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**A Lagrange multiplier-based finite element method for a reverse
osmosis model**

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A Lagrange multiplier-based finite element method for a reverse osmosis model ^{*}

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Abstract

In this work, we propose a Lagrange multiplier approach for a reverse osmosis model describing the flow of a saline solution through a channel bounded by semipermeable membranes. The model couples the incompressible Navier–Stokes equations with a convection–diffusion equation for the solute concentration, both through the convective term and through the membrane conditions: the normal velocity depends linearly on the concentration, the tangential velocity vanishes, and the concentration satisfies a Robin-type condition with a non-negativity constraint. In addition, Dirichlet conditions are prescribed at the inlet and directional do-nothing conditions at the outlet. We consider a velocity–pressure formulation for the fluid and a primal formulation for the concentration, imposing the Dirichlet-type conditions weakly through Lagrange multipliers. Then, by means of a fixed-point strategy and the Banach fixed-point theorem, we establish, under suitable smallness assumptions on the data, the well-posedness of the continuous problem and show that the concentration remains non-negative. Similarly, we prove the well-posedness of the associated Galerkin scheme for generic finite-dimensional subspaces satisfying suitable hypotheses, and derive a Strang-type a priori error estimate. In particular, Taylor–Hood elements for the velocity and pressure, continuous piecewise polynomials for the concentration, and piecewise polynomials on independent boundary partitions for the multipliers satisfy these hypotheses and yield optimal rates of convergence. Finally, numerical experiments confirm the theoretical results and illustrate the performance of the method in a realistic membrane channel.

Key words: reverse osmosis; desalination; Navier–Stokes equations; convection–diffusion equation; finite element method; Lagrange multipliers; positivity of concentration; error estimates

Mathematics Subject Classifications (2020): 65N15, 65N30, 76D05, 76M10

1 Introduction

Reverse osmosis (RO) is currently the dominant technology for seawater desalination and fresh-water production, accounting for nearly 85% of the installed capacity in large-scale desalination

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plants worldwide, mainly owing to its lower energy consumption compared with thermal methods such as multi-stage flash [28, 30]. From a physical point of view, RO membrane processes combine the incompressible flow of a saline solution through narrow feed channels with the advective and diffusive transport of the solute and the selective permeation of water and salt across the membranes. Reliable numerical models of these coupled phenomena are therefore essential, both to understand the mechanisms that determine membrane performance and to support the optimization of RO modules and the design of more efficient desalination systems.

Mathematically, these processes are typically modeled by the incompressible Navier–Stokes equations for the flow, coupled with a convection–diffusion equation for the solute concentration, where the two subsystems interact through the convective term and through concentration-dependent conditions on the membranes. From the analytical viewpoint, the latter coupling is particularly delicate: the normal velocity on the membranes is prescribed in terms of the unknown concentration, so that the Dirichlet datum for the flow is itself part of the solution and must be compatible with the inlet data at the corners of the channel. In addition, the nonlinear character of the equations and the need to preserve physically relevant properties, pose further challenges for both the analysis and the numerical approximation. In practice, most numerical studies have been restricted to single-channel configurations, mainly because of the high computational cost of simulating several channels, and have been carried out with commercial finite element packages, without a rigorous mathematical framework (see [7, 24, 26, 31, 32, 33] and the references therein).

Despite this interest, only a few contributions provide a rigorous numerical analysis of the coupled system. In [11], the authors proposed a finite element formulation coupling the incompressible Navier–Stokes equations with an advection–diffusion equation for the salt concentration, where the membrane conditions are imposed by means of Nitsche’s method; however, the analysis of both the continuous and discrete problems was left open. Later, in [6], a fully mixed Banach spaces-based formulation of the same model was analyzed, and a stable mixed finite element scheme with optimal a priori error estimates was derived. The main advantage of this approach is that it provides direct approximations of further variables of interest, such as the velocity and concentration gradients, although at the price of introducing additional unknowns, which increases the computational cost.

More recently, in [2], a three-field formulation in terms of the velocity, pressure and concentration was analyzed, where the model was modified by incorporating a non-negativity constraint on the concentration along the membranes, which allows proving the non-negativity of the concentration in the whole domain at the continuous level. In that work, the inlet concentration is assumed to be constant, which simplifies the treatment of the Dirichlet condition at the inlet. In addition, the Dirichlet conditions are imposed strongly, and the membrane condition involves the positive part of the concentration. As a consequence, the analysis relies on a lifting of the concentration-dependent boundary datum, defined through an auxiliary Stokes problem, and, at the discrete level, on an interpolation of this datum, which is not piecewise polynomial in general. In particular, the error analysis requires additional regularity of the auxiliary Stokes problem, and, since the lifted datum is approximated by a first-order interpolant, the resulting theoretical rates of convergence are limited to first order, independently of the polynomial degree of the discrete spaces.

Motivated by the above, in this work we propose a Lagrange multiplier approach for the model considered in [2], in which the Dirichlet-type conditions for the velocity, on the inlet and the membranes, and for the concentration, on the inlet, are imposed weakly. This approach

allows us to consider a general inlet concentration $\varphi_{\text{in}} \in H^{1/2}(\Gamma_{\text{in}})$, instead of the constant one assumed in [2], and to avoid both the auxiliary Stokes problem defining the lifting and the interpolation of the boundary datum, since the Dirichlet data enter the formulation only through the right-hand sides of the constraint equations. In addition, unlike [2], the membrane condition does not involve the positive part of the concentration, and the non-negativity of the concentration is ensured directly by the constraint imposed on the membranes. At the continuous level, we establish the well-posedness of the resulting formulation by means of a fixed-point strategy and the Banach fixed-point theorem, under suitable smallness assumptions on the data, and show that the concentration remains non-negative. At the discrete level, we introduce a Galerkin scheme based on generic finite-dimensional subspaces, derive suitable hypotheses on them ensuring its well-posedness, and establish a Strang-type a priori error estimate. Then, we show that Taylor–Hood elements for the velocity and pressure, continuous piecewise polynomials for the concentration, and piecewise polynomials on independent boundary partitions for the multipliers satisfy these hypotheses. As a consequence, and in contrast with [2], where the theoretical rates are limited to first order, we prove optimal rates of convergence of order $k + 1$ for any polynomial degree $k \geq 1$, without requiring additional regularity of auxiliary problems. Although we are not able to prove that the discrete concentration is non-negative, this property is consistently observed in our numerical experiments, which also confirm the theoretical rates of convergence.

The rest of the article is organized as follows. In the remainder of this section, we introduce the notation and preliminary theoretical results that will be used throughout the paper, and in Section 1.2 we present the model problem. In Section 2, we derive the variational formulation and establish its well-posedness, together with the non-negativity of the concentration. Next, in Section 3, we introduce the Galerkin scheme for generic finite-dimensional subspaces, derive suitable hypotheses ensuring its well-posedness, and establish the corresponding Strang-type a priori error estimate. In Section 4, we provide a concrete choice of finite element subspaces satisfying these hypotheses and derive the associated rates of convergence. Finally, in Section 5, we report numerical experiments that confirm the theoretical results and illustrate the performance of the method.

1.1 Preliminaries

Let $\Omega \subseteq \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ . Throughout the paper we use standard notation for Sobolev spaces $W^{m,p}(\Omega)$, where $p \in [1, \infty]$ and $m \geq 0$, and for the Lebesgue spaces $L^p(\Omega) := W^{0,p}(\Omega)$, equipped with the norms $\|\cdot\|_{W^{m,p}(\Omega)}$ and $\|\cdot\|_{L^p(\Omega)}$, respectively.

When $p = 2$, we write $H^m(\Omega)$ instead of $W^{m,2}(\Omega)$, and for simplicity we write $\mathbf{H}^m(\Omega) := [H^m(\Omega)]^2$ and $\mathbb{H}^m(\Omega) := [H^m(\Omega)]^{2 \times 2}$, whose corresponding norms and seminorms will be denoted by $\|\cdot\|_{m,\Omega}$ and $|\cdot|_{m,\Omega}$ in all cases.

Next, let $\gamma_0 : H^1(\Omega) \rightarrow L^2(\Gamma)$ denote the well-known trace operator, which satisfies

$$\|\varphi\|_{0,\Gamma} \leq C_\Gamma \|\varphi\|_{1,\Omega} \quad \forall \varphi \in H^1(\Omega), \quad (1.1)$$

for some $C_\Gamma > 0$. We define the trace space of $H^1(\Omega)$ as

$$H^{1/2}(\Gamma) := \gamma_0(H^1(\Omega)),$$

endowed with the norm

$$\|\psi\|_{1/2,\Gamma} := \inf \left\{ \|w\|_{1,\Omega} : w \in H^1(\Omega), \gamma_0(w) = \psi \right\}. \quad (1.2)$$

Alternatively, $H^{1/2}(\Gamma)$ can be endowed with the following norm, equivalent to $\|\cdot\|_{1/2,\Gamma}$ (see [22, Remark 4.2.3]):

$$\|u\|_{1/2,\Gamma} := \left(\int_{\Gamma} |u|^2 + \int_{\Gamma} \int_{\Gamma} \frac{|u(x) - u(y)|^2}{|x - y|^2} \right)^{1/2} \quad \forall u \in H^{1/2}(\Gamma). \quad (1.3)$$

On the other hand, for $0 < \eta \leq 1$ and following [15, 22], we define the Hölder boundary space on Γ by

$$C^{0,\eta}(\Gamma) := \left\{ g \in C^0(\Gamma) : |g(x) - g(y)| \leq M|x - y|^\eta \quad \forall x, y \in \Gamma, \text{ for some } M > 0 \right\},$$

endowed with the norm

$$\|u\|_{C^{0,\eta}(\Gamma)} := \left(\|u\|_{C^0(\Gamma)}^2 + |u|_{C^{0,\eta}(\Gamma)}^2 \right)^{1/2}, \quad (1.4)$$

where

$$\|u\|_{C^0(\Gamma)} := \sup_{x \in \Gamma} |u(x)| \quad \text{and} \quad |u|_{C^{0,\eta}(\Gamma)} := \sup_{\substack{x, y \in \Gamma \\ x \neq y}} \frac{|u(x) - u(y)|}{|x - y|^\eta}.$$

Then, given $f \in H^{1/2}(\Gamma)$ and $g \in C^{0,\eta}(\Gamma)$ with $0 < \eta \leq 1$, there holds $fg \in H^{1/2}(\Gamma)$ and (see [2, Lemma 1.1])

$$\|fg\|_{1/2,\Gamma} \leq c_\eta \|g\|_{C^{0,\eta}(\Gamma)} \|f\|_{1/2,\Gamma}. \quad (1.5)$$

In the sequel, the dual space of $H^{1/2}(\Gamma)$ is denoted by $H^{-1/2}(\Gamma)$ and we employ $\langle \cdot, \cdot \rangle_\Gamma$ to denote the duality pairing on $H^{-1/2}(\Gamma) \times H^{1/2}(\Gamma)$, which coincides with the $L^2(\Gamma)$ -inner product when applied to functions in $L^2(\Gamma)$. The vector-valued counterparts are denoted by $\mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{H}^{-1/2}(\Gamma)$, corresponding to $H^{1/2}(\Gamma)$ and $H^{-1/2}(\Gamma)$, respectively.

Furthermore, by the Sobolev embedding $H^{1/2}(\Gamma) \hookrightarrow L^4(\Gamma)$, we have

$$\|\psi\|_{L^4(\Gamma)} \leq \widehat{C}_\Gamma \|\psi\|_{1/2,\Gamma} \quad \forall \psi \in H^{1/2}(\Gamma), \quad (1.6)$$

with $\widehat{C}_\Gamma > 0$.

We now recall some definitions and technical results on extension operators (see, e.g., [13, 18]). To that end, we let $\Gamma_1, \Gamma_2 \subset \Gamma$, such that $\overline{\Gamma}_1 \cup \overline{\Gamma}_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$, and for each $j \in \{1, 2\}$, we define the trace operator

$$\gamma_j : H^1(\Omega) \rightarrow L^2(\Gamma_j), \quad \gamma_j(v) := \gamma_0(v)|_{\Gamma_j} \quad \text{a.e. on } \Gamma_j.$$

Then we let

$$H^{1/2}(\Gamma_j) := \gamma_j(H^1(\Omega)), \quad j \in \{1, 2\},$$

endowed with the norm

$$\begin{aligned} \|\phi\|_{1/2,\Gamma_j} &:= \inf \left\{ \|\psi\|_{1/2,\Gamma} : \psi \in H^{1/2}(\Gamma), \psi|_{\Gamma_j} = \phi \right\} \\ &= \inf \left\{ \|v\|_{1,\Omega} : v \in H^1(\Omega), \gamma_0(v)|_{\Gamma_j} = \phi \right\}. \end{aligned} \quad (1.7)$$

We also let

$$\mathbf{H}_{\Gamma_1}^1(\Omega) := \{v \in \mathbf{H}^1(\Omega) : \gamma_1(v) = 0\},$$

and recall that, owing to the well-known generalized Poincaré inequality (see, e.g., [25, 5.11.2]), the following estimate holds:

$$|v|_{1,\Omega} \geq C_P \|v\|_{1,\Omega} \quad \forall v \in \mathbf{H}_{\Gamma_1}^1(\Omega), \quad (1.8)$$

with $C_P > 0$.

Now, let

$$E_{0,\Gamma_2} : \mathbf{H}^{1/2}(\Gamma_2) \rightarrow \mathbf{L}^2(\Gamma),$$

be the extension operator given by

$$E_{0,\Gamma_2}(\xi) := \begin{cases} \xi & \text{on } \Gamma_2, \\ 0 & \text{on } \Gamma_1, \end{cases} \quad \forall \xi \in \mathbf{H}^{1/2}(\Gamma_2).$$

We then introduce the space

$$\mathbf{H}_{00}^{1/2}(\Gamma_2) := \{\xi \in \mathbf{H}^{1/2}(\Gamma_2) : E_{0,\Gamma_2}(\xi) \in \mathbf{H}^{1/2}(\Gamma)\},$$

endowed with the norm

$$\|\xi\|_{1/2,00,\Gamma_2} := \|E_{0,\Gamma_2}(\xi)\|_{1/2,\Gamma}, \quad (1.9)$$

where $\|\cdot\|_{1/2,\Gamma}$ is the norm defined in (1.2).

We also define the extension operator

$$E_{\Gamma_1} : \mathbf{H}^{1/2}(\Gamma_1) \longrightarrow \mathbf{H}^{1/2}(\Gamma), \quad \xi \longmapsto E_{\Gamma_1}(\xi) := z|_{\Gamma},$$

where $z \in \mathbf{H}^1(\Omega)$ is the unique solution of the boundary value problem

$$-\Delta z = 0 \text{ in } \Omega, \quad z = \xi \text{ on } \Gamma_1, \quad \nabla z \cdot \mathbf{n} = 0 \text{ on } \Gamma_2.$$

This operator satisfies

$$\|E_{\Gamma_1}(\xi)\|_{1/2,\Gamma} \leq C \|\xi\|_{1/2,\Gamma_1}, \quad (1.10)$$

where $C > 0$ is a constant independent of ξ . In addition, one can show (see [17, Lemma 2.2]) that for all $\varrho \in \mathbf{H}^{1/2}(\Gamma)$, there exist unique elements

$$\varrho_1 = \varrho|_{\Gamma_1} \in \mathbf{H}^{1/2}(\Gamma_1) \quad \text{and} \quad \varrho_2 := (\varrho - \varrho|_{\Gamma_1})|_{\Gamma_2} \in \mathbf{H}_{00}^{1/2}(\Gamma_2), \quad (1.11)$$

such that

$$\varrho = E_{\Gamma_1}(\varrho_1) + E_{0,\Gamma_2}(\varrho_2) \quad (1.12)$$

and

$$c_1(\|\varrho_1\|_{1/2,\Gamma_1} + \|\varrho_2\|_{1/2,00,\Gamma_2}) \leq \|\varrho\|_{1/2,\Gamma} \leq c_2(\|\varrho_1\|_{1/2,\Gamma_1} + \|\varrho_2\|_{1/2,00,\Gamma_2}), \quad (1.13)$$

with $c_1, c_2 > 0$ only depending on Ω . The latter implies that the following decomposition holds and is stable:

$$\mathbf{H}^{1/2}(\Gamma) = E_{\Gamma_1}(\mathbf{H}^{1/2}(\Gamma_1)) \oplus E_{0,\Gamma_2}(\mathbf{H}_{00}^{1/2}(\Gamma_2)).$$

Now, given $\xi \in H^{-1/2}(\Gamma)$ and $g \in L^2(\Gamma_2)$, we say that $\xi = g$ on Γ_2 in the weak sense if

$$\langle \xi, E_{0,\Gamma_2}(\phi) \rangle_\Gamma = \int_{\Gamma_2} g \phi, \quad \forall \phi \in H_{00}^{1/2}(\Gamma_2). \quad (1.14)$$

Using the preceding definition, we introduce the subspace of functionals in $H^{-1/2}(\Gamma)$ that vanish on Γ_2 in the weak sense:

$$H_{\Gamma_2}^{-1/2}(\Gamma) := \left\{ \xi \in H^{-1/2}(\Gamma) : \langle \xi, E_{0,\Gamma_2}(\phi) \rangle_\Gamma = 0, \quad \forall \phi \in H_{00}^{1/2}(\Gamma_2) \right\}. \quad (1.15)$$

Notice that, given $\lambda \in H^{-1/2}(\Gamma)$ satisfying (1.14), it admits the decomposition

$$\lambda = \lambda_1 + \lambda_2, \quad (1.16)$$

where $\lambda_2 \in H^{-1/2}(\Gamma)$ is defined by

$$\langle \lambda_2, \phi \rangle_\Gamma := \int_{\Gamma_2} g \phi, \quad \forall \phi \in H^{1/2}(\Gamma),$$

and $\lambda_1 := \lambda - \lambda_2$. Observe that from (1.14) it follows that

$$\langle \lambda_1, E_{0,\Gamma_2}(\psi) \rangle_\Gamma = \langle \lambda - \lambda_2, E_{0,\Gamma_2}(\psi) \rangle_\Gamma = 0, \quad \forall \psi \in H_{00}^{1/2}(\Gamma_2),$$

and hence

$$\lambda_1 \in H_{\Gamma_2}^{-1/2}(\Gamma).$$

In addition, the following identity holds:

$$\langle \lambda, \phi \rangle_\Gamma = \langle \lambda_1, \phi \rangle_\Gamma + \int_{\Gamma_2} g \phi, \quad \forall \phi \in H^{1/2}(\Gamma). \quad (1.17)$$

Finally, given $u \in H^1(\Omega)$ and $\varphi \in H^{1/2}(\Gamma_1)$, we say that $u = \varphi$ on Γ_1 if

$$\langle \xi, u \rangle_\Gamma = \langle \xi, E_{\Gamma_1}(\varphi) \rangle_\Gamma, \quad \forall \xi \in H_{\Gamma_2}^{-1/2}(\Gamma). \quad (1.18)$$

In particular, if $\varphi = 0$ on Γ_1 , we deduce that (1.18) implies that $u \in H_{\Gamma_1}^1(\Omega)$.

The following result provides a localization of the $H^{1/2}$ -norm on unions of consecutive boundary segments. For the sake of readability, its proof is postponed to Appendix A in Section 6.

Lemma 1.1 *Let $\Sigma_1, \Sigma_2 \subset \Gamma$ be two consecutive segments of the polygonal boundary Γ , sharing the common vertex $\mathbf{x}_1$. Given $f_1 \in H^{1/2}(\Sigma_1)$ and $f_2 \in H^{1/2}(\Sigma_2)$, define*

$$f = \begin{cases} f_1 & \text{on } \Sigma_1, \\ f_2 & \text{on } \Sigma_2. \end{cases}$$

Assume that the compatibility condition

$$\int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(f_1, f_2; \sigma)|^2}{\sigma} d\sigma < +\infty \quad (1.19)$$

holds, with

$$R_{\mathbf{x}_1}(f_1, f_2; \sigma) := f_2(\mathbf{x}_1(\sigma)) - f_1(\mathbf{x}_1(-\sigma)), \quad \text{for a.e. } \sigma \in (0, \delta_1),$$

where $\delta_1 > 0$ is chosen sufficiently small so that

$$\mathbf{x}_1(\sigma) \in \Sigma_2 \quad \text{and} \quad \mathbf{x}_1(-\sigma) \in \Sigma_1, \quad \forall \sigma \in (0, \delta_1),$$

and $\sigma \mapsto \mathbf{x}_1(\sigma)$, $\sigma \mapsto \mathbf{x}_1(-\sigma)$ are Lipschitz-continuous parametrizations of Σ_2 and Σ_1 , respectively, with Lipschitz-continuous inverses. Then

$$f \in \mathbf{H}^{1/2}(\Sigma_1 \cup \Sigma_2),$$

and there exists a constant $C > 0$, independent of f_1 and f_2 , such that

$$\|f\|_{1/2, \Sigma_1 \cup \Sigma_2} \leq C \left\{ \|f_1\|_{1/2, \Sigma_1} + \|f_2\|_{1/2, \Sigma_2} + \left(\int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(f_1, f_2; \sigma)|^2}{\sigma} d\sigma \right)^{1/2} \right\}. \quad (1.20)$$

We end this section by introducing some further notation. In what follows, $\mathbf{E}_{0, \Gamma_2}$ and $\mathbf{E}_{\Gamma_1}$ denote the vector-valued counterparts of E_{0, Γ_2} and E_{Γ_1} , respectively, acting component-wise, and γ_0 denotes the vector-valued version of γ_0 . Accordingly, we let

$$\mathbf{H}_{\Gamma_2}^{-1/2}(\Gamma) := \left\{ \boldsymbol{\xi} \in \mathbf{H}^{-1/2}(\Gamma) : \langle \boldsymbol{\xi}, \mathbf{E}_{0, \Gamma_2}(\boldsymbol{\varrho}) \rangle_{\Gamma} = 0 \quad \forall \boldsymbol{\varrho} \in \mathbf{H}_{00}^{1/2}(\Gamma_2) \right\}.$$

In addition, given a scalar function ϕ , we denote by ϕ^+ and ϕ^- its positive and negative parts, that is,

$$\phi^+ := \frac{1}{2}(\phi + |\phi|) \quad \text{and} \quad \phi^- := \frac{1}{2}(\phi - |\phi|), \quad (1.21)$$

so that $\phi = \phi^+ + \phi^-$.

Throughout the paper, $\mathbf{0}$ stands for a generic null vector and, when no ambiguity arises, $|\cdot|$ denotes the absolute value in $\mathbb{R}$ and the Euclidean norm in $\mathbb{R}^2$ or $\mathbb{R}^{2 \times 2}$. Moreover, $\|(\cdot, \cdot)\|$ denotes the norm of a product space, defined as the sum of the norms of its components; for instance, $\|(p, \boldsymbol{\lambda})\| := \|p\|_{0, \Omega} + \|\boldsymbol{\lambda}\|_{-1/2, \Gamma}$ for $(p, \boldsymbol{\lambda}) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$. Also, for any $S \subseteq \mathbb{R}^2$, scalar functions u, v , vector fields $\mathbf{u} = (u_i)_{i=1}^2$, $\mathbf{v} = (v_i)_{i=1}^2$, and tensors $\boldsymbol{\sigma} = (\sigma_{ij})_{i,j=1}^2$, $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1}^2$, we write

$$(u, v)_S := \int_S uv, \quad (\mathbf{u}, \mathbf{v})_S := \int_S \mathbf{u} \cdot \mathbf{v}, \quad (\boldsymbol{\sigma}, \boldsymbol{\tau})_S := \int_S \boldsymbol{\sigma} : \boldsymbol{\tau}.$$

Finally, from now on, $C > 0$, with or without subscripts, tildes or hats, denotes a generic constant, independent of the discretization parameters and of the physical parameters, which may take different values at different places. Similarly, given $A, B \geq 0$, we write $A \lesssim B$ (respectively, $A \gtrsim B$) whenever $A \leq C B$ (respectively, $A \geq C B$) for some such constant $C > 0$.

1.2 Model problem

Here we consider the nondimensionalized version of the model problem introduced in [2]. To that end, we let

$$\Omega := \left(0, \frac{a}{b}\right) \times (0, 1) \subset \mathbb{R}^2,$$

where $a, b > 0$ are fixed constants (the physical length and width of the channel, respectively) satisfying $a \gg b$, and let $\Gamma := \partial\Omega$ denote its boundary. We decompose Γ into the mutually disjoint portions

$$\Gamma_{\text{in}} := \{0\} \times (0, 1), \quad \Gamma_{\text{out}} := \{a/b\} \times (0, 1), \quad \Gamma_{\text{m}^+} := (0, a/b) \times \{1\}, \quad \Gamma_{\text{m}^-} := (0, a/b) \times \{0\}, \quad (1.22)$$

so that $\Gamma = \bar{\Gamma}_{\text{in}} \cup \bar{\Gamma}_{\text{m}^+} \cup \bar{\Gamma}_{\text{m}^-} \cup \bar{\Gamma}_{\text{out}}$, and we set $\Gamma_{\text{m}} := \Gamma_{\text{m}^+} \cup \Gamma_{\text{m}^-}$ and

$$\mathbf{x}^+ := (0, 1) = \bar{\Gamma}_{\text{in}} \cap \bar{\Gamma}_{\text{m}^+}, \quad \mathbf{x}^- := (0, 0) = \bar{\Gamma}_{\text{in}} \cap \bar{\Gamma}_{\text{m}^-}.$$

Along Γ , we denote by $\mathbf{n} := (n_1, n_2)^t$ the unit outward normal vector and by $\mathbf{t} := (-n_2, n_1)^t$ the associated counterclockwise tangent vector, as illustrated in Figure 1.1.

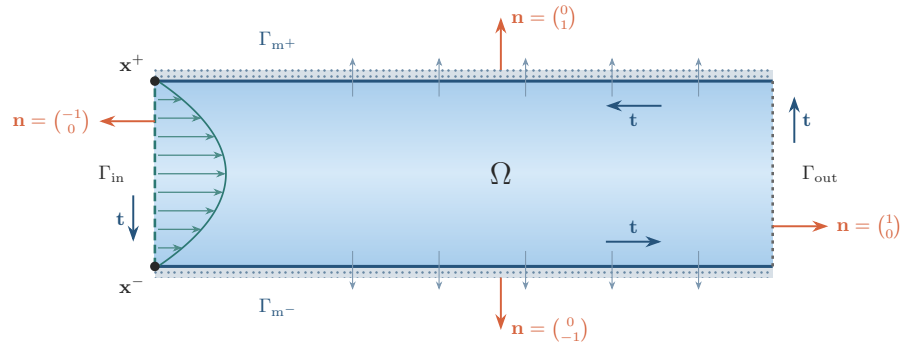


Figure 1.1: Sketch of the computational domain.

The model consists of the incompressible Navier–Stokes equations, governing the fluid flow, coupled with a convection–diffusion equation describing the transport of the solute. The coupling takes place through the convective term in the transport equation and through the membrane boundary conditions, where the fluid flux depends on the solute concentration. More precisely, we consider the following nondimensional system:

$$\begin{aligned} -\operatorname{div}\left(\frac{1}{\operatorname{Re}}\nabla\mathbf{u} - p\mathbf{I}\right) + \operatorname{div}(\mathbf{u} \otimes \mathbf{u}) &= \mathbf{0} \quad \text{in } \Omega, \quad \operatorname{div} \mathbf{u} = 0 \quad \text{in } \Omega, \\ \mathbf{u} &= \mathbf{u}_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \quad \mathbf{u} \cdot \mathbf{n} = a_1 - a_2 \varphi, \quad \mathbf{u} \cdot \mathbf{t} = 0 \quad \text{on } \Gamma_{\text{m}}, \\ \left(\frac{1}{\operatorname{Re}}\nabla\mathbf{u} - p\mathbf{I}\right)\mathbf{n} &= \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^- \mathbf{u} \quad \text{on } \Gamma_{\text{out}}, \end{aligned} \quad (1.23)$$

coupled with

$$\begin{aligned} -\frac{1}{\operatorname{Pe}}\Delta\varphi + \mathbf{u} \cdot \nabla\varphi &= 0 \quad \text{in } \Omega, \quad \varphi = \varphi_{\text{in}} \quad \text{on } \Gamma_{\text{in}}, \\ \left(\varphi\mathbf{u} - \frac{1}{\operatorname{Pe}}\nabla\varphi\right) \cdot \mathbf{n} &= a_3 \varphi, \quad \varphi \geq 0 \quad \text{on } \Gamma_{\text{m}}, \\ \frac{1}{\operatorname{Pe}}\nabla\varphi \cdot \mathbf{n} &= \frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^- \varphi \quad \text{on } \Gamma_{\text{out}}. \end{aligned} \quad (1.24)$$

Here, $\mathbf{I}$ denotes the identity matrix in $\mathbb{R}^{2 \times 2}$. The unknowns are the (nondimensional) fluid velocity $\mathbf{u}$, pressure p , and salt concentration φ . The given data are the Reynolds number $\operatorname{Re} > 0$, the Péclet number $\operatorname{Pe} > 0$, the membrane parameters $a_1, a_2, a_3 > 0$, the prescribed inlet

velocity $\mathbf{u}_{\text{in}} = (u_{1,\text{in}}, u_{2,\text{in}})^t \in \mathbf{H}^{1/2}(\Gamma_{\text{in}})$, and the prescribed inlet concentration $\varphi_{\text{in}} \in H^{1/2}(\Gamma_{\text{in}})$, with $\varphi_{\text{in}} \geq 0$ on Γ_{in} . For convenience, we set $\mathbf{a} := (a_1, a_2, a_3)$.

The Reynolds and Péclet numbers are defined by

$$\text{Re} := \frac{\rho U_0 L}{2\mu}, \quad \text{Pe} := \frac{U_0 L}{\kappa}, \quad (1.25)$$

where $U_0 > 0$ is the characteristic inlet velocity, $L := b$ is the characteristic length (the channel width), ρ is the fluid density, μ is the dynamic viscosity of the fluid, and κ is the salt diffusivity in water. The velocity, pressure and concentration have been scaled by U_0 , ρU_0^2 and a reference concentration $\varphi_0 > 0$, respectively, and the dimensionless membrane parameters a_1, a_2, a_3 are obtained from their physical counterparts as in [2].

Notice that, following [2] and unlike the formulations proposed in [6, 11], the model (1.24) explicitly incorporates the non-negativity constraint $\varphi \geq 0$ on Γ_{m} , which enables us to prove, in Lemma 2.3, that the concentration remains non-negative on Γ_{m} and throughout Ω . Furthermore, on Γ_{out} we impose directional do-nothing boundary conditions, which are known to provide improved numerical stability and performance; see [23, Section 2.4].

In what follows, we assume that $\mathbf{u}_{\text{in}}$ and φ_{in} satisfy the following conditions:

- (i) the limits $\varphi_{\text{in}}^\pm := \lim_{\mathbf{x} \rightarrow \mathbf{x}^\pm} \varphi_{\text{in}}(\mathbf{x})$ exist, and

$$\lim_{\mathbf{x} \rightarrow \mathbf{x}^\pm} u_{2,\text{in}}(\mathbf{x}) = \pm(a_1 - a_2 \varphi_{\text{in}}^\pm); \quad (1.26)$$

- (ii) the first component of the inlet velocity satisfies

$$u_{1,\text{in}} \in H_{00}^{1/2}(\Gamma_{\text{in}}); \quad (1.27)$$

- (iii) there exist $\eta \in (0, 1]$ and relatively open neighborhoods $\mathcal{U}^\pm \subseteq \Gamma_{\text{in}}$ of $\mathbf{x}^\pm$ such that

$$u_{2,\text{in}}|_{\mathcal{U}^\pm}, \varphi_{\text{in}}|_{\mathcal{U}^\pm} \in C^{0,\eta}(\mathcal{U}^\pm). \quad (1.28)$$

Condition (i) is a compatibility requirement between the inlet data and the membrane condition $\mathbf{u} \cdot \mathbf{n} = a_1 - a_2 \varphi$ at the corners $\mathbf{x}^\pm$, while (ii) is compatible with $\mathbf{u} \cdot \mathbf{t} = 0$ on Γ_{m} . Finally, (iii) will be used in Section 2, together with Lemma 1.1, to ensure that the boundary data can be extended to an element of $\mathbf{H}^{1/2}(\Gamma)$.

For instance, if $\varphi_{\text{in}} \geq 0$ is constant and

$$u_{2,\text{in}}(0, y) = (a_1 - a_2 \varphi_{\text{in}})(2y - 1) \quad \forall (0, y) \in \Gamma_{\text{in}},$$

then $\mathbf{u}_{\text{in}}$ satisfies (1.26) for any $u_{1,\text{in}} \in H_{00}^{1/2}(\Gamma_{\text{in}})$. Moreover, since $u_{2,\text{in}}$ is a polynomial and φ_{in} is constant, (1.28) holds for any $\eta \in (0, 1]$.

2 Continuous weak formulation and its well-posedness

In this section, we derive the weak formulation associated with (1.23) and (1.24) and analyze its well-posedness.

2.1 Weak formulation

Unlike [2], we first proceed as in [1] and notice that, since $(\frac{1}{\text{Re}}\nabla\mathbf{u} - p\mathbf{I})\mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$, the boundary condition on Γ_{out} in (1.23) can be expressed as

$$\left\langle \left(\frac{1}{\text{Re}}\nabla\mathbf{u} - p\mathbf{I} \right) \mathbf{n}, \mathbf{E}_{0,\Gamma_{\text{out}}}(\phi) \right\rangle_{\Gamma} = \frac{1}{2}((\mathbf{u} \cdot \mathbf{n})^- \mathbf{u}, \phi)_{\Gamma_{\text{out}}} \quad \forall \phi \in \mathbf{H}_{00}^{1/2}(\Gamma_{\text{out}}). \quad (2.1)$$

Then, according to (1.17), we introduce the additional unknown $\boldsymbol{\lambda} \in \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$ satisfying

$$-\left\langle \left(\frac{1}{\text{Re}}\nabla\mathbf{u} - p\mathbf{I} \right) \mathbf{n}, \boldsymbol{\psi} \right\rangle_{\Gamma} = \langle \boldsymbol{\lambda}, \boldsymbol{\psi} \rangle_{\Gamma} - \frac{1}{2}((\mathbf{u} \cdot \mathbf{n})^- \mathbf{u}, \boldsymbol{\psi})_{\Gamma_{\text{out}}} \quad \forall \boldsymbol{\psi} \in \mathbf{H}^{1/2}(\Gamma). \quad (2.2)$$

Thus, noting that $\mathbf{div}(\mathbf{u} \otimes \mathbf{u}) = (\mathbf{u} \cdot \nabla)\mathbf{u}$ since $\text{div } \mathbf{u} = 0$, multiplying the first equation of (1.23) by a test function $\mathbf{v} \in \mathbf{H}^1(\Omega)$, integrating by parts, and using (2.2), we obtain

$$\frac{1}{\text{Re}}(\nabla\mathbf{u}, \nabla\mathbf{v})_{\Omega} - \frac{1}{2}((\mathbf{u} \cdot \mathbf{n})^- \mathbf{u}, \mathbf{v})_{\Gamma_{\text{out}}} + \int_{\Omega} (\mathbf{u} \cdot \nabla)\mathbf{u} \cdot \mathbf{v} - \int_{\Omega} p \text{div } \mathbf{v} + \langle \boldsymbol{\lambda}, \mathbf{v} \rangle_{\Gamma} = 0 \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Next, we let

$$\mathbf{H}_{\varphi_{\text{in}}}^1(\Omega) := \{ \phi \in \mathbf{H}^1(\Omega) : \phi|_{\Gamma_{\text{in}}} = \varphi_{\text{in}} \},$$

and, for each $\phi \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$, we define the concentration-dependent boundary function

$$\tilde{\mathbf{u}}_{\text{in}}(\phi) := \begin{cases} \mathbf{u}_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ (a_1 - a_2 \phi)\mathbf{n} & \text{on } \Gamma_{\text{m}}. \end{cases} \quad (2.3)$$

As will be shown in Lemma 2.1, assumptions (1.26)–(1.28) and Lemma 1.1 imply that $\tilde{\mathbf{u}}_{\text{in}}(\phi) \in \mathbf{H}^{1/2}(\Gamma_{\text{out}}^c)$ for each such ϕ , where $\Gamma_{\text{out}}^c := \Gamma_{\text{in}} \cup \Gamma_{\text{m}}$. Moreover, the boundary conditions on Γ_{in} and Γ_{m} in (1.23) can be written as $\mathbf{u} = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$ on Γ_{out}^c . Therefore, the following redundant term can be introduced:

$$-\frac{1}{2} \int_{\Gamma_{\text{out}}^c} (\tilde{\mathbf{u}}_{\text{in}}(\varphi) \cdot \mathbf{n}) \mathbf{u} \cdot \mathbf{v} = -\frac{1}{2} \int_{\Gamma_{\text{out}}^c} (\tilde{\mathbf{u}}_{\text{in}}(\varphi) \cdot \mathbf{n}) \tilde{\mathbf{u}}_{\text{in}}(\varphi) \cdot \mathbf{v} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega). \quad (2.4)$$

This term is introduced to cancel the boundary contribution on Γ_{out}^c arising from the convective term, a fact that will be key to derive the well-posedness of the discrete problem (see Lemma 4.1 below).

In turn, the incompressibility constraint and the Dirichlet condition $\mathbf{u} = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$ on Γ_{out}^c are imposed weakly as

$$\int_{\Omega} q \text{div } \mathbf{u} = 0 \quad \forall q \in L^2(\Omega) \quad \text{and} \quad \langle \boldsymbol{\xi}, \mathbf{u} \rangle_{\Gamma} = \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_{\text{out}}^c}(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma} \quad \forall \boldsymbol{\xi} \in \mathbf{H}_{\Gamma_{\text{out}}^c}^{-1/2}(\Gamma).$$

Similarly, since $\frac{1}{\text{Pe}}\nabla\varphi \cdot \mathbf{n} \in \mathbf{H}^{-1/2}(\Gamma)$, the boundary conditions on $\Gamma_{\text{in}}^c := \Gamma_{\text{m}} \cup \Gamma_{\text{out}}$ in (1.24) can be written, according to (1.17), as

$$-\left\langle \frac{1}{\text{Pe}}\nabla\varphi \cdot \mathbf{n}, \phi \right\rangle_{\Gamma} = \langle \lambda, \phi \rangle_{\Gamma} + \int_{\Gamma_{\text{in}}^c} g \phi \quad \forall \phi \in \mathbf{H}^{1/2}(\Gamma),$$

with the additional unknown $\lambda \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ and

$$g := \begin{cases} a_3 \varphi - \varphi \mathbf{u} \cdot \mathbf{n} & \text{on } \Gamma_{\text{m}}, \\ -\frac{1}{2}(\mathbf{u} \cdot \mathbf{n})^- \varphi & \text{on } \Gamma_{\text{out}}. \end{cases}$$

Then, multiplying the first equation of (1.24) by a test function $\psi \in H^1(\Omega)$ and integrating by parts, we arrive at

$$\frac{1}{\text{Pe}} \int_{\Omega} \nabla \varphi \cdot \nabla \psi + a_3 \int_{\Gamma_{\text{m}}} \varphi \psi + \int_{\Omega} (\mathbf{u} \cdot \nabla \varphi) \psi - \int_{\Gamma_{\text{m}}} (\mathbf{u} \cdot \mathbf{n}) \varphi \psi - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{u} \cdot \mathbf{n})^- \varphi \psi + \langle \lambda, \psi \rangle_{\Gamma} = 0,$$

for all $\psi \in H^1(\Omega)$.

Analogously to (2.4), and using that $\mathbf{u} = \mathbf{u}_{\text{in}}$ and $\varphi = \varphi_{\text{in}}$ on Γ_{in} , we introduce the redundant term

$$-\frac{1}{2} \int_{\Gamma_{\text{in}}} (\mathbf{u}_{\text{in}} \cdot \mathbf{n}) \varphi \psi = -\frac{1}{2} \int_{\Gamma_{\text{in}}} (\mathbf{u}_{\text{in}} \cdot \mathbf{n}) \varphi_{\text{in}} \psi \quad \forall \psi \in H^1(\Omega), \quad (2.5)$$

which plays the same role for the transport equation. Finally, the Dirichlet condition $\varphi = \varphi_{\text{in}}$ on Γ_{in} is imposed weakly as

$$\langle \xi, \varphi \rangle_{\Gamma} = \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma} \quad \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma).$$

According to the above, the variational formulation of (1.23)–(1.24) reads: Find $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma))$ and $(\varphi, \lambda) \in H^1(\Omega) \times H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ such that $\varphi \geq 0$ on Γ_{m} , and

$$\begin{aligned} a_F(\mathbf{u}, \mathbf{v}) + O_F^{\varphi}(\mathbf{u}; \mathbf{u}, \mathbf{v}) + b_F(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F_{\varphi}(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b_F(\mathbf{u}, (q, \boldsymbol{\xi})) &= \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_{\text{out}}}(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma} & \forall (q, \boldsymbol{\xi}) \in \mathbf{L}^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma), \\ a_C(\varphi, \psi) + O_C(\mathbf{u}; \varphi, \psi) + b_C(\psi, \lambda) &= G(\psi) & \forall \psi \in H^1(\Omega), \\ b_C(\varphi, \xi) &= \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma} & \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma), \end{aligned} \quad (2.6)$$

where

$$\begin{aligned} a_F(\mathbf{u}, \mathbf{v}) &:= \frac{1}{\text{Re}} \int_{\Omega} \nabla \mathbf{u} : \nabla \mathbf{v}, & b_F(\mathbf{v}, (q, \boldsymbol{\xi})) &:= b_{F,1}(\mathbf{v}, q) + b_{F,2}(\mathbf{v}, \boldsymbol{\xi}), \\ b_{F,1}(\mathbf{v}, q) &:= - \int_{\Omega} q \operatorname{div} \mathbf{v}, & b_{F,2}(\mathbf{v}, \boldsymbol{\xi}) &:= \langle \boldsymbol{\xi}, \mathbf{v} \rangle_{\Gamma}, \\ a_C(\varphi, \psi) &:= \frac{1}{\text{Pe}} \int_{\Omega} \nabla \varphi \cdot \nabla \psi + a_3 \int_{\Gamma_{\text{m}}} \varphi \psi, & b_C(\psi, \xi) &:= \langle \xi, \psi \rangle_{\Gamma}, \\ F_{\varphi}(\mathbf{v}) &:= -\frac{1}{2} \int_{\Gamma_{\text{out}}}^c (\tilde{\mathbf{u}}_{\text{in}}(\varphi) \cdot \mathbf{n}) \tilde{\mathbf{u}}_{\text{in}}(\varphi) \cdot \mathbf{v}, & G(\psi) &:= -\frac{1}{2} \int_{\Gamma_{\text{in}}} (\mathbf{u}_{\text{in}} \cdot \mathbf{n}) \varphi_{\text{in}} \psi, \end{aligned}$$

whereas the forms O_F^{ϕ} and O_C are defined by

$$\begin{aligned} O_F^{\phi}(\mathbf{w}; \mathbf{u}, \mathbf{v}) &:= \int_{\Omega} (\mathbf{w} \cdot \nabla) \mathbf{u} \cdot \mathbf{v} - \frac{1}{2} \int_{\Gamma_{\text{out}}}^c (\tilde{\mathbf{u}}_{\text{in}}(\phi) \cdot \mathbf{n}) \mathbf{u} \cdot \mathbf{v} - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^- \mathbf{u} \cdot \mathbf{v}, \\ O_C(\mathbf{w}; \varphi, \psi) &:= \int_{\Omega} (\mathbf{w} \cdot \nabla \varphi) \psi - \frac{1}{2} \int_{\Gamma_{\text{in}}} (\mathbf{u}_{\text{in}} \cdot \mathbf{n}) \varphi \psi - \int_{\Gamma_{\text{m}}} (\mathbf{w} \cdot \mathbf{n}) \varphi \psi - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^- \varphi \psi, \end{aligned} \quad (2.7)$$

for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\varphi, \psi \in H^1(\Omega)$ and $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$.

2.2 Well-posedness and concentration positivity

We now proceed as in [2] and apply a fixed-point strategy to prove the existence and uniqueness of a solution to (2.6). To this end, we first establish the stability properties of the forms involved.

2.2.1 Stability properties

We begin by observing that a_F , a_C , $b_{F,1}$, $b_{F,2}$ and b_C satisfy

$$\begin{aligned} |a_F(\mathbf{u}, \mathbf{v})| &\leq \frac{1}{\text{Re}} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} & \forall \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega), \\ |a_C(\varphi, \psi)| &\leq L_{a,C} \left(\frac{1}{\text{Pe}} + a_3 \right) \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega} & \forall \varphi, \psi \in H^1(\Omega), \\ |b_{F,1}(\mathbf{v}, q)| &\lesssim \|\mathbf{v}\|_{1,\Omega} \|q\|_{0,\Omega} & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall q \in L^2(\Omega), \\ |b_{F,2}(\mathbf{v}, \boldsymbol{\xi})| &\lesssim \|\mathbf{v}\|_{1,\Omega} \|\boldsymbol{\xi}\|_{-1/2,\Gamma} & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall \boldsymbol{\xi} \in \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma), \\ |b_C(\psi, \xi)| &\lesssim \|\psi\|_{1,\Omega} \|\xi\|_{-1/2,\Gamma} & \forall \psi \in H^1(\Omega), \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma), \end{aligned} \quad (2.8)$$

with $L_{a,C} > 0$ depending only on Ω .

Regarding the forms O_F^ϕ and O_C , the Sobolev embedding $H^1(\Omega) \hookrightarrow L^4(\Omega)$ (see [27, Theorem 1.3.4]) yields

$$\|\mathbf{v}\|_{0,4,\Omega} \lesssim \|\mathbf{v}\|_{1,\Omega} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad (2.9)$$

and analogously for scalar functions, which, together with (1.1), (1.2) and (1.6), allows us to control both the volume and the boundary terms. Then, a straightforward application of Hölder's inequality gives

$$\begin{aligned} |O_F^\phi(\mathbf{w}; \mathbf{u}, \mathbf{v})| &\lesssim (\|\mathbf{w}\|_{1,\Omega} + \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}) \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \\ |O_C(\mathbf{w}; \varphi, \psi)| &\lesssim (\|\mathbf{w}\|_{1,\Omega} + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}) \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega}, \end{aligned} \quad (2.10)$$

for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\varphi, \psi \in H^1(\Omega)$ and $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$. Moreover, using the Lipschitz continuity of $t \mapsto t^-$, we obtain

$$\begin{aligned} |O_F^\phi(\mathbf{w}; \mathbf{u}, \mathbf{v}) - O_F^\phi(\mathbf{z}; \mathbf{u}, \mathbf{v})| &\lesssim \|\mathbf{w} - \mathbf{z}\|_{1,\Omega} \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \\ |O_C(\mathbf{w}; \varphi, \psi) - O_C(\mathbf{z}; \varphi, \psi)| &\lesssim \|\mathbf{w} - \mathbf{z}\|_{1,\Omega} \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega}, \end{aligned} \quad (2.11)$$

for all $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\varphi, \psi \in H^1(\Omega)$ and $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$.

Concerning the right-hand sides of (2.6), the duality pairing bound, together with (1.2), (1.7), (1.6) and (1.10), yields

$$\begin{aligned} |F_\phi(\mathbf{v})| &\lesssim \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}^2 \|\mathbf{v}\|_{1,\Omega} & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \forall \phi \in H_{\varphi_{\text{in}}}^1(\Omega), \\ |\langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_{\text{out}}^c}(\tilde{\mathbf{u}}_{\text{in}}(\phi)) \rangle_\Gamma| &\lesssim \|\boldsymbol{\xi}\|_{-1/2,\Gamma} \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c} & \forall \boldsymbol{\xi} \in \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma), \forall \phi \in H_{\varphi_{\text{in}}}^1(\Omega), \\ |\langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_\Gamma| &\lesssim \|\xi\|_{-1/2,\Gamma} \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} & \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma), \\ |G(\psi)| &\lesssim \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \|\psi\|_{1,\Omega} & \forall \psi \in H^1(\Omega). \end{aligned} \quad (2.12)$$

Note that $\|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}$ appears in the first estimate of (2.10) and in the first two estimates of (2.12). The following lemma provides an explicit bound for this quantity.

Lemma 2.1 *Given $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$, let $\tilde{\mathbf{u}}_{\text{in}}(\phi)$ be the function defined as in (2.3), with $\mathbf{u}_{\text{in}} = (u_{1,\text{in}}, u_{2,\text{in}})^t$ satisfying (1.26), (1.27) and (1.28). Then $\tilde{\mathbf{u}}_{\text{in}}(\phi) \in \mathbf{H}^{1/2}(\Gamma_{\text{out}}^c)$ and the following estimate holds*

$$\|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2, \Gamma_{\text{out}}^c} \lesssim \mathbf{C}_1(\text{data}) + a_2 \|\phi\|_{1, \Omega}, \quad (2.13)$$

with

$$\begin{aligned} \mathbf{C}_1(\text{data}) := & \|u_{1,\text{in}}\|_{1/2, 00, \Gamma_{\text{in}}} + \|u_{2,\text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(|a_1 - a_2 \varphi_{\text{in}}^\star| + |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right) \\ & + a_2 \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} + a_2 \sum_{\star \in \{+, -\}} \left(|\varphi_{\text{in}}^\star| + |\varphi_{\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right). \end{aligned} \quad (2.14)$$

In addition,

$$\|\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \lesssim a_2 \|\phi_1 - \phi_2\|_{1, \Omega}, \quad (2.15)$$

for all $\phi_1, \phi_2 \in H_{\varphi_{\text{in}}}^1(\Omega)$.

Proof. First, define

$$\tilde{u}_2 := \begin{cases} u_{2,\text{in}} & \text{on } \Gamma_{\text{in}}, \\ a_1 - a_2 \varphi_{\text{in}}^+ & \text{on } \Gamma_{\text{m}^+}, \\ -(a_1 - a_2 \varphi_{\text{in}}^-) & \text{on } \Gamma_{\text{m}^-}, \end{cases}$$

where $\varphi_{\text{in}}^\pm \in \mathbb{R}$ are the constants from (1.26).

In what follows we employ Lemma 1.1 (which extends straightforwardly to three consecutive sides) to prove that $\tilde{u}_2 \in H^{1/2}(\Gamma_{\text{out}}^c)$. To that end, we set

$$f_1 := u_{2,\text{in}} \text{ on } \Gamma_{\text{in}} \quad \text{and} \quad f_2 := \begin{cases} a_1 - a_2 \varphi_{\text{in}}^+ & \text{on } \Gamma_{\text{m}^+}, \\ -(a_1 - a_2 \varphi_{\text{in}}^-) & \text{on } \Gamma_{\text{m}^-}, \end{cases}$$

and let $\mathbf{x}^+(\sigma)$ denote the arclength parametrization of $\Gamma_{\text{in}} \cup \Gamma_{\text{m}^+}$ near $\mathbf{x}^+$, with $\mathbf{x}^+(\sigma) \in \Gamma_{\text{m}^+}$ and $\mathbf{x}^+(-\sigma) \in \mathcal{U}^+$ for all $\sigma \in (0, \delta^+)$, with $\delta^+ > 0$ sufficiently small, so that $\mathbf{x}^+(0) = \mathbf{x}^+$.

Using that $f_2 \equiv a_1 - a_2 \varphi_{\text{in}}^+$ is constant near $\mathbf{x}^+$ and $f_1 = u_{2,\text{in}} \in C^{0,\eta}(\mathcal{U}^+)$, for all $\sigma \in (0, \delta^+)$, it follows that

$$|f_1(\mathbf{x}^+(-\sigma)) - f_1(\mathbf{x}^+)| \leq |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^+)} \sigma^\eta \quad \text{and} \quad |f_2(\mathbf{x}^+(\sigma)) - f_2(\mathbf{x}^+)| = 0. \quad (2.16)$$

Moreover, the compatibility condition (1.26) gives exactly the pointwise matching

$$f_1(\mathbf{x}^+) = u_{2,\text{in}}(\mathbf{x}^+) = a_1 - a_2 \varphi_{\text{in}}^+ = f_2(\mathbf{x}^+). \quad (2.17)$$

Combining (2.16) and (2.17) via the triangle inequality, we obtain, for all $\sigma \in (0, \delta^+)$,

$$\begin{aligned} |f_2(\mathbf{x}^+(\sigma)) - f_1(\mathbf{x}^+(-\sigma))| & \leq |f_2(\mathbf{x}^+(\sigma)) - f_2(\mathbf{x}^+)| + |f_1(\mathbf{x}^+) - f_1(\mathbf{x}^+(-\sigma))| \\ & \leq |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^+)} \sigma^\eta. \end{aligned} \quad (2.18)$$

In this way,

$$\begin{aligned} \int_0^{\delta^+} \frac{|R_{\mathbf{x}^+}(f_1, f_2; \sigma)|^2}{\sigma} d\sigma &= \int_0^{\delta^+} \frac{|f_2(\mathbf{x}^+(\sigma)) - f_1(\mathbf{x}^+(-\sigma))|^2}{\sigma} d\sigma \\ &\leq |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^+)}^2 \int_0^{\delta^+} \sigma^{2\eta-1} d\sigma \\ &= \frac{(\delta^+)^{2\eta}}{2\eta} |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^+)}^2 < +\infty. \end{aligned} \quad (2.19)$$

An analogous computation at $\mathbf{x}^-$ gives

$$\int_0^{\delta^-} \frac{|R_{\mathbf{x}^-}(f_1, f_2; \sigma)|^2}{\sigma} d\sigma \leq \frac{(\delta^-)^{2\eta}}{2\eta} |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^-)}^2 < +\infty. \quad (2.20)$$

Hence, condition (1.19) of Lemma 1.1 holds at both $\mathbf{x}^+$ and $\mathbf{x}^-$, and therefore

$$\tilde{u}_2 \in H^{1/2}(\Gamma_{\text{out}}^c).$$

In addition, from (1.20), (2.19) and (2.20), the following estimate holds

$$\|\tilde{u}_2\|_{1/2, \Gamma_{\text{out}}^c} \lesssim \|u_{2,\text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(\|a_1 - a_2 \varphi_{\text{in}}^\star\|_{1/2, \Gamma_{\text{m}^\star}} + |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right). \quad (2.21)$$

However, since any constant on $\Gamma_{\text{m}^\star}$ is the trace of the same constant function in Ω , from (1.7) we deduce that

$$\|a_1 - a_2 \varphi_{\text{in}}^\star\|_{1/2, \Gamma_{\text{m}^\star}} \leq |\Omega|^{1/2} |a_1 - a_2 \varphi_{\text{in}}^\star|, \quad \forall \star \in \{+, -\},$$

which, together with (2.21), implies

$$\|\tilde{u}_2\|_{1/2, \Gamma_{\text{out}}^c} \lesssim \|u_{2,\text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(|a_1 - a_2 \varphi_{\text{in}}^\star| + |u_{2,\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right). \quad (2.22)$$

Now let

$$\tilde{\varphi}_{\text{in}} := \begin{cases} \varphi_{\text{in}} & \text{on } \Gamma_{\text{in}}, \\ \varphi_{\text{in}}^+ & \text{on } \Gamma_{\text{m}^+}, \\ \varphi_{\text{in}}^- & \text{on } \Gamma_{\text{m}^-}. \end{cases} \quad (2.23)$$

Employing Lemma 1.1, using that $\varphi_{\text{in}} \in C^{0,\eta}(\mathcal{U}^\pm)$ (cf. (1.28)), and proceeding analogously as for $\tilde{u}_2$, it is possible to prove that $\tilde{\varphi}_{\text{in}} \in H^{1/2}(\Gamma_{\text{out}}^c)$ and that the following estimate holds,

$$\begin{aligned} \|\tilde{\varphi}_{\text{in}}\|_{1/2, \Gamma_{\text{out}}^c} &\lesssim \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(\|\varphi_{\text{in}}^\star\|_{1/2, \Gamma_{\text{m}^\star}} + |\varphi_{\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right) \\ &\lesssim \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(|\varphi_{\text{in}}^\star| + |\varphi_{\text{in}}|_{C^{0,\eta}(\mathcal{U}^\star)} \right). \end{aligned} \quad (2.24)$$

Now, given $\eta \in (0, 1]$ being the parameter satisfying (1.28), we let $g \in C^{0,\eta}(\Gamma)$ be such that $g = 1$ on Γ_{m^+} and $g = -1$ on Γ_{m^-} , and given $\phi \in H^1(\Omega)$, we define

$$\mathbf{s}(\phi) := (E_{0, \Gamma_{\text{in}}}(u_{1,\text{in}}), \quad E_{\Gamma_{\text{out}}^c}(\tilde{u}_2 + a_2 \tilde{\varphi}_{\text{in}} g) - a_2 \gamma_0(\phi) g)^t \in \mathbf{H}^{1/2}(\Gamma). \quad (2.25)$$

After simple computations it is not difficult to see that if $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$, there hold

$$\mathbf{s}(\phi)|_{\Gamma_{\text{in}}} = \mathbf{u}_{\text{in}} \quad \text{on } \Gamma_{\text{in}} \quad \text{and} \quad \mathbf{s}(\phi)|_{\Gamma_{\text{m}}} = (a_1 - a_2 \phi) \mathbf{n} \quad \text{on } \Gamma_{\text{m}},$$

that is, $\mathbf{s}(\phi) = \tilde{\mathbf{u}}_{\text{in}}(\phi)$ on Γ_{out}^c for all $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$.

Using (1.2), (1.5), (1.7), (1.9), (1.10), (2.22) and (2.24), we deduce that

$$\begin{aligned}
\|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2, \Gamma_{\text{out}}^c} &\leq \|\mathbf{s}(\phi)\|_{1/2, \Gamma} \\
&\leq \|E_{0, \Gamma_{\text{in}}}(u_{1, \text{in}})\|_{1/2, \Gamma} + \|E_{\Gamma_{\text{out}}^c}(\tilde{u}_2 + a_2 \tilde{\varphi}_{\text{in}} g)\|_{1/2, \Gamma} + a_2 \|\gamma_0(\phi)g\|_{1/2, \Gamma} \\
&\lesssim \|u_{1, \text{in}}\|_{1/2, 00, \Gamma_{\text{in}}} + \|u_{2, \text{in}}\|_{1/2, \Gamma_{\text{in}}} + \sum_{\star \in \{+, -\}} \left(|a_1 - a_2 \varphi_{\text{in}}^{\star}| + |u_{2, \text{in}}|_{C^{0, \eta}(\mathcal{U}^{\star})} \right) \\
&\quad + a_2 \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} + a_2 \sum_{\star \in \{+, -\}} \left(|\varphi_{\text{in}}^{\star}| + |\varphi_{\text{in}}|_{C^{0, \eta}(\mathcal{U}^{\star})} \right) + a_2 \|\phi\|_{1, \Omega},
\end{aligned} \tag{2.26}$$

for all $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$, where, for the product $\tilde{\varphi}_{\text{in}} g$, we have applied (1.5) to $E_{\Gamma_{\text{out}}^c}(\tilde{\varphi}_{\text{in}})g$. This implies (2.13).

Now, for (2.15), given $\phi_1, \phi_2 \in H_{\varphi_{\text{in}}}^1(\Omega)$, we let $w := \phi_1 - \phi_2$ and first observe that

$$\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2) = \begin{cases} \mathbf{0} & \text{on } \Gamma_{\text{in}}, \\ -a_2 w \mathbf{n} & \text{on } \Gamma_{\text{m}}. \end{cases}$$

In turn, since $\phi_1 = \phi_2$ on Γ_{in} , it follows that $w \in H_{\Gamma_{\text{in}}}^1(\Omega)$. Then, with g as above, we let $\hat{\mathbf{s}}(w) := -a_2(0, \gamma_0(w)g)^t \in \mathbf{H}^{1/2}(\Gamma)$, and observe that $\hat{\mathbf{s}}(w) = \tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)$ on Γ_{out}^c . Consequently, from (1.7), (1.2) and (1.5), we obtain

$$\|\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \leq \|\hat{\mathbf{s}}(w)\|_{1/2, \Gamma} = a_2 \|\gamma_0(w)g\|_{1/2, \Gamma} \lesssim a_2 \|w\|_{1, \Omega},$$

which implies (2.15) and concludes the proof. $\square$

Notice that, using (2.15), adding and subtracting $(\tilde{\mathbf{u}}_{\text{in}}(\phi_2) \cdot \mathbf{n}) \tilde{\mathbf{u}}_{\text{in}}(\phi_1)$ inside the integral defining F_ϕ , and applying Hölder's inequality together with (1.2), (1.6) and (1.7), we obtain

$$\begin{aligned}
|F_{\phi_1}(\mathbf{v}) - F_{\phi_2}(\mathbf{v})| &\lesssim \left(\|\tilde{\mathbf{u}}_{\text{in}}(\phi_1)\|_{1/2, \Gamma_{\text{out}}^c} + \|\tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \right) \|\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \|\mathbf{v}\|_{1, \Omega} \\
&\lesssim a_2 \left(\|\tilde{\mathbf{u}}_{\text{in}}(\phi_1)\|_{1/2, \Gamma_{\text{out}}^c} + \|\tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \right) \|\phi_1 - \phi_2\|_{1, \Omega} \|\mathbf{v}\|_{1, \Omega},
\end{aligned} \tag{2.27}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\phi_1, \phi_2 \in H_{\varphi_{\text{in}}}^1(\Omega)$, with hidden constants depending only on Ω .

Now, we let $\mathbf{V}_F$ and V_C be the kernels of the bilinear forms b_F and b_C , respectively, that is

$$\mathbf{V}_F := \left\{ \mathbf{v} \in \mathbf{H}^1(\Omega) : b_F(\mathbf{v}, (q, \xi)) = 0 \quad \forall (q, \xi) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma) \right\}$$

and

$$V_C := \left\{ \psi \in H^1(\Omega) : b_C(\psi, \xi) = 0 \quad \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma) \right\}.$$

It is not difficult to see from the definition of b_F and b_C that

$$\mathbf{V}_F = \left\{ \mathbf{v} \in \mathbf{H}_{\Gamma_{\text{out}}^c}^1(\Omega) : \operatorname{div} \mathbf{v} = 0 \right\}, \tag{2.28}$$

and

$$V_C = H_{\Gamma_{\text{in}}}^1(\Omega). \tag{2.29}$$

Then, from the Poincaré inequality (1.8), we obtain

$$a_F(\mathbf{v}, \mathbf{v}) \gtrsim \frac{1}{\text{Re}} \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{V}_F, \quad (2.30)$$

and

$$a_C(\psi, \psi) \gtrsim \frac{1}{\text{Pe}} \|\psi\|_{1,\Omega}^2 + a_3 \|\psi\|_{0,\Gamma_m}^2 \quad \forall \psi \in V_C. \quad (2.31)$$

Now, given $\mathbf{w}, \mathbf{v} \in \mathbf{H}^1(\Omega)$, $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$ and $\psi \in H^1(\Omega)$, integrating by parts, the following identities hold:

$$\begin{aligned} O_F^\phi(\mathbf{w}; \mathbf{v}, \mathbf{v}) &= \frac{1}{2} \left(- \int_{\Omega} |\mathbf{v}|^2 \text{div}(\mathbf{w}) + \int_{\Gamma} (\mathbf{w} \cdot \mathbf{n}) |\mathbf{v}|^2 \right) \\ &\quad - \frac{1}{2} \int_{\Gamma_{\text{out}}^c} (\tilde{\mathbf{u}}_{\text{in}}(\phi) \cdot \mathbf{n}) |\mathbf{v}|^2 - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^- |\mathbf{v}|^2, \\ O_C(\mathbf{w}; \psi, \psi) &= - \frac{1}{2} \int_{\Omega} \psi^2 \text{div}(\mathbf{w}) + \frac{1}{2} \int_{\Gamma} \psi^2 (\mathbf{w} \cdot \mathbf{n}) - \frac{1}{2} \int_{\Gamma_{\text{in}}} (\mathbf{u}_{\text{in}} \cdot \mathbf{n}) \psi^2 \\ &\quad - \int_{\Gamma_m} (\mathbf{w} \cdot \mathbf{n}) \psi^2 - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^- \psi^2. \end{aligned} \quad (2.32)$$

In particular, from (2.30) and the first identity in (2.32), and since $\mathbf{v} = \mathbf{0}$ on Γ_{out}^c , for $\mathbf{v} \in \mathbf{V}_F$, we deduce that for any $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\Omega)$ satisfying $\text{div} \mathbf{w} = 0$ in Ω , there holds

$$a_F(\mathbf{v}, \mathbf{v}) + O_F^\phi(\mathbf{w}; \mathbf{v}, \mathbf{v}) \gtrsim \frac{1}{\text{Re}} \|\mathbf{v}\|_{1,\Omega}^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^+ |\mathbf{v}|^2 \gtrsim \frac{1}{\text{Re}} \|\mathbf{v}\|_{1,\Omega}^2 \quad \forall \mathbf{v} \in \mathbf{V}_F. \quad (2.33)$$

Similarly, from (2.31) and the second identity in (2.32) we obtain that for any $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$ such that $\phi \geq 0$ on Γ_m and $\mathbf{w} \in \mathbf{H}^1(\Omega)$ satisfying $\text{div} \mathbf{w} = 0$ in Ω and $\mathbf{w} \cdot \mathbf{n} = \tilde{\mathbf{u}}_{\text{in}}(\phi) \cdot \mathbf{n} = a_1 - a_2 \phi$ on Γ_m , and since $\psi = 0$ on Γ_{in} , for $\psi \in V_C$, there holds

$$\begin{aligned} a_C(\psi, \psi) + O_C(\mathbf{w}; \psi, \psi) &\geq \frac{C_P^2}{\text{Pe}} \|\psi\|_{1,\Omega}^2 + a_3 \|\psi\|_{0,\Gamma_m}^2 - \frac{1}{2} \int_{\Gamma_m} (\mathbf{w} \cdot \mathbf{n}) \psi^2 \\ &\geq \frac{C_P^2}{\text{Pe}} \|\psi\|_{1,\Omega}^2 + \left(a_3 - \frac{a_1}{2} \right) \|\psi\|_{0,\Gamma_m}^2 \quad \forall \psi \in V_C. \end{aligned} \quad (2.34)$$

We end this section by establishing the corresponding inf-sup estimates for b_F and b_C .

Lemma 2.2 *The following estimates hold:*

$$\sup_{\substack{\mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{v} \neq \mathbf{0}}} \frac{|b_F(\mathbf{v}, (q, \boldsymbol{\xi}))|}{\|\mathbf{v}\|_{1,\Omega}} \gtrsim (\|q\|_{0,\Omega} + \|\boldsymbol{\xi}\|_{-1/2,\Gamma}), \quad \forall (q, \boldsymbol{\xi}) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma), \quad (2.35)$$

and

$$\sup_{\substack{\psi \in H^1(\Omega) \\ \psi \neq 0}} \frac{|b_C(\psi, \xi)|}{\|\psi\|_{1,\Omega}} \gtrsim \|\xi\|_{-1/2,\Gamma}, \quad \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma). \quad (2.36)$$

Proof. Let $(q, \xi) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$. We first observe that, given $\mathbf{v} \in \mathbf{H}_{\Gamma_{\text{out}}}^1(\Omega)$, it follows that $\mathbf{v}|_{\Gamma} = \mathbf{E}_{0, \Gamma_{\text{out}}}(\mathbf{v}|_{\Gamma_{\text{out}}})$ with $\mathbf{v}|_{\Gamma_{\text{out}}} \in \mathbf{H}_{00}^{1/2}(\Gamma_{\text{out}})$, and thus $b_{F,2}(\mathbf{v}, \xi) = \langle \xi, \mathbf{v} \rangle_{\Gamma} = 0$. Then surjectivity of the operator

$$\text{div} : \mathbf{H}_0^1(\Omega) \rightarrow L_0^2(\Omega) := \left\{ q \in L^2(\Omega) : \int_{\Omega} q = 0 \right\},$$

the orthogonal decomposition $L^2(\Omega) = L_0^2(\Omega) \oplus \mathbb{R}$ and the fact that $\mathbf{H}_{\Gamma_{\text{out}}}^1(\Omega) \subseteq \mathbf{H}^1(\Omega)$, imply

$$S_F := \sup_{\substack{\mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{v} \neq \mathbf{0}}} \frac{|b_F(\mathbf{v}, (q, \xi))|}{\|\mathbf{v}\|_{1,\Omega}} \geq \sup_{\substack{\mathbf{v} \in \mathbf{H}_{\Gamma_{\text{out}}}^1(\Omega) \\ \mathbf{v} \neq \mathbf{0}}} \frac{|b_{F,1}(\mathbf{v}, q)|}{\|\mathbf{v}\|_{1,\Omega}} \geq C_1 \|q\|_{0,\Omega}, \quad (2.37)$$

with $C_1 > 0$ depending on Ω . In turn, applying the Cauchy-Schwarz inequality and proceeding analogously to [19, Section 2.4.4], we deduce that

$$S_F \geq \sup_{\substack{\mathbf{v} \in \mathbf{H}^1(\Omega) \\ \mathbf{v} \neq \mathbf{0}}} \frac{|\langle \xi, \mathbf{v} \rangle_{\Gamma}|}{\|\mathbf{v}\|_{1,\Omega}} - C_3 \|q\|_{0,\Omega} \geq C_2 \|\xi\|_{-1/2,\Gamma} - C_3 \|q\|_{0,\Omega}, \quad (2.38)$$

with $C_2, C_3 > 0$. In this way, combining (2.37) and (2.38), we obtain

$$\left(1 + \frac{2C_3}{C_1}\right) S_F \geq C_2 \|\xi\|_{-1/2,\Gamma} + C_3 \|q\|_{0,\Omega},$$

which clearly implies (2.35). We conclude the proof by noticing that the inf-sup condition (2.36) can be found in [19, Section 2.4.4]. We omit further details. $\square$

2.2.2 Positivity of the concentration

The following result establishes the non-negativity of the concentration in Ω .

Lemma 2.3 *Assume that $a_3 - \frac{1}{2}a_1 \geq 0$, and let $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$ be such that $\phi \geq 0$ on Γ_{in} , and $\mathbf{w} \in \mathbf{H}^1(\Omega)$ be such that $\text{div } \mathbf{w} = 0$ in Ω and $\mathbf{w} = \tilde{\mathbf{u}}_{\text{in}}(\phi)$ on Γ_{out}^c . If $(\varphi, \lambda) \in H^1(\Omega) \times H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ solves*

$$\begin{aligned} a_C(\varphi, \psi) + O_C(\mathbf{w}; \varphi, \psi) + b_C(\psi, \lambda) &= G(\psi) & \forall \psi \in H^1(\Omega), \\ b_C(\varphi, \xi) &= \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma} & \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma), \end{aligned} \quad (2.39)$$

then

$$\varphi \geq 0 \quad \text{a.e. in } \Omega \quad \text{and} \quad \varphi \geq 0 \quad \text{a.e. on } \Gamma_{\text{in}}. \quad (2.40)$$

Proof. From the second equation in (2.39) and (1.18), we have that $\varphi = \varphi_{\text{in}}$ on Γ_{in} , and since $\varphi_{\text{in}} \geq 0$, it follows that $\varphi^- = 0$ on Γ_{in} . In addition, similarly to [3, Lemma 5.2.24], one can show that $\varphi^- \in H^1(\Omega)$, with $\nabla \varphi^- = \nabla \varphi$ a.e. in $\{\mathbf{x} \in \Omega : \varphi(\mathbf{x}) < 0\}$ and $\nabla \varphi^- = \mathbf{0}$ elsewhere, and that $\gamma_0(\varphi^-) = (\gamma_0(\varphi))^-$. Hence $\varphi^- \in H_{\Gamma_{\text{in}}}^1(\Omega) = V_C$ and

$$\varphi \varphi^- = |\varphi^-|^2, \quad \nabla \varphi \cdot \nabla \varphi^- = |\nabla \varphi^-|^2 \quad \text{and} \quad \varphi^- \nabla \varphi = \varphi^- \nabla \varphi^- \quad \text{a.e. in } \Omega, \quad (2.41)$$

where the first identity also holds a.e. on Γ for the corresponding traces. Then, using (2.41), $\operatorname{div} \mathbf{w} = 0$ and $\varphi^- = 0$ on Γ_{in} , integrating by parts we deduce that

$$\int_{\Omega} (\mathbf{w} \cdot \nabla \varphi) \varphi^- = \frac{1}{2} \int_{\Omega} \mathbf{w} \cdot \nabla |\varphi^-|^2 = \frac{1}{2} \int_{\Gamma_{\text{m}} \cup \Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n}) |\varphi^-|^2. \quad (2.42)$$

Now, we take $\psi = \varphi^-$ in the first equation of (2.39). Since $\varphi^- = 0$ on Γ_{in} , we have $G(\varphi^-) = 0$ and, noticing that $\varphi^-|_{\Gamma} = E_{0, \Gamma_{\text{in}}^c}(\varphi^-|_{\Gamma_{\text{in}}^c})$ with $\varphi^-|_{\Gamma_{\text{in}}^c} \in H_{00}^{1/2}(\Gamma_{\text{in}}^c)$, we also have $b_C(\varphi^-, \lambda) = \langle \lambda, \varphi^- \rangle_{\Gamma} = 0$ (cf. (1.15)). Thus, from (2.41) and (2.42), and noting that the term on Γ_{in} in O_C vanishes, we obtain

$$\begin{aligned} 0 &= a_C(\varphi, \varphi^-) + O_C(\mathbf{w}; \varphi, \varphi^-) \\ &= \frac{1}{\text{Pe}} \int_{\Omega} |\nabla \varphi^-|^2 + a_3 \int_{\Gamma_{\text{m}}} |\varphi^-|^2 + \frac{1}{2} \int_{\Gamma_{\text{m}}} (\mathbf{w} \cdot \mathbf{n}) |\varphi^-|^2 - \int_{\Gamma_{\text{m}}} (\mathbf{w} \cdot \mathbf{n}) |\varphi^-|^2 \\ &\quad + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n}) |\varphi^-|^2 - \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^- |\varphi^-|^2 \\ &= \frac{1}{\text{Pe}} |\varphi^-|_{1, \Omega}^2 + \int_{\Gamma_{\text{m}}} \left(a_3 - \frac{1}{2} \mathbf{w} \cdot \mathbf{n} \right) |\varphi^-|^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^+ |\varphi^-|^2. \end{aligned}$$

In this way, recalling that $\mathbf{w} \cdot \mathbf{n} = \tilde{\mathbf{u}}_{\text{in}}(\phi) \cdot \mathbf{n} = a_1 - a_2 \phi$ on Γ_{m} , and using (1.8), we arrive at

$$0 \geq \frac{C_P^2}{\text{Pe}} \|\varphi^-\|_{1, \Omega}^2 + \left(a_3 - \frac{a_1}{2} \right) \|\varphi^-\|_{0, \Gamma_{\text{m}}}^2 + \frac{a_2}{2} \int_{\Gamma_{\text{m}}} \phi |\varphi^-|^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w} \cdot \mathbf{n})^+ |\varphi^-|^2.$$

Since $a_3 - \frac{a_1}{2} \geq 0$ and $\phi \geq 0$ on Γ_{m} , all the terms on the right-hand side are non-negative, and hence $\varphi^- = 0$ in $H^1(\Omega)$, which, together with the continuity of the trace operator, yields (2.40). $\square$

2.2.3 The fixed-point operator

In what follows, we introduce the associated fixed-point operator that will allow us to prove the well-posedness of problem (2.6) by means of Banach's fixed point theorem. First, similarly to [2, 4], we let

$$\begin{aligned} X &:= \left\{ (\mathbf{w}, \phi) \in \mathbf{H}^1(\Omega) \times H_{\varphi_{\text{in}}}^1(\Omega) : b_F(\mathbf{w}, (q, \xi)) = \langle \xi, \mathbf{E}_{\Gamma_{\text{out}}^c}(\tilde{\mathbf{u}}_{\text{in}}(\phi)) \rangle_{\Gamma}, \right. \\ &\quad \left. \forall (q, \xi) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma) \text{ and } \phi \geq 0 \text{ on } \Gamma_{\text{m}} \right\} \\ &= \left\{ (\mathbf{w}, \phi) \in \mathbf{H}^1(\Omega) \times H_{\varphi_{\text{in}}}^1(\Omega) : \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \text{ and } \mathbf{w} = \tilde{\mathbf{u}}_{\text{in}}(\phi) \text{ on } \Gamma_{\text{out}}^c \text{ and } \phi \geq 0 \text{ on } \Gamma_{\text{m}} \right\}, \end{aligned}$$

and define the auxiliary operator $\mathcal{L} : X \rightarrow H^1(\Omega)$ given by

$$\mathcal{L}(\mathbf{w}, \phi) := \varphi,$$

with $\varphi \in H^1(\Omega)$ being the first component of the unique solution (to be confirmed below) of the linearized problem: Find $(\varphi, \lambda) \in H^1(\Omega) \times H_{\Gamma_{\text{in}}^c}^{-1/2}(\Gamma)$, such that

$$\begin{aligned} a_C(\varphi, \psi) + O_C(\mathbf{w}; \varphi, \psi) + b_C(\psi, \lambda) &= G(\psi) & \forall \psi \in H^1(\Omega), \\ b_C(\varphi, \xi) &= \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma} & \forall \xi \in H_{\Gamma_{\text{in}}^c}^{-1/2}(\Gamma). \end{aligned} \quad (2.43)$$

In addition, we let $\mathcal{S} : \mathbf{H}^1(\operatorname{div}^0; \Omega) \times \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega) \rightarrow \mathbf{H}^1(\Omega)$, where $\mathbf{H}^1(\operatorname{div}^0; \Omega) := \{\mathbf{w} \in \mathbf{H}^1(\Omega) : \operatorname{div} \mathbf{w} = 0 \text{ in } \Omega\}$, be the operator given by

$$\mathcal{S}(\mathbf{w}, \phi) := \mathbf{u},$$

with $\mathbf{u} \in \mathbf{H}^1(\Omega)$ being the first component of the unique solution (to be confirmed below) of problem: Find $(\mathbf{u}, p, \boldsymbol{\lambda}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} a_F(\mathbf{u}, \mathbf{v}) + O_F^\phi(\mathbf{w}; \mathbf{u}, \mathbf{v}) + b_F(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F_\phi(\mathbf{v}) \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b_F(\mathbf{u}, (q, \boldsymbol{\xi})) &= \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\phi)) \rangle_\Gamma \quad \forall (q, \boldsymbol{\xi}) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma). \end{aligned} \quad (2.44)$$

Thus, we let $\mathcal{J} : X \rightarrow X$ be the operator defined by

$$\mathcal{J}(\mathbf{w}, \phi) := \left(\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi)), \mathcal{L}(\mathbf{w}, \phi) \right) \quad \forall (\mathbf{w}, \phi) \in X, \quad (2.45)$$

and realize that analyzing the well-posedness of problem (2.6) is equivalent to studying the unique solvability of the fixed point problem: Find $(\mathbf{u}, \varphi) \in X$, such that

$$\mathcal{J}(\mathbf{u}, \varphi) = (\mathbf{u}, \varphi). \quad (2.46)$$

As a first step, we verify that $\mathcal{J}$ is actually a well-defined operator, in the sense that both $\mathcal{S}$ and $\mathcal{L}$ are well-defined, that is, that problems (2.44) and (2.43) are uniquely solvable for each $(\mathbf{w}, \phi) \in X$. This is established next.

2.2.4 Well-definiteness of $\mathcal{J}$

We begin by mentioning that, due to the inf-sup condition of the bilinear form b_F provided in Lemma 2.2, the set X is meaningful, in the sense that, given any $\phi \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$ such that $\phi \geq 0$ on Γ_{m} , there exists at least one $\mathbf{w} \in \mathbf{H}^1(\Omega)$ such that $(\mathbf{w}, \phi) \in X$.

The following lemma establishes the well-definiteness of operator $\mathcal{L}$, whose proof relies on the well-posedness of problem (2.43).

Lemma 2.4 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ satisfy $a_3 - \frac{1}{2}a_1 \geq 0$ and let $(\mathbf{w}, \phi) \in X$. Then problem (2.43) has a unique solution $(\varphi, \lambda) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$, and hence $\mathcal{L}(\mathbf{w}, \phi) = \varphi$ is well defined. Moreover, $\mathcal{L}(\mathbf{w}, \phi) \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$, $\mathcal{L}(\mathbf{w}, \phi) \geq 0$ in Ω and on Γ_{m} , and the following estimate holds:*

$$\|\mathcal{L}(\mathbf{w}, \phi)\|_{1, \Omega} \lesssim \mathbf{C}_2(\text{data}) + \text{Pe} \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} \|\mathbf{w}\|_{1, \Omega}, \quad (2.47)$$

with

$$\mathbf{C}_2(\text{data}) := \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} \left[2 + \text{Pe} a_3 + 2 \text{Pe} \|\mathbf{u}_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} \right].$$

Proof. We begin by recalling from the definition of X that, given $(\mathbf{w}, \phi) \in X$, there holds:

$$\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{w} = \tilde{\mathbf{u}}_{\text{in}}(\phi) \text{ on } \Gamma_{\text{out}}^c,$$

which implies that

$$\operatorname{div} \mathbf{w} = 0 \text{ in } \Omega \quad \text{and} \quad \mathbf{w} \cdot \mathbf{n} = \tilde{\mathbf{u}}_{\text{in}}(\phi) \cdot \mathbf{n} = a_1 - a_2 \phi \text{ on } \Gamma_{\text{m}}.$$

Then, provided $a_3 - \frac{1}{2}a_1 \geq 0$, from (2.34) we obtain

$$a_C(\psi, \psi) + O_C(\mathbf{w}; \psi, \psi) \gtrsim \frac{1}{\text{Pe}} \|\psi\|_{1,\Omega}^2, \quad \forall \psi \in V_C,$$

that is, $a_C(\cdot, \cdot) + O_C(\mathbf{w}; \cdot, \cdot)$ is elliptic on V_C . This estimate, combined with the inf-sup condition (2.36) and the Babuška–Brezzi theory (see, e.g., [14, Theorem 2.34] and [19, Theorem 2.3]), implies the existence of a unique $(\varphi, \lambda) \in H^1(\Omega) \times H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ satisfying (2.43), for each $(\mathbf{w}, \phi) \in X$. This well-posedness result shows that the operator $\mathcal{L}$ is well defined. In addition, from the estimates in [14, Theorem 2.34], together with (2.8), (2.10) and the bounds for G and $\langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma}$ in (2.12), we deduce that

$$\|\varphi\|_{1,\Omega} \lesssim \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \left[\text{Pe} \|\mathbf{w}\|_{1,\Omega} + 2 \text{Pe} \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} + 2 + \text{Pe} a_3 \right], \quad (2.48)$$

$$\begin{aligned} \|\lambda\|_{-1/2,\Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3 + \|\mathbf{w}\|_{1,\Omega} + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \right) \\ &\times \left[\text{Pe} \|\mathbf{w}\|_{1,\Omega} + 2 \text{Pe} \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} + 2 + \text{Pe} a_3 \right] \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}, \end{aligned} \quad (2.49)$$

where the first estimate, in particular, implies (2.47). Finally, the second equation of (2.43) yields $\varphi \in H_{\varphi_{\text{in}}}^1(\Omega)$, and Lemma 2.3 implies that $\varphi \geq 0$ in Ω and on Γ_{m} , which concludes the proof. $\square$

Now we turn to prove the well-definiteness of operator $\mathcal{S}$.

Lemma 2.5 *Given $\mathbf{w} \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$, problem (2.44) has a unique solution $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times (L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma))$, and hence $\mathcal{S}(\mathbf{w}, \phi) = \mathbf{u}$ is well defined. In addition, if $\phi \geq 0$ on Γ_{m} , then $(\mathcal{S}(\mathbf{w}, \phi), \phi) \in X$, and the following estimate holds:*

$$\|\mathcal{S}(\mathbf{w}, \phi)\|_{1,\Omega} \lesssim 2\text{Re} \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}^2 + \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c} \left[2 + \text{Re} \|\mathbf{w}\|_{1,\Omega} \right]. \quad (2.50)$$

Proof. Given $\mathbf{w} \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$, from (2.33), the inf-sup condition (2.35), and the Babuška–Brezzi theory (see, e.g., [14, Theorem 2.34] and [19, Theorem 2.3]), we readily obtain the existence of a unique $(\mathbf{u}, p, \boldsymbol{\lambda}) \in \mathbf{H}^1(\Omega) \times L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$ satisfying (2.44), with $\mathbf{u}$ satisfying $\mathbf{u} = \mathcal{S}(\mathbf{w}, \phi)$, which shows that the operator $\mathcal{S}$ is well defined. In addition, since $\mathbf{u}$ satisfies the second equation of (2.44), it is clear that $(\mathcal{S}(\mathbf{w}, \phi), \phi) = (\mathbf{u}, \phi) \in X$.

Finally, combining the corresponding a priori estimates from [14, Theorem 2.34] with (2.8), (2.10) and (2.12), after simple computations we deduce that

$$\|\mathbf{u}\|_{1,\Omega} \lesssim 2\text{Re} \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}^2 + \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c} \left[2 + \text{Re} \|\mathbf{w}\|_{1,\Omega} \right], \quad (2.51)$$

$$\|(p, \boldsymbol{\lambda})\| \lesssim \frac{1}{\text{Re}} \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c} \left[2 + \text{Re} (\|\mathbf{w}\|_{1,\Omega} + \|\tilde{\mathbf{u}}_{\text{in}}(\phi)\|_{1/2,\Gamma_{\text{out}}^c}) \right]^2, \quad (2.52)$$

where the first estimate is precisely (2.50), which concludes the proof. $\square$

From Lemmas 2.4 and 2.5, it follows that $\mathcal{J}$ is well defined and that $\mathcal{J}(X) \subseteq X$. This result is established next.

Lemma 2.6 Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and let $(\mathbf{w}, \phi) \in X$. Then $\mathcal{J}(\mathbf{w}, \phi)$ is well defined and

$$\mathcal{J}(\mathbf{w}, \phi) \in X. \quad (2.53)$$

In addition, the following estimate holds:

$$\|\mathcal{J}(\mathbf{w}, \phi)\| \leq \mathcal{C}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data})\|\mathbf{w}\|_{1,\Omega} + \mathcal{C}_{\mathcal{J},3}(\text{data})\|\mathbf{w}\|_{1,\Omega}^2), \quad (2.54)$$

with $\mathcal{C}_{\mathcal{J},0} > 0$ independent of the data, and

$$\begin{aligned} \mathcal{C}_{\mathcal{J},1}(\text{data}) &:= 2(\mathbf{C}_1(\text{data}) + a_2\mathbf{C}_2(\text{data}))(\text{Re}(\mathbf{C}_1(\text{data}) + a_2\mathbf{C}_2(\text{data})) + 1) + \mathbf{C}_2(\text{data}), \\ \mathcal{C}_{\mathcal{J},2}(\text{data}) &:= \text{Re}(\mathbf{C}_1(\text{data}) + a_2\mathbf{C}_2(\text{data}))(1 + 4\text{Pe}a_2\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}) + \text{Pe}\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}(1 + 2a_2), \\ \mathcal{C}_{\mathcal{J},3}(\text{data}) &:= \text{Pe}\text{Re}a_2\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}(1 + 2\text{Pe}a_2\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}). \end{aligned} \quad (2.55)$$

Proof. Given $(\mathbf{w}, \phi) \in X$, Lemma 2.4 guarantees that $\mathcal{L}(\mathbf{w}, \phi)$ is well defined, that $\mathcal{L}(\mathbf{w}, \phi) \in H_{\varphi_{\text{in}}}^1(\Omega)$, and that $\mathcal{L}(\mathbf{w}, \phi) \geq 0$ on Γ_{m} . In turn, since $\mathbf{w} \in \mathbf{H}^1(\text{div}^0; \Omega)$, Lemma 2.5 applied to $(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi))$ implies that $\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi))$ is well defined and that $(\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi)), \mathcal{L}(\mathbf{w}, \phi)) \in X$, that is, $\mathcal{J}(\mathbf{w}, \phi) \in X$. Now, given $(\mathbf{w}, \phi) \in X$, combining (2.13) with (2.47) and (2.50), we obtain

$$\begin{aligned} \|\mathcal{J}(\mathbf{w}, \phi)\| &= \|\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi))\|_{1,\Omega} + \|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega} \\ &\lesssim 2\text{Re}\|\tilde{\mathbf{u}}_{\text{in}}(\mathcal{L}(\mathbf{w}, \phi))\|_{1/2,\Gamma_{\text{out}}^c}^2 + \|\tilde{\mathbf{u}}_{\text{in}}(\mathcal{L}(\mathbf{w}, \phi))\|_{1/2,\Gamma_{\text{out}}^c} \left[2 + \text{Re}\|\mathbf{w}\|_{1,\Omega} \right] \\ &\quad + \|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega} \\ &\lesssim 2\text{Re}(\mathbf{C}_1(\text{data}) + a_2\|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega})^2 + (\mathbf{C}_1(\text{data}) + a_2\|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega}) \left[2 + \text{Re}\|\mathbf{w}\|_{1,\Omega} \right] \\ &\quad + \|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega} \\ &\lesssim 2\text{Re} \left(\mathbf{C}_1(\text{data}) + a_2(\mathbf{C}_2(\text{data}) + \text{Pe}\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}\|\mathbf{w}\|_{1,\Omega}) \right)^2 \\ &\quad + \left(\mathbf{C}_1(\text{data}) + a_2(\mathbf{C}_2(\text{data}) + \text{Pe}\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}\|\mathbf{w}\|_{1,\Omega}) \right) \left[2 + \text{Re}\|\mathbf{w}\|_{1,\Omega} \right] \\ &\quad + \mathbf{C}_2(\text{data}) + \text{Pe}\|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}\|\mathbf{w}\|_{1,\Omega}, \end{aligned}$$

which, after straightforward algebraic manipulations, implies (2.54), and concludes the proof. $\square$

2.2.5 Well-posedness of the continuous problem

We now turn to proving the well-posedness of problem (2.6) by means of Banach's fixed-point theorem. To that end, we first establish Lipschitz-type estimates for the operators $\mathcal{L}$ and $\mathcal{S}$.

Lemma 2.7 Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ satisfy $a_3 - \frac{1}{2}a_1 \geq 0$. Then, for all $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in X$, there holds

$$\|\mathcal{L}(\mathbf{w}_1, \phi_1) - \mathcal{L}(\mathbf{w}_2, \phi_2)\|_{1,\Omega} \lesssim \text{Pe}\|\mathcal{L}(\mathbf{w}_2, \phi_2)\|_{1,\Omega}\|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega}. \quad (2.56)$$

Proof. Given $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in X$, we let $\varphi_1 = \mathcal{L}(\mathbf{w}_1, \phi_1)$ and $\varphi_2 = \mathcal{L}(\mathbf{w}_2, \phi_2)$. According to the definition of $\mathcal{L}$ (cf. (2.43)), it follows that there exist $\lambda_1, \lambda_2 \in H_{\Gamma_{\text{in}}^c}^{-1/2}(\Gamma)$ such that

$$\begin{aligned} a_C(\varphi_i, \psi) + O_C(\mathbf{w}_i; \varphi_i, \psi) + b_C(\psi, \lambda_i) &= G(\psi) \quad \forall \psi \in H^1(\Omega), \\ b_C(\varphi_i, \xi) &= \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma} \quad \forall \xi \in H_{\Gamma_{\text{in}}^c}^{-1/2}(\Gamma), \end{aligned}$$

for $i \in \{1, 2\}$. Subtracting both equations, we deduce that

$$\begin{aligned} a_C(\varphi_1 - \varphi_2, \psi) + O_C(\mathbf{w}_1; \varphi_1, \psi) - O_C(\mathbf{w}_2; \varphi_2, \psi) + b_C(\psi, \lambda_1 - \lambda_2) &= 0, \\ b_C(\varphi_1 - \varphi_2, \xi) &= 0, \end{aligned} \quad (2.57)$$

for all $\psi \in H^1(\Omega)$ and $\xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$. Then, noticing that the second equation implies that $\varphi_1 - \varphi_2 \in V_C = H_{\Gamma_{\text{in}}}^1(\Omega)$, taking $\psi = \varphi_1 - \varphi_2$ in the first equation of (2.57) and observing that $b_C(\varphi_1 - \varphi_2, \lambda_1 - \lambda_2) = 0$, adding and subtracting $O_C(\mathbf{w}_1; \varphi_2, \varphi_1 - \varphi_2)$, we deduce that

$$a_C(\varphi_1 - \varphi_2, \varphi_1 - \varphi_2) + O_C(\mathbf{w}_1; \varphi_1 - \varphi_2, \varphi_1 - \varphi_2) = O_C(\mathbf{w}_2; \varphi_2, \varphi_1 - \varphi_2) - O_C(\mathbf{w}_1; \varphi_2, \varphi_1 - \varphi_2).$$

Then, from (2.11), we get

$$|O_C(\mathbf{w}_2; \varphi_2, \varphi_1 - \varphi_2) - O_C(\mathbf{w}_1; \varphi_2, \varphi_1 - \varphi_2)| \lesssim \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} \|\varphi_2\|_{1,\Omega} \|\varphi_1 - \varphi_2\|_{1,\Omega}. \quad (2.58)$$

In this way, recalling that $(\mathbf{w}_1, \phi_1) \in X$, which implies that $\mathbf{w}_1 \cdot \mathbf{n} = \tilde{\mathbf{u}}_{\text{in}}(\phi_1) \cdot \mathbf{n} = a_1 - a_2 \phi_1$ on Γ_{m} , from (2.58) and (2.34) we deduce that

$$\frac{1}{\text{Pe}} \|\varphi_1 - \varphi_2\|_{1,\Omega}^2 \lesssim \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} \|\varphi_2\|_{1,\Omega} \|\varphi_1 - \varphi_2\|_{1,\Omega},$$

which implies the result. $\square$

The following lemma provides the corresponding Lipschitz-type estimate for $\mathcal{S}$.

Lemma 2.8 *For all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\phi_1, \phi_2 \in H_{\varphi_{\text{in}}}^1(\Omega)$, there holds*

$$\|\mathcal{S}(\mathbf{w}_1, \phi_1) - \mathcal{S}(\mathbf{w}_2, \phi_2)\|_{1,\Omega} \lesssim C_S \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|, \quad (2.59)$$

where

$$C_S = a_2 \left[1 + \text{Re}(\|\mathcal{S}(\mathbf{w}_1, \phi_1)\|_{1,\Omega} + \|\mathcal{S}(\mathbf{w}_2, \phi_2)\|_{1,\Omega} + \|\mathbf{w}_1\|_{1,\Omega}) \right] + \text{Re} \|\mathcal{S}(\mathbf{w}_2, \phi_2)\|_{1,\Omega}. \quad (2.60)$$

Proof. Given $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\phi_1, \phi_2 \in H_{\varphi_{\text{in}}}^1(\Omega)$, proceeding similarly to the proof of Lemma 2.7, we set $\mathbf{u}_1 := \mathcal{S}(\mathbf{w}_1, \phi_1)$ and $\mathbf{u}_2 := \mathcal{S}(\mathbf{w}_2, \phi_2)$, and let $(p_1, \lambda_1), (p_2, \lambda_2) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$ be the corresponding multipliers satisfying (2.44) for $i = 1, 2$. Since $\mathbf{u}_i = \tilde{\mathbf{u}}_{\text{in}}(\phi_i)$ on Γ_{out}^c (cf. the second equation of (2.44)), $\mathbf{u}_i$ is itself an $\mathbf{H}^1(\Omega)$ -extension of $\tilde{\mathbf{u}}_{\text{in}}(\phi_i)$, so that, by (1.7), we obtain

$$\|\tilde{\mathbf{u}}_{\text{in}}(\phi_i)\|_{1/2, \Gamma_{\text{out}}^c} \leq \|\mathbf{u}_i\|_{1,\Omega}, \quad \text{for } i = 1, 2. \quad (2.61)$$

Subtracting the two systems (2.44) for $i = 1, 2$, and adding and subtracting the term $O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v})$, we obtain

$$\begin{aligned} a_F(\delta \mathbf{u}, \mathbf{v}) + O_F^{\phi_1}(\mathbf{w}_1; \delta \mathbf{u}, \mathbf{v}) + b_F(\mathbf{v}, (\delta p, \delta \lambda)) &= \mathcal{F}(\mathbf{v}), \\ b_F(\delta \mathbf{u}, (q, \xi)) &= \mathcal{G}(q, \xi), \end{aligned} \quad (2.62)$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$ and $(q, \xi) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$, where $\delta \mathbf{u} := \mathbf{u}_1 - \mathbf{u}_2$, $\delta p := p_1 - p_2$, and $\delta \lambda := \lambda_1 - \lambda_2$, with

$$\begin{aligned} \mathcal{F}(\mathbf{v}) &:= F_{\phi_1}(\mathbf{v}) - F_{\phi_2}(\mathbf{v}) - \left(O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v}) - O_F^{\phi_2}(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v}) \right), \\ \mathcal{G}(q, \xi) &:= \left\langle \xi, \mathbf{E}_{\Gamma_{\text{out}}^c}(\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)) \right\rangle_{\Gamma}. \end{aligned}$$

According to the above, $(\delta \mathbf{u}, (\delta p, \delta \boldsymbol{\lambda}))$ is the solution of the saddle-point problem (2.62), with $a_F(\cdot, \cdot) + O_F^{\phi_1}(\mathbf{w}_1; \cdot, \cdot)$ elliptic on $\mathbf{V}_F$ (cf. (2.33)) and b_F satisfying the inf-sup condition (2.35). In what follows, we make use of the a priori estimates from [14, Theorem 2.34] to obtain the desired Lipschitz estimate. We begin by noticing that

$$|a_F(\mathbf{u}, \mathbf{v}) + O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}, \mathbf{v})| \lesssim \left(\frac{1}{\text{Re}} + \|\mathbf{w}_1\|_{1,\Omega} + \|\tilde{\mathbf{u}}_{\text{in}}(\phi_1)\|_{1/2,\Gamma_{\text{out}}^c} \right) \|\mathbf{u}\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega},$$

for all $\mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$. Now, to bound $\mathcal{F}$, given $\mathbf{v} \in \mathbf{H}^1(\Omega)$, adding and subtracting $O_F^{\phi_1}(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v})$, we first observe that

$$\begin{aligned} O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v}) - O_F^{\phi_2}(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v}) &= \left[O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v}) - O_F^{\phi_1}(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v}) \right] \\ &\quad - \frac{1}{2} \int_{\Gamma_{\text{out}}^c} \left[(\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)) \cdot \mathbf{n} \right] \mathbf{u}_2 \cdot \mathbf{v}. \end{aligned} \quad (2.63)$$

Then, using (2.11), (2.27), (2.15) and (2.61), together with (1.6), (1.2) and (1.7), we obtain

$$\begin{aligned} |\mathcal{F}(\mathbf{v})| &\leq |F_{\phi_1}(\mathbf{v}) - F_{\phi_2}(\mathbf{v})| + |O_F^{\phi_1}(\mathbf{w}_1; \mathbf{u}_2, \mathbf{v}) - O_F^{\phi_2}(\mathbf{w}_2; \mathbf{u}_2, \mathbf{v})| \\ &\lesssim (\|\mathbf{u}_1\|_{1,\Omega} + 2\|\mathbf{u}_2\|_{1,\Omega}) a_2 \|\phi_1 - \phi_2\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega} + \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} \|\mathbf{u}_2\|_{1,\Omega} \|\mathbf{v}\|_{1,\Omega}, \end{aligned}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$. In addition, using (2.15) we deduce that

$$|\mathcal{G}(q, \boldsymbol{\xi})| \lesssim \|\tilde{\mathbf{u}}_{\text{in}}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}}(\phi_2)\|_{1/2,\Gamma_{\text{out}}^c} \|\boldsymbol{\xi}\|_{-1/2,\Gamma} \lesssim a_2 \|\phi_1 - \phi_2\|_{1,\Omega} \|\boldsymbol{\xi}\|_{-1/2,\Gamma},$$

for all $(q, \boldsymbol{\xi}) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$. In this way, combining the corresponding a priori estimates from [14, Theorem 2.34] with the above bounds, after simple computations we deduce that

$$\|\delta \mathbf{u}\|_{1,\Omega} \lesssim a_2 \left[1 + \text{Re}(\|\mathbf{u}_1\|_{1,\Omega} + \|\mathbf{u}_2\|_{1,\Omega} + \|\mathbf{w}_1\|_{1,\Omega}) \right] \|\delta \phi\|_{1,\Omega} + \text{Re} \|\mathbf{u}_2\|_{1,\Omega} \|\delta \mathbf{w}\|_{1,\Omega}, \quad (2.64)$$

for all $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\phi_1, \phi_2 \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$, with $\delta \mathbf{w} := \mathbf{w}_1 - \mathbf{w}_2$ and $\delta \phi := \phi_1 - \phi_2$. This estimate implies (2.59), which concludes the proof. $\square$

Having established the Lipschitz-type estimates for $\mathcal{L}$ and $\mathcal{S}$, we are now in a position to provide the Lipschitz-type estimate for $\mathcal{J}$. To that end, given $\rho > 0$, we define

$$\mathcal{H}_\rho := \{(\mathbf{w}, \phi) \in X : \|(\mathbf{w}, \phi)\| \leq \rho\}.$$

Lemma 2.9 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$. For a given $\rho > 0$, assume further that the data are sufficiently small so that*

$$\mathcal{C}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data})\rho + \mathcal{C}_{\mathcal{J},3}(\text{data})\rho^2) \leq \rho, \quad (2.65)$$

with $\mathcal{C}_{\mathcal{J},0}, \mathcal{C}_{\mathcal{J},1}, \mathcal{C}_{\mathcal{J},2}, \mathcal{C}_{\mathcal{J},3}$ defined as in Lemma 2.6. Then $\mathcal{J}(\mathcal{H}_\rho) \subseteq \mathcal{H}_\rho$. In addition, the following estimate holds:

$$\|\mathcal{J}(\mathbf{w}_1, \phi_1) - \mathcal{J}(\mathbf{w}_2, \phi_2)\| \leq C_{\text{lip}} \mathcal{M}_{\mathcal{J}}(\rho) \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|, \quad (2.66)$$

for all $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathcal{H}_\rho$, with $C_{\text{lip}} > 0$ independent of the data and

$$\mathcal{M}_{\mathcal{J}}(\rho) := [a_2(1 + 3 \text{Re } \rho) + \text{Re } \rho](1 + \text{Pe } \rho) + \text{Pe } \rho. \quad (2.67)$$

Proof. Given $(\mathbf{w}, \phi) \in \mathcal{H}_\rho$, we have $\|\mathbf{w}\|_{1,\Omega} \leq \|(\mathbf{w}, \phi)\| \leq \rho$, so that, by (2.54) and the monotonicity of its right-hand side in $\|\mathbf{w}\|_{1,\Omega}$,

$$\begin{aligned} \|\mathcal{J}(\mathbf{w}, \phi)\| &\leq \mathcal{C}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data}) \|\mathbf{w}\|_{1,\Omega} + \mathcal{C}_{\mathcal{J},3}(\text{data}) \|\mathbf{w}\|_{1,\Omega}^2) \\ &\leq \mathcal{C}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data}) \rho + \mathcal{C}_{\mathcal{J},3}(\text{data}) \rho^2). \end{aligned}$$

Consequently, assumption (2.65) yields $\|\mathcal{J}(\mathbf{w}, \phi)\| \leq \rho$, that is, $\mathcal{J}(\mathbf{w}, \phi) \in \mathcal{H}_\rho$, which proves the first assertion. In particular, since $\mathcal{J}(\mathbf{w}, \phi) = (\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi)), \mathcal{L}(\mathbf{w}, \phi)) \in \mathcal{H}_\rho$ for all $(\mathbf{w}, \phi) \in \mathcal{H}_\rho$, both components satisfy

$$\|\mathcal{S}(\mathbf{w}, \mathcal{L}(\mathbf{w}, \phi))\|_{1,\Omega} \leq \rho \quad \text{and} \quad \|\mathcal{L}(\mathbf{w}, \phi)\|_{1,\Omega} \leq \rho, \quad \forall (\mathbf{w}, \phi) \in \mathcal{H}_\rho. \quad (2.68)$$

Now let $(\mathbf{w}_1, \phi_1), (\mathbf{w}_2, \phi_2) \in \mathcal{H}_\rho$, and set $\varphi_i := \mathcal{L}(\mathbf{w}_i, \phi_i)$ and $\mathbf{u}_i := \mathcal{S}(\mathbf{w}_i, \varphi_i)$ for $i = 1, 2$, so that $\mathcal{J}(\mathbf{w}_i, \phi_i) = (\mathbf{u}_i, \varphi_i)$. Applying Lemma 2.8 with $\mathbf{w}_1, \mathbf{w}_2 \in \mathbf{H}^1(\text{div}^0; \Omega)$ and $\varphi_1, \varphi_2 \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$ (cf. Lemma 2.4), and using (2.68) together with $\|\mathbf{w}_1\|_{1,\Omega} \leq \rho$ to bound the constant $C_{\mathcal{S}}$ in (2.60), we obtain

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \lesssim [a_2(1 + 3 \text{Re } \rho) + \text{Re } \rho] \|(\mathbf{w}_1, \varphi_1) - (\mathbf{w}_2, \varphi_2)\|.$$

On the other hand, by Lemma 2.7 and (2.68) (applied to $(\mathbf{w}_2, \phi_2)$),

$$\|\varphi_1 - \varphi_2\|_{1,\Omega} \lesssim \text{Pe} \|\varphi_2\|_{1,\Omega} \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\| \leq \text{Pe} \rho \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|.$$

Consequently,

$$\|(\mathbf{w}_1, \varphi_1) - (\mathbf{w}_2, \varphi_2)\| = \|\mathbf{w}_1 - \mathbf{w}_2\|_{1,\Omega} + \|\varphi_1 - \varphi_2\|_{1,\Omega} \lesssim (1 + \text{Pe} \rho) \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|,$$

and therefore

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} \lesssim [a_2(1 + 3 \text{Re } \rho) + \text{Re } \rho] (1 + \text{Pe} \rho) \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|.$$

Adding this to the estimate for $\|\varphi_1 - \varphi_2\|_{1,\Omega}$ above, we conclude that

$$\begin{aligned} \|\mathcal{J}(\mathbf{w}_1, \phi_1) - \mathcal{J}(\mathbf{w}_2, \phi_2)\| &= \|\mathbf{u}_1 - \mathbf{u}_2\|_{1,\Omega} + \|\varphi_1 - \varphi_2\|_{1,\Omega} \\ &\lesssim \left\{ [a_2(1 + 3 \text{Re } \rho) + \text{Re } \rho] (1 + \text{Pe} \rho) + \text{Pe} \rho \right\} \|(\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2)\|, \end{aligned}$$

which implies (2.66) with $\mathcal{M}_{\mathcal{J}}$ given by (2.67), and concludes the proof. $\square$

Finally, we present the main result of this section, namely, the well-posedness of problem (2.6).

Theorem 2.10 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and let $\rho > 0$ be such that*

$$C_{\text{lip}} \mathcal{M}_{\mathcal{J}}(\rho) < 1, \quad (2.69)$$

with C_{lip} and $\mathcal{M}_{\mathcal{J}}(\rho)$ as in Lemma 2.9. Assume further that the data are sufficiently small so that (2.65) holds. Then there exists a unique $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho$ such that $(\mathbf{u}, \varphi) = \mathcal{J}(\mathbf{u}, \varphi)$. Equivalently, there exist unique $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma))$ and $(\varphi, \lambda) \in \mathbf{H}^1(\Omega) \times \mathbf{H}_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ satisfying (2.6), with $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho$. Moreover, $\varphi \geq 0$ a.e. in Ω and on Γ_{in} , and there hold

$$\begin{aligned} \|(p, \boldsymbol{\lambda})\| &\lesssim \frac{1}{\text{Re}} (\mathbf{C}_1(\text{data}) + a_2 \rho) \left[2 + \text{Re}(\rho + \mathbf{C}_1(\text{data}) + a_2 \rho) \right]^2, \\ \|\lambda\|_{-1/2, \Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3 + \rho + \|\mathbf{u}_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} \right) \\ &\quad \times \left[\text{Pe} \rho + 2 \text{Pe} \|\mathbf{u}_{\text{in}}\|_{1/2, \Gamma_{\text{in}}} + 2 + \text{Pe} a_3 \right] \|\varphi_{\text{in}}\|_{1/2, \Gamma_{\text{in}}}. \end{aligned} \quad (2.70)$$

Proof. We first observe that $\mathcal{H}_\rho$ is a closed subset of $\mathbf{H}^1(\Omega) \times \mathbf{H}^1(\Omega)$, since X is closed (by the continuity of the trace operator and (2.15)), and that it is nonempty for data satisfying (2.65). Then, the well-posedness of problem (2.6) follows from Lemma 2.9, assumption (2.69), and Banach's fixed-point theorem. In addition, since $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho \subseteq X$, we have that $\operatorname{div} \mathbf{u} = 0$ in Ω , $\mathbf{u} = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$ on Γ_{out}^c and $\varphi \geq 0$ on Γ_{in} . Hence, since $\varphi = \mathcal{L}(\mathbf{u}, \varphi)$, Lemma 2.3 applied with $(\mathbf{w}, \phi) = (\mathbf{u}, \varphi)$ yields (2.40).

Now, for (2.70), we notice that, according to the definition of $\mathcal{J}$, the solution satisfies

$$(\mathbf{u}, \varphi) = \mathcal{J}(\mathbf{u}, \varphi) = \left(\mathcal{S}(\mathbf{u}, \mathcal{L}(\mathbf{u}, \varphi)), \mathcal{L}(\mathbf{u}, \varphi) \right),$$

so that $\varphi = \mathcal{L}(\mathbf{u}, \varphi)$ and $\mathbf{u} = \mathcal{S}(\mathbf{u}, \varphi)$, with $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho$ satisfying $\|\mathbf{u}\|_{1,\Omega} \leq \rho$ and $\|\varphi\|_{1,\Omega} \leq \rho$. Then, applying (2.52) with $\mathbf{w} = \mathbf{u}$ and $\phi = \varphi$, and using (2.13) to bound $\|\tilde{\mathbf{u}}_{\text{in}}(\varphi)\|_{1/2,\Gamma_{\text{out}}^c} \lesssim \mathbf{C}_1(\text{data}) + a_2\|\varphi\|_{1,\Omega} \leq \mathbf{C}_1(\text{data}) + a_2\rho$, together with $\|\mathbf{u}\|_{1,\Omega} \leq \rho$, we obtain

$$\|(p, \boldsymbol{\lambda})\| \lesssim \frac{1}{\operatorname{Re}} (\mathbf{C}_1(\text{data}) + a_2\rho) \left[2 + \operatorname{Re}(\rho + \mathbf{C}_1(\text{data}) + a_2\rho) \right]^2.$$

Similarly, applying (2.49) with $\mathbf{w} = \mathbf{u}$, and using $\|\mathbf{u}\|_{1,\Omega} \leq \rho$, we obtain

$$\|\boldsymbol{\lambda}\|_{-1/2,\Gamma} \lesssim \left(\frac{1}{\operatorname{Pe}} + a_3 + \rho + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \right) \left[\operatorname{Pe}\rho + 2\operatorname{Pe}\|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} + 2 + \operatorname{Pe}a_3 \right] \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}},$$

which together with the previous estimate gives (2.70) and concludes the proof. $\square$

3 Galerkin scheme

In this section, we introduce a conforming Galerkin scheme for the approximation of the solution to problem (2.6), based on arbitrary finite-dimensional subspaces, and derive suitable hypotheses on these subspaces ensuring the unique solvability of the associated discrete problem.

We begin by letting

$$\begin{aligned} \mathbf{H}_h &\subseteq \mathbf{H}^1(\Omega), \quad \mathbf{Q}_h \subseteq \mathbf{L}^2(\Omega), \quad \boldsymbol{\Lambda}_h \subseteq \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma) \cap \mathbf{L}^2(\Gamma), \\ \Psi_h &\subseteq \mathbf{H}^1(\Omega) \quad \text{and} \quad \Lambda_h \subseteq \mathbf{H}_{\Gamma_{\text{in}}^c}^{-1/2}(\Gamma) \cap \mathbf{L}^2(\Gamma), \end{aligned} \tag{3.1}$$

be arbitrary finite-dimensional subspaces.

Since φ_{in} is not necessarily the trace of a function in Ψ_h , the set $\Psi_h \cap \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$ may be empty, and hence $\tilde{\mathbf{u}}_{\text{in}}(\phi_h)$ is not well defined for $\phi_h \in \Psi_h$. To overcome this difficulty, we define

$$\tilde{\mathbf{u}}_{\text{in},h}(\phi) := \mathbf{s}(\phi)|_{\Gamma_{\text{out}}^c} \quad \forall \phi \in \mathbf{H}^1(\Omega), \tag{3.2}$$

with $\mathbf{s}(\phi)$ defined in (2.25). In addition, since the pair $(\mathbf{H}_h, \mathbf{Q}_h)$ will be chosen as an inf-sup stable pair, which does not necessarily yield divergence-free discrete velocities, we consider the following modified forms (see, e.g., [34]):

$$\begin{aligned} O_{F,h}^\phi(\mathbf{w}; \mathbf{u}, \mathbf{v}) &:= ((\mathbf{w} \cdot \nabla) \mathbf{u}, \mathbf{v})_\Omega + \frac{1}{2}((\operatorname{div} \mathbf{w}) \mathbf{u}, \mathbf{v})_\Omega - \frac{1}{2}((\tilde{\mathbf{u}}_{\text{in},h}(\phi) \cdot \mathbf{n}) \mathbf{u}, \mathbf{v})_{\Gamma_{\text{out}}^c} \\ &\quad - \frac{1}{2}((\mathbf{w} \cdot \mathbf{n})^- \mathbf{u}, \mathbf{v})_{\Gamma_{\text{out}}}, \end{aligned} \tag{3.3}$$

$$O_{C,h}(\mathbf{w}; \varphi, \psi) := O_C(\mathbf{w}; \varphi, \psi) + \frac{1}{2}((\operatorname{div} \mathbf{w}) \varphi, \psi)_\Omega,$$

for all $\mathbf{w}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\phi, \varphi, \psi \in H^1(\Omega)$.

Then, the discrete version of problem (2.6) reads: Find $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times (Q_h \times \boldsymbol{\Lambda}_h)$ and $(\varphi_h, \lambda_h) \in \Psi_h \times \Lambda_h$ such that

$$\begin{aligned} a_F(\mathbf{u}_h, \mathbf{v}_h) + O_{F,h}^{\varphi_h}(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h) + b_F(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F_{\varphi_h, h}(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{H}_h, \\ b_F(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in}, h}(\varphi_h))_{\Gamma_{\text{out}}^c} & \forall (q_h, \boldsymbol{\xi}_h) \in Q_h \times \boldsymbol{\Lambda}_h, \\ a_C(\varphi_h, \psi_h) + O_{C,h}(\mathbf{u}_h; \varphi_h, \psi_h) + b_C(\psi_h, \lambda_h) &= G(\psi_h) & \forall \psi_h \in \Psi_h, \\ b_C(\varphi_h, \xi_h) &= (\xi_h, \varphi_{\text{in}})_{\Gamma_{\text{in}}} & \forall \xi_h \in \Lambda_h, \end{aligned} \tag{3.4}$$

where

$$F_{\phi, h}(\mathbf{v}) := -\frac{1}{2} \left((\tilde{\mathbf{u}}_{\text{in}, h}(\phi) \cdot \mathbf{n}) \tilde{\mathbf{u}}_{\text{in}, h}(\phi), \mathbf{v} \right)_{\Gamma_{\text{out}}^c} \quad \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \quad \forall \phi \in H^1(\Omega).$$

The following section addresses the well-posedness of (3.4).

3.1 Analysis of the discrete problem

We begin by establishing some properties of $\tilde{\mathbf{u}}_{\text{in}, h}$ (cf. (3.2)). First, we notice that $\tilde{\mathbf{u}}_{\text{in}, h}(\phi) = \tilde{\mathbf{u}}_{\text{in}}(\phi)$ for all $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$. In addition, since the bound (2.26) for $\|\mathbf{s}(\phi)\|_{1/2, \Gamma}$ does not make use of the condition $\phi = \varphi_{\text{in}}$ on Γ_{in} , and since $\mathbf{s}(\phi_1) - \mathbf{s}(\phi_2) = -a_2(0, \gamma_0(\phi_1 - \phi_2)g)^t$ for all $\phi_1, \phi_2 \in H^1(\Omega)$, proceeding exactly as in the proof of Lemma 2.1, we obtain

$$\|\tilde{\mathbf{u}}_{\text{in}, h}(\phi)\|_{1/2, \Gamma_{\text{out}}^c} \leq \|\mathbf{s}(\phi)\|_{1/2, \Gamma} \lesssim \mathbf{C}_1(\text{data}) + a_2 \|\phi\|_{1, \Omega} \quad \forall \phi \in H^1(\Omega), \tag{3.5}$$

and

$$\|\tilde{\mathbf{u}}_{\text{in}, h}(\phi_1) - \tilde{\mathbf{u}}_{\text{in}, h}(\phi_2)\|_{1/2, \Gamma_{\text{out}}^c} \lesssim a_2 \|\phi_1 - \phi_2\|_{1, \Omega} \quad \forall \phi_1, \phi_2 \in H^1(\Omega). \tag{3.6}$$

Notice also that, for $\phi \in H_{\varphi_{\text{in}}}^1(\Omega)$ and $\mathbf{w} \in \mathbf{H}^1(\text{div}^0; \Omega)$ there hold $O_{F,h}^{\phi}(\mathbf{w}; \cdot, \cdot) = O_F^{\phi}(\mathbf{w}; \cdot, \cdot)$, $O_{C,h}(\mathbf{w}; \cdot, \cdot) = O_C(\mathbf{w}; \cdot, \cdot)$ and $F_{\phi, h} = F_{\phi}$, which ensures the consistency of (3.4) with (2.6).

Next, proceeding as in Section 2.2.1, that is, applying Hölder's inequality together with (2.9), (1.1), (1.2), (1.7) and (1.6), and using (3.5) and (3.6) in place of (2.13) and (2.15), we obtain the following discrete counterparts of (2.10) and (2.11):

$$|O_{F,h}^{\phi}(\mathbf{w}; \mathbf{u}, \mathbf{v})| \lesssim (\|\mathbf{w}\|_{1, \Omega} + \|\tilde{\mathbf{u}}_{\text{in}, h}(\phi)\|_{1/2, \Gamma_{\text{out}}^c}) \|\mathbf{u}\|_{1, \Omega} \|\mathbf{v}\|_{1, \Omega}, \tag{3.7}$$

$$|O_{C,h}(\mathbf{w}; \varphi, \psi)| \lesssim (\|\mathbf{w}\|_{1, \Omega} + \|\mathbf{u}_{\text{in}}\|_{1/2, \Gamma_{\text{in}}}) \|\varphi\|_{1, \Omega} \|\psi\|_{1, \Omega},$$

and

$$|O_{F,h}^{\phi}(\mathbf{w}; \mathbf{u}, \mathbf{v}) - O_{F,h}^{\phi}(\mathbf{z}; \mathbf{u}, \mathbf{v})| \lesssim \|\mathbf{w} - \mathbf{z}\|_{1, \Omega} \|\mathbf{u}\|_{1, \Omega} \|\mathbf{v}\|_{1, \Omega}, \tag{3.8}$$

$$|O_{C,h}(\mathbf{w}; \varphi, \psi) - O_{C,h}(\mathbf{z}; \varphi, \psi)| \lesssim \|\mathbf{w} - \mathbf{z}\|_{1, \Omega} \|\varphi\|_{1, \Omega} \|\psi\|_{1, \Omega},$$

for all $\mathbf{w}, \mathbf{z}, \mathbf{u}, \mathbf{v} \in \mathbf{H}^1(\Omega)$ and $\phi, \varphi, \psi \in H^1(\Omega)$. Similarly, the discrete counterparts of the first two estimates in (2.12) and of (2.27) read

$$\begin{aligned} |F_{\phi, h}(\mathbf{v})| &\lesssim \|\tilde{\mathbf{u}}_{\text{in}, h}(\phi)\|_{1/2, \Gamma_{\text{out}}^c}^2 \|\mathbf{v}\|_{1, \Omega}, \\ |(\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in}, h}(\phi))_{\Gamma_{\text{out}}^c}| &\leq \|\boldsymbol{\xi}_h\|_{-1/2, \Gamma} \|\mathbf{s}(\phi)\|_{1/2, \Gamma}, \\ |F_{\phi_1, h}(\mathbf{v}) - F_{\phi_2, h}(\mathbf{v})| &\lesssim a_2 \left(\sum_{i=1}^2 \|\tilde{\mathbf{u}}_{\text{in}, h}(\phi_i)\|_{1/2, \Gamma_{\text{out}}^c} \right) \|\phi_1 - \phi_2\|_{1, \Omega} \|\mathbf{v}\|_{1, \Omega}, \end{aligned} \tag{3.9}$$

for all $\mathbf{v} \in \mathbf{H}^1(\Omega)$, $\boldsymbol{\xi}_h \in \boldsymbol{\Lambda}_h$ and $\phi, \phi_1, \phi_2 \in H^1(\Omega)$.

We now state suitable hypotheses on the discrete spaces in (3.1) ensuring the well-posedness of (3.4). First, in order to guarantee the ellipticity of a_F and a_C on the corresponding discrete kernels, we assume that:

(H.1) $\mathbf{e}_{\Gamma_{\text{out}}^c}^i \in \boldsymbol{\Lambda}_h$ for $i \in \{1, 2\}$, where $\mathbf{e}_{\Gamma_{\text{out}}^c}^i := \mathbf{e}^i$ on Γ_{out}^c and $\mathbf{e}_{\Gamma_{\text{out}}^c}^i := \mathbf{0}$ on Γ_{out} , with $\mathbf{e}^i$ the i -th canonical basis vector of $\mathbb{R}^2$.

(H.2) $1_{\Gamma_{\text{in}}} \in \boldsymbol{\Lambda}_h$, where $1_{\Gamma_{\text{in}}} := 1$ on Γ_{in} and $1_{\Gamma_{\text{in}}} := 0$ on Γ_{in}^c .

Notice that (H.1) implies that the discrete kernel of the bilinear form b_F , namely

$$\mathbf{V}_{F,h} := \{\mathbf{v}_h \in \mathbf{H}_h : b_F(\mathbf{v}_h, (q_h, \boldsymbol{\xi}_h)) = 0, \quad \forall (q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h\}, \quad (3.10)$$

is contained in

$$\tilde{\mathbf{H}}_h := \{\mathbf{v}_h = (v_{h,1}, v_{h,2}) \in \mathbf{H}_h : (v_{h,i}, 1)_{\Gamma_{\text{out}}^c} = 0, \quad i = 1, 2\}. \quad (3.11)$$

Indeed, if $\mathbf{v}_h = (v_{h,1}, v_{h,2}) \in \mathbf{V}_{F,h}$, then

$$b_F(\mathbf{v}_h, (0, \mathbf{e}_{\Gamma_{\text{out}}^c}^i)) = \langle \mathbf{e}_{\Gamma_{\text{out}}^c}^i, \mathbf{v}_h \rangle_{\Gamma} = (v_{h,i}, 1)_{\Gamma_{\text{out}}^c} = 0, \quad i = 1, 2,$$

and therefore $\mathbf{V}_{F,h} \subseteq \tilde{\mathbf{H}}_h$. Then, by the generalized Poincaré inequality (see, e.g., [8, Lemma B.63]) $|\cdot|_{1,\Omega}$ and $\|\cdot\|_{1,\Omega}$ are equivalent norms on $\tilde{\mathbf{H}}_h$, and hence there exists $\tilde{\alpha}_F > 0$, depending only on Ω , such that

$$a_F(\mathbf{v}_h, \mathbf{v}_h) \geq \frac{\tilde{\alpha}_F}{\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2 \quad \forall \mathbf{v}_h \in \mathbf{V}_{F,h}. \quad (3.12)$$

Similarly, (H.2) implies that the discrete kernel of the bilinear form b_C , namely

$$V_{C,h} := \{\psi_h \in \Psi_h : b_C(\psi_h, \xi_h) = 0, \quad \forall \xi_h \in \boldsymbol{\Lambda}_h\}, \quad (3.13)$$

is contained in

$$\tilde{\Psi}_h := \{\psi_h \in \Psi_h : (\psi_h, 1)_{\Gamma_{\text{in}}} = 0\}. \quad (3.14)$$

Then, the generalized Poincaré inequality (see, e.g., [8, Lemma B.63]) implies that there exists $\tilde{\alpha}_C > 0$, depending only on Ω , such that

$$a_C(\psi_h, \psi_h) \geq \frac{\tilde{\alpha}_C}{\text{Pe}} \|\psi_h\|_{1,\Omega}^2 + a_3 \|\psi_h\|_{0,\Gamma_{\text{in}}}^2 \quad \forall \psi_h \in V_{C,h}. \quad (3.15)$$

In turn, to verify the discrete counterpart of (2.35), in what follows we assume that:

(H.3) $\mathcal{P}_0(\Omega) \subseteq \mathbf{Q}_h$, where $\mathcal{P}_0(\Omega)$ denotes the space of constant functions on Ω .

(H.4) There exists $\mathbf{v}_0 \in \mathbf{H}_{\Gamma_{\text{out}}^c}^1(\Omega)$ such that $\mathbf{v}_0 \in \mathbf{H}_{h,\Gamma_{\text{out}}^c} := \mathbf{H}_h \cap \mathbf{H}_{\Gamma_{\text{out}}^c}^1(\Omega)$ for all $h > 0$, and $(\mathbf{v}_0 \cdot \mathbf{n}, 1)_{\Gamma_{\text{out}}} \neq 0$.

(H.5) There exists $\tilde{\beta}_{F,1} > 0$, independent of the discretization parameter h , such that

$$\sup_{\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega) \setminus \{\mathbf{0}\}} \frac{b_{F,1}(\mathbf{v}_h, q_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \tilde{\beta}_{F,1} \|q_h\|_{0,\Omega}, \quad \forall q_h \in \mathbf{Q}_{h,0} := \mathbf{Q}_h \cap L_0^2(\Omega). \quad (3.16)$$

(H.6) There exists $\tilde{\beta}_{F,2} > 0$, independent of the discretization parameter h , such that

$$\sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \frac{b_{F,2}(\mathbf{v}_h, \boldsymbol{\xi}_h)}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \tilde{\beta}_{F,2} \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}, \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}_h. \quad (3.17)$$

Notice that (H.3) implies the orthogonal decomposition

$$\mathbf{Q}_h = \mathbf{Q}_{h,0} \oplus \mathbb{R}, \quad (3.18)$$

with $\mathbf{Q}_{h,0} = \mathbf{Q}_h \cap L_0^2(\Omega)$. The following result establishes the discrete inf-sup condition of the bilinear form b_F under hypotheses (H.3)–(H.6).

Lemma 3.1 *Assume that hypotheses (H.3)–(H.6) hold. Then, the following estimate holds:*

$$S_h := \sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq 0}} \frac{|b_F(\mathbf{v}_h, (q_h, \boldsymbol{\xi}_h))|}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \tilde{\beta}_F (\|q_h\|_{0,\Omega} + \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}), \quad \forall (q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h, \quad (3.19)$$

with $\tilde{\beta}_F > 0$, independent of h and the physical parameters.

Proof. Given $(q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h$, we first employ the decomposition (3.18) and write $q_h = q_{h,0} + \bar{q}_h$, where $q_{h,0} \in \mathbf{Q}_{h,0}$ and $\bar{q}_h \in \mathbb{R}$. Then, noticing that $b_{F,1}(\mathbf{v}_h, \bar{q}_h) = 0$ and $b_{F,2}(\mathbf{v}_h, \boldsymbol{\xi}_h) = 0$ for all $\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega)$, from (3.16) we deduce that

$$S_h \geq \sup_{\mathbf{v}_h \in \mathbf{H}_h \cap \mathbf{H}_0^1(\Omega) \setminus \{0\}} \frac{|b_{F,1}(\mathbf{v}_h, q_{h,0})|}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \tilde{\beta}_{F,1} \|q_{h,0}\|_{0,\Omega}. \quad (3.20)$$

Now, we let $\mathbf{v}_0 \in \mathbf{H}_{h,\Gamma_{\text{out}}^c}$ satisfying (H.4) and, noticing that $b_{F,2}(\mathbf{v}_0, \boldsymbol{\xi}_h) = 0$ since $\mathbf{v}_0 \in \mathbf{H}_{\Gamma_{\text{out}}^c}^1(\Omega)$ and $\boldsymbol{\xi}_h \in \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma)$, we observe that

$$S_h \geq \frac{|b_F(\mathbf{v}_0, (q_h, \boldsymbol{\xi}_h))|}{\|\mathbf{v}_0\|_{1,\Omega}} = \frac{|(q_{h,0}, \text{div } \mathbf{v}_0)_\Omega + \bar{q}_h (\mathbf{v}_0 \cdot \mathbf{n}, 1)_{\Gamma_{\text{out}}}|}{\|\mathbf{v}_0\|_{1,\Omega}} \geq c_0 |\bar{q}_h| - c_1 \|q_{h,0}\|_{0,\Omega}, \quad (3.21)$$

with $c_0 := \frac{|(\mathbf{v}_0 \cdot \mathbf{n}, 1)_{\Gamma_{\text{out}}}|}{\|\mathbf{v}_0\|_{1,\Omega}}$ and $c_1 > 0$ independent of h . Then, combining (3.20) and (3.21), we obtain

$$(1 + 2c_1 \tilde{\beta}_{F,1}^{-1}) S_h \geq c_0 |\bar{q}_h| + c_1 \|q_{h,0}\|_{0,\Omega},$$

which implies that

$$S_h \geq c_2 \|q_h\|_{0,\Omega}, \quad (3.22)$$

with $c_2 > 0$, independent of h .

On the other hand, using (H.6), we deduce that

$$S_h \geq \sup_{\mathbf{v}_h \in \mathbf{H}_h \setminus \{0\}} \frac{|b_{F,2}(\mathbf{v}_h, \boldsymbol{\xi}_h)|}{\|\mathbf{v}_h\|_{1,\Omega}} - c_3 \|q_h\|_{0,\Omega} \geq \tilde{\beta}_{F,2} \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma} - c_3 \|q_h\|_{0,\Omega}, \quad (3.23)$$

with $c_3 > 0$, independent of h .

Therefore, proceeding as in the proof of Lemma 2.2, we combine (3.22) and (3.23) to arrive at

$$S_h \gtrsim \|q_h\|_{0,\Omega} + \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}, \quad \forall (q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h, \quad (3.24)$$

which shows that b_F satisfies the discrete inf-sup condition (3.19), uniformly in h . $\square$

Similarly, the following hypothesis ensures the discrete counterpart of (2.36).

(H.7) There exists $\tilde{\beta}_C > 0$, independent of the discretization parameter h , such that

$$\sup_{\psi_h \in \Psi_h \setminus \{0\}} \frac{b_C(\psi_h, \boldsymbol{\xi}_h)}{\|\psi_h\|_{1,\Omega}} \geq \tilde{\beta}_C \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma}, \quad \forall \boldsymbol{\xi}_h \in \boldsymbol{\Lambda}_h. \quad (3.25)$$

Finally, to obtain the discrete counterparts of (2.33) and (2.34), we assume that:

(H.8) Let $\tilde{\rho} > 0$ be given. For each $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h \times \Psi_h$ such that $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$ and $\|\phi_h\|_{1,\Omega} \leq \tilde{\rho}$, there holds

$$\frac{\tilde{\alpha}_F}{\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}^c} ((\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)) \cdot \mathbf{n}) |\mathbf{v}_h|^2 \gtrsim \frac{1}{\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2, \quad \forall \mathbf{v}_h \in \mathbf{V}_{F,h}, \quad (3.26)$$

and, if in addition $(\mathbf{w}_h, \phi_h) \in X_h$, where

$$X_h := \{(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h \times \Psi_h : b_F(\mathbf{w}_h, (q_h, \boldsymbol{\xi}_h)) = (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in},h}(\phi_h))_{\Gamma_{\text{out}}^c}, \quad \forall (q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h\}, \quad (3.27)$$

there holds

$$\begin{aligned} \frac{\tilde{\alpha}_C}{\text{Pe}} \|\psi_h\|_{1,\Omega}^2 + \frac{1}{2} \int_{\Gamma_{\text{m}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 - \frac{1}{2} \int_{\Gamma_{\text{in}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 \\ + \frac{a_2}{2} \int_{\Gamma_{\text{m}}} \phi_h \psi_h^2 \gtrsim \frac{1}{\text{Pe}} \|\psi_h\|_{1,\Omega}^2 \quad \forall \psi_h \in V_{C,h}. \end{aligned} \quad (3.28)$$

Indeed, given $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h \times \Psi_h$ such that $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$ and $\|\phi_h\|_{1,\Omega} \leq \tilde{\rho}$, combining the first identity in (2.32) and (3.12) with (3.26), we obtain

$$a_F(\mathbf{v}_h, \mathbf{v}_h) + O_{F,h}^{\phi_h}(\mathbf{w}_h; \mathbf{v}_h, \mathbf{v}_h) \gtrsim \frac{1}{\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2, \quad \forall \mathbf{v}_h \in \mathbf{V}_{F,h}. \quad (3.29)$$

In addition, assuming that $a_3 - \frac{1}{2}a_1 \geq 0$, and given $(\mathbf{w}_h, \phi_h) \in X_h$ such that $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$ and $\|\phi_h\|_{1,\Omega} \leq \tilde{\rho}$, a combination of the second identity in (2.32) with (3.15) and (3.28) yields

$$a_C(\psi_h, \psi_h) + O_{C,h}(\mathbf{w}_h; \psi_h, \psi_h) \gtrsim \frac{1}{\text{Pe}} \|\psi_h\|_{1,\Omega}^2, \quad \forall \psi_h \in V_{C,h}. \quad (3.30)$$

Indeed, given $\psi_h \in V_{C,h}$, using the second identity in (2.32), we notice that

$$O_{C,h}(\mathbf{w}_h; \psi_h, \psi_h) = \frac{1}{2} \int_{\Gamma_{\text{in}}} ((\mathbf{w}_h - \mathbf{u}_{\text{in}}) \cdot \mathbf{n}) \psi_h^2 - \frac{1}{2} \int_{\Gamma_{\text{m}}} (\mathbf{w}_h \cdot \mathbf{n}) \psi_h^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}} (\mathbf{w}_h \cdot \mathbf{n})^+ \psi_h^2. \quad (3.31)$$

Then, noticing that $\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) \cdot \mathbf{n} = \mathbf{u}_{\text{in}} \cdot \mathbf{n}$ on Γ_{in} (since $\mathbf{n} = (-1, 0)^t$ there) and $\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) = (a_1 - a_2 \phi_h) \mathbf{n}$ on Γ_{m} (cf. (3.2)), and writing $\mathbf{w}_h \cdot \mathbf{n} = \tilde{\mathbf{u}}_{\text{in},h}(\phi_h) \cdot \mathbf{n} - (\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}$ on

$\Gamma_{\text{in}} \cup \Gamma_{\text{m}}$, (3.31) yields

$$\begin{aligned} O_{C,h}(\mathbf{w}_h; \psi_h, \psi_h) \geq & -\frac{a_1}{2} \int_{\Gamma_{\text{m}}} \psi_h^2 + \frac{a_2}{2} \int_{\Gamma_{\text{m}}} \phi_h \psi_h^2 \\ & + \frac{1}{2} \int_{\Gamma_{\text{m}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 - \frac{1}{2} \int_{\Gamma_{\text{in}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2, \end{aligned} \quad (3.32)$$

where we have dropped the non-negative term on Γ_{out} .

Adding (3.15) to (3.32), and using $a_3 - \frac{1}{2}a_1 \geq 0$ and (3.28), we obtain (3.30).

In what follows, we proceed similarly to the continuous case and employ a fixed-point argument to establish the existence and uniqueness of a solution to problem (3.4) under hypotheses (H.1)–(H.8). We begin by introducing the discrete version of the fixed-point operator $\mathcal{J}$.

3.1.1 The discrete fixed-point operator

Throughout this section, $\tilde{\rho} > 0$ denotes the radius in (H.8), and we let

$$\begin{aligned} \mathcal{H}_{\tilde{\rho},h} &:= \{(\mathbf{w}_h, \phi_h) \in X_h : \|(\mathbf{w}_h, \phi_h)\| \leq \tilde{\rho}\}, \\ \mathcal{B}_{\tilde{\rho},h} &:= \{(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h \times \Psi_h : \|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}, \|\phi_h\|_{1,\Omega} \leq \tilde{\rho}\}. \end{aligned} \quad (3.33)$$

Similarly to the continuous case, we define the auxiliary operator $\mathcal{L}_h : \mathcal{H}_{\tilde{\rho},h} \rightarrow \Psi_h$ given by

$$\mathcal{L}_h(\mathbf{w}_h, \phi_h) := \varphi_h,$$

with $\varphi_h \in \Psi_h$ the first component of the unique solution (to be confirmed below) of the linearized problem: Find $(\varphi_h, \lambda_h) \in \Psi_h \times \Lambda_h$ such that

$$\begin{aligned} a_C(\varphi_h, \psi_h) + O_{C,h}(\mathbf{w}_h; \varphi_h, \psi_h) + b_C(\psi_h, \lambda_h) &= G(\psi_h) \quad \forall \psi_h \in \Psi_h, \\ b_C(\varphi_h, \xi_h) &= (\xi_h, \varphi_{\text{in}})_{\Gamma_{\text{in}}} \quad \forall \xi_h \in \Lambda_h. \end{aligned} \quad (3.34)$$

In addition, we let $\mathcal{S}_h : \mathcal{B}_{\tilde{\rho},h} \rightarrow \mathbf{H}_h$ be the operator given by

$$\mathcal{S}_h(\mathbf{w}_h, \phi_h) := \mathbf{u}_h,$$

with $\mathbf{u}_h \in \mathbf{H}_h$ the first component of the unique solution (to be confirmed below) of the problem: Find $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times (\mathbf{Q}_h \times \boldsymbol{\Lambda}_h)$ such that

$$\begin{aligned} a_F(\mathbf{u}_h, \mathbf{v}_h) + O_{F,h}^{\phi_h}(\mathbf{w}_h; \mathbf{u}_h, \mathbf{v}_h) + b_F(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F_{\phi_h,h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h, \\ b_F(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in},h}(\phi_h))_{\Gamma_{\text{out}}^c} \quad \forall (q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h. \end{aligned} \quad (3.35)$$

Then, we let $\mathcal{J}_h : \mathcal{H}_{\tilde{\rho},h} \rightarrow \mathbf{H}_h \times \Psi_h$ be defined by

$$\mathcal{J}_h(\mathbf{w}_h, \phi_h) := \left(\mathcal{S}_h(\mathbf{w}_h, \mathcal{L}_h(\mathbf{w}_h, \phi_h)), \mathcal{L}_h(\mathbf{w}_h, \phi_h) \right) \quad \forall (\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}, \quad (3.36)$$

whenever the right-hand side is well defined, and notice that, analogously to the continuous case, solving (3.4) with $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho},h}$ is equivalent to finding a fixed point of $\mathcal{J}_h$ in $\mathcal{H}_{\tilde{\rho},h}$.

We continue by establishing the well-definedness of the operators $\mathcal{L}_h$ and $\mathcal{S}_h$.

Lemma 3.2 Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and assume that hypotheses (H.1)–(H.8) hold. Then:

(i) for each $(\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}$, problem (3.34) has a unique solution, and hence $\mathcal{L}_h(\mathbf{w}_h, \phi_h)$ is well defined; moreover,

$$\|\mathcal{L}_h(\mathbf{w}_h, \phi_h)\|_{1,\Omega} \lesssim \mathbf{C}_2(\text{data}) + \text{Pe} \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \|\mathbf{w}_h\|_{1,\Omega}; \quad (3.37)$$

(ii) for each $(\mathbf{w}_h, \phi_h) \in \mathcal{B}_{\tilde{\rho},h}$, problem (3.35) has a unique solution, and hence $\mathcal{S}_h(\mathbf{w}_h, \phi_h)$ is well defined; moreover, $(\mathcal{S}_h(\mathbf{w}_h, \phi_h), \phi_h) \in X_h$ and

$$\|\mathcal{S}_h(\mathbf{w}_h, \phi_h)\|_{1,\Omega} \lesssim 2 \text{Re} \|\tilde{\mathbf{u}}_{\text{in},h}(\phi_h)\|_{1/2,\Gamma_{\text{out}}^c}^2 + \|\tilde{\mathbf{u}}_{\text{in},h}(\phi_h)\|_{1/2,\Gamma_{\text{out}}^c} \left[2 + \text{Re} \|\mathbf{w}_h\|_{1,\Omega} \right]. \quad (3.38)$$

Proof. Given $(\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}$, by (3.30) the bilinear form $a_C(\cdot, \cdot) + O_{C,h}(\mathbf{w}_h; \cdot, \cdot)$ is elliptic on $V_{C,h}$, uniformly in h . This, together with the discrete inf-sup condition (3.25) and the Babuška–Brezzi theory (see, e.g., [14, Theorem 2.34]), implies that (3.34) has a unique solution. Similarly, given $(\mathbf{w}_h, \phi_h) \in \mathcal{B}_{\tilde{\rho},h}$, by (3.29) the bilinear form $a_F(\cdot, \cdot) + O_{F,h}^{\phi_h}(\mathbf{w}_h; \cdot, \cdot)$ is elliptic on $\mathbf{V}_{F,h}$, uniformly in h , which, together with (3.19), implies that (3.35) has a unique solution; in addition, the second equation of (3.35) shows that $(\mathcal{S}_h(\mathbf{w}_h, \phi_h), \phi_h) \in X_h$. Finally, (3.37) and (3.38) follow analogously to the proofs of Lemmas 2.4 and 2.5, now using (2.8), (2.12), (3.7) and (3.9). We omit further details. $\square$

Next, we establish the well-definedness of $\mathcal{J}_h$ and its self-mapping property.

Lemma 3.3 Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and assume that hypotheses (H.1)–(H.8) hold. Assume further that the data are sufficiently small so that

$$\tilde{\mathcal{C}}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data})\tilde{\rho} + \mathcal{C}_{\mathcal{J},3}(\text{data})\tilde{\rho}^2) \leq \tilde{\rho}, \quad (3.39)$$

with $\mathcal{C}_{\mathcal{J},1}, \mathcal{C}_{\mathcal{J},2}, \mathcal{C}_{\mathcal{J},3}$ as in (2.55) and $\tilde{\mathcal{C}}_{\mathcal{J},0} > 0$ independent of h and the data. Then $\mathcal{J}_h$ is well defined and $\mathcal{J}_h(\mathcal{H}_{\tilde{\rho},h}) \subseteq \mathcal{H}_{\tilde{\rho},h}$. In addition,

$$\|\mathcal{J}_h(\mathbf{w}_h, \phi_h)\| \leq \tilde{\mathcal{C}}_{\mathcal{J},0}(\mathcal{C}_{\mathcal{J},1}(\text{data}) + \mathcal{C}_{\mathcal{J},2}(\text{data}) \|\mathbf{w}_h\|_{1,\Omega} + \mathcal{C}_{\mathcal{J},3}(\text{data}) \|\mathbf{w}_h\|_{1,\Omega}^2), \quad (3.40)$$

for all $(\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}$.

Proof. Given $(\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}$, Lemma 3.2(i) guarantees that $\varphi_h := \mathcal{L}_h(\mathbf{w}_h, \phi_h)$ is well defined. Moreover, from (3.37), noticing that $\mathbf{C}_2(\text{data}) \leq \mathcal{C}_{\mathcal{J},1}(\text{data})$ and $\text{Pe} \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \leq \mathcal{C}_{\mathcal{J},2}(\text{data})$ (cf. (2.55)), and choosing $\tilde{\mathcal{C}}_{\mathcal{J},0}$ larger than the hidden constant in (3.37), we deduce from $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$ and (3.39) that $\|\varphi_h\|_{1,\Omega} \leq \tilde{\rho}$. Hence $(\mathbf{w}_h, \varphi_h) \in \mathcal{B}_{\tilde{\rho},h}$, and Lemma 3.2(ii) implies that $\mathcal{S}_h(\mathbf{w}_h, \varphi_h)$ is well defined and that $\mathcal{J}_h(\mathbf{w}_h, \phi_h) \in X_h$. Then, using (3.37), (3.38) and (3.5), and proceeding analogously to the proof of Lemma 2.6, we obtain (3.40). In addition, since $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$, from (3.40) and (3.39) we deduce that $\|\mathcal{J}_h(\mathbf{w}_h, \phi_h)\| \leq \tilde{\rho}$, which, together with the fact that $\mathcal{J}_h(\mathbf{w}_h, \phi_h) \in X_h$, yields $\mathcal{J}_h(\mathbf{w}_h, \phi_h) \in \mathcal{H}_{\tilde{\rho},h}$ and completes the proof. $\square$

3.1.2 Well-posedness of the discrete problem

In this subsection, we proceed analogously to Section 2.2.5 and prove the well-posedness of (3.4) by means of the Banach fixed-point theorem, for arbitrary discrete spaces satisfying (3.1) and hypotheses (H.1)–(H.8). We begin by providing Lipschitz-type estimates for the operators $\mathcal{L}_h$ and $\mathcal{S}_h$.

Lemma 3.4 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and assume that hypotheses (H.1)–(H.8) hold. Then, for all $(\mathbf{w}_{1,h}, \phi_{1,h}), (\mathbf{w}_{2,h}, \phi_{2,h}) \in \mathcal{H}_{\tilde{\rho},h}$, there holds*

$$\|\mathcal{L}_h(\mathbf{w}_{1,h}, \phi_{1,h}) - \mathcal{L}_h(\mathbf{w}_{2,h}, \phi_{2,h})\|_{1,\Omega} \lesssim \text{Pe} \|\mathcal{L}_h(\mathbf{w}_{2,h}, \phi_{2,h})\|_{1,\Omega} \|\mathbf{w}_{1,h} - \mathbf{w}_{2,h}\|_{1,\Omega}, \quad (3.41)$$

and, for all $(\mathbf{w}_{1,h}, \phi_{1,h}), (\mathbf{w}_{2,h}, \phi_{2,h}) \in \mathcal{B}_{\tilde{\rho},h}$, there holds

$$\|\mathcal{S}_h(\mathbf{w}_{1,h}, \phi_{1,h}) - \mathcal{S}_h(\mathbf{w}_{2,h}, \phi_{2,h})\|_{1,\Omega} \lesssim C_{\mathcal{S},h} \|(\mathbf{w}_{1,h}, \phi_{1,h}) - (\mathbf{w}_{2,h}, \phi_{2,h})\|, \quad (3.42)$$

where

$$C_{\mathcal{S},h} := a_2 \left[1 + \text{Re}(\|\mathcal{S}_h(\mathbf{w}_{1,h}, \phi_{1,h})\|_{1,\Omega} + \|\mathcal{S}_h(\mathbf{w}_{2,h}, \phi_{2,h})\|_{1,\Omega} + \|\mathbf{w}_{1,h}\|_{1,\Omega}) \right] + \text{Re} \|\mathcal{S}_h(\mathbf{w}_{2,h}, \phi_{2,h})\|_{1,\Omega}. \quad (3.43)$$

Proof. Analogous to the proofs of Lemmas 2.7 and 2.8, using now (3.29), (3.30), (3.19), (3.25), (3.8), (3.9) and (3.6). We omit further details. $\square$

As in the continuous case, we now show that $\mathcal{J}_h$ satisfies a Lipschitz-type estimate on $\mathcal{H}_{\tilde{\rho},h}$.

Lemma 3.5 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and assume that hypotheses (H.1)–(H.8) and (3.39) hold. Then*

$$\|\mathcal{J}_h(\mathbf{w}_{1,h}, \phi_{1,h}) - \mathcal{J}_h(\mathbf{w}_{2,h}, \phi_{2,h})\| \leq \tilde{C}_{\text{lip}} \mathcal{M}_{\mathcal{J}}(\tilde{\rho}) \|(\mathbf{w}_{1,h}, \phi_{1,h}) - (\mathbf{w}_{2,h}, \phi_{2,h})\| \quad (3.44)$$

for all $(\mathbf{w}_{1,h}, \phi_{1,h}), (\mathbf{w}_{2,h}, \phi_{2,h}) \in \mathcal{H}_{\tilde{\rho},h}$, with $\tilde{C}_{\text{lip}} > 0$ independent of h and the data, and $\mathcal{M}_{\mathcal{J}}$ as in (2.67).

Proof. Analogous to the proof of Lemma 2.9, using Lemmas 3.3 and 3.4 in place of Lemmas 2.6, 2.7 and 2.8, and noticing, as shown in the proof of Lemma 3.3, that $(\mathbf{w}_{i,h}, \mathcal{L}_h(\mathbf{w}_{i,h}, \phi_{i,h})) \in \mathcal{B}_{\tilde{\rho},h}$ for $i \in \{1, 2\}$, we obtain (3.44). We omit further details. $\square$

As in the continuous case, the well-posedness of (3.4) is a consequence of the previous results. This is established next.

Theorem 3.6 *Let $\mathbf{a} = (a_1, a_2, a_3) \in \mathbb{R}_+^3$ be such that $a_3 - \frac{1}{2}a_1 \geq 0$, and assume that hypotheses (H.1)–(H.8) hold, with $\tilde{\rho} > 0$ such that*

$$\tilde{C}_{\text{lip}} \mathcal{M}_{\mathcal{J}}(\tilde{\rho}) < 1, \quad (3.45)$$

with $\tilde{C}_{\text{lip}}$ and $\mathcal{M}_{\mathcal{J}}$ as in Lemma 3.5. Assume further that the data are sufficiently small so that (3.39) holds. Then there exists a unique $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho},h}$ such that $(\mathbf{u}_h, \varphi_h) = \mathcal{J}_h(\mathbf{u}_h, \varphi_h)$. Equivalently, there exist unique $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times (\mathbf{Q}_h \times \boldsymbol{\Lambda}_h)$ and $(\varphi_h, \lambda_h) \in \Psi_h \times \Lambda_h$ satisfying (3.4), with $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho},h}$, and there hold

$$\begin{aligned} \|(p_h, \boldsymbol{\lambda}_h)\| &\lesssim \frac{1}{\text{Re}} (\mathbf{C}_1(\text{data}) + a_2 \tilde{\rho}) \left[2 + \text{Re}(\tilde{\rho} + \mathbf{C}_1(\text{data}) + a_2 \tilde{\rho}) \right]^2, \\ \|\lambda_h\|_{-1/2,\Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3 + \tilde{\rho} + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \right) \\ &\quad \times \left[\text{Pe} \tilde{\rho} + 2 \text{Pe} \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} + 2 + \text{Pe} a_3 \right] \|\varphi_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}. \end{aligned} \quad (3.46)$$

Proof. We first notice that $\mathcal{H}_{\tilde{\rho},h}$ is a closed subset of the finite-dimensional space $\mathbf{H}_h \times \Psi_h$, and that it is nonempty for data satisfying (3.39). Indeed, by (3.19) there exists $\mathbf{w}_h \in \mathbf{H}_h$ such that $(\mathbf{w}_h, 0) \in X_h$ and $\|\mathbf{w}_h\|_{1,\Omega} \lesssim \|\mathbf{s}(0)\|_{1/2,\Gamma} \lesssim \mathbf{C}_1(\text{data})$ (cf. (3.5)). Hence, since $2\mathbf{C}_1(\text{data}) \leq \mathcal{C}_{\mathcal{J},1}(\text{data})$ (cf. (2.55)), and choosing $\tilde{\mathcal{C}}_{\mathcal{J},0}$ larger than the hidden constant in the latter estimate, assumption (3.39) yields $\|(\mathbf{w}_h, 0)\| = \|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$, that is, $(\mathbf{w}_h, 0) \in \mathcal{H}_{\tilde{\rho},h}$. Consequently, $\mathcal{H}_{\tilde{\rho},h}$ is a nonempty complete metric space, and the existence and uniqueness of a fixed point of $\mathcal{J}_h$ in $\mathcal{H}_{\tilde{\rho},h}$ follow from Lemmas 3.3 and 3.5, assumption (3.45), and Banach's fixed-point theorem. Finally, (3.46) follows by arguing exactly as in the proof of Theorem 2.10, now using $\|\mathbf{u}_h\|_{1,\Omega} \leq \tilde{\rho}$, $\|\varphi_h\|_{1,\Omega} \leq \tilde{\rho}$, the discrete counterparts of (2.49) and (2.52), and (3.5). We omit further details. $\square$

3.2 A priori error estimate

We now turn to establishing a Céa-type estimate for the Galerkin scheme (3.4). To this end, we apply the following Strang-type estimate for saddle-point problems, whose proof is postponed to Appendix B in Section 7.

Lemma 3.7 *Let $\mathbf{H}$ and $\mathbf{Q}$ be reflexive Banach spaces, and let $\{\mathbf{H}_h\}_{h>0}$ and $\{\mathbf{Q}_h\}_{h>0}$ be families of finite-dimensional subspaces of $\mathbf{H}$ and $\mathbf{Q}$, respectively. Let $a : \mathbf{H} \times \mathbf{H} \rightarrow \mathbb{R}$ and $b : \mathbf{H} \times \mathbf{Q} \rightarrow \mathbb{R}$ be bounded bilinear forms, with boundedness constants $\|a\|, \|b\|$, satisfying:*

- *There exists $\tilde{\alpha} > 0$, independent of h , such that*

$$a(v_h, v_h) \geq \tilde{\alpha} \|v_h\|_{\mathbf{H}}^2 \quad \forall v_h \in \text{Ker}(B_h) := \{v_h \in \mathbf{H}_h : b(v_h, q_h) = 0 \quad \forall q_h \in \mathbf{Q}_h\}. \quad (3.47)$$

- *There exists $\tilde{\beta} > 0$, independent of h , such that*

$$\sup_{\substack{v_h \in \mathbf{H}_h \\ v_h \neq 0}} \frac{|b(v_h, q_h)|}{\|v_h\|_{\mathbf{H}}} \geq \tilde{\beta} \|q_h\|_{\mathbf{Q}} \quad \forall q_h \in \mathbf{Q}_h. \quad (3.48)$$

Let $F \in \mathbf{H}'$, $G \in \mathbf{Q}'$, and, for each $h > 0$, let $F_h \in \mathbf{H}'_h$ and $G_h \in \mathbf{Q}'_h$. Let $(u, p) \in \mathbf{H} \times \mathbf{Q}$ and $(u_h, p_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ be such that

$$\begin{aligned} a(u, v) + b(v, p) &= F(v) \quad \forall v \in \mathbf{H}, & \text{and} & & a(u_h, v_h) + b(v_h, p_h) &= F_h(v_h) \quad \forall v_h \in \mathbf{H}_h, \\ b(u, q) &= G(q) \quad \forall q \in \mathbf{Q}, & & & b(u_h, q_h) &= G_h(q_h) \quad \forall q_h \in \mathbf{Q}_h. \end{aligned}$$

Then, the following estimates hold:

$$\begin{aligned} \|u - u_h\|_{\mathbf{H}} &\leq c_1 \inf_{v_h \in \mathbf{H}_h} \|u - v_h\|_{\mathbf{H}} + c_2 \inf_{q_h \in \mathbf{Q}_h} \|p - q_h\|_{\mathbf{Q}} + \frac{1}{\tilde{\alpha}} \sup_{\substack{y_h \in \mathbf{H}_h \\ y_h \neq 0}} \frac{|F(y_h) - F_h(y_h)|}{\|y_h\|_{\mathbf{H}}} \\ &\quad + c_3 \sup_{\substack{q_h \in \mathbf{Q}_h \\ q_h \neq 0}} \frac{|G(q_h) - G_h(q_h)|}{\|q_h\|_{\mathbf{Q}}}, \\ \|p - p_h\|_{\mathbf{Q}} &\leq c_4 c_1 \inf_{v_h \in \mathbf{H}_h} \|u - v_h\|_{\mathbf{H}} + c_5 \inf_{q_h \in \mathbf{Q}_h} \|p - q_h\|_{\mathbf{Q}} + c_3 \sup_{\substack{z_h \in \mathbf{H}_h \\ z_h \neq 0}} \frac{|F(z_h) - F_h(z_h)|}{\|z_h\|_{\mathbf{H}}} \\ &\quad + c_4 c_3 \sup_{\substack{q_h \in \mathbf{Q}_h \\ q_h \neq 0}} \frac{|G(q_h) - G_h(q_h)|}{\|q_h\|_{\mathbf{Q}}}, \end{aligned} \quad (3.49)$$

with $c_1 := (1 + \frac{\|a\|}{\alpha})(1 + \frac{\|b\|}{\beta})$, $c_3 := \frac{1}{\beta}(1 + \frac{\|a\|}{\alpha})$, $c_4 := \frac{\|a\|}{\beta}$, $c_5 := 1 + \frac{\|b\|}{\beta} + c_2 c_4$, and $c_2 := \frac{\|b\|}{\alpha}$ if $\text{Ker}(B_h) \not\subseteq \text{Ker}(B) := \{v \in H : b(v, q) = 0 \ \forall q \in Q\}$, and $c_2 := 0$ otherwise.

Remark 3.8 In the particular case $G_h = G|_{Q_h}$, the estimates in Lemma 3.7 reduce to

$$\begin{aligned} \|u - u_h\|_H &\leq c_1 \inf_{v_h \in H_h} \|u - v_h\|_H + c_2 \inf_{q_h \in Q_h} \|p - q_h\|_Q + \frac{1}{\tilde{\alpha}} \sup_{\substack{z_h \in H_h \\ z_h \neq 0}} \frac{|F(z_h) - F_h(z_h)|}{\|z_h\|_H}, \\ \|p - p_h\|_Q &\leq c_4 c_1 \inf_{v_h \in H_h} \|u - v_h\|_H + c_5 \inf_{q_h \in Q_h} \|p - q_h\|_Q + c_3 \sup_{\substack{z_h \in H_h \\ z_h \neq 0}} \frac{|F(z_h) - F_h(z_h)|}{\|z_h\|_H}, \end{aligned} \quad (3.50)$$

with $c_1, \dots, c_5$ as in Lemma 3.7.

Now, given $\mathbf{z} \in \mathbf{H}^1(\Omega)$ and $\zeta \in H^1(\Omega)$, we let $F_{\mathbf{z}, \zeta}^h : \mathbf{H}_h \rightarrow \mathbb{R}$ and $G_{\mathbf{z}, \zeta}^h : \Psi_h \rightarrow \mathbb{R}$ be the linear functionals given by

$$F_{\mathbf{z}, \zeta}^h(\mathbf{v}_h) := F_{\zeta, h}(\mathbf{v}_h) - O_{F, h}^\zeta(\mathbf{z}; \mathbf{z}, \mathbf{v}_h) \quad \text{and} \quad G_{\mathbf{z}, \zeta}^h(\psi_h) := G(\psi_h) - O_{C, h}(\mathbf{z}; \zeta, \psi_h), \quad (3.51)$$

and, if in addition $\zeta \in H_{\varphi_{\text{in}}}^1(\Omega)$, we let $F_{\mathbf{z}, \zeta} : \mathbf{H}^1(\Omega) \rightarrow \mathbb{R}$ and $G_{\mathbf{z}, \zeta} : H^1(\Omega) \rightarrow \mathbb{R}$ be given by

$$F_{\mathbf{z}, \zeta}(\mathbf{v}) := F_\zeta(\mathbf{v}) - O_F^\zeta(\mathbf{z}; \mathbf{z}, \mathbf{v}) \quad \text{and} \quad G_{\mathbf{z}, \zeta}(\psi) := G(\psi) - O_C(\mathbf{z}; \zeta, \psi). \quad (3.52)$$

With this notation, it follows from (2.6) and (3.4) that the continuous and discrete solutions $(\mathbf{u}, (p, \boldsymbol{\lambda}), \varphi, \lambda)$ and $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h), \varphi_h, \lambda_h)$ satisfy, respectively,

$$\begin{aligned} a_F(\mathbf{u}, \mathbf{v}) + b_F(\mathbf{v}, (p, \boldsymbol{\lambda})) &= F_{\mathbf{u}, \varphi}(\mathbf{v}) & \forall \mathbf{v} \in \mathbf{H}^1(\Omega), \\ b_F(\mathbf{u}, (q, \boldsymbol{\xi})) &= \langle \boldsymbol{\xi}, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_\Gamma & \forall (q, \boldsymbol{\xi}) \in L^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma), \\ a_C(\varphi, \psi) + b_C(\psi, \lambda) &= G_{\mathbf{u}, \varphi}(\psi) & \forall \psi \in H^1(\Omega), \\ b_C(\varphi, \xi) &= \langle \xi, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_\Gamma & \forall \xi \in H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma), \end{aligned} \quad (3.53)$$

and

$$\begin{aligned} a_F(\mathbf{u}_h, \mathbf{v}_h) + b_F(\mathbf{v}_h, (p_h, \boldsymbol{\lambda}_h)) &= F_{\mathbf{u}_h, \varphi_h}^h(\mathbf{v}_h) & \forall \mathbf{v}_h \in \mathbf{H}_h, \\ b_F(\mathbf{u}_h, (q_h, \boldsymbol{\xi}_h)) &= (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in}, h}(\varphi_h))_{\Gamma_{\text{out}}^c} & \forall (q_h, \boldsymbol{\xi}_h) \in Q_h \times \boldsymbol{\Lambda}_h, \\ a_C(\varphi_h, \psi_h) + b_C(\psi_h, \lambda_h) &= G_{\mathbf{u}_h, \varphi_h}^h(\psi_h) & \forall \psi_h \in \Psi_h, \\ b_C(\varphi_h, \xi_h) &= (\xi_h, \varphi_{\text{in}})_{\Gamma_{\text{in}}} & \forall \xi_h \in \Lambda_h. \end{aligned} \quad (3.54)$$

In this way, since the structure of problems (3.53) and (3.54) matches the abstract setting of Lemma 3.7, we apply this result in what follows to derive the desired a priori error estimate. We begin by establishing the following consistency-type estimate for $F_{\mathbf{u}, \varphi}$ and $F_{\mathbf{u}_h, \varphi_h}^h$.

Lemma 3.9 Assume that the hypotheses of Theorems 2.10 and 3.6 hold, that is, $\rho > 0$ and $\tilde{\rho} > 0$ satisfy (2.69) and (3.45), respectively, and the data are sufficiently small so that (2.65) and (3.39) hold. Let $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho$ and $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho}, h}$ be the unique velocity and concentration solutions of (2.6) and (3.4), respectively. Then, there exist positive constants $C_{F,1}, C_{F,2}$, independent of h and of the physical parameters, such that

$$\|F_{\mathbf{u}, \varphi} - F_{\mathbf{u}_h, \varphi_h}^h\|_{\mathbf{H}_h'} \leq C_{F,1}(\rho + \tilde{\rho}) \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + C_{F,2} a_2(\rho + \tilde{\rho} + \mathbf{C}_1(\text{data}) + a_2 \tilde{\rho}) \|\varphi - \varphi_h\|_{1, \Omega}. \quad (3.55)$$

Proof. Since $\operatorname{div} \mathbf{u} = 0$ in Ω and $\varphi \in \mathbf{H}_{\varphi_{\text{in}}}^1(\Omega)$, so that $\tilde{\mathbf{u}}_{\text{in},h}(\varphi) = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$, from the definitions of O_F^φ and $O_{F,h}^\varphi$ (cf. (2.7) and (3.3)) it is clear that

$$O_F^\varphi(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) = O_{F,h}^\varphi(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) \quad \text{and} \quad F_\varphi(\mathbf{v}_h) = F_{\varphi,h}(\mathbf{v}_h) \quad \forall \mathbf{v}_h \in \mathbf{H}_h. \quad (3.56)$$

Then, using (3.56), and adding and subtracting $O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h)$ and $O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u}_h, \mathbf{v}_h)$, we obtain, for all $\mathbf{v}_h \in \mathbf{H}_h$,

$$\begin{aligned} F_{\mathbf{u},\varphi}(\mathbf{v}_h) - F_{\mathbf{u}_h,\varphi_h}^h(\mathbf{v}_h) &= [F_{\varphi,h}(\mathbf{v}_h) - F_{\varphi_h,h}(\mathbf{v}_h)] - [O_{F,h}^\varphi(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) - O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h)] \\ &\quad - O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h) + \frac{1}{2} \int_{\Gamma_{\text{out}}^c} [(\tilde{\mathbf{u}}_{\text{in},h}(\varphi) - \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h)) \cdot \mathbf{n}] \mathbf{u}_h \cdot \mathbf{v}_h. \end{aligned} \quad (3.57)$$

Applying the Lipschitz-type estimate (3.8) and the stability estimate (3.7), together with Hölder's inequality, (2.9), (1.6), (1.2), (1.7), (3.9) and (3.6), we obtain the following estimates for each term on the right-hand side of (3.57):

$$\begin{aligned} |O_{F,h}^\varphi(\mathbf{u}; \mathbf{u}, \mathbf{v}_h) - O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u}, \mathbf{v}_h)| &\lesssim \|\mathbf{u}\|_{1,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}, \\ |O_{F,h}^\varphi(\mathbf{u}_h; \mathbf{u} - \mathbf{u}_h, \mathbf{v}_h)| &\lesssim \left(\|\mathbf{u}_h\|_{1,\Omega} + \|\tilde{\mathbf{u}}_{\text{in}}(\varphi)\|_{1/2,\Gamma_{\text{out}}^c} \right) \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}, \\ \left| \frac{1}{2} \int_{\Gamma_{\text{out}}^c} [(\tilde{\mathbf{u}}_{\text{in},h}(\varphi) - \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h)) \cdot \mathbf{n}] \mathbf{u}_h \cdot \mathbf{v}_h \right| &\lesssim a_2 \|\mathbf{u}_h\|_{1,\Omega} \|\varphi - \varphi_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}, \\ |F_{\varphi,h}(\mathbf{v}_h) - F_{\varphi_h,h}(\mathbf{v}_h)| &\lesssim a_2 \left(\|\tilde{\mathbf{u}}_{\text{in}}(\varphi)\|_{1/2,\Gamma_{\text{out}}^c} + \|\tilde{\mathbf{u}}_{\text{in},h}(\varphi_h)\|_{1/2,\Gamma_{\text{out}}^c} \right) \|\varphi - \varphi_h\|_{1,\Omega} \|\mathbf{v}_h\|_{1,\Omega}. \end{aligned} \quad (3.58)$$

Combining (3.57) and (3.58), and using that $\|\tilde{\mathbf{u}}_{\text{in}}(\varphi)\|_{1/2,\Gamma_{\text{out}}^c} \leq \|\mathbf{u}\|_{1,\Omega}$, which follows from (1.7) since $\mathbf{u} = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$ on Γ_{out}^c , together with (3.5) and the bounds $\|\mathbf{u}\|_{1,\Omega} \leq \rho$, $\|\mathbf{u}_h\|_{1,\Omega} \leq \tilde{\rho}$ and $\|\varphi_h\|_{1,\Omega} \leq \tilde{\rho}$, we obtain (3.55), which concludes the proof. $\square$

We continue by establishing the following consistency-type estimate for $G_{\mathbf{u},\varphi}$ and $G_{\mathbf{u}_h,\varphi_h}^h$.

Lemma 3.10 *Assume that the hypotheses of Theorems 2.10 and 3.6 hold, that is, $\rho > 0$ and $\tilde{\rho} > 0$ satisfy (2.69) and (3.45), respectively, and the data are sufficiently small so that (2.65) and (3.39) hold. Let $(\mathbf{u}, \varphi) \in \mathcal{H}_\rho$ and $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho},h}$ be the unique velocity and concentration solutions of (2.6) and (3.4), respectively. Then, there exist positive constants $C_{G,1}, C_{G,2}$, independent of h and of the physical parameters, such that*

$$\|G_{\mathbf{u},\varphi} - G_{\mathbf{u}_h,\varphi_h}^h\|_{\Psi_h'} \leq C_{G,1} \tilde{\rho} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + C_{G,2} (\rho + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}) \|\varphi - \varphi_h\|_{1,\Omega}. \quad (3.59)$$

Proof. Since $\operatorname{div} \mathbf{u} = 0$ in Ω , from the definitions of O_C and $O_{C,h}$ (cf. (2.7) and (3.3)) it is clear that

$$O_C(\mathbf{u}; \varphi, \psi_h) = O_{C,h}(\mathbf{u}; \varphi, \psi_h) \quad \forall \psi_h \in \Psi_h. \quad (3.60)$$

Then, using (3.60), and adding and subtracting $O_{C,h}(\mathbf{u}_h; \varphi_h, \psi_h)$, we obtain, for all $\psi_h \in \Psi_h$,

$$G_{\mathbf{u},\varphi}(\psi_h) - G_{\mathbf{u}_h,\varphi_h}^h(\psi_h) = [O_{C,h}(\mathbf{u}_h; \varphi_h, \psi_h) - O_{C,h}(\mathbf{u}; \varphi_h, \psi_h)] - O_{C,h}(\mathbf{u}; \varphi - \varphi_h, \psi_h). \quad (3.61)$$

Applying the stability estimate (3.7) and the Lipschitz-type estimate (3.8), we obtain

$$\begin{aligned} |O_{C,h}(\mathbf{u}_h; \varphi_h, \psi_h) - O_{C,h}(\mathbf{u}; \varphi_h, \psi_h)| &\lesssim \|\varphi_h\|_{1,\Omega} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \|\psi_h\|_{1,\Omega}, \\ |O_{C,h}(\mathbf{u}; \varphi - \varphi_h, \psi_h)| &\lesssim \left(\|\mathbf{u}\|_{1,\Omega} + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}} \right) \|\varphi - \varphi_h\|_{1,\Omega} \|\psi_h\|_{1,\Omega}. \end{aligned} \quad (3.62)$$

Then, combining (3.61) and (3.62), and using that $\|\mathbf{u}\|_{1,\Omega} \leq \rho$ and $\|\varphi_h\|_{1,\Omega} \leq \tilde{\rho}$, we obtain (3.59), which concludes the proof. $\square$

Finally, we provide the consistency-type estimate for $\langle \cdot, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma}$ and $(\cdot, \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h))_{\Gamma_{\text{out}}^c}$.

Lemma 3.11 *Assume that the hypotheses of Lemma 3.9 hold, and let $\varphi \in H_{\varphi_{\text{in}}}^1(\Omega)$ and $\varphi_h \in \Psi_h$ be the concentration solutions of (2.6) and (3.4), respectively. Then, there exists a positive constant C_{dir} , independent of h and of the physical parameters, such that*

$$\sup_{\substack{(q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h \\ (q_h, \boldsymbol{\xi}_h) \neq \mathbf{0}}} \frac{|\langle \boldsymbol{\xi}_h, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma} - (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h))_{\Gamma_{\text{out}}^c}|}{\|(q_h, \boldsymbol{\xi}_h)\|} \leq C_{\text{dir}} a_2 \|\varphi - \varphi_h\|_{1,\Omega}. \quad (3.63)$$

Proof. Let $(q_h, \boldsymbol{\xi}_h) \in \mathbf{Q}_h \times \boldsymbol{\Lambda}_h$. Since $\boldsymbol{\xi}_h \in \mathbf{L}^2(\Gamma)$ and $\boldsymbol{\xi}_h = \mathbf{0}$ on Γ_{out} (cf. (3.1)), and since $\tilde{\mathbf{u}}_{\text{in},h}(\varphi) = \tilde{\mathbf{u}}_{\text{in}}(\varphi)$, we have

$$\langle \boldsymbol{\xi}_h, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma} - (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h))_{\Gamma_{\text{out}}^c} = \langle \boldsymbol{\xi}_h, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in},h}(\varphi) - \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h)) \rangle_{\Gamma},$$

so that, using (1.10) and (3.6), we obtain

$$|\langle \boldsymbol{\xi}_h, \mathbf{E}_{\Gamma_{\text{out}}}^c(\tilde{\mathbf{u}}_{\text{in}}(\varphi)) \rangle_{\Gamma} - (\boldsymbol{\xi}_h, \tilde{\mathbf{u}}_{\text{in},h}(\varphi_h))_{\Gamma_{\text{out}}^c}| \lesssim a_2 \|\boldsymbol{\xi}_h\|_{-1/2,\Gamma} \|\varphi - \varphi_h\|_{1,\Omega}. \quad (3.64)$$

Since $\|\boldsymbol{\xi}_h\|_{-1/2,\Gamma} \leq \|(q_h, \boldsymbol{\xi}_h)\|$, dividing (3.64) by $\|(q_h, \boldsymbol{\xi}_h)\|$ and taking the supremum over $(q_h, \boldsymbol{\xi}_h) \in (\mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \setminus \{\mathbf{0}\}$ yields (3.63), which concludes the proof. $\square$

Having established the consistency-type estimate for the terms on the right-hand sides of (3.53)-(3.54), now we employ Lemma 3.7 to derive the Céa estimate for the Galerkin scheme (3.4).

Theorem 3.12 *Assume that the hypotheses of Theorems 2.10 and 3.6 hold, that is, $\rho > 0$ and $\tilde{\rho} > 0$ satisfy (2.69) and (3.45), respectively, and the data are sufficiently small so that (2.65) and (3.39) hold. Let $(\mathbf{u}, (p, \boldsymbol{\lambda})) \in \mathbf{H}^1(\Omega) \times (\mathbf{L}^2(\Omega) \times \mathbf{H}_{\Gamma_{\text{out}}}^{-1/2}(\Gamma))$ and $(\varphi, \lambda) \in H^1(\Omega) \times H_{\Gamma_{\text{in}}}^{-1/2}(\Gamma)$ be the unique solution of (2.6) with $(\mathbf{u}, \varphi) \in \mathcal{H}_{\rho}$, and let $(\mathbf{u}_h, (p_h, \boldsymbol{\lambda}_h)) \in \mathbf{H}_h \times (\mathbf{Q}_h \times \boldsymbol{\Lambda}_h)$ and $(\varphi_h, \lambda_h) \in \Psi_h \times \Lambda_h$ be the unique solution of (3.4) with $(\mathbf{u}_h, \varphi_h) \in \mathcal{H}_{\tilde{\rho},h}$. Assume further that*

$$A + E \leq \frac{1}{2} \quad \text{and} \quad B + H \leq \frac{1}{2}, \quad (3.65)$$

where

$$\begin{aligned} A &:= \frac{\text{Re}}{\tilde{\alpha}_F} C_{F,1}(\rho + \tilde{\rho}), & B &:= \frac{\text{Re}}{\tilde{\alpha}_F} C_{F,2} a_2(\rho + \tilde{\rho} + \mathbf{C}_1(\text{data}) + a_2 \tilde{\rho}) + \frac{1}{\tilde{\beta}_F} \left(1 + \frac{1}{\tilde{\alpha}_F}\right) C_{\text{dir}} a_2, \\ E &:= \frac{\text{Pe}}{\tilde{\alpha}_C} C_{G,1} \tilde{\rho}, & H &:= \frac{\text{Pe}}{\tilde{\alpha}_C} C_{G,2}(\rho + \|\mathbf{u}_{\text{in}}\|_{1/2,\Gamma_{\text{in}}}), \end{aligned} \quad (3.66)$$

with $C_{F,1}, C_{F,2}, C_{G,1}, C_{G,2}$ and C_{dir} as in Lemmas 3.9, 3.10 and 3.11, respectively. Then, there hold

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega} &\lesssim \text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \\ &\quad + (1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe dist}(\lambda, \Lambda_h), \end{aligned} \quad (3.67)$$

$$\begin{aligned} \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\| &\lesssim \frac{1}{\text{Re}} \left[\text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \right. \\ &\quad \left. + (1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe dist}(\lambda, \Lambda_h) \right], \end{aligned} \quad (3.68)$$

and

$$\begin{aligned} \|\lambda - \lambda_h\|_{-1/2, \Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3 \right) \left[\text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \right. \\ &\quad \left. + (1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe dist}(\lambda, \Lambda_h) \right], \end{aligned} \quad (3.69)$$

with hidden constants independent of h and of the physical parameters.

Proof. First, applying Lemma 3.7 to the first and second equations of (3.53) and (3.54), together with Lemmas 3.9 and 3.11, the ellipticity and inf-sup bounds (3.12) and (3.19), and the stability properties (2.8), we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} \leq C_1 \text{dist}(\mathbf{u}, \mathbf{H}_h) + C_2 \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) + A \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + B \|\varphi - \varphi_h\|_{1, \Omega}, \quad (3.70)$$

with $C_1, C_2 > 0$ independent of h and of the physical parameters. Analogously, noticing that $(\xi_h, \varphi_{\text{in}})_{\Gamma_{\text{in}}} = \langle \xi_h, E_{\Gamma_{\text{in}}}(\varphi_{\text{in}}) \rangle_{\Gamma}$ for all $\xi_h \in \Lambda_h$ (cf. (3.1)), we apply Lemma 3.7, in its simplified form (3.50), to the third and fourth equations of (3.53) and (3.54). Then, using Lemma 3.10, the ellipticity bound (3.15) (with the nonnegative term $a_3 \|\psi_h\|_{0, \Gamma_m}^2$ discarded), the inf-sup condition (3.25), and the stability properties (2.8), we obtain

$$\|\varphi - \varphi_h\|_{1, \Omega} \leq C_3(1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + C_4 \text{Pe dist}(\lambda, \Lambda_h) + E \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + H \|\varphi - \varphi_h\|_{1, \Omega}, \quad (3.71)$$

with $C_3, C_4 > 0$ independent of h and of the physical parameters. Then, denoting

$$\begin{aligned} D_1 &:= C_1 \text{dist}(\mathbf{u}, \mathbf{H}_h) + C_2 \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h), \\ D_2 &:= C_3(1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + C_4 \text{Pe dist}(\lambda, \Lambda_h), \end{aligned} \quad (3.72)$$

and summing up (3.70) and (3.71), we obtain

$$\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + \|\varphi - \varphi_h\|_{1, \Omega} \leq D_1 + D_2 + (A + E) \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + (B + H) \|\varphi - \varphi_h\|_{1, \Omega}.$$

In this way, since $A + E \leq \frac{1}{2}$ and $B + H \leq \frac{1}{2}$ (cf. (3.65)), we deduce that $\|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + \|\varphi - \varphi_h\|_{1, \Omega} \leq 2(D_1 + D_2)$, which yields

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + \|\varphi - \varphi_h\|_{1, \Omega} &\lesssim \text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{Re dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \\ &\quad + (1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe dist}(\lambda, \Lambda_h), \end{aligned}$$

that is, (3.67).

On the other hand, applying the second estimate in (3.49) to the first and second equations of (3.53) and (3.54), together with Lemmas 3.9 and 3.11, and the stability properties (2.8), we obtain

$$\begin{aligned} \|(p - p_h, \boldsymbol{\lambda} - \boldsymbol{\lambda}_h)\| &\lesssim \frac{1}{\text{Re}} \text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) + \|F_{\mathbf{u}, \varphi} - F_{\mathbf{u}_h, \varphi_h}^h\|_{\mathbf{H}'_h} \\ &\quad + \frac{C_{\text{dir}} a_2}{\text{Re}} \|\varphi - \varphi_h\|_{1, \Omega} \\ &\lesssim \frac{1}{\text{Re}} \text{dist}(\mathbf{u}, \mathbf{H}_h) + \text{dist}((p, \boldsymbol{\lambda}), \mathbf{Q}_h \times \boldsymbol{\Lambda}_h) \\ &\quad + \frac{1}{\text{Re}} \left[A \|\mathbf{u} - \mathbf{u}_h\|_{1, \Omega} + B \|\varphi - \varphi_h\|_{1, \Omega} \right]. \end{aligned} \quad (3.73)$$

Similarly, applying the second estimate in (3.50) to the third and fourth equations of (3.53) and (3.54), together with Lemma 3.10 and the stability properties (2.8), we obtain

$$\begin{aligned}
\|\lambda - \lambda_h\|_{-1/2,\Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3\right) \left[(1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe} \text{dist}(\lambda, \Lambda_h) \right. \\
&\quad \left. + \text{Pe} \|G_{\mathbf{u},\varphi} - G_{\mathbf{u}_h,\varphi_h}^h\|_{\Psi_h'} \right] \\
&\lesssim \left(\frac{1}{\text{Pe}} + a_3\right) \left[(1 + a_3 \text{Pe}) \text{dist}(\varphi, \Psi_h) + \text{Pe} \text{dist}(\lambda, \Lambda_h) + E \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} \right. \\
&\quad \left. + H \|\varphi - \varphi_h\|_{1,\Omega} \right].
\end{aligned} \tag{3.74}$$

Finally, using assumption (3.65) to bound $A, B, E, H \leq \frac{1}{2}$, substituting into (3.73) and (3.74) the bound for $\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega}$ established in (3.67), and noticing that the remaining terms in (3.73) and (3.74) are bounded by $\frac{1}{\text{Re}}$ and $\frac{1}{\text{Pe}} + a_3$ times the right-hand side of (3.67), respectively, we obtain (3.68) and (3.69), which concludes the proof. $\square$

4 A concrete choice of finite element spaces

In this section we provide a concrete example of finite element subspaces satisfying hypotheses (H.1)–(H.8). To this end, given an integer $k \geq 0$ and a set $S \subseteq \mathbb{R}^2$, which will be either a triangle or a segment, we denote by $P_k(S)$ the space of polynomial functions on S of degree $\leq k$.

4.1 Taylor–Hood and continuous piecewise polynomials

To approximate the velocity and pressure fields we consider the Taylor–Hood pair (see, e.g., [20]). More precisely, given an integer $k \geq 1$, we let

$$\begin{aligned}
\mathbf{H}_h &:= \{\mathbf{v}_h \in [C(\overline{\Omega})]^2 : \mathbf{v}_h|_K \in [P_{k+1}(K)]^2 \ \forall K \in \mathcal{T}_h\}, \\
Q_h &:= \{q_h \in C(\overline{\Omega}) : q_h|_K \in P_k(K) \ \forall K \in \mathcal{T}_h\}.
\end{aligned} \tag{4.1}$$

For the concentration field we use continuous piecewise polynomials of the same degree as the velocity,

$$\Psi_h := \{\psi_h \in C(\overline{\Omega}) : \psi_h|_K \in P_{k+1}(K) \ \forall K \in \mathcal{T}_h\}. \tag{4.2}$$

To approximate the multipliers $\boldsymbol{\lambda}$ and λ , and for the technical reasons detailed in [19, Section 4.3], we introduce partitions $\Gamma_{\tilde{h},\text{in}} := \{\bar{\Gamma}_1, \dots, \bar{\Gamma}_s\}$ and $\Gamma_{\tilde{h},\text{m}} := \{\bar{\Gamma}_{s+1}, \dots, \bar{\Gamma}_t\}$ of $\bar{\Gamma}_{\text{in}}$ and $\bar{\Gamma}_{\text{m}}$, respectively, which are independent of the triangulation of Ω , with $\tilde{h} := \max\{\tilde{h}_{\text{in}}, \tilde{h}_{\text{m}}\}$, where $\tilde{h}_{\text{in}}$ and $\tilde{h}_{\text{m}}$ denote the corresponding mesh sizes, that is,

$$\tilde{h}_{\text{in}} := \max_{j \in \{1, \dots, s\}} |\Gamma_j| \quad \text{and} \quad \tilde{h}_{\text{m}} := \max_{j \in \{s+1, \dots, t\}} |\Gamma_j|.$$

Given an integer $l \geq 0$, we set

$$\begin{aligned}
\Lambda_h^l &:= \left\{ \boldsymbol{\xi}_{\tilde{h}} \in \mathbf{L}^2(\Gamma) : \boldsymbol{\xi}_{\tilde{h}}|_S \in [P_l(S)]^2 \ \forall S \in \Gamma_{\tilde{h},\text{in}} \cup \Gamma_{\tilde{h},\text{m}}, \ \boldsymbol{\xi}_{\tilde{h}}|_{\Gamma_{\text{out}}} = \mathbf{0} \right\}, \\
\Lambda_h^l &:= \left\{ \xi_{\tilde{h}} \in L^2(\Gamma) : \xi_{\tilde{h}}|_S \in P_l(S) \ \forall S \in \Gamma_{\tilde{h},\text{in}}, \ \xi_{\tilde{h}}|_{\Gamma_{\text{in}}^c} = 0 \right\}.
\end{aligned} \tag{4.3}$$

It is straightforward to check that $\mathbf{\Lambda}_h^l$ and Λ_h^l satisfy (3.1), and that, since $l \geq 0$, hypotheses (H.1) and (H.2) hold. In turn, (H.3) is clearly satisfied by $\mathbf{Q}_h$, (H.4) follows with $\mathbf{v}_0$ constructed as in [10, Lemma 3.2], and (H.5) is classical for the Taylor–Hood pair. Moreover, proceeding analogously to [19, Lemma 4.7], one can show that, for $l = k$, there exist $\varepsilon_0, \widehat{\beta}_T, \widehat{\beta}_C > 0$, independent of h and $\tilde{h}$, such that, for each $h \leq \varepsilon_0 \tilde{h}$, there hold

$$\sup_{\substack{\mathbf{v}_h \in \mathbf{H}_h \\ \mathbf{v}_h \neq \mathbf{0}}} \frac{b_{F,2}(\mathbf{v}_h, \boldsymbol{\xi}_{\tilde{h}})}{\|\mathbf{v}_h\|_{1,\Omega}} \geq \widehat{\beta}_T \|\boldsymbol{\xi}_{\tilde{h}}\|_{-1/2,\Gamma} \quad \forall \boldsymbol{\xi}_{\tilde{h}} \in \mathbf{\Lambda}_{\tilde{h}}^k, \quad (4.4)$$

and

$$\sup_{\substack{\psi_h \in \Psi_h \\ \psi_h \neq 0}} \frac{b_C(\psi_h, \xi_{\tilde{h}})}{\|\psi_h\|_{1,\Omega}} \geq \widehat{\beta}_C \|\xi_{\tilde{h}}\|_{-1/2,\Gamma} \quad \forall \xi_{\tilde{h}} \in \Lambda_{\tilde{h}}^k, \quad (4.5)$$

that is, (H.6) and (H.7) hold.

Now, let

$$\begin{aligned} \mathbf{\Lambda}_{\tilde{h}}^l(\Gamma_\star) &:= \left\{ \boldsymbol{\xi}_{\tilde{h}} \in \mathbf{L}^2(\Gamma_\star) : \boldsymbol{\xi}_{\tilde{h}}|_S \in [\mathbf{P}_l(S)]^2 \quad \forall S \in \Gamma_{\tilde{h},\star} \right\}, \quad \star \in \{\text{in}, \text{m}\}, \\ \Lambda_{\tilde{h}}^l(\Gamma_{\text{in}}) &:= \left\{ \xi_{\tilde{h}} \in L^2(\Gamma_{\text{in}}) : \xi_{\tilde{h}}|_S \in \mathbf{P}_l(S) \quad \forall S \in \Gamma_{\tilde{h},\text{in}} \right\}, \end{aligned}$$

and denote by $\boldsymbol{\Pi}_{\tilde{h},\star}^l : \mathbf{L}^2(\Gamma_\star) \rightarrow \mathbf{\Lambda}_{\tilde{h}}^l(\Gamma_\star)$, $\star \in \{\text{in}, \text{m}\}$, and $\Pi_{\tilde{h},\text{in}}^l : L^2(\Gamma_{\text{in}}) \rightarrow \Lambda_{\tilde{h}}^l(\Gamma_{\text{in}})$ the respective orthogonal projections. Analogously to [19, Lemma 4.8], one shows the existence of $C_P^\star, c_P^{\text{in}} > 0$, $\star \in \{\text{in}, \text{m}\}$, independent of $\tilde{h}$, such that

$$\|\mathbf{z} - \boldsymbol{\Pi}_{\tilde{h},\star}^l(\mathbf{z})\|_{0,\Gamma_\star} \leq C_P^\star \tilde{h}^{1/2} \|\mathbf{z}\|_{1/2,\Gamma_\star} \quad \text{and} \quad \|z - \Pi_{\tilde{h},\text{in}}^l(z)\|_{0,\Gamma_{\text{in}}} \leq c_P^{\text{in}} \tilde{h}^{1/2} \|z\|_{1/2,\Gamma_{\text{in}}}, \quad (4.6)$$

for all $\mathbf{z} \in \mathbf{H}^{1/2}(\Gamma_\star)$, $\star \in \{\text{in}, \text{m}\}$, and all $z \in H^{1/2}(\Gamma_{\text{in}})$.

These estimates will be employed to prove that the estimates (3.26) and (3.28) hold. This is established in the following lemma.

Lemma 4.1 *Let $\tilde{\rho} > 0$ and set $M(\tilde{\rho}) := (1 + a_2)\tilde{\rho} + \mathbf{C}_1(\text{data})$. Then there exist $\varepsilon_1, \varepsilon_2 > 0$, independent of $h, \tilde{h}, \tilde{\rho}$ and the physical parameters, such that, if $\tilde{h}$ and a_2 satisfy*

$$\tilde{h} \leq \frac{\varepsilon_1}{\max\{\text{Re}, \text{Pe}\}^2 M(\tilde{\rho})^2} \quad \text{and} \quad a_2 \text{Pe} \tilde{\rho} \leq \varepsilon_2, \quad (4.7)$$

then hypothesis (H.8) holds.

Proof. Let $(\mathbf{w}_h, \phi_h) \in \mathbf{H}_h \times \Psi_h$ be such that $\|\mathbf{w}_h\|_{1,\Omega} \leq \tilde{\rho}$ and $\|\phi_h\|_{1,\Omega} \leq \tilde{\rho}$, and let $\mathbf{v}_h \in \mathbf{V}_{F,h}$. Since $b_F(\mathbf{v}_h, (0, \boldsymbol{\xi}_{\tilde{h}})) = (\boldsymbol{\xi}_{\tilde{h}}, \mathbf{v}_h)_{\Gamma_{\text{out}}^c} = 0$ for all $\boldsymbol{\xi}_{\tilde{h}} \in \mathbf{\Lambda}_{\tilde{h}}^l$, and since the extension by zero of any element of $\mathbf{\Lambda}_{\tilde{h}}^l(\Gamma_\star)$ belongs to $\mathbf{\Lambda}_{\tilde{h}}^l$, we deduce that $\boldsymbol{\Pi}_{\tilde{h},\star}^l(\mathbf{v}_h|_{\Gamma_\star}) = \mathbf{0}$ for $\star \in \{\text{in}, \text{m}\}$. Hence, by (4.6), (1.7) and (1.2),

$$\|\mathbf{v}_h\|_{0,\Gamma_{\text{out}}^c} \leq \sum_{\star \in \{\text{in}, \text{m}\}} \|(\mathbf{I} - \boldsymbol{\Pi}_{\tilde{h},\star}^l)(\mathbf{v}_h)\|_{0,\Gamma_\star} \lesssim \tilde{h}^{1/2} \|\mathbf{v}_h\|_{1,\Omega}. \quad (4.8)$$

Then, by Hölder's inequality, (1.6), (1.2), (4.8) and (3.5), we obtain

$$\begin{aligned} \left| \int_{\Gamma_{\text{out}}^c} ((\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)) \cdot \mathbf{n}) |\mathbf{v}_h|^2 \right| &\leq (\|\mathbf{w}_h\|_{L^4(\Gamma)} + \|\mathbf{s}(\phi_h)\|_{L^4(\Gamma)}) \|\mathbf{v}_h\|_{L^4(\Gamma)} \|\mathbf{v}_h\|_{0,\Gamma_{\text{out}}^c} \\ &\leq C_1 \tilde{h}^{1/2} M(\tilde{\rho}) \|\mathbf{v}_h\|_{1,\Omega}^2, \end{aligned}$$

with $C_1 > 0$ independent of h , $\tilde{h}$ and the physical parameters, and therefore

$$\frac{\tilde{\alpha}_F}{\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2 + \frac{1}{2} \int_{\Gamma_{\text{out}}^c} ((\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)) \cdot \mathbf{n}) |\mathbf{v}_h|^2 \geq \left\{ \frac{\tilde{\alpha}_F}{\text{Re}} - \frac{1}{2} C_1 \tilde{h}^{1/2} M(\tilde{\rho}) \right\} \|\mathbf{v}_h\|_{1,\Omega}^2, \quad (4.9)$$

which, under the first condition in (4.7), is bounded below by $\frac{\tilde{\alpha}_F}{2\text{Re}} \|\mathbf{v}_h\|_{1,\Omega}^2$, that is, (3.26) holds.

Assume now, in addition, that $(\mathbf{w}_h, \phi_h) \in X_h$. Since $\boldsymbol{\xi}_{\tilde{h}}|_{\Gamma_{\text{out}}} = \mathbf{0}$ for all $\boldsymbol{\xi}_{\tilde{h}} \in \boldsymbol{\Lambda}_{\tilde{h}}^l$ (cf. (4.3)), the definition of X_h yields

$$(\boldsymbol{\xi}_{\tilde{h}}, \mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h))_{\Gamma_{\text{out}}^c} = 0 \quad \forall \boldsymbol{\xi}_{\tilde{h}} \in \boldsymbol{\Lambda}_{\tilde{h}}^l,$$

which, since the extension by zero of any element of $\boldsymbol{\Lambda}_{\tilde{h}}^l(\Gamma_\star)$ belongs to $\boldsymbol{\Lambda}_{\tilde{h}}^l$, implies that

$$\boldsymbol{\Pi}_{\tilde{h},\star}^l((\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h))|_{\Gamma_\star}) = \mathbf{0}, \quad \star \in \{\text{in}, \text{m}\}. \quad (4.10)$$

Then, applying the first estimate in (4.6), together with (1.7), (1.2) and (3.5), we obtain, for $\star \in \{\text{in}, \text{m}\}$,

$$\begin{aligned} \|\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)\|_{0,\Gamma_\star} &= \|(\mathbf{I} - \boldsymbol{\Pi}_{\tilde{h},\star}^l)(\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h))\|_{0,\Gamma_\star} \leq C_P^\star \tilde{h}^{1/2} \|\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)\|_{1/2,\Gamma_\star} \\ &\leq C_P^\star \tilde{h}^{1/2} (\|\mathbf{w}_h\|_{1,\Omega} + \|\mathbf{s}(\phi_h)\|_{1/2,\Gamma}) \lesssim \tilde{h}^{1/2} M(\tilde{\rho}). \end{aligned} \quad (4.11)$$

Hence, given $\psi_h \in V_{C,h}$, by Hölder's inequality, (1.6), (1.2) and (4.11), we get

$$\begin{aligned} \frac{1}{2} \left| \int_{\Gamma_{\text{m}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 - \int_{\Gamma_{\text{in}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 \right| \\ \leq \frac{1}{2} \sum_{\star \in \{\text{in}, \text{m}\}} \|\mathbf{w}_h - \tilde{\mathbf{u}}_{\text{in},h}(\phi_h)\|_{0,\Gamma_\star} \|\psi_h\|_{L^4(\Gamma)}^2 \leq \frac{1}{2} C_2 \tilde{h}^{1/2} M(\tilde{\rho}) \|\psi_h\|_{1,\Omega}^2, \end{aligned}$$

with $C_2 > 0$ independent of h , $\tilde{h}$ and the physical parameters. In addition, by (1.1) and (1.6), there holds

$$\frac{a_2}{2} \left| \int_{\Gamma_{\text{m}}} \phi_h \psi_h^2 \right| \leq \frac{a_2}{2} \|\phi_h\|_{0,\Gamma} \|\psi_h\|_{L^4(\Gamma)}^2 \leq \frac{1}{2} C_\Gamma \hat{C}_\Gamma^2 a_2 \tilde{\rho} \|\psi_h\|_{1,\Omega}^2.$$

Consequently, the left-hand side of (3.28) is bounded below by

$$\begin{aligned} \frac{\tilde{\alpha}_C}{\text{Pe}} \|\psi_h\|_{1,\Omega}^2 + \frac{1}{2} \int_{\Gamma_{\text{m}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 - \frac{1}{2} \int_{\Gamma_{\text{in}}} ((\tilde{\mathbf{u}}_{\text{in},h}(\phi_h) - \mathbf{w}_h) \cdot \mathbf{n}) \psi_h^2 \\ + \frac{a_2}{2} \int_{\Gamma_{\text{m}}} \phi_h \psi_h^2 \geq \left\{ \frac{\tilde{\alpha}_C}{\text{Pe}} - \frac{1}{2} C_2 \tilde{h}^{1/2} M(\tilde{\rho}) - \frac{1}{2} C_\Gamma \hat{C}_\Gamma^2 a_2 \tilde{\rho} \right\} \|\psi_h\|_{1,\Omega}^2, \end{aligned} \quad (4.12)$$

which, under (4.7), is bounded below by $\frac{\tilde{\alpha}_C}{2\text{Pe}}\|\psi_h\|_{1,\Omega}^2$, that is, (3.28) holds. Here, it suffices to take $\varepsilon_1 := \min\{\tilde{\alpha}_F/C_1, \tilde{\alpha}_C/(2C_2)\}^2$ and $\varepsilon_2 := \tilde{\alpha}_C/(2C_\Gamma\hat{C}_\Gamma^2)$, which concludes the proof. $\square$

Having verified hypotheses (H.1)–(H.8), Theorems 3.6 and 3.12 guarantee that (3.4) is well posed and convergent.

We now turn to deriving the theoretical rate of convergence. To that end, we recall that the particular choice of spaces introduced above satisfies the following approximation properties:

- ($\mathbf{AP}_h^{\mathbf{u}}$) There exists $C > 0$, independent of h , such that, for all $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$, $\inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{1,\Omega} \leq Ch^{k+1}\|\mathbf{u}\|_{k+2,\Omega}$.
- ($\mathbf{AP}_h^p$) There exists $C > 0$, independent of h , such that, for all $p \in \mathbf{H}^{k+1}(\Omega)$, $\inf_{q_h \in \mathbf{Q}_h} \|p - q_h\|_{0,\Omega} \leq Ch^{k+1}\|p\|_{k+1,\Omega}$.
- ($\mathbf{AP}_h^\varphi$) There exists $C > 0$, independent of h , such that, for all $\varphi \in \mathbf{H}^{k+2}(\Omega)$, $\inf_{\psi_h \in \Psi_h} \|\varphi - \psi_h\|_{1,\Omega} \leq Ch^{k+1}\|\varphi\|_{k+2,\Omega}$.
- ($\mathbf{AP}_h^\lambda$) There exists $C > 0$, independent of $\tilde{h}$, such that, for all $\lambda \in \mathbf{H}^{k+1/2}(\Gamma)$, $\inf_{\xi_{\tilde{h}} \in \Lambda_{\tilde{h}}^k} \|\lambda - \xi_{\tilde{h}}\|_{-1/2,\Gamma} \leq C\tilde{h}^{k+1}\|\lambda\|_{k+1/2,\Gamma}$.
- ($\mathbf{AP}_{\tilde{h}}^\lambda$) There exists $C > 0$, independent of $\tilde{h}$, such that, for all $\lambda \in \mathbf{H}^{k+1/2}(\Gamma)$, $\inf_{\xi_{\tilde{h}} \in \Lambda_{\tilde{h}}^k} \|\lambda - \xi_{\tilde{h}}\|_{-1/2,\Gamma} \leq C\tilde{h}^{k+1}\|\lambda\|_{k+1/2,\Gamma}$.

The following theorem establishes the well-posedness of problem (3.4) with the spaces $\mathbf{H}_h$, $\mathbf{Q}_h$, Ψ_h , Λ_h^k and $\Lambda_{\tilde{h}}^k$ defined in (4.1), (4.2) and (4.3), and provides the corresponding theoretical rates of convergence.

Theorem 4.2 *Let $k \geq 1$ be an integer, let $\mathbf{H}_h, \mathbf{Q}_h, \Psi_h$ be as in (4.1)–(4.2), and let $\Lambda_h^k, \Lambda_{\tilde{h}}^k$ be as in (4.3) with $l = k$. Assume that $h \leq \varepsilon_0\tilde{h}$, with ε_0 such that (4.4) and (4.5) hold, and that $\tilde{h}$ and a_2 satisfy (4.7), with $\tilde{\rho}$ as in Theorem 3.6. Assume further that the hypotheses of Theorem 3.12 hold. If the exact solution satisfies $\mathbf{u} \in \mathbf{H}^{k+2}(\Omega)$, $p \in \mathbf{H}^{k+1}(\Omega)$, $\varphi \in \mathbf{H}^{k+2}(\Omega)$, $\lambda \in \mathbf{H}^{k+1/2}(\Gamma)$, and $\lambda \in \mathbf{H}^{k+1/2}(\Gamma)$, then there exists a unique $(\mathbf{u}_h, (p_h, \lambda_h), \varphi_h, \lambda_h) \in \mathbf{H}_h \times (\mathbf{Q}_h \times \Lambda_h^k) \times \Psi_h \times \Lambda_{\tilde{h}}^k$ solving (3.4), satisfying the following rates of convergence:*

$$\begin{aligned} \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega} + \|\varphi - \varphi_h\|_{1,\Omega} &\lesssim h^{k+1} \mathcal{E}_1(\mathbf{u}, p, \varphi) + \tilde{h}^{k+1} \mathcal{E}_2(\lambda, \lambda), \\ \|(p - p_h, \lambda - \lambda_h)\| &\lesssim \frac{1}{\text{Re}} [h^{k+1} \mathcal{E}_1(\mathbf{u}, p, \varphi) + \tilde{h}^{k+1} \mathcal{E}_2(\lambda, \lambda)], \\ \|\lambda - \lambda_h\|_{-1/2,\Gamma} &\lesssim \left(\frac{1}{\text{Pe}} + a_3\right) [h^{k+1} \mathcal{E}_1(\mathbf{u}, p, \varphi) + \tilde{h}^{k+1} \mathcal{E}_2(\lambda, \lambda)], \end{aligned} \quad (4.13)$$

where

$$\begin{aligned} \mathcal{E}_1(\mathbf{u}, p, \varphi) &:= \|\mathbf{u}\|_{k+2,\Omega} + (1 + a_3 \text{Pe}) \|\varphi\|_{k+2,\Omega} + \text{Re} \|p\|_{k+1,\Omega}, \\ \mathcal{E}_2(\lambda, \lambda) &:= \text{Re} \|\lambda\|_{k+1/2,\Gamma} + \text{Pe} \|\lambda\|_{k+1/2,\Gamma}, \end{aligned}$$

and the hidden constants are independent of $h, \tilde{h}$ and the physical parameters.

Proof. The well-posedness of (3.4) follows from Theorem 3.6, since hypotheses (H.1)–(H.8) have been verified above (cf. (4.4), (4.5) and Lemma 4.1). In turn, estimate (4.13) follows directly from (3.67)–(3.69), together with the approximation properties ($\mathbf{AP}_h^{\mathbf{u}}$), ($\mathbf{AP}_h^p$), ($\mathbf{AP}_h^\varphi$), ($\mathbf{AP}_h^\lambda$), and ($\mathbf{AP}_{\tilde{h}}^\lambda$). We omit further details. $\square$

5 Computational results

In this section, we present a series of numerical experiments to assess the accuracy and performance of the proposed finite element method introduced in Section 3. The numerical study is organized into two main parts. First, we investigate the convergence behavior of the method for different finite element configurations through a systematic mesh-refinement study. The numerical errors and experimental convergence rates are reported for the different unknowns of the coupled problem.

In the second part, we consider a membrane channel representing a reverse osmosis configuration and investigate the effect of different spacer arrangements on the computed solution. Four configurations are considered, ranging from a single *Central* spacer and a *Non-centered* spacer to configurations involving three spacers, namely the *Submerged* and *Zig-zag* arrangements. The numerical results are analyzed from two complementary perspectives. First, the velocity, pressure, and salt concentration fields are examined throughout the computational domain. Second, the behavior of the relevant quantities along the membrane is analyzed to characterize the local influence of the spacer arrangement on the flow and transport processes.

5.1 Convergence and stability test

In our first example, we illustrate the accuracy of our method by considering a manufactured exact solution defined on the unit square $\Omega = (0, 1) \times (0, 1)$, whose boundary is decomposed as in (1.22). We consider a Reynolds $\text{Re} = 1$ and Péclet number $\text{Pe} = 1$, and adjust the terms on the right-hand side of (2.6) so that the exact solution is given by the functions

$$\mathbf{u} = \begin{pmatrix} 2x^2y(2y-1)(y-1)(x-1)^2 \\ -2xy^2(y-1)^2(2x-1)(x-1) \end{pmatrix}, \quad p = \cos(\pi x) \exp(y) \quad \text{and} \\ \varphi = \frac{1}{a_2}((1-2y)(-2xy^2(y-1)^2(2x-1)(x-1)) + a_1).$$

Since the manufactured solution is smooth, the restriction of the exact Lagrange multiplier for the velocity $\boldsymbol{\lambda}$ to Γ_{out}^c reduces to $\boldsymbol{\lambda}|_{\Gamma_{\text{out}}^c} = -(\frac{1}{\text{Re}}\nabla\mathbf{u} - p\mathbf{I})\mathbf{n}|_{\Gamma_{\text{out}}^c}$. Analogously, the exact Lagrange multiplier for the concentration satisfies $\lambda|_{\Gamma_{\text{in}}} = -\frac{1}{\text{Pe}}\nabla\varphi \cdot \mathbf{n}|_{\Gamma_{\text{in}}}$. The approximation errors for the velocity and pressure are denoted by

$$\mathbf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}, \quad \mathbf{e}(p) := \|p - p_h\|_{0,\Omega}, \quad \mathbf{e}(\varphi) := \|\varphi - \varphi_h\|_{1,\Omega}.$$

Now, since the natural $H^{-1/2}(\Gamma)$ - and $\mathbf{H}^{-1/2}(\Gamma)$ -norms of the corresponding Lagrange multipliers are not directly computable, their errors are measured in the L^2 - and $\mathbf{L}^2$ -norms on the corresponding boundary portions, namely,

$$\mathbf{e}(\lambda) := \|\lambda - \lambda_h\|_{0,\Gamma_{\text{in}}} \quad \text{and} \quad \mathbf{e}(\boldsymbol{\lambda}) := \|\boldsymbol{\lambda} - \boldsymbol{\lambda}_h\|_{0,\Gamma_{\text{out}}^c}.$$

Given two successive mesh sizes h and h' , the experimental order of convergence is computed as

$$r(*) := \frac{\log(\mathbf{e}(*)/\mathbf{e}'(*))}{\log(h/h')}, \quad * \in \{\mathbf{u}, p, \varphi, \boldsymbol{\lambda}, \lambda\},$$

where $\mathbf{e}(*)$ and $\mathbf{e}'(*)$ denote the corresponding errors for h and h' .

Besides the Taylor–Hood family introduced in Section 4.1, the velocity and pressure can also be approximated by means of the MINI-element (see, e.g., [20]). To this end, for each $K \in \mathcal{T}_h$, we let $b_K := \mu_1 \mu_2 \mu_3$ be the cubic bubble function on K , where μ_1, μ_2, μ_3 denote the barycentric coordinates of K , and set $P_{1,b}(K) := P_1(K) \oplus \text{span}\{b_K\}$. Then, the MINI-element consists of the pair $(\mathbf{H}_h, Q_h)$ given by

$$\begin{aligned}\mathbf{H}_h &:= \left\{ \mathbf{v}_h \in [C(\bar{\Omega})]^2 : \mathbf{v}_h|_K \in [P_{1,b}(K)]^2 \quad \forall K \in \mathcal{T}_h \right\}, \\ Q_h &:= \left\{ q_h \in C(\bar{\Omega}) : q_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h \right\}.\end{aligned}\tag{5.1}$$

In this case, the concentration and the multipliers $\boldsymbol{\lambda}$ and λ are approximated by

$$\Psi_h := \left\{ \psi_h \in C(\bar{\Omega}) : \psi_h|_K \in P_1(K) \quad \forall K \in \mathcal{T}_h \right\}, \quad \boldsymbol{\Lambda}_h^0 \quad \text{and} \quad \Lambda_h^0,\tag{5.2}$$

respectively, with $\boldsymbol{\Lambda}_h^0$ and Λ_h^0 as in (4.3) with $l = 0$. Proceeding as in Section 4.1, one can show that these spaces satisfy hypotheses **(H.1)**–**(H.8)**, provided that $h \leq \varepsilon_0 \tilde{h}$ and (4.7) holds. Then, from Theorem 3.12 and the approximation properties of these spaces (see, e.g., [20]), we deduce that under minimal regularity assumptions, the total error is of order $O(h) + O(\tilde{h})$, where the contributions $O(h)$ and $O(\tilde{h})$ come from the approximation of $(\mathbf{u}, p, \varphi)$ and of $(\boldsymbol{\lambda}, \lambda)$, respectively.

In all the experiments, the partitions $\Gamma_{\tilde{h},\text{in}}$ and $\Gamma_{\tilde{h},\text{m}}$ used to define the multiplier spaces are obtained by merging pairs of consecutive boundary edges of $\mathcal{T}_h$, so that $\tilde{h} = 2h$, in agreement with the condition $h \leq \varepsilon_0 \tilde{h}$ required by the analysis (cf. Section 4.1).

The convergence histories for the two parameter configurations are reported in Tables 5.1 and 5.2. The first experiment corresponds to $\mathbf{a} = (2.0, 0.2, 4.5)$, which satisfies the theoretical assumption $a_3 > 0.5 a_1$. In contrast, the second experiment considers $\mathbf{a} = (3.0, 2.0, 0.1)$, for which the hypothesis of Lemma 2.3 is not satisfied. Nevertheless, both numerical experiments exhibit the expected asymptotic behavior for all finite element families considered. The MINI+ $P_1 + P_0 + P_0$ approximation attains the expected first-order convergence for the velocity, pressure, concentration, and both Lagrange multipliers, while the pressure displays the well-known superconvergent behavior of order $O(h^{1.5})$, in agreement with the analysis reported in [12]. From the viewpoint of computational cost, this family therefore provides an efficient low-order approximation.

For the Taylor–Hood families, the results confirm the optimal convergence rates predicted by the theory. The $P_2 - P_1 - P_2 - P_1 - P_1$ approximation yields second-order convergence for the velocity, concentration, and both Lagrange multipliers, together with second-order convergence for the pressure. Increasing the polynomial degree to the $P_3 - P_2 - P_3 - P_2 - P_2$ family further improves the accuracy, producing third-order convergence for all variables while preserving the expected asymptotic behavior. These results indicate that the higher-order Taylor–Hood discretizations provide a substantial accuracy gain at the expense of a larger number of degrees of freedom.

5.2 Membrane channel with different spacer configurations

This experiment aims to test the proposed scheme in a membrane channel unit whose length is chosen to allow the development of the flow before interacting with the spacer. We consider

Table 5.1: Example 5.1. Convergence history of the scheme in the square domain for $\mathbf{a} = (2.0, 0.2, 4.5)$.

h	DoFs	$\mathbf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathbf{e}(p)$	$r(p)$	$\mathbf{e}(\varphi)$	$r(\varphi)$	$\mathbf{e}(\boldsymbol{\lambda})$	$r(\boldsymbol{\lambda})$	$\mathbf{e}(\lambda)$	$r(\lambda)$
<i>MINI: $P_{1,b} - P_1 - P_1 - P_0 - P_0$</i>											
0.0202	40595	2.1451e-03	1.00	8.0231e-04	1.52	4.0212e-03	1.00	5.5671e-02	1.00	6.6125e-03	1.01
0.0177	52795	1.8759e-03	1.00	6.5482e-04	1.52	3.5196e-03	1.00	4.8707e-02	1.00	5.7776e-03	1.01
0.0157	66595	1.6668e-03	1.00	5.4751e-04	1.52	3.1292e-03	1.00	4.3292e-02	1.00	5.1306e-03	1.01
0.0141	81995	1.4996e-03	1.00	4.6659e-04	1.52	2.8166e-03	1.00	3.8961e-02	1.00	4.6143e-03	1.01
0.0129	98995	1.3629e-03	1.00	4.0379e-04	1.52	2.5609e-03	1.00	3.5418e-02	1.00	4.1926e-03	1.00
<i>Taylor-Hood: $P_2 - P_1 - P_2 - P_1 - P_1$</i>											
0.0202	66346	3.76183e-05	2.08	1.11671e-04	2.02	9.47230e-05	2.00	7.57596e-04	1.99	2.34458e-04	2.03
0.0177	86226	2.85031e-05	2.07	8.52328e-05	2.02	7.25169e-05	2.00	5.80969e-04	1.99	1.78881e-04	2.02
0.0157	108706	2.23362e-05	2.06	6.71792e-05	2.02	5.72915e-05	2.00	4.59604e-04	1.99	1.40963e-04	2.02
0.0141	133786	1.79720e-05	2.06	5.43069e-05	2.02	4.64012e-05	2.00	3.72643e-04	1.99	1.13946e-04	2.02
0.0129	161466	1.47712e-05	2.05	4.48080e-05	2.02	3.83442e-05	2.00	3.08212e-04	1.99	9.40177e-05	2.01
<i>High-order Taylor-Hood: $P_3 - P_2 - P_3 - P_2 - P_2$</i>											
0.0202	155937	3.9018e-07	3.01	6.7896e-07	3.00	9.8069e-07	3.01	3.6629e-06	3.00	4.8991e-06	2.98
0.0177	203017	2.6107e-07	3.01	4.5466e-07	3.00	6.5590e-07	3.00	2.4532e-06	3.00	3.2921e-06	2.98
0.0157	256297	1.8319e-07	3.01	3.1922e-07	3.00	4.6039e-07	2.99	1.7226e-06	3.00	2.3178e-06	2.98
0.0141	315777	1.3344e-07	3.01	2.3265e-07	3.00	3.3582e-07	2.98	1.2556e-06	3.00	1.6933e-06	2.98
0.0129	381457	1.0020e-07	3.01	1.7476e-07	3.00	2.5281e-07	2.96	9.4322e-07	3.00	1.2749e-06	2.97

the dimensionless computational domain $\Omega := (0, \frac{a}{b}) \times (0, 1)$, where $a = 20$ mm and $b = 2$ mm, so that $\Omega = (0, 10) \times (0, 1)$. The scaling by the channel width b allows us to work with a fixed transverse dimension while preserving the relevant geometric features of the membrane channel.

The numerical simulations presented in Figures 5.1, 5.2, and 5.3 were performed using 1,358,310 degrees of freedom (DoFs). The velocity and pressure fields were approximated using the Taylor–Hood finite element pair $P_2 - P_1$, while the concentration was approximated using P_2 elements. The Lagrange multipliers were discretized using discontinuous P_1 elements on the boundary. The numerical results in the computational channel, presented in Figures 5.1, 5.2, and 5.3, were visualized using ParaView. In contrast, the quantities evaluated along the membrane, presented in Figures 5.4–5.6, were obtained and visualized using a custom Python script based on the `matplotlib.pyplot` library.

The channel contains impermeable circular spacers arranged in different configurations, which modify the local flow field and, consequently, the transport of salt near the membrane. For each spacer configuration, we denote by Γ_w the boundary of the solid spacers. On Γ_w , we impose

$$\mathbf{u} = \mathbf{0}, \quad \left(\varphi \mathbf{u} - \frac{1}{\text{Pe}} \nabla \varphi \right) \cdot \mathbf{n} = 0 \quad \text{on } \Gamma_w.$$

The first condition imposes zero velocity on the spacer boundary, reflecting the fact that the spacers act as solid obstacles. The second condition imposes zero salt flux through the solid spacers.

We consider four different spacer configurations to investigate the effect of the spacer arrangement on the velocity, pressure, and salt concentration fields. In the first configuration, referred to as the *Central spacer* configuration, a single circular spacer is placed at $(x, y) = (5, 0.5)$. In the second configuration, referred to as the *Non-centered spacer* configuration, the spacer is

Table 5.2: Example 5.1. Convergence history of the scheme in the square domain for $\mathbf{a} = (3.0, 2.0, 0.1)$.

h	DoFs	$\mathbf{e}(\mathbf{u})$	$r(\mathbf{u})$	$\mathbf{e}(p)$	$r(p)$	$\mathbf{e}(\varphi)$	$r(\varphi)$	$\mathbf{e}(\boldsymbol{\lambda})$	$r(\boldsymbol{\lambda})$	$\mathbf{e}(\lambda)$	$r(\lambda)$
<i>MINI: $P_{1,b} - P_1 - P_1 - P_0 - P_0$</i>											
0.0202	40595	2.1452e-03	1.00	8.0354e-04	1.52	4.0212e-03	1.00	5.5670e-02	1.00	6.6117e-03	1.01
0.0177	52795	1.8760e-03	1.00	6.5570e-04	1.52	3.5196e-03	1.00	4.8707e-02	1.00	5.7771e-03	1.01
0.0157	66595	1.6668e-03	1.00	5.4816e-04	1.52	3.1291e-03	1.00	4.3292e-02	1.00	5.1302e-03	1.01
0.0141	81995	1.4996e-03	1.00	4.6709e-04	1.52	2.8166e-03	1.00	3.8961e-02	1.00	4.6140e-03	1.01
0.0129	98995	1.3629e-03	1.00	4.0419e-04	1.52	2.5609e-03	1.00	3.5418e-02	1.00	4.1924e-03	1.00
<i>Taylor-Hood: $P_2 - P_1 - P_2 - P_1 - P_1$</i>											
0.0202	66346	3.7605e-05	2.08	1.1164e-04	2.02	9.4727e-05	2.00	7.5581e-04	1.99	2.3451e-04	2.03
0.0177	86226	2.8495e-05	2.07	8.5212e-05	2.02	7.2519e-05	2.00	5.7971e-04	1.99	1.7891e-04	2.02
0.0157	108706	2.2331e-05	2.06	6.7165e-05	2.02	5.7293e-05	2.00	4.5869e-04	1.99	1.4098e-04	2.02
0.0141	133786	1.7968e-05	2.06	5.4297e-05	2.02	4.6402e-05	2.00	3.7195e-04	1.99	1.1396e-04	2.02
0.0129	161466	1.4768e-05	2.05	4.4801e-05	2.02	3.8345e-05	2.00	3.0768e-04	1.99	9.4024e-05	2.02
<i>High-order Taylor-Hood: $P_3 - P_2 - P_3 - P_2 - P_2$</i>											
0.0202	155937	3.9018e-07	3.01	6.7896e-07	3.00	9.8074e-07	3.01	3.6629e-06	3.00	4.9008e-06	2.98
0.0177	203017	2.6107e-07	3.01	4.5466e-07	3.00	6.5591e-07	3.01	2.4532e-06	3.00	3.2930e-06	2.98
0.0157	256297	1.8319e-07	3.01	3.1922e-07	3.00	4.6038e-07	3.00	1.7226e-06	3.00	2.3183e-06	2.98
0.0141	315777	1.3344e-07	3.01	2.3265e-07	3.00	3.3580e-07	2.98	1.2556e-06	3.00	1.6936e-06	2.98
0.0129	381457	1.0020e-07	3.01	1.7476e-07	3.00	2.5278e-07	2.96	9.4322e-07	3.00	1.2751e-06	2.97

displaced toward the lower membrane, with its center located at $(x, y) = (5, 0.21)$, allowing us to assess the effect of reducing the gap between the spacer and the membrane. The third configuration, referred to as the *Submerged* configuration, consists of three circular spacers aligned along the horizontal direction, with their centers located at $(x, y) = (2.5, 0.5)$, $(x, y) = (5, 0.5)$, and $(x, y) = (7.5, 0.5)$. Finally, the fourth configuration, referred to as the *Zig-zag* configuration, consists of three circular spacers arranged in a staggered pattern, with their centers located at $(x, y) = (2.5, 0.21)$, $(x, y) = (5, 0.79)$, and $(x, y) = (7.5, 0.21)$. This arrangement induces an asymmetric flow structure and promotes the interaction of recirculation regions between consecutive spacers. For all configurations, the radius of each spacer is $r = 0.2$.

The boundary Γ is decomposed according to the partition introduced in (1.22). For the numerical experiments, we consider the coupled problem (2.6), with the inlet velocity prescribed by the following (nondimensional) Berman-type profile (see, e.g., [5]):

$$\mathbf{u}_{\text{in}}(0, y) = \begin{pmatrix} \frac{3}{2}(1 - \zeta^2) \left[1 - \frac{\text{Re}}{420}(2 - 7\zeta^2 - 7\zeta^4) \right] \\ v_w \left[\frac{1}{2}\zeta(3 - \zeta^2) - \frac{\text{Re}}{280}\zeta(2 - 3\zeta^2 + \zeta^6) \right] \end{pmatrix} \quad \forall (0, y) \in \Gamma_{\text{in}}, \quad (5.3)$$

where $\zeta := 2y - 1$ and $v_w := a_1 - a_2$ denotes the (nondimensional) permeate velocity, where we take $\varphi_0 := \varphi_{\text{in}}$ as the reference concentration, so that the nondimensional inlet concentration equals 1. Notice that $u_{1,\text{in}}$ vanishes at $y = 0$ and $y = 1$, so that (1.27) holds, and that $u_{2,\text{in}}(0, 1) = v_w$ and $u_{2,\text{in}}(0, 0) = -v_w$, so that the compatibility condition (1.26) is satisfied. In addition, since $u_{2,\text{in}}$ is a polynomial and φ_{in} is constant, (1.28) holds for any $\eta \in (0, 1]$.

The physical parameters associated with the scaled computational domain are reported in Table 5.3. These include the membrane permeability, transmembrane pressure, salt permeability,

fluid properties, temperature, inlet salt concentration, and the derived parameters appearing in the scaled formulation. These quantities are kept fixed throughout the numerical experiments, while the spacer arrangement and the characteristic inlet velocity are varied according to the test under consideration.

The characteristic inlet velocity is varied according to $U_0 \in \{30, 45, 60\} \text{ mm s}^{-1}$, with the corresponding value of U_0 used in the inlet profile (5.3). The parameters a_1 , a_2 , and a_3 are defined consistently with the scaling associated with the computational domain $\Omega = (0, a/b) \times (0, 1)$, and their corresponding values are reported in Table 5.3.

The results shown in Figures 5.1–5.3 illustrate the behavior of the velocity, pressure, and salt concentration fields, respectively, for the four spacer configurations at an inlet velocity of $U_0 = 60 \text{ mm s}^{-1}$. Overall, the results show that the arrangement of the spacers significantly affects the local flow structure and, consequently, the convective transport of salt throughout the channel.

Figure 5.1 shows that the Central spacer configuration produces an approximately symmetric flow structure around the spacer. The flow accelerates as it passes through the regions on either side of the obstacle, followed by the development of a recirculation region downstream. In the Non-centered spacer configuration, the displacement of the spacer toward the lower membrane reduces the available gap between the obstacle and the wall, leading to a more asymmetric flow structure and a recirculation region concentrated downstream of the spacer. In the Submerged configuration, the three spacers generate a sequence of acceleration and recirculation regions, with the wakes produced by consecutive spacers interacting along the channel. Finally, in the Zig-zag configuration, the alternating arrangement of the spacers with respect to the membranes produces a more complex flow structure, with recirculation regions developing alternately near the upper and lower membranes. These results illustrate the strong dependence of the flow structure on the geometric arrangement of the spacers.

The corresponding pressure distributions are shown in Figure 5.2. For the Central configuration, the spacer produces an increase in pressure upstream of the obstacle, followed by a localized decrease downstream and a subsequent pressure recovery. The Non-centered configuration exhibits a more asymmetric pressure distribution, with a pronounced variation in the region where the gap between the spacer and the lower membrane is reduced. In the Submerged configuration, the three spacers generate a sequence of pressure variations along the channel, reflecting the successive acceleration, deceleration, and recovery of the flow around the obstacles. A similar behavior is observed in the Zig-zag configuration, although the alternating position of the spacers produces pressure variations near both membranes. Thus, the pressure field reflects the successive interaction between the flow and the solid obstacles.

The salt concentration distributions are presented in Figure 5.3. In all four configurations, the concentration exhibits a relatively uniform distribution in the central region of the channel, while pronounced variations develop in the vicinity of the membranes and around the spacers. For the Central configuration, the effect of the spacer is mainly localized around the obstacle and in its downstream region. In the Non-centered configuration, the proximity of the spacer to the lower membrane produces a more localized perturbation of the concentration field near the obstacle. In the Submerged configuration, the three spacers generate a sequence of concentration perturbations associated with the recirculation regions developing downstream of each obstacle. Finally, the Zig-zag configuration produces alternating concentration variations near the upper and lower membranes, following the arrangement of the spacers. These results indicate that the modification of the flow structure induced by the spacers is directly reflected in the salt

distribution throughout the channel.

The behavior of the relevant quantities along the membrane is examined next. Figures 5.4–5.6 provide a more detailed description of the local response of the permeable membrane to the different spacer arrangements. The permeate velocity shown in Figure 5.4 exhibits localized variations at the positions of the spacers. The Central configuration presents a single pronounced perturbation, whereas the Non-centered configuration exhibits a sharper local variation due to the proximity of the spacer to the membrane. The Submerged and Zig-zag configurations produce multiple peaks and valleys, corresponding to the successive interactions between the spacers and the membrane. In particular, the Zig-zag arrangement leads to alternating local variations along the membrane, consistent with the staggered position of the spacers.

The pressure distributions along the membrane, shown in Figure 5.5, exhibit corresponding local variations at the spacer locations. The Central and Non-centered configurations display a single dominant pressure variation, whereas the Submerged and Zig-zag configurations produce successive pressure drops and recoveries along the membrane. These repeated variations reflect the influence of the individual spacers and their interaction with the surrounding flow.

The salt concentration along the membrane is shown in Figure 5.6. The Central configuration exhibits a relatively smooth variation along the membrane, with a localized perturbation near the spacer. The Non-centered configuration produces a more pronounced local concentration variation in the vicinity of the spacer. For the Submerged and Zig-zag configurations, the concentration exhibits repeated oscillations associated with the successive spacers. In the Zig-zag configuration, these variations alternate along the membrane according to the staggered arrangement of the obstacles.

Overall, the numerical results show that the spacer arrangement plays an important role in determining the local flow, pressure, and salt concentration along the membrane. The Central spacer configuration leads to relatively smooth variations in the membrane quantities, with a localized perturbation near the spacer. By contrast, the Non-centered configuration results in more pronounced local changes in the permeate velocity and salt concentration, which can be attributed to the reduced distance between the spacer and the membrane. The Submerged and Zig-zag configurations generate successive variations along the membrane as a consequence of the interaction between the multiple spacers and the surrounding flow. In particular, the Zig-zag arrangement produces alternating local perturbations along the membrane, consistent with the staggered configuration of the spacers. These numerical observations are in agreement with the behavior reported in [2, 11].

Furthermore, the computed salt concentration remains positive throughout the computational domain and along the membrane for all the configurations considered, even though the assumption $a_3 - \frac{1}{2}a_1 \geq 0$ of Lemma 2.3 is not satisfied in this setting. This suggests that the latter is a technical requirement of the analysis rather than a limitation of the method.

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Table 5.3: Physical and derived parameters adopted in the numerical simulations.

Symbol	Description	Value (units)
<i>Membrane properties</i>		
A	Water permeability	$3.75 \times 10^{-6} \text{ (mm}^2 \text{ s kg}^{-1}\text{)}$
B	Salt permeability	$5.56 \times 10^{-6} \text{ (mm s}^{-1}\text{)}$
ΔP	Transmembrane pressure difference	$5572.875 \text{ (kg mm}^{-1} \text{ s}^{-2}\text{)}$
<i>Fluid and transport properties</i>		
μ	Dynamic viscosity	$8.9 \times 10^{-7} \text{ (kg mm}^{-1} \text{ s}^{-1}\text{)}$
ρ	Fluid density	$1027.2 \times 10^{-9} \text{ (kg mm}^{-3}\text{)}$
κ	Salt diffusivity	$1.611 \times 10^{-3} \text{ (mm}^2 \text{ s}^{-1}\text{)}$
<i>Thermodynamic and operating parameters</i>		
i	Van't Hoff factor	2
R	Ideal gas constant	$8.314 \times 10^6 \text{ (kg mm}^2 \text{ s}^{-2} \text{ mol}^{-1} \text{ K}^{-1}\text{)}$
T	Temperature	298.0 (K)
U_0	Characteristic inlet velocity	30–60 (mm s ^{−1})
φ_{in}	Inlet salt concentration	$600 \times 10^{-9} \text{ (mol mm}^{-3}\text{)}$
<i>Nondimensional parameters</i>		
a_1		$A\Delta P/U_0$
a_2		$AiRT\varphi_{\text{in}}/U_0$
a_3		B/U_0

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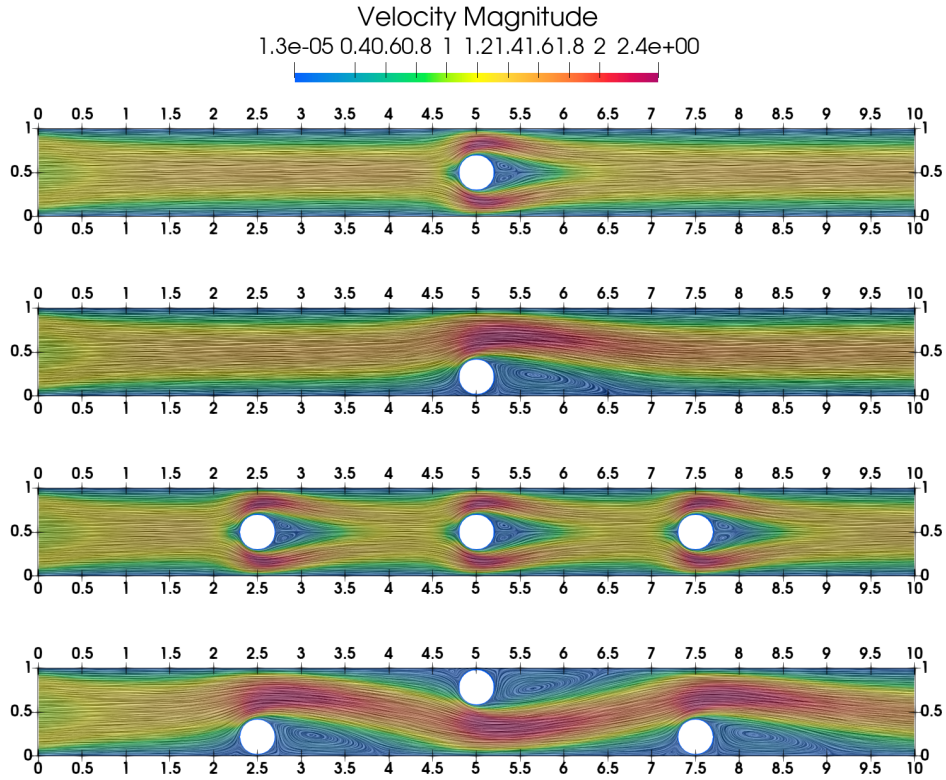


Figure 5.1: Comparison of velocity magnitude and streamline patterns at an inlet velocity of $U_0 = 60 \text{ mm s}^{-1}$. The first panel corresponds to the Central spacer configuration, the second panel to the Non-centered spacer configuration, and the third and fourth panels to the Submerged and Zig-zag configurations, respectively.

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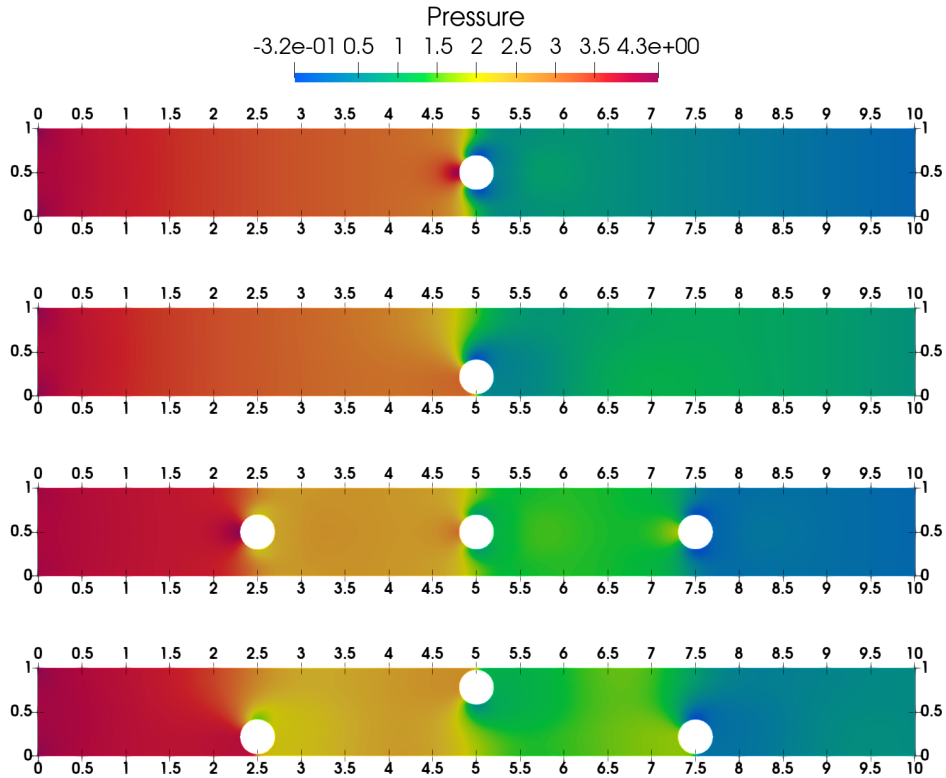


Figure 5.2: Comparison of pressure distributions at an inlet velocity of $U_0 = 60 \text{ mm s}^{-1}$. The first panel corresponds to the Central spacer configuration, the second panel to the Non-centered spacer configuration, and the third and fourth panels to the Submerged and Zig-zag configurations, respectively.

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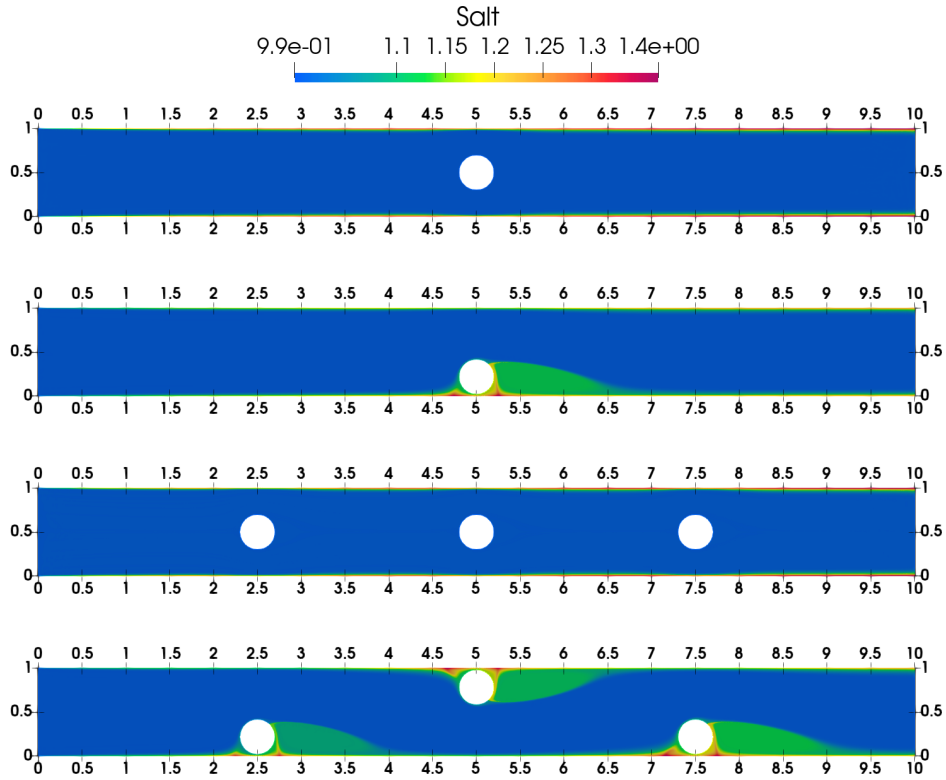


Figure 5.3: Comparison of salt concentration distributions at an inlet velocity of $U_0 = 60 \text{ mm s}^{-1}$. The first panel corresponds to the Central spacer configuration, the second panel to the Non-centered spacer configuration, and the third and fourth panels to the Submerged and Zig-zag configurations, respectively.

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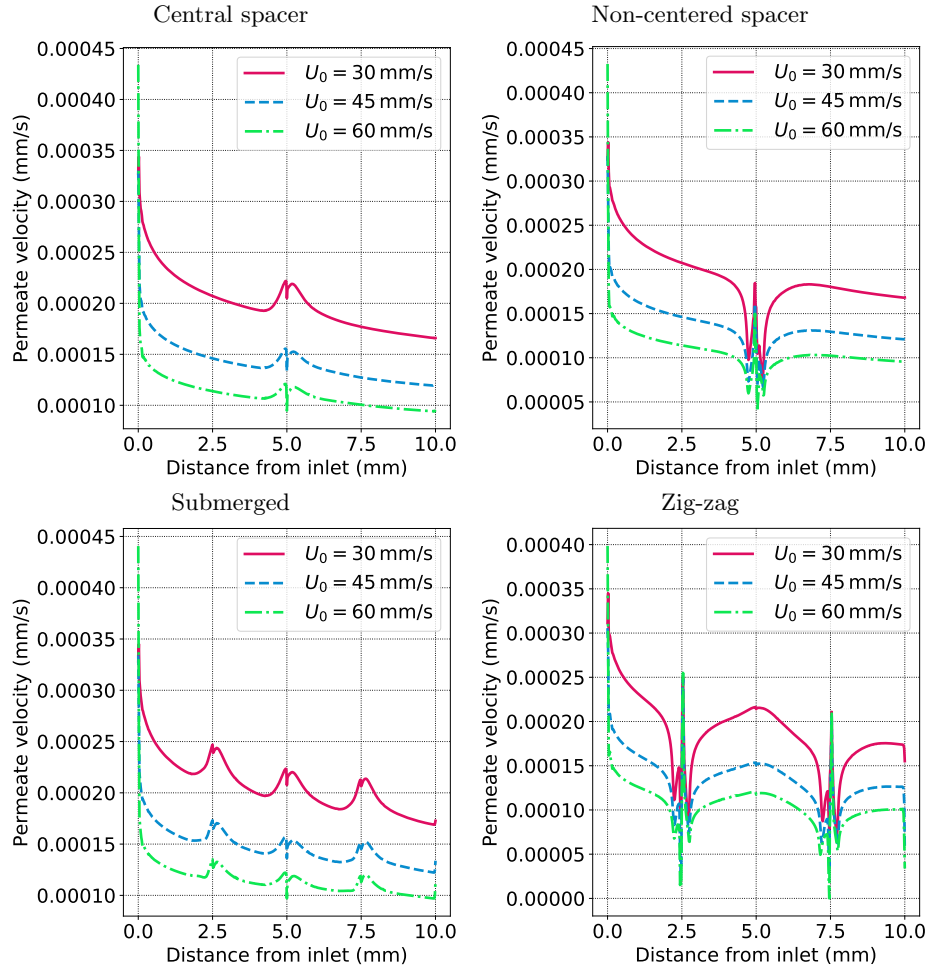


Figure 5.4: Permeate velocity along the bottom membrane for $U_0 = 60 \text{ mm s}^{-1}$ and the four spacer configurations.

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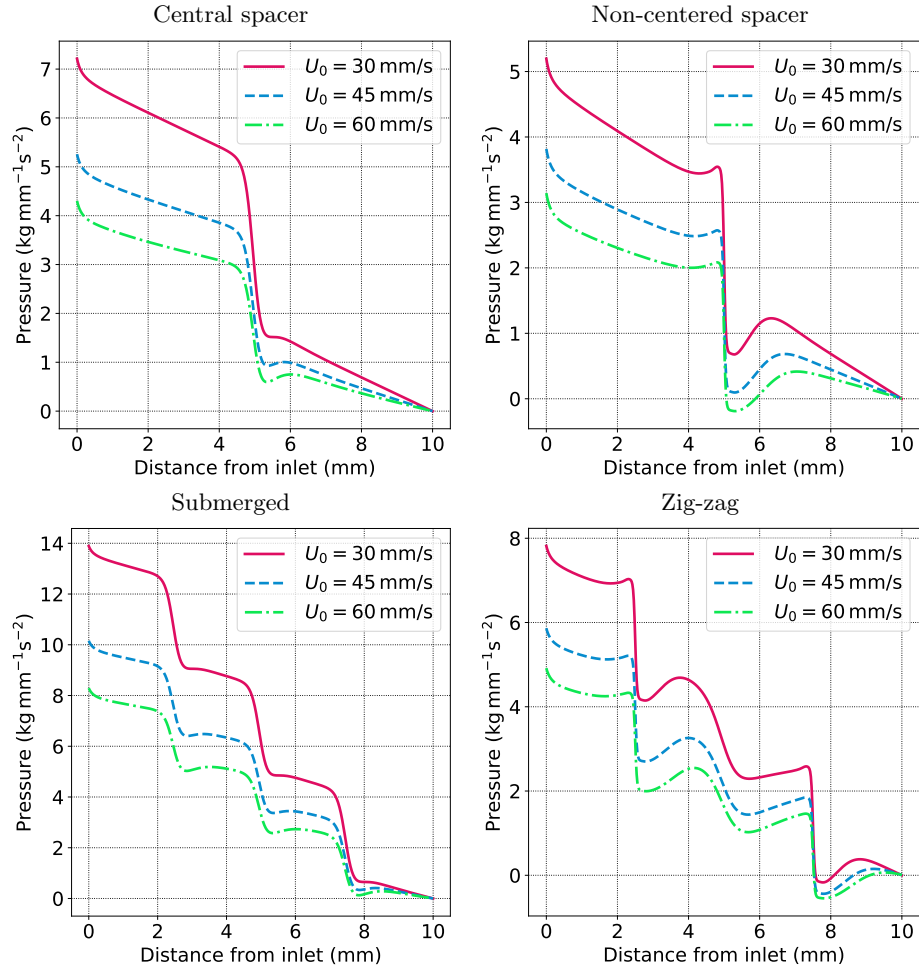


Figure 5.5: Pressure along the bottom membrane for $U_0 = 60 \text{ mm s}^{-1}$ and the four spacer configurations.

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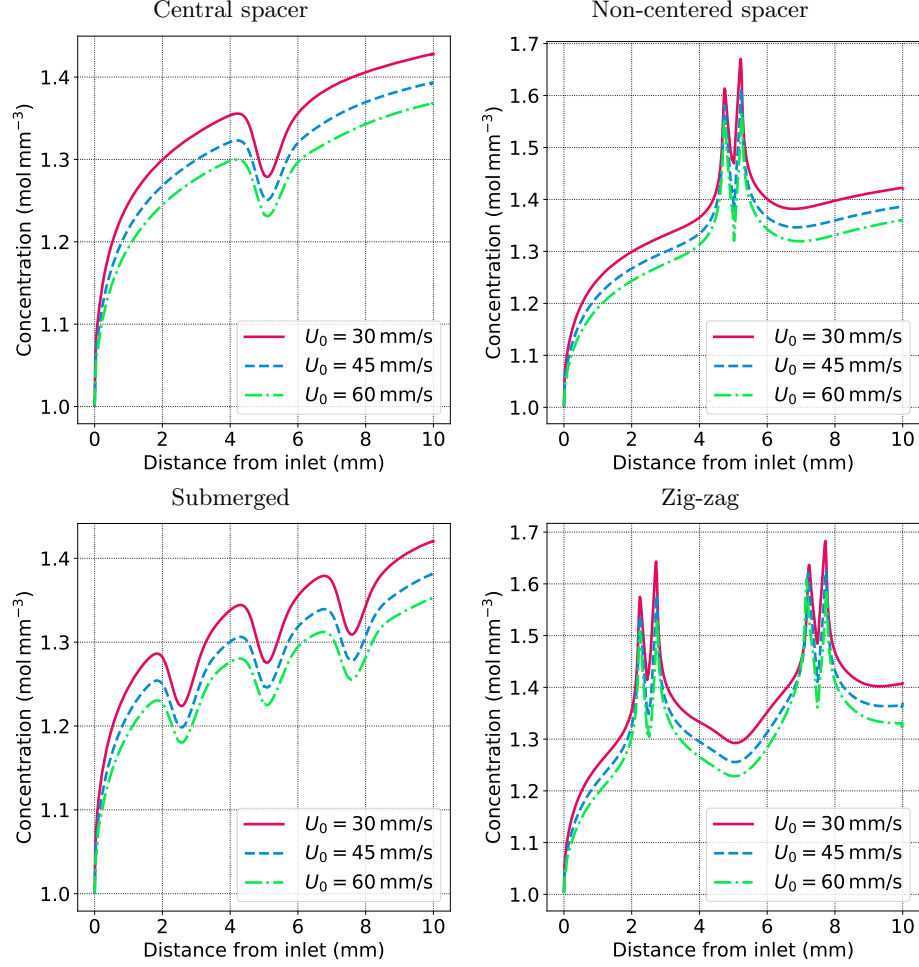


Figure 5.6: Salt concentration along the bottom membrane for $U_0 = 60 \text{ mm s}^{-1}$ and the four spacer configurations.

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6 Appendix A. Proof of Lemma 1.1

Let Σ_1 , Σ_2 and $\mathbf{x}_1$ be as in Lemma 1.1, and set

$$\Sigma_3 := \Gamma \setminus (\overline{\Sigma}_1 \cup \overline{\Sigma}_2), \quad \mathbf{x}_2 := \overline{\Sigma}_2 \cap \overline{\Sigma}_3 \quad \text{and} \quad \mathbf{x}_3 := \overline{\Sigma}_3 \cap \overline{\Sigma}_1.$$

From [21, Theorem 1.5.2.3] it follows that the operator

$$T : H^1(\Omega) \rightarrow (\mathcal{C}, \|\cdot\|_{\mathcal{C}}), \quad T(v) := (\gamma_1(v), \gamma_2(v), \gamma_3(v)),$$

is linear, bounded, and surjective, where

$$\mathcal{C} := \left\{ (\phi_1, \phi_2, \phi_3) \in \prod_{j=1}^3 H^{1/2}(\Sigma_j) : \int_0^{\delta_j} \frac{|R_{\mathbf{x}_j}(\phi_j, \phi_{j+1}; \sigma)|^2}{\sigma} d\sigma < +\infty \quad \forall j \in \{1, 2, 3\} \right\},$$

with

$$R_{\mathbf{x}_j}(\phi_j, \phi_{j+1}; \sigma) := \phi_{j+1}(\mathbf{x}_j(\sigma)) - \phi_j(\mathbf{x}_j(-\sigma)), \quad \text{for a.e. } \sigma \in (0, \delta_j), \quad j \in \{1, 2, 3\},$$

and

$$\|(\phi_1, \phi_2, \phi_3)\|_{\mathcal{C}} = \sum_{j=1}^3 \left\{ \|\phi_j\|_{1/2, \Sigma_j} + \left(\int_0^{\delta_j} \frac{|R_{\mathbf{x}_j}(\phi_j, \phi_{j+1}; \sigma)|^2}{\sigma} d\sigma \right)^{1/2} \right\},$$

where, by convention, $\phi_{j+1} = \phi_1$ when $j = 3$. Here, $\delta_j > 0$ is chosen sufficiently small so that

$$\mathbf{x}_j(\sigma) \in \Sigma_{j+1} \quad \text{and} \quad \mathbf{x}_j(-\sigma) \in \Sigma_j, \quad \forall \sigma \in (0, \delta_j),$$

where $\sigma \mapsto \mathbf{x}_j(\sigma)$ and $\sigma \mapsto \mathbf{x}_j(-\sigma)$ are Lipschitz-continuous parametrizations of Σ_{j+1} and Σ_j , respectively, with Lipschitz-continuous inverses. We also adopt the convention $\Sigma_{j+1} = \Sigma_1$ when $j = 3$.

Given $f_1 \in H^{1/2}(\Sigma_1)$ and $f_2 \in H^{1/2}(\Sigma_2)$, let

$$f = \begin{cases} f_1 & \text{on } \Sigma_1, \\ f_2 & \text{on } \Sigma_2. \end{cases}$$

To prove that $f \in H^{1/2}(\Sigma_1 \cup \Sigma_2)$, we introduce the auxiliary operator

$$\widehat{T} : H^1(\Omega) \rightarrow (\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}}), \quad \widehat{T}(v) := (\gamma_1(v), \gamma_2(v)),$$

where

$$\widehat{\mathcal{C}} := \left\{ (\phi_1, \phi_2) \in H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) : \int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma)|^2}{\sigma} d\sigma < +\infty \right\},$$

endowed with the norm

$$\|(\phi_1, \phi_2)\|_{\widehat{\mathcal{C}}} := \sum_{j=1}^2 \|\phi_j\|_{1/2, \Sigma_j} + \left(\int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma)|^2}{\sigma} d\sigma \right)^{1/2}. \quad (6.1)$$

We first prove that $\widehat{T}$ is surjective. Indeed, given $(\phi_1, \phi_2) \in \widehat{\mathcal{C}}$, we can choose $\phi_3 \in H^{1/2}(\Sigma_3)$ such that

$$(\phi_1, \phi_2, \phi_3) \in \mathcal{C}.$$

Hence, by the surjectivity of T , there exists $v \in H^1(\Omega)$ such that

$$T(v) = (\phi_1, \phi_2, \phi_3).$$

It follows that

$$\widehat{T}(v) = (\gamma_1(v), \gamma_2(v)) = (\phi_1, \phi_2),$$

and therefore $\widehat{T}$ is also surjective.

Now, by the compatibility assumption (1.19), we have that $(f_1, f_2) \in \widehat{\mathcal{C}}$. Hence, the surjectivity of $\widehat{T}$ ensures the existence of $w \in H^1(\Omega)$ such that

$$\widehat{T}(w) = (f_1, f_2).$$

By the definition of γ_j , this implies

$$\gamma_0(w)|_{\Sigma_j} = f_j, \quad j = 1, 2,$$

and consequently

$$\gamma_0(w)|_{\Sigma_1 \cup \Sigma_2} = f.$$

Therefore, by the definition of $H^{1/2}(\Sigma_1 \cup \Sigma_2)$,

$$f \in H^{1/2}(\Sigma_1 \cup \Sigma_2).$$

Now we turn to prove (1.20) by means of the Bounded Inverse Theorem (see [29, Theorem 3.8]). To that end, we let

$$\widetilde{T} : H^1(\Omega)/\text{Ker}(\widehat{T}) \rightarrow (\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}})$$

be the operator defined by

$$\widetilde{T}([v]) := \widehat{T}(v),$$

where $[v]$ denotes the equivalence class of v with respect to the equivalence relation

$$v \sim u \iff v - u \in \text{Ker}(\widehat{T}).$$

It is not difficult to see that

$$\text{Ker}(\widehat{T}) = H_{\Sigma_3^c}^1(\Omega) := \{v \in H^1(\Omega) : \gamma_0(v) = 0 \text{ a.e. on } \Sigma_3^c := \Sigma_1 \cup \Sigma_2\}.$$

By the continuity and surjectivity of $\widehat{T}$, it follows that $\widetilde{T}$ is also continuous and surjective. In addition, since

$$[\mathbf{0}] = \left\{v \in H^1(\Omega) : v - \mathbf{0} = v \in \text{Ker}(\widehat{T})\right\} = \text{Ker}(\widehat{T}),$$

we readily deduce that

$$\text{Ker}(\widetilde{T}) = \{[\mathbf{0}]\},$$

and hence $\widetilde{T}$ is injective. In turn, since $H^1(\Omega)$ is a Hilbert space and $\text{Ker}(\widehat{T})$ is a closed subspace of $H^1(\Omega)$, it is well known (see, e.g., [29, Theorem 3.13]) that the quotient space

$$H^1(\Omega)/\text{Ker}(\widehat{T})$$

is a Banach space. Therefore, in order to apply the Bounded Inverse Theorem to $\widetilde{T}$, it only remains to prove that $(\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}})$ is a Banach space. To this end, we introduce the product space

$$X := H^{1/2}(\Sigma_1) \times H^{1/2}(\Sigma_2) \times L^2((0, \delta_1); \sigma^{-1} d\sigma),$$

where

$$L^2((0, \delta_1); \sigma^{-1} d\sigma) := \left\{q : (0, \delta_1) \rightarrow \mathbb{R} : \int_0^{\delta_1} \frac{|q(\sigma)|^2}{\sigma} d\sigma < +\infty\right\}.$$

Endowed with the norm

$$\|q\|_{0,\sigma^{-1}} := \left(\int_0^{\delta_1} \frac{|q(\sigma)|^2}{\sigma} d\sigma \right)^{1/2},$$

the space $L^2((0, \delta_1); \sigma^{-1} d\sigma)$ is a Hilbert space and, in particular, a Banach space.

We endow X with the norm

$$\|(\phi_1, \phi_2, q)\|_X := \|\phi_1\|_{1/2, \Sigma_1} + \|\phi_2\|_{1/2, \Sigma_2} + \|q\|_{0, \sigma^{-1}}.$$

Since each factor of X is a Banach space, $(X, \|\cdot\|_X)$ is also a Banach space.

We now define the operator

$$\Phi : (\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}}) \longrightarrow (X, \|\cdot\|_X)$$

by

$$\Phi(\phi_1, \phi_2) := (\phi_1, \phi_2, R_{\mathbf{x}_1}(\phi_1, \phi_2; \cdot)).$$

By the definitions of the norms $\|\cdot\|_X$ and $\|\cdot\|_{\widehat{\mathcal{C}}}$, for each $(\phi_1, \phi_2) \in \widehat{\mathcal{C}}$, we have

$$\begin{aligned} \|\Phi(\phi_1, \phi_2)\|_X &= \|\phi_1\|_{1/2, \Sigma_1} + \|\phi_2\|_{1/2, \Sigma_2} + \left(\int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma)|^2}{\sigma} d\sigma \right)^{1/2} \\ &= \|(\phi_1, \phi_2)\|_{\widehat{\mathcal{C}}}. \end{aligned}$$

Hence, Φ is an isometric isomorphism from $\widehat{\mathcal{C}}$ onto its range $\Phi(\widehat{\mathcal{C}})$.

Now, let $\{(\phi_{1,n}, \phi_{2,n})\}_{n \in \mathbb{N}} \subseteq \widehat{\mathcal{C}}$ be a Cauchy sequence. Since Φ is an isometry, it follows that

$$\{\Phi(\phi_{1,n}, \phi_{2,n})\}_{n \in \mathbb{N}} = \{(\phi_{1,n}, \phi_{2,n}, q_n)\}_{n \in \mathbb{N}},$$

where

$$q_n(\sigma) := R_{\mathbf{x}_1}(\phi_{1,n}, \phi_{2,n}; \sigma) \quad \text{for a.e. } \sigma \in (0, \delta_1),$$

is a Cauchy sequence in $(X, \|\cdot\|_X)$. Since X is a Banach space, there exists

$$(\phi_1, \phi_2, q) \in X$$

such that

$$(\phi_{1,n}, \phi_{2,n}, q_n) \rightarrow (\phi_1, \phi_2, q) \quad \text{in } X.$$

Equivalently,

$$\phi_{j,n} \rightarrow \phi_j \quad \text{in } H^{1/2}(\Sigma_j), \quad j = 1, 2,$$

and

$$q_n \rightarrow q \quad \text{in } L^2((0, \delta_1); \sigma^{-1} d\sigma). \quad (6.2)$$

In particular, by the continuous embedding $H^{1/2}(\Sigma_j) \hookrightarrow L^2(\Sigma_j)$, we also have

$$\phi_{j,n} \rightarrow \phi_j \quad \text{in } L^2(\Sigma_j), \quad j = 1, 2.$$

Hence, by [9, Theorem 4.9], and after successively extracting subsequences, there exists a common subsequence, still denoted by $\{n_k\}_{k \in \mathbb{N}}$, such that

$$\phi_{j,n_k} \rightarrow \phi_j \quad \text{a.e. on } \Sigma_j, \quad j = 1, 2.$$

Then, since $\sigma \rightarrow \mathbf{x}_1(\sigma)$ and $\sigma \rightarrow \mathbf{x}_1(-\sigma)$ are Lipschitz-continuous with Lipschitz-continuous inverses, it follows from [16, Theorem 2.8] that

$$\phi_{2,n_k}(\mathbf{x}_1(\sigma)) \rightarrow \phi_2(\mathbf{x}_1(\sigma)) \quad \text{and} \quad \phi_{1,n_k}(\mathbf{x}_1(-\sigma)) \rightarrow \phi_1(\mathbf{x}_1(-\sigma))$$

for a.e. $\sigma \in (0, \delta_1)$. Therefore,

$$\begin{aligned} \lim_{k \rightarrow \infty} q_{n_k}(\sigma) &= \lim_{k \rightarrow \infty} [\phi_{2,n_k}(\mathbf{x}_1(\sigma)) - \phi_{1,n_k}(\mathbf{x}_1(-\sigma))] \\ &= \phi_2(\mathbf{x}_1(\sigma)) - \phi_1(\mathbf{x}_1(-\sigma)) \\ &= R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma) \quad \text{for a.e. } \sigma \in (0, \delta_1). \end{aligned}$$

In addition, by (6.2) and [9, Theorem 4.9], there exists a further subsequence $\{q_{n_{k_\ell}}\}_{\ell \in \mathbb{N}} \subseteq \{q_{n_k}\}_{k \in \mathbb{N}}$ such that

$$q_{n_{k_\ell}}(\sigma) \rightarrow q(\sigma) \quad \text{for a.e. } \sigma \in (0, \delta_1).$$

On the other hand, from the preceding convergence result, we have

$$q_{n_{k_\ell}}(\sigma) \rightarrow R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma) \quad \text{for a.e. } \sigma \in (0, \delta_1).$$

Hence, by uniqueness of the limit,

$$q(\sigma) = R_{\mathbf{x}_1}(\phi_1, \phi_2; \sigma) = \phi_2(\mathbf{x}_1(\sigma)) - \phi_1(\mathbf{x}_1(-\sigma)) \quad \text{for a.e. } \sigma \in (0, \delta_1).$$

Since

$$q \in L^2((0, \delta_1); \sigma^{-1} d\sigma),$$

it follows that $(\phi_1, \phi_2) \in \widehat{\mathcal{C}}$ and

$$\Phi(\phi_1, \phi_2) = (\phi_1, \phi_2, q).$$

Moreover,

$$\begin{aligned} \|(\phi_{1,n}, \phi_{2,n}) - (\phi_1, \phi_2)\|_{\widehat{\mathcal{C}}} &= \|\phi_{1,n} - \phi_1\|_{1/2, \Sigma_1} + \|\phi_{2,n} - \phi_2\|_{1/2, \Sigma_2} \\ &\quad + \|q_n - q\|_{0, \sigma^{-1}} \longrightarrow 0 \quad \text{as } n \rightarrow \infty. \end{aligned}$$

Therefore, every Cauchy sequence in $(\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}})$ converges to an element of $\widehat{\mathcal{C}}$, and hence $(\widehat{\mathcal{C}}, \|\cdot\|_{\widehat{\mathcal{C}}})$ is a Banach space.

Consequently, $\widetilde{T}$ satisfies the hypotheses of the Bounded Inverse Theorem. Hence, there exists a constant $C > 0$ such that

$$\begin{aligned} \|[w]\|_{H^1(\Omega)/\text{Ker}(\widehat{T})} &\leq C \|\widetilde{T}([w])\|_{\widehat{\mathcal{C}}} \\ &= C \|\widehat{T}(w)\|_{\widehat{\mathcal{C}}} = C \|(\gamma_1(w), \gamma_2(w))\|_{\widehat{\mathcal{C}}} = C \|(f_1, f_2)\|_{\widehat{\mathcal{C}}}. \end{aligned} \tag{6.3}$$

On the other hand, recalling that $\gamma_i(w) = f_i$, $i = 1, 2$, and using the definition of the quotient norm, we obtain

$$\begin{aligned} \|[w]\|_{H^1(\Omega)/\text{Ker}(\widehat{T})} &= \inf_{v \in \text{Ker}(\widehat{T})} \|w + v\|_{1, \Omega} \\ &= \inf \left\{ \|w'\|_{1, \Omega} : \widehat{T}(w') = \widehat{T}(w) \right\} \\ &= \inf \left\{ \|w'\|_{1, \Omega} : \gamma_1(w') = f_1, \gamma_2(w') = f_2 \right\} \\ &= \inf \left\{ \|w'\|_{1, \Omega} : \gamma_0(w')|_{\Sigma_1 \cup \Sigma_2} = f \right\} \\ &= \|f\|_{1/2, \Sigma_1 \cup \Sigma_2}. \end{aligned}$$

Combining this identity with (6.3) and (6.1), we conclude that

$$\|f\|_{1/2, \Sigma_1 \cup \Sigma_2} \leq C \left\{ \|f_1\|_{1/2, \Sigma_1} + \|f_2\|_{1/2, \Sigma_2} + \left(\int_0^{\delta_1} \frac{|R_{\mathbf{x}_1}(f_1, f_2; \sigma)|^2}{\sigma} d\sigma \right)^{1/2} \right\},$$

which proves (1.20) and completes the proof.

7 Appendix B. Proof of Lemma 3.7

Given $v_h \in \mathbf{H}_h$, we first recall that (3.48) implies that the operator $B_h : \mathbf{H}_h \rightarrow \mathbf{Q}'_h$, $B_h(v_h) := b(v_h, \cdot)$, is surjective. Then, proceeding analogously to [14, Theorem 2.34], there exists $r_h \in \mathbf{H}_h$ such that

$$b(r_h, q_h) = b(u - v_h, q_h) \quad \forall q_h \in \mathbf{Q}_h \quad \text{and} \quad \tilde{\beta} \|r_h\|_{\mathbf{H}} \leq \|b\| \|u - v_h\|_{\mathbf{H}}. \quad (7.1)$$

Setting $w_h := r_h + v_h$, we get $b(w_h, q_h) = b(u, q_h) = G(q_h)$ for all $q_h \in \mathbf{Q}_h$, and

$$\|u - w_h\|_{\mathbf{H}} \leq \left(1 + \frac{\|b\|}{\tilde{\beta}}\right) \|u - v_h\|_{\mathbf{H}}. \quad (7.2)$$

By the surjectivity of B_h again, there exists $s_h \in \mathbf{H}_h$ such that

$$b(s_h, q_h) = G_h(q_h) - G(q_h) \quad \forall q_h \in \mathbf{Q}_h \quad \text{and} \quad \tilde{\beta} \|s_h\|_{\mathbf{H}} \leq \sup_{\substack{q_h \in \mathbf{Q}_h \\ q_h \neq 0}} \frac{|G(q_h) - G_h(q_h)|}{\|q_h\|_{\mathbf{Q}}}. \quad (7.3)$$

Setting $\hat{w}_h := w_h + s_h$, we have $b(\hat{w}_h, q_h) = G_h(q_h) = b(u_h, q_h)$ for all $q_h \in \mathbf{Q}_h$, so that $y_h := u_h - \hat{w}_h \in \text{Ker}(B_h)$, and, by (7.2), (7.3) and the triangle inequality,

$$\|u - \hat{w}_h\|_{\mathbf{H}} \leq \|u - w_h\|_{\mathbf{H}} + \|s_h\|_{\mathbf{H}} \leq \eta_h(v_h) := \left(1 + \frac{\|b\|}{\tilde{\beta}}\right) \|u - v_h\|_{\mathbf{H}} + \frac{1}{\tilde{\beta}} \sup_{\substack{q_h \in \mathbf{Q}_h \\ q_h \neq 0}} \frac{|G(q_h) - G_h(q_h)|}{\|q_h\|_{\mathbf{Q}}}. \quad (7.4)$$

Since $y_h \in \text{Ker}(B_h)$, it follows that

$$a(u_h, y_h) = F_h(y_h) - b(y_h, p_h) = F_h(y_h).$$

On the other hand, for any $q_h \in \mathbf{Q}_h$, using again that $y_h \in \text{Ker}(B_h)$, we have $b(y_h, p) = b(y_h, p - q_h)$, so that

$$a(u, y_h) = F(y_h) - b(y_h, p - q_h).$$

In this way, adding and subtracting suitable terms, we deduce that

$$a(u_h - \hat{w}_h, y_h) = a(u_h, y_h) - a(u, y_h) + a(u - \hat{w}_h, y_h) = [F_h(y_h) - F(y_h)] + b(y_h, p - q_h) + a(u - \hat{w}_h, y_h), \quad (7.5)$$

which, together with (3.47) and (7.4), yields

$$\tilde{\alpha} \|u_h - \hat{w}_h\|_{\mathbf{H}} \leq \sup_{\substack{y_h \in \mathbf{H}_h \\ y_h \neq 0}} \frac{|F(y_h) - F_h(y_h)|}{\|y_h\|_{\mathbf{H}}} + \|b\| \|p - q_h\|_{\mathbf{Q}} + \|a\| \eta_h(v_h) \quad \forall q_h \in \mathbf{Q}_h. \quad (7.6)$$

Then, the triangle inequality $\|u - u_h\|_H \leq \|u - \hat{w}_h\|_H + \|\hat{w}_h - u_h\|_H$, combined with (7.4) and (7.6), and taking the infimum over $v_h \in H_h$ and $q_h \in Q_h$ independently, yields the first estimate in (3.49).

For the second estimate, we notice that, for all $v_h \in H_h$ and $q_h \in Q_h$,

$$b(v_h, q_h - p_h) = b(v_h, q_h - p) + b(v_h, p - p_h) = b(v_h, q_h - p) + a(u_h - u, v_h) + F(v_h) - F_h(v_h). \quad (7.7)$$

Applying the discrete inf-sup condition (3.48) to the left-hand side of (7.7), we obtain

$$\tilde{\beta} \|q_h - p_h\|_Q \leq \|b\| \|p - q_h\|_Q + \|a\| \|u - u_h\|_H + \sup_{\substack{z_h \in H_h \\ z_h \neq 0}} \frac{|F(z_h) - F_h(z_h)|}{\|z_h\|_H}. \quad (7.8)$$

Then, combining (7.8) with the triangle inequality $\|p - p_h\|_Q \leq \|p - q_h\|_Q + \|q_h - p_h\|_Q$, we get

$$\|p - p_h\|_Q \leq \left(1 + \frac{\|b\|}{\tilde{\beta}}\right) \|p - q_h\|_Q + c_4 \|u - u_h\|_H + \frac{1}{\tilde{\beta}} \sup_{\substack{z_h \in H_h \\ z_h \neq 0}} \frac{|F(z_h) - F_h(z_h)|}{\|z_h\|_H},$$

which, together with the first estimate in (3.49), and noticing that $\frac{1}{\tilde{\beta}} + \frac{c_4}{\alpha} = c_3$, yields the second estimate in (3.49) after taking the infimum over $q_h \in Q_h$. This concludes the proof.

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