UNIVERSIDAD DE CONCEPCIÓN



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PREPRINT 2026-37

SERIE DE PRE-PUBLICACIONES

UNIFIED APPROACH FOR THE CONTINUOUS SOLVABILITY ANALYSIS OF NONLINEAR VARIATIONAL PROBLEMS IN BANACH SPACES

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Abstract. We develop a unified framework for analyzing the solvability of nonlinear saddle point problems and their perturbed variants, arising from variational formulations in Banach spaces of various models in continuum mechanics. We first extend the classical Babuška–Brezzi theorem to the nonlinear setting, establishing necessary and sufficient conditions for the well-posedness of a general nonlinear saddle point problem. Drawing on a result from nonlinear functional analysis, we then apply this extension to a class of problems in which the nonlinear operator satisfies a strict monotonicity condition. Finally, we address the unique solvability of perturbed and nested nonlinear saddle point problems.

Keywords: nonlinear Babuška–Brezzi theory; perturbed nonlinear Babuška–Brezzi theory; nested nonlinear Babuška–Brezzi theory; well-posedness; a priori estimates

MSC 2020: 47J30, 47H05, 46B10, 46B25, 65J15

1. Introduction

In the mathematical modeling and numerical simulation of fluid flow in porous media and free-flow regions, it is common to encounter saddle-point and perturbed saddle-point problems involving nonlinear forms. This is particularly the case for mixed formulations of the Navier–Stokes and Darcy–Forchheimer equations, as well as related coupled problems, which have been studied, for instance, in [1, 2, 3, 4, 5]. To the best of the authors' knowledge, the first contribution to the well-posedness analysis of problems of this kind is due to Scheuer [7]. More precisely, that work addresses saddle-point systems in which the nonlinear form induces an operator $A : X \to X'$, with X being a Banach space, satisfying, among other assumptions, a *strict monotonicity* condition of the form

$$[A(u) - A(v),\, u - v] \geq \alpha \|u - v\|_X^p \tag{1.1}$$

for all u, v belonging to a suitably chosen affine subspace of X, where $\alpha > 0$ and $p \geq 2$. Several works have subsequently extended this result to different settings arising in computational fluid dynamics. For instance, in [3], the authors established a generalization in which X is the Cartesian product of two Banach spaces X_1 and X_2, and the term on the right-hand side of (1.1) is replaced by the sum of the norms of the X_1 and X_2 components

This research was supported by ANID-Chile through the projects Centro de Modelamiento Matemático (FB210005) and Fondecyt 1250937; by Grupo de Investigación en Análisis Numérico y Cálculo Científico (GIANuC^2), Universidad Católica de la Santísima Concepción; and by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción.

of $u - v$, raised to the powers $p_1 \geq 2$ and $p_2 \geq 2$, respectively. On the other hand, the well-posedness of a family of perturbed nonlinear saddle-point problems was analyzed in [1] under a *strict monotonicity* condition imposed on the entire Banach space. This abstract result was then applied to establish the well-posedness of a nested nonlinear saddle-point problem arising from a mixed formulation of the coupled Stokes and Darcy–Forchheimer equations, in which the inner problem is both perturbed and nonlinear. We also mention [4], where the corresponding theory was extended to a more general class of nested nonlinear saddle-point problems. Although the analysis in [4] does not constitute a generalization of Scheuer's work, both approaches share several strategies and tools from nonlinear functional analysis.

Motivated by the preceding discussion, the purpose of the present paper is to develop a unified framework for the analysis of nonlinear saddle-point problems that encompasses the existing results from the aforementioned works. To this end, we first establish a nonlinear counterpart of the classical Babuška–Brezzi theory, providing necessary and sufficient conditions for the existence and uniqueness of a solution to a nonlinear saddle-point problem. We then extend the strict monotonicity condition to the case where X is the Cartesian product of finitely many Banach spaces, and employ this generalization to analyze both nonlinear saddle-point problems and their perturbed counterparts. By combining these results, we also establish the unique solvability of a nested nonlinear saddle-point problem. Furthermore, our analysis allows us to dispense with the uniform convexity assumption imposed in all the aforementioned works.

The remainder of this paper is organized as follows. In the next three sections, we establish the well-posedness of three classes of nonlinear saddle-point problems. We begin in Section 2 with a general nonlinear saddle-point problem, for which we derive necessary and sufficient conditions for well-posedness, and then apply these results to a particular case frequently encountered in the literature. Next, in Section 3, we analyze a class of perturbed nonlinear saddle-point problems. Finally, in Section 4, we employ our nonlinear counterpart of the classical Babuška–Brezzi theory to establish the unique solvability of a double saddle-point problem. The corresponding Galerkin schemes and their associated a priori error estimates will be addressed in a forthcoming companion paper.

2. Nonlinear Babuška–Brezzi theory

In what follows, given a Banach space X and its (topological) dual space X', we denote by $[\cdot, \cdot]$ the duality pairing between them. We first consider the nonlinear saddle-point problem of finding $(\sigma, u) \in X \times Y$ such that

$$\begin{aligned} [A(\sigma),\, \tau] + [B(\tau),\, u] &= [F,\, \tau] \qquad \forall\, \tau \in X\,, \\ [B(\sigma),\, v] \qquad\qquad &= [G,\, v] \qquad \forall\, v \in Y\,, \end{aligned} \tag{2.1}$$

where X and Y are Banach spaces, $B : X \to Y'$ is a bounded linear operator, $A : X \to X'$ is a nonlinear operator, and $F \in X'$ and $G \in Y'$ are given functionals. Our aim is to extend the classical Babuška–Brezzi theorem (see, for instance, [6, Theorem 2.34]) to the present setting, in which A is not necessarily a bounded linear operator. The following results provide necessary and sufficient conditions for the well-posedness of this problem. To this end, we let $V := N(B)$ denote the null space of B.

We first recall that if Y is reflexive and B satisfies the inf-sup condition, that is, there exists a constant $\beta > 0$ such that

$$\sup_{\substack{\tau \in X \\ \tau \neq 0}} \frac{[B(\tau),\, v]}{\|\tau\|_X} \geq \beta \|v\|_Y \qquad \forall\, v \in Y, \tag{2.2}$$

then there exists $\bar{\sigma} \in X$ such that (see [6, Lemma A.40])

$$B(\bar{\sigma}) = G \qquad \text{and} \qquad \beta \|\bar{\sigma}\|_X \leq \|G\|_{Y'}\,. \tag{2.3}$$

This allows us to introduce the following reduced problem: Find $\sigma_0 \in V$ such that

$$[A(\sigma_0 + \bar{\sigma}),\, \tau] = [F,\, \tau] \qquad \forall\, \tau \in V\,. \tag{2.4}$$

As in the linear case, there is a close connection between the solvability of the reduced problem and the well-posedness of the original saddle-point problem.

Lemma 2.1. *Suppose that Y is reflexive and that the inf-sup condition* (2.2) *holds. If $\sigma_0 \in V$ is a solution of the reduced problem* (2.4)*, then there exists a unique $u \in Y$ such that $(\sigma, u) \in X \times Y$, with $\sigma = \sigma_0 + \bar{\sigma}$ and $\bar{\sigma}$ satisfying* (2.3)*, is a solution of* (2.1)*. Moreover, there holds: $\beta \|u\|_Y \leq \|F\|_{X'} + \|A(\sigma)\|_{X'}$.*

Proof. We first observe that σ_0 is a solution of (2.4) if and only if $F - A(\sigma_0 + \bar{\sigma})$ belongs to V°. On the other hand, the inf-sup condition implies that $B' : Y'' \to V^\circ$ is an isomorphism (cf. [6, Lemmas A.30 and A.38]). Hence, since Y is reflexive, there exists a unique $u \in Y$ such that

$$B'(J_Y(u)) = F - A(\sigma_0 + \bar{\sigma}) \qquad \text{and} \qquad \beta \|u\|_Y \leq \|F\|_{X'} + \|A(\sigma_0 + \bar{\sigma})\|_{X'}\,,$$

where $J_Y : Y \to Y''$ denotes the canonical isometric isomorphism characterizing the reflexivity of Y. The first identity can be equivalently written as

$$[A(\sigma_0 + \bar{\sigma}),\, \tau] + [B(\tau),\, u] = [F,\, \tau] \qquad \forall\, \tau \in X.$$

Finally, setting $\sigma := \sigma_0 + \bar{\sigma}$, we deduce from the preceding identity, together with the fact that $B(\sigma) = B(\bar{\sigma}) = G$, that the pair (σ, u) is a solution of (2.1), which completes the proof. □

The equivalence between the reduced and the original problem is completely described by the next result.

Theorem 2.2. *Suppose that Y is reflexive. The problem* (2.1) *has a unique solution for all $F \in X'$ and $G \in Y'$ if and only if*

i) *there exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\tau \in X \\ \tau \neq 0}} \frac{[B(\tau),\, v]}{\|\tau\|_X} \geq \beta \|v\|_Y \qquad \forall\, v \in Y,$$

ii) *given $\bar{\sigma} \in X$ and $F_0 \in V'$, there exists a unique $\eta \in V$ such that*

$$[A(\eta + \bar{\sigma}), \tau] = [F_0, \tau] \qquad \forall\, \tau \in V\,.$$

Proof. We first assume that i) and ii) hold, and let $F \in X'$ and $G \in Y'$ be arbitrary. Then, Lemma 2.1 immediately yields the existence of a solution to (2.1). To establish uniqueness, suppose that $(\sigma_1, u_1), (\sigma_2, u_2) \in X \times Y$ are two solutions of (2.1). Subtracting the corresponding equations, we obtain

$$\begin{aligned} [A(\sigma_2) - A(\sigma_1), \tau] + [B(\tau), u_2 - u_1] &= 0 \qquad \forall\, \tau \in X, \\ [B(\sigma_2 - \sigma_1), v] &= 0 \qquad \forall\, v \in Y. \end{aligned}$$

The second equation implies that $\sigma_2 - \sigma_1 \in V$. In turn, taking $\tau \in V$ in the first equation yields

$$[A(\sigma_2) - A(\sigma_1), \tau] = 0 \qquad \forall\, \tau \in V,$$

or, equivalently,

$$A(\sigma_1 + (\sigma_2 - \sigma_1)) = A(\sigma_2) = A(\sigma_1) \quad \text{in } V'.$$

Hence, since $\sigma_2 - \sigma_1 \in V$ and the operator $A(\cdot + \sigma_1) : V \to V'$ is injective by ii), we conclude that $\sigma_2 - \sigma_1 = 0$. Consequently, the first equation reduces to

$$[B(\tau), u_2 - u_1] = 0 \qquad \forall\, \tau \in X,$$

which, together with the inf-sup condition i), implies that $u_1 = u_2$. Thus, $(\sigma_1, u_1) = (\sigma_2, u_2)$, proving the uniqueness of the solution. Conversely, we now show that conditions i) and ii) are necessary. To this end, suppose that (2.1) admits a unique solution for every $F \in X'$ and $G \in Y'$. It follows immediately from the second equation of (2.1) that B is surjective. Consequently, B' is injective and has closed range, which implies the inf-sup condition i) (cf. [6, Lemma A.40]). Next, to establish ii), let $\bar{\sigma} \in X$ and $F_0 \in V'$ be arbitrary. By the Hahn–Banach extension theorem (cf. [6, Theorem A.16]), there exists $\widetilde{F} \in X'$ such that $\widetilde{F}|_V = F_0$. By assumption, there exists a unique pair $(\sigma, u) \in X \times Y$ satisfying

$$\begin{aligned} [A(\sigma), \tau] + [B(\tau), u] &= [\widetilde{F}, \tau] \qquad \forall\, \tau \in X, \\ [B(\sigma), v] &= [B(\bar{\sigma}), v] \qquad \forall\, v \in Y. \end{aligned}$$

The second equation implies that $\sigma - \bar{\sigma} \in V$, and hence we can write $\sigma = \eta + \bar{\sigma}$ for some $\eta \in V$. Testing the first equation with $\tau \in V$, we obtain

$$[A(\eta + \bar{\sigma}), \tau] = [F_0, \tau] \qquad \forall\, \tau \in V.$$

This proves the existence of a solution to the reduced problem in ii). It remains to establish its uniqueness. To this end, suppose that $\eta_1, \eta_2 \in V$ satisfy

$$[A(\bar{\sigma} + \eta_1) - A(\bar{\sigma} + \eta_2), \tau] = 0 \qquad \forall\, \tau \in V.$$

Equivalently, $A(\bar{\sigma}+\eta_1) - A(\bar{\sigma}+\eta_2) \in V^\circ$. Since B is surjective, we have $V^\circ = R(B')$. Moreover, the reflexivity of Y implies that there exists $\bar{u} \in Y$ such that

$$B'(J_Y(\bar{u})) = A(\bar{\sigma}+\eta_1) - A(\bar{\sigma}+\eta_2).$$

Now, consider the nonlinear saddle-point problem of finding $(\sigma, u) \in X \times Y$ such that

$$\begin{aligned} [A(\sigma),\, \tau] + [B(\tau),\, u] &= [A(\bar{\sigma}+\eta_1),\, \tau] \qquad && \forall\, \tau \in X, \\ [B(\sigma),\, v] &= [B(\bar{\sigma}),\, v] && \forall\, v \in Y. \end{aligned}$$

It is readily seen that both $(\bar{\sigma}+\eta_1, 0)$ and $(\bar{\sigma}+\eta_2, \bar{u})$ are solutions of this problem. Therefore, the assumed uniqueness implies that $\eta_1 = \eta_2$, which proves the injectivity of the operator $A(\cdot + \bar{\sigma}) : V \to V'$. Together with the existence result established above, this yields ii) and completes the proof. □

As discussed previously, it is not difficult to verify that, when A is a bounded linear operator, condition ii) is equivalent to the conditions imposed on A in the classical Babuška–Brezzi theorem. Hence, in this particular case, Theorem 2.2 reduces to the classical linear Babuška–Brezzi theorem. However, in the general nonlinear setting, we are unable to establish the continuous dependence of the solution on the data.

We now aim to provide sufficient conditions ensuring the bijectivity of a nonlinear operator, so that condition ii) of the preceding theorem is satisfied. To this end, given a Banach space X, we recall that a nonlinear operator $A : X \to X'$ is said to be *bounded* if it maps bounded subsets of X into bounded subsets of X', and *monotone* if

$$[A(v) - A(w),\, v - w] \geq 0 \qquad \forall\, v, w \in X.$$

In addition, A is said to be *coercive* if

$$\frac{[A(v),\, v]}{\|v\|_X} \to \infty \quad \text{as} \quad \|v\|_X \to \infty.$$

We now recall the following classical existence result for nonlinear operator equations.

Lemma 2.3. *If $A : X \to X'$ is continuous, monotone, bounded and coercive and X is a separable and reflexive Banach space, then A is surjective.*

Proof. It is a particular case of Corollary 2.2 of [8]. □

As announced, the next lemma lists some properties that are sufficient for condition ii) of Theorem 2.2 to hold that appear frequently in applications (cf. [1], [3] and [4]).

Lemma 2.4. *Let $\mathbf{X}$ be a closed subspace of $X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are separable and reflexive Banach spaces, let $\mathbf{V}$ be a closed linear subspace of $\mathbf{X}$, and let $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ be a nonlinear operator. Assume that*

i) *there exist constants $\gamma > 0$ and $p_1, p_2, \ldots, p_n \geq 2$ such that*

$$\|\mathbf{A}(\boldsymbol{\sigma}) - \mathbf{A}(\boldsymbol{\tau})\|_{\mathbf{X}'} \leq \gamma \sum_{j=1}^{n} \left\{ \|\sigma_j - \tau_j\|_{X_j} + \|\sigma_j - \tau_j\|_{X_j} \left(\|\sigma_j\|_{X_j} + \|\tau_j\|_{X_j} \right)^{p_j - 2} \right\}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)$ in $\mathbf{X}$,

ii) *for each* $\boldsymbol{\zeta} := (\zeta_1, \ldots, \zeta_n) \in \mathbf{X}$, *the operator* $\mathbf{A}(\cdot + \boldsymbol{\zeta}) : \mathbf{V} \to \mathbf{V}'$ *is* strictly monotone, *in the following sense: there exists a constant* $\alpha > 0$, *independent of* $\boldsymbol{\zeta}$, *such that, with the same exponents* $p_1, p_2, \ldots, p_n$ *as in* i), *there holds*

$$[\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\tau} + \boldsymbol{\zeta}), \boldsymbol{\sigma} - \boldsymbol{\tau}] \geq \alpha \sum_{j=1}^{n} \|\sigma_j - \tau_j\|_{X_j}^{p_j}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *and* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)$ *in* $\mathbf{V}$.

Then, for every $\boldsymbol{\zeta} \in \mathbf{X}$ *and* $\mathbf{F} \in \mathbf{X}'$, *there exists a unique* $\boldsymbol{\sigma} \in \mathbf{V}$ *such that*

$$[\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}), \boldsymbol{\tau}] = [\mathbf{F}, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{V}.$$

Moreover, there exists a constant $C > 0$, *depending only on* α, γ, n, *and* $p_1, p_2, \ldots, p_n$, *such that*

$$\|\boldsymbol{\sigma}\|_{\mathbf{X}} \leq C \max_{k \in \{1,2,\ldots,n\}} \left(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \sum_{j=1}^{n} \left\{ \|\zeta_j\|_{X_j} + \|\zeta_j\|_{X_j}^{p_j - 1} \right\} \right)^{\frac{1}{p_k - 1}} . \tag{2.5}$$

Proof. Fix $\boldsymbol{\zeta} \in \mathbf{X}$ as indicated and define $\bar{\mathbf{A}} : \mathbf{V} \to \mathbf{V}'$ by

$$\bar{\mathbf{A}}(\boldsymbol{\sigma}) := \mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}) \qquad \forall\, \boldsymbol{\sigma} \in \mathbf{V}.$$

It follows from ii) that $\bar{\mathbf{A}}$ is strictly monotone and, hence, injective. Furthermore, i) implies that $\bar{\mathbf{A}}$ is continuous. Next, using that $\left\{ \sum_{j=1}^{n} a_j \right\}^p \leq n^{p-1} \sum_{j=1}^{n} a_j^p$, for all $a_1, a_2, \ldots, a_n \geq 0$ and $p \in [1, +\infty)$, we deduce that there exists a constant $c_1 > 0$, depending only on γ and $p_1, p_2, \ldots, p_n$, such that

$$\begin{aligned}
\|\bar{\mathbf{A}}(\boldsymbol{\sigma})\|_{\mathbf{V}'} &\leq \|\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'} + \|\mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'} \\
&\leq \gamma \sum_{j=1}^{n} \left\{ \|\sigma_j\|_{X_j} + \|\sigma_j\|_{X_j} \left(\|\sigma_j + \zeta_j\|_{X_j} + \|\zeta_j\|_{X_j} \right)^{p_j - 2} \right\} + \|\mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'} \\
&\leq c_1 \sum_{j=1}^{n} \left\{ \|\sigma_j\|_{X_j} + \|\sigma_j\|_{X_j}^{p_j - 1} + \|\sigma_j\|_{X_j} \|\zeta_j\|_{X_j}^{p_j - 2} \right\} + \|\mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'}
\end{aligned}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbf{V}$. This shows that $\bar{\mathbf{A}}$ is bounded. On the other hand, for every $\boldsymbol{\sigma} \in \mathbf{V}$, $\boldsymbol{\sigma} \neq 0$, we have

$$\begin{aligned}
\frac{[\bar{\mathbf{A}}(\boldsymbol{\sigma}), \boldsymbol{\sigma}]}{\|\boldsymbol{\sigma}\|_{\mathbf{X}}} &= \frac{[\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\zeta}), \boldsymbol{\sigma}]}{\|\boldsymbol{\sigma}\|_{\mathbf{X}}} + \frac{[\mathbf{A}(\boldsymbol{\zeta}), \boldsymbol{\sigma}]}{\|\boldsymbol{\sigma}\|_{\mathbf{X}}} \\
&\geq \frac{\alpha}{\|\boldsymbol{\sigma}\|_{\mathbf{X}}} \sum_{j=1}^{n} \|\sigma_j\|_{X_j}^{p_j} - \|\mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'}.
\end{aligned}$$

Since $p_j \geq 2$ for all $j \in \{1, 2, \ldots, n\}$, the preceding inequality implies that $\bar{\mathbf{A}}$ is coercive. Consequently, an application of Lemma 2.3 shows that $\bar{\mathbf{A}} = \mathbf{A}(\cdot + \boldsymbol{\zeta})$ is surjective. Together

with its injectivity, this proves that $\bar{\mathbf{A}}$ is bijective. It remains to establish the a priori estimate. Indeed, let $\mathbf{F} \in \mathbf{X}'$ and let $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbf{V}$ be the unique solution of

$$[\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}), \boldsymbol{\tau}] = [\mathbf{F}, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{V}.$$

Then, proceeding as in the proof of the boundedness of $\bar{\mathbf{A}}$, we obtain

$$\begin{aligned}
\alpha \sum_{j=1}^{n} \|\sigma_j\|_{X_j}^{p_j} &\leq [\mathbf{A}(\boldsymbol{\sigma} + \boldsymbol{\zeta}) - \mathbf{A}(\boldsymbol{\zeta}), \boldsymbol{\sigma}] = [\mathbf{F} - \mathbf{A}(\boldsymbol{\zeta}), \boldsymbol{\sigma}] \\
&\leq \Big(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(\boldsymbol{\zeta})\|_{\mathbf{X}'} \Big) \|\boldsymbol{\sigma}\|_{\mathbf{X}} \\
&\leq c_2 \|\boldsymbol{\sigma}\|_{\mathbf{X}} \left(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \sum_{j=1}^{n} \Big\{ \|\zeta_j\|_{X_j} + \|\zeta_j\|_{X_j}^{p_j - 1} \Big\} \right),
\end{aligned}$$

where $c_2 > 0$ depends only on γ and $p_1, p_2, \ldots, p_n$. In particular, choosing $j_0 \in \{1, 2, \ldots, n\}$ such that $\|\sigma_{j_0}\|_{X_{j_0}} = \max\limits_{j \in \{1,2,\ldots,n\}} \|\sigma_j\|_{X_j}$, we deduce that

$$\|\sigma_{j_0}\|_{X_{j_0}}^{p_{j_0}} \leq \frac{c_2}{\alpha} \|\boldsymbol{\sigma}\|_{\mathbf{X}} \left(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \sum_{j=1}^{n} \Big\{ \|\zeta_j\|_{X_j} + \|\zeta_j\|_{X_j}^{p_j - 1} \Big\} \right).$$

It follows (2.5) with a constant $C > 0$ depending only on α, γ, n, and $p_1, p_2, \ldots, p_n$. □

We are now in position to state a particular application of Theorem 2.2 when the nonlinear operator is strictly monotone in the sense of the previous lemma.

Theorem 2.5. *Let $\mathbf{X}$ be a closed subspace of $X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are separable and reflexive Banach spaces, and let Y be a reflexive Banach space. Let $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ be a nonlinear operator and $\mathbf{B} : \mathbf{X} \to Y'$ a bounded linear operator, with $\mathbf{V} := N(\mathbf{B})$ denoting its nullspace. Assume that*

i) *there holds* i) *from Lemma* 2.4,
ii) *there holds* ii) *from Lemma* 2.4,
iii) *there exists a constant $\beta > 0$ such that*

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbf{X} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}), v]}{\|\boldsymbol{\tau}\|_{\mathbf{X}}} \geq \beta \|v\|_Y \qquad \forall\, v \in Y\,.$$

Then, for every $\mathbf{F} \in \mathbf{X}'$ and $G \in Y'$, there exists a unique pair $(\boldsymbol{\sigma}, u) \in \mathbf{X} \times Y$ such that

$$\begin{aligned}
[\mathbf{A}(\boldsymbol{\sigma}), \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}), u] &= [\mathbf{F}, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{X}, \\
[\mathbf{B}(\boldsymbol{\sigma}), v] &= [G, v] \qquad \forall\, v \in Y.
\end{aligned} \tag{2.6}$$

Moreover, there exist constants $C_1, C_2 > 0$, depending only on α, β, γ, n, and $p_1, p_2, \ldots, p_n$, such that

$$\|\boldsymbol{\sigma}\|_{\mathbf{X}} \leq C_1 \max_{k \in \{1,2,\ldots,n\}} \mathcal{N}(\mathbf{F}, G)^{\frac{1}{p_k - 1}}, \quad \text{and} \tag{2.7}$$

$$\|u\|_Y \leq C_2 \max\left\{ \max_{k\in\{1,2,\ldots,n\}} \mathcal{N}(\mathbf{F},G)^{\frac{1}{p_k-1}}, \max_{j,k\in\{1,2,\ldots,n\}} \mathcal{N}(\mathbf{F},G)^{\frac{p_j-1}{p_k-1}} \right\}, \tag{2.8}$$

where

$$\mathcal{N}(\mathbf{F},G) := \|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|G\|_{Y'} + \sum_{j=1}^{n} \|G\|_{Y'}^{p_j-1} \quad \forall\, (\mathbf{F},G) \in \mathbf{X}' \times Y'\,.$$

Proof. The existence and uniqueness of the solution follow directly from Theorem 2.2 and Lemma 2.4. Thus, we focus on establishing the a priori estimates for the solution in terms of the data. Let $(\boldsymbol{\sigma},u) \in \mathbf{X}\times Y$ be the unique solution of (2.6). According to Lemma 2.1 and Theorem 2.2, $\boldsymbol{\sigma}$ can be decomposed as $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0$, where

$$\|\bar{\boldsymbol{\sigma}}\|_{\mathbf{X}} \leq \frac{1}{\beta}\|G\|_{Y'}$$

and $\boldsymbol{\sigma}_0 \in \mathbf{V}$ is the unique solution of

$$[\mathbf{A}(\boldsymbol{\sigma}_0 + \bar{\boldsymbol{\sigma}}),\, \boldsymbol{\tau}] = [\mathbf{F},\, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{V}.$$

Moreover, Lemma 2.4 implies that

$$\|\boldsymbol{\sigma}_0\|_{\mathbf{X}} \leq C \max_{k\in\{1,2,\ldots,n\}} \left(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \sum_{j=1}^{n} \left\{ \|\bar{\sigma}_j\|_{X_j} + \|\bar{\sigma}_j\|_{X_j}^{p_j-1} \right\} \right)^{\frac{1}{p_k-1}},$$

where $\bar{\boldsymbol{\sigma}} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$. Combining this estimate with the preceding bound for $\bar{\boldsymbol{\sigma}}$, we deduce that there exists a constant $c > 0$, depending only on α, β, γ, n, and $p_1, p_2, \ldots, p_n$, such that

$$\|\boldsymbol{\sigma}_0\|_{\mathbf{X}} \leq c \max_{k\in\{1,2,\ldots,n\}} \mathcal{N}(\mathbf{F},G)^{\frac{1}{p_k-1}}.$$

Hence, recalling that $\boldsymbol{\sigma} = \bar{\boldsymbol{\sigma}} + \boldsymbol{\sigma}_0$, we conclude that there exists a constant $C_1 > 0$, with the same dependence as c, such that

$$\|\boldsymbol{\sigma}\|_{\mathbf{X}} \leq \|\boldsymbol{\sigma}_0\|_{\mathbf{X}} + \|\bar{\boldsymbol{\sigma}}\|_{\mathbf{X}} \leq C_1 \max_{k\in\{1,2,\ldots,n\}} \mathcal{N}(\mathbf{F},G)^{\frac{1}{p_k-1}},$$

which is precisely the required estimate (2.7). On the other hand, Lemma 2.1 yields

$$\|u\|_Y \leq \frac{1}{\beta}\Big(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(\boldsymbol{\sigma})\|_{\mathbf{X}'} \Big). \tag{2.9}$$

In addition, applying i) with $\boldsymbol{\tau} = 0$, we obtain

$$\|\mathbf{A}(\boldsymbol{\sigma})\|_{\mathbf{X}'} \leq \gamma \sum_{j=1}^{n} \left\{ \|\sigma_j\|_{X_j} + \|\sigma_j\|_{X_j}^{p_j-1} \right\} + \|\mathbf{A}(0)\|_{\mathbf{X}'}. \tag{2.10}$$

The inequalities (2.9) and (2.10), together with the previously established estimate for $\|\boldsymbol{\sigma}\|_{\mathbf{X}}$, yield (2.8), thus completing the proof. □

A particular case of the preceding theorem, corresponding to $n = 2$, with X_1 and X_2 assumed to be uniformly convex and the *hemicontinuity* of $\mathbf{A}$ imposed as an additional hypothesis, was established in [3, Theorem 3.1]. This result was then applied to the coupled Navier–Stokes/Darcy–Forchheimer problem, where Ω_S and Ω_D are bounded, simply connected polygonal domains in R^2 such that $\partial\Omega_S \cap \partial\Omega_D = \Sigma \neq \varnothing$ and $\Omega_S \cap \Omega_D = \varnothing$, with Γ_S and Γ_D denoting $\partial\Omega_S \setminus \bar{\Sigma}$ and $\partial\Omega_D \setminus \bar{\Sigma}$, respectively.

3. Perturbed nonlinear Babuška–Brezzi problems

Let $\mathbf{X}$ be a closed subspace of $X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are Banach spaces, and let Y be another Banach space. Given $\mathbf{F} \in \mathbf{X}'$ and $G \in Y'$, we now consider the so-called *perturbed* nonlinear Babuška–Brezzi problems, which consist of finding $(\boldsymbol{\sigma}, u) \in \mathbf{X} \times Y$ such that

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}),\, \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}),\, u] &= [\mathbf{F},\, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{X}, \\ [\mathbf{B}(\boldsymbol{\sigma}),\, v] - [\mathbf{C}(u),\, v] &= [G,\, v] \qquad \forall\, v \in Y, \end{aligned} \tag{3.1}$$

where $\mathbf{A} : \mathbf{X} \to \mathbf{X}'$ is a nonlinear operator, $\mathbf{B} \in \mathcal{L}(\mathbf{X}, Y')$, and $\mathbf{C} \in \mathcal{L}(Y, Y')$. Our approach consists of two steps. First, under suitable assumptions, we solve the first equation for $\boldsymbol{\sigma}$ in terms of u. Then, substituting the resulting expression into the second equation, we seek to determine u.

Lemma 3.1. *Let $X_1, X_2, \ldots, X_n$ be separable and reflexive Banach spaces. Assume that*

i) *there holds* i) *from Lemma* 2.4,

ii) *the operator* $\mathbf{A}$ *is* strictly monotone *in the following sense: there exists a constant $\alpha > 0$ such that, with the same exponents $p_1, p_2, \ldots, p_n$ as in* i), *there holds*

$$[\mathbf{A}(\boldsymbol{\sigma}) - \mathbf{A}(\boldsymbol{\tau}),\, \boldsymbol{\sigma} - \boldsymbol{\tau}] \geq \alpha \sum_{j=1}^{n} \|\sigma_j - \tau_j\|_{X_j}^{p_j}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)$ in $\mathbf{X}$.

Then, for every $u \in Y$, there exists a unique $\boldsymbol{\sigma}(u) \in \mathbf{X}$ such that

$$[\mathbf{A}(\boldsymbol{\sigma}(u)),\, \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}),\, u] = [\mathbf{F},\, \boldsymbol{\tau}] \qquad \forall\, \boldsymbol{\tau} \in \mathbf{X}.$$

Proof. By Lemma 2.4, the operator $\mathbf{A}$ is bijective. Hence, for every $u \in Y$, there exists a unique $\boldsymbol{\sigma}(u) \in \mathbf{X}$ such that $\mathbf{A}(\boldsymbol{\sigma}(u)) = \mathbf{F} - \mathbf{B}'(J_Y(u))$, which completes the proof. □

We now seek to establish the existence of a solution to (3.1). To this end, we introduce the nonlinear operator $\mathbf{T} : Y \to Y'$ defined by

$$\mathbf{T}(u) := -\mathbf{B}(\boldsymbol{\sigma}(u)) + \mathbf{C}(u) \qquad \forall\, u \in Y,$$

and show that it is surjective under suitable assumptions.

Lemma 3.2. *Assume that the hypotheses of Lemma* 3.1 *hold and that* $\mathbf{C}$ *is* positive semidefinite, *that is,* $[\mathbf{C}(v), v] \geq 0 \quad \forall\, v \in Y$. *Then, the operator* $\mathbf{T}$ *is continuous, bounded, and monotone.*

Proof. Let $u_1, u_2 \in Y$. Since

$$\mathbf{A}(\boldsymbol{\sigma}(u_1)) - \mathbf{A}(\boldsymbol{\sigma}(u_2)) = -\mathbf{B}'(J_Y(u_1 - u_2)),$$

the strict monotonicity of $\mathbf{A}$ implies that

$$\begin{aligned} \alpha \sum_{j=1}^{n} \|s_j - r_j\|_{X_j}^{p_j} &\leq \|\mathbf{B}'(J_Y(u_1 - u_2))\|_{\mathbf{X}'} \|\boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)\|_{\mathbf{X}} \\ &\leq \|\mathbf{B}\|\, \|u_1 - u_2\|_Y \|\boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)\|_{\mathbf{X}}\,, \end{aligned}$$

where $\boldsymbol{\sigma}(u_1) = (s_1, s_2, \ldots, s_n)$ and $\boldsymbol{\sigma}(u_2) = (r_1, r_2, \ldots, r_n)$. Proceeding as in the proof of Lemma 2.4, we deduce that there exists a constant $c_1 > 0$ such that

$$\|\boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)\|_{\mathbf{X}} \leq c_1 \max_{j \in \{1,2,\ldots,n\}} \|u_1 - u_2\|_Y^{\frac{1}{p_j - 1}}.$$

On the other hand, the definition of $\mathbf{T}$ yields

$$\|\mathbf{T}(u_1) - \mathbf{T}(u_2)\|_{Y'} \leq \|\mathbf{B}\| \, \|\boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)\|_{\mathbf{X}} + \|\mathbf{C}\| \, \|u_1 - u_2\|_Y.$$

Combining the last two estimates, we conclude that $\mathbf{T}$ is continuous. Moreover, since

$$\|\mathbf{T}(u)\|_{Y'} \leq \|\mathbf{T}(u) - \mathbf{T}(0)\|_{Y'} + \|\mathbf{T}(0)\|_{Y'} \qquad \forall\, u \in Y,$$

the same estimates show that $\mathbf{T}$ maps bounded subsets of Y into bounded subsets of Y', and hence is bounded. It remains to establish the monotonicity of $\mathbf{T}$. To this end, using its definition and the identity satisfied by $\boldsymbol{\sigma}(u_1)$ and $\boldsymbol{\sigma}(u_2)$, we obtain

$$\begin{aligned}
&[\mathbf{T}(u_1) - \mathbf{T}(u_2),\, u_1 - u_2] \\
&\quad = [\mathbf{A}(\boldsymbol{\sigma}(u_1)) - \mathbf{A}(\boldsymbol{\sigma}(u_2)), \boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)] \,+\, [\mathbf{C}(u_1 - u_2),\, u_1 - u_2] \\
&\quad \geq\, [\mathbf{A}(\boldsymbol{\sigma}(u_1)) - \mathbf{A}(\boldsymbol{\sigma}(u_2)), \boldsymbol{\sigma}(u_1) - \boldsymbol{\sigma}(u_2)] \,\geq\, \alpha \sum_{j=1}^{n} \|s_j - r_j\|_{X_j}^{p_j} \,\geq\, 0\,,
\end{aligned}$$

where we have used the positive semidefiniteness of $\mathbf{C}$ and the strict monotonicity of $\mathbf{A}$. This proves that $\mathbf{T}$ is monotone and completes the proof. □

The following result establishes the surjectivity of $\mathbf{T}$, which will subsequently be used to prove the existence of a solution to the perturbed problem.

Lemma 3.3. *Assume that the hypotheses of Lemmas* 3.1 *and* 3.2 *hold. Suppose, in addition, that Y is separable and reflexive, and that $\mathbf{B}$ satisfies the inf-sup condition*

$$\sup_{\substack{\boldsymbol{\tau} \in \mathbf{X} \\ \boldsymbol{\tau} \neq 0}} \frac{[\mathbf{B}(\boldsymbol{\tau}),\, v]}{\|\boldsymbol{\tau}\|_{\mathbf{X}}} \geq \beta \|v\|_Y \qquad \forall\, v \in Y, \tag{3.2}$$

for some constant $\beta > 0$. Then, the operator $\mathbf{T}$ is coercive and surjective.

Proof. Let $u \in Y$, $u \neq 0$. Since $\mathbf{C}$ is positive semidefinite, we have

$$\begin{aligned}
\frac{[\mathbf{T}(u),\, u]}{\|u\|_Y} &\geq \frac{[\mathbf{A}(\boldsymbol{\sigma}(u)),\, \boldsymbol{\sigma}(u)]}{\|u\|_Y} - \frac{[\mathbf{F},\, \boldsymbol{\sigma}(u)]}{\|u\|_Y} \\
&= \frac{[\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{A}(\boldsymbol{\sigma}(0)),\, \boldsymbol{\sigma}(u)]}{\|u\|_Y} \\
&= \frac{[\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{A}(\boldsymbol{\sigma}(0)),\, \boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(0)]}{\|u\|_Y} + \frac{[\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{F},\, \boldsymbol{\sigma}(0)]}{\|u\|_Y}.
\end{aligned}$$

To establish the coercivity of $\mathbf{T}$, we first observe that, writing $\boldsymbol{\sigma}(u) = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, assumption i) of Lemma 3.1, together with the inf-sup condition for $\mathbf{B}$, yields

$$
\begin{aligned}
\beta \, \|u\|_Y \, &\leq \, \|\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{F}\|_{\mathbf{X}'} \, \leq \, \|\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{F}\|_{\mathbf{X}'} \\
&\leq \, \gamma \sum_{j=1}^{n} \left\{ \|\sigma_j\|_{X_j} + \|\sigma_j\|_{X_j}^{p_j - 1} \right\} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{F}\|_{\mathbf{X}'} \, ,
\end{aligned} \tag{3.3}
$$

In particular, this implies that $\|\boldsymbol{\sigma}(u)\|_{\mathbf{X}} \to \infty$ as $\|u\|_Y \to \infty$. Next, writing $\boldsymbol{\sigma}(0) = (r_1, r_2, \ldots, r_n)$, we have by ii) of Lemma 3.1 and the previous bound that

$$
\begin{aligned}
&\frac{[\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{A}(\boldsymbol{\sigma}(0)), \, \boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(0)]}{\|u\|_Y} \\
&\qquad \geq \, \frac{\alpha\beta \sum_{j=1}^{n} \|\sigma_j - r_j\|_{X_j}^{p_j}}{\gamma \sum_{j=1}^{n} \left\{ \|\sigma_j\|_{X_j} + \|\sigma_j\|_{X_j}^{p_j - 1} \right\} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{F}\|_{\mathbf{X}'}} \, ,
\end{aligned}
$$

Since $p_j \geq 2$ for all $j \in \{1, 2, \ldots, n\}$, the right-hand side tends to infinity as $\|\boldsymbol{\sigma}(u)\|_{\mathbf{X}} \to \infty$. On the other hand, the defining equation for $\boldsymbol{\sigma}(u)$ implies that

$$
\frac{[\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{F}, \, \boldsymbol{\sigma}(0)]}{\|u\|_Y} = -\frac{[\mathbf{B}(\boldsymbol{\sigma}(0)), \, u]}{\|u\|_Y} \geq -\|\mathbf{B}\| \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}} \, ,
$$

where the lower bound is independent of u. Combining the preceding estimates, we conclude that $\dfrac{[\mathbf{T}(u), \, u]}{\|u\|_Y} \to \infty$ as $\|u\|_Y \to \infty$, which proves that $\mathbf{T}$ is coercive. Finally, this property, together with the continuity, boundedness, and monotonicity established in Lemma 3.2, shows that $\mathbf{T}$ satisfies the hypotheses of Lemma 2.3. Consequently, $\mathbf{T}$ is surjective, which completes the proof. □

Since $\mathbf{T}$ is surjective, there exists $u \in Y$ such that $\mathbf{T}(u) = -G$, or, equivalently,

$$
[\mathbf{B}(\boldsymbol{\sigma}(u)), \, v] - [\mathbf{C}(u), \, v] = [G, \, v] \qquad \forall \, v \in Y.
$$

On the other hand, by the definition of $\boldsymbol{\sigma}(u)$, the pair $(\boldsymbol{\sigma}(u), u)$ also satisfies

$$
[\mathbf{A}(\boldsymbol{\sigma}(u)), \, \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}), \, u] = [\mathbf{F}, \, \boldsymbol{\tau}] \qquad \forall \, \boldsymbol{\tau} \in \mathbf{X}.
$$

Consequently, $(\boldsymbol{\sigma}(u), u) \in \mathbf{X} \times Y$ is a solution of the perturbed problem (3.1).

Lemma 3.4. *The solution of the perturbed problem* (3.1) *constructed above is unique.*

Proof. Suppose that $(\boldsymbol{\sigma}_1, u_1), (\boldsymbol{\sigma}_2, u_2) \in \mathbf{X} \times Y$ are two solutions of (3.1). Subtracting the corresponding equations, we obtain

$$
\begin{aligned}
[\mathbf{A}(\boldsymbol{\sigma}_1) - \mathbf{A}(\boldsymbol{\sigma}_2), \, \boldsymbol{\tau}] + [\mathbf{B}(\boldsymbol{\tau}), \, u_1 - u_2] &= 0 \qquad \forall \, \boldsymbol{\tau} \in \mathbf{X}, \\
[\mathbf{B}(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2), \, v] \qquad - [\mathbf{C}(u_1 - u_2), \, v] &= 0 \qquad \forall \, v \in Y.
\end{aligned}
$$

Choosing $\boldsymbol{\tau} = \boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2$ and $v = u_1 - u_2$ in the first and second equations, respectively, subtracting the resulting identities, and then invoking the strict monotonicity of $\mathbf{A}$, the positive semidefiniteness of $\mathbf{C}$, and the inf-sup condition for $\mathbf{B}$, we conclude the proof. □

We are now in a position to state the main result of this section.

Theorem 3.5. *Suppose that $X_1, X_2, \ldots, X_n$ and Y are separable and reflexive Banach spaces. Assume that*

i) *there holds* i) *from Lemma* 3.1
ii) *there holds* ii) *from Lemma* 3.1
iii) $\mathbf{B}$ *satisfies the inf-sup condition* (3.2) *with constant* $\beta > 0$,
iv) *the operator* $\mathbf{C}$ *is positive semidefinite, that is,* $[\mathbf{C}(v),\, v] \geq 0 \;\; \forall\, v \in Y$.

Then, for every $\mathbf{F} \in \mathbf{X}'$ *and* $G \in Y'$, *there exists a unique solution* $(\boldsymbol{\sigma}, u) \in \mathbf{X} \times Y$ *to problem* (3.1). *Moreover, there exist constants* $C_1, C_2 > 0$, *depending only on* α, β, γ, n, *and* $p_1, p_2, \ldots, p_n$, *such that, defining*

$$\mathcal{K}_1(\mathbf{F}, G) := \|G\|_{Y'} + \max_{j \in \{1,2,\ldots,n\}} \{\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'}\}^{\frac{1}{p_j - 1}}\,,$$

there hold the a priori estimates:

$$\|u\|_Y \,\leq\, C_1 \max_{j \in \{1,2,\ldots,n\}} \Big\{ \mathcal{K}_1(\mathbf{F}, G)^{\frac{1}{p_j - 1}},\; \mathcal{K}_1(\mathbf{F}, G),\; \mathcal{K}_1(\mathbf{F}, G)^{p_j - 1} \Big\}\,, \quad \textit{and} \tag{3.4}$$

$$\begin{aligned} \|\boldsymbol{\sigma}\|_{\mathbf{X}} \,\leq\, C_2 \max \Big\{ & \max_{j \in \{1,2,\ldots,n\}} \mathcal{K}_1(\mathbf{F}, G)^{\frac{1}{(p_j - 1)^2}}, \max_{j \in \{1,2,\ldots,n\}} \mathcal{K}_1(\mathbf{F}, G)^{\frac{1}{p_j - 1}}, \\ & \max_{j,k \in \{1,2,\ldots,n\}} \mathcal{K}_1(\mathbf{F}, G)^{\frac{p_j - 1}{p_k - 1}} \Big\}. \end{aligned} \tag{3.5}$$

Proof. By the preceding results, the problem admits a unique solution, and it only remains to establish the corresponding a priori estimates. To this end, we first observe that, for the constructed solution $(\boldsymbol{\sigma}(u), u) \in \mathbf{X} \times Y$, writing $\boldsymbol{\sigma}(u) = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ and $\boldsymbol{\sigma}(0) = (r_1, r_2, \ldots, r_n)$, we have

$$\alpha \sum_{j=1}^{n} \|r_j\|_{X_j}^{p_j} \,\leq\, [\mathbf{A}(\boldsymbol{\sigma}(0)) - \mathbf{A}(0),\, \boldsymbol{\sigma}(0)] \,\leq\, \Big(\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} \Big)\, \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}}\,,$$

from which it follows that there exists a constant $c_1 > 0$ such that

$$\|\boldsymbol{\sigma}(0)\|_{\mathbf{X}} \leq c_1 \max_{j \in \{1,2,\ldots,n\}} \Big\{ \|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{A}(0)\|_{\mathbf{X}'} \Big\}^{\frac{1}{p_j - 1}}\,.$$

Furthermore, we have

$$\begin{aligned} [\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{A}(\boldsymbol{\sigma}(0)),\, \boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(0)] \,&=\, [\mathbf{A}(\boldsymbol{\sigma}(u)) - \mathbf{F},\, \boldsymbol{\sigma}(u) - \boldsymbol{\sigma}(0)] \\ &=\, [-G + \mathbf{B}(\boldsymbol{\sigma}(0)),\, u] - [\mathbf{C}(u),\, u]\,, \end{aligned}$$

so that, using the positive semidefiniteness of $\mathbf{C}$, we deduce that, for each $j \in \{1, 2, \ldots, n\}$,

$$\|\sigma_j - r_j\|_{X_j}^{p_j} \,\leq\, \frac{1}{\alpha} \Big(\|G\|_{Y'} + \|\mathbf{B}\| \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}} \Big) \|u\|_Y \,\leq\, c_2^{p_j} \mathcal{K}_1(\mathbf{F}, G) \|u\|_Y\,,$$

for some constant $c_2 > 0$. Consequently,

$$\|\sigma_j\|_{X_j} \,\leq\, c_2 \mathcal{K}_1(\mathbf{F}, G)^{1/p_j} \|u\|_Y^{1/p_j} + \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}} \quad \forall\, j \in \big\{1, 2, \ldots, n\big\}\,.$$

Combining this estimate with (3.3), we infer that there exists a constant $c_3 > 0$ such that

$$\|u\|_Y \leq c_3 \left(\sum_{j=1}^{n} \left\{ \mathcal{K}_1(\mathbf{F}, G)^{\frac{1}{p_j}} \|u\|_Y^{\frac{1}{p_j}} + \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}} + \mathcal{K}_1(\mathbf{F}, G)^{\frac{p_j - 1}{p_j}} \|u\|_Y^{\frac{p_j - 1}{p_j}} \right.\right.$$
$$\left.\left. + \quad \|\boldsymbol{\sigma}(0)\|_{\mathbf{X}}^{p_j - 1} \right\} + \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{F}\|_{\mathbf{X}'} \right).$$

Applying Young's inequality and the preceding bound for $\|\boldsymbol{\sigma}(0)\|_{\mathbf{X}}$, we obtain

$$\|u\|_Y \leq \frac{1}{2}\|u\|_Y + c_4 \left(\sum_{j=1}^{n} \left\{ \mathcal{K}_1(\mathbf{F}, G)^{\frac{1}{p_j - 1}} + \mathcal{K}_1(\mathbf{F}, G) + \mathcal{K}_1(\mathbf{F}, G)^{p_j - 1} \right\} \right.$$
$$\left. + \quad \|\mathbf{A}(0)\|_{\mathbf{X}'} + \|\mathbf{F}\|_{\mathbf{X}'} \right),$$

for some constant $c_4 > 0$. Hence, absorbing the first term on the right-hand side into the left-hand side, we arrive at (3.4). It remains to derive the corresponding upper bound for $\boldsymbol{\sigma}(u)$. To this end, applying the strict monotonicity condition ii), we deduce that there exists a constant $c_6 > 0$ such that

$$\|\boldsymbol{\sigma}(u)\|_{\mathbf{X}} \leq c_6 \max_{j \in \{1,2,\ldots,n\}} \{\|\mathbf{F}\|_{\mathbf{X}'} + \|\mathbf{B}\| \|u\|_Y + \|\mathbf{A}(0)\|_{\mathbf{X}'}\}^{\frac{1}{p_j - 1}},$$

which, along with (3.4), yields the required estimate (3.5). Finally, we observe that all the constants introduced in the preceding estimates are such that C_1 and C_2 depend only on α, β, γ, n, and $p_1, p_2, \ldots, p_n$. This completes the proof. □

4. Nested nonlinear Babuška–Brezzi problems

In Theorem 2.2, we established necessary and sufficient conditions for the unique solvability of a problem of the form (2.1). We now consider the case in which the nonlinear operator appearing in that problem is itself induced by another nonlinear Babuška–Brezzi problem. Problems of this kind have been studied in [1] and [4]. Here, however, we provide a proof that does not require any of the underlying spaces to be uniformly convex.

Theorem 4.1. *Let $\mathbf{X} := X_1 \times X_2 \times \cdots \times X_n$, where $X_1, X_2, \ldots, X_n$ are separable and reflexive Banach spaces, and let Y be another separable and reflexive Banach space. Set $\mathbf{M}_1 := \mathbf{X} \times Y$, and let M_2 be an additional reflexive Banach space. Consider the problem: Find $((\boldsymbol{\sigma}, u), \psi) \in \mathbf{M}_1 \times M_2$ such that*

$$\begin{aligned} &[\mathbf{A}((\boldsymbol{\sigma}, u)), (\boldsymbol{\tau}, v)] + [\mathbf{B}((\boldsymbol{\tau}, v)), \psi] = [\mathbf{F}, (\boldsymbol{\tau}, v)] \qquad \forall\, (\boldsymbol{\tau}, v) \in \mathbf{M}_1\,, \\ &[\mathbf{B}((\boldsymbol{\sigma}, u)), \varphi] \qquad\qquad\qquad\quad = [G, \varphi] \qquad \forall\, \varphi \in M_2\,, \end{aligned} \tag{4.1}$$

where $\mathbf{F} = (F_1, F_2) \in \mathbf{M}_1'$, $G \in M_2'$, $\mathbf{B} \in \mathcal{L}(\mathbf{M}_1, M_2')$, and $\mathbf{A} : \mathbf{M}_1 \to \mathbf{M}_1'$ is the nonlinear operator defined by

$$\mathbf{A}((\boldsymbol{\sigma}, u)) := \begin{bmatrix} a & b' \circ J_Y \\ b & -c \end{bmatrix} \begin{bmatrix} \boldsymbol{\sigma} \\ u \end{bmatrix}$$

for all $(\boldsymbol{\sigma}, u) \in \mathbf{M}_1 = \mathbf{X} \times Y$, where $b \in \mathcal{L}(\mathbf{X}, Y')$, $c \in \mathcal{L}(Y, Y')$, and $a : \mathbf{X} \to \mathbf{X}'$ is a nonlinear operator. Suppose that there exist subspaces $\mathbf{V}_1$ and V_2 of $\mathbf{X}$ and Y, respectively, such that the kernel of $\mathbf{B}$ admits the decomposition $N(\mathbf{B}) = \mathbf{V}_1 \times V_2$. Additionally, assume that

i) *there exists a constant* $\widehat{\beta} > 0$ *such that*

$$\sup_{\substack{(\boldsymbol{\tau},v)\in\mathbf{M}_1\\(\boldsymbol{\tau},v)\neq 0}} \frac{[\mathbf{B}((\boldsymbol{\tau},v)),\,\varphi]}{\|(\boldsymbol{\tau},v)\|_{\mathbf{M}_1}} \geq \widehat{\beta}\|\varphi\|_{M_2} \qquad \forall\,\varphi\in M_2\,,$$

ii) *there exist constants* $\gamma > 0$ *and* $p_1, p_2, \ldots, p_n \geq 2$ *such that*

$$\|a(\boldsymbol{\sigma}) - a(\boldsymbol{\tau})\|_{\mathbf{X}'} \leq \gamma \sum_{j=1}^{n} \left\{ \|\sigma_j - \tau_j\|_{X_j} + \|\sigma_j - \tau_j\|_{X_j} \left(\|\sigma_j\|_{X_j} + \|\tau_j\|_{X_j} \right)^{p_j - 2} \right\}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *and* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)$ *in* $\mathbf{X}$,

iii) *for each fixed* $\boldsymbol{\zeta} \in \mathbf{X}$, *the operator* $a(\cdot + \boldsymbol{\zeta}) : \mathbf{V}_1 \to \mathbf{V}_1'$ *is* strictly monotone *in the following sense: there exists a constant* $\alpha > 0$, *independent of* $\boldsymbol{\zeta}$, *such that, with the same exponents* $p_1, p_2, \ldots, p_n$ *as in* ii), *there holds*

$$[a(\boldsymbol{\sigma} + \boldsymbol{\zeta}) - a(\boldsymbol{\tau} + \boldsymbol{\zeta}),\, \boldsymbol{\sigma} - \boldsymbol{\tau}] \;\geq\; \alpha \sum_{j=1}^{n} \|\sigma_j - \tau_j\|_{X_j}^{p_j}$$

for all $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \ldots, \sigma_n)$ *and* $\boldsymbol{\tau} = (\tau_1, \tau_2, \ldots, \tau_n)$ *in* $\mathbf{V}_1$,

iv) *there exists a constant* $\beta > 0$ *such that*

$$\sup_{\substack{\boldsymbol{\tau}\in\mathbf{V}_1\\ \boldsymbol{\tau}\neq 0}} \frac{[b(\boldsymbol{\tau}),\, v]}{\|\boldsymbol{\tau}\|_{\mathbf{X}}} \;\geq\; \beta\,\|v\|_Y \qquad \forall\, v \in V_2\,,$$

v) *the operator* c *is positive semidefinite, that is,* $[c(v), v] \geq 0 \;\; \forall v \in V_2$.

Then, problem (4.1) *admits a unique solution* $((\boldsymbol{\sigma}, u), \psi) \in \mathbf{M}_1 \times M_2$. *Moreover, there exist constants* $C_1, C_2, C_3 > 0$ *such that, defining*

$$\begin{aligned}\mathcal{K}_2(\mathbf{F}, G) \;:=\; \|F_2\|_{Y'} + \|G\|_{M_2'} + \max_{k\in\{1,2,\ldots,n\}} \Big\{ &\|F_1\|_{\mathbf{X}'} + \|G\|_{M_2'} \\ &+ \sum_{j=1}^{n} \|G\|_{M_2'}^{p_j - 1} + \|a(0)\|_{\mathbf{X}'} \Big\}^{\frac{1}{p_k - 1}}\,,\end{aligned}$$

the following a priori estimates hold:

$$\|\boldsymbol{\sigma}\|_{\mathbf{X}} \;\leq\; C_1 \max_{j,k\in\{1,2,\ldots,n\}} \left\{ \mathcal{K}_2(\mathbf{F}, G)^{\frac{1}{(p_j - 1)^2}},\; \mathcal{K}_2(\mathbf{F}, G)^{\frac{1}{p_j - 1}},\; \mathcal{K}_2(\mathbf{F}, G)^{\frac{p_j - 1}{p_k - 1}} \right\}, \tag{4.2}$$

$$\|u\|_Y \;\leq\; C_2 \max_{j\in\{1,2,\ldots,n\}} \left\{ \mathcal{K}_2(\mathbf{F}, G)^{\frac{1}{p_j - 1}},\; \mathcal{K}_2(\mathbf{F}, G),\; \mathcal{K}_2(\mathbf{F}, G)^{p_j - 1} \right\}, \quad \textit{and} \tag{4.3}$$

$$\begin{aligned}\|\psi\|_{M_2} \;\leq\; C_3 \max_{j,k\in\{1,2,\ldots,n\}} \Big\{ &\mathcal{K}_2(\mathbf{F}, G)^{\frac{1}{(p_j - 1)^2}},\; \mathcal{K}_2(\mathbf{F}, G)^{\frac{p_j - 1}{(p_k - 1)^2}}, \\ &\mathcal{K}_2(\mathbf{F}, G)^{\frac{p_j - 1}{p_k - 1}},\; \mathcal{K}_2(\mathbf{F}, G)^{\frac{(p_j - 1)^2}{p_k - 1}} \Big\}.\end{aligned} \tag{4.4}$$

Proof. By Theorem 2.2, problem (4.1) admits a unique solution if and only if the corresponding problem restricted to the kernel of $\mathbf{B}$ admits a unique solution for each admissible lifting of the constraint. More precisely, by Lemma 2.1 and the inf-sup condition for $\mathbf{B}$, there exists $(\bar{\boldsymbol{\sigma}}, \bar{u}) \in \mathbf{M}_1$ such that

$$[\mathbf{B}((\bar{\boldsymbol{\sigma}}, \bar{u})), \varphi] = [G, \varphi] \quad \forall\, \varphi \in M_2, \quad \text{and} \quad \|(\bar{\boldsymbol{\sigma}}, \bar{u})\|_{\mathbf{M}_1} \leq \frac{1}{\widehat{\beta}} \|G\|_{M_2'}\,.$$

For this fixed lifting, the associated problem on $N(\mathbf{B}) = \mathbf{V}_1 \times V_2$ reads: Find $(\boldsymbol{\sigma}_0, u_0) \in \mathbf{V}_1 \times V_2$ such that

$$\begin{aligned} &[a(\boldsymbol{\sigma}_0 + \bar{\boldsymbol{\sigma}}), \boldsymbol{\tau}] + [b(\boldsymbol{\tau}), u_0] = [F_1 - b'(J_Y(\bar{u})), \boldsymbol{\tau}] && \forall\, \boldsymbol{\tau} \in \mathbf{V}_1 \\ &[b(\boldsymbol{\sigma}_0), v] \qquad - [c(u_0), v] = [F_2 - b(\bar{\boldsymbol{\sigma}}) + c(\bar{u}), v] && \forall\, v \in V_2. \end{aligned} \tag{4.5}$$

We notice that this is a problem of the form (3.1), with the nonlinear operator $a(\cdot + \bar{\boldsymbol{\sigma}})$ restricted to $\mathbf{V}_1$, and with b and c replaced by their corresponding restrictions to $\mathbf{V}_1$ and V_2. In particular, the inf-sup condition in assumption iv) is precisely the one required for the operator induced by b from $\mathbf{V}_1$ into V_2'. Moreover, assumptions ii) and iii) ensure that the shifted operator $a(\cdot + \bar{\boldsymbol{\sigma}})$ satisfies the continuity and strict monotonicity hypotheses of Theorem 3.5, while assumption v) provides the required positive semidefiniteness of c. Consequently, Theorem 3.5 guarantees that problem (4.5) admits a unique solution. It then follows from Theorem 2.2 that problem (4.1) also admits a unique solution. It remains to establish the corresponding a priori estimates. For this purpose, we recall that the solution $((\boldsymbol{\sigma}, u), \psi) \in \mathbf{M}_1 \times M_2$ can be decomposed as $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + \bar{\boldsymbol{\sigma}}$ and $u = u_0 + \bar{u}$, where $(\boldsymbol{\sigma}_0, u_0) \in \mathbf{V}_1 \times V_2$ is the solution of (4.5). We first observe that, by assumption ii), there exists a constant $c_1 > 0$ such that

$$\begin{aligned} \|a(\bar{\boldsymbol{\sigma}})\|_{\mathbf{X}'} &\leq \|a(0)\|_{\mathbf{X}'} + \gamma \sum_{j=1}^{n} \left\{ \|\bar{\sigma}_j\|_{X_j} + \|\bar{\sigma}_j\|_{X_j}^{p_j - 1} \right\} \\ &\leq \|a(0)\|_{\mathbf{X}'} + c_1 \left\{ \|G\|_{M_2'} + \sum_{j=1}^{n} \|G\|_{M_2'}^{p_j - 1} \right\}, \end{aligned}$$

where $\bar{\boldsymbol{\sigma}} = (\bar{\sigma}_1, \bar{\sigma}_2, \ldots, \bar{\sigma}_n)$ and we have used the bound for $(\bar{\boldsymbol{\sigma}}, \bar{u})$. Next, recalling the definition of $\mathcal{K}_1$ in Theorem 3.5, and adapting it to the restricted problem (4.5), we deduce that there exists a constant $c_2 > 0$ such that

$$\begin{aligned} \mathcal{K}_1(F_1 - b'(J_Y(\bar{u})),\, F_2 - b(\bar{\boldsymbol{\sigma}}) + c(\bar{u})) \leq c_2 \Bigg\{ &\|F_2\|_{Y'} + \|G\|_{M_2'} \\ + \max_{k \in \{1,2,\ldots,n\}} &\left(\|F_1\|_{\mathbf{X}'} + \|G\|_{M_2'} + \sum_{j=1}^{n} \|G\|_{M_2'}^{p_j - 1} + \|a(0)\|_{\mathbf{X}'} \right)^{\frac{1}{p_k - 1}} \Bigg\}. \end{aligned}$$

Here, the term involving the nonlinear operator at zero is $\|a(\bar{\boldsymbol{\sigma}})\|_{\mathbf{V}_1'}$, rather than $\|a(0)\|_{\mathbf{X}'}$, and the preceding estimate has been used to bound it in terms of the data. Thus, by the definition of $\mathcal{K}_2(\mathbf{F}, G)$, we obtain

$$\mathcal{K}_1(F_1 - b'(J_Y(\bar{u})),\, F_2 - b(\bar{\boldsymbol{\sigma}}) + c(\bar{u})) \leq c_2\, \mathcal{K}_2(\mathbf{F}, G)\,.$$

Applying Theorem 3.5 to (4.5), we conclude that there exist constants $c_3, c_4 > 0$ such that

$$\|\boldsymbol{\sigma}_0\|_{\mathbf{X}} \leq c_3 \max_{j,k\in\{1,2,\ldots,n\}} \left\{ \mathcal{K}_2(\mathbf{F},G)^{\frac{1}{(p_j-1)^2}},\ \mathcal{K}_2(\mathbf{F},G)^{\frac{1}{p_j-1}},\ \mathcal{K}_2(\mathbf{F},G)^{\frac{p_j-1}{p_k-1}} \right\}, \quad \text{and}$$
$$\|u_0\|_Y \leq c_4 \max_{j\in\{1,2,\ldots,n\}} \left\{ \mathcal{K}_2(\mathbf{F},G)^{\frac{1}{p_j-1}},\ \mathcal{K}_2(\mathbf{F},G),\ \mathcal{K}_2(\mathbf{F},G)^{p_j-1} \right\}.$$

Since $\|\boldsymbol{\sigma}\|_{\mathbf{X}} \leq \|\boldsymbol{\sigma}_0\|_{\mathbf{X}} + \|\bar{\boldsymbol{\sigma}}\|_{\mathbf{X}}$ and $\|u\|_Y \leq \|u_0\|_Y + \|\bar{u}\|_Y$, and the lifting is bounded in terms of $\|G\|_{M_2'}$, we obtain the first two asserted estimates (4.2) and (4.3). Finally, the proof of Theorem 2.2 shows that

$$\|\psi\|_{M_2} \leq \frac{1}{\widehat{\beta}} \Big(\|\mathbf{F}\|_{\mathbf{M}_1'} + \|\mathbf{A}((\boldsymbol{\sigma},u))\|_{\mathbf{M}_1'} \Big). \tag{4.6}$$

On the other hand, the boundedness of b and c, together with assumption ii), implies that there exists a constant $c_5 > 0$ such that

$$\begin{aligned} \|\mathbf{A}((\boldsymbol{\sigma},u))\|_{\mathbf{M}_1'} &\leq \|a(\boldsymbol{\sigma})\|_{\mathbf{X}'} + \|b\|\|\boldsymbol{\sigma}\|_{\mathbf{X}} + \|b'\|\|u\|_Y + \|c\|\|u\|_Y \\ &\leq c_5 \Big(\|a(0)\|_{\mathbf{X}'} + \|\boldsymbol{\sigma}\|_{\mathbf{X}} + \max_{j\in\{1,2,\ldots,n\}} \|\boldsymbol{\sigma}\|_{\mathbf{X}}^{p_j-1} + \|u\|_Y \Big). \end{aligned} \tag{4.7}$$

The estimates (4.6) and (4.7), together with the previously established bounds for $\boldsymbol{\sigma}$ and u and the corresponding powers of $\mathcal{K}_2(\mathbf{F},G)$, yield (4.4), thus completing the proof. □

In [1, Theorem 3.13], a particular case of the preceding result was established for $n = 2$, with X_1 and X_2 uniformly convex Banach spaces and $\mathbf{B}$ independent of the second component of its argument. That theorem was then applied to establish the well-posedness of a coupled Stokes/Darcy–Forchheimer model. Another particular case of Theorem 4.1 was established in [4, Theorem 3.4], again for $n = 2$ and with X_1 and X_2 uniformly convex Banach spaces. The latter result was then employed to establish the well-posedness of a coupled Navier–Stokes/Darcy–Forchheimer model.

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