_1^K, p_2^K, p_3^K \rangle \quad \forall K \in \mathcal{T}_h \right\}.$$

To approximate the transverse displacements, we use the usual piecewise-linear continuous finite element space:

$$W_h := \left\{ v_h \in \mathrm{H}_0^1(\Omega) : v_h|_K \in \mathbb{P}_1(K) \quad \forall K \in \mathcal{T}_h \right\}.$$

Finally, for the shear stress, we use the lowest-order rotated Raviart-Thomas space:

$$\Gamma_h := \left\{ \phi \in \mathrm{H}_0(\mathrm{rot};\Omega) :\ \phi|_K \in \mathbb{P}_0^2 \oplus (x_2, -x_1)\mathbb{P}_0 \quad \forall K \in \mathcal{T}_h \right\}.$$

We consider as reduction operator the rotated Raviart-Thomas interpolant

$$R:\ \mathrm{H}^1(\Omega)^2 \cap \mathrm{H}_0(\mathrm{rot};\Omega) \to \Gamma_h,$$

which is uniquely determined by

$$\int_\ell R\phi \cdot \tau_\ell = \int_\ell \phi \cdot \tau_\ell$$

for every edge ℓ of the triangulation, τ_ℓ being a unit vector tangent to ℓ. It is well-known that

$$\|R\phi\|_{0,\Omega} \leq C \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2, \tag{3.4.1}$$

$$\|\phi - R\phi\|_{0,\Omega} \leq Ch \|\phi\|_{1,\Omega} \qquad \forall \phi \in \mathrm{H}^1(\Omega)^2. \tag{3.4.2}$$

Moreover, the operator R can be extended continuously to $\mathrm{H}^s(\Omega)^2 \cap \mathrm{H}_0(\mathrm{rot};\Omega)$ for any $s > 0$ and it is also well known that, for all $v \in \mathrm{H}^{1+s}(\Omega) \cap \mathrm{H}_0^1(\Omega)$,

$$R(\nabla v) = \nabla v_{\mathbf{I}}, \tag{3.4.3}$$

where $v_{\mathbf{I}} \in W_h$ is the standard piecewise-linear Lagrange interpolant of v (which is well defined because $\mathrm{H}^{1+s}(\Omega) \subset \mathcal{C}(\bar{\Omega})$ for all $s > 0$).

The discretization of Problem 3.2.1 reads as follows:

Problem 3.4.1 *Find $\lambda_h \in \mathbb{R}$ and $0 \neq (\beta_h, w_h) \in H_h \times W_h$ such that*

$$\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h (\boldsymbol{\sigma} \nabla w_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2} (\nabla w_h - R\beta_h). \end{cases}$$

Notice that this leads to a nonconforming method, since consistency terms arise because of the reduction operator R. The final goal of this paper is to prove that the smallest (in absolute value) eigenvalues λ_h converge to the smallest (in absolute value) eigenvalues λ of Problem 3.2.1. We will also prove convergence of the corresponding eigenfunctions and error estimates.

Our first step is to obtain a characterization of the solutions to Problem 3.4.1.

Lemma 3.4.1 *Let* $Y_h := \left\{ w_h \in W_h : (\boldsymbol{\sigma}\nabla w_h, \nabla v_h)_{0,\Omega} = 0 \quad \forall v_h \in W_h \right\}$. *Then Problem 3.4.1 has exactly* $\dim W_h - \dim Y_h$ *eigenvalues, repeated according to their respective multiplicities. All of them are real and nonzero.*

Proof. We eliminate γ_h in Problem 3.4.1 to write it as follows:

$$a(\beta_h, \eta_h) + \frac{\kappa}{t^2} (\nabla w_h - R\beta_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda_h (\boldsymbol{\sigma}\nabla w_h, \nabla v_h)_{0,\Omega} \qquad \forall (\eta_h, v_h) \in H_h \times W_h. \tag{3.4.4}$$

Taking particular bases of H_h and W_h, this problem can be written in matrix form as follows:

$$\boldsymbol{\mathcal{A}} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix} = \lambda_h \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix}, \tag{3.4.5}$$

where $\boldsymbol{\beta}_h$ and $\boldsymbol{w}_h$ denote the vectors whose entries are the components in those basis of β_h and w_h, respectively. The matrix $\boldsymbol{\mathcal{A}}$ is symmetric and positive definite because the bilinear form on the left-hand side of (3.4.4) is elliptic in $\mathrm{H}^1_0(\Omega)^2 \times \mathrm{H}^1_0(\Omega)$ (cf. [22]). Consequently, $\lambda_h \neq 0$ and, since $\mathbf{E}$ is also symmetric, $\lambda_h \in \mathbb{R}$. Now, (3.4.5) holds true if and only if

$$\begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix} = \mu_h \boldsymbol{\mathcal{A}} \begin{bmatrix} \boldsymbol{\beta}_h \\ \boldsymbol{w}_h \end{bmatrix}$$

with $\lambda_h = 1/\mu_h$ and $\mu_h \neq 0$. The latter is a well-posed generalized eigenvalue problem with $\dim W_h - \dim \mathrm{Ker}(\mathbf{E})$ nonzero eigenvalues. Thus, we conclude the lemma by noting that $\mathbf{E}\boldsymbol{w}_h = \mathbf{0}$ if and only if $w_h \in Y_h$. □

Remark 3.4.2 If $(\lambda_h, \beta_h, w_h)$ is a solution of Problem 3.4.1, then

$$\boldsymbol{w}_h^{\mathrm{t}} \mathbf{E} \boldsymbol{w}_h = (\boldsymbol{\sigma}\nabla w_h, \nabla w_h)_{0,\Omega} \neq 0.$$

In fact, this follows by left multiplying both sides of (3.4.5) by $(\boldsymbol{\beta}_h^{\mathrm{t}}, \boldsymbol{w}_h^{\mathrm{t}})$ and using the positive definiteness of $\boldsymbol{\mathcal{A}}$.

As in the continuous case, we introduce for the analysis the *discrete solution operator*

$$T_{th} : \mathrm{H}^1_0(\Omega) \to W_h \hookrightarrow \mathrm{H}^1_0(\Omega),$$
$$f \mapsto w_h,$$

where w_h is the second component of the solution (β_h, w_h) to the corresponding *discrete source problem*:

Given $f \in \mathrm{H}_0^1(\Omega)$, *find* $(\beta_h, w_h) \in H_h \times W_h$ *such that*

$$\begin{cases} a(\beta_h, \eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma} \nabla f, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2} (\nabla w_h - R\beta_h) . \end{cases} \tag{3.4.6}$$

Existence and uniqueness of the solution to problem (3.4.6) follow easily (see [22]). Moreover, the nonzero eigenvalues of T_{th} are given by $\mu_h := 1/\lambda_h$, with λ_h being the eigenvalues of Problem 3.4.1, and the corresponding eigenfunctions coincide.

Remark 3.4.3 The solution to (3.4.6) is a finite element approximation of the solution to (3.2.6). However, given a generic $f \in \mathrm{H}_0^1(\Omega)$, the usual convergence rate in terms of positive powers of the mesh-size h does not hold in this case, because the solution to (3.2.6) is not sufficiently smooth. Indeed, the right-hand side is not regular enough, since $\operatorname{div}(\boldsymbol{\sigma} \nabla f) \notin \mathrm{L}^2(\Omega)$. Now, whenever f is more regular, for instance assuming $f \in \mathrm{H}^2(\Omega)$, by taking into account the regularity of $\boldsymbol{\sigma}$ (cf. (3.2.3)), the convergence results of [22] can be applied to obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t \|\gamma - \gamma_h\|_{0,\Omega} + \|w - w_h\|_{1,\Omega} \leq Ch \|f\|_{2,\Omega} .$$

3.4.1 Auxiliary results

In what follows we will prove several auxiliary results which will be used in the following section to prove convergence and error estimates for our spectral approximation. The first of them is the following lemma which shows that the operator T_{th} defined above is bounded uniformly in t and h.

Lemma 3.4.4 *There exists* $C > 0$ *such that* $\|T_{th}\| \leq C$ *for all* $t > 0$ *and all* $h > 0$.

Proof. Let $f \in \mathrm{H}_0^1(\Omega)$ and (β_h, w_h) be the solution to problem (3.4.6). Taking $(\eta_h, v_h) = (\beta_h, w_h)$ as test function in (3.4.6), we obtain

$$a(\beta_h, \beta_h) + \kappa^{-1} t^2 \|\gamma_h\|_{0,\Omega}^2 \leq \|\boldsymbol{\sigma}\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega} .$$

Hence, from the ellipticity of $a(\cdot,\cdot)$,

$$\|\beta_h\|_{1,\Omega}^2 + \kappa^{-1}t^2 \|\gamma_h\|_{0,\Omega}^2 \le C \|\boldsymbol{\sigma}\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega} .$$

Therefore, using the definition of γ_h (cf. (3.4.6)) and (3.4.1),

$$\|\nabla w_h\|_{0,\Omega}^2 = \left\|\kappa^{-1}t^2\gamma_h + R\beta_h\right\|_{0,\Omega}^2 \le C \|\boldsymbol{\sigma}\|_{\infty,\Omega} \|\nabla f\|_{0,\Omega} \|\nabla w_h\|_{0,\Omega} ,$$

which allows us to conclude the proof. $\square$

Next, we will adapt the theory developed in [18, 19] for non-compact operators to our case. With this aim, we will prove the following properties:

P1. $\|T_0 - T_{th}\|_h := \sup\limits_{\substack{f_h\in W_h\\ f_h\neq 0}} \dfrac{\|(T_0 - T_{th}) f_h\|_{1,\Omega}}{\|f_h\|_{1,\Omega}} \to 0, \qquad \text{as } (h,t)\to(0,0);$

P2. $\forall u \in \mathrm{H}_0^1(\Omega) \qquad \inf\limits_{v_h\in W_h} \|u - v_h\|_{1,\Omega} \to 0, \qquad \text{as } h\to 0.$

From now on, we will use the operator norm $\|\cdot\|_h$ as defined in property P1.

We focus on property P1, since property P2 follows from standard approximation results. We notice first that

$$\|T_0 - T_{th}\|_h \le \|T_0 - T_t\|_h + \|T_t - T_{th}\|_h , \tag{3.4.7}$$

where T_t is the operator defined in (3.2.5). Since $W_h \subset \mathrm{H}_0^1(\Omega)$, from Lemma 3.3.6 we deduce that for all $h > 0$

$$\|T_0 - T_t\|_h \le Ct. \tag{3.4.8}$$

Regarding the other term in the right-hand side of (3.4.7), we aim at proving the following result.

Proposition 3.4.5 *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Then we have*

$$\|T_t - T_{th}\|_h \le C\,(h+t) .$$

The proof of Proposition 3.4.5 will be given at the end of this section. With this aim, we consider problems (3.2.6) and (3.4.6) with source term in W_h, namely:

Given $f_h \in W_h$, find $(\beta, w) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega)$ such that

$$\begin{cases} a(\beta,\eta) + (\gamma, \nabla v - \eta)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f_h, \nabla v)_{0,\Omega} & \forall (\eta, v) \in \mathrm{H}_0^1(\Omega)^2 \times \mathrm{H}_0^1(\Omega), \\ \gamma = \dfrac{\kappa}{t^2}(\nabla w - \beta). \end{cases} \tag{3.4.9}$$

Given $f_h \in W_h$, find $(\beta_h, w_h) \in H_h \times W_h$ such that

$$\begin{cases} a(\beta_h,\eta_h) + (\gamma_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma}\nabla f_h, \nabla v_h)_{0,\Omega} & \forall (\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h = \dfrac{\kappa}{t^2}(\nabla w_h - R\beta_h). \end{cases} \tag{3.4.10}$$

We need some results concerning the solutions of these problems. First, we apply the Helmholtz decomposition (3.2.7) to the term γ from (3.4.9):

$$\gamma = \nabla\psi + \operatorname{curl} p, \qquad \psi \in \mathrm{H}_0^1(\Omega),\ p \in \mathrm{H}^1(\Omega)/\mathbb{R}. \tag{3.4.11}$$

Then, we apply Theorem 3.2.1 and (3.2.9), to obtain the following a priori estimate for the solution to problem (3.4.9):

$$\|\psi\|_{1,\Omega} + \|\beta\|_{2,\Omega} + \|w\|_{1,\Omega} + \|p\|_{1,\Omega} + t\,\|p\|_{2,\Omega} + \|\gamma\|_{0,\Omega} \le C\,\|f_h\|_{1,\Omega}. \tag{3.4.12}$$

The following result shows that, for $f_h \in W_h$, w and ψ are actually smoother and an inverse estimate which will be used to prove Proposition 3.4.5.

Lemma 3.4.6 *Let w be defined by problem* (3.4.9) *and ψ as in* (3.4.11)*. Then $w, \psi \in \mathrm{H}^{1+s}(\Omega)$ for all $s \in (0, \frac{1}{2})$. Moreover, if the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, then*

$$\|\psi\|_{1+s,\Omega} \le Ch^{-s}\,\|f_h\|_{1,\Omega}.$$

Proof. Recall the equivalence between problems (3.4.9) and (3.2.8), the latter with source term f_h instead of f. From the first equation of (3.2.8) we have that ψ is the weak solution of

$$\begin{cases} \Delta\psi = \operatorname{div}(\boldsymbol{\sigma}\nabla f_h) \in \mathrm{H}^{-1}(\Omega), \\ \psi = 0 \qquad \text{on } \partial\Omega. \end{cases} \tag{3.4.13}$$

Since f_h is a continuous piecewise linear function, we have that $f_h \in \mathrm{H}^{1+s}(\Omega)\ \forall s \in (0, \frac{1}{2})$. Therefore, the assumption (3.2.3) implies $\boldsymbol{\sigma}\nabla f_h \in \mathrm{H}^s(\Omega)^2$. Hence, $\mathrm{div}(\boldsymbol{\sigma}\nabla f_h) \in \mathrm{H}^{s-1}(\Omega)$. Then, from standard regularity results for problem (3.4.13), $\psi \in \mathrm{H}^{1+s}(\Omega)\ \forall s \in (0, \frac{1}{2})$ and

$$\|\psi\|_{1+s,\Omega} \leq C\, \|\mathrm{div}(\boldsymbol{\sigma}\nabla f_h)\|_{s-1,\Omega} \leq C\, \|f_h\|_{1+s,\Omega}\,.$$

If the family of meshes is quasi-uniform, then the inverse inequality $\|f_h\|_{1+s,\Omega} \leq Ch^{-s}\, \|f_h\|_{1,\Omega}$ holds true and from this and the estimate above we obtain

$$\|\psi\|_{1+s,\Omega} \leq Ch^{-s}\, \|f_h\|_{1,\Omega}\,.$$

On the other hand, from the last equation of (3.2.8) we have that

$$\left(\nabla\left(w - \kappa^{-1}t^2\psi\right), \nabla\xi\right)_{0,\Omega} = (\beta, \nabla\xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega).$$

Therefore, $(w - \kappa^{-1}t^2\psi)$ is the weak solution to the problem

$$\begin{cases} \Delta\left(w - \kappa^{-1}t^2\psi\right) = \mathrm{div}\,\beta \in \mathrm{L}^2(\Omega), \\ (w - \kappa^{-1}t^2\psi) = 0 \qquad \text{on } \partial\Omega. \end{cases}$$

Hence, $(w - \kappa^{-1}t^2\psi) \in \mathrm{H}^2(\Omega)$ (recall Ω is convex) and $w = (w - \kappa^{-1}t^2\psi) + \kappa^{-1}t^2\psi \in \mathrm{H}^{1+s}(\Omega)$ for all $s \in (0, \frac{1}{2})$. Thus the proof is complete. $\square$

The following lemma is the key point to prove Proposition 3.4.5.

Lemma 3.4.7 *If* (β, w, γ) *and* (β_h, w_h, γ_h) *as in* (3.4.9) *and* (3.4.10)*, respectively, then*

$$\|\beta - \beta_h\|_{1,\Omega} + t\, \|\gamma - \gamma_h\|_{0,\Omega} \leq C\,(h + t)\, \|f_h\|_{1,\Omega}\,.$$

Proof. It has been proved in [22] (see Example 4.1 from this reference) that there exists $\widetilde{\beta} \in H_h$ satisfying

$$R\widetilde{\beta} = R\beta,$$
$$\|\beta - \widetilde{\beta}\|_{1,\Omega} \leq Ch\, \|\beta\|_{2,\Omega}\,.$$

Let

$$\widetilde{\gamma} := \frac{\kappa}{t^2}(\nabla w_{\mathbf{I}} - R\widetilde{\beta}),$$

where $w_{\mathbf{I}} \in W_h$ is the Lagrange interpolant of w, which is well defined because of Lemma 3.4.6. Notice that by virtue of (3.4.3) and the equation above,

$$\widetilde{\gamma} = R\gamma.$$

It has also been proved in [22] that

$$\|\widetilde{\beta} - \beta_h\|_{1,\Omega} + t\,\|\widetilde{\gamma} - \gamma_h\|_{0,\Omega} \leq C\left(\|\widetilde{\beta} - \beta\|_{1,\Omega} + t\,\|\widetilde{\gamma} - \gamma\|_{0,\Omega} + h\,\|\gamma\|_{0,\Omega}\right).$$

Hence, by adding and subtracting $\widetilde{\beta}$ and $\widetilde{\gamma} = R\gamma$, we obtain

$$\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} \leq C\left(\|\beta - \widetilde{\beta}\|_{1,\Omega} + t\,\|\gamma - R\gamma\|_{0,\Omega} + h\,\|\gamma\|_{0,\Omega}\right).$$

The first and last term in the right-hand side above are already bounded. To estimate the second one, we use (3.4.11), Lemma 3.4.6, and (3.4.3), to obtain

$$\|\gamma - R\gamma\|_{0,\Omega} \leq \|\nabla\psi - \nabla\psi_{\mathbf{I}}\|_{0,\Omega} + \|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega}\,. \tag{3.4.14}$$

Next, from standard error estimates for the Lagrange interpolant, we have

$$\|\nabla\psi - \nabla\psi_{\mathbf{I}}\|_{0,\Omega} \leq Ch^s\,\|\psi\|_{1+s,\Omega}\,,$$

whereas from (3.4.2) and the fact that $p \in \mathrm{H}^2(\Omega)$ (cf. (3.4.12))

$$\|\operatorname{curl} p - R(\operatorname{curl} p)\|_{0,\Omega} \leq Ch\,\|p\|_{2,\Omega}\,.$$

Thus, by using Lemma 3.4.6, we conclude

$$\begin{aligned}\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} &\leq C\left(h\,\|\beta\|_{2,\Omega} + t\,\|f_h\|_{1,\Omega} + th\,\|p\|_{2,\Omega} + h\,\|\gamma\|_{0,\Omega}\right)\\ &\leq C\,(h+t)\,\|f_h\|_{1,\Omega}\,,\end{aligned}$$

where we have used (3.4.12) for the last inequality. The proof is complete. $\square$

We are now in a position to prove Proposition 3.4.5.

Proof of Proposition 3.4.5. Let (β, w, γ) and (β_h, w_h, γ_h) be as in (3.4.9) and (3.4.10), respectively. We need to prove that

$$\|w - w_h\|_{1,\Omega} \leq C\,(h+t)\,\|f_h\|_{1,\Omega}\,.$$

Since

$$\nabla w - \nabla w_h = \kappa^{-1} t^2 (\gamma - \gamma_h) + (\beta - R\beta_h),$$

adding and subtracting $R\beta$, we obtain

$$\|\nabla w - \nabla w_h\|_{0,\Omega} \leq \kappa^{-1} t^2 \|\gamma - \gamma_h\|_{0,\Omega} + \|\beta - R\beta\|_{0,\Omega} + \|R(\beta - \beta_h)\|_{0,\Omega}. \qquad (3.4.15)$$

Hence, using Poincaré inequality, (3.4.1), Lemma 3.4.7, (3.4.2), and (3.4.12), we have

$$\|w - w_h\|_{1,\Omega} \leq C (h + t) \|f_h\|_{1,\Omega}.$$

The proof is complete. □

We end this section by proving property P1.

Lemma 3.4.8 *Suppose that the family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform. Then we have*

$$\|T_0 - T_{th}\|_h \leq C (h + t).$$

Proof. The assertion follows immediately from estimate (3.4.7), by using (3.4.8) and Proposition 3.4.5. □

3.5 Convergence and error estimates

In this section we will adapt the arguments from [19] to prove error estimates for the approximate eigenvalues and eigenfunctions. Throughout this section, we will assume that the family of meshes $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform, so that property P1 holds true, although such assumption is not actually necessary in some particular cases (see the appendix below).

Our first goal is to prove that, provided the plate is sufficiently thin, the numerical method does not introduce spurious modes with eigenvalues interspersed among the relevant ones of T_t (namely, around $\mu_t^{(k)}$ for small k). Let us remark that such a spectral pollution could be in principle expected from the fact that T_t has a non-trivial essential spectrum. However, that this is not the case is an immediate consequence of the following theorem, which is essentially identical to Lemma 1 from [18].

Theorem 3.5.1 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. There exist strictly positive constants h_0, t_0, and C such that, $\forall h < h_0$ and $\forall t < t_0$, there holds $F \cap \mathrm{Sp}(T_{th}) = \emptyset$ and*

$$\|R_z(T_{th})\|_h \leq C \qquad \forall z \in F.$$

Proof. The same arguments used to prove Lemma 3.3.8 (but using Lemma 3.4.8 instead of Lemma 3.3.6) allow us to show an estimate analogous to (3.3.11), namely, for all $w_h \in W_h$ and all $z \in F$,

$$\|(zI - T_{th})\, w_h\|_{1,\Omega} \geq \|(zI - T_0)\, w_h\|_{1,\Omega} - \|(T_0 - T_{th})\, w_h\|_{1,\Omega} \geq \frac{1}{2C_1} \|w_h\|_{1,\Omega},$$

provided h and t are small enough. Since W_h is finite dimensional, the inequality above implies that $(zI - T_{th})\,|_{W_h}$ is invertible and, hence, $z \notin \mathrm{Sp}(T_{th}|_{W_h})$. Now, $\mathrm{Sp}(T_{th}) = \mathrm{Sp}(T_{th}|_{W_h}) \cup \{0\}$ (see, for instance, [9, Lemma 4.1]) and, for $z \in F$, $z \neq 0$. Thus, $z \notin \mathrm{Sp}(T_{th})$ either. Then $(zI - T_{th})$ is invertible too and

$$\|R_z(T_{th})\|_h = \|\,(zI - T_{th})^{-1}\,\|_h \leq 2C_1 \qquad \forall z \in F.$$

The proof is complete. □

We have already proved in Theorem 3.3.1 that the essential spectrum of T_t is confined to the real interval $(-\kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega})$. The spectrum of T_t outside this interval consists of finite multiplicity isolated eigenvalues of ascent one, which converge to eigenvalues of T_0, as t goes to zero (cf. Theorem 3.3.9). The eigenvalue of T_t with physical significance is the largest in modulus, $\mu_t^{(1)}$, which corresponds to the limit of elastic stability that leads to buckling effects. This eigenvalue is typically simple and converges to a simple eigenvalue of T_0, as t tends to zero. Because of this, for simplicity, from now on we restrict our analysis to simple eigenvalues.

Let $\mu_0 \neq 0$ be an eigenvalue of T_0 with multiplicity $m = 1$. Let D be a closed disk centered at μ_0 with boundary Γ such that $0 \notin D$ and $D \cap \mathrm{Sp}(T_0) = \{\mu_0\}$. Let $t_0 > 0$ be small enough, so that for all $t < t_0$:

- D contains only one eigenvalue μ_t of T_t, which we already know is simple (cf. Lemma 3.3.7) and

- D does not intersect the real interval $(-\kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega}, \kappa^{-1}t^2\|\boldsymbol{\sigma}\|_{\infty,\Omega})$, which contains the essential spectrum of T_t.

According to Theorem 3.5.1 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0$, $\Gamma \subset \rho(T_{th})$. Moreover, proceeding as in [18, Section 2], from properties P1 and P2 it follows that, for h small enough, T_{th} has exactly one eigenvalue $\mu_{th} \in D$. The theory in [19] could be adapted too, to prove error estimates for the eigenvalues and eigenfunctions of T_{th} to those of T_0 as h and t go to zero. However, our goal is not this one, but to prove that μ_{th} converges to μ_t as h goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [19].

Let $\Pi_h : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$ be the projector with range W_h defined for all $u \in \mathrm{H}_0^1(\Omega)$ by

$$\left(\nabla\left(\Pi_h u - u\right), \nabla v_h\right)_{0,\Omega} = 0 \qquad \forall v_h \in W_h.$$

The projector Π_h is bounded uniformly on h, namely, $\|\Pi_h u\|_{1,\Omega} \leq \|u\|_{1,\Omega}$, and the following error estimate is well known:

$$\|\Pi_h u - u\|_{1,\Omega} \leq Ch\,\|u\|_{2,\Omega} \qquad \forall u \in \mathrm{H}^2(\Omega). \tag{3.5.1}$$

Let us define

$$B_{th} := T_{th}\Pi_h : \mathrm{H}_0^1(\Omega) \to W_h \hookrightarrow \mathrm{H}_0^1(\Omega).$$

It is clear that T_{th} and B_{th} have the same nonzero eigenvalues and corresponding eigenfunctions. Furthermore, we have the following result (cf. [19, Lemma 1]).

Lemma 3.5.2 *There exist h_0, t_0, and C such that*

$$\|R_z(B_{th})\| \leq C \qquad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

Proof. Since B_{th} is compact it suffices to verify that $\|(zI - B_{th})\,u\|_{1,\Omega} \geq C\,\|u\|_{1,\Omega}$ for all $u \in \mathrm{H}_0^1(\Omega)$ and $z \in \Gamma$. Taking into account that $0 \notin \Gamma$ and using Theorem 3.5.1, we have

$$\|u\|_{1,\Omega} \leq \|\Pi_h u\|_{1,\Omega} + \|u - \Pi_h u\|_{1,\Omega} \leq C\,\|(zI - T_{th})\,\Pi_h u\|_{1,\Omega} + |z|^{-1}\,\|z\,(u - \Pi_h u)\|_{1,\Omega}\,.$$

By using properties of the projector Π_h, we obtain

$$\begin{aligned}\|u\|_{1,\Omega} &\leq C\,\|(zI - B_{th})\,\Pi_h u\|_{1,\Omega} + |z|^{-1}\,\|z\,(u - \Pi_h u) - B_{th}(u - \Pi_h u)\|_{1,\Omega}\\ &= C\,\|\Pi_h(zI - B_{th})u\|_{1,\Omega} + |z|^{-1}\,\|(I - \Pi_h)\,(zI - B_{th})\,u\|_{1,\Omega}\\ &\leq C\,\|(zI - B_{th})\,u\|_{1,\Omega}\,.\end{aligned}$$

Thus we end the proof. □

Next, we introduce:

- $E_t : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$, the spectral projector of T_t corresponding to the isolated eigenvalue μ_t, namely,
$$E_t := \frac{1}{2\pi i}\int_\Gamma R_z(T_t)\,dz;$$
- $F_{th} : \mathrm{H}_0^1(\Omega) \to \mathrm{H}_0^1(\Omega)$, the spectral projector of B_{th} corresponding to the eigenvalue μ_{th}, namely,
$$F_{th} := \frac{1}{2\pi i}\int_\Gamma R_z(B_{th})\,dz.$$

As a consequence of Lemma 3.5.2, the spectral projectors F_{th} are bounded uniformly in h and t, for h and t small enough. Notice that $E_t(\mathrm{H}_0^1(\Omega))$ is the eigenspace of T_t associated to μ_t and $F_{th}(\mathrm{H}_0^1(\Omega))$ the eigenspace of B_{th} (and hence of T_{th}, too) associated to μ_{th}. According to our assumptions, $E_t(\mathrm{H}_0^1(\Omega))$ and $F_{th}(\mathrm{H}_0^1(\Omega))$ are both one dimensional. The following estimate (cf. [19, Lemma 3]) will be used to prove convergence of the eigenspaces.

Lemma 3.5.3 *There exist positive constants h_0, t_1, and C, such that for all $h < h_0$ and for all $t < t_1$,*

$$\|\,(E_t - F_{th})\,|_{E_t(\mathrm{H}_0^1(\Omega))}\| \leq C\|\,(T_t - B_{th})\,|_{E_t(\mathrm{H}_0^1(\Omega))}\| \leq Ch.$$

Proof. The first inequality is proved using the same arguments of [19, Lemma 3] and Lemmas 3.3.8 and 3.5.2. For the other estimate, fix $w \in E_t(\mathrm{H}_0^1(\Omega))$. From Proposition 3.3.10, Remark 3.4.3, Lemma 3.4.4, and (3.5.1), we have

$$\begin{aligned}\|(T_t - B_{th})\,w\|_{1,\Omega} &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|(T_{th} - B_{th})w\|_{1,\Omega}\\ &\leq \|(T_t - T_{th})w\|_{1,\Omega} + \|T_{th}\|\,\|(I - \Pi_h)\,w\|_{1,\Omega}\\ &\leq Ch\,\|w\|_{2,\Omega}\,.\end{aligned}$$

Therefore, by using (3.3.13), we conclude the proof. □

To prove an error estimate for the eigenspaces, we also need the following result.

Lemma 3.5.4 *Let*

$$\Lambda_{th} := F_{th}|_{E_t(\mathrm{H}_0^1(\Omega))} : E_t(\mathrm{H}_0^1(\Omega)) \to F_{th}(\mathrm{H}_0^1(\Omega)).$$

For h and t small enough, the operator Λ_{th} is invertible and

$$\left\| \Lambda_{th}^{-1} \right\| \leq C,$$

with C independent of h and t.

Proof. See the proof of Theorem 1 in [19]. □

We recall the definition of the gap $\hat{\delta}$ between two closed subspaces Y and Z of $\mathrm{H}_0^1(\Omega)$:

$$\hat{\delta}(Y, Z) := \max \{\delta(Y, Z), \delta(Z, Y)\},$$

where

$$\delta(Y, Z) := \sup_{\substack{y \in Y \\ \|y\|_{1,\Omega}=1}} \left(\inf_{z \in Z} \|y - z\|_{1,\Omega} \right).$$

The following theorem shows that the eigenspace of T_{th} (which coincides with that of B_{th}) approximate the eigenspace of T_t with optimal order.

Theorem 3.5.5 *There exist constants h_0, t_1, and C, such that, for all $h < h_0$ and for all $t < t_1$, there holds*

$$\hat{\delta}\left(F_{th}(\mathrm{H}_0^1(\Omega)), E_t(\mathrm{H}_0^1(\Omega))\right) \leq Ch.$$

Proof. It follows by arguing exactly as in the proof of Theorem 1 from [19], and using Lemmas 3.5.3 and 3.5.4. □

Next, we prove a preliminary sub-optimal error estimate for $|\mu_t - \mu_{th}|$, which will be improved below (cf. Theorem 3.5.8).

Lemma 3.5.6 *There exists a positive constant C such that, for h and t small enough,*

$$|\mu_t - \mu_{th}| \leq Ch.$$

Proof. We define the following operators:

$$\widehat{T}_t := T_t|_{E_t(\mathrm{H}^1_0(\Omega))} : E_t(\mathrm{H}^1_0(\Omega)) \to E_t(\mathrm{H}^1_0(\Omega)),$$
$$\widehat{B}_{th} := \Lambda_{th}^{-1} B_{th} \Lambda_{th} : E_t(\mathrm{H}^1_0(\Omega)) \to E_t(\mathrm{H}^1_0(\Omega)).$$

The operator $\widehat{T}_t$ has a unique eigenvalue μ_t of multiplicity $m = 1$, while the unique eigenvalue of $\widehat{B}_{th}$ is μ_{th}.

Let $v \in E_t(\mathrm{H}^1_0(\Omega))$. Since $\left(\Lambda_{th}^{-1} F_{th} - I\right) T_t|_{E_t(\mathrm{H}^1_0(\Omega))} = 0$ and B_{th} commutes with its spectral projector F_{th}, we have

$$(\widehat{T}_t - \widehat{B}_{th})v = (T_t - B_{th})\, v + \left(\Lambda_{th}^{-1} F_{th} - I\right)(T_t - B_{th})\, v.$$

Therefore, using Lemmas 3.5.3 and 3.5.4 and the fact that $\|F_{th}\|$ is bounded uniformly in h and t, for h and t small enough, we obtain

$$\|(\widehat{T}_t - \widehat{B}_{th})v\|_{1,\Omega} \leq \|(T_t - B_{th})\, v\|_{1,\Omega} + \left\|\left(\Lambda_{th}^{-1} F_{th} - I\right)(T_t - B_{th})\, v\right\|_{1,\Omega} \leq Ch\, \|v\|_{1,\Omega}\,.$$

Hence, the lemma follows from the fact that $\widehat{T}_t = \mu_t I$ and $\widehat{B}_{th} = \mu_{th} I$. $\square$

Since the eigenvalue $\mu_t \neq 0$ of T_t corresponds to an eigenvalue $\lambda = 1/\mu_t$ of Problem 3.2.1, Lemma 3.5.6 leads to an error estimate for the approximation of λ as well. However, the order of convergence is $O(h)$ as in this lemma. We now aim at improving this result. Let $\lambda_h := 1/\mu_{th}$, w_h, β_h and γ_h be such that $(\lambda_h, w_h, \beta_h, \gamma_h)$ is a solution of Problem 3.4.1, with $\|w_h\|_{1,\Omega} = 1$. According to Theorem 3.5.5, there exists a solution $(\lambda, w, \beta, \gamma)$ to Problem 3.2.1, with $\|w\|_{1,\Omega} = 1$, such that

$$\|w - w_h\|_{1,\Omega} \leq Ch.$$

The following lemma will be used to prove a double order of convergence for the corresponding eigenvalues, but it is interesting by itself, too. In fact, it shows optimal order convergence for the rotations of the vibration modes.

Lemma 3.5.7 *Let (λ, w, β) be a solution of Problem 3.2.1, with $\|w\|_{1,\Omega} = 1$, and $(\lambda_h, w_h, \beta_h)$ a solution of Problem 3.4.1, with $\|w_h\|_{1,\Omega} = 1$, such that*

$$\|w - w_h\|_{1,\Omega} \leq Ch. \tag{3.5.2}$$

Let γ and γ_h be as defined in Problems 3.2.1 and 3.4.1, respectively. Then for h and t small enough there holds

$$\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} \leq Ch. \tag{3.5.3}$$

Proof. Let $\hat{w}_h \in W_h$, $\hat{\beta}_h \in H_h$ and $\hat{\gamma}_h$ be the solution of the auxiliary problem:

$$\begin{cases} a(\hat{\beta}_h, \eta_h) + (\hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = \lambda\,(\boldsymbol{\sigma}\nabla w, \nabla v_h)_{0,\Omega} & \forall(\eta_h, v_h) \in H_h \times W_h, \\ \hat{\gamma}_h = \dfrac{\kappa}{t^2}(\nabla \hat{w}_h - R\hat{\beta}_h). \end{cases}$$

This problem is the finite element discretization of Problem 3.2.1, with source term $f = \lambda w \in \mathrm{H}^2(\Omega) \cap \mathrm{H}^1_0(\Omega)$. Then, from Remark 3.4.3, (3.3.13), and the fact that $\|w_h\|_{1,\Omega} = 1$, we obtain the following error estimate:

$$\|\beta - \hat{\beta}_h\|_{1,\Omega} + t\,\|\gamma - \hat{\gamma}_h\|_{0,\Omega} + \|w - \hat{w}_h\|_{1,\Omega} \leq Ch\,|\lambda|\,\|w\|_{2,\Omega} \leq Ch\,|\lambda|\,. \tag{3.5.4}$$

On the other hand, from Problem 3.4.1, we have that $(\beta_h - \hat{\beta}_h, w_h - \hat{w}_h) \in H_h \times W_h$ satisfies

$$\begin{cases} a(\beta_h - \hat{\beta}_h, \eta_h) + (\gamma_h - \hat{\gamma}_h, \nabla v_h - R\eta_h)_{0,\Omega} = (\boldsymbol{\sigma}\nabla\left(\lambda_h w_h - \lambda w\right), \nabla v_h)_{0,\Omega} \\ \hfill \forall(\eta_h, v_h) \in H_h \times W_h, \\ \gamma_h - \hat{\gamma}_h = \dfrac{\kappa}{t^2}(\nabla\left(w_h - \hat{w}_h\right) - R(\beta_h - \hat{\beta}_h)). \end{cases}$$

Taking $\eta_h = \beta_h - \hat{\beta}_h$ and $v_h = w_h - \hat{w}_h$ in the system above, from the ellipticity of $a(\cdot,\cdot)$, we obtain

$$\begin{aligned} &\|\beta_h - \hat{\beta}_h\|^2_{1,\Omega} + \kappa^{-1}t^2\,\|\gamma_h - \hat{\gamma}_h\|^2_{0,\Omega} \\ &\qquad \leq C\,\|\lambda_h w_h - \lambda w\|_{1,\Omega}\,\|w_h - \hat{w}_h\|_{1,\Omega} \\ &\qquad \leq C\left(|\lambda_h|\,\|w - w_h\|_{1,\Omega} + |\lambda - \lambda_h|\,\|w\|_{1,\Omega}\right)\left(\|w - w_h\|_{1,\Omega} + \|w - \hat{w}_h\|_{1,\Omega}\right) \\ &\qquad \leq Ch^2, \end{aligned}$$

where we have used Lemma 3.5.6 and estimates (3.5.2) and (3.5.4) for the last inequality. Therefore, we have

$$\|\beta_h - \hat{\beta}_h\|_{1,\Omega} + t\,\|\gamma_h - \hat{\gamma}_h\|_{0,\Omega} \leq Ch.$$

Thus, the lemma follows from this estimate and (3.5.4). □

We are now in a position to prove an optimal double-order error estimate for the eigenvalues.

Theorem 3.5.8 *There exist positive constants h_0, t_1, and C such that, $\forall h < h_0$ and $\forall t < t_1$,*

$$|\lambda - \lambda_h| \leq Ch^2.$$

Proof. We adapt to our case a standard argument for eigenvalue problems (see [6, Lemma 9.1]). Let $(\lambda, \beta, w, \gamma)$ and $(\lambda_h, \beta_h, w_h, \gamma_h)$ be as in Lemma 3.5.7. We will use the bilinear forms A and B defined in (3.3.1) and (3.3.2), respectively, as well as the bilinear form A_h defined in $H_h \times W_h$ as follows:

$$A_h((\beta_h, w_h), (\eta_h, v_h)) := a(\beta_h, \eta_h) + \frac{\kappa}{t^2} (\nabla w_h - R\beta_h, \nabla v_h - R\eta_h)_{0,\Omega}.$$

With this notation, Problems 3.2.1 and 3.4.1 can be written as follows:

$$\begin{aligned} A((\beta, w), (\eta, v)) &= \lambda B((\beta, w), (\eta, v)), \\ A_h((\beta_h, w_h), (\eta_h, v_h)) &= \lambda_h B((\beta_h, w_h), (\eta_h, v_h)). \end{aligned}$$

From these equations, straightforward computations lead to

$$\begin{aligned} (\lambda_h - \lambda)\, B((\beta_h, w_h), (\beta_h, w_h)) = {} & A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ & - \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) \\ & + \left[A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h))\right]. \end{aligned} \tag{3.5.5}$$

Next, we define $\bar{\gamma}_h := \frac{\kappa}{t^2} (\nabla w_h - \beta_h)$. Recalling that $R\nabla w_h = \nabla w_h$ (cf. (3.4.3)), from the definition of γ_h (cf. Problem 3.4.1) we have that $\gamma_h = R\bar{\gamma}_h$. On the other hand, from the definition of A and A_h we write

$$\begin{aligned} A((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)) &= a(\beta - \beta_h, \beta - \beta_h) + \kappa^{-1} t^2 \|\gamma - \bar{\gamma}_h\|^2_{0,\Omega}, \\ A_h((\beta_h, w_h), (\beta_h, w_h)) - A((\beta_h, w_h), (\beta_h, w_h)) &= \kappa^{-1} t^2 \left(\|R\bar{\gamma}_h\|^2_{0,\Omega} - \|\bar{\gamma}_h\|^2_{0,\Omega} \right). \end{aligned}$$

Therefore,

$$
\begin{aligned}
(\lambda_h - \lambda)\, B((\beta_h, w_h), (\beta_h, w_h)) = a(\beta - \beta_h, \beta - \beta_h)& \\
+ \kappa^{-1} t^2 \left(\|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 \right)& \\
- \lambda B((\beta - \beta_h, w - w_h), (\beta - \beta_h, w - w_h)).&
\end{aligned}
$$

The first and the third term in the right-hand side above are easily bounded by virtue of (3.5.2) and (3.5.3). For the second term, we write

$$
\begin{aligned}
\|\gamma - \bar{\gamma}_h\|_{0,\Omega}^2 + \|R\bar{\gamma}_h\|_{0,\Omega}^2 - \|\bar{\gamma}_h\|_{0,\Omega}^2 &= \|\gamma - R\bar{\gamma}_h\|_{0,\Omega}^2 - 2\,(\gamma, \bar{\gamma}_h - R\bar{\gamma}_h)_{0,\Omega} \\
&= \|\gamma - \gamma_h\|_{0,\Omega}^2 + \frac{2\kappa}{t^2}\,(\gamma, \beta_h - R\beta_h)_{0,\Omega}\,.
\end{aligned}
\tag{3.5.6}
$$

For $\beta \in \mathrm{H}^2(\Omega)^2 \cap \mathrm{H}_0^1(\Omega)$, we denote by $\beta^{\mathbf{I}} \in \mathrm{H}_h$ the standard Clément interpolant of β, which satisfies

$$
\left\|\beta^{\mathbf{I}}\right\|_{1,\Omega} \le C\,\|\beta\|_{1,\Omega} \qquad \text{and} \qquad \left\|\beta - \beta^{\mathbf{I}}\right\|_{1,\Omega} \le Ch\,\|\beta\|_{2,\Omega}\,. \tag{3.5.7}
$$

It follows that

$$
\begin{aligned}
(\gamma, \beta_h - R\beta_h)_{0,\Omega} &= \left(\gamma, \left(\beta_h - \beta^{\mathbf{I}}\right) - R(\beta_h - \beta^{\mathbf{I}})\right)_{0,\Omega} + \left(\gamma, \beta^{\mathbf{I}} - R\beta^{\mathbf{I}}\right)_{0,\Omega} \\
&\le \|\gamma\|_{0,\Omega} \left\| \left(\beta_h - \beta^{\mathbf{I}}\right) - R(\beta_h - \beta^{\mathbf{I}}) \right\|_{0,\Omega} + \left(\gamma, \beta^{\mathbf{I}} - R\beta^{\mathbf{I}}\right)_{0,\Omega}\,.
\end{aligned}
$$

Thus, using (3.4.2) and Lemma 3.3 from [21], we obtain

$$
\begin{aligned}
(\gamma, \beta_h - R\beta_h)_{0,\Omega} &\le Ch\,\|\gamma\|_{0,\Omega} \left\|\beta_h - \beta^{\mathbf{I}}\right\|_{1,\Omega} + Ch^2\,\|\operatorname{div}\gamma\|_{0,\Omega}\,\|\beta\|_{1,\Omega} \\
&\le Ch\,\|\gamma\|_{0,\Omega} \left(\|\beta - \beta_h\|_{1,\Omega} + \left\|\beta - \beta^{\mathbf{I}}\right\|_{1,\Omega} \right) + Ch^2\,\|\operatorname{div}\gamma\|_{0,\Omega}\,\|\beta\|_{1,\Omega}\,,
\end{aligned}
$$

and from Lemma 3.5.7, (3.5.7), and Proposition 3.3.10, we have

$$
(\gamma, \beta_h - R\beta_h)_{0,\Omega} \le Ch\,\|\gamma\|_{0,\Omega} \left(Ch + Ch\,\|\beta\|_{2,\Omega} \right) + Ch^2\,|\lambda|\,\|w\|_{2,\Omega}\,\|\beta\|_{1,\Omega} \le Ch^2\,|\lambda|\,.
$$

Finally, we use this estimate, (3.5.5), (3.5.6), and the fact that $B((\beta_h, w_h), (\beta_h, w_h)) = (\boldsymbol{\sigma}\nabla w_h, \nabla w_h)_{0,\Omega} \ne 0$ (cf. Remark 3.4.2) to obtain

$$
|\lambda - \lambda_h| \le C\, \frac{\|\beta - \beta_h\|_{1,\Omega}^2 + \|w - w_h\|_{1,\Omega}^2 + \kappa^{-1} t^2\,\|\gamma - \gamma_h\|_{0,\Omega}^2 + Ch^2\,|\lambda|}{|B((\beta_h, w_h), (\beta_h, w_h))|}.
$$

Consequently, from Lemma 3.5.7,

$$|\lambda - \lambda_h| \leq Ch^2$$

and we conclude the proof. $\square$

3.6 Numerical results

In this section we report some numerical experiments carried out with our method applied to Problem 3.2.1. We recall that the buckling coefficients can be directly computed from the eigenvalues of Problem 3.2.1: $\lambda_{\mathrm{b}} = \lambda t^2$.

For all the computations we have taken $\Omega := (0, 6)\times(0, 4)$ (all the lengths are measured in meters) and typical parameters of steel: the Young modulus has been chosen $E = 1.44 \times 10^{11}$ Pa and the Poisson ratio $\nu = 0.30$. The shear correction factor has been taken $k = 5/6$.

We have used uniform meshes as those shown in Fig. 3.1; the meaning of the refinement parameter N can be easily deduced from this figure. Notice that $h \sim N^{-1}$.

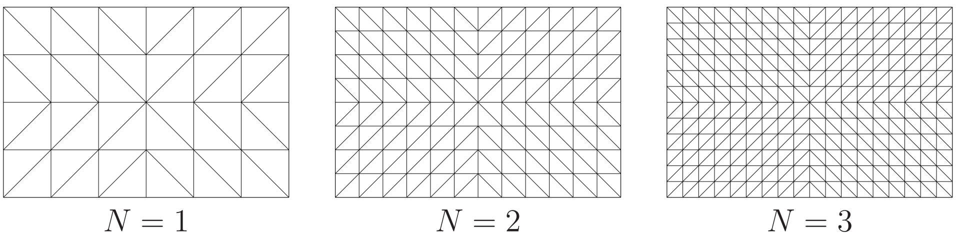


Figure 3.1: Rectangular plate. Uniform meshes.

3.6.1 Uniformly compressed rectangular plate

For this test we have used $\boldsymbol{\sigma} = \boldsymbol{I}$, which corresponds to a uniformly compressed plate.

Simply supported plate

First, we have considered a simply supported plate, because analytical solutions are available in this case (see [38, 40]). Even though our theoretical analysis has been developed only for clamped plates, we think that the results of Sections 3.4 and 3.5 should hold true for more general boundary conditions, as well. The results that follow give some numerical evidence of this.

In Table 3.1 we report the four lowest eigenvalues (λ_i, $i = 1, 2, 3, 4$) computed by our method with four different meshes ($N = 2, 4, 8, 16$) for a a simply supported plate with thickness $t = 0.001$. The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. The last column shows the exact eigenvalues.

Table 3.1: Lowest eigenvalues λ_i (multiplied by 10^{-10}) of a uniformly compressed simply supported plate with thickness $t = 0.001$.

Eigenvalue	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated	Exact
λ_1	1.1793	1.1759	1.1752	1.1750	2.14	1.1750	1.1749
λ_2	2.2638	2.2602	2.2596	2.2595	2.68	2.2595	2.2595
λ_3	3.7293	3.6441	3.6224	3.6170	1.98	3.6151	3.6152
λ_4	4.1573	4.0892	4.0726	4.0685	2.03	4.0672	4.0671

It can be seen from Table 3.1 that the method converges to the exact values with an optimal quadratic order.

Figure 3.2 shows the transverse displacements for the principal buckling mode computed with the finest mesh ($N = 16$).

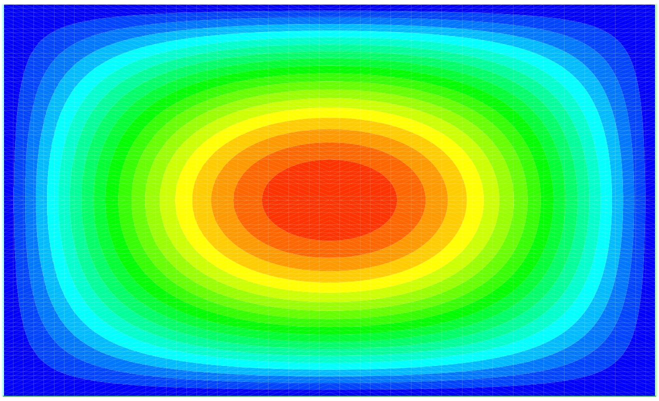


Figure 3.2: Uniformly compressed simply supported plate; principal buckling mode.

Clamped plate

In Table 3.2 we present the results for the lowest eigenvalue of a uniformly compressed clamped rectangular plate, with varying thickness. We have used the same meshes as in the previous test. Again, we have computed the orders of convergence, and more accurate values obtained by a least-squares fitting. In the last row we report for each mesh the limit values as t goes to zero obtained by extrapolation.

Table 3.2: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of uniformly compressed clamped plates with varying thickness.

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	3.4031	3.3440	3.3293	3.3258	2.02	3.3246
$t = 0.01$	3.4324	3.3723	3.3571	3.3533	1.99	3.3520
$t = 0.001$	3.4327	3.3726	3.3574	3.3536	1.99	3.3522
$t = 0.0001$	3.4327	3.3726	3.3574	3.3536	1.98	3.3522
$t = 0$ (extrap.)	3.4327	3.3726	3.3574	3.3536	1.99	3.3523

Figure 3.3 shows the transverse displacements for the principal buckling mode, for

$t = 0.1$, and the finest mesh ($N = 16$).

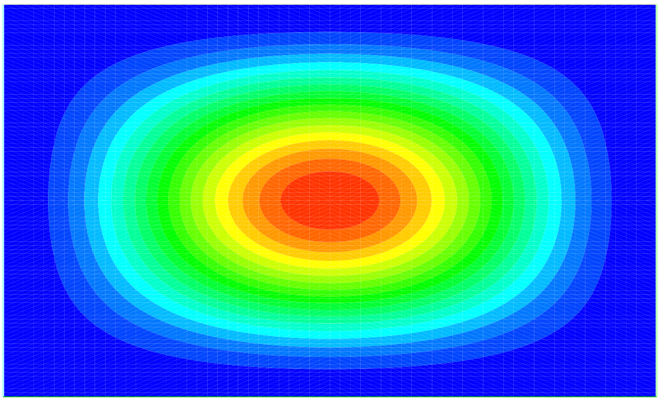


Figure 3.3: Uniformly compressed clamped plate; principal buckling mode.

According to Lemma 3.3.7, the values on the last row of Table 3.2 should correspond to the lowest eigenvalues of a Kirchhoff-Love uniformly compressed clamped plate with thickness $t = 1$. As a further test, we have also computed the latter, by using the methods analyzed in [16] and [36]. We show the obtained results in Table 3.3, where an excellent agreement with the last row of Table 3.2 can be appreciated.

Table 3.3: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a uniformly compressed clamped thin plate (Kirchhoff-Love model) computed with the methods from [16] and [36].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[16]	3.3718	3.3611	3.3573	3.3555	1.97	3.3523
[36]	3.3514	3.3519	3.3521	3.3522	1.95	3.3523

It is clear that the results from the Reissner-Mindlin model do not deteriorate as the plate thickness become smaller, which confirms that our method is *locking-free*.

3.6.2 Clamped plate uniformly compressed in one direction

We have used for this test

$$\boldsymbol{\sigma} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

which corresponds to a plate uniformly compressed in one direction. Notice that in this test $\boldsymbol{\sigma}$ is only positive semi-definite. Table 3.4 shows the same quantities as Table 3.2 in this case.

Table 3.4: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of clamped plates with varying thickness, uniformly compressed in one direction.

Thickness	$N = 2$	$N = 4$	$N = 8$	$N = 16$	Order	Extrapolated
$t = 0.1$	6.7969	6.7274	6.7104	6.7066	2.05	6.7052
$t = 0.01$	6.8825	6.8143	6.7971	6.7930	2.00	6.7915
$t = 0.001$	6.8834	6.8151	6.7980	6.7939	2.00	6.7924
$t = 0.0001$	6.8834	6.8152	6.7980	6.7939	2.00	6.7924
$t = 0$ (extrap.)	6.8834	6.8152	6.7980	6.7939	2.00	6.7924

Figure 3.4 shows the principal buckling mode for $t = 0.1$ and the finest mesh ($N = 16$).

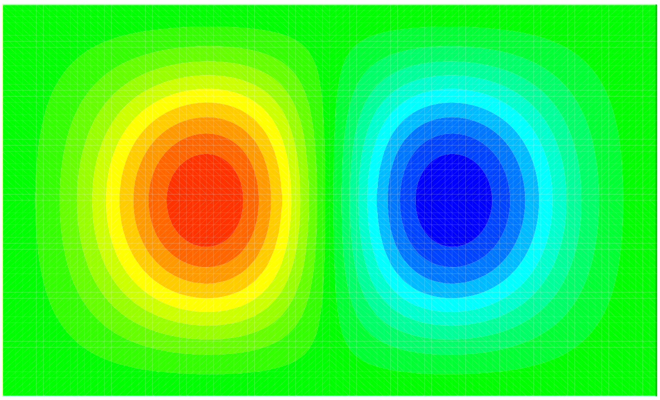


Figure 3.4: Clamped plate uniformly compressed in one direction; principal buckling mode.

Finally, Table 3.5 shows the same quantities as Table 3.3 in this case. Once more, an excellent agreement with the values extrapolated from the Reissner-Mindlin model (last row of Table 3.4) can be clearly appreciated.

Table 3.5: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a clamped thin plate (Kirchhoff-Love model) uniformly compressed in one direction, computed with the methods from [16] and [36].

Method	$N=8$	$N=12$	$N=16$	$N=20$	Order	Extrapolated
[16]	6.8450	6.8158	6.8056	6.8009	2.00	6.7925
[36]	6.7904	6.7913	6.7917	6.7920	1.92	6.7926

3.6.3 Shear loaded clamped plate

In this case we have used

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

which corresponds to a uniform shear load. Notice that $\boldsymbol{\sigma}$ is indefinite in this test. The numerical results are reported in Table 3.6, Figure 3.5, and Table 3.7, using the same pattern as the previous tests.

Table 3.6: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of shear loaded clamped plates with varying thickness.

Thickness	$N=4$	$N=8$	$N=12$	$N=16$	Order	Extrapolated
$t=0.1$	9.4306	9.2179	9.1783	9.1645	1.99	9.1464
$t=0.01$	9.6098	9.3923	9.3514	9.3371	1.98	9.3184
$t=0.001$	9.6116	9.3942	9.3533	9.3389	1.98	9.3202
$t=0.0001$	9.6117	9.3942	9.3533	9.3389	1.98	9.3202
$t=0$ (extrap.)	9.6117	9.3942	9.3533	9.3389	1.98	9.3202

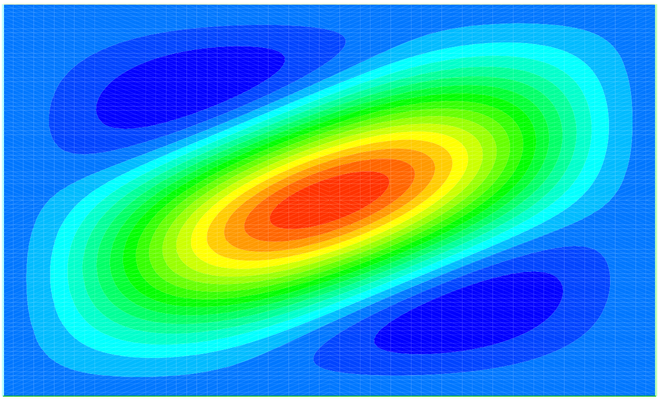


Figure 3.5: Shear loaded clamped plate; principal buckling mode.

Table 3.7: Lowest eigenvalue λ_1 (multiplied by 10^{-10}) of a shear loaded clamped thin plate (Kirchhoff-Love model) computed with the methods from [16] and [36].

Method	$N = 8$	$N = 12$	$N = 16$	$N = 20$	Order	Extrapolated
[16]	9.4625	9.3840	9.3563	9.3435	1.98	9.3203
[36]	9.3660	9.3408	9.3319	9.3278	1.99	9.3204

In all cases, an excellent agreement between the numerical experiments and the theoretical results detailed in Section 3.5 can be noticed and the method appears thoroughly locking-free.

3.7 Appendix. Uniformly compressed plates

The aim of this appendix is to show that the results of Sections 3.3, 3.4, and 3.5 can be refined when $\boldsymbol{\sigma} = \boldsymbol{I}$, which corresponds to a uniformly compressed plate. In this case, we are able to give a better characterization of the spectrum of T_t and to prove the spectral approximation without assuming that the family of meshes is *quasi-uniform*.

3.7.1 Spectral characterization

We have the following counterpart of Theorem 3.3.1, showing that the spectrum of T_t is simply a shift of the spectrum of a compact operator.

Theorem 3.7.1 *Suppose that $\boldsymbol{\sigma} = \boldsymbol{I}$. For all $t > 0$, the spectrum of T_t satisfies*

$$\mathrm{Sp}(T_t) = \mathrm{Sp}(G) + \kappa^{-1}t^2,$$

where G is the compact operator defined in (3.3.5).

Proof. The first equation of (3.2.8) leads in this case to $\psi = f$, due to the fact that $\boldsymbol{\sigma} = \boldsymbol{I}$. Therefore, (3.2.8) reduces to

$$\begin{cases} a(\beta,\eta) - (\operatorname{curl} p, \eta)_{0,\Omega} = (\nabla f, \eta)_{0,\Omega} & \forall \eta \in \mathrm{H}_0^1(\Omega)^2, \\ -(\beta, \operatorname{curl} q)_{0,\Omega} - \kappa^{-1}t^2 (\operatorname{curl} p, \operatorname{curl} q)_{0,\Omega} = 0 & \forall q \in \mathrm{H}^1(\Omega)/\mathbb{R}, \\ (\nabla w, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} + \kappa^{-1}t^2 (\nabla f, \nabla \xi)_{0,\Omega} & \forall \xi \in \mathrm{H}_0^1(\Omega). \end{cases} \tag{3.7.1}$$

Next, recall that G is defined in (3.3.5) as the operator mapping $f \mapsto u$, with $u \in \mathrm{H}_0^1(\Omega)$ such that

$$(\nabla u, \nabla \xi)_{0,\Omega} = (\beta, \nabla \xi)_{0,\Omega} \qquad \forall \xi \in \mathrm{H}_0^1(\Omega),$$

where $\beta \in \mathrm{H}_0^1(\Omega)^2$ is determined in this case by the first two equations from (3.7.1). Therefore, the third equation from (3.7.1) yields $T_t = G + \kappa^{-1}t^2 I$. Since G has been already shown to be compact, this allows us to conclude the theorem. □

As a consequence of this theorem, $\mathrm{Sp}(T_t) = \{\kappa^{-1}t^2\} \cup \{\mu_n : \ n \in \mathbb{N}\}$, with μ_n being a sequence of finite-multiplicity eigenvalues converging to $\kappa^{-1}t^2$. Therefore, in this particular case, the *essential spectrum* of T_t reduces to a unique point: $\kappa^{-1}t^2$.

3.7.2 Spectral approximation

In this particular case, we will improve the error estimate shown in Section 3.4 in that we will not need to assume quasi-uniformity of the meshes. Indeed, this property was used above only to prove Proposition 3.4.5. Instead, we have now the following result.

Proposition 3.7.2 *Suppose that $\boldsymbol{\sigma} = \boldsymbol{I}$. Then, for any regular family of triangular meshes $\{\mathcal{T}_h\}_{h>0}$, there exists $C > 0$ such that, for all $t > 0$,*

$$\|T_t - T_{th}\|_h \leq Ch.$$

Proof. We will simply sketch the proof, since it follows exactly the same steps as that of Proposition 3.4.5. First, we notice that in the decomposition (3.4.11) we have $\psi = f_h \in W_h$ (cf. problem (3.4.13) with $\boldsymbol{\sigma} = \boldsymbol{I}$).

As a consequence, we infer that the term $\|\nabla\psi - \nabla\psi_{\mathbf{I}}\|_{0,\Omega}$ in (3.4.14) vanishes. Hence, the last estimate in the proof of Lemma 3.4.7 changes into

$$\|\beta - \beta_h\|_{1,\Omega} + t\,\|\gamma - \gamma_h\|_{0,\Omega} \leq C\left(h\,\|\beta\|_{2,\Omega} + th\,\|p\|_{2,\Omega} + h\,\|\gamma\|_{0,\Omega}\right) \leq Ch\,\|f_h\|_{1,\Omega}\,.$$

By using the above estimate in the proof of Proposition 3.4.5 (in particular in (3.4.15)), we obtain

$$\|(T_t - T_{th})\,f_h\|_{1,\Omega} = \|w - w_h\|_{1,\Omega} \leq Ch\,\|f_h\|_{1,\Omega}\,,$$

from which we conclude the proof. □

As a consequence of Proposition 3.7.2, we can improve Lemma 3.4.8. In fact, now for t small enough there holds directly

$$\|T_t - T_{th}\|_h \leq Ch,$$

with a constant C independent of h and t. By using this instead of property P1, we could give somewhat simpler proofs for the error estimates from Section 3.5. However the final results, Theorems 3.5.1, 3.5.5, and 3.5.8 are the same, although now valid for any regular family of triangular meshes, without the need of being quasi-uniform.

Chapter 4

A locking-free finite element method for the buckling problem of a non-homogeneous Timoshenko beam

4.1 Introduction

This paper deals with the numerical approximation of the buckling problem of a non-homogeneous beam modeled by Timoshenko equations. Structural components with continuous and discontinuous variations of the geometry and the physical parameters are common in buildings and bridges as well as in aircrafts, cars, ships, etc. For that reason, it is important to know the limit of elastic stability of this kind of structures.

On the other hand, it is very well known that standard finite element methods applied to models of thin structures, like beams, rods and plates, are subject to the so-called locking phenomenon. This means that they produce very unsatisfactory results when the thickness is small with respect to the other dimensions of the structure. To avoid locking, the techniques most used are based on reduced integration or mixed formulations (see [23] and references therein).

In this paper, we present a rigorous thorough analysis of a low order finite element method to compute the buckling coefficients and modes of a non-homogeneous Timoshenko beam, the method was introduced for source problems on homogeneous beams by Arnold in [4], and was recently analized for the vibration problem of a rod in [28] (which covers the vibration problem of the Timoshenko beam).

The main drawback that appears in the formulation of the problem is the fact that the solution operator (whose eigenvalues are the reciprocals of the buckling coefficients) is non-compact. Among other consequences, we have that this operator has a non-trivial essential spectrum, which is a potencial source of spectral pollution in the numerical methods. Thus, our first task will be to prove that the eigenvalues corresponding to the limit of elastic stability (i.e., the smallest buckling coefficients) can be isolated from the essential spectrum of the solution operator, at least for sufficiently thin beams. Let us mention that similar arguments were used in [33] for Reissner-Mindlin plates.

To study the convergence of the proposed method and obtain error estimates, we will adapt the classical theory developed for non-compact operators in [18, 19]. We will obtain optimal order error estimates for the approximation of the buckling modes and a double order for the buckling coefficients, all these estimates being uniform in the beam thickness.

This approach follows the strategy used in [33] for buckling problem of plates. However, the one-dimensional character of the present problem allows us to give simpler proofs valid on a more general context. In particular, the results of this paper are valid for non-homogeneous beams, whose physical and geometrical properties may be even discontinuous at a finite number of points.

The outline of this paper is as follows: In Section 4.2, we introduce the buckling problem and a non-compact linear operator whose spectrum is related with the solution of the buckling problem. We end this section with some preliminary regularity results. In Section 4.3 we provide a thorough spectral characterization of this operator; its eigenvalues and eigenfunctions are proved to converge to the corresponding ones of the limit problem (an Euler-Bernoulli beam) as the thickness goes to zero. Additional regularity results are also proved. In Section 4.4 we introduce a finite element discretization with piecewise polinomials of low order. In Section 4.5 optimal order of convergence for the eigenfunctions and a double order for the eigenvalues are proved; all these error estimates are proved to be independent of the thickness of the beam, which allows us to conclude that the method is locking-free. Finally, in Section 4.6, we report some numerical tests which confirm the theoretical order of the error and allow us to assess the performance of the proposed method.

Throughout the paper we will use standard notations for Sobolev spaces, norms and seminorms. Moreover, we will denote with c and C, with or without subscripts, tildes or hats, a generic constant independent of the mesh parameter h and the plate thickness t, which may take different values in different occurrences.

4.2 Timoshenko beam model.

Let us consider an elastic beam which satisfies the Timoshenko hypotheses for the admissible displacements. The deformation of the beam is described in terms of the vertical displacement w and the rotation of the vertical fibers β. Let x be the coordinate in the axial direction. Moreover, we assume that the geometry and the physical parameters of the beam may change along the axial direction.

The buckling problem for a clamped Timoshenko beam loaded by a constant compressive (positive) load P, reads as follows:

Find $\lambda_{\mathbf{c}} \in \mathbb{R}$ *and* $0 \neq (\beta(x), w(x)) \in V := H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} E(x)\mathbb{I}(x)\beta'(x)\eta'(x)\,dx + \int_{\mathrm{I}} G(x)A(x)k_c(\beta(x) - w'(x))(\eta(x) - v'(x))\,dx = \lambda_{\mathbf{c}} \int_{\mathrm{I}} Pw'(x)v'(x)\,dx \tag{4.2.1}$$

for all $(\eta(x), v(x)) \in V$, where $\mathrm{I} := (0, L)$, L being the length of the beam, $E(x)$ the Young modulus, $\mathbb{I}(x)$ the moment of inertia of the cross-section, $A(x)$ the area of the cross-section and $G(x) := E(x)/(2(1 + \nu(x)))$ the shear modulus, with $\nu(x)$ the Poisson ratio, and k_c a correction factor. We consider that $E(x)$, $\mathbb{I}(x)$, $A(x)$ and $\nu(x)$ are piecewice smooth in I, the most usual case being when all those coefficients are piecewise constant. Moreover, primes denote derivative with respect to the x-coordinate.

The eigenvalues of the problem above are called the *buckling coefficients* and the eigenfunctions the *buckling modes*. We recall that the limit of elastic stability correspond to the smallest buckling coefficient $\lambda_{\mathbf{c}}$.

Remark 4.2.1 *The buckling problem above can be formally obtained from the three-dimensional linear elasticity equations as follows (see [17, 45]): The first step is to consider the beam as a three-dimensional structure. Then, the beam is supposed inextensible and only deformation in the plane* (x, z) *is allowed. According to the Timoshenko hypotheses, the admissible displacements at each point of the beam are of the form* $\mathbf{u}(x, y, z) = (z\beta(x), 0, w(x))$*. Test and trial displacements of this form are taken in the variational formulation of the buckling problem for the three-dimensional structure. By integrating over the cross-sections, multiplying the shear term by a correcting factor* k_c *and eliminating a higher order shear term in the right hand side, one arrives at problem* (4.2.1) *(see [39] for the same problem for a homogeneous beam).*

For very thin structures, it is well known that standard finite element procedures, when used in formulations such as (4.2.1), are subject to numerical locking, a phenomenon induced by the difference of magnitude between the coefficients in front of the different terms (see [4]). The appropriate framework for analysing this difficulty is obtained by rescaling formulation (4.2.1) so as to identify a well-posed sequence of problems in the limit as the thickness becomes infinitely small. With this aim, we introduce the following nondimensional parameter, characteristic of the thickness of the beam,

$$t^2 := \frac{1}{L}\int_{\mathrm{I}} \frac{\mathbb{I}(x)}{A(x)L^2}\,dx, \tag{4.2.2}$$

which we assume may take values in the range $(0, t_{\max}]$.

We define

$$\lambda := \frac{\lambda_{\mathbf{c}}}{t^3}, \quad \hat{\mathbb{I}}(x) := \frac{\mathbb{I}(x)}{t^3}, \quad \hat{A}(x) := \frac{A(x)}{t}, \quad \mathbb{E}(x) := E(x)\hat{\mathbb{I}}(x) \quad \text{and} \quad \kappa(x) := G(x)\hat{A}(x)k_c,$$

and assume that there exist $\underline{\mathbb{E}}, \overline{\mathbb{E}}, \underline{\kappa}, \overline{\kappa} \in \mathbb{R}^+$ such that

$$\begin{aligned} \overline{\mathbb{E}} \geq \mathbb{E}(x) \geq \underline{\mathbb{E}} > 0 \quad \forall x \in \mathrm{I}, \\ \overline{\kappa} \geq \kappa(x) \geq \underline{\kappa} > 0 \quad \forall x \in \mathrm{I}. \end{aligned} \tag{4.2.3}$$

Furthermore, because of the assumption on the physical and geometrical parameters, we have that $\mathbb{E}(x)$ and $\kappa(x)$ are piecewise smooth. More precisely, there exists a partition $0 = s_0 < \cdots < s_n = L$, of the interval I, where s_i, $i = 1, \ldots, n-1$ are the points of possible discontinuities of $\mathbb{E}(x)$ and $\kappa(x)$. If we denote $S_i := (s_i - s_{i-1})$, then, we assume that $\mathbb{E}_i(x) := \mathbb{E}(x)|_{S_i} \in W^{1,\infty}(S_i)$ and $\kappa_i(x) := \kappa(x)|_{S_i} \in W^{1,\infty}(S_i)$, $i = 1, \ldots, n$.

Then, problem (4.2.1) can be equivalently written as follows, where from now on we omit the dependence on the axial variable x:

Find $\lambda \in \mathbb{R}$ and $0 \neq (\beta, w) \in V$ such that

$$\int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \frac{1}{t^2}\int_{\mathrm{I}} \kappa(\beta - w')(\eta - v')\,dx = \lambda \int_{\mathrm{I}} P w' v'\,dx \quad \forall(\eta, v) \in V. \tag{4.2.4}$$

Note that all the eigenvalues of (4.2.4) are strictly positive, because of the symmetry and positiveness of the bilinear forms.

Finally, introducing the scaled shear stress $\gamma := \dfrac{\kappa}{t^2}(\beta - w')$, problem (4.2.4) can be written as follows:

Problem 4.2.1 Find $\lambda \in \mathbb{R}^+$ and $0 \neq (\beta, w) \in V$ such that

$$\left\{\begin{array}{l} \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} \gamma(\eta - v')\,dx = \lambda \int_{\mathrm{I}} P w' v'\,dx \quad \forall(\eta, v) \in V, \\ \gamma = \dfrac{\kappa}{t^2}(\beta - w'). \end{array}\right. \tag{4.2.5}$$

The goal of this paper is to propose and analyse a finite element method to solve Problem 4.2.1. In particular, the aim is to obtain accurate approximations of the smallest eigenvalues λ (which correspond to the buckling coefficients $\lambda_{\mathbf{c}} = \lambda t^3$) and the corresponding eigenfuctions or buckling modes.

In the rest of the section, we will introduce an operator whose spectrum will be related with that of Problem 4.2.1 and will prove some regularity results which will be used in the sequel. With this aim, first, we consider the following source problem associated with the spectral Problem 4.2.1:

Given $f \in H_0^1(\mathrm{I})$, find $(\beta, w) \in V$ such that

$$\left\{\begin{array}{l} \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} \gamma(\eta - v')\,dx = \int_{\mathrm{I}} P f' v'\,dx \quad \forall(\eta, v) \in V, \\ \gamma = \dfrac{\kappa}{t^2}(\beta - w'), \end{array}\right. \tag{4.2.6}$$

and introduce the following bounded linear operator called the *solution operator*:

$$T_t : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I}),$$
$$f \mapsto w,$$

where (β, w) is the unique solution of problem (4.2.6).

It is easy to check that (μ, w), with $\mu \neq 0$ is an eigenpair of T_t (i.e., $T_t w = \mu w$, $w \neq 0$) if and only if there exists $\beta \in H_0^1(\mathrm{I})$ such that (λ, β, w) with $\lambda = \frac{1}{\mu}$ being a solution of Problem 4.2.1. We recall that these eigenvalues are strictly positive. Let us recall that our aim is to approximate the smallest eigenvalues of Problem 4.2.1, which correspond to the largest eigenvalues of the operator T_t.

We note that T_t is self-adjoint with respect to the inner product $\int_{\mathrm{I}} Pu'v'\,dx$ in $H_0^1(\mathrm{I})$. In fact, for f, $g \in H_0^1(\mathrm{I})$, let (w, β) and (v, η) be the solutions of (4.2.6) with source terms f and g, respectively. Therefore, $w = T_t f$ and $v = T_t g$ and

$$\begin{aligned}\int_{\mathrm{I}} Pf'(T_t g)'\,dx &= \int_{\mathrm{I}} Pf'v'\,dx \\ &= \int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} \frac{\kappa}{t^2}(\beta - w')(\eta - v')\,dx = \int_{\mathrm{I}} Pg'w'\,dx = \int_{\mathrm{I}} Pg'(T_t f)'\,dx.\end{aligned}$$

Now, considering the following decomposition for the shear stress:

$$\gamma = \psi' + k, \tag{4.2.7}$$

with $\psi \in H_0^1(\mathrm{I})$ and $k := \frac{1}{L}\int_{\mathrm{I}} \gamma \in \mathbb{R}$. Replacing (4.2.7) in the first equation of (4.2.6) and testing with $(\eta, v) = (0, \psi + Pf) \in V$, we obtain

$$\psi = -Pf. \tag{4.2.8}$$

Thus, we have that problem (4.2.6) and the following problem are equivalent:

Given $f \in H_0^1(\mathrm{I})$, find $(\beta, k, w) \in H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ such that

$$\left\{\begin{aligned} &\int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} k\eta\,dx = \int_{\mathrm{I}} Pf'\eta\,dx \quad \forall \eta \in H_0^1(\mathrm{I}), \\ &\int_{\mathrm{I}} \beta q\,dx - t^2 \int_{\mathrm{I}} \frac{kq}{\kappa}\,dx = -t^2 \int_{\mathrm{I}} \frac{Pf'q}{\kappa}\,dx \quad \forall q \in \mathbb{R}, \\ &\int_{\mathrm{I}} \kappa w'\xi'\,dx = \int_{\mathrm{I}} \kappa\beta\xi'\,dx + t^2 \int_{\mathrm{I}} Pf'\xi'\,dx \quad \forall \xi \in H_0^1(\mathrm{I}). \end{aligned}\right. \tag{4.2.9}$$

For this problem, we have the following stability result:

Theorem 4.2.2 *For any $t \in [0, t_{max}]$ and $f \in H_0^1(\mathrm{I})$, there exists a unique triple $(\beta, k, w) \in H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ solving (4.2.9). Moreover, there exists a constant C independent of t and f, such that*

$$\|\beta\|_{1,\mathrm{I}} + |k| + \|w\|_{1,\mathrm{I}} \leq C\|f\|_{1,\mathrm{I}}.$$

Proof. For all $t \in (0, t_{\max}]$ we can apply Theorem 5.1 of [4] to obtain that there exists a unique solution $(\beta, k) \in H_0^1(\mathrm{I}) \times \mathbb{R}$ of problem $(4.2.9)_{1-2}$, moreover,

$$\|\beta\|_{1,\mathrm{I}} + |k| \leq C\|f'\|_{0,\mathrm{I}},$$

where the constant C is independent of t. If $t = 0$ the clasical theory for mixed formulations considered in [11] can be applied to obtain the same result.

Finally, we obtain by the Lax-Milgram's lemma, that there exists a unique solution $w \in H_0^1(\mathrm{I})$ of problem $(4.2.9)_3$, and taking $\xi = w$, we get

$$\|w\|_{1,\mathrm{I}} \leq C(\|\beta\|_{0,\mathrm{I}} + \|f'\|_{0,\mathrm{I}}) \leq C\|f\|_{1,\mathrm{I}}.$$

This completes the proof. $\square$

Consequently, by virtue of (4.2.7) and (4.2.8), and the equivalence between problems (4.2.6) and (4.2.9), we have that there exists C independent of t and f such that

$$\|\beta\|_{1,\mathrm{I}} + \|w\|_{1,\mathrm{I}} + \|\gamma\|_{0,\mathrm{I}} \leq C\|f\|_{1,\mathrm{I}}. \tag{4.2.10}$$

We end this section with the following result which shows additional regularity of the rotation β from the solution of (4.2.6).

Proposition 4.2.3 *Let (β, w) be the solution of problem (4.2.6). Then $\beta|_{S_i} \in H^2(S_i)$, $i = 1, \ldots, n$, and*

$$\left(\sum_{i=1}^{n} \|\beta''\|_{0,S_i}^2\right)^{1/2} \leq C\|f\|_{1,\mathrm{I}} \left(1 + \max_{1\leq i\leq n} \|\mathbb{E}_i'\|_{\infty,S_i}\right).$$

Proof. Testing, $(4.2.6)_1$, with $(\eta, 0)$, we obtain

$$\int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} \gamma\eta\,dx = 0 \quad \forall\eta \in H_0^1(\mathrm{I}).$$

For all $i = 1, \ldots, n$, we take $\eta \in \mathcal{D}(S_i)$, to get

$$(\mathbb{E}_i\beta')' = \gamma \;\text{ in }\; S_i,$$

namely,

$$\beta''|_{S_i} = \frac{\gamma|_{S_i} - \mathbb{E}_i'\beta'|_{S_i}}{\mathbb{E}_i}.$$

Hence $\beta|_{S_i} \in H^2(S_i)$ and by virtue of (4.2.3),

$$\|\beta''\|_{0,S_i} \leq C(\|\gamma\|_{0,S_i} + \|\mathbb{E}_i'\|_{\infty,S_i}\|\beta'\|_{0,S_i}) \quad \forall i = 1, \ldots, n.$$

Finally, summing over i and using (4.2.10), we conclude the proof. □

4.3 Spectral characterization.

The aim of this section is give a thorough spectral characterization for the operator T_t introduced in Section 4.2, to study the spectral properties of T_t as t goes to zero (limit problem), and to show an additional regularity result for the eigenfunctions of Problem 4.2.1

4.3.1 Description of the spectrum.

In this section, we will show that the operator T_t is non-compact. In fact, this operator has a non-trivial essential spectrum which is well separated from its largest eigenvalues; which as we stated above are the relevant ones in practice. With this end, we recall some basic properties about spectral theory.

Given a generic linear bounded operator $T : X \to X$, defined on a Hilbert space X, we denote the spectrum of T by $\mathrm{Sp}(T) := \{z \in \mathbb{C} : (zI - T) \text{ is not invertible}\}$ and by $\rho(T) := \mathbb{C} \setminus \mathrm{Sp}(T)$ the resolvent set of T. Moreover, for any $z \in \rho(T)$, $R_z(T) := (zI - T)^{-1} : X \to X$ denotes the resolvent operator of T corresponding to z.

We define the following components of the spectrum as in [17].

(1) Discrete spectrum

$$\mathrm{Sp_d}(T) := \{z \in \mathbb{C} : \mathrm{Ker}(zI - T) \neq \{0\} \text{ and } (zI - T) : X \to X \text{ is Fredholm}\}.$$

(2) Essential spectrum

$$\mathrm{Sp_e}(T) := \{z \in \mathbb{C} : \ (zI - T) : \ X \to X \text{ is not Fredholm}\}.$$

Then, the self-adjointness of T_t yields the following result (see [17, Theorem 3.3]).

Theorem 4.3.1 *The spectrum of T_t decomposes as follows:* $\mathrm{Sp}(T_t) = \mathrm{Sp_d}(T_t) \cup \mathrm{Sp_e}(T_t)$. *Moreover, if $\mu \in \mathrm{Sp_d}(T_t)$, then, μ is an isolated eigenvalue of finite multiplicity.*

Our next goal is to show that the essential spectrum of T_t is well separated from the largest eigenvalues. With this aim, first we prove the following result.

Lemma 4.3.2 *Let (β, w) be the solution of problem* (4.2.6) *with source term $f \in H_0^1(\mathrm{I})$. Let $u \in H_0^1(\mathrm{I})$ be the unique solution of the following problem:*

$$\int_{\mathrm{I}} \kappa u' v' \, dx = \int_{\mathrm{I}} \kappa \beta v' \, dx \quad \forall v \in H_0^1(\mathrm{I}). \tag{4.3.1}$$

Then, $u|_{S_i} \in H^2(S_i)$, $i = 1, \dots, n$. Moreover,

$$\left(\sum_{i=1}^{n} \|u''\|_{0,S_i}^2 \right)^{1/2} \leq C \|f\|_{1,\mathrm{I}} \left(1 + \max_{1 \leq i \leq n} \|\kappa_i'\|_{\infty,S_i} \right).$$

Proof. Notice that the existence of a unique u solution of (4.3.1) is guaranteed by (4.2.3) and Lax-Milgram's lemma. Taking $v = u$ in (4.3.1), from (4.2.10) and the Poincaré inequality, we obtain

$$\|u\|_{1,\mathrm{I}} \leq C \|\beta\|_{0,\mathrm{I}} \leq C \|f\|_{1,\mathrm{I}}. \tag{4.3.2}$$

For all $i = 1, \ldots, n$, we take $v \in \mathcal{D}(S_i)$, to get

$$(\kappa_i u')' = (\kappa_i \beta)' \text{ in } S_i,$$

namely,

$$u''|_{S_i} = \frac{\kappa_i' \beta|_{S_i} + \kappa_i \beta'|_{S_i} - \kappa_i' u'|_{S_i}}{\kappa_i}.$$

By virtue of (4.2.3), we have

$$\|u''\|_{0,S_i} \leq C\|\beta'\|_{0,S_i} + C\|\kappa_i'\|_{\infty,S_i}(\|\beta\|_{0,S_i} + \|u'\|_{0,S_i}).$$

Summing over i and using (4.2.10) and (4.3.2), we conclude the proof. □

The following result shows that the essential spectrum of T_t is confined to a real interval proportional to t^2; we note that the thinner the beam, the smaller the interval containing the essential spectrum.

Proposition 4.3.3 $\mathrm{Sp_e}(T_t) \subset \left[\dfrac{t^2 P}{\overline{\kappa}}, \dfrac{t^2 P}{\underline{\kappa}}\right]$.

Proof. Let $\mu \notin \left[\frac{t^2 P}{\overline{\kappa}}, \frac{t^2 P}{\underline{\kappa}}\right]$. We have to show that $(\mu I - T_t)$ is a Fredholm operator. To prove this, it is enough to show that there exists a compact operator $G : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I})$ such that $(\mu I - T_t + G)$ is invertible. We define G as follows: for $f \in \mathrm{H}_0^1(\Omega)\mathrm{I}$, let $G(f) = u$, with u as in Lemma 4.3.2. By standard arguments, it follows that the subspace of $H_0^1(\mathrm{I})$ with second derivative piecewise in $L^2(I)$ is compactly included in $H_0^1(I)$. Therefore, using Lemma 4.3.2, we deduce that G is a compact operator.

Thus, there only remains to prove that $(\mu I - T_t + G) : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I})$ is invertible. First, notice that given $f, v \in H_0^1(\mathrm{I})$, $v = (\mu I - T_t + G)f$ if and only if

$$\int_{\mathrm{I}} \kappa v' \xi' \, dx = \int_{\mathrm{I}} \kappa \left[(\mu I - T_t + G)f\right]' \xi' \, dx \quad \forall \xi \in H_0^1(\mathrm{I}).$$

Now, for $f \in H_0^1(\mathrm{I})$, let (β, k, w) be the solution of problem (4.2.9), so that $w = T_t f$, and let u be the solution of problem (4.3.1), so that $u = Gf$. Hence, from (4.3.1) and problem $(4.2.9)_3$, we have that

$$\begin{aligned}\int_{\mathrm{I}} \kappa \left[(\mu I - T_t + G)f\right]' \xi' \, dx &= \int_{\mathrm{I}} \kappa(\mu f' - w' + u')\xi' \, dx \\ &= \int_{\mathrm{I}} \kappa(\mu f' - w' + \beta)\xi' \, dx \\ &= \int_{\mathrm{I}} \kappa \left(\mu - \frac{t^2 P}{\kappa}\right) f'\xi' \, dx.\end{aligned}$$

Therefore, $v = (\mu I - T_t + G)f$ if and only if

$$\int_{\mathrm{I}} \kappa v'\xi' \, dx = \int_{\mathrm{I}} \kappa \left(\mu - \frac{t^2 P}{\kappa}\right) f'\xi' \, dx. \tag{4.3.3}$$

Then, if $\mu \notin \left[\frac{t^2P}{\overline{\kappa}}, \frac{t^2P}{\underline{\kappa}}\right]$, we have that for each $v \in H_0^1(\mathrm{I})$ there exists a unique $f \in H_0^1(\mathrm{I})$ such that (4.3.3) holds true; therefore $(\mu I - T_t)$ is Fredholm operator and the proof is complete. $\square$

The following theorem is an immediate consequence of Theorem 4.3.1 and Proposition 4.3.3.

Theorem 4.3.4 *The spectrum* $\mathrm{Sp}(T_t)$ *decomposes into:*

- $\mathrm{Sp_d}(T_t)$*, which consists of finite multiplicity real positive eigenvalues.*
- $\mathrm{Sp_e}(T_t)$*, the essential spectrum.*

Moreover, for all $\mu \in \mathrm{Sp}(T_t)$ *such that* $\mu \notin \left[\frac{t^2P}{\overline{\kappa}}, \frac{t^2P}{\underline{\kappa}}\right]$, $\mu \in \mathrm{Sp_d}(T_t)$.

4.3.2 Limit problem.

In this section we study the convergence properties of the operator T_t as t goes to zero. With this end, we introduce the so-called limit problem:

Given $f \in H_0^1(\mathrm{I})$*, find* $(\beta_0, w_0, \gamma_0) \in V \times L^2(\mathrm{I})$ *such that*

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta_0'\eta' \, dx + \int_{\mathrm{I}} \gamma_0(\eta - v') \, dx = \int_{\mathrm{I}} P f' v' \, dx \quad \forall (\eta, v) \in V, \\ \beta_0 - w_0' = 0. \end{cases} \tag{4.3.4}$$

This is a mixed formulation of the following well-posed problem, which corresponds to the source problem associated with the buckling of an Euler-Bernoulli beam:

Find $w_0 \in H_0^2(\mathrm{I})$ *such that*

$$\int_{\mathrm{I}} \mathbb{E} w_0'' v''\, dx = \int_{\mathrm{I}} P f' v'\, dx \quad \forall v \in H_0^2(\mathrm{I}). \tag{4.3.5}$$

On the other hand, we have that the proof of Theorem 4.2.2 holds for $t = 0$, too. Thus, problem (4.3.4) has a unique solution $(\beta_0, w_0, \gamma_0) \in V \times L^2(\mathrm{I})$ and there exists C such that

$$\|\beta_0\|_{1,\mathrm{I}} + \|w_0\|_{1,\mathrm{I}} + \|\gamma_0\|_{0,\mathrm{I}} \le C\|f\|_{1,\mathrm{I}}. \tag{4.3.6}$$

Moreover, w_0 is the solution of problem (4.3.5) and $\|w_0\|_{2,\mathrm{I}} \le C\|f\|_{1,\mathrm{I}}$.

Let T_0 be the following bounded linear operator

$$\begin{aligned} T_0 :\ H_0^1(\mathrm{I}) &\to H_0^1(\mathrm{I}), \\ f &\mapsto w_0, \end{aligned}$$

where (β_0, w_0, γ_0) is the unique solution of problem (4.3.4). Since $w_0 \in H_0^2(\mathrm{I})$, the operator T_0 is compact and hence its spectrum satisfies $\mathrm{Sp}(T_0) = \{0\} \cup \{\mu_n :\ n \in \mathbb{N}\}$, where $\{\mu_n\}_{n\in\mathbb{N}}$ is a sequence of positive eigenvalues which converges to 0. The multiplicity of each non-zero eigenvalue is finite and its ascent is 1.

The following lemma states the convergence in norm of T_t to T_0.

Lemma 4.3.5 *There exists a constant C, independent of t, such that*

$$\|(T_t - T_0) f\|_{1,\mathrm{I}} \le Ct\|f\|_{1,\mathrm{I}},$$

for all $f \in H_0^1(\mathrm{I})$.

Proof. Subtracting (4.3.4) from (4.2.6), we obtain

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}(\beta' - \beta_0')\eta'\, dx + \int_{\mathrm{I}} (\gamma - \gamma_0)(\eta - v')\, dx = 0 \quad \forall (\eta, v) \in V, \\ \gamma = \dfrac{\kappa}{t^2}\left[(\beta - \beta_0) - (w' - w_0')\right], \end{cases}$$

and taking $\eta = \beta - \beta_0$ and $v = w - w_0$, we obtain

$$\int_{\mathrm{I}} \mathbb{E}(\beta' - \beta_0')(\beta' - \beta_0')\,dx = -\int_{\mathrm{I}} \frac{t^2}{\kappa}\gamma(\gamma - \gamma_0)\,dx.$$

Now, using the Poincaré inequality, (4.2.10) and (4.3.6), we have

$$\|\beta - \beta_0\|_{1,\mathrm{I}}^2 \leq Ct^2(\|\gamma\|_{0,\mathrm{I}} + \|\gamma_0\|_{0,\mathrm{I}})\|\gamma\|_{0,\mathrm{I}} \leq Ct^2\|f\|_{1,\mathrm{I}}^2,$$

which implies

$$\|\beta - \beta_0\|_{1,\mathrm{I}} \leq \; Ct\|f\|_{1,\mathrm{I}}. \tag{4.3.7}$$

Finally, observe that

$$(w' - w_0') = (\beta - \beta_0) - \frac{t^2}{\kappa}\gamma.$$

Thus, using the Poincaré inequality and (4.2.3), we obtain

$$\|w - w_0\|_{1,\mathrm{I}} \leq C(\|\beta - \beta_0\|_{0,\mathrm{I}} + t^2\|\gamma\|_{0,\mathrm{I}}),$$

which together with (4.3.7), and again the a priori estimate (4.2.10) allow us to conclude the proof. $\square$

As a consequence of this lemma, standard properties about the separation of isolated parts of the spectrum (see [31], for instance) yield the following result.

Lemma 4.3.6 *Let μ_0 be an eigenvalue of T_0 of multiplicity m. Let D be any disc in the complex plane centered at μ_0 and containing no other element of the spectrum of T_0. Then, there exists $t_0 > 0$ such that, $\forall t < t_0$, D contains exactly m isolated eigenvalues of T_t (repeated according to their respective multiplicities). Consequently, each eigenvalue μ_0 of T_0 is a limit of isolated eigenvalues μ_t of T_t, as t goes to zero.*

Our next goal is to show that the largest eigenvalues of T_t converge to the largest eigenvalues of T_0 as t goes to zero. With this aim, we prove first the following lemma. Here and thereafter, we will use $\|\cdot\|$ to denote the operator norm induced by the $H^1(\mathrm{I})$ norm.

Lemma 4.3.7 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. Then there exist strictly positive constants t_0 and C such that, $\forall t < t_0$, $F \cap \mathrm{Sp}(T_t) = \emptyset$ and*

$$\|R_z(T_t)\| := \sup_{\substack{w \in H_0^1(\mathrm{I}) \\ w \neq 0}} \frac{\|R_z(T_t)w\|_{1,\Omega}}{\|w\|_{1,\Omega}} \leq C \qquad \forall z \in F.$$

Proof. The proof is identical to that of Lemma 3.8 from [33] and makes use of Theorem 4.3.4 to localize the essential spectrum. □

Since T_0 is a compact operator, its nonzero eigenvalues are isolated and we can order them as follows:

$$\mu_0^{(1)} \geq \mu_0^{(2)} \geq \cdots \geq \mu_0^{(k)} \geq \cdots$$

where each eigenvalue is repeated as many times as its corresponding multiplicity. According to Lemma 4.3.6, for t sufficiently small there exist eigenvalues of T_t close to each $\mu_0^{(k)}$. On the other hand, according to Theorem 4.3.4, the essential spectrum of T_t is confined in the interval $\left[\frac{t^2 P}{\overline{\kappa}}, \frac{t^2 P}{\underline{\kappa}}\right]$. Therefore, at least for t sufficiently small, the largest points of the spectrum of T_t have to be isolated eigenvalues. Hence we order them as we did with those of T_0:

$$\mu_t^{(1)} \geq \mu_t^{(2)} \geq \cdots \geq \mu_t^{(k)} \geq \cdots$$

The following theorem, whose proof is similar to that of Theorem 3.9 from [33], shows that the k-th eigenvalue of T_t converge to the k-th eigenvalue of T_0 as t goes to zero.

Theorem 4.3.8 *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as defined above. For all $k \in \mathbb{N}$, $\mu_t^{(k)} \to \mu_0^{(k)}$ as $t \to 0$.*

4.3.3 Additional regularity of the eigenfunctions.

The aim of this section is to prove additional regularity for the eigenfunctions of Problem 4.2.1. More precisely, we have the following result.

Lemma 4.3.9 *Let $\mu_t^{(k)}$, $k \in \mathbb{N}$, $t \geq 0$, be as in Theorem 4.3.8. Let $(\lambda, \beta, w, \gamma)$ be a solution of Problem 4.2.1 with $\lambda = 1/\mu_t^{(k)}$. Then, there exists $t_0 > 0$ such that, for all*

$t < t_0$, $\beta|_{S_i}, w|_{S_i} \in H^2(S_i)$, $i = 1, \dots, n$, and there holds

$$\left(\sum_{i=1}^{n} \|\beta''\|_{0,S_i}^2 \right)^{1/2} \leq C\lambda \|w\|_{1,\mathrm{I}}\,, \tag{4.3.8}$$

$$\left(\sum_{i=1}^{n} \|w''\|_{0,S_i}^2 \right)^{1/2} \leq C\lambda \|w\|_{1,\mathrm{I}}\,, \tag{4.3.9}$$

with C a positive constant independent of t.

Proof. Using the decomposition (4.2.7) in Problem 4.2.1, we obtain that

$$\psi = -\lambda P w.$$

Moreover, (4.2.9) holds true with f substituted by λw and Theorem (4.2.2) leads in our case to

$$\|\beta\|_{1,\mathrm{I}} + |k| + \|w\|_{1,\mathrm{I}} \leq C\lambda \|w\|_{1,\mathrm{I}}. \tag{4.3.10}$$

Thus, repeating the arguments used in the proof of Proposition 4.2.3, we immediately obtain (4.3.8).

Now, from problem $(4.2.9)_3$ with f substituted by λw as above, we have

$$\int_{\mathrm{I}} \left(\kappa - \lambda t^2 P \right) w' \xi' \, dx = \int_{\mathrm{I}} \kappa \beta \xi' \, dx \quad \forall \xi \in H_0^1(\mathrm{I}).$$

For all $i = 1, \dots, n$, we take $\xi \in \mathcal{D}(S_i)$, to obtain

$$\left[\left(\kappa_i - \lambda t^2 P \right) w' \right]' = (\kappa_i \beta)' \ \text{ in } \ S_i,$$

and consequently,

$$w''|_{S_i} = \frac{\kappa_i \beta'|_{S_i} + \kappa_i' \beta|_{S_i} - \kappa_i' w'|_{S_i}}{(\kappa_i - \lambda t^2 P)}.$$

Choosing t_0 such that $\forall t < t_0$, $\lambda t^2 P \leq (\underline{\kappa}/2)$, and using (4.2.3), we obtain

$$\|w''\|_{0,S_i} \leq \frac{2}{\underline{\kappa}} \left(\|\kappa_i'\|_{\infty,S_i} \|w'\|_{0,S_i} + \|\kappa_i\|_{\infty,S_i} \|\beta'\|_{0,S_i} + \|\kappa_i'\|_{\infty,S_i} \|\beta\|_{0,S_i} \right).$$

Summing over i, using Poincaré inequality, and (4.3.10), we get

$$\left(\sum_{i=1}^{n} \|w''\|_{0,S_i}^2\right)^{1/2} \leq C\lambda \|w\|_{1,\mathrm{I}}.$$

Thus, we conclude the proof. $\square$

4.4 Spectral approximation.

For the numerical approximation, we consider a family of partitions of I

$$\mathcal{T}_h := 0 = x_0 < \cdots < x_N = L,$$

which are refinements of the initial partition $0 = s_0 < \cdots < s_n = L$. We denote $\mathrm{I}_j = (x_j - x_{j-1})$, $j = 1, \ldots, N$, and the maximun subinterval length is denoted $h := \max_{1\leq j\leq N} \mathrm{I}_j$. Notice that for any mesh $\mathcal{T}_h$, each I_j is contained in some subinterval S_i, $i = 1, \ldots, n$, where the coefficients are smooth.

To approximate the transverse displacement and the rotations, we consider the space of piecewise linear continuous finite elements:

$$W_h := \{v_h \in H_0^1(\mathrm{I}) \ : \ v_h|_{\mathrm{I}_j} \in \mathbb{P}_1, \ j = 1, \ldots, N, \ v_h(0) = v_h(L) = 0\}.$$

To approximate the shear stress, we will use the space of piecewise constant functions:

$$Q_h := \{v_h \in L^2(\mathrm{I}) \ : \ v_h|_{\mathrm{I}_j} \in \mathbb{P}_0, \ j = 1, \ldots, N\}.$$

We consider the L^2-proyector onto Q_h:

$$\begin{aligned} \mathcal{P} : L^2(\mathrm{I}) &\to Q_h, \\ v \mapsto \mathcal{P}(v) &:= \bar{v} \ : \ \int_{\mathrm{I}} (v - \bar{v}) q_h = 0 \quad \forall q_h \in Q_h. \end{aligned}$$

The discretization of Problem 4.2.1 reads as follows:

Problem 4.4.1 Find $\lambda_h \in \mathbb{R}^+$ and $0 \neq (\beta_h, w_h) \in V_h := W_h \times W_h$ and $\gamma_h \in Q_h$ such that

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta_h' \eta_h' \, dx + \int_{\mathrm{I}} \gamma_h(\eta_h - v_h') \, dx = \lambda_h \int_{\mathrm{I}} P w_h' v_h' \, dx \quad \forall (\eta_h, v_h) \in V_h, \\ \displaystyle\int_{\mathrm{I}} (\beta_h - w_h') s_h \, dx - t^2 \int_{\mathrm{I}} \frac{\gamma_h s_h}{\kappa} \, dx = 0 \quad \forall s_h \in Q_h. \end{cases} \tag{4.4.1}$$

As in the continuous case, we introduce the solution operator

$$T_{th} : W_h \to W_h,$$
$$f \mapsto w_h,$$

where $(\beta_h, w_h, \gamma_h) \in V_h \times Q_h$ is the solution of the corresponding discrete source problem:

Given $f \in W_h$, find $(\beta_h, w_h, \gamma_h) \in V_h \times Q_h$ such that

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}\beta_h' \eta_h' \, dx + \int_{\mathrm{I}} \gamma_h(\eta_h - v_h') \, dx = \int_{\mathrm{I}} P f' v_h' \, dx \quad \forall (\eta_h, v_h) \in V_h, \\ \displaystyle\int_{\mathrm{I}} (\beta_h - w_h') s_h \, dx - t^2 \int_{\mathrm{I}} \frac{\gamma_h s_h}{\kappa} \, dx = 0 \quad \forall s_h \in Q_h. \end{cases} \tag{4.4.2}$$

Clearly, the nonzero eigenvalues of T_{th} are given by $\mu_h := 1/\lambda_h$, with λ_h being the nonzero eigenvalues of Problem 4.4.1, and the corresponding eigenfunctions coincide.

By adding equations (4.4.2), because of the symmetry of the resulting bilinear forms, T_{th} is self-adjoint with respect to the inner product $\int_{\mathrm{I}} P f' g' \, dx$ in $H_0^1(\mathrm{I})$.

We will prove the following spectral characterization for Problem 4.4.1:

Lemma 4.4.1 *Problem 4.4.1 has exactly* $\dim W_h$ *eigenvalues, repeated accordingly to their respective multiplicities. All of them are real and positive.*

Proof. Taking particular bases of W_h and Q_h, Problem 4.4.1 can be written as follows:

$$\begin{bmatrix} \mathbf{A} & 0 & \mathbf{B} \\ 0 & 0 & \mathbf{C} \\ \mathbf{B}^{\mathrm{t}} & \mathbf{C}^{\mathrm{t}} & -\mathbf{D} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \\ \boldsymbol{\gamma}_h \end{bmatrix} = \lambda_h \begin{bmatrix} 0 & 0 & 0 \\ 0 & \mathbf{E} & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \\ \boldsymbol{\gamma}_h \end{bmatrix}, \tag{4.4.3}$$

where $\boldsymbol{\beta}_h$, $\mathbf{w}_h$, and $\boldsymbol{\gamma}_h$ denote the vectors whose entries are the components in those basis of β_h, w_h, and γ_h, respectively. Matrices $\mathbf{A}$, $\mathbf{D}$ and $\mathbf{E}$ are symmetric and positive definite. From the last row of (4.4.3), we have that

$$\boldsymbol{\gamma}_h = \mathbf{D}^{-1}(\mathbf{B}^{\mathrm{t}}\boldsymbol{\beta}_h + \mathbf{C}^{\mathrm{t}}\mathbf{w}_h),$$

thus, defining

$$\mathcal{A} := \begin{bmatrix} \mathbf{A} + \mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{\mathrm{t}} & \mathbf{B}\mathbf{D}^{-1}\mathbf{C}^{\mathrm{t}} \\ \mathbf{C}\mathbf{D}^{-1}\mathbf{B}^{\mathrm{t}} & \mathbf{C}\mathbf{D}^{-1}\mathbf{C}^{\mathrm{t}} \end{bmatrix},$$

problem (4.4.3) can be written as follows:

$$\mathcal{A} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \end{bmatrix} = \lambda_h \begin{bmatrix} 0 & 0 \\ 0 & \mathbf{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \end{bmatrix}. \tag{4.4.4}$$

The matrix $\mathcal{A}$ is a positive definite. In fact,

$$\begin{aligned} [\boldsymbol{\beta}_h^{\mathrm{t}} \ \ \mathbf{w}_h^{\mathrm{t}}]\mathcal{A} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \end{bmatrix} =& \boldsymbol{\beta}_h^{\mathrm{t}}\mathbf{A}\boldsymbol{\beta}_h + \boldsymbol{\beta}_h^{\mathrm{t}}\mathbf{B}\mathbf{D}^{-1}\mathbf{B}^{\mathrm{t}}\boldsymbol{\beta}_h + 2\boldsymbol{\beta}_h^{\mathrm{t}}\mathbf{B}\mathbf{D}^{-1}\mathbf{C}^{\mathrm{t}}\mathbf{w}_h + \mathbf{w}_h^{\mathrm{t}}\mathbf{C}\mathbf{D}^{-1}\mathbf{C}^{\mathrm{t}}\mathbf{w}_h \\ =& \boldsymbol{\beta}_h^{\mathrm{t}}\mathbf{A}\boldsymbol{\beta}_h + (\mathbf{B}^{\mathrm{t}}\boldsymbol{\beta}_h + \mathbf{C}^{\mathrm{t}}\mathbf{w}_h)^{\mathrm{t}}\mathbf{D}^{-1}(\mathbf{B}^{\mathrm{t}}\boldsymbol{\beta}_h + \mathbf{C}^{\mathrm{t}}\mathbf{w}_h) \geq 0. \end{aligned}$$

Hence $\mathcal{A}$ is non-negative definite. Moreover, the expression above vanishes if and only if $\boldsymbol{\beta}_h = \mathbf{0}$ and $(\mathbf{B}^{\mathrm{t}}\boldsymbol{\beta}_h + \mathbf{C}^{\mathrm{t}}\mathbf{w}_h) = \mathbf{0}$, namely, $\boldsymbol{\beta}_h = \mathbf{0}$ and $\mathbf{C}^{\mathrm{t}}\mathbf{w}_h = \mathbf{0}$. Now, $\mathbf{C}^{\mathrm{t}}\mathbf{w}_h = \mathbf{0}$ implies that $\int_{\mathrm{I}_j} w_h' = 0$, $j = 1, \ldots, N$, then $w_h(x_{j-1}) = w_h(x_j)$, $j = 1, \ldots, N$. But, $w_h(x_0) = w_h(x_N) = 0$. Hence, $w_h(x_j) = 0$, $j = 1, \ldots, N-1$, and $w_h \in W_h$. Therefore, $w_h = 0$ and we conclude that $\mathcal{A}$ is positive definite.

Consequently, from (4.4.4) $\lambda_h \neq 0$ and, since $\mathbf{E}$ is symmetric and positive definite, $\lambda_h \in \mathbb{R}^+$. Moreover, (4.4.4) holds true if and only if

$$\begin{bmatrix} 0 & 0 \\ 0 & \mathbf{E} \end{bmatrix} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \end{bmatrix} = \mu_h \mathcal{A} \begin{bmatrix} \boldsymbol{\beta}_h \\ \mathbf{w}_h \end{bmatrix},$$

with $\lambda_h = \frac{1}{\mu_h}$ and $\mu_h \neq 0$. The latter problem is a well posed generalized eigenvalue problem with $\dim W_h$ non-zero eigenvalues. Thus we conclude the proof. □

Remark 4.4.2 *As a consequence of the above lemma the second component of any eigenfunction* (β_h, w_h) *of Problem 4.4.1 can not vanish. In fact, from* (4.4.4)*, we have*

$$\int_{\mathrm{I}} P w_h' w_h' \, dx = \mathbf{w}_h^{\mathrm{t}} \mathbf{E} \mathbf{w}_h > 0.$$

Since T_t is not compact, in the next section we will adapt the theory from [18, 19] to prove convergence of our spectral approximation and nonexistence of spurious modes, as well as to obtain error estimates. To do this, the remainder of this section is devoted to prove the following properties:

P1. There holds:

$$\|T_t - T_{th}\|_h := \sup_{\substack{f_h \in W_h \\ f_h \neq 0}} \frac{\|(T_t - T_{th})f_h\|_{1,\mathrm{I}}}{\|f_h\|_{1,\mathrm{I}}} \to 0, \quad \text{as} \quad h \to 0.$$

P2. $\forall u \in H_0^1(\mathrm{I})$, there holds:

$$\inf_{v_h \in W_h} \|u - v_h\|_{1,\mathrm{I}} \to 0, \quad \text{as} \quad h \to 0.$$

P2 is a consequence of the fact that $\mathcal{D}(\mathrm{I})$ is a dense subspace of $H_0^1(\mathrm{I})$ and standard approximation results for finite element spaces.

To prove property P1, we consider the following auxiliary problems:

Given $f_h \in W_h$, find $(\tilde{\beta}, \tilde{w}, \tilde{\gamma}) \in V \times L^2(\mathrm{I})$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathrm{I}} \mathbb{E}\tilde{\beta}'\eta'\,dx + \int_{\mathrm{I}} \tilde{\gamma}(\eta - v')\,dx = \int_{\mathrm{I}} P f_h' v'\,dx \quad \forall(\eta, v) \in V, \\ \displaystyle\int_{\mathrm{I}} (\tilde{\beta} - \tilde{w}')s\,dx - t^2 \int_{\mathrm{I}} \frac{\tilde{\gamma}s}{\kappa}\,dx = 0 \quad \forall s \in L^2(\mathrm{I}). \end{array} \right. \tag{4.4.5}$$

Given $f_h \in W_h$, find $(\tilde{\beta}_h, \tilde{w}_h, \tilde{\gamma}_h) \in V_h \times Q_h$ such that

$$\left\{ \begin{array}{l} \displaystyle\int_{\mathrm{I}} \mathbb{E}\tilde{\beta}_h'\eta_h'\,dx + \int_{\mathrm{I}} \tilde{\gamma}_h(\eta_h - v_h')\,dx = \int_{\mathrm{I}} P f_h' v_h'\,dx \quad \forall(\eta_h, v_h) \in V_h, \\ \displaystyle\int_{\mathrm{I}} (\tilde{\beta}_h - \tilde{w}_h')s_h\,dx - t^2 \int_{\mathrm{I}} \frac{\tilde{\gamma}_h s_h}{\kappa}\,dx = 0 \quad \forall s_h \in Q_h. \end{array} \right. \tag{4.4.6}$$

An estimate analogous to (4.2.10) also holds for problem (4.4.5):

$$\|\tilde{\beta}\|_{1,\mathrm{I}} + \|\tilde{w}\|_{1,\mathrm{I}} + \|\tilde{\gamma}\|_{0,\mathrm{I}} \leq C\|f_h\|_{1,\mathrm{I}}. \tag{4.4.7}$$

Using the following decompositions for $\tilde{\gamma}$ and $\tilde{\gamma}_h$,

$$\tilde{\gamma} = \tilde{\psi}' + \tilde{k}, \quad \text{and} \quad \tilde{\gamma}_h = \tilde{\psi}_h' + \tilde{k}_h, \tag{4.4.8}$$

with $\tilde{\psi} \in H_0^1(\mathrm{I})$, $\tilde{\psi}_h \in W_h$ and $\tilde{k}, \tilde{k}_h \in \mathbb{R}$, we have that the previous problems are respectively equivalent to the following ones:

Given $f_h \in W_h$, find $(\tilde{\psi}, \tilde{\beta}, \tilde{k}, \tilde{w}) \in H_0^1(\mathrm{I}) \times H_0^1(\mathrm{I}) \times \mathbb{R} \times H_0^1(\mathrm{I})$ such that

$$\left\{ \begin{aligned} &\int_{\mathrm{I}} \tilde{\psi}' v' \, dx = - \int_{\mathrm{I}} P f_h' v' \, dx \quad \forall v \in H_0^1(\mathrm{I}), \\ &\int_{\mathrm{I}} \mathbb{E} \tilde{\beta}' \eta' \, dx + \int_{\mathrm{I}} \tilde{k} \eta \, dx = - \int_{\mathrm{I}} \tilde{\psi}' \eta \, dx \quad \forall \eta \in H_0^1(\mathrm{I}), \\ &\int_{\mathrm{I}} \tilde{\beta} q \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{k} q}{\kappa} \, dx = t^2 \int_{\mathrm{I}} \frac{\tilde{\psi}' q}{\kappa} \, dx \quad \forall q \in \mathbb{R}, \\ &\int_{\mathrm{I}} \tilde{w}' \xi' \, dx = \int_{\mathrm{I}} \tilde{\beta} \xi' \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{\psi}' \xi'}{\kappa} \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{k} \xi'}{\kappa} \, dx \quad \forall \xi \in H_0^1(\mathrm{I}). \end{aligned} \right. \tag{4.4.9}$$

Given $f_h \in W_h$, find $(\tilde{\psi}_h, \tilde{\beta}_h, \tilde{k}_h, \tilde{w}_h) \in W_h \times W_h \times \mathbb{R} \times W_h$ such that

$$\left\{ \begin{aligned} &\int_{\mathrm{I}} \tilde{\psi}_h' v_h' \, dx = - \int_{\mathrm{I}} P f_h' v_h' \, dx \quad \forall v_h \in W_h, \\ &\int_{\mathrm{I}} \mathbb{E} \tilde{\beta}_h' \eta_h' \, dx + \int_{\mathrm{I}} \tilde{k}_h \eta_h \, dx = - \int_{\mathrm{I}} \tilde{\psi}_h' \eta_h \, dx \quad \forall \eta_h \in W_h, \\ &\int_{\mathrm{I}} \tilde{\beta}_h q_h \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{k}_h q_h}{\kappa} \, dx = t^2 \int_{\mathrm{I}} \frac{\tilde{\psi}_h' q_h}{\kappa} \, dx \quad \forall q_h \in \mathbb{R}, \\ &\int_{\mathrm{I}} \tilde{w}_h' \xi_h' \, dx = \int_{\mathrm{I}} \tilde{\beta}_h \xi_h' \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{\psi}_h' \xi_h'}{\kappa} \, dx - t^2 \int_{\mathrm{I}} \frac{\tilde{k}_h \xi_h'}{\kappa} \, dx \quad \forall \xi_h \in W_h. \end{aligned} \right. \tag{4.4.10}$$

First of all we prove that $\tilde{\psi}$ and $\tilde{\psi}_h$ coincide.

Lemma 4.4.3 *The solution $\tilde{\psi}$ of problem* $(4.4.9)_1$ *and the solution $\tilde{\psi}_h$ of problem* $(4.4.10)_1$ *satisfy*

$$\tilde{\psi} = \tilde{\psi}_h \ \ \textit{in} \ \ \mathrm{I}.$$

Proof. Testing the first equation from (4.4.9) with $v \in \mathcal{D}(\mathrm{I}_j)$, we obtain that $\tilde{\psi}'' = -(P f_h')' = 0$ in I_j, $j = 1, \ldots, N$. Hence $\tilde{\psi} \in W_h$ is also the solution of the first equation in (4.4.10). Namely, $\tilde{\psi} = \tilde{\psi}_h$. □

Using this lemma, we have that problem $(4.4.10)_{2-3}$ is the finite element discretization of problem $(4.4.9)_{2-3}$. Then, from standard approximation for mixed problems (see [11, Proposition 2.11]), we obtain

$$\|\tilde{\beta} - \tilde{\beta}_h\|_{1,\mathrm{I}} + |\tilde{k} - \tilde{k}_h| \leq \inf_{\eta_h \in W_h} \|\tilde{\beta} - \eta_h\|_{1,\mathrm{I}} \leq \|\beta - \beta^I\|_{1,\mathrm{I}},$$

where $\beta^I \in W_h$ is the Lagrange interpolant of $\tilde{\beta}$. Using Proposition 4.2.3 applied to problem (4.4.5), we have that

$$\begin{aligned}
\|\tilde{\beta} - \beta^I\|_{1,\mathrm{I}} \leq & \left(\sum_{j=1}^{N} \|\tilde{\beta} - \beta^I\|^2_{1,\mathrm{I}_j} \right)^{1/2} \\
\leq & \left(\sum_{j=1}^{N} C h_j^2 \|\tilde{\beta}''\|^2_{0,\mathrm{I}_j} \right)^{1/2} \\
\leq & Ch \left(\sum_{i=1}^{n} \|\tilde{\beta}''\|^2_{0,S_i} \right)^{1/2} \leq Ch\|f_h\|_{1,\mathrm{I}}.
\end{aligned}$$

Thus,

$$\|\tilde{\beta} - \tilde{\beta}_h\|_{1,\mathrm{I}} + |\tilde{k} - \tilde{k}_h| \leq Ch\|f_h\|_{1,\mathrm{I}}. \tag{4.4.11}$$

Then, from (4.4.8), Lemma 4.4.3 and the estimate above, we have

$$\|\tilde{\gamma} - \tilde{\gamma}_h\|_{0,\mathrm{I}} = \|(\tilde{\psi}' + \tilde{k}) - (\tilde{\psi}'_h + \tilde{k}_h)\|_{0,\mathrm{I}} \leq C|\tilde{k} - \tilde{k}_h| \leq Ch\|f_h\|_{1,\mathrm{I}}. \tag{4.4.12}$$

On the other hand, from $(4.4.5)_2$, we obtain

$$\tilde{w}' = \tilde{\beta} - t^2\kappa^{-1}\tilde{\gamma},$$

and from $(4.4.6)_2$,

$$\tilde{w}'_h = \mathcal{P}\left(\tilde{\beta}_h - t^2\kappa^{-1}\tilde{\gamma}_h\right) = \mathcal{P}(\tilde{\beta}_h) - t^2\mathcal{P}\left(\kappa^{-1}\tilde{\gamma}_h\right).$$

Then,

$$\|\tilde{w}' - \tilde{w}'_h\|_{0,\mathrm{I}} \leq \|\tilde{\beta} - \mathcal{P}(\tilde{\beta}_h)\|_{0,\mathrm{I}} + t^2\|\kappa^{-1}\tilde{\gamma} - \mathcal{P}(\kappa^{-1}\tilde{\gamma}_h)\|_{0,\mathrm{I}}. \tag{4.4.13}$$

Now,

$$\|\tilde{\beta} - \mathcal{P}(\tilde{\beta}_h)\|_{0,\mathrm{I}} \leq \|\tilde{\beta} - \mathcal{P}(\tilde{\beta})\|_{0,\mathrm{I}} + \|\mathcal{P}(\tilde{\beta} - \tilde{\beta}_h)\|_{0,\mathrm{I}} \leq Ch\|f_h\|_{1,\mathrm{I}}, \tag{4.4.14}$$

the last inequality because of (4.4.7) and (4.4.11).

On the other hand, on each subinterval I_j, $j = 1, \ldots, N$, since $\tilde{\gamma}_h$ is piecewise constant,

$$\begin{aligned}\|\kappa^{-1}\tilde{\gamma} - \mathcal{P}(\kappa^{-1}\tilde{\gamma}_h)\|_{0,\mathrm{I}_j} \leq & \|\left(\kappa^{-1} - \mathcal{P}(\kappa^{-1})\right)\tilde{\gamma}\|_{0,\mathrm{I}_j} + \|\mathcal{P}(\kappa^{-1})(\tilde{\gamma} - \tilde{\gamma}_h)\|_{0,\mathrm{I}_j} \\ \leq & \|\kappa^{-1} - \mathcal{P}(\kappa^{-1})\|_{\infty,\mathrm{I}_j}\|\tilde{\gamma}\|_{0,\mathrm{I}_j} + \|\mathcal{P}(\kappa^{-1})\|_{\infty,\mathrm{I}_j}\|\tilde{\gamma} - \tilde{\gamma}_h\|_{0,\mathrm{I}_j}.\end{aligned}$$

Moreover, it is simple to prove that

$$\|\kappa^{-1} - \mathcal{P}(\kappa^{-1})\|_{\infty,\mathrm{I}_j} \leq h_j\|\kappa^{-1}\|_{1,\infty,\mathrm{I}_j} \leq Ch,$$

with C depending on $\underline{\kappa}$ and $\|\kappa\|_{1,\infty,\mathrm{I}_j}$, and

$$\|\mathcal{P}(\kappa^{-1})\|_{\infty,\mathrm{I}_j} \leq \|\kappa^{-1}\|_{\infty,\mathrm{I}_j} \leq \underline{\kappa}^{-1}.$$

Hence, from (4.4.7) and (4.4.12), the last three inequalities yield

$$\|\kappa^{-1}\tilde{\gamma} - \mathcal{P}(\kappa^{-1}\tilde{\gamma}_h)\|_{0,\mathrm{I}} \leq Ch\|f_h\|_{1,\mathrm{I}}. \tag{4.4.15}$$

Therefore, from (4.4.13), (4.4.14), (4.4.15) and Poincaré inequality, we obtain

$$\|(T_t - T_{th})f_h\|_{1,\mathrm{I}} = \|\tilde{w} - \tilde{w}_h\|_{1,\mathrm{I}} \leq Ch\|f_h\|_{1,\mathrm{I}}.$$

Consequently, we have proved the following result.

Lemma 4.4.4 *P1 holds true; moreover,*

$$\|T_t - T_{th}\|_h \leq Ch.$$

4.5 Convergence and error estimates.

In this section we will adapt the arguments from [18, 19] to prove convergence of our spectral approximation and nonexistence of spurious modes, as well as to obtain error estimates for the approximate eigenvalues and eigenfunctions.

Our first goal is to prove that the numerical method does not introduce spurious eigenvalues interspersed among the relevant ones of T_t (namely, around $\mu_t^{(k)}$ for small

k), provided the beam is sufficiently thin. Let us remark that such a spectral pollution could be in principle expected from the fact that T_t has a nontrivial essential spectrum. However, that this is not the case is an immediate consequence of the following theorem, which is essentially identical to Lemma 1 from [18].

Theorem 4.5.1 *Let $F \subset \mathbb{C}$ be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. There exist strictly positive constants h_0, t_0, and C such that, $\forall h < h_0$ and $\forall t < t_0$, there holds $F \cap \mathrm{Sp}(T_{th}) = \emptyset$ and*

$$\|R_z(T_{th})\|_h \leq C \qquad \forall z \in F.$$

Proof. Let F be a closed set such that $F \cap \mathrm{Sp}(T_0) = \emptyset$. As an inmediate consequence of Lemma 4.3.7, we have that for all $w \in H_0^1(\mathrm{I})$, for all $z \in F$, and for all $t < t_0$,

$$\|w\|_{1,\mathrm{I}} \leq C\|(zI - T_t)w\|_{1,\mathrm{I}}.$$

From Lemma 4.4.4 we have for h small enough

$$\|(T_t - T_{th})w_h\|_{1,\mathrm{I}} \leq \frac{1}{2C}\|w_h\|_{1,\mathrm{I}} \quad \forall w_h \in W_h,$$

then, for $w_h \in W_h$ and $z \in F$, we have

$$\|(zI - T_{th})w_h\|_{1,\mathrm{I}} \geq \|(zI - T_t)w_h\|_{1,\mathrm{I}} - \|(T_t - T_{th})w_h\|_{1,\mathrm{I}} \geq \frac{1}{2C}\|w_h\|_{1,\mathrm{I}}.$$

Since W_h is finite dimensional, we deduce that $(zI - T_{th})$ is invertible and, hence, $z \notin \mathrm{Sp}(T_{th})$. Moreover,

$$\|R_z(T_{th})\|_h = \|\,(zI - T_{th})^{-1}\,\|_h \leq 2C \qquad \forall z \in F.$$

The proof is complete. $\square$

We have already proved in Theorem 4.3.4 that the essential spectrum of T_t is confined to the real interval $\left[\frac{t^2P}{\overline{\kappa}}, \frac{t^2P}{\underline{\kappa}}\right]$. The spectrum of T_t outside this interval consists of finite multiplicity isolated eigenvalues of ascent one, which converge to eigenvalues of T_0, as t goes to zero (cf. Theorem 4.3.8).

The eigenvalue of T_t with physical significance is the largest in modulus, $\mu_t^{(1)}$, which corresponds to the critical load that leads to buckling effects. This eigenvalue is typically simple and converges to a simple eigenvalue of T_0, as t tends to zero. Because of this, for simplicity, from now on we restrict our analysis to simple eigenvalues.

Let $\mu_0 \neq 0$ be an eigenvalue of T_0 with multiplicity $m = 1$. Let D be a closed disk centered at μ_0 with boundary Γ such that $0 \notin D$ and $D \cap \mathrm{Sp}(T_0) = \{\mu_0\}$. Let $t_0 > 0$ be small enough, so that for all $t < t_0$:

- D contains only one eigenvalue of T_t, which we already know is simple (cf. Lemma 4.3.6) and
- D does not intersect the real interval $\left[\frac{t^2 P}{\overline{\kappa}}, \frac{t^2 P}{\underline{\kappa}}\right]$, which contains the essential spectrum of T_t.

According to Theorem 4.5.1 there exist $t_0 > 0$ and $h_0 > 0$ such that $\forall t < t_0$ and $\forall h < h_0$, $\Gamma \subset \rho(T_{th})$. Moreover, proceeding as in [18, Section 2], from properties P1 and P2 it follows that, for h small enough, T_{th} has exactly one eigenvalue $\mu_{th} \in D$. In principle, the theory in [19] could be used to prove error estimates for the eigenvalues and eigenfunctions of T_{th} to those of T_t as h goes to zero. However, proceeding in this way, we would not be able to prove that the constant in the resulting error estimates are independent of t and, consequently, that the proposed method is locking-free. Thus, our goal will be to prove that μ_{th} converges to μ_t as h goes to zero, with $t < t_0$ fixed, and to provide the corresponding error estimates for eigenvalues and eigenfunctions. With this aim, we will modify accordingly the theory from [19].

Let $\Pi_h : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I})$ be the standard elliptic projector with range W_h defined by

$$\int_{\mathrm{I}} (\Pi_h u - u)' v_h' = 0 \quad \forall v_h \in W_h.$$

Notice that Π_h is bounded uniformly on h (namely, $\|\Pi_h u\|_{1,\mathrm{I}} \leq \|u\|_{1,\mathrm{I}}$) and the following classical error estimate holds true

$$\|\Pi_h u - u\|_{1,\mathrm{I}} \leq Ch \left(\sum_{i=1}^{n} \|u''\|_{0,S_i}^2 \right)^{1/2} \qquad \forall u \in H_0^1(\mathrm{I}) : u|_{S_i} \in H^2(S_i),\ i = 1, \dots, n. \quad (4.5.1)$$

Let us define

$$B_{th} := T_{th} \Pi_h : H_0^1(\mathrm{I}) \to W_h.$$

It is clear that T_{th} and B_{th} have the same eigenvalues and corresponding eigenfunctions.

Let $E_t : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I})$ be the spectral projector of T_t relative to the isolated eigenvalue μ_t. Let $F_{th} : H_0^1(\mathrm{I}) \to H_0^1(\mathrm{I})$ be the spectral projector of B_{th} relative to its eigenvalues μ_{th}.

Lemma 4.5.2 *There exist strictly positive constants h_0, t_0 and C such that*

$$\|R_z(B_{th})\| \leq C \quad \forall h < h_0, \quad \forall t < t_0, \quad \forall z \in \Gamma.$$

Proof. It is identical to that of Lemma 5.2 from [33]. □

Consequently, for h and t small enough, the spectral projectors F_{th} are bounded uniformly in h and t.

Lemma 4.5.3 *There exist stricly positive constants h_0, t_1 and C such that $\forall h < h_0$ and $\forall t < t_1$,*

$$\|(E_t - F_{th})|_{E_t(H_0^1(\mathrm{I}))}\| \leq C\|(T_t - B_{th})|_{E_t(H_0^1(\mathrm{I}))}\| \leq Ch.$$

Proof. The proof of the first inequality follows from the same arguments of Lemma 3 from [19], and Lemmas 4.3.7 and 4.5.2. For the other inequality, let $w \in E_t(H_0^1(\mathrm{I}))$. We have

$$\begin{aligned}
\|(T_t - B_{th})w\|_{1,\mathrm{I}} &\leq \|(T_t - T_t\Pi_h)w\|_{1,\mathrm{I}} + \|(T_t\Pi_h - B_{th})w\|_{1,\mathrm{I}} \\
&\leq \|T_t\|\|(I - \Pi_h)w\|_{1,\mathrm{I}} + \|T_t - T_{th}\|_h\|\Pi_h w\|_{1,\mathrm{I}} \\
&\leq Ch\left[\left(\sum_{i=1}^n \|w''\|_{0,S_i}^2\right)^{1/2} + \|w\|_{1,\mathrm{I}}\right] \\
&\leq Ch\|w\|_{1,\mathrm{I}},
\end{aligned}$$

where we have used Lemma 4.4.4, (4.5.1) and (4.3.9). □

Now, we are in position to prove an optimal order error estimate for the eigenfunctions. We recall the definition of the gap $\hat{\delta}$ between two closed subspaces Y and Z of $H_0^1(\mathrm{I})$, let

$$\delta(Y, Z) := \sup_{\substack{y \in Y \\ \|y\|_{1,\mathrm{I}}=1}} \left(\inf_{z \in Z} \|y - z\|_{1,\mathrm{I}}\right)$$

and

$$\hat{\delta}(Y,Z) := \max\{\delta(Y,Z),\delta(Z,Y)\}.$$

Theorem 4.5.4 *There exist strictly positive constants h_0, t_1 and C such that, for $h < h_0$ and $t < t_1$,*

$$\hat{\delta}\left(F_{th}(H_0^1(\mathrm{I})),E_t(H_0^1(\mathrm{I}))\right) \leq Ch.$$

Proof. The proof follows by arguing exactly as in the proof of Theorem 1 in [19], and using Lemma 4.5.3. □

Our final goal is to obtain an error estimate for the approximate eigenvalues. First, by repeating the same steps as in the proof of Lemma 5.6 from [33] we are able to prove the following preliminary estimate.

Lemma 4.5.5 *There exist strictly positive constants h_0, t_1 and C such that, for $h < h_0$ and $t < t_1$,*

$$|\mu_t - \mu_{th}| \leq Ch.$$

The error estimates for the eigenvalues $\mu_t \neq 0$ of T_t and μ_{th} of T_{th} yield analogous estimates for the eigenvalues $\lambda = 1/\mu_t$ and $\lambda_h = 1/\mu_{th}$. However, the order of convergence in Lemma 4.5.5 is not optimal. Our next goal is improve this order. Let w_h, β_h and γ_h be such that $(\lambda_h,w_h,\beta_h,\gamma_h)$ is a solution of Problem 4.4.1 with $\|w_h\|_{1,\mathrm{I}} = 1$. According to Theorem 4.5.4, there exists a solution (λ,w,β,γ) of Problem 4.2.1 with $\|w\|_{1,\mathrm{I}} = 1$ such that $\|w-w_h\|_{1,\mathrm{I}} \leq Ch$. The following lemma, will be used to prove a double order of convergence for the corresponding eigenvalues.

Lemma 4.5.6 *Let (λ,w,β,γ) be a solution of Problem 4.2.1 and $(\lambda_h,w_h,\beta_h,\gamma_h)$ be a solution of Problem 4.4.1 with $\|w\|_{1,\mathrm{I}} = 1$, $\|w_h\|_{1,\mathrm{I}} = 1$ and such that*

$$\|w-w_h\|_{1,\mathrm{I}} \leq Ch. \tag{4.5.2}$$

Then, for h and t small enough, there holds

$$\|\beta-\beta_h\|_{1,\mathrm{I}} + \|\gamma-\gamma_h\|_{0,\mathrm{I}} \leq Ch.$$

Proof. Let $(\hat{w}, \hat{\beta}) \in V$ be the solution of the auxiliary problem

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}\hat{\beta}'\eta'\,dx + \int_{\mathrm{I}} \hat{\gamma}(\eta - v')\,dx = \lambda_h \int_{\mathrm{I}} P w_h' v'\,dx \quad \forall(\eta, v) \in V, \\ \hat{\gamma} = \dfrac{\kappa}{t^2}(\hat{\beta} - \hat{w}'). \end{cases} \tag{4.5.3}$$

Notice that (4.4.1) can be seen as a discretization of the problem above. The arguments in the proof of Lemma 4.4.4 can be repeated, using (4.4.11) and (4.4.12) with $f_h = \lambda_h w_h$, to show that the solutions of (4.5.3) and (4.4.1) satisfy

$$\|\hat{\beta} - \beta_h\|_{1,\mathrm{I}} + \|\hat{\gamma} - \gamma_h\|_{0,\mathrm{I}} \le Ch\lambda_h\|w_h\|_{1,\mathrm{I}} \le Ch\lambda, \tag{4.5.4}$$

the last inequality because $\lambda_h \to \lambda$ as a consequence of Lemma 4.5.5.

On the other hand, using (4.2.5) and (4.5.3), we have

$$\begin{cases} \displaystyle\int_{\mathrm{I}} \mathbb{E}(\beta' - \hat{\beta}')\eta'\,dx + \int_{\mathrm{I}} (\gamma - \hat{\gamma})(\eta - v')\,dx = \int_{\mathrm{I}} P(\lambda w' - \lambda_h w_h')v'\,dx \quad \forall(\eta, v) \in V, \\ \gamma - \hat{\gamma} = \dfrac{\kappa}{t^2}((\beta - \hat{\beta}) - (w' - \hat{w}')). \end{cases}$$

Now, from the estimate (4.2.10) applied to the problem above, we obtain

$$\begin{aligned} \|\beta - \hat{\beta}\|_{1,\mathrm{I}} + \|\gamma - \hat{\gamma}\|_{0,\mathrm{I}} &\le C\|\lambda w - \lambda_h w_h\|_{1,\mathrm{I}} \\ &\le C(\lambda\|w - w_h\|_{1,\mathrm{I}} + |\lambda - \lambda_h|\|w_h\|_{1,\mathrm{I}}). \end{aligned}$$

Therefore, using Lemma 4.5.5 and (4.5.2), we have

$$\|\beta - \hat{\beta}\|_{1,\mathrm{I}} + \|\gamma - \hat{\gamma}\|_{0,\mathrm{I}} \le Ch. \tag{4.5.5}$$

Hence, the result follows from triangular inequality and the estimates (4.5.4) and (4.5.5). $\square$

Now we are in a position to prove a double order of convergence for the eigenvalues.

Theorem 4.5.7 *There exist strictly positive constants h_0, t_1 and C such that, for $h < h_0$ and $t < t_1$,*

$$|\lambda - \lambda_h| \le Ch^2.$$

Proof. We adapt to our case a standard argument for eigenvalue problems (cf. [6, Lemma 9.1]). Let $(\lambda, \beta, w, \gamma)$ and $(\lambda_h, \beta_h, w_h, \gamma_h)$ be as in Lemma 4.5.6. We consider the following bilinear forms defined by

$$A((w,\beta,\gamma),(v,\eta,s)) := \int_{\mathrm{I}} \mathbb{E}\beta'\eta'\,dx + \int_{\mathrm{I}} \gamma(\eta - v')\,dx + \int_{\mathrm{I}} s(\beta - w')\,dx - t^2 \int_{\mathrm{I}} \frac{\gamma s}{\kappa}\,dx$$

$$B((w,\beta,\gamma),(v,\eta,s)) := \int_{\mathrm{I}} Pw'v'\,dx.$$

Using this notation, Problems 4.2.1 and 4.4.1 can be respectively written as follows:

$$A((w,\beta,\gamma),(v,\eta,s)) = \lambda B((w,\beta,\gamma),(v,\eta,s)),$$

$$A((w_h,\beta_h,\gamma_h),(v_h,\eta_h,s_h)) = \lambda_h B((w_h,\beta_h,\gamma_h),(v_h,\eta_h,s_h)).$$

Defining $U := (w,\beta,\gamma)$ and $U_h := (w_h,\beta_h,\gamma_h)$, it is straightforward to show that

$$(\lambda_h - \lambda)B(U_h,U_h) = A(U - U_h, U - U_h) - \lambda B(U - U_h, U - U_h).$$

Therefore, using that $B(U_h,U_h) = \int_{\mathrm{I}} P|w_h'|^2\,dx \neq 0$ (cf. Remark 4.4.2) and Lemma 4.5.6, we obtain

$$|\lambda - \lambda_h| \leq Ch^2.$$

Thus we end the proof. □

4.6 Numerical results.

We report in this section the results of some numerical tests computed with a MATLAB code implementing the finite element method described above.

In all cases we consider a clamped beam subjected to a compresssive load $P = 1$ and uniform meshes of N elements, with different values of N. We have taken the following physical parameters (typical of steel):

- Elastic moduli: $E = 30 \times 10^6$,

- Poisson coefficient: $\nu = 0.25$,
- Correction factor: $k_c = 5/6$.

4.6.1 Test 1: Uniform beam with analytical solution.

The aim of this first test is to validate the computer code by solving a problem with known analytical solution. With this purpose, we will compare the exact buckling coefficients of a beam as that shown in Figure 4.1 (undeformed beam) with those computed with the method analized in this paper.

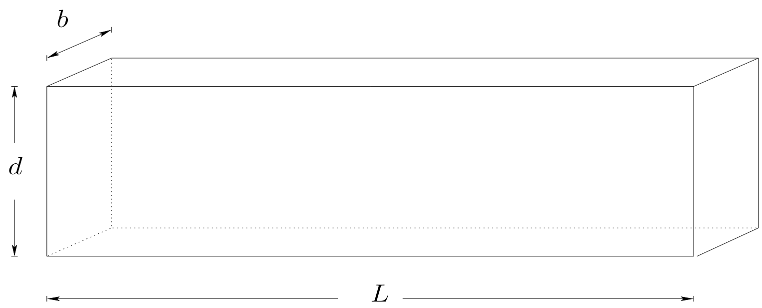


Figure 4.1: Undeformed uniform beam.

Let b and d as shown in Figure 4.1. For this kind of beam, we have that $\mathbb{I} = \frac{bd^3}{12}$ and $A = bd$ are constant. In this case (4.2.1) is equivalent to find $\beta, w \in H_0^1(\mathrm{I})$ solution of

$$\begin{cases} -E\mathbb{I}\beta'' + GAk_c(\beta - w') = 0, \\ GAk_c(\beta - w')' = -\lambda_{\mathbf{c}} w''. \end{cases} \tag{4.6.1}$$

The problem above leads to the following non-standard boundary value problem:

$$\begin{cases} \beta''' + \omega^2 \beta' = 0, \\ \beta(0) = \beta(L) = 0, \\ -E\mathbb{I}(\beta'(L) - \beta'(0)) + GAk_c \displaystyle\int_0^L \beta \, dx = 0, \end{cases} \tag{4.6.2}$$

where

$$\omega^2 := \frac{\lambda_{\mathbf{c}} GAk_c}{E\mathbb{I}(GAk_c - \lambda_{\mathbf{c}})}. \tag{4.6.3}$$

Once β is determined, w can be obtained by solving

$$\begin{cases} w'' = \left(\dfrac{GAk_c}{GAk_c - \lambda_{\mathbf{c}}}\right)\beta', \\ w(0) = w(L) = 0. \end{cases}$$

By imposing the boundary conditions on the general solution of the differential equation in $(4.6.2)_1$, we obtain that ω has to be the solution of the following nonlinear equation:

$$L\sin(L\omega) - 2\left(\frac{EI}{GAk_c}\omega + \frac{1}{\omega}\right)(1 - \cos(L\omega)) = 0.$$

We have solved numerically this equation and used (4.6.3), to obtain the exact values of $\lambda_{\mathbf{c}}$.

In Table 4.1 we report the four lowest eigenvalues ($\lambda_{\mathbf{c}}^{(i)},\ i = 1,2,3,4$) computed by our method with four diferent meshes ($N = 10, 20, 30, 40$). We have taken a total length $L = 100$, and a square cross section of side-length $b = d = 5$. The table includes computed orders of convergence, as well as more accurate values extrapolated by means of a least-squares fitting. Furthermore, the last column shows the exact eigenvalues.

Table 4.1: Lowest eigenvalue $\lambda_{\mathbf{c}}^{(i)}$ (multiplied by 10^{-7}) of a uniform beam.

	$N = 10$	$N = 20$	$N = 30$	$N = 40$	Order	Extrapolated	Exact
$\lambda_{\mathbf{c}}^{(1)}$	0.642863	0.611794	0.606318	0.604421	2.08	0.602162	0.601997
$\lambda_{\mathbf{c}}^{(2)}$	1.375703	1.236684	1.213566	1.205649	2.16	1.196744	1.195600
$\lambda_{\mathbf{c}}^{(3)}$	2.914531	2.387288	2.306884	2.279802	2.30	2.253185	2.245754
$\lambda_{\mathbf{c}}^{(4)}$	4.801022	3.536107	3.361391	3.303593	2.45	3.253759	3.231672

It can be seen from Table 4.1 that the computed buckling coefficients converge to the exact ones with an optimal quadratic order.

We show in Figure 4.2 the deformed transversal section of the beam for the first four buckling modes.

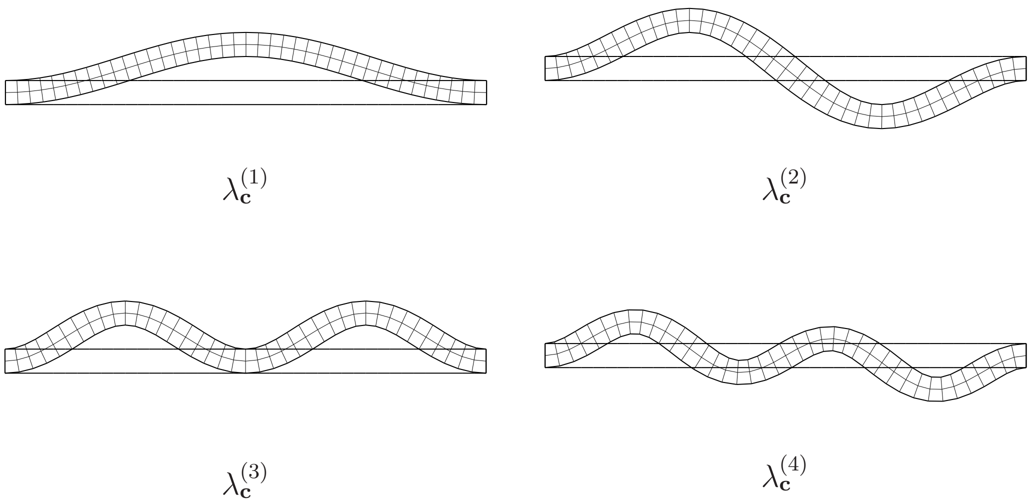


Figure 4.2: Uniform beam; four lowest buckling modes.

4.6.2 Test 2: Rigidly joined beams.

The aim of this test is to apply the method analized in this paper to a beam of rectangular section with area varying along its axis. With this purpose, we consider a composed beam formed by two rigidly joined beams as shown in Figure 4.3. Moreover, we will assess the performance of the method as the thickness d approaches to zero.

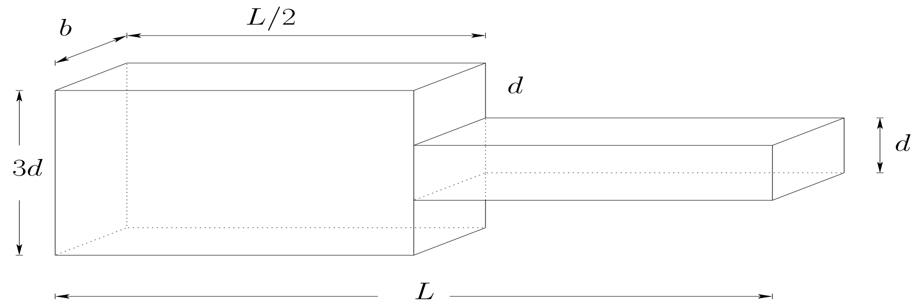


Figure 4.3: Rigidly joined beams.

Let b and d be as shown in Figure 4.3. We have taken $L = 100$ and $b = 3$, so that the area of the cross section and the moment of inertia are:

$$A(x) = \begin{cases} 9d, & 0 \le x \le 50, \\ 3d, & 50 < x \le 100. \end{cases} \qquad \mathbb{I}(x) = \begin{cases} \frac{27d^3}{4}, & 0 \le x \le 50, \\ \frac{d^3}{4}, & 50 < x \le 100. \end{cases}$$

We have taken meshes with an even number of elements N, so that the point $x = L/2$ is always a node as required by the theory.

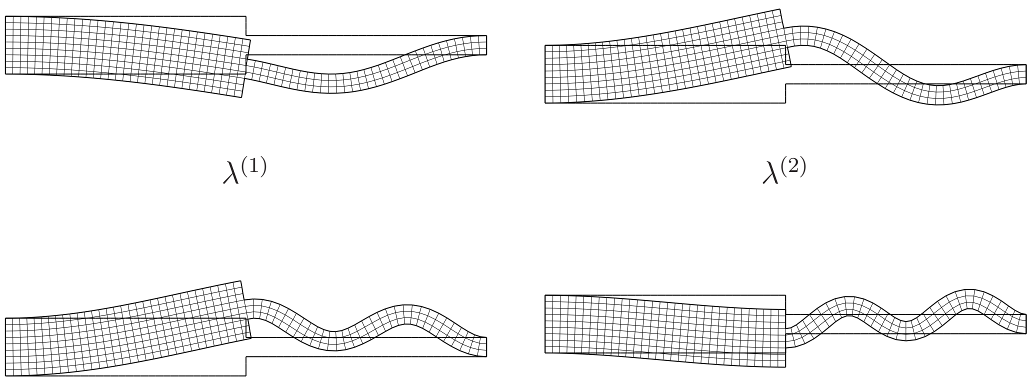
In Table 4.2 we present the results for the lowest scaled buckling coefficient $\lambda^{(1)} = \lambda_{\mathbf{c}}^{(1)}/t^3$, with varying thickness d and different meshes. According to (4.2.2), in this case we take $t^2 = \frac{5d^2}{8L^2}$, so that $\lambda^{(1)}$ has a limit as d goes to zero. Again, we have computed the orders of convergence, and more accurate values obtained by a least-squares fitting. Furthermore, in the last row we also report for each mesh the limit values as d goes to zero obtained by extrapolation.

Table 4.2: Computed lowest scaled buckling coefficients $\lambda^{(1)}$ (multiplied by 10^{-10}) of a composed beam with varing thickness d.

Thickness	$N = 8$	$N = 16$	$N = 32$	$N = 64$	Order	Extrap.
$d = 4$	22.667732	19.570170	18.789287	18.594783	1.99	18.527905
$d = 0.4$	23.702364	20.438746	19.611856	19.405572	1.98	19.332297
$d = 0.04$	23.713096	20.447761	19.620395	19.413989	1.98	19.340691
$d = 0.004$	23.713181	20.447850	19.620485	19.414041	1.98	19.340765
$d = 0$ (Extrap.)	23.713235	20.447881	19.620510	19.414090	1.98	19.340799

These result show that the our method does not deteriorate when the thickness parameter becomes small, i.e., the method is locking free.

We show in Figure 4.4 the deformed transversal section of the beam for the first four buckling modes.



$\lambda^{(1)}$ $\lambda^{(2)}$

$\lambda^{(3)}$ $\lambda^{(4)}$

Figure 4.4: Rigidly joined beams; four lowest buckling modes.

4.6.3 Test 3: Beam with a smoothly varying cross-section.

The aim of this final test is to apply the method analized in this paper to a beam of rectangular section with area and moment of inertia defined by a smooth function along its axis. With this purpose, we consider a beam as that shown in Figure 4.5. We will assess again the performance of the method as the thickness d approaches to zero.

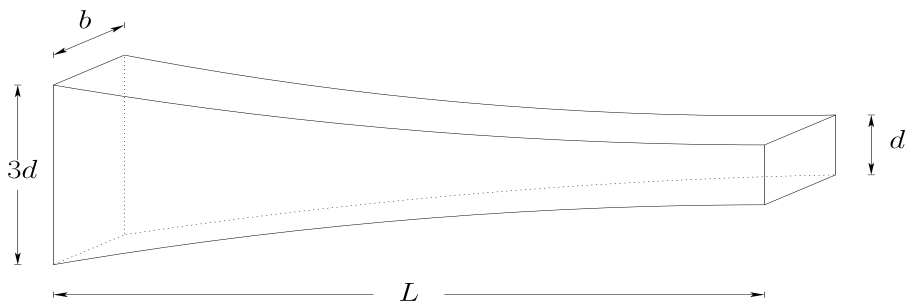


Figure 4.5: Smoothly varying cross-section beam.

Let b and d be as shown in Figure 4.5. We have taken $L = 100$, $b = 3$ and the equation of the top and botton surfaces of the beam are

$$z = \pm \frac{150d}{2x + 100}, \quad 0 \leq x \leq 100,$$

so that the area of the cross section and the moment of inertia are defined as follows:

$$A(x) = \frac{900d}{2x + 100}, \qquad \mathbb{I}(x) = \frac{1}{4}\left(\frac{300d}{2x + 100}\right)^3, \qquad 0 \leq x \leq 100.$$

In Table 4.3 we report the results for the lowest scaled buckling coefficient $\lambda^{(1)} = \lambda_{\mathbf{c}}^{(1)}/t^3$, with varying thickness d and different meshes. According to (4.2.2), in this case we take $t^2 = \frac{75d^2}{2L^2(L+50)}$, so that $\lambda^{(1)}$ has a limit as d goes to zero. Again, we have computed the orders of convergence, and more accurate values obtained by a least-squares fitting. Furthermore, in the last row we also report for each mesh the limit values as d goes to zero obtained by extrapolation.

Table 4.3: Computed lowest scaled buckling coefficients $\lambda^{(1)}$ (multiplied by 10^{-10}) of a smoothly varying cross-section beam with varing thickness d.

Thickness	$N=10$	$N=20$	$N=30$	$N=40$	Order	Extrap.
$d=4$	83.524954	77.384182	76.297239	75.920330	2.07	75.465288
$d=0.4$	87.106303	80.498122	79.331886	78.927724	2.06	78.436615
$d=0.04$	87.143633	80.530498	79.363423	78.958974	2.07	78.467482
$d=0.004$	87.143970	80.530779	79.363716	78.959322	2.07	78.467788
$d=0$ (Extrap.)	87.144068	80.530899	79.363824	78.959393	2.07	78.467886

We show in Figure 4.6 the deformed transversal section of the beam for the first four buckling modes.

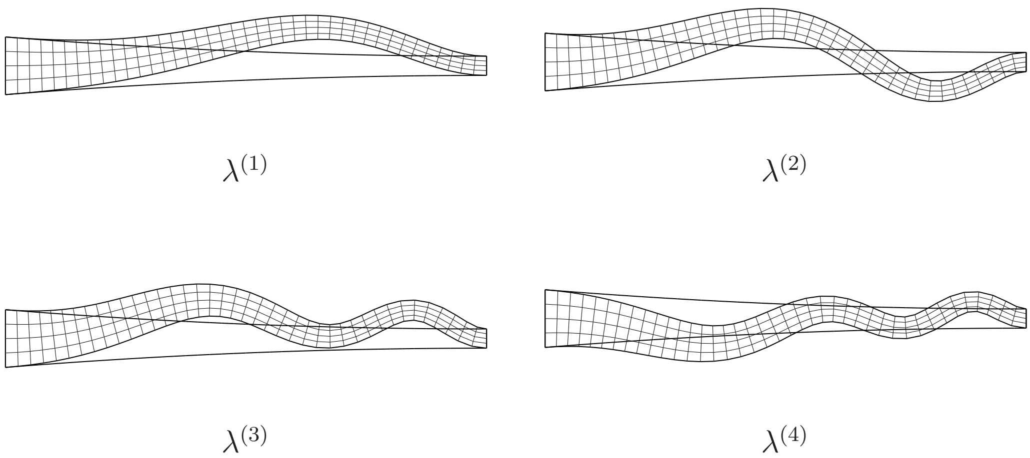


Figure 4.6: Smoothly varying cross-section beam; four lowest buckling modes.

Chapter 5

Conclusiones y trabajo futuro

En este capítulo se presenta un resumen de los principales aportes de esta tesis y una descripción del trabajo futuro a desarrollar.

5.1 Conclusiones

1. Se estudió el problema de pandeo y el problema de vibraciones de una placa poligonal empotrada no necesariamente convexa modelada por las ecuaciones de Kirchhoff-Love. Usando elementos finitos lineales a trozos y continuos para todas las variables de la formulación en momentos de ambos problemas, mediante la teoría espectral para operadores compactos, se obtuvieron órdenes óptimos de convergencia $\mathcal{O}(h)$ para los desplazamientos transversales de los modos de pandeo y de vibración y un orden $\mathcal{O}(h^t)$ para las variables secundarias del modelo, donde $t \in (\frac{1}{2}, 1]$ depende de la regularidad Sobolev del dominio para los problemas bilaplaciano y de Laplace (si el dominio es convexo entonces $t = 1$). Además, se obtuvo un orden $\mathcal{O}(h^{2t})$ para la aproximación de los coeficientes de pandeo y para las frecuencias de vibración. Se presentaron resultados numéricos que confirman los resultados teóricos obtenidos.

2. Se estudió el problema de pandeo de una placa elástica modelada por las ecuaciones de Reissner-Mindlin. Se dio una caracterización espectral completa de este problema. Adaptando la teoría espectral clásica para operadores no compactos desarrollada por Descloux, Nassif y Rappaz, se obtuvieron órdenes óptimos de convergencia para las autofunciones y un doble orden de convergencia para los autovalores. Para la aproximación por elementos finitos se usaron elementos DL3. Se demostró que el método propuesto es libre de bloqueo y se presentaron resultados numéricos que confirman los resultados teóricos obtenidos. Cabe mencionar que estos resultados son los primeros que incluyen el análisis numérico de un método de elementos finitos para el problema de pandeo de placas Reissner-Mindlin.

3. Se estudió un método de elementos finitos para el problema de pandeo de una viga no homogénea modelada por las ecuaciones de Timoshenko. Se demostraron órdenes óptimos de convergencia para las autofunciones (desplazamiento, rotaciones y esfuerzos de corte) y un orden doble para los autovalores (coeficientes de pandeo). Se demostró que el método es libre de bloqueo. Por último incluimos también resultados numéricos que muestran el buen comportamiento del método. Cabe mencionar que son muy pocos los artículos que incluyen el análisis numérico de vigas no homogéneas.

5.2 Trabajo futuro

1. Extender los resultados obtenidos en los Capítulos 2, 3 y 4 considerando condiciones de contorno más generales.

2. Estudiar otros métodos de elementos finitos para el problema de pandeo y vibraciones de placas Kirchhoff y Reissner-Mindlin.

3. Estudiar métodos de elementos finitos para el problema de pandeo y vibraciones de otro tipo de estructuras delgadas.

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