UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

A network of conservation laws with discontinuous fluxes and source terms modeling countercurrent decantation

Juan Barajas-Calonge, Raimund Bürger, Stefan Diehl, Luis M. Villada

PREPRINT 2026-23

SERIE DE PRE-PUBLICACIONES

A NETWORK OF CONSERVATION LAWS WITH DISCONTINUOUS FLUXES AND SOURCE TERMS MODELING COUNTERCURRENT DECANTATION*

JUAN BARAJAS-CALONGE†, RAIMUND BÜRGER‡, STEFAN DIEHL§, AND LUIS MIGUEL VILLADA¶

Abstract. Countercurrent decantation (CCD) is an industrial separation process used to wash a slurry (solid-liquid suspension) by allowing the wash liquid and the slurry to flow in opposite directions through a series of clarifier-thickeners (CTs). The solid-liquid two-phase flow dynamics in a CCD circuit can be represented by a system of N CT models, where the effluent and underflow streams are fed into neighboring units in opposite directions. The slurry to be treated and the wash water are fed into the first and the Nth unit, respectively. The final CCD circuit model is formulated as a system of N weakly coupled, nonlinear first-order conservation laws with discontinuous flux and smooth source terms combined with N ordinary differential equations representing mixing tanks. The unknowns are the solids volume fraction in each CT, in the connecting pipes, and in the mixing tanks. An L^1-stability result for weak entropy solutions for the CCD model is established, which in particular implies uniqueness. It is demonstrated that a monotone finite difference scheme, based on the Engquist–Osher numerical flux, converges to the unique weak entropy solution. Finally, optimal steady states of the CCD circuit are characterized, and numerical simulations are presented.

Key words. Countercurrent decantation, network of conservation laws, solid-liquid separation, weak entropy solution, monotone difference scheme, compactness, well-posedness

MSC codes. 35L65, 65M06, 76T20

1. Introduction.

1.1. Scope. Countercurrent decantation (CCD) is an industrial separation process used to wash a slurry (solid-liquid suspension) by allowing the wash liquid and the slurry to flow in opposite directions through a series of continuously operated settling tanks, so-called clarifier-thickeners (CTs). Common applications of CCD include hydrometallurgy (to recover gold, copper, uranium, and other valuable minerals from leached ores); chemical processing (to remove impurities from chemical precipitates); and the food industry (to extract soluble materials from organic matter). The principle of countercurrent flow states that when water is to act upon solids, both are made to pass, in contact, in opposite directions, so that at each end the strongest or most potent portion of either is acting upon the weakest or most exhausted portion of the other. A CCD circuit consists of a sequence of N of CTs, also called stages, denoted CT_i, $i = 1, \dots, N$; see Fig. 1. In each stage, solids settle to the bottom and leave as underflow, which is fed into CT_{i+1}, while the clarified liquid leaves through the effluent and is fed into CT_{i-1}. The feed slurry enters CT_1, while clean wash water

*July 20, 2026

Funding: We acknowledge support by ANID (Chile) through Fondecyt project 1250676 (to R. B.) and Centro de Modelamiento Matemático (CMM) (BASAL project FB210005) (to R. B. and L. M. V.), and Becas Doctorado Nacional 2022, folio 21221387 (to J. B.-C.).

†GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and Departamento de Ciencias Naturales y Tecnología, Universidad de Aysén, Lillo 667, Coyhaique, Chile (juan.barajas2001@alumnos.ubiobio.cl).

‡CI²MA and Departamento de Ingeniería Matemática, Facultad de Ciencias Físicas y Matemáticas, Universidad de Concepción, Casilla 160-C, Concepción, Chile (rburger@ing-mat.udec.cl).

§Centre for Mathematical Sciences, Lund University, P.O. Box 118, S-221 00 Lund, Sweden (stefan.diehl@math.lth.se).

¶GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile and CI²MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile (lvillada@ubiobio.cl).

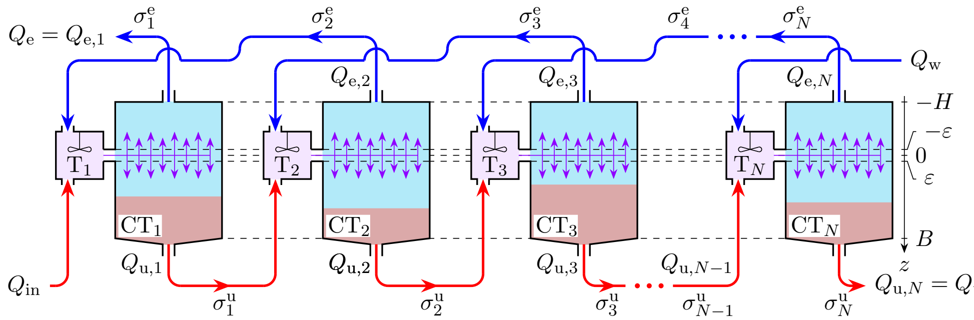

FIG. 1. *A CCD circuit composed of N clarifier-thickeners* CT_i *connected to mixing tanks* T_i. *Red arrows indicate the flow direction of slurry (solids), and blue arrows the wash-water flow.*

enters the final unit, CT_N, where the solids are already largely washed.

A well-studied mathematical model of a CT is given by a spatially one-dimensional nonlinear scalar conservation law with discontinuous flux and a possibly singular source term, where the unknown function is the solids volume fraction in the interior of the unit, the effluent, and the underflow pipes [6,7,9,11]. We herein model a CCD circuit based on N versions of the known CT model, modified to include effluent and underflow pipes of finite length to connect with neighboring CTs, a mixing tank that receives the incoming streams from neighboring units and converts them into the feed stream (see Fig. 2(a)), and a distributed feed source within each CT.

The governing model can be summarized as follows. We assume that the interior of each CT_i occupies the depth interval $z \in [-H, B]$ and has constant cross-sectional area A, that its effluent tube has length σ_i^{e} and cross-sectional area A_i^{e}, and that σ_i^{u} and A_i^{u} are the analogous constants for the underflow tube; see Fig. 1. Thus, each CT_i corresponds to the z-interval $\mathcal{I}_i := [-H - \sigma_i^{\mathrm{e}}, B + \sigma_i^{\mathrm{u}}]$ and a piecewise constant cross-sectional area $A_i = A_i(z)$. Furthermore, we assume that $Q_{\mathrm{u},i}(t)$ and $Q_{\mathrm{e},i}(t)$ are the underflow and effluent flows leaving CT_i, $q_{\mathrm{u},i}(t) := Q_{\mathrm{u},i}(t)/A$, $q_{\mathrm{e},i}(t) := Q_{\mathrm{e},i}(t)/A$, and that $b(\phi) = \phi v(\phi)$ is the non-negative batch hindered-settling function, where $v(\phi)$ is the hindered-settling velocity and ϕ is the solids volume fraction. We denote by $\phi_i(z,t)$ the solids volume fraction in CT_i and define the conserved variables $u_i(z,t) := A_i(z)\phi_i(z,t)$, $i = 1,\dots,N$, along with $\tilde{b}(u_i(z,t)) := b(u_i(z,t)/A_i(z))$. We denote by $\mu(z)$ a compactly supported smooth function of unit mass centered around $z = 0$ and by $h(z)$ its primitive, and define $S_i(z,t)$ to be the source term for CT_i (specified later), which depends on $Q_{\mathrm{e},i+1}(t)$ (or $Q_{\mathrm{w}}(t)$ for $i = N$) and on $Q_{\mathrm{u},i-1}(t)$ (or $Q_{\mathrm{in}}(t)$ for $i = 1$), and the corresponding concentrations $\phi_{\mathrm{u},i-1}(t) := \phi_{i-1}(B + \sigma_{i-1}^{\mathrm{u}}, t)$, etc., via the solution of an ordinary differential equation (ODE) representing the solids mass balance of mixing tank T_i. The governing PDE model can be expressed as a system of N conservation laws with discontinuous flux weakly coupled via source terms

(1.1)
$$\partial_t u_i + \partial_z\bigl(g_i(u_i,z,t)\bigr) = S_i(z,t), \quad z \in \mathcal{I}_i, \quad t > 0, \quad i = 1,\dots N, \quad \text{where}$$
$$g_i(u_i,z,t) := \begin{cases} -(Q_{\mathrm{e},i}(t)/A_i^{\mathrm{e}})u_i, & -H - \sigma_i^{\mathrm{e}} < z < -H, \\ \bigl((h(z)-1)q_{\mathrm{e},i}(t) + h(z)q_{\mathrm{u},i}(t)\bigr)u_i + A\tilde{b}(u_i), & -H < z < B, \\ (Q_{\mathrm{u},i}(t)/A_i^{\mathrm{u}})u_i, & B < z < B + \sigma_i^{\mathrm{u}}, \end{cases}$$

supplied with suitable initial conditions and control functions. Clearly, solutions of this initial value problem (IVP) will in general be discontinuous.

It is the purpose of the present work to introduce a concept of weak entropy solution to this IVP and to prove a stability result for weak entropy solutions that in particular implies uniqueness. Moreover, we develop a monotone finite difference scheme consistent with (1.1) and prove its convergence to the unique weak entropy solution, thereby establishing the well-posedness of the model. We also analyze steady states in a CCD circuit, which leads to sufficient conditions on the control functions for the existence of stationary solutions for so-called optimal operation of the CCD circuit. The difference scheme is applied to simulate the transient dynamics of the system, including the recovery of optimal steady-state operation after a perturbation.

1.2. Related work. Engineering descriptions of CCD circuits and their applications can be found in [12, 19, 20] and [21, Ch. 16] (see also the references therein). For the formulation, analysis, and numerical approximation of CT models governed by conservation laws with discontinuous flux and a singular feed source, as well as extensions to suspensions that form compressible sediments (leading to an additional degenerate diffusion term that is not considered herein), we refer to [6, 7, 9]. Our analysis also builds on results concerning the well-posedness and numerical analysis of conservation laws and related equations with fluxes that depend discontinuously on the spatial variable; which include [5,8,17,18]. The present analysis is particularly motivated by models of multilane vehicular traffic analyzed by Holden and Risebro [16] and Goatin and Rossi [14]. Their structure resembles that of (1.1). In both settings, N continua are considered in parallel (traffic lanes in [14, 16]), the evolution of the density in each continuum is governed by a scalar conservation law (the Lighthill–Whitham–Richards model in the traffic setting), and the continua are weakly coupled through source terms describing lane changes. The principal novelty of [14], compared with [16], is the inclusion of spatial discontinuities in both the speed law and the number of lanes. In both cases, the resulting model is a weakly coupled system of conservation laws similar to (1.1). However, in contrast to the models in [14, 16] we herein consider a collection of finite spatial domains (CT units) $\mathcal{I}_1, \dots, \mathcal{I}_N$, and the source term describing the transfer of solids from one unit to the other depends on the solution values at the endpoint of each domain. Thus, we need to ensure that these values are well defined as traces. This property is ensured since the governing model is linear on intervals of positive length adjacent to each of the endpoints of each interval $\mathcal{I}_i$, in conjunction with a technique of providing local spatial BV bounds of approximate solutions (originally introduced in [5]) that applies to z-intervals that avoid flux discontinuities, but that may include one of the endpoints of $\mathcal{I}_i$.

1.3. Outline of the paper. The remainder of the paper is organized as follows. In Sect. 2, we formulate the CCD model, discuss the choice of feed and underflow bulk flows as control variables, describe the mixing tank associated with each CT unit, state the assumptions on the model functions and initial data, introduce a notion of entropy weak solutions, and prove an L^1-stability result implying uniqueness. Sect. 3 presents a monotone finite-difference scheme for the governing IVP and proves its convergence to the unique entropy solution. In Sect. 4, we investigate stationary solutions of the CCD model. After reviewing the corresponding idealized model with a point source, we characterize optimal steady-state solutions for the smoothed model and for the complete CCD circuit. Sect. 5 presents two numerical examples for a CCD circuit with $N = 3$ CTs, illustrating the attainment and recovery of optimal steady-state operation. Finally, Sect. 6 contains some concluding remarks.

2. A model of a countercurrent decantation (CCD) circuit. We consider a CCD circuit consisting of $N \geq 2$ clarifier-thickeners $\mathrm{CT}_1, \dots, \mathrm{CT}_N$, each equipped with a stirred mixing tank $\mathrm{T}_1, \dots, \mathrm{T}_N$ at its feed inlet (see Fig. 1).

2.1. Bulk flows and control variables. The external feed flow Q_{in} and solids volume fraction ϕ_{in} are prescribed. The wash water flow Q_{w} and the underflows $Q_{\mathrm{u},1}, \dots, Q_{\mathrm{u},N}$ are considered to be $N+1$ control variables. The wash water solids volume fraction is usually $\phi_{\mathrm{w}} = 0$, but may be nonzero if the wash water is recycled. We assume that all volumetric flows are non-negative and time dependent. We assume conservation of volume across each CT, so that the feed, effluent, and underflow satisfy

$$Q_{\mathrm{f},i} = Q_{\mathrm{e},i} + Q_{\mathrm{u},i}, \quad i = 1, \dots, N. \tag{2.1}$$

We define the auxiliary variables of inflow $Q_{\mathrm{u},0} \coloneqq Q_{\mathrm{in}}$ and $Q_{\mathrm{e},N+1} \coloneqq Q_{\mathrm{w}}$, and outflow $Q_{\mathrm{u}} \coloneqq Q_{\mathrm{u},N}$ and $Q_{\mathrm{e}} \coloneqq Q_{\mathrm{e},1}$, and use analogous notation for the corresponding solids volume fractions. The volume conservation at each feed inlet is expressed as

$$Q_{\mathrm{f},i} = Q_{\mathrm{u},i-1} + Q_{\mathrm{e},i+1}, \quad i = 1, \dots, N. \tag{2.2}$$

For simplicity, we assume that all CTs are identical and have the constant cross-sectional area A (however, the pipe lengths and diameters may be different). Each volumetric flow into and out of CT_i is then related to a bulk velocity $q_{\mathrm{f},i} \coloneqq Q_{\mathrm{f},i}/A$, etc. The $2N$ equations (2.1) and (2.2) uniquely determine the $2N$ unknown feed and effluent bulk velocities $q_{\mathrm{f},i}$ and $q_{\mathrm{e},i}$, $i = 1, \dots, N$, for prescribed input flows and control variables. Equations (2.1) and (2.2) imply $q_{\mathrm{e},i} = q_{\mathrm{e},i+1} + q_{\mathrm{u},i-1} - q_{\mathrm{u},i}$ for $i = 1, \dots, N$. Solving the recursion backward from CT_N, we obtain the effluent bulk velocities, which we require to be nonnegative:

$$\begin{aligned} q_{\mathrm{e},N} &= q_{\mathrm{e},N+1} + q_{\mathrm{u},N-1} - q_{\mathrm{u},N} = q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},N-1} \geq 0, \\ q_{\mathrm{e},N-1} &= (q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},N-1}) + q_{\mathrm{u},N-2} - q_{\mathrm{u},N-1} = q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},N-2} \geq 0, \quad \dots \\ q_{\mathrm{e},1} &= q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},0} = q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{in}} \geq 0. \end{aligned} \tag{2.3}$$

The last equality immediately yields global volume conservation.

Lemma 2.1. *Global volume conservation holds:* $q_{\mathrm{w}} + q_{\mathrm{in}} = q_{\mathrm{e}} + q_{\mathrm{u}}$.

For the nonnegative feed bulk velocities, we obtain the explicit expressions

$$\begin{aligned} q_{\mathrm{f},1} &= q_{\mathrm{u},1} + q_{\mathrm{e},1} = q_{\mathrm{u},1} + q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{in}}, \\ q_{\mathrm{f},2} &= q_{\mathrm{u},2} + q_{\mathrm{e},2} = q_{\mathrm{u},2} + q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},1}, \quad \dots \\ q_{\mathrm{f},N} &= q_{\mathrm{u},N} + q_{\mathrm{e},N} = q_{\mathrm{u}} + q_{\mathrm{w}} - q_{\mathrm{u}} + q_{\mathrm{u},N-1}. \end{aligned} \tag{2.4}$$

2.2. CCD model in final form. We recall that each CT_i is associated with the cross-sectional area

$$A_i(z) = \begin{cases} A_i^{\mathrm{e}} & \text{for } -H - \sigma_i^{\mathrm{e}} < z < -H, \\ A & \text{for } -H \leq z \leq B, \\ A_i^{\mathrm{u}} & \text{for } B < z < B + \sigma_i^{\mathrm{u}}. \end{cases}$$

We assume that $\max_{i,z} A_i(z) = A$. The feed volume fraction $\phi_{\mathrm{f},i}(t)$ to CT_i depends on the effluents and underflows of neighboring CTs, as will be detailed below. Each

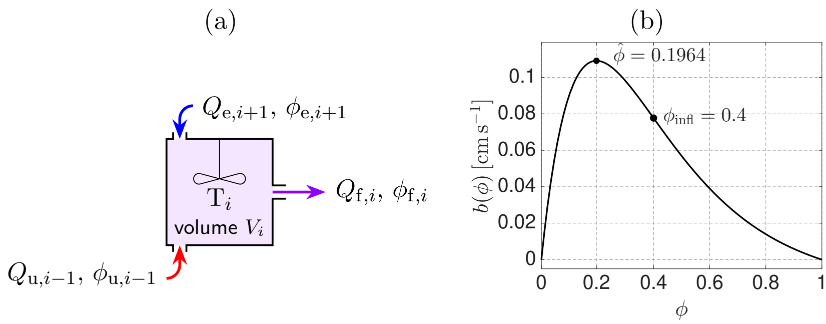

FIG. 2. *(a) Completely mixed tank* T_i *with volume* V_i *whose outlet stream is fed into* CT_i*, (b) function* $b(\phi) = \phi v(\phi) = v_{\mathrm{St}}\phi(\mathrm{e}^{-r_{\mathrm{V}}\phi} - \mathrm{e}^{r_{\mathrm{V}}\phi_{\max}})/(1-\mathrm{e}^{-r_{\mathrm{V}}\phi_{\max}})$*, where* $v_{\mathrm{St}} = 1.5\,\mathrm{cm\,s^{-1}}$ *is the Stokes velocity,* $r_{\mathrm{V}} = 5$ *is a dimensionless constant, and* $\phi_{\max} = 1.0$ *is the maximum concentration. The maximum point* $\hat{\phi}$ *and the inflection point* ϕ_{infl} *are marked. We set* $b(\phi) := 0$ *outside* $[0,1]$.

CT_i is modeled by the conservation law with discontinuous flux and source term

(2.5)
$$\partial_t\bigl(A_i(z)\phi_i\bigr) + \partial_z\bigl(\tilde{g}_i(\phi_i,z,t)\bigr) = \mu(z)Q_{\mathrm{f},i}(t)\phi_{\mathrm{f},i}(t), \quad z \in \mathcal{I}_i, \quad t>0,$$
$$\tilde{g}_i(\phi_i,z,t) := \begin{cases} -Q_{\mathrm{e},i}(t)\phi_i, & -H-\sigma_i^{\mathrm{e}} < z < -H, \\ \bigl((h(z)-1)Q_{\mathrm{e},i}(t) + h(z)Q_{\mathrm{u},i}(t)\bigr)\phi_i + Ab(\phi_i), & -H < z < B, \\ Q_{\mathrm{u},i}(t)\phi_i, & B < z < B + \sigma_i^{\mathrm{u}}, \end{cases}$$

where $\mu \in C_{\mathrm{c}}^1(\mathbb{R})$ is assumed to satisfy $\mu \geq 0$, $\operatorname{supp}\mu \subset (-\varepsilon,\varepsilon)$ for some $\varepsilon \in (0,\min\{B,H\})$, and $\int_{\mathbb{R}}\mu(z)\,\mathrm{d}z = 1$. Moreover, we define $h(z) := \int_{-\infty}^z \mu(\zeta)\,\mathrm{d}\zeta$.

To model the coupling between the units we denote by $\phi_{\mathrm{e},i}(t) := \phi_i(-H-\sigma_i^{\mathrm{e}},t)$ and $\phi_{\mathrm{u},i}(t) := \phi_i(B+\sigma_i^{\mathrm{u}},t)$ the effluent and underflow volume fractions of CT_i at time t, measured at the end of the corresponding tubes. For CT_i we assume that the streams coming from the neighboring units enter a completely mixed tank T_i with volume V_i whose outlet stream with solid volume fraction $\phi_{\mathrm{f},i}(t)$ is fed into CT_i; see Fig. 2(a). We denote the solid volume per time unit entering T_i by

$$\mathcal{M}_i(t) := \phi_{\mathrm{u},i-1}(t)Q_{\mathrm{u},i-1}(t) + \phi_{\mathrm{e},i+1}(t)Q_{\mathrm{e},i+1}(t), \quad i = 1,\dots,N. \tag{2.6}$$

Then $\phi_{\mathrm{f},i}(t)$, $i=1,\dots,N$, is the solution to the ODE

$$V_i\frac{\mathrm{d}\phi_{\mathrm{f},i}}{\mathrm{d}t} = \mathcal{M}_i(t) - \phi_{\mathrm{f},i}(t)Q_{\mathrm{f},i}(t), \quad t>0, \tag{2.7}$$

given the initial datum $\phi_{\mathrm{f},i}(0)$, $i=1,\dots,N$. The solution is

$$\phi_{\mathrm{f},i}(t) = \frac{1}{E_i(t)}\left(\phi_{\mathrm{f},i}(0) + \frac{1}{V_i}\int_0^t E_i(\tau)\mathcal{M}_i(\tau)\,\mathrm{d}\tau\right), \; E_i(t) := \exp\left(\int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau/V_i\right).$$

Besides the conserved variables $u_i(z,t) = A_i(z)\phi_i(z,t)$, $i=1,\dots,N$ (cf. Sect. 1.1), we also define $u_{\mathrm{f},i}(t) := A\phi_{\mathrm{f},i}(t)$, $u_{\mathrm{w}}(t) := A\phi_{\mathrm{w}}(t)$ and $S_i(z,t) := \mu(z)q_{\mathrm{f},i}(t)u_{\mathrm{f},i}(t)$. Then we obtain the final governing model, which is a coupled ODE-PDE system consisting of N PDEs (1.1) coupled with the N ODEs (2.7) supplied with the initial conditions

$$\begin{aligned} u_i(z,0) = u_i^0(z) = A_i(z)\phi_i(z,0) = A_i(z)\phi_i^0(z), \quad z \in \mathcal{I}_i; \\ u_{\mathrm{f},i}(0) = u_{\mathrm{f},i}^0 = A\phi_{\mathrm{f},i}^0, \quad i = 1,\dots,N. \end{aligned} \tag{2.8}$$

2.3. Assumptions. We assume that $\phi_i^0(z) = u_i^0(z)/A_i(z)$ satisfies $\phi_i^0 \in L^1(\mathcal{I}_i) \cap L^\infty(\mathcal{I}_i;[0,1]) \cap BV(\mathcal{I}_i)$ for all $i = 1,\dots,N$, and that $\phi_{\mathrm{in}}, \phi_{\mathrm{w}} \in L^\infty([0,T];[0,1]) \cap L^1([0,T]) \cap BV([0,T])$ and $Q_{\mathrm{in}}, Q_{\mathrm{w}}, Q_{\mathrm{u},1},\dots,Q_{\mathrm{u},N} \in L^\infty([0,T];\mathbb{R}_0^+) \cap L^1([0,T]) \cap BV([0,T])$. (Analogous properties hold for u_{in}, u_{w}, u_i^0, $q_{\mathrm{u},i}$ for $i = 1,\dots,N$, etc.) For simplicity, we set $\phi_{\max} = 1$. Furthermore, we require that the constitutive function satisfies $b(\phi) = 0$ for $\phi \le 0$ and $\phi \ge 1$, $b(\phi) > 0$ for $0 < \phi < 1$, that there exists $\hat\phi \in (0,1)$ such that $b'(\phi) > 0$ for $0 < \phi < \hat\phi$, $b'(\hat\phi) = 0$, that $b'(\phi) < 0$ for $\hat\phi < \phi < 1$, and that b has an inflection point $\phi_{\mathrm{infl}} \in (0,1)$. A sample function $b(\phi)$, which will also be used for the numerical examples of Sect. 5, is provided in Fig. 2(b).

2.4. Weak entropy solution. Motivated by [14, Def. 1.2] we define a weak entropy solution of (1.1), (2.8), where $\Pi_{i,T} \coloneqq \mathcal{I}_i \times [0,T]$ for $T > 0$, $i = 1,\dots,N$.

Definition 2.1 (Weak entropy solution). *The functions $u_i = A_i(z)\phi_i$ with $\phi_i \in L^\infty(\Pi_{i,T};[0,1])$ for $i = 1,\dots,N$, are said to form a* weak entropy solution *to the IVP (1.1), (2.8) if the following conditions are satisfied:*

(i) The following (trace) functions exist and assume values in $[0,1]$:

$$\begin{aligned} &\phi_{\mathrm{u},i}(t) = \lim_{z\to(B+\sigma_i^{\mathrm{u}})^-} \phi_i(z,t) = \lim_{z\to(B+\sigma_i^{\mathrm{u}})^-} u_i(z,t)/A_i^{\mathrm{u}}, \quad i = 1,\dots,N-1,\\ \text{and}\quad &\phi_{\mathrm{e},i}(t) = \lim_{z\to(-H-\sigma_i^{\mathrm{e}})^+} \phi_i(z,t) = \lim_{z\to(-H-\sigma_i^{\mathrm{e}})^+} u_i(z,t)/A_i^{\mathrm{e}}, \quad i = 2,\dots,N,\end{aligned}$$

Moreover, they have bounded variation, i.e., there exists a constant C_1 such that $\mathrm{TV}_{[0,T]}\,\phi_{\mathrm{u},i} \le C_1$ for $i = 1,\dots,N-1$ and $\mathrm{TV}_{[0,T]}\,\phi_{\mathrm{e},i} \le C_1$ for $i = 2,\dots,N$.

(ii) For any test function $\varphi_i \in C_{\mathrm{c}}^1([0,T) \times \mathcal{I}_i;\mathbb{R})$ there holds

$$\int_0^T \int_{\mathcal{I}_i} \bigl(u_i\partial_t\varphi_i + g_i(u_i,z,t)\partial_z\varphi_i + S_i(z,t)\varphi_i\bigr)\,\mathrm{d}z\,\mathrm{d}t + \int_{\mathcal{I}_i} u_i^0(z)\varphi_i(z,0)\,\mathrm{d}z = 0. \tag{2.9}$$

(iii) For any test function $\varphi_i \in C_{\mathrm{c}}^1([0,T) \times \mathcal{I}_i;\mathbb{R}^+)$ and any $k \in \mathbb{R}$ there holds

$$\begin{aligned}&\int_0^T \int_{\mathcal{I}_i} \Bigl(|u_i - k|\partial_t\varphi_i + \operatorname{sgn}(u_i - k)\bigl((g_i(u_i,z,t) - g_i(k,z,t))\partial_z\varphi_i + S_i(z,t)\varphi\bigr)\Bigr)\,\mathrm{d}z\,\mathrm{d}t\\ &+ \int_{\mathcal{I}_i} |u_i^0 - k|\varphi_i(z,0)\,\mathrm{d}z + \int_0^T \sum_{\zeta\in\{-H,B\}} \bigl|g_i(k,\zeta^+,t) - g_i(k,\zeta^-,t)\bigr|\varphi_i(\zeta,t)\,\mathrm{d}t \ge 0.\end{aligned}$$

2.5. Stability and uniqueness of weak entropy solutions. The following theorem ensures that solutions to the IVP (1.1), (2.8) depend L^1 Lipschitz continuously on the initial data; hence, they are unique.

Theorem 2.1. *Assume that $\boldsymbol{u} \coloneqq (u_1,\dots,u_N)$ and $\boldsymbol{v} \coloneqq (v_1,\dots,v_N)$ are weak entropy solutions to the IVP (1.1), (2.8) with the respective initial functions u_i^0 and v_i^0, $i = 1,\dots,N$. We also denote by $\phi_{\mathrm{f},i}$, ϕ_{in} and ϕ_{w} the feed functions associated with $\boldsymbol{u}$, and by $\psi_{\mathrm{f},i}$, ψ_{in} and ψ_{w} those associated with $\boldsymbol{v}$. Then we have the following properties.*

(i) There holds an ordering principle in the following sense: if $\phi_{\mathrm{f},i}(0) \le \psi_{\mathrm{f},i}(0)$, and for all $s \in [0,T]$, $\phi_{\mathrm{in}}(s) \le \psi_{\mathrm{in}}(s)$ and $\phi_{\mathrm{w}}(s) \le \psi_{\mathrm{w}}(s)$; and in addition $u_i(z,0) \le v_i(z,0)$ for all z, then $u_i(z,t) \le v_i(z,t)$ for all $(z,t) \in \Pi_{i,T}$, and all $i = 1,\dots,N$.

(ii) Let $V \coloneqq \min\{V_1,\dots,V_N\}$ *and* $\hat{Q}_{\mathrm{f}}(t) \coloneqq \max\{Q_{\mathrm{f},1}(t),\dots,Q_{\mathrm{f},N}(t)\}$*. Then*

$$
\begin{aligned}
&\sum_{i=1}^{N}\big\|u_i(\cdot,t)-v_i(\cdot,t)\big\|_{L^1(\mathcal{I}_i)} \leq \exp\biggl(\int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V}\,\mathrm{d}\tau\biggr)\times \\
&\times\Biggl(\sum_{i=1}^{N}\big\|u_i(\cdot,0)-v_i(\cdot,0)\big\|_{L^1(\mathcal{I}_i)} + \sum_{i=1}^{N}\big|\phi_{\mathrm{f},i}(0)-\psi_{\mathrm{f},i}(0)\big|\int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V}\int_0^\tau \Bigl(Q_{\mathrm{in}}(s)\big|\phi_{\mathrm{in}}(s)-\psi_{\mathrm{in}}(s)\big| + Q_{\mathrm{w}}(s)\big|\phi_{\mathrm{w}}(s)-\psi_{\mathrm{w}}(s)\big|\Bigr)\,\mathrm{d}s\,\mathrm{d}\tau\Biggr)
\end{aligned}
\tag{2.10}
$$

for a.e. $t \in [0,T]$*. In particular, if we choose* $u_i(z,0)=v_i(z,0)$ *for almost all* $z\in\mathcal{I}_i$*,* $\phi_{\mathrm{f},i}(0)=\psi_{\mathrm{f},i}(0)$*, and* $\phi_{\mathrm{in}}(s)=\psi_{\mathrm{in}}(s)$ *and* $\phi_{\mathrm{w}}(s)=\psi_{\mathrm{w}}(s)$ *for almost all* $s\in[0,T]$*, we obtain uniqueness of an entropy weak solution.*

Before proving Theorem 2.1 we recall an auxiliary result. If $\omega \in C_0^\infty$ satisfies $\omega \geq 0$, $\operatorname{supp}\omega \subset (-1,1)$, and $\|\omega\|_{L^1(\mathbb{R})}=1$, then $\omega_h(z) \coloneqq \omega(z/h)/h$ is a standard mollifier with support in $(-h,h)$, $h>0$. For $h>0$ and $a<b$ we then define $\rho_h(z) \coloneqq \int_{-\infty}^z \omega_h(\xi)\,\mathrm{d}\xi$, $\mu_{h,a}(z) \coloneqq 1-\rho_h(z-(a+2h))$, and $\nu_{h,b}(z) \coloneqq \rho_h(z-(b-2h))$. Then $|\mu'_{h,a}(z)|, |\nu'_{h,b}(z)| \leq C_\mu/h$ with a constant C_μ that is independent of h. Moreover, these functions have the following well-known property (cf., e.g., [4]):

Proposition 2.1. *Let* $v \in L^1((0,T);L^\infty(a,b))$*. If the traces* $\gamma_a v \coloneqq (\gamma v)(a,t)$ *and* $\gamma_b v \coloneqq (\gamma v)(b,t)$ *exist a.e. on* $(0,T)$*, then for* $\varphi \in C^\infty((a,b)\times(0,T))$ *there holds*

$$
\int_a^b\int_0^T \partial_z\bigl(\varphi(z,t)(\mu_{h,a}(z)+\nu_{h,b}(z))\bigr)v(z,t)\,\mathrm{d}t\,\mathrm{d}z \xrightarrow{h\downarrow 0} \int_0^T\bigl(\varphi(b,t)\gamma_b v-\varphi(a,t)\gamma_a v\bigr)\,\mathrm{d}t.
$$

Proof of Theorem 2.1. We partially follow the proof of [14, Th. 2.9], which in turn is partially based on treatments in [8,16,18]. We fix $i\in\{1,\dots,N\}$. Then, following [18, Th. A.1 and (2.22)] we deduce that for any $\varphi_i \in C_{\mathrm{c}}^1((0,T)\times\mathcal{I}_i\backslash\{-H,B\};\mathbb{R}^+)$,

$$
\begin{aligned}
-\int_0^T\int_{\mathcal{I}_i}\Bigl(&|u_i-v_i|\partial_t\varphi_i + \operatorname{sgn}(u_i-v_i)\bigl(g_i(u_i,z,t)-g_i(v_i,z,t)\bigr)\partial_z\varphi_i \\
&+\operatorname{sgn}(u_i-v_i)\bigl(S_i(z,t)-\tilde{S}_i(z,t)\bigr)\varphi_i\Bigr)\,\mathrm{d}z\,\mathrm{d}t \leq 0,
\end{aligned}
\tag{2.11}
$$

where $S_i(z,t)$ and $\tilde{S}_i(z,t)$ denote the source terms pertaining to $\boldsymbol{u}$ and $\boldsymbol{v}$, respectively. Noticing that u_i and v_i satisfy item (ii) in Definition 2.1 and following [16, Th. 3.3], we subtract from (2.11) equation (2.9) for u_i and add (2.9) for v_i to obtain

$$
\begin{aligned}
-\int_0^T\int_{\mathcal{I}_i}\Bigl(&(u_i-v_i)^+\partial_t\varphi_i + H(u_i-v_i)\bigl(g_i(u_i,z,t)-g_i(v_i,z,t)\bigr)\partial_z\varphi_i \\
&+H(u_i-v_i)\bigl(S_i(z,t)-\tilde{S}_i(z,t)\bigr)\varphi_i\Bigr)\,\mathrm{d}z\,\mathrm{d}t \leq 0
\end{aligned}
\tag{2.12}
$$

for any $\varphi_i \in C_{\mathrm{c}}^1((0,T)\times\mathcal{I}_i\backslash\{-H,B\};\mathbb{R}^+)$, and where $a^+ \coloneqq \max\{a,0\}$ and $a^- \coloneqq \min\{a,0\}$. Following a procedure similar to that of [18, Th. 2.1] we may extend (2.12) to $\Phi_i \in C^1_{\mathrm{c}}(\mathcal{I}_i \times (0,T);\mathbb{R}^+)$ and obtain

$$\begin{aligned}
&-\int_0^T \int_{\mathcal{I}_i} \Big((u_i - v_i)^+ \partial_t \Phi_i + H(u_i - v_i)\big(g_i(u_i,z,t) - g_i(v_i,z,t)\big)\partial_z \Phi_i \\
&\qquad + H(u_i - v_i)\big(S_i(z,t) - \tilde{S}_i(z,t)\big)\Phi_i \Big)\,\mathrm{d}z\,\mathrm{d}t \\
&\leq E := \int_0^T \sum_{\zeta \in \{-H,B\}} \big[H(u_i - v_i)\big(g_i(u_i,z,t) - g_i(v_i,z,t)\big)\big]_{z=\zeta^-}^{z=\zeta^+} \Phi_i(\zeta,t)\,\mathrm{d}t
\end{aligned} \tag{2.13}$$

for all $\Phi_i \in C^1_{\mathrm{c}}(\mathcal{I}_i \times (0,T);\mathbb{R}^+)$. Notice that since $A_i^{\mathrm{e}} < A$ and $A_i^{\mathrm{u}} < A$, the fluxes

$$g_i\big(u,(-H)^-,t\big) = -\big(Q_{\mathrm{e},i}(t)/A_i^{\mathrm{e}}\big)u \quad \text{and} \quad g_i\big(u,(-H)^+,t\big) = -\big(Q_{\mathrm{e},i}(t)/A\big)u + A\tilde{b}(u)$$

defined by (1.1) and adjacent to $\zeta = -H$ satisfy the "crossing condition" (cf. [6,18]), which states that if $f(u)$ and $g(u)$ denote the fluxes to the right and to the left adjacent to a discontinuity (considering, e.g., $u_t + (H(x)f(u) + (1 - H(x))g(u))_x = 0$), then these fluxes either do not cross or if they do, then the graph of $g(u)$ lies above that of $f(u)$ to the left of any intersection point. Similarly, the fluxes

$$g_i(u,B^-,t) = \big(Q_{\mathrm{u},i}(t)/A\big)u + A\tilde{b}(u) \quad \text{and} \quad g_i(u,B^+,t) = \big(Q_{\mathrm{u},i}(t)/A_i^{\mathrm{u}}\big)u$$

adjacent to $\zeta = B$ also satisfy the crossing condition, which implies that $E \leq 0$ (cf., e.g., [6, Th. 2.1]). We again follow [16, Th. 3.3] and for $\tau \in (0,T]$ and sufficiently small h choose $\Phi = \chi_{h,[0,\tau]}(t)(1 - \mu_h(z) - \nu_h(z))$, where $\chi_{h,[0,\tau]} \approx \chi_{[0,\tau]}$. Since

$$\begin{aligned}
&-\int_0^T \int_{\mathcal{I}_i} H(u_i - v_i)\big(g_i(u_i,z,t) - g_i(v_i,z,t)\big)\partial_z\big(1 - \mu_h(z) - \nu_h(z)\big)\tilde{\chi}_{h,[0,\tau]}\,\mathrm{d}z\,\mathrm{d}t \\
&\xrightarrow{h\downarrow 0} \int_0^\tau \left(\frac{Q_{\mathrm{u},i}(t)}{A_i^{\mathrm{u}}}\big(u_i^{\mathrm{u}}(t) - v_i^{\mathrm{u}}(t)\big)^+ + \frac{Q_{\mathrm{e},i}(t)}{A_i^{\mathrm{e}}}\big(u_i^{\mathrm{e}}(t) - v_i^{\mathrm{e}}(t)\big)^+ \right)\mathrm{d}t =: \tilde{\Theta}_i(\tau),
\end{aligned}$$

we deduce from (2.13) and $E \leq 0$ that $\Theta_i(t) := \int_{\mathcal{I}_i} (u_i(z,t) - v_i(z,t))^+ \,\mathrm{d}z + \tilde{\Theta}_i(t)$ and $R_i(t) := \int_0^t \int_{\mathcal{I}_i} H(u_i - v_i)(S_i(z,\tau) - \tilde{S}_i(z,\tau))\,\mathrm{d}z\,\mathrm{d}\tau$ satisfy the inequality

$$\Theta_i(t) \leq \int_{\mathcal{I}_i} \big(u_i(z,0) - v_i(z,0)\big)^+ \,\mathrm{d}z + R_i(t) = \Theta_i(0) + R_i(t). \tag{2.14}$$

Clearly, considering that always $E_i(\tau) \geq 1$, we obtain

$$\begin{aligned}
R_i(t) &\leq \int_0^t \int_{\mathcal{I}_i} \big(S_i(z,\tau) - \tilde{S}_i(z,\tau)\big)^+ \,\mathrm{d}z\,\mathrm{d}\tau = \int_0^t Q_{\mathrm{f},i}(\tau)\big(\phi_{\mathrm{f},i}(\tau) - \psi_{\mathrm{f},i}(\tau)\big)^+ \,\mathrm{d}\tau \\
&\leq \int_0^t Q_{\mathrm{f},i}(\tau)\bigg(\big(\phi_{\mathrm{f},i}(0) - \psi_{\mathrm{f},i}(0)\big)^+ + \frac{1}{V_i}\int_0^\tau \bigg(\frac{Q_{\mathrm{u},i-1}(s)}{A_{i-1}^{\mathrm{u}}}\big(u_{i-1}^{\mathrm{u}}(s) - v_{i-1}^{\mathrm{u}}(s)\big)^+ \\
&\qquad + \frac{Q_{\mathrm{e},i+1}(s)}{A_{i+1}^{\mathrm{e}}}\big(u_{i+1}^{\mathrm{e}}(s) - v_{i+1}^{\mathrm{e}}(s)\big)^+ \bigg)\,\mathrm{d}s\bigg)\,\mathrm{d}\tau.
\end{aligned}$$

Let us now define $\Theta(t) := \Theta_1(t) + \dots + \Theta_N(t)$ and $R(t) := R_1(t) + \dots + R_N(t)$. Then

$$R(t) \leq \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \int_0^\tau \sum_{i=1}^N \bigg(\frac{Q_{\mathrm{u},i-1}(s)}{A_{i-1}^{\mathrm{u}}}\big(u_{i-1}^{\mathrm{u}}(s) - v_{i-1}^{\mathrm{u}}(s)\big)^+$$

$$
\begin{aligned}
&\qquad + \frac{Q_{\mathrm{e},i+1}(s)}{A^{\mathrm{e}}_{i+1}} \bigl(u^{\mathrm{e}}_{i+1}(s) - v^{\mathrm{e}}_{i+1}(s)\bigr)^+\Bigr) \,\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \sum_{i=1}^{N} \bigl(\phi_{\mathrm{f},i}(0) - \psi_{\mathrm{f},i}(0)\bigr)^+ \int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau \\
&\le \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \int_0^\tau \sum_{i=1}^{N} \biggl(\frac{Q_{\mathrm{u},i}(s)}{A^{\mathrm{u}}_i} \bigl(u^{\mathrm{u}}_i(s) - v^{\mathrm{u}}_i(s)\bigr)^+ + \frac{Q_{\mathrm{e},i}(s)}{A^{\mathrm{e}}_i} \bigl(u^{\mathrm{e}}_i(s) - v^{\mathrm{e}}_i(s)\bigr)^+ \biggr) \,\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \int_0^\tau \Bigl(Q_{\mathrm{in}}(s)\bigl(\phi_{\mathrm{in}}(s) - \psi_{\mathrm{in}}(s)\bigr)^+ + Q_{\mathrm{w}}(s)\bigl(\phi_{\mathrm{w}}(s) - \psi_{\mathrm{w}}(s)\bigr)^+ \Bigr) \,\mathrm{d}s\,\mathrm{d}\tau \\
&\quad + \sum_{i=1}^{N} \bigl(\phi_{\mathrm{f},i}(0) - \psi_{\mathrm{f},i}(0)\bigr)^+ \int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau.
\end{aligned}
$$

Thus, summing (2.14) from $i = 1$ to N, we arrive at the inequality

$$
\begin{aligned}
\Theta(t) &\le \Theta(0) + \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \Theta(\tau)\,\mathrm{d}\tau + \sum_{i=1}^{N} \bigl(\phi_{\mathrm{f},i}(0) - \psi_{\mathrm{f},i}(0)\bigr)^+ \int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \int_0^\tau \Bigl(Q_{\mathrm{in}}(s)\bigl(\phi_{\mathrm{in}}(s) - \psi_{\mathrm{in}}(s)\bigr)^+ + Q_{\mathrm{w}}(s)\bigl(\phi_{\mathrm{w}}(s) - \psi_{\mathrm{w}}(s)\bigr)^+ \Bigr) \,\mathrm{d}s\,\mathrm{d}\tau.
\end{aligned}
$$

By Grönwall's inequality we obtain

(2.15)

$$
\begin{aligned}
\Theta(t) &\le \exp\biggl(\int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V}\,\mathrm{d}\tau \biggr) \biggl(\Theta(0) + \sum_{i=1}^{N} \bigl(\phi_{\mathrm{f},i}(0) - \psi_{\mathrm{f},i}(0)\bigr)^+ \int_0^t Q_{\mathrm{f},i}(\tau)\,\mathrm{d}\tau \\
&\quad + \int_0^t \frac{\hat{Q}_{\mathrm{f}}(\tau)}{V} \int_0^\tau \Bigl(Q_{\mathrm{in}}(s)\bigl(\phi_{\mathrm{in}}(s) - \psi_{\mathrm{in}}(s)\bigr)^+ + Q_{\mathrm{w}}(s)\bigl(\phi_{\mathrm{w}}(s) - \psi_{\mathrm{w}}(s)\bigr)^+ \Bigr) \,\mathrm{d}s\,\mathrm{d}\tau \biggr).
\end{aligned}
$$

If the feed control functions associated with $\boldsymbol{u}$ and $\boldsymbol{v}$ are ordered in the sense stated in item (i), then (2.15) reduces to $\Theta(t) \le \Theta(0) \exp\bigl(\int_0^t (\hat{Q}_{\mathrm{f}}(\tau)/V)\mathrm{d}\tau\bigr)$. Consequently, if in addition $u_i(z, 0) \le v_i(z, 0)$ for all z and all $i = 1, \dots, N$, then $\Theta(0) = 0$, and we get $\Theta(t) = 0$ for all $t \in [0, T]$, i.e., $u_i(z, t) \le v_i(z, t)$ for all z and t. This concludes the proof of item (i). Finally, there holds a second version of (2.15) with $\boldsymbol{u}$ and $\boldsymbol{v}$ interchanged. Suppose that $\hat{\Theta}(t)$ is the version of $\Theta(t)$ for that case. Summing (2.15) and its analog for $\hat{\Theta}(t)$ and noticing that $\sum_{i=1}^{N} \|u_i(\cdot, 0) - v_i(\cdot, 0)\|_{L^1(\mathcal{I}_i)} = \Theta(0) + \hat{\Theta}(0)$ as well as $\sum_{i=1}^{N} \|u_i(\cdot, t) - v_i(\cdot, t)\|_{L^1(\mathcal{I}_i)} \le \Theta(t) + \hat{\Theta}(t)$ we obtain (2.10). ◻

3. Numerical method. We discretize all intervals $\mathcal{I}_i$ by cells $I_j := [z_{j-\frac{1}{2}}, z_{j+\frac{1}{2}})$, $j \in \mathbb{Z}$, with cell interfaces $z_{j\mp\frac{1}{2}} := (j \mp \frac{1}{2})\Delta z$ and centers $z_j := j\Delta z$. Similarly, the time interval $[0, T]$ is discretized as $t_n = n\Delta t$ for $n = 0, \dots, M$, where $M = \lfloor T/\Delta t \rfloor + 1$. We take a spatial meshwidth Δz and a time step Δt, subject to the CFL condition

$$
\text{(CFL)} \qquad \Delta t \max\biggl(\frac{2}{\Delta z} \max_{1 \le i \le N} \biggl\{ \frac{\|Q_{\mathrm{e},i}\|_\infty}{A^{\mathrm{e}}_i}, \frac{\|Q_{\mathrm{u},i}\|_\infty}{A^{\mathrm{u}}_i} \biggr\} + \frac{\|\tilde{b}'\|_\infty}{\Delta z}, \max_{1 \le i \le N} \frac{\|Q_{\mathrm{f},i}\|_\infty}{2V_i} \biggr) \le \frac{1}{2}
$$

In the remainder of the present Sect. 3, it is always assumed that (CFL) is in effect and that $\lambda = \Delta t/\Delta z$ is kept constant.

3.1. Discretization of the CCD model. The points $z=-H$ and $z=B$ are assumed belong to one cell $I_{j_{-H}}$ and I_{j_B}, respectively. Similarly, the endpoints of the pipes at $z=-H-\sigma_i^{\mathrm{e}}$ and $z=B+\sigma_i^{\mathrm{u}}$ are located within $I_{j_{\mathrm{e},i}}$ and $I_{j_{\mathrm{u},i}}$, respectively, where $j_{\mathrm{e},i}<j_{-H}$ and $j_{\mathrm{u},i}>j_B$. We discretize the initial condition (2.8) by

$$u_{i,j}^0 := \frac{1}{\Delta z}\int_{z_{j-\frac12}}^{z_{j+\frac12}} u_i^0(z)\,\mathrm{d}z, \quad j\in\mathcal{J}_i := \{j_{\mathrm{e},i},\dots,j_{\mathrm{u},i}\}, \quad i=1,\dots,N; \tag{3.1}$$

the approximate values $u_{i,j}^n$ for $n=1,\dots,M$ are understood as analogous cell averages. We discretize given time-dependent functions by $u_{\mathrm{f}}^n := \frac{1}{\Delta t}\int_{t_n}^{t_{n+1}} u_{\mathrm{f}}(t)\,\mathrm{d}t$, etc., and all z-dependent parameters on a grid that is staggered with respect to that of $u_{i,j}^n$, following, e.g., [6,7,17]). Thus, we define $j_\varepsilon := \lceil \varepsilon/\Delta z\rceil$ and $\mu_j := \frac{1}{\Delta z}\int_{I_j}\mu(z)\,\mathrm{d}z$, where we notice that $\mu_j=0$ if $j<-j_\varepsilon$ or $j>j_\varepsilon$ and $\mu_j\le\|\mu\|_\infty$ for all j, along with

$$h_{j+\frac12} := h(z_{j+\frac12}) = \int_{-H}^{z_{j+\frac12}} \mu(z)\,\mathrm{d}z = \Delta z(\mu_{j_{-H}}+\dots+\mu_j), \tag{3.2}$$

and introduce the parameters

$$\gamma_{i,j+\frac12}^{1,n} := h_{j+\frac12}(q_{\mathrm{e},i}^n+q_{\mathrm{u},i}^n), \quad \gamma_{j+\frac12}^2 := \begin{cases} A & \text{if } j=j_{-H},\dots,j_B,\\ 0 & \text{otherwise,}\end{cases} \tag{3.3}$$

$$\gamma_{i,j+\frac12}^{3,n} := \begin{cases} -Q_{\mathrm{e},i}^n/A_i^{\mathrm{e}}, & j=j_{\mathrm{e},i}-1,\dots,j_{-H}-1,\\ -q_{\mathrm{e},i}^n, & j=j_{-H},\dots,j_B,\\ Q_{\mathrm{u},i}^n/A_i^{\mathrm{u}}-q_{\mathrm{e},i}^n-q_{\mathrm{u},i}^n, & j=j_B+1,\dots,j_{\mathrm{u},i},\end{cases} \tag{3.4}$$

Finally, we discretize the ODE (2.7) by a first-order explicit Euler step, i.e.,

$$u_{\mathrm{f},i}^n = \begin{cases} u_{\mathrm{f},i}^0, & n=0,\\ u_{\mathrm{f},i}^{n-1}+\Delta t(A\mathcal{M}_i^{n-1}-u_{\mathrm{f},i}^{n-1}Q_{\mathrm{f},i}^{n-1})/V_i, & n=1,\dots,M,\end{cases} \tag{3.5}$$

and use $u_{\mathrm{f},i}^n$ to define the approximate source term

$$S_{i,j}^n := \mu_j u_{\mathrm{f},i}^n q_{\mathrm{f},i}^n = (h_{j+\frac12}-h_{j-\frac12})u_{\mathrm{f},i}^n q_{\mathrm{f},i}^n/\Delta z. \tag{3.6}$$

3.2. Marching formula. The numerical flux is a combination of the upwind flux for the linear part and the Engquist–Osher flux for the nonlinear portion involving $\tilde b(u)$. To define it we introduce the Engquist–Osher numerical flux [13]

$$\mathcal{B}(\gamma^2,\gamma^3;u,v) := \int_0^u \max\bigl\{0,\gamma^3+\gamma^2\tilde b'(s)\bigr\}\,\mathrm{d}s + \int_0^v \min\bigl\{0,\gamma^3+\gamma^2\tilde b'(s)\bigr\}\,\mathrm{d}s,$$

where we recall that $\tilde b(0)=0$. Considering that $\gamma_{i,j+\frac12}^{1,n}\ge 0$, we introduce

$$\mathcal{F}_{i,j+\frac12}^n(u,v) := \gamma_{i,j+\frac12}^{1,n}u + \mathcal{B}\bigl(\gamma_{j+\frac12}^2,\gamma_{i,j+\frac12}^{3,n};u,v\bigr). \tag{3.7}$$

Collecting the ingredients, recalling the difference operators acting on the spatial index $\Delta_\pm u_{i,j} := \pm(u_{i,j\pm1}-u_{i,j})$, and defining $\lambda := \Delta t/\Delta z$, we obtain the scheme

$$\begin{aligned} u_{i,j}^{n+1} &= u_{i,j}^n - \lambda\Delta_-\mathcal{F}_{i,j+\frac12}^n\bigl(u_{i,j}^n,u_{i,j+1}^n\bigr)+\Delta t\,S_{i,j}^n\\ &=: \mathcal{H}_{i,j}^n(\boldsymbol{u}^n,\boldsymbol{u}^{n-1}), \quad j\in\mathcal{J}_i, \quad i=1,\dots,N, \quad n=0,\dots,M-1.\end{aligned} \tag{3.8}$$

where $\boldsymbol{u}^n := \{u_{i,j}^n : j \in \mathcal{J}_i, i = 1,\dots,N\}$, and $u_{\mathrm{f},i}^n$ in (3.6) is given by (3.5). (Note that $\mathcal{H}_{i,j}^0$ depends on $\boldsymbol{u}^0$ only.) Here it is assumed that the coupling conditions (2.1) and (2.2) are in effect. In view of the regularity assumptions in Sect. 2.3 on ϕ_i^0 (equivalently, u_i^0) there exists a constant C_2 independent of $(\Delta z, \Delta t)$ such that

$$\sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \bigl|\lambda\Delta_-\mathcal{F}_{i,j+\frac{1}{2}}^0\bigl(u_{i,j}^0, u_{i,j+1}^0\bigr) - \Delta t\, S_{i,j}^0\bigr| \leq C_2 \quad \text{for all } i = 1,\dots,N.$$

The piecewise constant approximate solutions $u_1^\Delta,\dots,u_N^\Delta$ are given by

$$u_i^\Delta(z,t) := \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \sum_{n=0}^{M-1} \chi_{I_j}(z)\chi_{[t_n,t_{n+1})}(t)u_{i,j}^n, \quad i = 1,\dots,N, \quad (z,t) \in \Pi_{i,T}, \tag{3.9}$$

where χ_Ω refers to the characteristic function of the set Ω. To go back to the original independent volume fractions, we first define

$$A_{i,j} := \begin{cases} A_i^{\mathrm{e}}, & j = j_{\mathrm{e},i} - 1,\dots,j_{-H} - 1, \\ A, & j = j_{-H},\dots,j_B, \\ A_i^{\mathrm{u}}, & j = j_B + 1,\dots,j_{\mathrm{u},i} \end{cases} \qquad j \in \mathcal{J}_i, \quad i = 1,\dots,N.$$

then $\phi_{i,j}^n := u_{i,j}^n / A_{i,j}$, and finally the piecewise constant approximate solution $\phi_i^\Delta(z,t)$ in analogy with (3.9).

THEOREM 3.1. *The scheme is conservative in the following sense. We define the volume of solids, the so-called inventory, contained in the discrete version of* CT_i *(including the tank* T_i*) at time* t_n *by* $\mathcal{S}_i^n := \sum_{j\in\mathcal{J}_i} \Delta z A_{i,j}\phi_{i,j}^n + V_i\phi_{\mathrm{f},i}^n$*, and the inventory of the whole CCD circuit by* $\mathcal{S}^n := \mathcal{S}_1^n + \dots + \mathcal{S}_N^n$*. Then*

$$\mathcal{S}^{n+1} - \mathcal{S}^n = \Delta t\bigl(Q_{\mathrm{w}}^n\phi_{\mathrm{w}}^n - Q_{\mathrm{e}}^n\phi_{\mathrm{e}}^n + Q_{\mathrm{in}}^n\phi_{\mathrm{in}}^n - Q_{\mathrm{u}}^n\phi_{\mathrm{u}}^n\bigr). \tag{3.10}$$

Proof. Using that $\sum_{j\in\mathcal{J}_i}\mu_j = \frac{1}{\Delta z}$ along with (3.5) and (3.6) we get

$$\begin{aligned} \mathcal{S}_i^{n+1} - \mathcal{S}_i^n &= -\Delta t\mathcal{F}_{i,j_{\mathrm{u},i}+\frac{1}{2}}^n\bigl(u_{j_{\mathrm{u},i}}^n, u_{j_{\mathrm{u},i}+1}^n\bigr) + \Delta t\mathcal{F}_{i,j_{\mathrm{e},i}-\frac{1}{2}}^n\bigl(u_{j_{\mathrm{e},i}-1}^n, u_{j_{\mathrm{e},i}}^n\bigr) \\ &\quad + \Delta z\Delta t\sum_{j\in\mathcal{J}_i}\mu_j u_{\mathrm{f},i}^n q_{\mathrm{f},i}^n + V_i\bigl(\phi_{\mathrm{f},i}^{n+1} - \phi_{\mathrm{f},i}^n\bigr) \\ &= \Delta t\bigl(-Q_{\mathrm{u},i}^n\phi_{j_{\mathrm{u},i}}^n - Q_{\mathrm{e},i}^n\phi_{j_{\mathrm{e},i}}^n + Q_{\mathrm{f},i}^n\phi_{\mathrm{f},i}^n + \mathcal{M}_i^n - Q_{\mathrm{f},i}^n\phi_{\mathrm{f},i}^n\bigr) \\ &= \Delta t\bigl(-Q_{\mathrm{u},i}^n\phi_{j_{\mathrm{u},i}}^n - Q_{\mathrm{e},i}^n\phi_{j_{\mathrm{e},i}}^n + Q_{\mathrm{u},i-1}^n\phi_{j_{\mathrm{u},i-1}}^n + Q_{\mathrm{e},i+1}^n\phi_{j_{\mathrm{e},i+1}}^n\bigr). \end{aligned}$$

Summing the last identity from $i = 1$ to N and applying cancellations, we obtain

$$\mathcal{S}^{n+1} - \mathcal{S}^n = \Delta t\bigl(Q_{\mathrm{e},N+1}^n\phi_{\mathrm{e},N+1}^n - Q_{\mathrm{e},1}^n\phi_{\mathrm{e},1}^n + Q_{\mathrm{u},0}^n\phi_{\mathrm{u},0}^n - Q_{\mathrm{u},N}^n\phi_{\mathrm{u},N}^n\bigr),$$

which implies (3.10) in light of the conventions in Sect. 2.1. ∎

3.3. Convergence of the scheme. In what follows, we derive a series of properties and compactness estimates of the scheme (3.8) that will eventually allow us to appeal to standard arguments to prove convergence of approximate solutions to the unique entropy solution of the IVP (1.1), (2.8).

LEMMA 3.1. *The scheme* (3.8) *is monotone, i.e.,*

$$\partial u_{i,j}^{n+1}/\partial u_{l,k}^n \geq 0 \quad \text{for all } i,l = 1,\dots,N;\ j \in \mathcal{J}_i;\ \text{and } k \in \mathcal{J}_l. \tag{3.11}$$

Proof. The source term $S^n_{i,j}$ (cf. (3.5), (3.6)) is a linear combination of components of $\boldsymbol{u}^{n-1}$ with nonnegative coefficients if $1 - \Delta t\, Q^{n-1}_{\mathrm{f},i}/V_i \geq 0$, which follows from (CFL); hence, $\partial S^n_{i,j}/\partial u^m_{l,k} \geq 0$ for all indices listed in (3.11). Thus, to prove monotonicity of (3.8), it suffices to recall that if (CFL) holds, then for all $i = 1,\dots,N$ the scheme $\eta^{n+1}_{i,j} = \eta^n_{i,j} - \lambda\Delta_- \mathcal{F}^n_{i,j+\frac12}(\eta^n_{i,j}, \eta^n_{i,j+1})$ is monotone since both the upwind and Engquist–Osher numerical fluxes in (3.7) are increasing in u and decreasing in v. ∎

Lemma 3.2. *The scheme* (3.8) *satisfies an invariant region principle, i.e., if* $0 \leq \phi^m_{i,j} = u^m_{i,j}/A_{i,j} \leq 1$ *for all* $i = 1,\dots,N$, $j \in \mathcal{J}_i$, *and* $m = n-1, n$ *(if* $n \geq 1$, *otherwise* $m = 0$*), then* $0 \leq \phi^{n+1}_{i,j} = u^{n+1}_{i,j}/A_{i,j} \leq 1$ *for all* $i = 1,\dots,N$ *and* $j \in \mathcal{J}_i$.

Proof. We denote by $\mathbf{0}$ and $\boldsymbol{A}$ vectors of the same format as $\boldsymbol{u}^n$ with all entries equal to 0 and $A_{i,j}$, respectively. By the assumptions on $b(\phi)$ there holds $\mathcal{H}^n_{i,j}(\mathbf{0},\mathbf{0}) = 0$ for all i, j. Furthermore, $\mathcal{B}_{i,j+1/2}(\gamma^2,\gamma^3; A_{i,j}, A_{i,j}) = \gamma^{3,n}_{i,j+1/2} A_{i,j}$ and (2.1) imply

$$\mathcal{F}^n_{i,j+\frac12}(A_{i,j}, A_{i,j}) = \begin{cases} -Q^n_{\mathrm{e},i}, & j = j_{\mathrm{e},i}-1,\dots,j_{-H}-1, \\ h_{j+\frac12} Q^n_{\mathrm{f},i} - Q^n_{\mathrm{e},i}, & j = j_{-H},\dots,j_B, \\ Q^n_{\mathrm{u},i}, & j = j_B+1,\dots,j_{\mathrm{u},i}, \end{cases}$$

hence $\mathcal{F}^n_{i,j+\frac12}(A_{i,j}, A_{i,j}) = h_{j+\frac12} Q^n_{\mathrm{f},i} - Q^n_{\mathrm{e},i}$ for all $j \in \mathcal{J}_i$. In light of (3.2), (3.5), (2.2), and Lemma 3.1, we now obtain for $j \in \mathcal{J}_i$, $i = 1,\dots,N$,

$$\begin{aligned} 0 = \mathcal{H}_{i,j}(\mathbf{0},\mathbf{0}) &\leq u^{n+1}_{i,j} = \mathcal{H}_{i,j}(\boldsymbol{u}^n, \boldsymbol{u}^{n-1}) \leq \mathcal{H}_{i,j}(\boldsymbol{A},\boldsymbol{A}) \\ &= A_{i,j} - \lambda(h_{j+\frac12} - h_{j-\frac12}) Q^n_{\mathrm{f},i} \\ &\quad + \Delta t\, \mu_j q^n_{\mathrm{f},i}\big(A + (\Delta t/V_i)\big(A(Q^{n-1}_{\mathrm{u},i-1} + Q^{n-1}_{\mathrm{e},i+1}) - A Q^{n-1}_{\mathrm{f},i}\big)\big) \\ &= A_{i,j} - \Delta t \mu_j Q^n_{\mathrm{f},i} + \Delta t \mu_j q^n_{\mathrm{f},i}(A + 0) = A_{i,j}. \end{aligned}$$ ∎

For simplicity, from now on we assume that $Q_{\mathrm{u},i}$, $Q_{\mathrm{e},i}$, $Q_{\mathrm{f},i}$, Q_{in}, and Q_{w} are constant in time. The proofs in the remainder of this section can be adapted to the case of time-dependent Q-flows at the cost of additional terms involving time differences of Q-terms, which we will not pursue herein.

Lemma 3.3 (L^1 Lipschitz continuity in time). *There exists a constant* C_3 *that does not depend on* Δt *or* Δz *such that*

$$\Delta z \sum_{j\in\mathcal{J}_i} \big|u^{n+1}_{i,j} - u^n_{i,j}\big| \leq C_3 \Delta t \quad \text{for all } i = 1,\dots,N. \tag{3.12}$$

Proof. We define $\theta^{n-\frac12}_{i,j} := u^n_{i,j} - u^{n-1}_{i,j}$ for $i = 1,\dots,N$, $j \in \mathcal{J}_i$, and $n = 1,\dots,M$, and now fix $i \in \{1,\dots,N\}$. Subtracting from (3.8) the scheme for $u^n_{i,j}$ and defining

$$\begin{aligned} b^{n-\frac12}_{i,j+1} &:= \int_0^1 \big(\gamma^2_{i,j+1/2} + \gamma^3_{j+1/2}\tilde{b}'\big)^- \big(\theta u^n_{i,j+1} + (1-\theta) u^{n-1}_{i,j+1}\big)\,\mathrm{d}\theta \leq 0, \\ c^{n-\frac12}_{i,j} &:= \gamma^1_{i,j+1/2} + \int_0^1 \big(\gamma^2_{i,j+1/2} + \gamma^3_{j+1/2}\tilde{b}'\big)^+ \big(\theta u^n_{i,j} + (1-\theta) u^{n-1}_{i,j}\big)\,\mathrm{d}\theta \geq 0, \end{aligned}$$

we obtain

$$\begin{aligned} \theta^{n+\frac12}_{i,j} &= \theta^{n-\frac12}_{i,j} - \lambda c^{n-\frac12}_{i,j}\theta^{n-\frac12}_{i,j} + \lambda c^{n-\frac12}_{i,j-1}\theta^{n-\frac12}_{i,j-1} + \lambda b^{n-\frac12}_{i,j}\theta^{n-\frac12}_{i,j} - \lambda b^{n-\frac12}_{i,j+1}\theta^{n-\frac12}_{i,j+1} \\ &\quad + \Delta t\big(S^n_{i,j} - S^{n-1}_{i,j}\big) \\ &= \big\{1 - \lambda c^{n-\frac12}_{i,j} + \lambda b^{n-\frac12}_{i,j}\big\}\theta^{n-\frac12}_{i,j} + \lambda c^{n-\frac12}_{i,j-1}\theta^{n-\frac12}_{i,j-1} - \lambda b^{n-\frac12}_{i,j+1}\theta^{n-\frac12}_{i,j+1} \end{aligned}$$

$$+ \Delta t\big(S_{i,j}^{n} - S_{i,j}^{n-1}\big)$$

for all $j \in \mathcal{J}_i$. Since $\{\dots\} \geq 0$ due to (CFL), we obtain

$$\begin{aligned}\big|\theta_{i,j}^{n+\frac{1}{2}}\big| \leq \big(1 - \lambda c_{i,j}^{n-\frac{1}{2}} + \lambda b_{i,j}^{n-\frac{1}{2}}\big)\big|\theta_{i,j}^{n-\frac{1}{2}}\big| + \lambda c_{i,j-1}^{n-\frac{1}{2}}\big|\theta_{i,j-1}^{n-\frac{1}{2}}\big| - \lambda b_{i,j+1}^{n-\frac{1}{2}}\big|\theta_{i,j+1}^{n-\frac{1}{2}}\big| \\ + \Delta t\big|S_{i,j}^{n} - S_{i,j}^{n-1}\big| \quad \text{for all } j \in \mathcal{J}_i.\end{aligned} \tag{3.13}$$

Moreover, in view of (3.5) and (3.6) there exists a constant $C_4 \geq 0$ such that

$$\big|S_{i,j}^{n} - S_{i,j}^{n-1}\big| = \Delta t\, \mu_j \frac{q_{\mathrm{f},i}A}{V_i}\big|Q_{\mathrm{u},i-1}^{n-1}\phi_{\mathrm{u},i-1}^{n-1} + Q_{\mathrm{e},i+1}^{n-1}\phi_{\mathrm{e},i+1}^{n-1} - Q_{\mathrm{f},i}^{n-1}\phi_{\mathrm{f},i}^{n-1}\big| \leq C_4\mu_j\Delta t$$

Consequently, summing (3.13) from $j = j_{\mathrm{e},i}$ to $j = j_{\mathrm{u},i}$ and then taking into account that $c_{i,j_{\mathrm{e},i}-1}^{n-\frac{1}{2}} = 0$ and $b_{i,j_{\mathrm{u},i}+1}^{n-\frac{1}{2}} = 0$ as well as $\mu_{-j_\varepsilon} + \dots + \mu_{j_\varepsilon} = 1/\Delta z$, we get

$$\begin{aligned}\sum_{j\in\mathcal{J}_i}\big|\theta_{i,j}^{n+\frac{1}{2}}\big| &\leq \sum_{j\in\mathcal{J}_i}\big|\theta_{i,j}^{n-\frac{1}{2}}\big| - \lambda c_{i,j_{\mathrm{u},i}}^{n-\frac{1}{2}}\big|\theta_{i,j_{\mathrm{u},i}}^{n-\frac{1}{2}}\big| + \lambda b_{i,j_{\mathrm{e},i}}^{n-\frac{1}{2}}\big|\theta_{i,j_{\mathrm{e},i}}^{n-\frac{1}{2}}\big| + C_4\lambda\Delta t \\ &\leq \sum_{j\in\mathcal{J}_i}\big|\theta_{i,j}^{n-\frac{1}{2}}\big| + C_4\lambda\Delta t \leq \sum_{j\in\mathcal{J}_i}\big|\theta_{i,j}^{n-\frac{3}{2}}\big| + C_4\lambda 2\Delta t \leq \cdots \\ &\leq \sum_{j\in\mathcal{J}_i}\big|\theta_{i,j}^{\frac{1}{2}}\big| + C_4\lambda n\Delta t \leq C_5 + C_4\lambda T =: C_6.\end{aligned}$$

This proves (3.12) if we take $C_3 = C_6/\lambda$. ∎

LEMMA 3.4. *Let $\Sigma := \{-H, B\}$. For any family of intervals $[a_i, b_i] \subset [-H - \sigma_i^{\mathrm{e}}, B + \sigma_i^{\mathrm{u}}]$ such that $[a_i, b_i] \cap \Sigma = \varnothing$ for all $i = 1, \dots, N$ let us fix $r > 0$ such that $2r < \operatorname{dist}(\Sigma, [a_i, b_i])$ for all $i = 1, \dots, N$, and without loss of generality we assume that $r > \Delta z$ for all Δz of interest. Then for any $i = 1, \dots, N$ and $n = 1, \dots, M$ the following estimate holds, where the constant C_7 is independent of Δt and Δz:*

$$\sum_{i=1}^{N}\sum_{j\in J_{a,b,i}}\big|u_{i,j+1}^{n} - u_{i,j}^{n}\big| \leq C_7, \quad \text{where } J_{a,b,i} := \{j \in \mathcal{J}_i \mid a \leq z_j \leq b\}.$$

Proof. We fix $i \in \{1, \dots, N\}$, select $[a, b] := [a_i, b_i]$ and define the cell-index sets

$$\mathcal{A}_{\Delta,i} := \big\{j \in \mathcal{J}_i \mid z_j \in [a - r - \Delta z, a]\big\} \quad \text{and} \quad \mathcal{B}_{\Delta,i} := \big\{j \in \mathcal{J}_i \mid z_j \in [b, b + r + \Delta z]\big\}.$$

By construction, for all i each set $\mathcal{A}_{\Delta,i}$ and $\mathcal{B}_{\Delta,i}$ has at least two elements, and $|\mathcal{A}_{\Delta,i}|\Delta z \geq r$, $|\mathcal{B}_{\Delta,i}|\Delta z \geq r$. Moreover, Lemma 3.3 implies that there exists a constant K such that for all $i = 1, \dots, N$ there holds

$$\Delta z \sum_{j\in\mathcal{A}_{\Delta,i}}\sum_{n=0}^{M-1}\big|\theta_{i,j}^{n+\frac{1}{2}}\big| \leq K, \quad \Delta z \sum_{j\in\mathcal{B}_{\Delta,i}}\sum_{n=0}^{M-1}\big|\theta_{i,j}^{n+\frac{1}{2}}\big| \leq K. \tag{3.14}$$

For all $i = 1, \dots, N$ we now choose $j_{a,i} \in \mathcal{A}_{\Delta,i}$ and $j_{b,i}$ with $j_{b,i} + 1 \in \mathcal{B}_{\Delta,i}$ such that

$$\sum_{n=0}^{M-1}\big|\theta_{i,j_{a,i}}^{n+\frac{1}{2}}\big| = \min_{j\in\mathcal{A}_{\Delta,i}}\sum_{n=0}^{M-1}\big|\theta_{i,j}^{n+\frac{1}{2}}\big| \quad \text{and} \quad \sum_{n=0}^{M-1}\big|\theta_{i,j_{b,i}+1}^{n+\frac{1}{2}}\big| = \min_{j\in\mathcal{B}_{\Delta,i}}\sum_{n=0}^{M-1}\big|\theta_{i,j}^{n+\frac{1}{2}}\big|.$$

Consequently, (3.14) implies that

$$\sum_{n=0}^{M-1} \left|\theta_{i,j_{a,i}}^{n+\frac{1}{2}}\right| \leq \frac{K}{|\mathcal{A}_{\Delta,i}|\Delta z} \leq \frac{K}{r} \quad \text{and} \quad \sum_{n=0}^{M-1} \left|\theta_{i,j_{b,i}+1}^{n+\frac{1}{2}}\right| \leq \frac{K}{|\mathcal{B}_{\Delta,i}|\Delta z} \leq \frac{K}{r}. \tag{3.15}$$

The choice of $j_{a,i}$ and $j_{b,i}$ implies that for all $j_{a,i} \leq j \leq j_{b,i}$, the parameters $\gamma^2_{j\pm\frac{1}{2}}$ and $\gamma^3_{i,j\pm\frac{1}{2}}$ belong to the same case of (3.3) and of (3.4), respectively. For these j, and in view of (3.6), we may rewrite (3.8) in the form

$$u_{i,j}^{n+1} = u_{i,j}^n + C_{i,j+\frac{1}{2}}^n \Delta_+ u_{i,j}^n - D_{i,j-\frac{1}{2}}^n \Delta_- u_{i,j}^n, \tag{3.16}$$

where for all $i = 1, \dots, N$ we define $C_{i,j+\frac{1}{2}}^n := 0$ if $\Delta_+ u_{i,j}^n = 0$ and

$$C_{i,j+\frac{1}{2}}^n := \lambda\big(g_i\big(u_{i,j}^n, z_{j+\frac{1}{2}}, t_n\big) - \mathcal{F}_{i,j+\frac{1}{2}}\big(u_{i,j}^n, u_{i,j+1}^n\big) + u_{\mathrm{f},i}^n q_{\mathrm{f},i} h_{j+\frac{1}{2}}\big)/\Delta_+ u_{i,j}^n$$

otherwise, and $D_{i,j-\frac{1}{2}}^n := 0$ if $\Delta_- u_{i,j}^n = 0$ and

$$D_{i,j-\frac{1}{2}}^n := \lambda\big(g_i\big(u_{i,j}^n, z_{j-\frac{1}{2}}, t_n\big) - \mathcal{F}_{i,j-\frac{1}{2}}\big(u_{i,j-1}^n, u_{i,j}^n\big) + u_{\mathrm{f},i}^n q_{\mathrm{f},i} h_{j-\frac{1}{2}}\big)/\Delta_- u_{i,j}^n$$

otherwise. The monotonicity of the scheme (3.8) and (CFL) imply that

$$C_{i,j+\frac{1}{2}}^n \geq 0, \quad D_{i,j+\frac{1}{2}}^n \geq 0, \quad C_{i,j+\frac{1}{2}}^n + D_{i,j+\frac{1}{2}}^n \leq 1. \tag{3.17}$$

According to the incremental form (3.16), we get

$$\begin{aligned} \Delta_+ u_{i,j}^{n+1} = \Delta_+ u_{i,j}^n &+ C_{j+\frac{3}{2}}^{i,n} \Delta_+ u_{i,j+1}^n - C_{j+\frac{1}{2}}^{i,n} \Delta_+ u_{i,j}^n \\ &- D_{j+\frac{1}{2}}^{i,n} \Delta_+ u_{i,j}^n + D_{j-\frac{1}{2}}^{i,n} \Delta_- u_{i,j}^n; \end{aligned} \tag{3.18}$$

notice that for $j = j_{a,i}$ and $j = j_{b,i}$ we obtain the respective equations

$$\begin{aligned} \Delta_+ u_{i,j_{a,i}}^{n+1} &= \Delta_+ u_{i,j_{a,i}}^n + C_{j_{a,i}+\frac{3}{2}}^{i,n} \Delta_+ u_{i,j_{a,i}+1}^n - D_{i,j_{a,i}+\frac{1}{2}}^n \Delta_+ u_{i,j_{a,i}}^n - \big(u_{i,j_{a,i}}^{n+1} - u_{i,j_{a,i}}^n\big), \\ \Delta_+ u_{i,j_{b,i}}^{n+1} &= \Delta_+ u_{i,j_{b,i}}^n - C_{j_{b,i}+\frac{1}{2}}^{i,n} \Delta_+ u_{i,j_{b,i}}^n + D_{j_{b,i}-\frac{1}{2}}^{i,n} \Delta_- u_{i,j_{b,i}}^n + \big(u_{i,j_{b,i}+1}^{n+1} - u_{i,j_{b,i}+1}^n\big). \end{aligned}$$

Taking absolute values and summing over j in (3.18) and utilizing the properties (3.17) to proceed as in the proof of Harten's lemma [15, Lemma 2.2] along with the last two equations to deal with the boundary contributions, and by reasoning exactly as in the proof of [5, Lemma 4.2], we arrive at the inequality

$$\sum_{j=j_{a,i}}^{j_{b,i}} \left|\Delta_+ u_{i,j}^{n+1}\right| \leq \sum_{j=j_{a,i}}^{j_{b,i}} \left|\Delta_+ u_{i,j}^n\right| + \left|u_{i,j_{a,i}}^{n+1} - u_{i,j_{a,i}}^n\right| + \left|u_{i,j_{b,i}+1}^{n+1} - u_{i,j_{b,i}+1}^n\right|.$$

Proceeding by induction and using (3.15) we obtain for $1 \leq n \leq M$

$$\sum_{j=j_{a,i}}^{j_{b,i}} \left|\Delta_+ u_{i,j}^n\right| \leq \sum_{j=j_{a,i}}^{j_{b,i}} \left|\Delta_+ u_{i,j}^0\right| + \frac{2K}{r}.$$

The proof is completed by the observation that $[a, b] \subseteq [z_{j_{a,i}}, z_{j_{b,i}+1}]$ along with the assumption that u_i^0 has bounded variation. $\square$

Let $V_a^b(g)$ be the total variation of a function $z \mapsto g(z)$ over the interval (a,b). Therefore, by using the results cited before, we can deduce the following lemma.

LEMMA 3.5. *Fix $i \in \{1,\dots,N\}$ and let $[a,b]$ be an interval such that $[a,b]\cap\Sigma = \varnothing$. Then the piecewise constant functions u_i^Δ given by (3.9) satisfy the uniform bound $V_a^b(u_i^\Delta(\cdot,t)) \leq C(a,b)$, where $C(a,b)$ is independent of Δz, Δt, and t for $t \in [0,T]$.*

Notice that for given $i \in \{1,\dots,N\}$, Lemma 3.5 includes the choices $a = -H-\sigma_i^{\mathrm{e}}$, $a<b<-H$ and $b = B+\sigma_i^{\mathrm{u}}$, $B<a<b$.

LEMMA 3.6. *There exist constants C_8 and C_9 such that for all $i = 1,\dots,N$,*

$$\sum_{n=0}^{M-1} \bigl|u_{i,j_{\mathrm{u},i}}^{n+1} - u_{i,j_{\mathrm{u},i}}^{n}\bigr| \leq C_8 \quad \text{and} \quad \sum_{n=0}^{M-1} \bigl|u_{i,j_{\mathrm{e},i}}^{n+1} - u_{i,j_{\mathrm{e},i}}^{n}\bigr| \leq C_9. \tag{3.19}$$

Proof. We only prove the first inequality in (3.19); the proof of the second is analogous. Let us fix $i \in \{1,\dots,N\}$ and consider a fixed interval $[a,b]$ with $B < a < b = B+\sigma_i^{\mathrm{u}}$. Let $j = j_{a,i}$ be such that $[z_{j_{a,i}+1}, z_{j_{\mathrm{u},i}}] \subseteq [a,b] \subseteq [z_{j_{a,i}}, z_{j_{\mathrm{u},i}}]$. Then Lemma 3.5 implies that there exists a constant C_{10} such that

$$\sum_{j=j_{a,i}}^{j_{\mathrm{u},i}-1} \bigl|\Delta_+ u_{i,j}^n\bigr| \leq C_{10}.$$

To simplify notation, in this proof we omit the index i in $u_{i,j}^n$, $\theta_{i,j}^{n+\frac{1}{2}}$ (defined as in the proof of Lemma 3.3), $j_{\mathrm{u},i}$, and $j_{a,i}$, and define $q \coloneqq \lambda Q_{\mathrm{u},i}/A_{\mathrm{u},i}$. Notice that by (CFL), $0 \leq q < 1$. We can now write

$$\begin{aligned} \theta_{j_\mathrm{u}}^{n+\frac{1}{2}} &= -q\Delta_+ u_{j_\mathrm{u}-1}^n, \quad \theta_{j_\mathrm{u}}^{n+\frac{3}{2}} = -q\Delta_+ u_{j_\mathrm{u}-1}^{n+1} = -q(1-q)\Delta_+ u_{j_\mathrm{u}-1}^n - q^2\Delta_+ u_{j_\mathrm{u}-2}^n, \\ \theta_{j_\mathrm{u}}^{n+\frac{5}{2}} &= -q\Delta_+ u_{j_\mathrm{u}-1}^{n+2} = -q(1-q)^2\Delta_+ u_{j_\mathrm{u}-1}^n - 2q^2(1-q)\Delta_+ u_{j_\mathrm{u}-2}^n - q^3\Delta_+ u_{j_\mathrm{u}-3}^n \end{aligned}$$

and so on, as long as all spatial differences of values of $\boldsymbol{u}^n$ are taken within $[a,b]$. Thus, defining $c_{k,l} \coloneqq q^{l+1}(1-q)^{k-l}\binom{k}{l}$, we obtain

$$\theta_{j_\mathrm{u}}^{n+k+\frac{1}{2}} = -\sum_{l=0}^{k} c_{k,l}\Delta_+ u_{j_\mathrm{u}-l-1}^n \quad \text{for } k = 0,1,\dots,j_\mathrm{u}-j_\mathrm{a}-1.$$

Let $J \coloneqq j_\mathrm{u} - j_\mathrm{a} - 1$. Defining $\delta_l \coloneqq \sum_{k=l}^{J} c_{k,l}$, we now get

$$\sum_{k=0}^{J} \bigl|\theta_{j_\mathrm{u}}^{n+k+\frac{1}{2}}\bigr| \leq \sum_{k=0}^{J}\sum_{l=0}^{k} c_{k,l}\bigl|\Delta_+ u_{j_\mathrm{u}-l-1}^n\bigr| = \sum_{l=0}^{J} \delta_l \bigl|\Delta_+ u_{j_\mathrm{u}-l-1}^n\bigr|. \tag{3.20}$$

Setting $p \coloneqq k-l$ and using the known limit of the binomial series we obtain

$$\delta_l = q^{l+1}\sum_{p=0}^{J-l}\binom{p+l}{p}(1-q)^p < q^{l+1}\sum_{p=0}^{\infty}\binom{p+l}{p}(1-q)^p = q^{l+1}\cdot q^{-(l+1)} = 1.$$

Thus, from (3.20) we deduce that

$$\sum_{k=0}^{J} \bigl|\theta_{j_\mathrm{u}}^{n+k+\frac{1}{2}}\bigr| \leq \sum_{l=0}^{J} \bigl|\Delta_+ u_{j_\mathrm{u}-l-1}^n\bigr| = C_{10}. \tag{3.21}$$

Since $J+1 = j_{\mathrm{u}} - j_a \leq \lfloor (b-a)/\Delta z \rfloor$, we observe that at most

$$K := \left\lceil \frac{\lfloor T/\Delta t \rfloor + 1}{\lfloor (b-a)/\Delta z \rfloor} \right\rceil \leq \frac{1}{\lambda} \frac{T+\Delta t}{b-a-\Delta z}$$

versions of (3.21) are required so that we may write

$$\sum_{k=0}^{M-1} \bigl|\theta_{j_{\mathrm{u}}}^{n+\frac{1}{2}}\bigr| = \sum_{s=0}^{K-1} \sum_{r=0}^{J} \bigl|\theta_{j_{\mathrm{u}}}^{sL+r+\frac{1}{2}}\bigr| \leq K C_{10}$$

(where we set $u_j^n = u_j^M$ if $n > M$ if necessary). Since K may be chosen constant if Δt and Δz are sufficiently small, we obtain (3.19) with $C_8 = K C_{10}$. ∎

Lemmas 3.2 to 3.6 allow us to prove the following result.

THEOREM 3.2. *Suppose that the functions $u_1^{\Delta}, \dots, u_N^{\Delta}$ are defined by (3.9), where the values $u_{i,j}^n$ are computed by (3.8). Let $\Delta t, \Delta z \to 0$, with $\lambda = \Delta t/\Delta z = \mathrm{const}$. Then for each $i \in \{1, \dots, N\}$ the function u_i^{Δ} converges in $L^1_{\mathrm{loc}}(\Pi_{i,T})$ and a.e. in $\Pi_{i,T}$ to a function $u_i \in L^{\infty}(\Pi_{i,T}; [0, A])$ such that $\phi_i(z,t) = u_i(z,t)/A_i(z) \in [0,1]$ for almost all $z \in \mathcal{I}_i$ and $t \in [0,T]$. The functions $u_1, \dots, u_N$ form a weak solution of the IVP (1.1), (2.8) in the sense that for all $i = 1, \dots, N$, items (i) and (ii) of Definition 2.1 is satisfied. Moreover, each limit function u_i belongs to $C^0([0,T]; L^1(\mathcal{I}_i; [0,A]))$.*

Proof. The proof follows the first part of that of [14, Theorem 2.8], which in turn is based on those of [8, Theorem 5.1] and [17, Theorem 5.1]. Lemma 3.2 ensures that $u_i^{\Delta} \in L^{\infty}$ for all $i = 1, \dots, N$, $t \in [0,T]$ and $z \in \mathcal{I}_i$; in particular $u_i^{\Delta}(z,t) \in [0, A_i(z)]$. For each $i = 1, \dots, N$, Lemma 3.3 proves the L^1-continuity in time of the sequence u_i^{Δ} while Lemmas 3.4 and 3.5 guarantee a bound on the spatial total variation of u_i^{Δ} on any interval $[a,b]$ not containing any point of Σ. Standard compactness results then imply that for any interval $[a,b]$ with $[a,b] \cap \Sigma = \varnothing$ there exists a subsequence, still denoted by u_i^{Δ}, that converges in $L^1([a,b] \times [0,T]; [0,A])$. Consider now a countable set of intervals $[a_m, b_m]$ such that $\bigcup_{m \in \mathbb{N}} [a_m, b_m] = \mathcal{I}_i \backslash \Sigma$. By a standard diagonal process we may extract a sequence, still denoted by u_i^{Δ}, that converges in $L^1_{\mathrm{loc}}(\Pi_{i,T}; [0,A])$ and a.e. to a function $u_i \in L^{\infty}(\Pi_{i,T}; [0,A])$ (more precisely, $\phi_i(z,t) = u_i(z,t)/A_i(z) \in [0,1]$ a.e.). Moreover, let us define for all $i = 1, \dots, N$ the piecewise constant in time functions $\phi_{\mathrm{u},i}^{\Delta}$ and $\phi_{\mathrm{e},i}^{\Delta}$ with the property that $\phi_{\mathrm{u},i}^{\Delta}(t) = u_{j_{\mathrm{u},i}}^n / A_i^{\mathrm{u}}$ and $\phi_{\mathrm{e},i}^{\Delta}(t) = u_{j_{\mathrm{e},i}}^n / A_i^{\mathrm{e}}$ for $t \in [t_n, t_{n+1})$. By Lemma 3.2 we know that these functions assume values in $[0,1]$ for all $t \in [0,T]$, and Lemma 3.6 ensures that $\mathrm{TV}_{[0,T]}(\phi_{\mathrm{u},i}^{\Delta})$ and $\mathrm{TV}_{[0,T]}(\phi_{\mathrm{e},i}^{\Delta})$ are uniformly bounded. Thus, by Helly's selection theorem, and extracting another subsequence if necessary, we know that there exist functions $\phi_{\mathrm{e},i}, \phi_{\mathrm{u},i} \in L^{\infty}([0,T]; [0,1]) \cap BV[0,T]$ such that $\phi_{\mathrm{e},i}^{\Delta} \to \phi_{\mathrm{e},i}$ and $\phi_{\mathrm{u},i}^{\Delta} \to \phi_{\mathrm{u},i}$ a.e. on $[0,T]$ as $\Delta z, \Delta t \to 0$.

To see that these limit functions coincide with corresponding traces of the limit functions u_i we focus on the interval $(B, B+\sigma_i^{\mathrm{u}}]$ of each unit (the interval $[-H-\sigma_i^{\mathrm{e}}, -H)$ is handled by completely analogous arguments). Let us fix i and select $a_i \in (B, B+\sigma_i^{\mathrm{u}}]$. In view of Lemma 3.5 the restriction of each function u_i to $\Pi_{i,T}^{\mathrm{u}} := [a_i, B+\sigma_i^{\mathrm{u}}] \times [0,T]$ has bounded spatial variation on $\Pi_{i,T}^{\mathrm{u}}$ for all $t \in [0,T]$, and therefore has a trace at $B+\sigma_i^{\mathrm{u}}$ for almost all $t \in [0,T]$. To see that this trace coincides with the limit function constructed above, we proceed by an argument similar to that of the proof of [2, Lemma 4.3]. Choosing $\varphi_i(z,t) := \Phi(t)\nu_h(z)$, where $\Phi \in C_0^{\infty}([0,T])$ and $\nu_h(z) = \nu_{h,B+\sigma_i^{\mathrm{u}}}(z)$ (cf. Proposition 2.1) with h sufficiently small such that $\operatorname{supp} \nu_h \subset (B, B+\sigma_i^{\mathrm{u}}]$, multiplying (3.8) by $\int_{I_j} \varphi_i(z,t_n)\,\mathrm{d}z$, summing the result over $n = 0, \dots, M-1$, applying summation by parts and omitting the index i in $u_{i,j}^n$

(for simplicity) we get

$$\begin{aligned}
0 &= \sum_{n=0}^{M-1}\Big(u_{j_{\mathrm{u},i}}^{n+1}-u_{j_{\mathrm{u},i}}^{n}-\lambda\frac{Q_{\mathrm{u},i}}{A_i^{\mathrm{u}}}\big(u_{j_{\mathrm{u},i}}^{n}-u_{j_{\mathrm{u},i}-1}^{n}\big)\Big)\Phi(t_n)\int_{I_{j_{\mathrm{u},i}}}\nu_h(z)\,\mathrm{d}z\\
&= \sum_{n=0}^{M-1}\big(u_{j_{\mathrm{u},i}}^{n+1}-u_{j_{\mathrm{u},i}}^{n}\big)\Phi(t_n)\int_{I_{j_{\mathrm{u},i}}}\nu_h(z)\,\mathrm{d}z-\Delta t\frac{Q_{\mathrm{u},i}}{A_i^{\mathrm{u}}}\sum_{n=0}^{M-1}u_{j_{\mathrm{u}}}^{n}\Phi(t_n)\frac{1}{\Delta z}\int_{I_{j_{\mathrm{u},i}}}\nu_h(z)\,\mathrm{d}z\\
&\quad+\Delta t\frac{Q_{\mathrm{u},i}}{A_i^{\mathrm{u}}}\sum_{n=0}^{M-1}\sum_{j=j_B+1}^{j_{\mathrm{u},i}}u_{j-1}^{n}\Phi(t_n)\int_{I_j}\frac{\nu_h(z)-\nu_h(z-\Delta z)}{\Delta z}\,\mathrm{d}z.
\end{aligned}$$

Applying a summation by parts to the first term on the right-hand side and using that Φ has bounded variation, we see that this term vanishes as $\Delta z\to 0$. Thus, we obtain for $\Delta z\to 0$, after multiplying by $A_i^{\mathrm{u}}/Q_{\mathrm{u},i}$,

$$0=-\int_0^T u_{j_{\mathrm{u},i}}(t)\Phi(t)\,\mathrm{d}t+\int_0^T\int_B^{B+\sigma_i^{\mathrm{u}}}u(z,t)\Phi(t)\nu_h'(z)\,\mathrm{d}z\,\mathrm{d}t.$$

Invoking Proposition 2.1 we obtain the desired result. Thus, the limit functions u_i^{u} satisfy item (i) of Definition 2.1. A Lax-Wendroff-type argument similar to that of [17, Th. 3.1] suffices to prove that the limit functions also satisfy item (ii) of Definition 2.1. Finally, $u_i\in C^0([0,T];L^1(\mathcal{I}_i;[0,A]))$ follows from Lemma 3.3. ∎

Next, we need to prove that the limit functions $u_1,\dots,u_N$ constitute an entropy solution. To this end we establish a discrete entropy inequality.

LEMMA 3.7. *The approximate values $u_{i,j}^n$ generated by (3.8) satisfy the following discrete entropy inequality for all $i=1,\dots,N$, $j\in\mathcal{J}_i$, $n=0,\dots,M-1$, and $k\in\mathbb{R}$:*

$$\begin{aligned}
&\big|u_{i,j}^{n+1}-k\big|-\big|u_{i,j}^{n}-k\big|+\lambda\Delta_-G_{i,j+\frac12}^k\big(u_{i,j}^n,u_{i,j+1}^n\big)\\
&\quad-\lambda\big|\mathcal{F}_{i,j+\frac12}^n(k,k)-\mathcal{F}_{i,j-\frac12}^n(k,k)\big|-\Delta t\,\mathrm{sgn}\big(u_{i,j}^{n+1}-k\big)S_{i,j}^n\le 0,
\end{aligned}\tag{3.22}$$

with the numerical entropy flux $G_{i,j+\frac12}^k(a,b):=\mathcal{F}_{i,j+\frac12}^n(a\vee k,b\vee k)-\mathcal{F}_{i,j+\frac12}^n(a\wedge k,b\wedge k)$.

Proof. We fix $i\in\{1,\dots,N\}$, $j\in\mathcal{J}_i$, and define

$$\mathcal{H}_{i,j}(a,b,c):=b-\lambda\big(\mathcal{F}_{i,j+\frac12}^n(b,c)-\mathcal{F}_{i,j-\frac12}^n(a,b)\big).$$

Clearly, $u_{i,j}^{n+1}-\Delta tS_{i,j}^n=\mathcal{H}_{i,j}(u_{i,j-1}^n,u_{i,j}^n,u_{i,j+1}^n)$. Since the scheme is monotone, for all k there holds

$$\begin{aligned}
\mathcal{H}_{i,j}\big(u_{i,j-1}^n\vee k,u_{i,j}^n\vee k,u_{i,j+1}^n\vee k\big)&\ge\mathcal{H}_{i,j}\big(u_{i,j-1}^n,u_{i,j}^n,u_{i,j+1}^n\big)\vee\mathcal{H}_{i,j}(k,k,k),\\
-\mathcal{H}_{i,j}\big(u_{i,j-1}^n\wedge k,u_{i,j}^n\wedge k,u_{i,j+1}^n\wedge k\big)&\ge-\big(\mathcal{H}_{i,j}\big(u_{i,j-1}^n,u_{i,j}^n,u_{i,j+1}^n\big)\wedge\mathcal{H}_{i,j}(k,k,k)\big).
\end{aligned}$$

Summing these inequalities and considering that $a\vee b-a\wedge b=|a-b|$ as well as that

$$\begin{aligned}
&\mathcal{H}_{i,j}\big(u_{i,j-1}^n\vee k,u_{i,j}^n\vee k,u_{i,j+1}^n\vee k\big)-\mathcal{H}_{i,j}\big(u_{i,j-1}^n\wedge k,u_{i,j}^n\wedge k,u_{i,j+1}^n\wedge k\big)\\
&=\big|u_{i,j}^n-k\big|-\lambda\Delta_-G_{i,j+\frac12}^k\big(u_{i,j}^n,u_{i,j+1}^n\big),\\
&\mathcal{H}_{i,j}\big(u_{i,j-1}^n,u_{i,j}^n,u_{i,j+1}^n\big)\vee\mathcal{H}_{i,j}(k,k,k)-\mathcal{H}_{i,j}\big(u_{i,j-1}^n,u_{i,j}^n,u_{i,j+1}^n\big)\wedge\mathcal{H}_{i,j}(k,k,k)\\
&=\big|u_{i,j}^{n+1}-\Delta tS_{i,j}^n-k+\lambda\big(\mathcal{F}_{i,j+\frac12}^n(k,k)-\mathcal{F}_{i,j-\frac12}^n(k,k)\big)\big|\\
&\ge\big|u_{i,j}^{n+1}-k\big|-\Delta t\,\mathrm{sgn}\big(u_{i,j}^{n+1}-k\big)S_{i,j}^n-\lambda\big|\mathcal{F}_{i,j+\frac12}^n(k,k)-\mathcal{F}_{i,j-\frac12}^n(k,k)\big|,
\end{aligned}$$

where in last step we have used that $|a+b|\ge|a|+\mathrm{sgn}(a)b$, we arrive at (3.22). ∎

THEOREM 3.3. *Under the conditions of Theorem 3.2 the functions $u_1, \dots, u_N$ satisfy item (iii) of Definition 2.1.*

Combining Theorems 3.2 and 3.3, which are proven here under the assumption that the Q-flows are constant in time (see the remark before Lemma 3.3) and we can therefore write $g_i(u,z)$ instead of $g_i(u,z,t)$, we deduce that a weak entropy solution in the sense of Definition 2.1 exists. In view of Theorem 2.1 we can say that the IVP (1.1), (2.8) is well posed, and that in the sense of Theorem 3.2, the approximate solutions produced by the scheme (3.8) converge to the unique entropy solution of (1.1), (2.8).

Proof of Theorem 3.3. The proof is similar to the last part of the proof of [14, Theorem 2.8]. We fix $i \in \{1, \dots, N\}$, multiply (3.22) by $\Delta z \varphi_{i,j}^n = \Delta z \varphi_i(z_j, t_n)$ and sum over $j \in \mathcal{J}_i$ and $n = 0, \dots, M-1$ to obtain an inequality $\mathcal{E}_1 + \dots + \mathcal{E}_4 \leq 0$, where

$$\begin{aligned}
\mathcal{E}_1 &:= \Delta z \sum_{n=0}^{M-1} \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \bigl(\bigl|u_{i,j}^{n+1} - k\bigr| - \bigl|u_{i,j}^n - k\bigr|\bigr) \varphi_{i,j}^n, \\
\mathcal{E}_2 &:= \Delta t \sum_{n=0}^{M-1} \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \bigl(\Delta_- G_{i,j+\frac{1}{2}}^k \bigl(u_{i,j}^n, u_{i,j+1}^n\bigr)\bigr) \varphi_{i,j}^n, \\
\mathcal{E}_3 &:= -\Delta t \sum_{n=0}^{M-1} \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \bigl|\mathcal{F}_{i,j+\frac{1}{2}}^n(k,k) - \mathcal{F}_{i,j-\frac{1}{2}}^n(k,k)\bigr| \varphi_{i,j}^n, \\
\mathcal{E}_4 &:= -\Delta t \Delta z \sum_{n=0}^{M-1} \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}} \operatorname{sgn}\bigl(u_{i,j}^{n+1} - k\bigr) S_{i,j}^n \varphi_{i,j}^n.
\end{aligned}$$

Summing by parts, letting $\Delta z \to 0^+$ (where we always assume that $\lambda = \Delta t / \Delta z$ is kept constant) and applying the dominated convergence theorem we get

$$\mathcal{E}_1 \overset{\Delta z \to 0^+}{\longrightarrow} -\int_{\mathcal{I}_i} \bigl|u_i^0(z) - k\bigr| \varphi_i(z,0) \,\mathrm{d}z - \int_0^T \int_{\mathcal{I}_i} \bigl|u_i(z,t) - k\bigr| \partial_t \varphi_i(z,t) \,\mathrm{d}z \,\mathrm{d}t. \tag{3.23}$$

Furthermore, applying a summation by parts, and since $\varphi_i(-H - \sigma_i^{\mathrm{e}}, t) = 0$ and $\varphi_i(B + \sigma_i^{\mathrm{u}}, t) = 0$ along with $\partial_z \varphi_i(-H - \sigma_i^{\mathrm{e}}, t) = 0$ and $\partial_z \varphi_i(B + \sigma_i^{\mathrm{u}}, t) = 0$, we get

$$\begin{aligned}
\mathcal{E}_2 = &-\Delta t \Delta z \sum_{n=0}^{M-1} \sum_{j=j_{\mathrm{e},i}}^{j_{\mathrm{u},i}-1} G_{i,j+\frac{1}{2}}^k \bigl(u_{i,j}^n, u_{i,j+1}^n\bigr) \frac{\varphi_{i,j+1}^n - \varphi_{i,j}^n}{\Delta z} \\
&+ \Delta t \Delta z \sum_{n=0}^{M-1} G_{i,j_{\mathrm{u},i}+\frac{1}{2}}^k \bigl(u_{i,j_{\mathrm{u},i}}^n, u_{i,j_{\mathrm{u},i}+1}^n\bigr) \frac{\varphi_{i,j_{\mathrm{u},i}}^n - \varphi_i(B + \sigma_i^{\mathrm{u}}, t)}{\Delta z} \\
&- \Delta t \Delta z \sum_{n=0}^{M-1} G_{i,j_{\mathrm{e},i}-\frac{1}{2}}^k \bigl(u_{i,j_{\mathrm{e},i}-1}^n, u_{i,j_{\mathrm{e},i}}^n\bigr) \frac{\varphi_{i,j_{\mathrm{e},i}}^n - \varphi_i(-H - \sigma_i^{\mathrm{e}}, t)}{\Delta z}.
\end{aligned} \tag{3.24}$$

Since by the properties of the function φ_i, the two last terms on the right-hand side of (3.24) are $\mathcal{O}(\Delta z)$ quantities, we obtain by the dominated convergence theorem

$$\mathcal{E}_2 \overset{\Delta z \to 0^+}{\longrightarrow} -\int_0^T \int_{\mathcal{I}_i} \operatorname{sgn}\bigl(u_i(z,t) - k\bigr) \bigl(g_i(u_i, z) - g_i(k, z)\bigr) \partial_z \varphi_i(z,t) \,\mathrm{d}z \,\mathrm{d}t. \tag{3.25}$$

Since only the summands for $j=j_{-H}$ and $j=j_B$ in the definition of $\mathcal{E}_3$ do not vanish,

$$\begin{aligned}\mathcal{E}_3 &= -\Delta t\sum_{n=0}^{M-1}\sum_{j\in\{j_{-H},j_B\}}\bigl|\mathcal{F}^n_{i,j+\frac12}(k,k)-\mathcal{F}^n_{i,j-\frac12}(k,k)\bigr|\varphi^n_{i,j}\\ &= -\Delta t\sum_{n=0}^{M-1}\sum_{j\in\{j_{-H},j_B\}}\bigl|g_i(k,z_{j+\frac12})-g_i(k,z_{j-\frac12})\bigr|\varphi^n_{i,j}\\ &\xrightarrow{\Delta z\to 0^+} -\int_0^T\sum_{\zeta\in\{-H,B\}}\bigl|g_i(k,\zeta^+)-g_i(k,\zeta^-)\bigr|\varphi_i(\zeta,t)\,\mathrm{d}t.\end{aligned}\tag{3.26}$$

Finally, again by the dominated convergence theorem,

$$\mathcal{E}_4 \xrightarrow{\Delta z\to 0^+} -\int_0^T\int_{\mathcal{I}_i}\operatorname{sgn}(u_i-k)S_i(z,t)\varphi_i(z,t)\,\mathrm{d}z\,\mathrm{d}t.\tag{3.27}$$

Combining (3.23), (3.25), (3.26), and (3.27), we deduce that $\mathcal{E}_1+\dots+\mathcal{E}_4\le 0$ implies that u_i satisfies item (iii) of Definition 2.1. ∎

4. Optimal steady states of a CCD circuit. We study stationary solutions of the PDE model (2.5) for the solids volume fraction $\phi_i(z)$ in each clarifier-thickener CT_i and its outlet pipes. Unlike the models considered in the authors' previous works [6,7,9,11], the feed inlet is represented here by a smooth source rather than a point source. After reviewing the idealized point-source model in Sect. 4.1, for which all steady-state solutions have been completely characterized [10], we analyze the corresponding smoothed model in Sect. 4.2, with particular emphasis on optimal steady states (defined below). Finally, Sect. 4.3 characterizes the corresponding optimal steady states of the complete CCD circuit.

4.1. Steady states of a single idealized CT. We consider (2.5) with $h(z)$ replaced by the Heaviside function $\bar{H}(z)$ and $\mu(z)$ by $\delta(z)=\bar{H}'(z)$ for a CT_i and suppress the index 'i'. It is well known that the stationary outlet volume fractions $\phi_{\mathrm{e}}:-\phi(-H^-)$ and $\phi_{\mathrm{u}}:-\phi(B^+)$ are independent of the pipe cross-sectional areas A^{e} and A^{u} [9,10]. Indeed, flux continuity at $z=B$ gives $Ab(\phi(B^-))+Q_{\mathrm{u}}\phi(B^-)=Q_{\mathrm{u}}\phi_{\mathrm{u}}$, and hence (provided $q_{\mathrm{u}}>0$), $\phi_{\mathrm{u}}=\phi(B^-)+b(\phi(B^-))/q_{\mathrm{u}}$. Thus, the underflow volume fraction ϕ_{u} depends only on the bottom trace $\phi(B^-)$ and the bulk velocity q_{u} in the tank. An analogous formula holds for ϕ_{e}. The complete categorization of the steady states of a CT, including ϕ_{e} and ϕ_{u}, can be represented in a single flux diagram, referred to as an operating chart, where the outlet pipes have the bulk velocities q_{e} and q_{u} (other bulk velocities arise in the CCD problem, but here we only determine the outlet volume fractions). This corresponds to the following model for an idealized CT with a constant feed flux $s_{\mathrm{f}}:=q_{\mathrm{f}}\phi_{\mathrm{f}}$:

$$\begin{aligned}\partial_t\phi-q_{\mathrm{e}}\partial_z\phi&=0, & z&<-H,\\ \partial_t\phi+\partial_z\bigl(f_{\mathrm{L}}(\phi)(1-\bar{H}(z))+f_{\mathrm{R}}(\phi)\bar{H}(z)\bigr)&=\delta(z)s_{\mathrm{f}}, & -H&<z<B,\\ \partial_t\phi+q_{\mathrm{u}}\partial_z\phi&=0, & z&>B,\end{aligned}\tag{4.1}$$

where $\delta(z)=\bar{H}'(z)$, and $f_{\mathrm{L}}(\phi):=b(\phi)-q_{\mathrm{e}}\phi$ and $f_{\mathrm{R}}(\phi):=b(\phi)+q_{\mathrm{u}}\phi$ are the flux functions in the clarification $(-H<z<0)$ and thickening zone $(0<z<B)$, respectively. A steady-state solution $\phi(z)$ to (4.1) is a piecewise constant function of z. For the CCD process, we are interested only in CTs in optimal operation: the

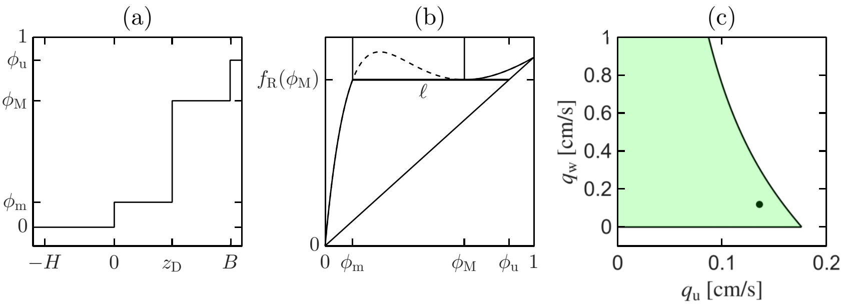

FIG. 3. *(a) A stationary solution of an idealized CT in optimal operation with a discontinuity at $z_D \in (0, B)$. (b) Operating chart for steady-state solutions based on the graph of f_R (dashed). (c) The green region represents the inequalities (4.14). The marked point corresponds to the optimal steady state shown in the simulations in Sect. 5, with input data $q_{in} = 0.5977\,\mathrm{cm\,s^{-1}}$ and $\phi_{in} = 0.2$.*

clarification zone and the effluent contain no solids and the thickening zone contains a discontinuity; see Fig. 3(a).

To state sufficient conditions for such a solution, we recall the following. The inflection point of f_R is independent of q_u, so we can define $\bar{q}_u := -b'(\phi_{max})$ and $\bar{\bar{q}}_u := -b'(\phi_{infl})$. These are the values of q_u for which f_R' vanishes at ϕ_{max} and ϕ_{infl}, respectively. For $\bar{q}_u < q_u < \bar{\bar{q}}_u$, f_R has a local minimizer $\phi_M > \phi_{infl}$. Then $0 = f_R'(\phi_M) = q_u + b'(\phi_M)$. Thus, ϕ_M depends on q_u. Since b' is strictly increasing on $(\phi_{infl}, \phi_{max})$, we define

$$\phi_M = \phi_M(q_u) := \begin{cases} \phi_{max}, & 0 \le q_u \le \bar{q}_u, \\ ((b|_{(\phi_{infl},\phi_{max})})')^{-1}(-q_u), & \bar{q}_u < q_u < \bar{\bar{q}}_u, \\ \phi_{infl}, & q_u \ge \bar{\bar{q}}_u. \end{cases}$$

Given ϕ_M, let $\phi_m = \phi_m(q_u)$ be the unique concentration satisfying $f_R(\phi_m) = f_R(\phi_M)$, where $0 \le \phi_m \le \phi_{infl}$. For clarity, we henceforth write $f_R(\phi; q_u)$ when the dependence on q_u is relevant. From [10, Lemma 2], we recall the following result.

LEMMA 4.1. *The following properties hold: $\phi_M'(q_u) = 0$ if $q_u < \bar{q}_u$, $\phi_M'(q_u) < 0$ if $\bar{q}_u < q_u < \bar{\bar{q}}_u$, and $\phi_M'(q_u) = 0$ if $q_u > \bar{\bar{q}}_u$, along with $\mathrm{d}f_R(\phi_M(q_u); q_u)/\mathrm{d}q_u = \phi_M(q_u)$.*

The operating chart of an idealized CT is based on the graph of f_R; see Fig. 3(b). All qualitatively distinct steady states are categorized by the location of the feed point (ϕ_f, s_f) in this chart; see [10, Th. 1]. If the feed point lies on line segment ℓ in the chart, that is, if $(\phi_f, s_f) \in \ell := \{(\phi, f_R(\phi_M)) : \phi_m < \phi \le \phi_u = f_R(\phi_M)/q_u\}$, then the following optimal-operation stationary solution to (4.1) exists:

$$\phi(z) = \begin{cases} 0, & z < 0, \\ \phi_m, & 0 < z < z_D, \\ \phi_M, & z_D < z < B, \\ \phi_u = f_R(\phi_M)/q_u, & z > B. \end{cases}$$

This solution is unique up to the location of the discontinuity at the depth $z_D \in (0, B)$; see Fig. 3(a). The conditions for optimal operation are

$$s_f = f_R(\phi_M(q_u)), \tag{4.2}$$

$$\phi_{\mathrm{m}}(q_{\mathrm{u}}) < \phi_{\mathrm{f}}. \tag{4.3}$$

Condition (4.2) states that all solids are transported through the thickening zone. In fact, by Lemma 4.1, (4.2) determines q_{u} uniquely; the right-hand side is an increasing function of $q_{\mathrm{u}} \in [0, \bar{\bar{q}}_{\mathrm{u}}]$ and takes values in $[0, f_{\mathrm{R}}(\phi_{\mathrm{M}}(\bar{\bar{q}}_{\mathrm{u}}); \bar{\bar{q}}_{\mathrm{u}})] = [0, f_{\mathrm{R}}(\phi_{\mathrm{infl}}; \bar{\bar{q}}_{\mathrm{u}})]$.

4.2. Optimal steady states in a single CT with smoothed feed inlet. We now construct an optimal steady state for the model (2.5) with a smoothed feed inlet. The idealized system (4.1) is replaced by

$$\partial_t \phi - q_{\mathrm{e}} \partial_z \phi = 0, \qquad z < -H, \tag{4.4}$$

$$\partial_t \phi + \partial_z \big(f_{\mathrm{L}}(\phi)(1 - h(z)) + f_{\mathrm{R}}(\phi) h(z) \big) = h'(z) s_{\mathrm{f}}, \quad -H < z < B, \tag{4.5}$$

$$\partial_t \phi + q_{\mathrm{u}} \partial_z \phi = 0, \qquad z > B, \tag{4.6}$$

and seek a piecewise C^1 solution $\phi = \phi(z)$ for a CT in optimal operation. For some constant flux C the solution must satisfy, for all $-H < z < B$,

$$f_{\mathrm{L}}(\phi(z))(1 - h(z)) + (f_{\mathrm{R}}(\phi(z)) - s_{\mathrm{f}}) h(z) = C. \tag{4.7}$$

Optimal operation requires $\phi(z) = 0$ for $z \le -\varepsilon$. Eq. (4.7) implies that $C = 0$. We then obtain that the flux in the thickening zone is (let $z > \varepsilon$) $f_{\mathrm{R}}(\phi(z)) = s_{\mathrm{f}}$. Under condition (4.2), there are two roots: ϕ_{m} and ϕ_{M}. The entropy condition for scalar conservation laws yields a possible existence of a discontinuity from ϕ_{m} to ϕ_{M} at any $z_{\mathrm{D}} \in (\varepsilon, B)$. We assume $0 < q_{\mathrm{u}} < \bar{\bar{q}}_{\mathrm{u}}$, so that the discontinuity is non-degenerate.

It remains to construct the solution $\phi(z) = \phi_\varepsilon(z)$ in $|z| < \varepsilon$ satisfying (4.5) with $\phi_\varepsilon(-\varepsilon) = 0$ and $\phi_\varepsilon(\varepsilon) = \phi_{\mathrm{m}}$. Since $f_{\mathrm{R}}(\phi) - f_{\mathrm{L}}(\phi) = q_{\mathrm{f}} \phi$ and $s_{\mathrm{f}} = q_{\mathrm{f}} \phi_{\mathrm{f}}$, Equation (4.7) with $C = 0$ can be written $f_{\mathrm{L}}(\phi) + h(z) q_{\mathrm{f}} (\phi - \phi_{\mathrm{f}}) = 0$, or equivalently,

$$h(z) = \mathcal{H}(\phi), \quad \text{where} \quad \mathcal{H}(\phi) := f_{\mathrm{L}}(\phi) / (q_{\mathrm{f}}(\phi_{\mathrm{f}} - \phi)). \tag{4.8}$$

We have $\mathcal{H}(0) = 0$ and $\mathcal{H}(\phi_{\mathrm{m}}) = 1$ (use (4.2) and the definition of ϕ_{m}). To show that $\mathcal{H}(\phi)$ is strictly increasing on $[0, \phi_{\mathrm{m}}]$, we differentiate

$$\mathcal{H}'(\phi) = N(\phi) / (q_{\mathrm{f}}(\phi_{\mathrm{f}} - \phi)^2), \quad \text{where} \quad N(\phi) := (\phi_{\mathrm{f}} - \phi) f_{\mathrm{L}}'(\phi) + f_{\mathrm{L}}(\phi),$$

and see that the numerator N satisfies $N'(\phi) = (\phi_{\mathrm{f}} - \phi) b''(\phi) < 0$ for $0 < \phi < \phi_{\mathrm{m}} < \phi_{\mathrm{f}}$ and $N(\phi_{\mathrm{m}}) = (\phi_{\mathrm{f}} - \phi_{\mathrm{m}}) f_{\mathrm{L}}'(\phi_{\mathrm{m}}) + f_{\mathrm{L}}(\phi_{\mathrm{m}}) = (\phi_{\mathrm{f}} - \phi_{\mathrm{m}})(f_{\mathrm{L}}'(\phi_{\mathrm{m}}) + q_{\mathrm{f}}) = (\phi_{\mathrm{f}} - \phi_{\mathrm{m}}) f_{\mathrm{R}}'(\phi_{\mathrm{m}}) > 0$. Consequently, $N(\phi) > 0$ on $[0, \phi_{\mathrm{m}}]$ and, hence $\mathcal{H}'(\phi) > 0$ on $[0, \phi_{\mathrm{m}}]$. We can therefore define $\phi_\varepsilon(z) := \mathcal{H}^{-1}(h(z))$ for $|z| \le \varepsilon$. The implicit-function theorem gives $\phi_\varepsilon \in C^1([-\varepsilon, \varepsilon])$ and $\phi_\varepsilon'(z) = h'(z) / \mathcal{H}'(\phi_\varepsilon(z)) > 0$ for $|z| < \varepsilon$.

Theorem 4.1. *If $0 < q_{\mathrm{u}} < \bar{\bar{q}}_{\mathrm{u}}$, (4.2) and (4.3) hold, then the following optimal stationary solution to (4.4)–(4.6) exists:*

$$\phi(z) = \begin{cases} 0, & z < -\varepsilon, \\ \phi_\varepsilon(z), & -\varepsilon \le z \le \varepsilon, \\ \phi_{\mathrm{m}}, & \varepsilon < z < z_{\mathrm{D}}, \\ \phi_{\mathrm{M}}, & z_{\mathrm{D}} < z < B, \\ \phi_{\mathrm{u}} = f_{\mathrm{R}}(\phi_{\mathrm{M}})/q_{\mathrm{u}}, & B < z, \end{cases} \tag{4.9}$$

where $\phi_\varepsilon(z) \in C^1$ is a strictly increasing function satisfying (4.7) with $\phi_\varepsilon(-\varepsilon) = 0$ and $\phi_\varepsilon(\varepsilon) = \phi_{\mathrm{m}}$. The solution $\phi(z)$ is unique up to the location of $z_{\mathrm{D}} \in (\varepsilon, B)$.

4.3. Optimal steady states in a CCD circuit. Given the external inlet volume fraction $\phi_{\rm in}$ and velocity $q_{\rm in}>0$, and no solids in the wash water, $\phi_{\rm w}=0$, we will obtain conditions for the control bulk velocities $q_{{\rm u},1},\dots,q_{{\rm u},N}$ and $q_{\rm w}$ so that every CT is in optimal operation. We assume that $0<q_{{\rm u},i}<\bar{\bar{q}}_{\rm u}$ for all i (see Theorem 4.1), which is obtained if the cross-sectional area A is sufficiently large.

With $\phi_{{\rm e},i}=0$ in optimal operation, all solids are transported through the thickening zones of all CTs and the stirred tanks (cf. (2.6) and (2.7)):

$$s_{\rm in}\coloneqq q_{\rm in}\phi_{\rm in}=q_{{\rm f},i}\phi_{{\rm f},i}=q_{{\rm u},i}\phi_{{\rm u},i},\quad i=1,\dots,N. \tag{4.10}$$

Thus, the feed flux to each CT_i is $s_{\rm f}=q_{{\rm f},i}\phi_{{\rm f},i}=s_{\rm in}$. We define every $q_{{\rm u},i}$ as the unique solution of (4.2); $q_{\rm u}\coloneqq q_{{\rm u},1}=\dots=q_{{\rm u},N}$. Finally, we verify that we may choose $q_{\rm w}$ such that all bulk velocities are non-negative and (4.3) holds for each CT_i. By the last equality in (4.10), $\phi_{\rm u}\coloneqq\phi_{{\rm u},i}=s_{\rm in}/q_{\rm u}$ for all i, and (2.3) and (2.4) yield

$$q_{{\rm e},1}=q_{\rm e}=q_{\rm w}+q_{\rm in}-q_{\rm u}\geq 0;\quad q_{{\rm e},i}=q_{\rm w}\geq 0\quad\text{for } i=2,\dots,N, \tag{4.11}$$

$$q_{{\rm f},1}=q_{\rm w}+q_{\rm in};\quad q_{{\rm f},i}=q_{\rm w}+q_{\rm u}\quad\text{for } i=2,\dots,N. \tag{4.12}$$

Consequently, (4.10) implies that

$$\phi_{{\rm f},1}=s_{\rm in}/(q_{\rm w}+q_{\rm in});\quad \phi_{{\rm f},i}=s_{\rm in}/(q_{\rm w}+q_{\rm u})\quad\text{for } i=2,\dots,N. \tag{4.13}$$

Condition (4.3) implies the following together with (4.10) and (4.12):

$$\phi_{\rm m}(q_{\rm u})<\min\{\phi_{{\rm f},1},\phi_{{\rm f},2}\}=s_{\rm in}/\max\{q_{{\rm f},1},q_{{\rm f},2}\}=s_{\rm in}/(q_{\rm w}+\max\{q_{\rm in},q_{\rm u}\}),$$

which gives an upper bound for $q_{\rm w}$. A lower bound follows from (4.11) so that

$$\max\{0,q_{\rm u}-q_{\rm in}\}\leq q_{\rm w}<s_{\rm in}/\phi_{\rm m}(q_{\rm u})-\max\{q_{\rm in},q_{\rm u}\}. \tag{4.14}$$

We conclude the results, and recall that $f_{\rm R}(\phi)=\tilde{g}_i(\phi,z)/A$ for $\varepsilon<z<B$ in (2.5).

THEOREM 4.2. *Consider a CCD circuit with $N\geq 2$, constant inputs $q_{\rm in}>0$, $\phi_{\rm in}\in(0,\phi_{\max}]$ and $\phi_{\rm w}=0$ such that $s_{\rm in}\coloneqq q_{\rm in}\phi_{\rm in}\in(0,f_{\rm R}(\phi_{\rm infl};\bar{\bar{q}}_{\rm u}))$. If every $q_{{\rm u},i}\in(0,\bar{\bar{q}}_{\rm u})$ is the unique solution to (4.2) and $q_{\rm w}$ satisfies (4.14), then there exists an optimal steady-state solution of the form (4.9) in every CT_i with $\phi_{{\rm u},i}=s_{\rm in}/q_{\rm u}$, $i=1,\dots,N$ and $\phi_{{\rm f},i}$ satisfies (4.13). The CCD steady state is unique up to the locations of the discontinuities in the thickening zones.*

5. Numerical simulations. For $\varepsilon\in(0,\min\{B,H\})$ we define $\mu(z)$ via the classical Friedrichs mollifier, i.e., $\mu(z)\coloneqq\varphi(z)/I$, where $\varphi(z)\coloneqq\exp(-1/(1-(z/\varepsilon)^2))$ for $|z|<\varepsilon$ and $\varphi(z)\coloneqq 0$ otherwise, and $I\coloneqq\int_{\mathbb{R}}\varphi(z)\,{\rm d}z$. Then μ is discretized by $\mu_j\coloneqq\varphi(z_j)/(\Delta z(\varphi(z_{-j_\varepsilon})+\dots+\varphi(z_{j_\varepsilon})))$ for $j\in\mathcal{J}$.

5.1. Computation of numerical error. We compute reference solutions for error tests in Example 1 below and approximate L^1 errors at different times for the scheme as follows. We denote by $(\phi_{i,j}^K(t))_{j=1}^K$ and $(\phi_{i,j}^{\rm ref}(t))_{j=1}^{K_{\rm ref}}$ the numerical solution for CT_i at time t calculated with K and $K_{\rm ref}$ cells, respectively. Notice that all data are given in the variables ϕ_i, however, the scheme is computed in terms of the variables u_i; that is, we set $\phi_{i,j}^K=u_{i,j}^K/A_{i,j}$, for $i=1,\dots,N$ and $j=1,\dots,K$, where we have renumbered the cells starting from $j=1$ to simplify notation. We compute

$$\tilde{\phi}_{i,j}^{\rm ref}(t)=\frac{1}{R}\sum_{k=1}^{R}\phi_{i,R(j-1)+k}^{\rm ref}(t),\quad R\coloneqq K_{\rm ref}/K.$$

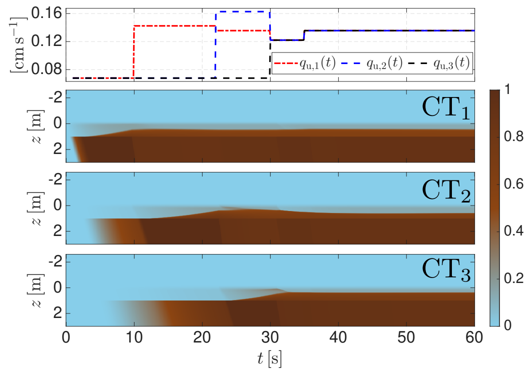

FIG. 4. *Example 1: All three CTs contain initially water (blue). Darker color is higher solids volume fraction; cf. Fig. 5. The pipe lengths between the CTs cause time delays in filling them up.*

The total approximate L^1 error of the numerical solution on the K-cell grid at time t and numerical order of convergence are given by

$$e_K^{\mathrm{tot}}(t) := \Delta z \sum_{i=1}^{N} \sum_{j=1}^{K} \left| \tilde{\phi}_{i,j}^{\mathrm{ref}}(t) - \phi_{i,j}^{K}(t) \right|, \qquad \theta_K(t) := \log_2 \left(e_K^{\mathrm{tot}}(t) / e_{2K}^{\mathrm{tot}}(t) \right).$$

We compute the error by comparing the numerical solution obtained with $K = 10 \cdot 2^{\ell}$, $\ell = 1, \dots, 5$, grid cells with a reference solution obtained with $K_{\mathrm{ref}} = 1280$ cells.

5.2. Two examples. We consider $N = 3$ laboratory-size CTs with $H = 0.6\,\mathrm{m}$, $B = 1\,\mathrm{m}$, $A = 83.65\,\mathrm{cm}^2$ and $A_i^{\mathrm{u}} = A_i^{\mathrm{e}} = 3.14\,\mathrm{cm}^2$, $i = 1,2,3$. The length of all underflow and overflow pipes is $\sigma_i^{\mathrm{u}} = \sigma_i^{\mathrm{e}} = 2\,\mathrm{m}$, hence $\mathcal{I}_i = \mathcal{I} := [-2.6, 3]\,\mathrm{m}$, $i = 1,2,3$. The volumes of the stirred tanks are $V_i = 100\,\mathrm{cm}^3$. The function $b(\phi)$ is specified in Figure 2(b). These data are the same as in [3] and imply $\phi_{\mathrm{infl}} = 0.4$. We consider constant inputs: $\phi_{\mathrm{in}} = 0.2$, $\phi_{\mathrm{w}} = 0$, $Q_{\mathrm{in}} = 50\,\mathrm{cm}^3\,\mathrm{s}^{-1}$ and $Q_{\mathrm{w}} = 10\,\mathrm{cm}^3\,\mathrm{s}^{-1}$, whence $q_{\mathrm{in}} = 0.5977\,\mathrm{cm\,s}^{-1}$ and $q_{\mathrm{w}} = 0.1195\,\mathrm{cm\,s}^{-1}$. With these values, the first prerequisite of Theorem 4.2 is satisfied: $s_{\mathrm{f}} = 0.1195 \in (0, f_{\mathrm{R}}(\phi_{\mathrm{infl}}; \bar{\bar{q}}_{\mathrm{u}})) = (0, 0.1635)\,\mathrm{cm\,s}^{-1}$. The constants of the corresponding optimal steady-state solution are

$$\begin{aligned} &q_{\mathrm{u}} = q_{\mathrm{u},i} = 0.1356 \le \bar{\bar{q}}_{\mathrm{u}} = 0.2146, \quad q_{\mathrm{f},1} = 0.7173, \quad q_{\mathrm{f},2} = 0.2552, \\ &\phi_{\mathrm{f},1} = 0.1667, \quad \phi_{\mathrm{f},2} = 0.4685, \quad q_{\mathrm{e},1} = 0.5817, \quad q_{\mathrm{e},2} = q_{\mathrm{w}}, \quad \phi_{\mathrm{u}} = 0.8815 \end{aligned}$$

(with units as above). The constraint (4.14) for q_{w} is satisfied (see Fig. 3(c)):

$$0 \le q_{\mathrm{w}} = 0.1195 < s_{\mathrm{in}}/\phi_{\mathrm{m}}(q_{\mathrm{u}}) - \max\{q_{\mathrm{in}}, q_{\mathrm{u}}\} = 0.2837.$$

In Example 1, we consider $\varepsilon = 0.25$ (Scenario 1) and $\varepsilon = 0.5$ (Scenario 2). Initially, there is only water in the CCD circuit, i.e., $\phi_i(z,0) = 0$, $z \in \mathcal{I}$, $i = 1,2,3$. Then we manipulate the control variables $q_{\mathrm{u},i}$ as shown in Fig. 4 so the tanks fill up and eventually reach the optimal steady-state solution described above. For both scenarios, we perform convergence tests at two time points; see Table 1. We simulate until $T = 100\,\mathrm{s}$; see Fig. 5, which also demonstrates the convergence of the scheme. In Example 2 we start with a concentration profile at the end of Example 1. We change the control variables $q_{\mathrm{u},i}$ so that the system becomes overloaded; see Fig. 6. At $t = 120\,\mathrm{s}$, we choose $q_{\mathrm{u}} = 0.1356\,\mathrm{cm\,s}^{-1}$ to recover optimal operation.

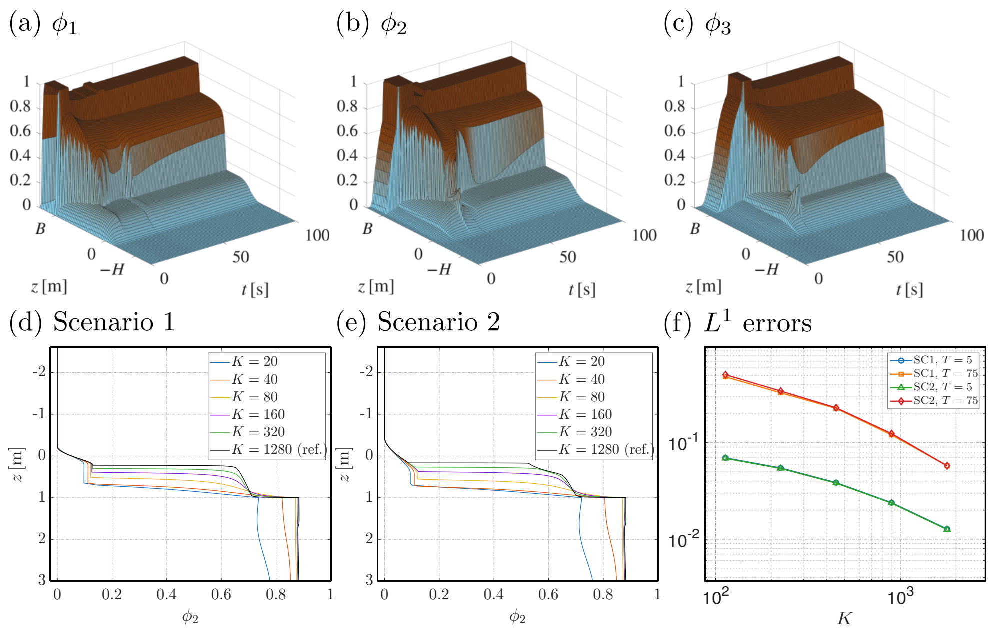

FIG. 5. *Example 1: (a)–(c) simulated solids volume fractions in the three CTs and in parts of the outlet pipes, (d), (e) numerical solution at $T = 75$ s for (d) $\varepsilon = 0.25$ m and (e) $\varepsilon = 0.5$ m, (f) approixmate L^1 errors as a function of the number of grid cells K for the two scenarios.*

TABLE 1
Example 1 : L^1 errors and numerical order in Scenarios 1 and 2 of the scheme at times $T = 5$ s and $T = 75$ s. The reference solution is computed by with $K_{\mathrm{ref}} = 1280$.

K	e_K^{tot}	θ_K	e_K^{tot}	θ_K	e_K^{tot}	θ_K	e_K^{tot}	θ_K
	Scenario 1, $T = 5$ s		Scenario 1, $T = 75$ s		Scenario 2, $T = 5$ s		Scenario 2, $T = 75$ s	
20	6.97e-02	—	4.83e-01	—	6.89e-02	—	5.08e-01	—
40	5.47e-02	0.349	3.29e-01	0.550	5.41e-02	0.348	3.44e-01	0.562
80	3.85e-02	0.505	2.27e-01	0.537	3.81e-02	0.504	2.30e-01	0.577
160	2.39e-02	0.689	1.20e-01	0.912	2.36e-02	0.689	1.24e-01	0.889
320	1.27e-02	0.908	5.76e-02	1.066	1.25e-02	0.910	5.76e-02	1.109

6. Conclusions. We have formulated and analyzed a model of a CCD circuit consisting of N clarifier-thickeners (CTs) coupled in series, where the effluent from each CT is fed to its predecessor and the underflow to its successor. Each CT, including its outlet pipes, is modeled by a one-dimensional scalar hyperbolic conservation law with discontinuous flux. The conservation laws are weakly coupled through the merging of the two incoming streams feeding each CT. In many industrial plants, these streams are first mixed in a small continuously stirred tank, whose outlet forms the feed to the next CT. These mixing tanks, which have finite volume, are included in the model and provide a smoothing effect. Furthermore, whereas the feed to a CT has often been modeled by a point source, we employ a smooth source with support of width 2ε, which converges to the Dirac measure as $\varepsilon \to 0$. This feature is preferred by practitioners, and our mathematical results are independent of the parameter ε.

We have established the well-posedness of the resulting coupled PDE–ODE system

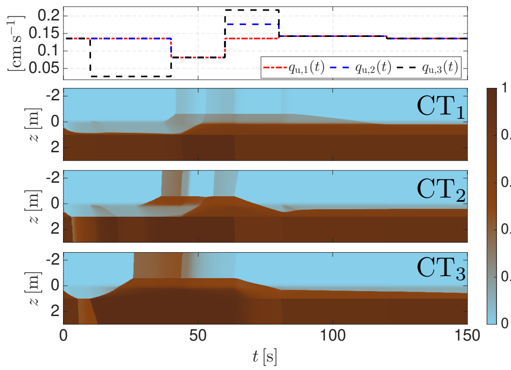

FIG. 6. *Example 2: After varying bulk underflow velocities that cause an overflow, the CCD circuit is recovered to an optimal operating state.*

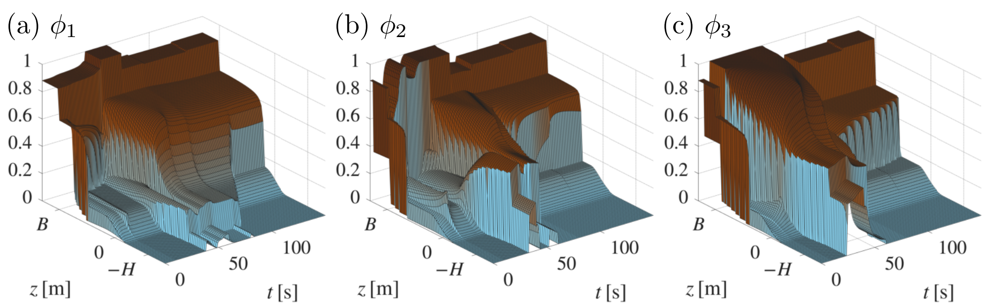

FIG. 7. *Example 2: Volume fractions in the three CTs and in parts of the outlet pipes.*

with respect to the notion of weak solution introduced in Definition 2.1. Theorem 2.1 provides both a ordering principle and an L^1-Lipschitz stability estimate with respect to the initial data, implying uniqueness. Existence is obtained by proving convergence of a monotone finite difference scheme under a CFL condition. Moreover, Theorem 3.1 shows that the scheme is globally conservative with respect to the total volume of the solids in the entire CCD circuit. The analysis can be extended to vessels with continuously varying cross-sectional area; see [1].

Optimal operation of a CCD circuit means that no solids leave through the effluent streams, so that all solids are transported through the underflow streams at the highest possible concentration. For given input data $(q_{\mathrm{in}}, \phi_{\mathrm{in}})$, we have characterized the choices of the control variables $q_{\mathrm{u},1}, \dots, q_{\mathrm{u},N}$ and q_{w} for which an optimal steady state exists. In this regime, every effluent stream is free of solids, and the first CT plays a distinctive role. The feed solids concentration is diluted in the first mixing tank, so that $\phi_{\mathrm{f},1} < \phi_{\mathrm{in}}$, whereas the feeds to the remaining CTs satisfy $\phi_{\mathrm{f},2} = \dots = \phi_{\mathrm{f},N} > \phi_{\mathrm{in}}$. Example 2 in Sect. 5 illustrates that solids may overflow during transient operation. To avoid such overflows and to control the sharp discontinuity in the thickening zone at steady state, one may employ the regulator proposed in [11].

In mineral processing a CCD circuit treats a suspension of ore particles in a leach solution consisting of water, acid, and dissolved metal. The present work focuses on the hydrodynamics of the two-phase solid-liquid separation. A more comprehensive model would also describe the transport of the dissolved liquid components. Notice that, whereas optimal steady-state operation requires specific values of the underflow control variables $q_{\mathrm{u},1},\dots,q_{\mathrm{u},N}$, the wash-water flow rate q_{w} may vary within the interval given by (4.14); see also Fig. 3(c). This additional degree of freedom could be exploited to optimize the recovery of dissolved metals while minimizing their loss in the overall underflow. Such a multiphase model will be investigated in future work.

REFERENCES

[1] R. Bürger, J. Careaga, and S. Diehl, *Entropy solutions of a scalar conservation law modeling sedimentation in vessels with varying cross-sectional area*, SIAM J. Appl. Math., 77 (2017), pp. 789–811.

[2] R. Bürger, A. Coronel, and M. Sepúlveda, *A semi-implicit monotone difference scheme for an initial-boundary value problem of a strongly degenerate parabolic equation modeling sedimentation-consolidation processes*, Math. Comp., 75 (2006), pp. 91–112.

[3] R. Bürger, S. Diehl, M. C. Martí, and Y. Vásquez, *A difference scheme for a triangular system of conservation laws with discontinuous flux modeling three-phase flows*, Netw. Heterog. Media, 18 (2023), pp. 140–190.

[4] R. Bürger, S. Evje, and K. H. Karlsen, *On strongly degenerate convection-diffusion problems modeling sedimentation-consolidation processes*, J. Math. Anal. Appl., 247 (2000), pp. 517–556.

[5] R. Bürger, A. García, K. H. Karlsen, and J. D. Towers, *A family of numerical schemes for kinematic flows with discontinuous flux*, J. Engrg. Math., 60 (2008), pp. 387–425.

[6] R. Bürger, K. H. Karlsen, N. H. Risebro, and J. D. Towers, *Well-posedness in BV_t and convergence of a difference scheme for continuous sedimentation in ideal clarifier-thickener units*, Numer. Math., 97 (2004), pp. 25–65.

[7] R. Bürger, K. H. Karlsen, and J. D. Towers, *A model of continuous sedimentation of flocculated suspensions in clarifier-thickener units*, SIAM J. Appl. Math., 65 (2005), pp. 882–940.

[8] R. Bürger, K. H. Karlsen, and J. D. Towers, *An Engquist-Osher-type scheme for conservation laws with discontinuous flux adapted to flux connections*, SIAM J. Numer. Anal., 47 (2009), pp. 1684–1712.

[9] S. Diehl, *A conservation law with point source and discontinuous flux function modelling continuous sedimentation*, SIAM J. Appl. Math., 56 (1996), pp. 388–419.

[10] S. Diehl, *Operating charts for continuous sedimentation. I. Control of steady states*, J. Engrg. Math., 41 (2001), pp. 117–144.

[11] S. Diehl, *A regulator for continuous sedimentation in ideal clarifier-thickener units*, J. Eng. Math., 60 (2008), pp. 265–291.

[12] R. C. Dunne, S. K. Kawatra, and C. A. Young, *SME Mineral Processing & Extractive Metallurgy Handbook*, Society for Mining, Metallurgy, and Exploration, Englewood, CO, USA, 2019.

[13] B. Engquist and S. Osher, *One-sided difference approximations for nonlinear conservation laws*, Math. Comp., 36 (1981), pp. 321–351.

[14] P. Goatin and E. Rossi, *A multilane macroscopic traffic flow model for simple networks*, SIAM J. Appl. Math., 79 (2019), pp. 1967–1989.

[15] A. Harten, *High resolution schemes for hyperbolic conservation laws*, J. Comput. Phys., 49 (1983), pp. 357–393.

[16] H. Holden and N. H. Risebro, *Models for dense multilane vehicular traffic*, SIAM J. Math. Anal., 51 (2019), pp. 3694–3713.

[17] K. H. Karlsen, N. H. Risebro, and J. D. Towers, *Upwind difference approximations for degenerate parabolic convection-diffusion equations with a discontinuous coefficient*, IMA J. Numer. Anal., 22 (2002), pp. 623–664.

[18] K. H. Karlsen, N. H. Risebro, and J. D. Towers, *L^1 stability for entropy solutions of nonlinear degenerate parabolic convection-diffusion equations with discontinuous coefficients*, Skr. K. Nor. Vidensk. Selsk., (2003), pp. 1–49.

[19] M. L. McCaslin and J. Johnson, *Liquid–solid separation in gold processing*, in Gold Ore Processing, M. D. Adams, ed., Elsevier Science, Amsterdam, 2016, ch. 18, pp. 279–298.

[20] B. E. Raahauge and F. S. Williams (eds.), *Smelter Grade Alumina from Bauxite*, Springer, Cham, Switzerland, 2022.

[21] J. D. Seader, E. J. Henley, and D. K. Roper, *Separation Process Principles. Chemical and Biochemical Operations*, Wiley, Hoboken, NJ, USA, third ed., 2011.

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

PRE-PUBLICACIONES 2026

2026-12 SERGIO CAUCAO, GABRIEL N. GATICA, LUIS F. GATICA, CRISTIAN INZUNZA: *A priori and a posteriori error analysis of a mixed FEM for stationary convective Brinkman-Forchheimer flows with variable porosity*

2026-13 JESSIKA CAMAÑO, RICARDO OYARZÚA, KATHERINE ROJO, SEGUNDO VILLA-FUENTES: *A mixed finite element method based on pseudostress and stream-function for the Navier–Stokes problem in 2D*

2026-14 RAIMUND BÜRGER, CIPRIANO ESCALANTE, ENRIQUE D. FERNÁNDEZ NIETO, JORGE MOYA: *A two-dimensional multilayer shallow water model of tsunami-forest interaction*

2026-15 ALONSO J. BUSTOS, SERGIO CAUCAO: *A posteriori error analysis of two mixed formulations for a coupled Brinkman–Forchheimer and convection-diffusion-reaction system*

2026-16 GABRIEL N. GATICA, SALIM MEDDAHI, KEVIN W. PUCHA-ATAN, RICARDO RUIZ-BAIER: *A Banach spaces-based fully mixed finite element method for the thermo-electro-hydrodynamic Boussinesq problem*

2026-17 TOMÁS BARRIOS, EDWIN BEHRENS, ROMMEL BUSTINZA, JOSE M. CASCON: *A stabilized displacement - stress formulation for a linear elasticity problema with mixed boundary conditions*

2026-18 GONZALO A. BENAVIDES, SERGIO CAUCAO, GABRIEL N. GATICA, YURI D. SOBRAL: *New Banach spaces-based mixed finite element methods for steady-state flows of magnetic fluids*

2026-19 FAHIM ASLAM, MARCELO CAVALCANTI, ZAYD HAJJEJ, YOU HAIYANG LIN BO, MAURICIO SEPÚLVEDA: *Blow-up of a Lamé system with fractional damping and infinite memory: theoretical and numerical study*

2026-20 GABRIEL N. GATICA, SALIM MEDDAHI: *A hybridizable discontinuous Galerkin method for the elasto–acoustic transmission problem*

2026-21 IDULFO ARROCHA, RAIMUND BÜRGER, MARÍA GÓMEZ, YOLANDA VÁSQUEZ: *Modeling and numerical simulation of the colonization of plant growth-promoting bacteria in the rice rhizosphere*

2026-22 ABRAHAM J. ARENAS, JUAN BARAJAS-CALONGE, GILBERTO GONZÁLEZ-PARRA, LUIS M. VILLADA: *Convergence analysis of a nonstandard finite volume method for a diffusion-production-destruction model*

2026-23 JUAN BARAJAS-CALONGE, RAIMUND BÜRGER, STEFAN DIEHL, LUIS M. VILLADA: *A network of conservation laws with discontinuous fluxes and source terms modeling countercurrent decantation*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: DIRECTOR, CENTRO DE INVESTIGACIÓN EN INGENIERÍA MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN, CHILE, TEL.: 41-2661324, o bien, visitar la página web del centro: `http://www.ci2ma.udec.cl`






Centro de Investigación en
Ingeniería Matemática (CI²MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl