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**Analytic and Gevrey regularity for a hybrid thermoelastic
Euler–Bernoulli beam with singular interactions**

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ANALYTIC AND GEVREY REGULARITY FOR A HYBRID THERMOELASTIC EULER–BERNOULLI BEAM WITH SINGULAR INTERACTIONS

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ABSTRACT. We investigate the regularity and stability properties of a hybrid Euler–Bernoulli beam consisting of elastic and thermoelastic components, with type III thermoelasticity, discontinuous material coefficients, and singular pointwise interactions modeled by Dirac distributions and their derivatives. The beam is dynamically coupled at one endpoint to a rigid tip body with mass, rotational inertia, and an offset center of mass, giving rise to a PDE–ODE system with dynamic boundary conditions.

Using frequency-domain methods and high-frequency resolvent estimates, we show that when the thermoelastic mechanism acts throughout the beam, the associated C_0 -semigroup is analytic, despite the presence of singular interactions and the dynamic coupling with the tip body. In contrast, when the thermoelastic effects are confined to a proper subinterval, analyticity is lost and the semigroup is of Gevrey class 2. In both configurations, the system is exponentially stable. These results show that the transition from analyticity to Gevrey regularity induced by the localization of the thermoelastic mechanism persists in the presence of both singular interactions and dynamic tip-body coupling.

1. INTRODUCTION AND MAIN RESULTS

Thermoelasticity provides a fundamental framework for describing the interaction between mechanical deformation and thermal effects in elastic structures. Classical theories based on Fourier’s law of heat conduction lead to parabolic thermal equations and therefore predict an infinite speed of thermal propagation. Generalized thermoelastic theories have been developed to overcome this limitation. Among them, the type III theory introduced by Green and Naghdi [7, 8, 9] is particularly relevant, since it allows thermal waves to propagate with finite speed while retaining a dissipative mechanism through the thermal constitutive law.

The coupling between elastic structures and type III heat conduction has been extensively studied from the viewpoints of well-posedness, stability, and semigroup regularity. The situation becomes substantially more delicate when the thermoelastic mechanism acts only on a proper subregion of the structure, as occurs naturally in composite materials, layered structures, and systems with localized thermal devices. From the mathematical point of view, localization destroys the uniform interaction between the elastic and thermal variables and may significantly affect both the long-time dynamics and the smoothing properties of the associated semigroup. In particular, analyticity may be lost when the thermoelastic mechanism is restricted to a subinterval, while a weaker Gevrey regularization may persist.

Additional difficulties arise in the presence of concentrated interactions. Pointwise dampers, feedback devices, and concentrated forces are naturally represented by Dirac distributions, while moment-type actions lead to derivatives of Dirac distributions. Such terms cannot be treated simply as regular lower-order perturbations, since they generate jump conditions for the bending moment and shear force and lead naturally to a transmission formulation.

The purpose of this paper is to investigate the regularity and stability of such a localized thermoelastic structure when these singular interactions are combined with a genuinely dynamic boundary coupling. More precisely, we consider an Euler–Bernoulli beam with discontinuous elastic coefficients, type III thermoelasticity acting on a prescribed subinterval, singular pointwise

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interactions, and a rigid tip body dynamically coupled to the right endpoint. The configuration is illustrated in Figure 1.

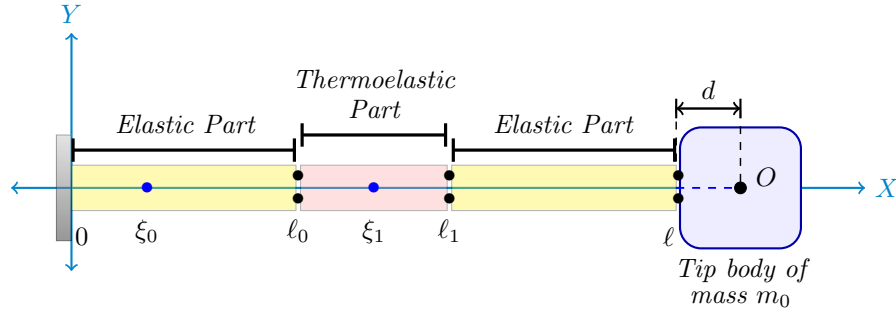


FIGURE 1. Localized thermoelastic beam with singular interactions at ξ_0 and ξ_1 and with a tip body of mass m_0 .

The governing equations are

$$(1.1) \quad \rho u_{tt} + M_{xx} + \beta_0 u_t(\xi_0) \delta_{\xi_0} + \beta_1 u_t(\xi_1) \delta_{\xi_1} - \beta_2 u_{xt}(\xi_1) \frac{d}{dx} \delta_{\xi_1} = 0, \quad \text{in } (0, \ell) \times \mathbb{R}_0^+,$$

$$(1.2) \quad c \theta_{tt} - Q_x - m u_{xxt} + \beta_3 \theta_t(\xi_1) \delta_{\xi_1} = 0, \quad \text{in } (\ell_0, \ell_1) \times \mathbb{R}_0^+,$$

where u denotes the transverse displacement, θ the thermal displacement, and δ_ξ the Dirac distribution concentrated at ξ . The bending moment and heat flux are given by

$$M = \alpha(x) u_{xx} + m(x) \theta_t, \quad Q = \kappa \theta_x + \kappa_0 \theta_{xt}.$$

At $x = \ell$, a rigid body of mass m_0 and moment of inertia J is attached to the beam, with its center of mass located at a distance d from the attachment point. If $w(t)$ and $z(t)$ denote, respectively, the transverse displacement and rotation of the beam tip, the boundary conditions are given by

$$u(0, t) = u_x(0, t) = 0, \quad u(\ell, t) = w(t), \quad u_x(\ell, t) = z(t), \quad \theta(\ell_0, t) = \theta(\ell_1, t) = 0, \quad \forall t > 0,$$

The balance of forces and moments at the tip yields the dynamic boundary conditions

$$(1.3) \quad m_0 w_{tt} + m_0 d z_{tt} + \gamma w_t - (\alpha u_{xxx} + m \theta_{xt})(\ell, t) = 0,$$

$$(1.4) \quad m_0 d w_{tt} + (J + m_0 d^2) z_{tt} + d \gamma w_t + d \gamma^* z_t + \alpha u_{xx}(\ell, t) = 0.$$

Notice that the thermal contribution to the dynamic boundary conditions is present whenever the thermoelastic region reaches the right endpoint of the beam, that is, when $\ell_1 = \ell$. This includes both the fully thermoelastic case, $\ell_0 = 0$ and $\ell_1 = \ell$, and the partially thermoelastic case, $0 < \ell_0 < \ell_1 = \ell$. On the other hand, if $\ell_1 < \ell$, the neighborhood of the tip is purely elastic, so that $m = 0$ near $x = \ell$ and the thermal term $m \theta_{xt}(\ell, t)$ does not appear in the dynamic boundary condition. For related beam models with dynamic tip bodies, we refer to [16, 17, 18].

The offset d has a direct mechanical significance: it acts as a lever arm coupling the translational and rotational motions of the attached body. In particular, for $d > 0$ the tip kinetic energy is

$$E_{\text{tip}}(t) = \frac{1}{2} m_0 |w_t|^2 + m_0 d w_t z_t + \frac{1}{2} (J + m_0 d^2) |z_t|^2.$$

Consequently, the tip-body dynamics form part of the principal energy of the system and cannot be treated merely as a lower-order boundary perturbation.

The model therefore combines several mechanisms that are individually nontrivial: transmission across discontinuous material interfaces, localized type III thermoelasticity, singular interactions generated by Dirac distributions and their derivatives, and a finite-dimensional mechanical system coupled through the high-order boundary traces of the beam. The main analytical challenge is to propagate the regularization generated in the thermoelastic region through the elastic components and up to the translational and rotational degrees of freedom of the tip body.

Our main result shows that the regularity transition induced by the localization of the thermoelastic mechanism persists in the presence of the combined effects of transmission, singular interactions, and dynamic tip-body coupling. In the fully thermoelastic configuration, the associated C_0 -semigroup is analytic, whereas in the localized configuration it is of Gevrey class 2. More precisely, a frequency-domain analysis of the generator yields, for sufficiently large $|\lambda|$,

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^{-1}$$

in the fully thermoelastic case, while

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq C|\lambda|^{-1/2}$$

when the thermoelastic mechanism is localized. In particular, the latter estimate improves the Gevrey regularity previously obtained in [6], where Gevrey class 8 was established for a nonhybrid beam with localized thermoelasticity and without singular interactions. Thus, the present result not only extends the analysis to a substantially more complex hybrid setting, but also improves the Gevrey regularity from class 8 to class 2.

The resolvent estimates are obtained by combining localized energy arguments with multiplier and trace estimates designed to control the transmission terms, the singular interactions, and the dynamic boundary variables. Besides providing enhanced temporal regularity and an instantaneous smoothing effect, the Gevrey property ensures that the type of the semigroup coincides with the spectral bound of its generator. Consequently, a strictly negative spectral bound yields exponential stability.

We also examine the robustness of this regularity under bounded lower-order perturbations and show that the Gevrey property is preserved. Finally, a structure-preserving finite-element approximation is introduced. The discretization incorporates the point interactions and tip-body dynamics at the variational level and preserves the dissipative structure of the continuous problem. Numerical computations of the discrete resolvent along the imaginary axis exhibit a high-frequency behavior consistent with the $|\lambda|^{-1/2}$ estimate obtained for the continuous localized system.

The remainder of the paper is organized as follows. Section 2 introduces the hybrid thermoelastic model, its transmission formulation, and the associated semigroup framework. Section 3 is devoted to the high-frequency resolvent analysis and the proofs of analyticity and Gevrey regularity, together with the preservation of Gevrey regularity under bounded lower-order perturbations. Finally, Section 4 develops a conforming finite-element approximation of the hybrid system and presents the corresponding numerical experiments.

2. THE HYBRID THERMOELASTIC BEAM MODEL

Let I denote the beam domain, excluding the material interfaces and the singular interaction points, and let $I_T \subseteq I$ denote the region where the thermoelastic mechanism is active. More precisely,

$$I =]0, \xi_0[\cup]\xi_0, \ell_0[\cup]\ell_0, \xi_1[\cup]\xi_1, \ell_1[\cup]\ell_1, \ell[, \quad I_T =]\ell_0, \xi_1[\cup]\xi_1, \ell_1[.$$

Then, the model (1.1)–(1.2) can be equivalently written as the following transmission problem:

$$(2.5) \quad \rho u_{tt} + M_{xx} = 0, \quad (x, t) \in I \times \mathbb{R}_0^+,$$

$$(2.6) \quad c\theta_{tt} - Q_x - \tilde{m}u_{xt} = 0, \quad (x, t) \in I_T \times \mathbb{R}_0^+.$$

Furthermore, we will assume that $\rho, c, \kappa, \kappa_0$ are positive constants, α is a positive function and m a non negative function.

$$(2.7) \quad \alpha(x) = \begin{cases} \alpha_1, & x \in (0, \ell_0) \\ \alpha_2, & x \in (\ell_0, \ell_1) \\ \alpha_3, & x \in (\ell_1, \ell) \end{cases}, \quad m(x) = \begin{cases} 0, & x \in (0, \ell_0) \\ m_1, & x \in (\ell_0, \ell_1) \\ 0, & x \in (\ell_1, \ell) \end{cases}.$$

Verifying the transmission and singularity conditions

$$(2.8) \quad \begin{aligned} \llbracket M \rrbracket_{\ell_i} &= \llbracket M_x \rrbracket_{\ell_i} = 0, & \llbracket M_x \rrbracket_{\xi_i} &= -\beta_i u_t(\xi_i, t), \quad i = 0, 1. \\ \llbracket M \rrbracket_{\xi_1} &= \beta_2 u_{xt}(\xi_1, t), & \llbracket Q \rrbracket_{\xi_1} &= \beta_3 \theta_t(\xi_1, t), \end{aligned}$$

where the jump of f at the point $x = b$ is denoted by $\llbracket f \rrbracket_b$ and is given by

$$\llbracket f \rrbracket_b := f(b^+) - f(b^-).$$

We consider the following boundary conditions:

$$(2.9) \quad u(0, t) = u_x(0, t) = 0, \quad u(\ell, t) = w(t), \quad u_x(\ell, t) = z(t), \quad \theta(\ell_0, t) = \theta(\ell_1, t) = 0, \quad \forall t > 0,$$

together with the initial conditions

$$(2.10) \quad \begin{aligned} u(x, 0) &= u_0(x), & u_t(x, 0) &= u_1(x), & \forall x \in]0, \ell[, \\ \theta(x, 0) &= \theta_0(x), & \theta_t(x, 0) &= \theta_1(x), & \forall x \in]\ell_0, \ell_1[. \end{aligned}$$

Therefore, system (2.5)–(2.10), coupled with the dynamic boundary equations (1.3) and (1.4), provides a transmission formulation equivalent to the original system (1.1)–(1.4), subject to the boundary and initial conditions (2.9) and (2.10).

Remark 2.1. *The model (1.1) - (1.2) and the model (2.5) - (2.6) their have a variational formulation equivalent. Furthermore, for the both model we have the energy*

$$E(t) = \frac{1}{2} \int_0^\ell \rho |u_t|^2 + \alpha |u_{xx}|^2 dx + \frac{1}{2} \int_{\ell_0}^{\ell_1} c |\theta_t|^2 + \kappa |\theta_x|^2 dx + \frac{1}{2} (m_0 |w_t|^2 + 2m_0 dw_t z_t + (J + m_0 d^2) |z_t|^2),$$

that verifies

$$\begin{aligned} \frac{d}{dt} E(t) &= -\kappa_0 \int_{\ell_0}^{\ell_1} |\theta_{xt}|^2 dx - \beta_0 |u_t(\xi_0, t)|^2 - \beta_1 |u_t(\xi_1, t)|^2 - \beta_2 |u_{xt}(\xi_1, t)|^2 - \beta_3 |\theta_t(\xi_1, t)|^2 \\ &\quad - \gamma |w_t|^2 - d\gamma w_t z_t - d\gamma^* |z_t|^2. \end{aligned}$$

Where this condition must be satisfied

$$d\gamma \leq 4\gamma^*$$

Now, we present the main result of this work

Theorem 2.2. *Let $S(t)$ be the C_0 -semigroup associated with system (2.5)–(2.10) coupled to the equations (1.3) and (1.4). Then:*

- (1) *If $]\ell_0, \ell_1[=]0, \ell[$, then $S(t)$ is analytic.*
- (2) *If $]\ell_0, \ell_1[\subsetneq]0, \ell[$, then $S(t)$ is of Gevrey class 2.*
- (3) *In all cases, $S(t)$ is not compact.*

It is worth emphasizing that the localized configuration includes two different situations at the tip. If $0 < \ell_0 < \ell_1 = \ell$, the thermoelastic component reaches the tip and the dynamic boundary condition retains the thermal coupling. If $\ell_1 < \ell$, the tip lies in the purely elastic region and this thermal contribution disappears. Nevertheless, both configurations lead to the same Gevrey regularity established below.

In that follows we assume, without loss of generality, that $\rho = c = 1$.

Denoting by $u_t = v$, $\theta_t = \Theta$, $w_t = W$, $z_t = Z$ and $U = (u, v, \theta, \Theta, W, Z)$. We define the phase space $\mathcal{H}$ as

$$\mathcal{H} = V \times L^2(0, \ell) \times H_0^1(I_T) \times L^2(I_T) \times \mathbb{C}^2,$$

where

$$V = \{g \in H^2(0, \ell) : g(0) = g_x(0) = 0\}.$$

With the norm

$$\|U\|_{\mathcal{H}}^2 = \int_0^\ell |v|^2 + \alpha |u_{xx}|^2 dx + \int_{\ell_0}^{\ell_1} |\Theta|^2 + \kappa |\theta_x|^2 dx + m_0 |W|^2 + 2m_0 dWZ + (J + m_0 d^2) |Z|^2.$$

For simplify the notation we call $U = (\mathcal{U}, \mathcal{V})$ with $\mathcal{U} = (u, v, \theta, \Theta)$ and $\mathcal{V} = (W, Z)$. We note that

$$\frac{d}{dt} \mathcal{V} = -K^{-1} B \mathcal{V} + K^{-1} \Gamma$$

where

$$K = \begin{pmatrix} m_0 & m_0 d \\ m_0 d & J + m_0 d^2 \end{pmatrix}, \quad B = \begin{pmatrix} \gamma & 0 \\ d\gamma & d\gamma^* \end{pmatrix}, \quad \Gamma = \begin{pmatrix} \alpha u_{xxx}(\ell) + m\Theta_x(\ell) \\ -\alpha u_{xx}(\ell) \end{pmatrix}$$

Now, we introduce the operator A , as

$$(2.11) \quad AU = \begin{pmatrix} v \\ -M_{xx} \\ \Theta \\ Q_x + mv_{xx} \\ -K^{-1}B\mathcal{V} + K^{-1}\Gamma \end{pmatrix},$$

with domain $D(A)$ given by

$$(2.12) \quad D(A) = \{U \in \mathcal{H}; v \in V, \Theta \in H_0^1(I_T), M \in H^2(I), Q \in H^1(I_T), \text{ satisfying (2.8) and (2.9)}\}.$$

It is straightforward to verify that A is a dissipative operator. Indeed,

$$(2.13) \quad \begin{aligned} \operatorname{Re}(AU, U)_{\mathcal{H}} = & - \int_{\ell_0}^{\ell_1} \kappa_0 |\Theta_x|^2 dx - \beta_0 |v(\xi_0)|^2 - \beta_1 |v(\xi_1)|^2 - \beta_2 |v_x(\xi_1)|^2 - \beta_3 |\Theta(\xi_1)|^2 \\ & - \gamma |W|^2 - d\gamma WZ - d\gamma^* |Z|^2, \end{aligned}$$

provided that $d\gamma \leq 4\gamma^*$.

We now consider the resolvent equation

$$(i\lambda I - A)U = F.$$

In terms of its components, the corresponding resolvent system is given by

$$(2.14) \quad i\lambda u - v = f_1,$$

$$(2.15) \quad i\lambda v + (\alpha u_{xx} + m\Theta)_{xx} = f_2,$$

$$(2.16) \quad i\lambda \theta - \Theta = f_3,$$

$$(2.17) \quad i\lambda \Theta - (\kappa \theta_x + \kappa_0 \Theta_x)_x - mv_{xx} = f_4,$$

$$(2.18) \quad i\lambda (m_0 W + m_0 dZ) - (\alpha u_{xxx} + m\Theta_x)(\ell) - \gamma W = m_0 f_5 + m_0 df_6,$$

$$(2.19) \quad i\lambda (m_0 dW + (J + m_0 d^2)Z) + \alpha u_{xx}(\ell) - d\gamma W - d\gamma^* Z = m_0 df_5 + (J + m_0 d^2)f_6.$$

The above system satisfies the transmission and singularity conditions

$$(2.20) \quad \begin{aligned} \llbracket M \rrbracket_{\ell_i} &= \llbracket M_x \rrbracket_{\ell_i} = 0, & \llbracket M_x \rrbracket_{\xi_i} &= -\beta_i v(\xi_i), \quad i = 0, 1, \\ \llbracket M \rrbracket_{\xi_1} &= \beta_2 v_x(\xi_1), & \llbracket Q \rrbracket_{\xi_1} &= \beta_3 \Theta(\xi_1), \end{aligned}$$

together with the boundary conditions

$$(2.21) \quad u(0) = u_x(0) = 0, \quad u(\ell) = w, \quad u_x(\ell) = z, \quad \theta(\ell_0) = \theta(\ell_1) = 0.$$

Taking the real part of the inner product in $\mathcal{H}$ of the resolvent equation with U , and using (2.13), we obtain

$$(2.22) \quad \begin{aligned} & \int_{\ell_0}^{\ell_1} \kappa_0 |\Theta_x|^2 dx + \beta_0 |v(\xi_0)|^2 + \beta_1 |v(\xi_1)|^2 + \beta_2 |v_x(\xi_1)|^2 + \beta_3 |\Theta(\xi_1)|^2 + \gamma |W|^2 \\ & + d\gamma WZ + d\gamma^* |Z|^2 = \operatorname{Re}(U, F)_{\mathcal{H}}. \end{aligned}$$

Since $d\gamma \leq 4\gamma^*$, it follows from the above identity that

$$(2.23) \quad \begin{aligned} & \int_{\ell_0}^{\ell_1} |\Theta_x|^2 dx + |v(\xi_0)|^2 + |v(\xi_1)|^2 + |v_x(\xi_1)|^2 + |\Theta(\xi_1)|^2 + |W|^2 + |Z|^2 \\ & \leq c \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}, \end{aligned}$$

for some positive constant c .

Under the above assumptions, we obtain the following result.

Theorem 2.3. *The operator $\mathcal{A}$ defined by (2.11)–(2.12) is the infinitesimal generator of a contraction C_0 -semigroup on $\mathcal{H}$.*

Proof. We apply the Lumer–Phillips theorem. Since $\mathcal{A}$ is dissipative, it remains to prove that $\mathcal{A}$ is maximal dissipative. For this purpose, it is enough to show that

$$0 \in \varrho(\mathcal{A}).$$

Equivalently, for every $F = (f_1, \dots, f_6) \in \mathcal{H}$, we shall prove that there exists a unique $U \in D(\mathcal{A})$ satisfying

$$-\mathcal{A}U = F.$$

Setting $\lambda = 0$ in (2.14)–(2.19), we immediately obtain

$$v = -f_1, \quad \Theta = -f_3,$$

and consequently

$$W = v(\ell) = -f_1(\ell), \quad Z = v_x(\ell) = -f_{1,x}(\ell).$$

Thus, it remains to determine the displacement u and the thermal variable θ .

Let

$$V = \{\varphi \in H^2(0, \ell) : \varphi(0) = \varphi_x(0) = 0\}, \quad H_T = H_0^1(\ell_0, \ell_1).$$

We introduce the bilinear forms

$$a_m(u, \varphi) = \int_0^\ell \alpha(x) u_{xx} \varphi_{xx} dx, \quad u, \varphi \in V,$$

and

$$a_T(\theta, \psi) = \kappa \int_{\ell_0}^{\ell_1} \theta_x \psi_x dx, \quad \theta, \psi \in H_T.$$

Using $v = -f_1$ and $\Theta = -f_3$, the weak mechanical equation can be written as

$$\begin{aligned} a_m(u, \varphi) = & \int_0^\ell f_2 \varphi dx - \int_{\ell_0}^{\ell_1} m f_3 \varphi_{xx} dx - \beta_0 f_1(\xi_0) \varphi(\xi_0) - \beta_1 f_1(\xi_1) \varphi(\xi_1) \\ & + \beta_2 f_{1,x}(\xi_1) \varphi_x(\xi_1) + \left(m_0 f_5 + m_0 df_6 - m f_{3,x}(\ell) - \gamma f_1(\ell) \right) \varphi(\ell) \\ (2.24) \quad & + \left(m_0 df_5 + (J + m_0 d^2) f_6 - d\gamma f_1(\ell) - d\gamma^* f_{1,x}(\ell) \right) \varphi_x(\ell), \end{aligned}$$

for every $\varphi \in V$, where terms involving m at the endpoint are understood according to the corresponding trace of the thermoelastic component.

Similarly, the thermal equation is written in weak form as

$$\begin{aligned} a_T(\theta, \psi) = & -\kappa_0 \int_{\ell_0}^{\ell_1} f_{3,x} \psi_x dx + m \int_{\ell_0}^{\ell_1} f_{1,xx} \psi dx + \int_{\ell_0}^{\ell_1} f_4 \psi dx \\ (2.25) \quad & - \beta_3 f_3(\xi_1) \psi(\xi_1), \end{aligned}$$

for every $\psi \in H_T$.

Since $\alpha(x) \geq \alpha_* > 0$ and $\kappa > 0$, the forms a_m and a_T are continuous and coercive on V and H_T , respectively. Indeed, by the Poincaré inequalities on these spaces,

$$a_m(\varphi, \varphi) \geq c \|\varphi\|_{H^2(0, \ell)}^2, \quad a_T(\psi, \psi) \geq c \|\psi\|_{H^1(\ell_0, \ell_1)}^2.$$

Moreover, all the terms on the right-hand sides of (2.24) and (2.25) define continuous linear functionals. In particular, the point evaluations are continuous in $H^2(0, \ell)$ and $H^1(\ell_0, \ell_1)$ in one space dimension.

Therefore, the Lax–Milgram theorem yields unique

$$u \in V, \quad \theta \in H_0^1(\ell_0, \ell_1)$$

satisfying (2.24)–(2.25).

It remains to verify that the solution actually belongs to $D(\mathcal{A})$. From the variational identities, first by taking test functions compactly supported in each subinterval determined by $\ell_0, \ell_1, \xi_0, \xi_1$,

and then integrating by parts, we recover the differential equations together with the transmission and jump conditions in (2.8). Consequently,

$$M = \alpha u_{xx} + m\Theta \in H^2(I), \quad Q = \kappa\theta_x + \kappa_0\Theta_x \in H^1(I_T),$$

in the piecewise sense prescribed by the definition of $D(\mathcal{A})$, and the boundary terms in (2.24) recover precisely the two tip-body equations at $x = \ell$.

Hence

$$U = (u, -f_1, \theta, -f_3, -f_1(\ell), -f_{1,x}(\ell)) \in D(\mathcal{A})$$

and satisfies

$$-\mathcal{A}U = F.$$

Therefore $-\mathcal{A}$ is onto. Since the homogeneous problem has only the trivial solution, the solution is unique, and thus

$$0 \in \varrho(\mathcal{A}).$$

Hence $\mathcal{A}$ is maximal dissipative. The Lumer–Phillips theorem then implies that $\mathcal{A}$ generates a contraction C_0 -semigroup on $\mathcal{H}$. $\square$

Next, we will enunciate an interpolation theorem used in the next section

Theorem 2.4. *Suppose that w and $w^{(m)} \in L^2(a, b)$. Then $w^{(j)} \in L^2(a, b)$ for $j = 1, \dots, m-1$, where $w^{(j)} = \frac{d^j w}{dx^j}$. Moreover, for any $0 < \epsilon \leq \epsilon_0$, we have*

$$(2.26) \quad \|w^{(j)}\|_{L^2} \leq c\epsilon \|w^{(m)}\|_{L^2} + c\epsilon^{-\frac{j}{m-j}} \|w\|_{L^2},$$

which implies

$$(2.27) \quad \|w^{(j)}\|_{L^2} \leq c\|w\|_{L^2} + c\|w\|_{L^2}^{1-\frac{j}{m}} \|w^{(m)}\|_{L^2}^{\frac{j}{m}}.$$

Proof. Inequality (2.26) follows from Theorem 4.14 in [1]. Define $\|u\|_m := \|u\|_{L^2} + \|u^{(m)}\|_{L^2}$. Choosing ϵ such that

$$\epsilon = \epsilon_0 \left(\frac{\|u\|_{L^2}}{\|u\|_m} \right)^{\frac{m-j}{m}}$$

yields

$$\|u^{(j)}\|_{L^2} \leq C\|u\|_{L^2}^{1-\frac{j}{m}} \|u\|_m^{\frac{j}{m}}.$$

Substituting this into (2.26) and recalling the definition of $\|\cdot\|_m$ gives (2.27). $\square$

Corollary 2.5. *Let $w \in H^2(a, b)$. Then*

$$(2.28) \quad \|w\|_{\infty} = \sup_{s \in [a, b]} |w(s)| \leq c\|w\|_{L^2} + c\|w\|_{L^2}^{3/4} \|w_{xx}\|_{L^2}^{1/4},$$

$$(2.29) \quad \|w_x\|_{\infty} = \sup_{s \in [a, b]} |w_x(s)| \leq c\|w\|_{L^2} + c\|w\|_{L^2}^{1/4} \|w_{xx}\|_{L^2}^{3/4}.$$

Proof. Applying Theorem 2.4 with $m = 2$ and $j = 1$ gives

$$\|w_x\|_{L^2} \leq c\|w\|_{L^2}^{1/2} (\|w\|_{L^2} + \|w_{xx}\|_{L^2})^{1/2}.$$

The fundamental theorem of calculus yields, for any $s \in [a, b]$,

$$|w_x(s)|^2 \leq |b-a| \|w_x\|_{L^2} \|w_{xx}\|_{L^2} \leq c\|w_x\|_{L^2} (\|w\|_{L^2} + \|w_{xx}\|_{L^2}).$$

Substituting the interpolation estimate for $\|w_x\|_{L^2}$ gives

$$|w_x(s)|^2 \leq c\|w\|_{L^2}^{1/2} (\|w\|_{L^2} + \|w_{xx}\|_{L^2})^{3/2},$$

from which (2.29) follows. The estimate (2.28) is obtained similarly. $\square$

3. ANALYTICITY AND GEVREY REGULARITY

we will denote the energy over thermoelastic part by

$$\mathfrak{E}^2 = \int_{\ell_0}^{\ell_1} |v|^2 + |u_{xx}|^2 + |\Theta|^2 + |\theta_x|^2 dx$$

and $\mathfrak{R}$ to any function that verifies

$$\mathfrak{R}^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}$$

Furthermore, the norm $\|\cdot\|_{L^2}$ means the L^2 -norm sometimes to simplify we denote $\|\cdot\|$. Our starting point is the following lemma.

Lemma 3.1. *The solution to (2.14)–(2.21) satisfies*

$$(3.1) \quad \|v_x\| \leq c|\lambda|^{1/2} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/2} \|F\|_{\mathcal{H}}^{1/2},$$

$$(3.2) \quad \|v\|_{\infty}, \|M\|_{\infty} \leq c|\lambda|^{1/4} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{3/4} \|F\|_{\mathcal{H}}^{1/4},$$

$$(3.3) \quad \|v_x\|_{\infty}, \|M_x\|_{\infty} \leq c|\lambda|^{3/4} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/4} \|F\|_{\mathcal{H}}^{3/4},$$

$$(3.4) \quad \|Q\|_{\infty} \leq c|\lambda|^{1/2} \mathfrak{R}^{1/2} \mathfrak{E}^{1/2} + c\mathfrak{R}^{1/2} \|F\|_{\mathcal{H}}^{1/2},$$

$$(3.5) \quad \|M_x\|, \|u_{xxx}\| \leq c|\lambda|^{1/2} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/2} \|F\|_{\mathcal{H}}^{1/2}.$$

Moreover, on any subset of the thermal interval I_T , the norm $\|U\|_{\mathcal{H}}$ can be replaced by $\mathfrak{E}$ in all the above estimates.

Proof. We begin with estimate (3.1). Using (2.14) and a standard interpolation inequality, we obtain

$$\begin{aligned} \|v_x\| &\leq c\|v\|^{1/2} \|v_{xx}\|^{1/2} \\ &\leq c\|v\|^{1/2} \|i\lambda u_{xx} - f_{1,xx}\|^{1/2} \\ &\leq c|\lambda|^{1/2} \|v\|^{1/2} \|u_{xx}\|^{1/2} + c\|v\|^{1/2} \|f_{1,xx}\|^{1/2}, \end{aligned}$$

which yields (3.1). Similarly, for M_x we get

$$\begin{aligned} \|M_x\| &\leq c\|M\|^{1/2} \|M_{xx}\|^{1/2} \\ &\leq c\|M\|^{1/2} \|i\lambda v - f_2\|^{1/2} \\ &\leq c|\lambda|^{1/2} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/2} \|F\|_{\mathcal{H}}^{1/2}. \end{aligned}$$

This implies (3.5). Next, we derive the L^{∞} estimates. Using inequality (2.29) from Corollary 2.5 applied to $w = M$, and equation (2.15), we obtain

$$\begin{aligned} \|M\|_{\infty} &\leq c\|M\|^{3/4} \|M_{xx}\|^{1/4} \\ &\leq c\|U\|_{\mathcal{H}}^{3/4} \|i\lambda v + f_2\|^{1/4} \\ &\leq c|\lambda|^{1/4} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{3/4} \|F\|_{\mathcal{H}}^{1/4}. \end{aligned}$$

A similar argument yields

$$\begin{aligned} \|v\|_{\infty} &\leq c\|v\|^{3/4} \|v_{xx}\|^{1/4} \\ &\leq c\|U\|_{\mathcal{H}}^{3/4} \|i\lambda u_{xx} + f_{1,xx}\|^{1/4} \\ &\leq c|\lambda|^{1/4} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{3/4} \|F\|_{\mathcal{H}}^{1/4}, \end{aligned}$$

which proves (3.2). For M_x , we write

$$\begin{aligned} \|M_x\|_{\infty} &\leq c\|M_x\|^{1/2} \|M_{xx}\|^{1/2} \\ &\leq c\|M\|^{1/4} \|M_{xx}\|^{3/4} \\ &\leq c\|U\|_{\mathcal{H}}^{1/4} \|i\lambda v - f_2\|^{3/4} \\ &\leq c|\lambda|^{3/4} \|U\|_{\mathcal{H}} + c\|U\|_{\mathcal{H}}^{1/4} \|F\|_{\mathcal{H}}^{3/4}. \end{aligned}$$

An analogous estimate holds for $\|v_x\|_\infty$, yielding (3.3).

Finally, using inequality (2.28) of Corollary 2.5 applied to $w = Q$, together with equation (2.17), we obtain

$$\begin{aligned}\|Q\|_\infty &\leq c\|Q\|^{1/2}\|Q_x\|^{1/2} \\ &\leq c\mathfrak{R}^{1/2}(|\lambda|\mathfrak{E} + \|F\|_{\mathcal{H}})^{1/2} \\ &\leq c|\lambda|^{1/2}\mathfrak{R}^{1/2}\mathfrak{E}^{1/2} + c\mathfrak{R}^{1/2}\|F\|_{\mathcal{H}}^{1/2},\end{aligned}$$

which proves (3.4). $\square$

Remark 3.2. Let f and g be functions that may have a discontinuity at $x = b$, and assume that they are sufficiently smooth on $(0, b)$ and (b, ℓ) . Then the following integration by parts formula holds:

$$\begin{aligned}\int_0^\ell f_x g \, dx &= fg|_0^\ell + f(b^-)g(b^-) - f(b^+)g(b^+) - \int_0^\ell f g_x \, dx \\ &= fg|_0^\ell - \llbracket fg \rrbracket_b - \int_0^\ell f g_x \, dx,\end{aligned}$$

the above expression can be rewritten as

$$\int_0^\ell f_x g \, dx = fg|_0^\ell - \llbracket f \rrbracket_b g(b^-) - \llbracket g \rrbracket_b f(b^+) - \int_0^\ell f g_x \, dx.$$

Lemma 3.3. The solution of (2.14) - (2.21) verifies

$$(3.6) \quad \int_{\ell_0}^{\ell_1} \kappa |\theta_x|^2 dx \leq \frac{1}{|\lambda|^2} \mathfrak{R}^2 + \frac{1}{|\lambda|^2} \|F\|_{\mathcal{H}}^2,$$

$$(3.7) \quad \int_{\ell_0}^{\ell_1} |\Theta|^2 dx \leq \epsilon \mathfrak{E}^2 + \frac{c}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|_{\mathcal{H}}^2.$$

for λ large.

Proof. Using (2.16) and (2.23) we arrive to (3.6). On the other hand, multiplying (2.17) by $\overline{i\lambda\Theta}$, we get

$$|\lambda|^2 \int_{\ell_0}^{\ell_1} |\Theta|^2 dx - \int_{\ell_0}^{\ell_1} (\kappa \theta_{xx} - \kappa_0 \Theta_{xx}) \overline{i\lambda\Theta} \, dx - m \int_{\ell_0}^{\ell_1} v_{xx} \overline{i\lambda\Theta} \, dx = \int_{\ell_0}^{\ell_1} f_4 \overline{i\lambda\Theta} \, dx$$

Performing an integration by parts using the jump condition (2.20) and taking the real part we get

$$|\lambda|^2 \int_{\ell_0}^{\ell_1} |\Theta|^2 dx = \underbrace{\int_{\ell_0}^{\ell_1} \kappa |\Theta_x|^2 dx + \operatorname{Re} \int_{\ell_0}^{\ell_1} \kappa f_{3,x} \overline{\Theta_x} dx}_{\leq \mathfrak{R}^2 + c\|F\|_{\mathcal{H}}^2} - \underbrace{\operatorname{Re} \int_{\ell_0}^{\ell_1} m v_x \overline{i\lambda\Theta_x} dx}_J + \underbrace{\operatorname{Re} \int_{\ell_0}^{\ell_1} f_4 \overline{i\lambda\Theta} dx}_{\leq \epsilon \|\lambda\Theta\|^2 + c_\epsilon \|F\|^2}.$$

Where we used that $\operatorname{Re} (\llbracket Q \rrbracket_{\xi_1} \overline{i\lambda\Theta}(\xi_1)) = 0$. Using (3.1) to estimate J we get

$$\begin{aligned}|J| &\leq c|\lambda| \|v_x\| |\mathfrak{R}| \\ &\leq c|\lambda|^{3/2} \mathfrak{E} \mathfrak{R} + c|\lambda| \mathfrak{R}^2.\end{aligned}$$

from where we have

$$\frac{1}{|\lambda|^2} |J| \leq \frac{c}{|\lambda|^{1/2}} \mathfrak{E} \mathfrak{R} + \frac{c}{|\lambda|} \mathfrak{R}^2 \leq \epsilon \mathfrak{E}^2 + \frac{c}{|\lambda|} \mathfrak{R}^2$$

Hence inequality (3.7) follows for λ large. $\square$

Lemma 3.4. Over the thermoelastic interval, for any $\epsilon > 0$ there exists a constant $c_\epsilon > 0$ such that any solution to system (2.14)–(2.21) satisfies

$$(3.8) \quad \int_{\ell_0}^{\ell_1} |u_{xx}|^2 dx \leq \epsilon \mathfrak{E}^2 + \frac{c_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{c_\epsilon}{|\lambda|^2} \|F\|^2.$$

Proof. Using equations (2.17) and (2.14), we obtain

$$\Theta - \frac{1}{i\lambda}(\kappa\theta_x + \kappa_0\Theta_x)_x - mu_{xx} = \frac{1}{i\lambda}(f_4 + mf_{1,xx}).$$

Multiplying the above identity by $\overline{u_{xx}}$ and applying Remark 3.2, we arrive at

$$\begin{aligned} \int_{\ell_0}^{\ell_1} m|u_{xx}|^2 dx &= \int_{\ell_0}^{\ell_1} \Theta \overline{u_{xx}} dx + \underbrace{\frac{1}{i\lambda} \llbracket Q \overline{u_{xx}} \rrbracket_{\xi_1} + \frac{1}{i\lambda} (\kappa\theta_x + \kappa_0\Theta_x) \overline{u_{xx}}}_{:=I_0} \Big|_{\ell_0}^{\ell_1} \\ (3.9) \quad &+ \underbrace{\int_{\ell_0}^{\ell_1} \frac{1}{i\lambda} (\kappa\theta_x + \kappa_0\Theta_x) \overline{u_{xxx}} dx}_{:=I_1} + \underbrace{\int_{\ell_0}^{\ell_1} \frac{1}{i\lambda} (mf_{1,xx} - f_4) \overline{u_{xx}} dx}_{\leq \epsilon \|u_{xx}\|^2 + \frac{C_\epsilon}{|\lambda|^2} \|F\|^2}. \end{aligned}$$

We first estimate the term I_0 so we have

$$\begin{aligned} |I_0| &\leq \frac{c}{|\lambda|} \|\kappa\theta_x + \kappa_0\Theta_x\|_\infty \|u_{xx}\|_\infty \\ &\leq \underbrace{\frac{c}{|\lambda|} \|\kappa\theta_x + \kappa_0\Theta_x\|_\infty \|M\|_\infty}_{:=I_3} + \underbrace{\frac{c}{|\lambda|} \|\kappa\theta_x + \kappa_0\Theta_x\|_\infty \|\Theta\|_\infty}_{:=I_4}. \end{aligned}$$

Using estimates (3.2) and (3.4), we deduce

$$\begin{aligned} |I_3| &\leq \frac{c}{|\lambda|} \left(c|\lambda|^{1/2} \mathfrak{R}^{1/2} \mathfrak{E}^{1/2} + c \mathfrak{R}^{1/2} \|F\|_{\mathcal{H}}^{1/2} \right) \left(c|\lambda|^{1/4} \mathfrak{E} + c \mathfrak{E}^{3/4} \|F\|^{1/4} \right) \\ &\leq \frac{c}{|\lambda|^{1/4}} \mathfrak{R}^{1/2} \mathfrak{E}^{3/2} + \frac{c}{|\lambda|^{1/2}} \mathfrak{R}^{1/2} \mathfrak{E}^{5/4} \|F\|_{\mathcal{H}}^{1/4} + \frac{c}{|\lambda|^{3/4}} \mathfrak{R}^{1/2} \mathfrak{E} \|F\|_{\mathcal{H}}^{1/2} + \frac{c}{|\lambda|} \mathfrak{R}^{1/2} \mathfrak{E}^{3/4} \|F\|_{\mathcal{H}}^{3/4} \\ &\leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{C_\epsilon}{|\lambda|^{4/3}} \mathfrak{R}^{4/3} \|F\|_{\mathcal{H}}^{2/3} + \frac{C_\epsilon}{|\lambda|^{3/2}} \mathfrak{R} \|F\|_{\mathcal{H}} + \frac{C_\epsilon}{|\lambda|^{8/5}} \mathfrak{R}^{4/5} \|F\|_{\mathcal{H}}^{6/5} \\ (3.10) \quad &\leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{C_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2. \end{aligned}$$

Moreover, using the estimate $\|\Theta\|_\infty \leq c\|\Theta\|^{1/2}\|\Theta_x\|^{1/2} \leq c\mathfrak{E}^{1/2}\mathfrak{R}^{1/2}$, we obtain

$$\begin{aligned} |I_4| &\leq \frac{c}{|\lambda|} \left(c|\lambda|^{1/2} \mathfrak{R}^{1/2} \mathfrak{E}^{1/2} + c \mathfrak{R}^{1/2} \|F\|_{\mathcal{H}}^{1/2} \right) \mathfrak{E}^{1/2} \mathfrak{R}^{1/2} \\ (3.11) \quad &\leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{C_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2. \end{aligned}$$

Combining (3.10) and (3.11), we conclude

$$(3.12) \quad |I_0| \leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{C_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2.$$

Next, using (2.23) and (3.5), we estimate

$$\begin{aligned} |I_1| &\leq \frac{c}{|\lambda|} \mathfrak{R} \|u_{xxx}\|_{L^2} \\ (3.13) \quad &\leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2. \end{aligned}$$

Finally, inserting the estimates for I_0 and I_1 into (3.9), and using Lemma 3.3, we obtain (3.8). $\square$

In the next lemma, we complete the estimate of the energy over $]\ell_0, \ell_1[$ by deriving an appropriate bound for v .

Lemma 3.5. *Over the thermoelastic interval, there exists a constant $c > 0$ such that the solution to system (2.14)–(2.21) satisfies*

$$(3.14) \quad \mathfrak{E}^2 \leq \frac{c}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|^2.$$

Proof. Multiplying equation (2.15) by $\frac{1}{i\lambda}\bar{v}$ and integrating over $[\ell_0, \ell_1]$, we obtain

$$\begin{aligned} \int_{\ell_0}^{\ell_1} |v|^2 dx + \frac{1}{i\lambda} \llbracket M_x \rrbracket_{\xi_1} \bar{v}(\xi_1) - \frac{1}{i\lambda} \llbracket M \rrbracket_{\xi_1} \overline{v_x}(\xi_1) + \frac{1}{i\lambda} M_x \bar{v} \Big|_{\ell_0}^{\ell_1} - \frac{1}{i\lambda} M \overline{v_x} \Big|_{\ell_0}^{\ell_1} \\ - \int_{\ell_0}^{\ell_1} M \frac{1}{i\lambda} \bar{v}_{xx} dx = \int_{\ell_0}^{\ell_1} f_2 \frac{1}{i\lambda} \bar{v} dx. \end{aligned}$$

Since

$$\operatorname{Re} \left(\frac{1}{i\lambda} \llbracket M_x \rrbracket_{\xi_1} \bar{v}(\xi_1) - \frac{1}{i\lambda} \llbracket M \rrbracket_{\xi_1} \overline{v_x}(\xi_1) \right) = 0,$$

taking real parts yields

$$(3.15) \quad \int_{\ell_0}^{\ell_1} |v|^2 dx = - \underbrace{\frac{1}{i\lambda} M_x \bar{v} \Big|_{\ell_0}^{\ell_1}}_{:=J_1} + \underbrace{\frac{1}{i\lambda} M \overline{v_x} \Big|_{\ell_0}^{\ell_1}}_{:=J_2} + \underbrace{\int_{\ell_0}^{\ell_1} M \frac{1}{i\lambda} \bar{v}_{xx} dx}_{:=J_3} + \underbrace{\int_{\ell_0}^{\ell_1} f_2 \frac{1}{i\lambda} \bar{v} dx}_{:=\epsilon \|v\|^2 + \frac{C_\epsilon}{|\lambda|^2} \|F\|^2}.$$

We now estimate the point term. Using (3.2), (3.3) and remembering the estimates (3.7) and (3.8), we obtain

$$(3.16) \quad |J_1| \leq \epsilon \|v\|_{L^2}^2 + \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|^2,$$

similarly,

$$(3.17) \quad |J_2| \leq \epsilon \|v\|_{L^2}^2 + \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|^2.$$

For estimate J_3 , we use (3.7) and (3.8)

$$(3.18) \quad J_3 \leq \int_{\ell_0}^{\ell_1} M(\alpha \bar{u}_{xx} - \frac{1}{i\lambda} \overline{f_{1,xx}}) dx \leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|^2,$$

Substituting (3.16)–(3.18) into (3.15), we conclude that

$$\int_{\ell_0}^{\ell_1} |v|^2 dx \leq \epsilon \mathfrak{E}^2 + \frac{C_\epsilon}{|\lambda|} \mathfrak{R}^2 + \frac{c}{|\lambda|^2} \|F\|^2.$$

Finally, combining this estimate with Lemma 3.3 and Lemma 3.4, the result follows. $\square$

3.1. Observability inequalities. Let us introduce the functional

$$\mathcal{I}(x) = \frac{1}{2} (|v(x)|^2 + \alpha |u_{xx}(x)|^2)$$

we have the following result

Lemma 3.6. *Let (a, b) be either $(0, \ell_0)$, (ℓ_1, ℓ) , or any subinterval of (ℓ_0, ℓ_1) that does not contain the singular point ξ_1 . Let*

$$q(x) = x - r, \quad r \in \{a, b\},$$

so that q vanishes at one of the endpoints of (a, b) . Then, for every $\epsilon > 0$ and $|\lambda|$ sufficiently large, any solution U of the resolvent system (2.14)–(2.21) satisfies

$$\left| - \int_a^b q(x) \mathcal{I}'(x) dx + \int_a^b \alpha |u_{xx}|^2 dx \right| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{C_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2.$$

Proof. Multiplying equation (2.15) by $q \frac{1}{i\lambda} \bar{v}_x$ and integrating over (a, b) , we obtain

$$- \int_a^b q v \bar{v}_x dx - \frac{1}{i\lambda} \int_a^b M_{xx} q \bar{v}_x dx = - \frac{1}{i\lambda} \int_a^b f_2 q \bar{v}_x dx.$$

Since $q'(x) = 1$ and $q(r) = 0$ at one of the endpoints, using the resolvent equation and integrating by parts, we get

$$\begin{aligned} & - \int_a^b q v \overline{v_x} dx - \underbrace{\frac{1}{i\lambda} q M_x \overline{v_x}}_{J_1} \Big|_a^b + \underbrace{\frac{1}{i\lambda} M \overline{v_x}}_{J_2} \Big|_a^b + \underbrace{\frac{q(\xi_0)}{i\lambda} \llbracket M_x \rrbracket_{\xi_0} \overline{v_x}(\xi_0)}_{:=J_3} \\ & - \frac{1}{i\lambda} \int_a^b M \overline{v_{xx}} dx + \frac{1}{i\lambda} \int_a^b q M_x \overline{v_{xx}} dx = - \underbrace{\frac{1}{i\lambda} \int_a^b f_2 q \overline{v_x} dx}_{J_4}. \end{aligned}$$

Here, J_3 is present only when $\xi_0 \in (a, b)$. Moreover, since $q(r) = 0$, only the endpoint different from r contributes to J_1 .

Using (2.14), we have

$$\begin{aligned} - \frac{1}{i\lambda} \int_a^b M \overline{v_{xx}} dx &= \int_a^b M \overline{u_{xx}} dx - \frac{1}{i\lambda} \int_a^b M \overline{f_{1,xx}} dx \\ &= \int_a^b \alpha |u_{xx}|^2 dx + \underbrace{\int_a^b m \Theta \overline{u_{xx}} dx - \frac{1}{i\lambda} \int_a^b M \overline{f_{1,xx}} dx}_{:=R_1}, \end{aligned}$$

and

$$\begin{aligned} \frac{1}{i\lambda} \int_a^b q M_x \overline{v_{xx}} dx &= - \int_a^b q M_x \overline{u_{xx}} dx - \frac{1}{i\lambda} \int_a^b q M_x \overline{f_{1,xx}} dx \\ &= - \int_a^b q \alpha u_{xxx} \overline{u_{xx}} dx - \underbrace{\int_a^b q m \Theta_x \overline{u_{xx}} dx + \frac{1}{i\lambda} \int_a^b q M_x \overline{f_{1,xx}} dx}_{:=R_2}. \end{aligned}$$

Therefore,

$$- \int_a^b q v \overline{v_x} dx - \int_a^b q \alpha u_{xxx} \overline{u_{xx}} dx + R_2 - J_1 + J_2 + J_3 + \int_a^b \alpha |u_{xx}|^2 dx + R_1 = J_4.$$

Taking the real part and recalling that

$$\mathcal{I}(x) = \frac{1}{2} (|v(x)|^2 + \alpha |u_{xx}(x)|^2),$$

we obtain

$$(3.19) \quad - \int_a^b q \frac{d}{dx} \mathcal{I}(x) dx + \int_a^b \alpha |u_{xx}|^2 dx = \operatorname{Re} (-R_1 - R_2 + J_1 - J_2 - J_3 + J_4).$$

We now estimate the terms on the right-hand side. Using (3.2), (3.3), and the estimates over the thermoelastic interval I_T , we have

$$\begin{aligned} |M_x(\ell_i)|, |v_x(\ell_i)| &\leq c|\lambda|^{3/4} \mathfrak{E} + c\mathfrak{E}^{1/4} \|F\|^{3/4}, \\ |M(\ell_i)| &\leq c|\lambda|^{1/4} \mathfrak{E} + c\mathfrak{E}^{3/4} \|F\|^{1/4}. \end{aligned}$$

Since q vanishes at one of the endpoints, these estimates give

$$(3.20) \quad |J_1| \leq c|\lambda|^{1/2} \mathfrak{E}^2 + \frac{c}{|\lambda|} \mathfrak{E}^{1/2} \|F\|^{3/2},$$

$$(3.21) \quad |J_2| \leq c\mathfrak{E}^2 + \frac{c}{|\lambda|^2} \|F\|^2.$$

For J_3 , using $\llbracket M_x \rrbracket_{\xi_0} = -\beta_0 v(\xi_0)$, (3.3), and the dissipation estimate, we obtain

$$\begin{aligned} |J_3| &\leq \frac{c}{|\lambda|} |v(\xi_0) v_x(\xi_0)| \\ &\leq \frac{c}{|\lambda|} \Re \left(|\lambda|^{3/4} \|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/4} \|F\|^{3/4} \right), \end{aligned}$$

and hence

$$(3.22) \quad |J_3| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|^{1/2}} \mathfrak{R}^2.$$

Using (3.1), we also have

$$(3.23) \quad \begin{aligned} |J_4| &\leq \frac{c}{|\lambda|^{1/2}} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{c}{|\lambda|} \|U\|_{\mathcal{H}}^{1/2} \|F\|_{\mathcal{H}}^{3/2} \\ &\leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2. \end{aligned}$$

For the remaining terms, the Cauchy–Schwarz inequality, (3.5), and Lemma 3.5 yield

$$(3.24) \quad |R_1| \leq c\mathfrak{E}^2 + \frac{c}{|\lambda|} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}},$$

$$(3.25) \quad |R_2| \leq c\mathfrak{R}\mathfrak{E} + \frac{c}{|\lambda|^{1/2}} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{c}{|\lambda|} \|U\|_{\mathcal{H}}^{1/2} \|F\|_{\mathcal{H}}^{3/2}.$$

Moreover, Lemma 3.5 gives

$$\mathfrak{R}\mathfrak{E} \leq \frac{c}{|\lambda|^{1/2}} \mathfrak{R}^2 + \frac{c}{|\lambda|} \mathfrak{R} \|F\|_{\mathcal{H}}.$$

Consequently, using $\mathfrak{R}^2 \leq C \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}}$ and Young's inequality in (3.20)–(3.25), we obtain

$$|J_1| + |J_2| + |J_3| + |J_4| + |R_1| + |R_2| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2.$$

Substituting this estimate into (3.19) gives

$$\left| -\int_a^b q(x) \mathcal{I}'(x) dx + \int_a^b \alpha |u_{xx}|^2 dx \right| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2,$$

which completes the proof. $\square$

Lemma 3.7. *For every $\epsilon > 0$, there exists a constant $c_\epsilon > 0$ such that any solution U of the resolvent system (2.14)–(2.21) satisfies the following estimates:*

(1) *If the thermoelastic mechanism acts on the entire beam, that is, $] \ell_0, \ell_1[=]0, \ell[$, then*

$$(3.26) \quad m_0 |W|^2 + 2m_0 d \operatorname{Re}(W \bar{Z}) + (J + m_0 d^2) |Z|^2 \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2.$$

(2) *If the thermoelastic mechanism is localized in a proper subinterval of the beam, that is, $] \ell_0, \ell_1[\subsetneq]0, \ell[$, then*

$$(3.27) \quad m_0 |W|^2 + 2m_0 d \operatorname{Re}(W \bar{Z}) + (J + m_0 d^2) |Z|^2 \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2.$$

Remark 3.8. *Notice that case 2 includes both the configuration $0 < \ell_0 < \ell_1 = \ell$, in which the thermoelastic region reaches the tip, and the configuration $\ell_1 < \ell$, in which the tip lies in the purely elastic part of the beam. Estimate (3.27) holds in both configurations.*

Proof. Multiplying equation (2.18) by $\bar{W}$ and (2.19) by $\bar{Z}$ and adding, we get

$$(3.28) \quad \underbrace{\frac{1}{i\lambda} (\gamma |W|^2 + d\gamma W \bar{Z} + d\gamma^* |Z|^2)}_{I_3} + \underbrace{\frac{1}{i\lambda} (m_0 f_5 + m_0 d f_6) \bar{W} + \frac{1}{i\lambda} (m_0 d f_5 + (J + m_0 d^2) f_6) \bar{Z}}_{I_4} = \underbrace{\frac{1}{i\lambda} (\alpha u_{xxx} + m\Theta_x)(\ell) \bar{W}}_{I_1} - \underbrace{\frac{1}{i\lambda} \alpha u_{xx}(\ell) \bar{Z}}_{I_2}.$$

Using (2.23) and Young's inequality and taking $|\lambda|$ sufficiently large we get

$$|I_3| + |I_4| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2,$$

In case 1, the endpoint $x = \ell$ belongs to the thermoelastic region. Recall that, on the thermal interval I_T , the estimates of Lemma 3.1 remain valid with $\|U\|_{\mathcal{H}}$ replaced by $\mathfrak{E}$. Therefore, using (3.2), (3.3), and Lemma 3.5, we obtain

$$|I_1| + |I_2| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2.$$

Combining this estimate with (3.28), we obtain (3.26).

In case 2, using (3.2) and (3.3), we obtain

$$|I_1|, |I_2| \leq \frac{c}{|\lambda|} \left(|\lambda|^{3/4} \|U\|_{\mathcal{H}} + \|U\|_{\mathcal{H}}^{1/4} \|F\|_{\mathcal{H}}^{3/4} \right) (|W| + |Z|).$$

Using (2.23) to estimate W and Z , and then applying Lemma 3.6, we obtain, for $|\lambda|$ sufficiently large,

$$(3.29) \quad |I_1| + |I_2| \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c_\epsilon}{|\lambda|} \|F\|_{\mathcal{H}}^2.$$

Finally, combining (3.29) with (3.28) yields (3.27), and the proof is complete. $\square$

Lemma 3.9 (Lack of compactness). *The operator $\mathcal{A}$ does not have compact resolvent. Consequently, the semigroup generated by $\mathcal{A}$ is not compact.*

Proof. Set

$$\lambda_0 = -\frac{\kappa}{\kappa_0}, \quad \kappa + \lambda_0 \kappa_0 = 0.$$

We construct a singular Weyl sequence associated with λ_0 . Choose an open interval

$$J \Subset (\ell_0, \ell_1), \quad \xi_1 \notin \overline{J},$$

and let $0 \neq \chi \in C_c^\infty(J)$. Since the coefficients are constant on J , denote by α_2 and m_1 the corresponding values of α and m . Define

$$\psi_n(x) = n^{-3} \chi(x) e^{inx}, \quad \theta_n = \psi_n'',$$

and

$$u_n = -\frac{m_1 \lambda_0}{\alpha_2} \psi_n, \quad v_n = \lambda_0 u_n, \quad \Theta_n = \lambda_0 \theta_n, \quad W_n = Z_n = 0.$$

All these functions are extended by zero outside J .

Since their supports are strictly contained in the thermoelastic region and avoid the singular point ξ_1 , all boundary, transmission, and singular interaction conditions are automatically satisfied. Moreover,

$$M_n = \alpha_2 u_{n,xx} + m_1 \Theta_n = 0$$

and

$$Q_n = \kappa \theta_{n,x} + \kappa_0 \Theta_{n,x} = (\kappa + \lambda_0 \kappa_0) \theta_{n,x} = 0.$$

Hence $U_n \in D(\mathcal{A})$.

By construction,

$$\lambda_0 u_n - v_n = 0, \quad \lambda_0 \theta_n - \Theta_n = 0,$$

while

$$\lambda_0 v_n + M_{n,xx} = \lambda_0^2 u_n$$

and

$$\lambda_0 \Theta_n - Q_{n,x} - m_1 v_{n,xx} = \lambda_0^2 \left(1 + \frac{m_1^2}{\alpha_2} \right) \theta_n.$$

Since

$$\|\psi_n\|_{L^2} = O(n^{-3}), \quad \|\theta_n\|_{L^2} = O(n^{-1}), \quad \|\theta_{n,x}\|_{L^2} = O(1),$$

we have

$$\|U_n\|_{\mathcal{H}} \asymp 1, \quad \|(\lambda_0 I - \mathcal{A})U_n\|_{\mathcal{H}} \longrightarrow 0.$$

After normalization, we may therefore assume that

$$\|U_n\|_{\mathcal{H}} = 1.$$

Moreover, the oscillatory character of the sequence implies

$$U_n \rightharpoonup 0 \quad \text{in } \mathcal{H}.$$

Suppose, by contradiction, that $\mathcal{A}$ has compact resolvent. Choose $\mu \in \varrho(\mathcal{A})$ and set

$$R(\mu, \mathcal{A}) = (\mu I - \mathcal{A})^{-1}.$$

Since $R(\mu, \mathcal{A})$ is compact and $U_n \rightharpoonup 0$,

$$R(\mu, \mathcal{A})U_n \longrightarrow 0 \quad \text{strongly in } \mathcal{H}.$$

On the other hand,

$$R(\mu, \mathcal{A})(\lambda_0 I - \mathcal{A})U_n = U_n + (\lambda_0 - \mu)R(\mu, \mathcal{A})U_n.$$

Both the left-hand side and the second term on the right-hand side converge strongly to zero. Consequently,

$$U_n \longrightarrow 0 \quad \text{strongly in } \mathcal{H},$$

contradicting $\|U_n\|_{\mathcal{H}} = 1$. Therefore $\mathcal{A}$ does not have compact resolvent. $\square$

Proof of Theorem 2.2. We estimate In case of $] \ell_0, \ell_1[=]0, \ell[$, Lemma 3.1 applied to $\mathfrak{E} = \|U\|_{\mathcal{H}}$ together with Lemma 3.7 implies that

$$\|U\|_{\mathcal{H}}^2 \leq \frac{c_\epsilon}{|\lambda|} \|U\|_{\mathcal{H}} \|F\|_{\mathcal{H}} + \frac{c_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2, \quad \Rightarrow \quad \|U\|_{\mathcal{H}}^2 \leq \frac{c_\epsilon}{|\lambda|^2} \|F\|_{\mathcal{H}}^2.$$

From where the analyticity follows.

In case of $] \ell_0, \ell_1[\subsetneq]0, \ell[$,

Using Lemma 3.6 over the intervals $]a, \ell_1[$ and $] \ell_0, b[$ for $\xi_1 < a < \ell_1$ and $\ell_0 < b < \xi_1$, we get

$$\begin{aligned} q(\ell_0)I(\ell_0) &\leq \int_{\ell_0}^b I(x) dx + \int_{\ell_0}^b \alpha |u_{xx}|^2 dx + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \\ &\leq c\mathfrak{E}^2 + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \end{aligned}$$

Similarly

$$q(\ell_1)I(\ell_1) \leq c\mathfrak{E}^2 + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2,$$

Applying Lemma 3.6 over the intervals $]0, \ell_0[$ and $] \ell_1, \ell[$ we get

$$\begin{aligned} \int_0^{\ell_0} I(x) dx &\leq q(\ell_0)I(\ell_0) + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \\ &\leq c\mathfrak{E}^2 + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \\ \int_{\ell_1}^{\ell} I(x) dx &\leq q(\ell_1)I(\ell_1) + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \\ &\leq c\mathfrak{E}^2 + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2, \end{aligned}$$

And furthermore, using (3.27) we get

$$\|U\|_{\mathcal{H}}^2 \leq \int_0^{\ell_0} I(x) dx + \mathfrak{E}^2 + \int_{\ell_1}^{\ell} I(x) dx \leq c\mathfrak{E}^2 + \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2,$$

Using Lemma 3.5 we arrive to

$$\|U\|_{\mathcal{H}}^2 \leq \epsilon \|U\|_{\mathcal{H}}^2 + \frac{c}{|\lambda|} \|F\|_{\mathcal{H}}^2,$$

Hence the semigroup is of Gevrey class 2. Finally, item 3) follows from Lemma 3.9 and from [11, Theorem 4.29].

Remark 3.10. If we insert the first derivate of Dirac Delta in the elastic part produce a discontinuity of first order in u_{xx} , when using integration by part in Lemma (3.6) we obtain the term $q(\xi_0)[M]_{\xi_0}$ that we can estimate by means the dissipation, as

$$q(\xi_0)[M]_{\xi_0} \leq c\mathfrak{R}^2.$$

There implies that loss all regularizing properties of semigroup.

Furthermore, if all terms are estimate as in this article and adding this last estimate, for λ large we obtain

$$\|U\|_{\mathcal{H}}^2 \leq c\|F\|_{\mathcal{H}}^2,$$

by [15], the semigroup decays exponentially.

As a direct consequence of the bounded perturbation argument, the Gevrey regularity obtained above remains unchanged under bounded lower-order perturbations. The argument is based on the classical bounded perturbation theory for semigroups; see, for instance, [13].

Corollary 3.11. Let $\mathcal{A}$ be the infinitesimal generator of a C_0 -semigroup of Gevrey class μ , and let $\mathcal{B} \in \mathcal{L}(\mathcal{H})$. Then $\mathcal{A} + \mathcal{B}$, with

$$D(\mathcal{A} + \mathcal{B}) = D(\mathcal{A}),$$

generates a C_0 -semigroup of the same Gevrey class μ .

Proof. Since $\mathcal{B}$ is bounded and the Gevrey regularity of the semigroup generated by $\mathcal{A}$ yields the high-frequency resolvent estimate

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|\lambda|^{1/\mu}}, \quad |\lambda| \gg 1,$$

we have

$$\|\mathcal{B}(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C\|\mathcal{B}\|_{\mathcal{L}(\mathcal{H})}}{|\lambda|^{1/\mu}}.$$

Consequently,

$$\|\mathcal{B}(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{L}(\mathcal{H})} < 1$$

for $|\lambda|$ sufficiently large. Hence

$$I - \mathcal{B}(i\lambda I - \mathcal{A})^{-1}$$

is invertible, and its inverse is uniformly bounded for large $|\lambda|$. Using the factorization

$$i\lambda I - (\mathcal{A} + \mathcal{B}) = [I - \mathcal{B}(i\lambda I - \mathcal{A})^{-1}](i\lambda I - \mathcal{A}),$$

we obtain

$$(i\lambda I - (\mathcal{A} + \mathcal{B}))^{-1} = (i\lambda I - \mathcal{A})^{-1}[I - \mathcal{B}(i\lambda I - \mathcal{A})^{-1}]^{-1}.$$

Therefore,

$$\|(i\lambda I - (\mathcal{A} + \mathcal{B}))^{-1}\|_{\mathcal{L}(\mathcal{H})} \leq \frac{C}{|\lambda|^{1/\mu}}, \quad |\lambda| \gg 1.$$

Thus, the high-frequency resolvent estimate characterizing the Gevrey class μ is preserved under the bounded perturbation, and the conclusion follows. $\square$

We can now apply Corollary 3.11 to the hybrid thermoelastic system. In particular, Theorem 2.2 remains valid after the addition of the following bounded lower-order terms.

Theorem 3.12. Let $A, B, C, D, E, F, G \in \mathbb{R}$, and let $T(t)$ denote the C_0 -semigroup associated with the perturbed system

$$\begin{aligned} \rho u_{tt} + M_{xx} + \gamma_0 u_t(\xi_0)\delta_{\xi_0} + \gamma_1 u_t(\xi_1)\delta_{\xi_1} - \gamma_2 u_{xt}(\xi_1)\frac{d}{dx}\delta_{\xi_1} &= Au_{xx} + Bu_t + Cu, \\ c\theta_{tt} - (\kappa\theta_x + \kappa_0\theta_{xt})_x - mu_{xxt} + \gamma_3\theta_t(\xi_1)\delta_{\xi_1} &= Du_{xx} + Eu_t + Fu + G\theta_x, \end{aligned}$$

subject to the transmission and boundary conditions (2.8)–(2.10). Then:

- (1) If $(\ell_0, \ell_1) = (0, \ell)$, then $T(t)$ is analytic.
- (2) If $(\ell_0, \ell_1) \subsetneq (0, \ell)$, then $T(t)$ is of Gevrey class 2.
- (3) In either case, $T(t)$ is not compact.

4. NUMERICAL ILLUSTRATIONS

In this section we introduce a finite element approximation for the transmission problem (2.5)–(2.10), coupled with the tip-body equations (1.3)–(1.4). The approximation is based on a primal conforming finite element method for the Euler–Bernoulli displacement [3, 4, 10] and on the Newmark scheme for the time discretization [12]. The resolvent approximation and the numerical experiments will be discussed in subsequent subsections.

4.1. Spatial approximation using Hermite polynomials. Let $0 = x_0 < x_1 < \dots < x_N = \ell$, be a partition of $[0, \ell]$ such that the points $\xi_0, \ell_0, \xi_1, \ell_1$ belong to the set of nodes. This assumption is important in order to incorporate the transmission conditions and the singular interactions without introducing any regularization of the Dirac distributions. We define

$$V_h = \{\varphi_h \in C^1([0, \ell]); \varphi_h|_{[x_j, x_{j+1}]} \in \mathbb{P}_3, \quad \varphi_h(0) = \varphi_{h,x}(0) = 0\} \subset V,$$

where $\mathbb{P}_3$ denotes the space of polynomials of degree less than or equal to three. For the thermal variable we use

$$\Theta_h = \{\psi_h \in H_0^1(\ell_0, \ell_1); \psi_h|_{[x_j, x_{j+1}]} \in \mathbb{P}_k, \quad k = 1 \text{ or } 2\}.$$

We also introduce the tip trace operator $\mathcal{T}\varphi = \begin{pmatrix} \varphi(\ell) \\ \varphi_x(\ell) \end{pmatrix}$. For $\varphi, \eta \in V_h$ and $\psi, \chi \in \Theta_h$, we define

$$\begin{aligned} m_u(\eta, \varphi) &= \int_0^\ell \rho \eta \varphi dx + (K\mathcal{T}\eta, \mathcal{T}\varphi)_{\mathbb{R}^2}, & a_u(\eta, \varphi) &= \int_0^\ell \alpha \eta_{xx} \varphi_{xx} dx, \\ d_u(\eta, \varphi) &= \beta_0 \eta(\xi_0) \varphi(\xi_0) + \beta_1 \eta(\xi_1) \varphi(\xi_1) + \beta_2 \eta_x(\xi_1) \varphi_x(\xi_1) + (B\mathcal{T}\eta, \mathcal{T}\varphi)_{\mathbb{R}^2}, \end{aligned}$$

and

$$\begin{aligned} m_\theta(\chi, \psi) &= \int_{\ell_0}^{\ell_1} c \chi \psi dx, & a_\theta(\chi, \psi) &= \int_{\ell_0}^{\ell_1} \kappa \chi_x \psi_x dx, \\ d_\theta(\chi, \psi) &= \int_{\ell_0}^{\ell_1} \kappa_0 \chi_x \psi_x dx + \beta_3 \chi(\xi_1) \psi(\xi_1), & b(\psi, \varphi) &= \int_{\ell_0}^{\ell_1} m \psi \varphi_{xx} dx. \end{aligned}$$

The semidiscrete problem reads as follows: find

$$u_h : [0, \infty) \rightarrow V_h, \quad \theta_h : [0, \infty) \rightarrow \Theta_h$$

such that, for all $\varphi_h \in V_h$ and $\psi_h \in \Theta_h$,

$$(4.30) \quad m_u(u_{h,tt}, \varphi_h) + a_u(u_h, \varphi_h) + d_u(u_{h,t}, \varphi_h) + b(\theta_{h,t}, \varphi_h) = 0,$$

$$(4.31) \quad m_\theta(\theta_{h,tt}, \psi_h) + a_\theta(\theta_h, \psi_h) + d_\theta(\theta_{h,t}, \psi_h) - b(\psi_h, u_{h,t}) = 0.$$

The terms involving ξ_0 and ξ_1 are well-defined because these points are nodes of the mesh. Thus the singularities are treated exactly at the discrete level as point interactions.

Let $\{\Phi_i\}_{i=1}^{N_u}$ be a basis of V_h and $\{\Psi_j\}_{j=1}^{N_\theta}$ a basis of Θ_h . Writing

$$u_h(t) = \sum_{i=1}^{N_u} q_i(t) \Phi_i, \quad \theta_h(t) = \sum_{j=1}^{N_\theta} r_j(t) \Psi_j,$$

we obtain the matrix system

$$(4.32) \quad M_u \ddot{q} + D_u \dot{q} + A_u q + C \dot{r} = 0,$$

$$(4.33) \quad M_\theta \ddot{r} + D_\theta \dot{r} + A_\theta r - C^T \dot{q} = 0.$$

Equivalently, for $Y = (q, r)^T$, system (4.32)–(4.33) can be written as

$$\mathbb{M} \ddot{Y} + \mathbb{D} \dot{Y} + \mathbb{A} Y = 0,$$

where

$$\mathbb{M} = \begin{pmatrix} M_u & 0 \\ 0 & M_\theta \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} A_u & 0 \\ 0 & A_\theta \end{pmatrix},$$

and

$$\mathbb{D} = \begin{pmatrix} D_u & C \\ -C^T & D_\theta \end{pmatrix}.$$

The skew-symmetric structure of the coupling blocks reflects the conservative nature of the thermoelastic interaction, while the matrices D_u and D_θ contain the dissipative terms.

The semidiscrete energy is given by

$$E_h(t) = \frac{1}{2}m_u(u_{h,t}, u_{h,t}) + \frac{1}{2}a_u(u_h, u_h) + \frac{1}{2}m_\theta(\theta_{h,t}, \theta_{h,t}) + \frac{1}{2}a_\theta(\theta_h, \theta_h).$$

Taking $\varphi_h = u_{h,t}$ and $\psi_h = \theta_{h,t}$ in (4.30)–(4.31), and adding the resulting identities, we obtain

$$\frac{d}{dt}E_h(t) = -d_u(u_{h,t}, u_{h,t}) - d_\theta(\theta_{h,t}, \theta_{h,t}).$$

Consequently, under the same condition $d\gamma \leq 4\gamma^*$, the semidiscrete energy is non-increasing.

4.2. Newmark time approximation. We now introduce the time discretization. Let $\Delta t > 0$ and $t_n = n\Delta t$. We denote by Y^n , V^n and A^n the approximations of $Y(t_n)$, $\dot{Y}(t_n)$ and $\ddot{Y}(t_n)$, respectively.

We use the trapezoidal Newmark method with parameters $\beta_N = 1/4$ and $\gamma_N = 1/2$. With this choice, the method satisfies

$$\frac{Y^{n+1} - Y^n}{\Delta t} = \frac{V^{n+1} + V^n}{2}.$$

Testing the fully discrete equation with $(V^{n+1} + V^n)/2$ yields the discrete energy identity

$$E_h^{n+1} - E_h^n = -\Delta t \left[d_u \left(\frac{u_h^{n+1} - u_h^n}{\Delta t}, \frac{u_h^{n+1} - u_h^n}{\Delta t} \right) + d_\theta \left(\frac{\theta_h^{n+1} - \theta_h^n}{\Delta t}, \frac{\theta_h^{n+1} - \theta_h^n}{\Delta t} \right) \right],$$

where

$$E_h^n = \frac{1}{2}m_u(V_u^n, V_u^n) + \frac{1}{2}a_u(u_h^n, u_h^n) + \frac{1}{2}m_\theta(V_\theta^n, V_\theta^n) + \frac{1}{2}a_\theta(\theta_h^n, \theta_h^n).$$

Therefore,

$$E_h^{n+1} \leq E_h^n, \quad n \geq 0,$$

which is the discrete counterpart of the continuous energy dissipation law.

4.3. First example. In this first numerical example we illustrate the dissipative behavior of the fully discrete scheme. The parameters used in the simulation are summarized in Table 1.

Group	Parameters
Geometry	$L = 1, \quad \xi_0 = 0.20, \quad \ell_0 = 0.35, \quad \xi_1 = 0.55, \quad \ell_1 = 0.75$
Mechanical coefficients	$\rho = 1, \quad \alpha_1 = 1, \quad \alpha_2 = 2, \quad \alpha_3 = 1$
Thermal coefficients	$c = 1, \quad m_1 = 0.25, \quad \kappa = 0.15, \quad \kappa_0 = 0.05$
Localized damping	$\beta_0 = 0.50, \quad \beta_1 = 0.50, \quad \beta_2 = 0.10, \quad \beta_3 = 0.20$
Tip body	$m_0 = 1, \quad d = 0.05, \quad J = 0.05, \quad \gamma = 0.20, \quad \gamma^* = 0.20$
Spatial discretization	$n_{\text{sub}} = 160, \quad N_u = 1600, \quad N_\theta = 319, \quad N_{\text{dof}} = 1919$
Time discretization	$T = 400, \quad \Delta t = 0.005, \quad N_t = 80000, \quad \beta_N = \frac{1}{4}, \quad \gamma_N = \frac{1}{2}$
Energy fitting interval	Second half of the simulation
Estimated decay rate	$\omega_h \approx 0.00100337$

TABLE 1. Parameters used in the first numerical example.

The condition $d\gamma \leq 4\gamma^*$ is satisfied for the above choice of parameters. The localized thermoelastic region is the interval (ℓ_0, ℓ_1) , while the singular mechanical interactions are located at ξ_0 and ξ_1 .

The initial data are chosen as smooth functions compatible with the essential boundary conditions:

$$u_0(x) = 0.02x^2(3L - 2x), \quad u_1(x) = 0.02x^2(L - x)^2,$$

and

$$\theta_0(x) = 0.01 \sin\left(\frac{\pi(x - \ell_0)}{\ell_1 - \ell_0}\right), \quad \theta_1(x) = 0.01 \sin\left(\frac{2\pi(x - \ell_0)}{\ell_1 - \ell_0}\right), \quad x \in (\ell_0, \ell_1).$$

We do not impose all the strong transmission conditions defining $D(\mathcal{A})$, since the purpose of this example is to illustrate the dissipative behavior of the fully discrete scheme for finite-energy data.

For the Hermite approximation, the nodal values of u_0 and u_1 , as well as their derivatives, are used. In particular,

$$u'_0(x) = 0.12x(L - x), \quad u'_1(x) = 0.04x(L - x)(L - 2x).$$

The mesh is constructed by using 160 elements on each one of the five subintervals $[0, \xi_0]$, $[\xi_0, \ell_0]$, $[\ell_0, \xi_1]$, $[\xi_1, \ell_1]$, and $[\ell_1, L]$. Thus the displacement is approximated by 800 cubic Hermite elements on the whole beam. Since each Hermite node has two degrees of freedom, corresponding to u_h and $u_{h,x}$, and since the clamped conditions at $x = 0$ are imposed strongly, the number of mechanical degrees of freedom is $N_u = 2(800 + 1) - 2 = 1600$. The thermal variable is approximated by continuous P_1 finite elements only on the interval (ℓ_0, ℓ_1) . This interval contains 320 elements, and after imposing the homogeneous boundary conditions at ℓ_0 and ℓ_1 , the number of thermal degrees of freedom is $N_\theta = 321 - 2 = 319$. Therefore, the total number of degrees of freedom is

$$N_{\text{dof}} = N_u + N_\theta = 1919.$$

Figure 2 shows the normalized numerical energy $E_h(t)/E_h(0)$ in semilogarithmic scale. Since the exponential regime is not necessarily visible during the initial transient, the exponential fit is computed using the second half of the simulation. The estimated decay rate is

$$\omega_h \approx 0.00100337.$$

The almost linear behavior in semilogarithmic scale is consistent with the exponential stability predicted by the theoretical results.

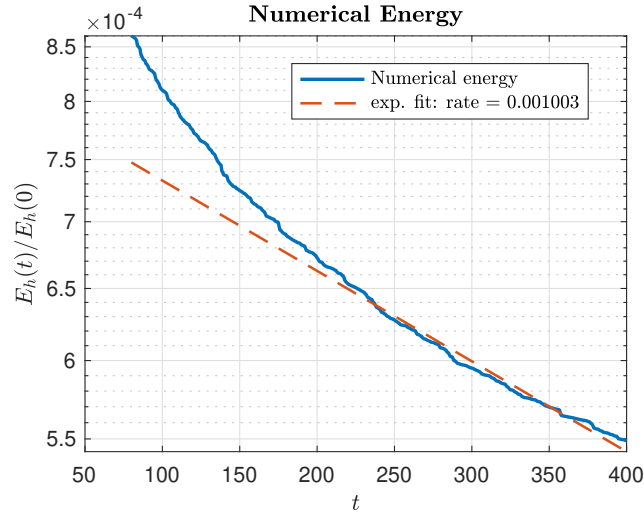
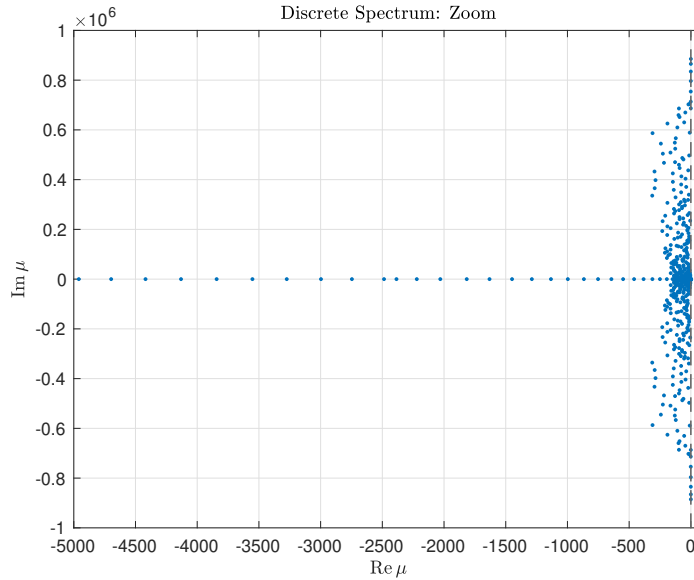


FIGURE 2. Numerical energy in semilogarithmic scale.

FIGURE 3. Discrete spectrum for $n_{\text{sub}} = 20$.

4.4. Discrete spectrum. We also compute the spectrum of the finite-dimensional system associated with the semidiscrete approximation. This experiment is intended only as a qualitative illustration of the location of the discrete eigenvalues and of their relation with the dissipative character of the model.

The semidiscrete system can be written as

$$(4.34) \quad \mathbb{M}\ddot{Y} + \mathbb{D}Y + \mathbb{A}Y = 0,$$

where $Y = (q, r)^T$, and where $\mathbb{M}$, $\mathbb{D}$, and $\mathbb{A}$ denote the mass, damping and stiffness matrices, respectively. Introducing:

$$(4.35) \quad Z = \begin{pmatrix} Y \\ \dot{Y} \end{pmatrix}, \quad \mathbb{G} = \begin{pmatrix} I & 0 \\ 0 & \mathbb{M} \end{pmatrix}, \quad \mathbb{H} = \begin{pmatrix} 0 & I \\ -\mathbb{A} & -\mathbb{D} \end{pmatrix}.$$

we obtain the generalized first-order system

$$(4.36) \quad \mathbb{G}\dot{Z} = \mathbb{H}Z,$$

Thus, the discrete eigenvalues are obtained from the generalized eigenvalue problem

$$\mathbb{H}Z = \mu \mathbb{G}Z.$$

For this spectral experiment we use a coarser mesh than in the previous time evolution example. More precisely, we take $n_{\text{sub}} = 20$, which gives $N_u = 200$, $N_\theta = 39$, $N_{\text{dof}} = 239$. Therefore, the first-order generalized eigenvalue problem has size $2N_{\text{dof}} = 478$. All eigenvalues are computed using the standard dense generalized eigenvalue solver.

The computed spectrum is shown in Figure 3. The eigenvalues are located in the left half-plane, consistently with the dissipative structure of the semidiscrete system. For this mesh, the largest real part of the discrete spectrum is

$$s(\mathcal{A}_h) := \max_{\mu_h \in \sigma(\mathcal{A}_h)} \text{Re } \mu_h \approx -5.651033 \times 10^{-2}.$$

Here $s(\mathcal{A}_h)$ denotes the spectral abscissa of the finite-dimensional discrete generator. Hence, a modal solution associated with μ_h behaves as $e^{\mu_h t}$, while the corresponding energy contribution behaves as $e^{2\text{Re } \mu_h t}$. This gives the spectral estimate for the energy decay rate

$$\omega_h^{\text{spec}} := -2s(\mathcal{A}_h) \approx 1.130207 \times 10^{-1}.$$

This quantity should not be confused with the essential spectrum of the continuous operator. The latter is an infinite-dimensional notion, whereas the present computation concerns only the point spectrum of the finite-dimensional semidiscrete generator. The relation between spectral bounds and semigroup growth is classical; see, for instance, [11, 13]. We also recall that for non-normal matrices the spectral abscissa describes the asymptotic modal behavior, but it may not fully characterize transient dynamics [14].

For finer meshes, the computation of the full spectrum becomes sensitive to high-frequency modes of the discretized pencil. Some of these modes may appear with small positive real part in the raw eigenvalue computation, although they are not consistent with the discrete energy identity. Since removing such modes would require additional filtering criteria, we restrict the present spectral plot to the coarser mesh $N_{\text{dof}} = 239$, and use it only as qualitative evidence of the spectral location in the left half-plane.

4.5. Discrete resolvent norm. We now compute the norm of the discrete resolvent along the imaginary axis. This experiment is motivated by the resolvent estimates used in the proof of Gevrey regularity. More precisely, for the localized thermoelastic case, the theoretical result is consistent with a resolvent behavior of the form

$$\|(i\lambda I - \mathcal{A})^{-1}\|_{\mathcal{H}} \leq C|\lambda|^{-1/2}, \quad |\lambda| \gg 1,$$

which corresponds to Gevrey regularity of class 2. The discrete generator is $\mathcal{A}_h = \mathbb{G}^{-1}\mathbb{H}$. For each $\lambda > 0$, the discrete resolvent is therefore given by

$$(i\lambda I - \mathcal{A}_h)^{-1} = (i\lambda \mathbb{G} - \mathbb{H})^{-1}\mathbb{G}.$$

The norm is computed in the discrete energy space. We introduce the energy matrix

$$\mathbb{W} = \begin{pmatrix} \mathbb{A} & 0 \\ 0 & \mathbb{M} \end{pmatrix},$$

so that $\|Z\|_{E_h}^2 = Z^*\mathbb{W}Z$. The resolvent norm induced by this energy is

$$\|(i\lambda I - \mathcal{A}_h)^{-1}\|_{E_h} = \left\| \mathbb{W}^{1/2}(i\lambda \mathbb{G} - \mathbb{H})^{-1}\mathbb{G}\mathbb{W}^{-1/2} \right\|_2.$$

Hence, for each frequency λ , we compute the largest singular value of the matrix

$$B_\lambda = \mathbb{W}^{1/2}(i\lambda \mathbb{G} - \mathbb{H})^{-1}\mathbb{G}\mathbb{W}^{-1/2}.$$

In the numerical implementation, this singular value is computed without forming B_λ explicitly. Instead, matrix-vector products with B_λ and B_λ^* are evaluated by solving the complex linear systems associated with $i\lambda \mathbb{G} - \mathbb{H}$, and the largest singular value is obtained by an iterative Lanczos bidiagonalization method; see, for instance, [2, 5].

The computation was performed for $\lambda \in [10^0, 10^5]$, using 1000 logarithmically spaced values and total DOF $N_{\text{DOF}} = 3839$. Figure 4 shows, on the left, the discrete energy norm of the resolvent, and, on the right, the scaled quantity

$$\lambda^{1/2}\|(i\lambda I - \mathcal{A}_h)^{-1}\|_{E_h}.$$

The left panel shows that the resolvent norm decreases on average as the frequency increases, although several peaks appear. These peaks are expected in a finite-dimensional frequency response computation, since they correspond to frequencies close to discrete spectral components of the semidiscrete system. The right panel shows that the scaled quantity

$$\lambda^{1/2}\|(i\lambda I - \mathcal{A}_h)^{-1}\|_{E_h}$$

remains essentially bounded over the computed range of frequencies, up to the oscillations produced by the discrete spectrum. This behavior is consistent with the estimate

$$\|(i\lambda I - \mathcal{A}_h)^{-1}\|_{E_h} \lesssim C\lambda^{-1/2},$$

which is the expected signature of Gevrey regularity of class 2 in the localized thermoelastic case.

We emphasize that this computation is not intended as a proof of the Gevrey regularity, since the semidiscrete system is finite dimensional. Nevertheless, it provides numerical evidence that

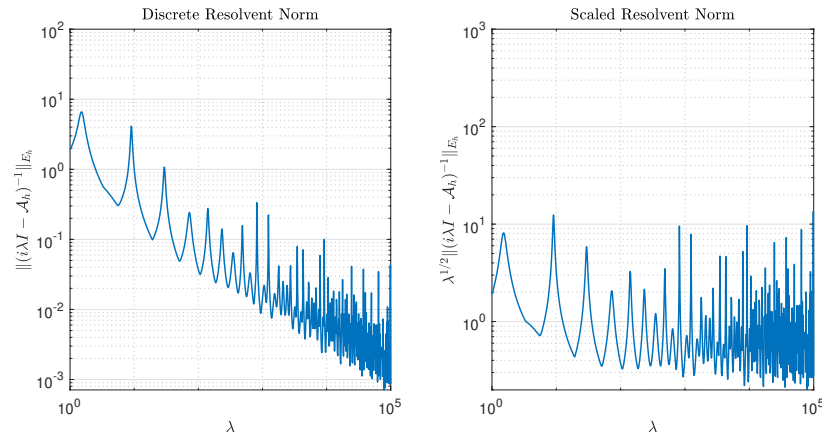


FIGURE 4. Discrete resolvent norm and scaled resolvent norm.

the frequency response of the discrete model is compatible with the resolvent behavior predicted by the continuous theory.

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CONFLICT OF INTEREST

This work does not have any conflicts of interest.

REFERENCES

- [1] Adams, R. A. (1975). *Sobolev Spaces*. New York: Academic Press.
- [2] Baglama, J., & Reichel, L. (2005). Augmented implicitly restarted Lanczos bidiagonalization methods. *SIAM Journal on Scientific Computing*, **27**(1), 19–42.
- [3] Brenner, S. C., & Scott, L. R. (2008). *The Mathematical Theory of Finite Element Methods*. Third edition. New York: Springer.
- [4] Ciarlet, P. G. (1978). *The Finite Element Method for Elliptic Problems*. Amsterdam: North-Holland.
- [5] Golub, G. H., & Van Loan, C. F. (2013). *Matrix Computations*. Fourth edition. Baltimore: Johns Hopkins University Press.
- [6] Gómez, G., Muñoz, J., & Ochoa, E. (2023). Gevrey semigroup of the type III localized thermoelastic model. *Journal of Differential Equations*, **347**, 282–309. <https://doi.org/10.1016/j.jde.2022.11.053>
- [7] Green, A. E., & Naghdi, P. M. (1991). A re-examination of the basic postulates of thermomechanics. *Proceedings of the Royal Society of London A*, **432**, 171–194. <https://doi.org/10.1098/rspa.1991.0012>
- [8] Green, A. E., & Naghdi, P. M. (1992). On undamped heat waves in an elastic solid. *Journal of Thermal Stresses*, **15**, 253–264. <https://doi.org/10.1080/01495739208946136>
- [9] Green, A. E., & Naghdi, P. M. (1995). A unified procedure for construction of theories of deformable media. I. Classical continuum physics, II. Generalized continua, III. Mixtures of interacting continua. *Proceedings of the Royal Society of London A*, **448**, 335–356, 357–377, 379–388. <https://doi.org/10.1098/rspa.1995.0022>
- [10] Hughes, T. J. R. (1987). *The Finite Element Method: Linear Static and Dynamic Finite Element Analysis*. Englewood Cliffs: Prentice-Hall.
- [11] Klaus-Jochen, E., & Nagel, R. (2000). *One-Parameter Semigroups for Linear Evolution Equations*. New York: Springer-Verlag. <https://doi.org/10.1007/b97696>
- [12] Newmark, N. M. (1959). A method of computation for structural dynamics. *Journal of the Engineering Mechanics Division, ASCE*, **85**(EM3), 67–94.
- [13] Pazy, A. (1983). *Semigroups of Linear Operators and Applications to Partial Differential Equations*. New York: Springer-Verlag. <https://doi.org/10.1007/978-1-4612-5561-1>
- [14] Trefethen, L. N., & Embree, M. (2005). *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*. Princeton: Princeton University Press.

- [15] Prüss Jan (1984). On the spectrum of C_0 -semigroups. *American Mathematical Society*, **284**, 847–857.
- [16] K. T. Andrews, M. Shillor, Vibrations of a Beam with a Damping Tip Body Math. Mathematical and Computer Modelling, 35,1033-1042,(2002).
- [17] K. T. Andrews, J. R. Fernández, M. Shillor, A thermoviscoelastic beam with a tip body. Computational Mechanics, 33, 225–234, (2004).
- [18] Liu, Kangsheng and Liu, Zhuangyi; Boundary stabilization of a nonhomogeneous beam with rotatory inertia at the tip, *Journal of Computational and Applied Mathematics Vol. 114, (1), pages 1- 10, (2000)*. doi:10.1016/S0377-0427(99)00284-8

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