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Blow-Up of a Lamé System with Fractional Damping and Infinite Memory: Theoretical and Numerical Study.

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Abstract

As a continuation of the previous work on the Lamé system by Aslam et al. [28], where global existence and exponential stability of solutions were established, this paper is devoted to the analysis of finite-time blow-up phenomena. Specifically, we consider the same system as in [28] under suitable conditions on the relaxation functions and focus on the case of low initial energy, for which we prove a finite-time blow-up result.

To complement the theoretical results, we design a finite volume method combined with a Newmark-type time discretization to approximate the system numerically. The numerical experiments corroborate the theoretical decay rates, capture the blow-up dynamics, and demonstrate the efficiency of the proposed schemes.

Keywords: Lamé system, Fractional damping, Infinite memory, Blow-up.

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1 Introduction

The study of dynamical systems arising in elasticity theory has attracted considerable attention over the past decades due to their rich mathematical structure and significant applications in engineering and material sciences. In particular, the Lamé system, which models the displacement field in isotropic elastic media, plays a fundamental role in describing wave propagation phenomena. When additional physical effects such as damping, memory, and nonlinear sources are incorporated, the resulting models exhibit complex qualitative behaviors including stability, decay, and blow-up of solutions.

In recent years, increasing attention has been devoted to the analysis of evolution equations involving fractional derivatives. These operators provide an effective framework for modeling hereditary and anomalous dissipation mechanisms that cannot be captured by classical integer-order derivatives. Fractional damping has been successfully used to describe viscoelastic materials and wave propagation in complex media; see, for instance, [1, 2, 3, 4, 5, 6, 7, 8].

In parallel, systems with *infinite memory* have been extensively investigated, as they naturally arise in viscoelasticity where the current state depends on the entire past history of the system [9, 14, 15, 16, 17, 18]. The interplay between fractional damping and memory effects leads to new analytical challenges and deeper insights into the dissipative structure of such systems.

On the other hand, nonlinear source terms, especially of logarithmic type, have proven to be particularly interesting due to their critical growth and delicate analytical properties. Logarithmic nonlinearities appear in various physical models and often lead to competing effects between dissipation and energy production. This competition may result in either global existence or finite-time blow-up of solutions depending on the initial energy and structural parameters; see [7, 19, 20, 21]. The investigation of blow-up phenomena is of special importance as it corresponds to the formation of singularities and the breakdown of the physical model in finite time.

For Lamé-type systems, several results have been established concerning well-posedness and stability under different damping mechanisms. In particular, exponential and general decay results for Lamé systems with memory or delay terms and nonlinear damping can be found in [22, 23, 24, 26, 27]. More recently, the incorporation of fractional damping into Lamé systems has led to new developments, including stability and decay estimates as well as qualitative properties of solutions [24, 28, 29, 30, 31].

However, the combined effect of fractional damping, infinite memory, and logarithmic nonlinearities in Lamé systems remains less explored, particularly regarding the occurrence of blow-up and the interaction between these mechanisms.

The purpose of this paper is to contribute to this direction by studying a Lamé system endowed with a nonlinear logarithmic source term, a fractional damping operator of Caputo type, and an infinite memory term. The model under consideration captures the combined influence of hereditary effects and nonlocal dissipation, leading to a highly nontrivial dynamical behavior. Our analysis is motivated by recent works on wave and Lamé equations with similar features [7, 8, 18], where the competition between damping and source terms plays a crucial role. Note that the system under consideration was previously studied in [28], where the authors established results on local existence, global existence, and stability. The present work extends these contributions in two directions. First, we investigate the finite-time blow-up of solutions corresponding to low initial energy under suitable conditions on the relaxation functions. Second, we develop a detailed numerical approximation of the model by combining a finite volume discretization with a Newmark-type time integration scheme. Particular attention is given to the treatment of the fractional derivative and infinite memory terms through appropriate auxiliary variables, enabling the scheme to preserve the dissipative structure of the system. The numerical simulations illustrate both the energy decay in the stable regime and the occurrence of finite-time blow-up, thereby confirming the theoretical results.

More precisely, we consider the following system:

$$\begin{cases} w'' - \Delta_e w + \displaystyle\int_0^{+\infty} r(s)\Delta w(t-s)ds + \partial_t^{\eta,\varrho} w(t) \\ \qquad\qquad = w|w|^{p-2}\ln|w|, \text{ in } \Omega \times (0,\infty), \\ w = 0, \text{ on } \partial\Omega \times (0,\infty), \\ w(x,t) = w_0(x,t) \text{ in } \Omega \times (-\infty, 0], \end{cases} \tag{1.1}$$

where Ω is bounded domain of $\mathbb{R}^3$ with a smooth boundary $\partial\Omega$,

$$w = (w_1, w_2, w_3)^T, \quad \Delta_e w = \mu\Delta w + (\mu+\lambda)\nabla div w,$$

$$w|w|^{p-2}\ln|w| = \left(w_1|w_1|^{p-2}\ln|w_1|,\, w_2|w_2|^{p-2}\ln|w_2|,\, w_3|w_3|^{p-2}\ln|w_3|\right)^T,$$

and

$$r(s) = \begin{pmatrix} r_1(s) & 0 & 0 \\ 0 & r_2(s) & 0 \\ 0 & 0 & r_3(s) \end{pmatrix}.$$

The functions $r_i, i = 1,2,3$, and the constant p will be specified later.
Note that the operator Δ is the Laplacian operator and Δ_e is the elasticity operator which is a 3×3 matrix-valued differential operator. The parameters λ and μ are the Lamé constants, meeting the following requirements:

$$\mu > 0, \quad \mu + \lambda \geq 0. \tag{1.2}$$

The symbol $\partial_t^{\eta,\varrho}$ denotes the modified Caputo fractional derivative, which is defined, in [2, 33], by

$$\partial_t^{\eta,\varrho} w(t) = \frac{1}{\Gamma(1-\eta)} \int_0^t (t-\tau)^{-\eta} e^{-\varrho(t-\tau)} w_\tau(\tau) d\tau, \qquad 0 < \eta < 1, \varrho \geq 0.$$

Throughout this paper, we assume the following conditions:

(H1) The functions r_i: $\mathbb{R}_+ \to \mathbb{R}_+, i = 1,2,3$, are differentiable, non-increasing and satisfy

$$\mu > \gamma_i, \; \forall\, 1 \leq i \leq 3, \quad \textit{where} \quad \gamma_i = \int_0^{+\infty} r_i(s)ds > 0.$$

(H2) There exists a positive constant $\theta > 0$ such that

$$r_i'(t) \leq -\theta r_i(t), \quad \forall\, t \geq 0.$$

(H3) The constant p is such that $2 < p < 4$.

The paper is organized as follows. Section 2 presents the preliminaries and recalls previously established results; detailed proofs can be found in [28]. Section 3 is devoted to the blow-up analysis of (1.1). Section 4 addresses the numerical approximation of the system, where we illustrate both the decay of energy in the stable regime and the occurrence of finite-time blow-up.

2 Preliminaries and Previous results

Let recall the following lemmas.

Lemma 2.1 ([1]). *Let ϖ be the function:*

$$\varpi(\sigma) = |\sigma|^{\frac{(2\eta-1)}{2}}, \quad \sigma \in \mathbb{R}, \quad 0 < \eta < 1,$$

and $b = \frac{\sin(\eta\pi)}{\pi}$. Then the relation between the system's input U and output O

$$\begin{cases} \partial_t \phi(\sigma, t) + (\sigma^2 + \varrho)\phi(x, \sigma, t) - U(x, t)\varpi(\sigma) = 0, & \sigma \in \mathbb{R},\, t > 0,\, \varrho \geq 0, \\ \phi(x, \sigma, 0) = 0, \\ O(t) := b \displaystyle\int_{-\infty}^{+\infty} \phi(\sigma, t)\varpi(\sigma) d\sigma \end{cases} \tag{2.1}$$

is given by

$$O := I^{1-\eta,\varrho} U,$$

where

$$\phi = (\phi_1, \phi_2, \phi_3)^T$$

and

$$I^{\eta,\varrho} u(t) := \frac{1}{\Gamma(\eta)} \int_0^t (t-\tau)^{\eta-1} e^{-\varrho(t-\tau)} u(\tau) d\tau.$$

Lemma 2.2 ([24]). *For all $\lambda \in D_\varrho = \mathbb{C} \backslash\,]-\infty, -\varrho]$, we have*

$$A_\lambda := \int_{-\infty}^{+\infty} \frac{\varpi^2(\sigma)}{\lambda + \varrho + \sigma^2} d\sigma = \frac{\pi}{\sin(\eta\pi)} (\lambda + \varrho)^{\eta-1}.$$

Now, similarly to [15], we define the variable ν by:

$$\nu(x, s) = w(x, t) - w(x, t-s), \tag{2.2}$$

The variable ν represents the relative history of w and fulfills the following equation:

$$\nu_t(x, s) - w_t(x, t) + \nu_s(x, s) = 0, \quad x \in \Omega, \;\; t, s > 0. \tag{2.3}$$

By using Lemma 2.1 and (2.3), the system (1.1) can be rewritten as follows:

$$
\begin{cases}
w'' - (\mu Id - \displaystyle\int_0^{+\infty} r(s)ds\ )\Delta w - (\mu+\lambda)\nabla div(w)) - \int_0^{+\infty} r(s)\Delta\nu(x,s)ds \\
\qquad\qquad +b\displaystyle\int_{-\infty}^{+\infty} \phi(x,\sigma,t)\varpi(\sigma)d\sigma = w|w|^{p-2}\ln|w|,\ x\in\Omega,\ t>0, \\
\partial_t\phi(x,\sigma,t) + (\sigma^2+\varrho)\phi(x,\sigma,t) - w_t(x,t)\varpi(\sigma) = 0,\ \sigma\in\mathbb{R},\ t>0,\ \varrho\geq 0, \\
\partial_t\nu(x,s) + \partial_s\nu(x,s) = \partial_t w(x,t),\ x\in\Omega,\ t,\ s>0, \\
w(x,t) = \nu(x,s) = 0,\ x\in\partial\Omega,\ t,\ s>0, \\
w(x,0) = w_0(x,0), \quad w_t(x,0) = \partial_t w_0(x,0),\ x\in\Omega, \\
\nu(x,0) = 0, \quad \nu_0(x,s) = \nu(x,0,s) = w_0(x,0) - w_0(x,-s),\ x\in\Omega,\ t,\ s>0, \\
\phi(x,\sigma,0) = 0,\ \sigma\in\mathbb{R}.
\end{cases} \tag{2.4}
$$

where

$$
Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Now, let $u = w_t$ and consider $Y = (w, u, \phi, \nu)$, then the problem (2.4) is equivalent to:

$$
\begin{cases}
Y'(t) = AY(t) + B(Y(t)), \\
Y(0) = Y_0 = (w_0, w_1, 0, \nu_0),
\end{cases} \tag{2.5}
$$

and

$$
B(Y) = (0, |w|^{p-2}w\ln|w|, 0, 0)^T, \tag{2.6}
$$

and the linear operator A is defined by

$$
AY = \begin{pmatrix}
u \\
\Delta_e w - (\displaystyle\int_0^{+\infty} r(s)ds)\Delta w - b\int_{-\infty}^{+\infty}\phi(\sigma)\varpi(\sigma)d\sigma + \int_0^{+\infty} r(s)\Delta\nu ds \\
\varpi(\sigma)u - (\sigma^2+\varrho)\phi(\sigma) \\
u - \partial_s\nu
\end{pmatrix}.
$$

Define $\mathscr{H}$ the state space (energy space) by:

$$
\mathscr{H} = \quad (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times (L^2(\Omega\times\mathbb{R}))^3 \times L_r^2(\mathbb{R}_+, H_0^1(\Omega)),
$$

with

$$
L_r^2(\mathbb{R}_+, H_0^1(\Omega)) \quad = \quad \Big\{ w = (w_1, w_2, w_3)^T : \mathbb{R}_+ \to (H_0^1(\Omega))^3, \\
\qquad\qquad \int_0^{+\infty} r_i(s)||\nabla w_i(s)||_2^2 ds < \infty,\ i = 1,2,3 \Big\},
$$

which is endowed with the following inner product:

$$\langle u, w\rangle_{L^2_r(\mathbb{R}_+, H^1_0(\Omega))} = \sum_{i=1}^{3} \int_0^{+\infty} r_i(s) \int_\Omega \nabla u_i(s) \nabla w_i(s) dx ds,$$

for $w = (w_1, w_2, w_3)^T,\ u = (u_1, u_2, u_3)^T \in L^2_r(\mathbb{R}_+, H^1_0(\Omega))$.
Let $\phi = (\phi_1, \phi_2, \phi_3)$ and $\widetilde{\phi} = (\widetilde{\phi}_1, \widetilde{\phi}_2, \widetilde{\phi}_3)$. Then, for $W = (w_1, w_2, \phi, w_3)^T$ and $U = (u_1, u_2, \widetilde{\phi}, u_3)^T \in \mathscr{H}$, the Hilbert space $\mathscr{H}$ will be equipped by the following inner product

$$\begin{aligned} \langle U, W\rangle_{\mathscr{H}} &= \int_\Omega \Big[\sum_{i=1}^{3} ((\mu - \gamma_i) \nabla u_1^i \nabla w_1^i + u_2^i w_2^i) dx + (\lambda + \mu) \int_\Omega div u_1 div w_1] dx \\ &\quad + b \sum_{i=1}^{3} \int_\Omega \int_{-\infty}^{+\infty} \phi_i \widetilde{\phi}_i d\xi dx + \langle u_3, w_3\rangle_{L^2_r(\mathbb{R}_+, H^1_0(\Omega))}, \end{aligned}$$

where $w_i = (w_i^1, w_i^2, w_i^3)^T,\ u_i = (u_i^1, u_i^2, u_i^3)^T$, and $\gamma_i = \displaystyle\int_0^{\infty} r_i(s) ds,\ i = 1, 2, 3.$
The domain $D(A)$ of A is given by

$$D(A) = \begin{Bmatrix} (w, u, \phi, \nu)^T \in \mathscr{H} : w \in \big(H^2(\Omega)\big)^3,\ u \in \big(H^1_0(\Omega)\big)^3, \\ \varpi(\sigma) u(x) - (\sigma^2 + \varrho)\phi(\sigma) \in \big(L^2(\Omega \times \mathbb{R})\big)^3, \\ |\sigma|\phi \in \big(L^2(\Omega \times \mathbb{R})\big)^3 \\ \partial_s \nu \in L^2_r(\mathbb{R}_+, H^1_0(\Omega)),\ \nu(., 0) = 0 \end{Bmatrix}.$$

Theorem 2.3. *(Local existence result)[[28]] Assume that (H1)-(H3) hold true. Then, for any $Y_0 \in \mathscr{H}$, the problem (2.5) has a unique local solution $Y \in C([0, T); \mathscr{H})$. Moreover, if $Y_0 \in D(A)$, then $Y \in C^1([0, T); \mathscr{H}) \cap C([0, T); D(A))$.*

Theorem 2.4. *(Global existence result)[28] For any $U_0 \in \mathcal{H}$ satisfying*

$$\begin{cases} \chi = \frac{2C^*_{p+l}}{p(\mu - \max\limits_{1\le i\le 3}\{\gamma_i\})} \left(\frac{2p}{(p-2)(\mu - \max\limits_{1\le i\le 3}\{\gamma_i\})} E(0) \right)^{\frac{p-2+l}{2}} < 1, \\ \mathbb{I}(0) > 0, \end{cases} \tag{2.7}$$

we have that the solution w is global, where

$$\begin{aligned} \mathbb{I}(t) = &\sum_{i=1}^{3} (\mu - \gamma_i) \int_\Omega |\nabla w_i|^2 dx + (\lambda + \mu) \sum_{i=1}^{3} \int_\Omega |div w_i|^2 dx + \sum_{i=1}^{3} b \int_\Omega \int_{-\infty}^{+\infty} |\phi_i(\sigma, t)|^2 d\sigma dx \\ &- \sum_{i=1}^{3} \int_\Omega |w_i|^p \ln |w_i| dx + \frac{1}{2} r \circ \nabla \nu, \end{aligned} \tag{2.8}$$

and C^*_{p+l} is the embedding constant of $H^1_0(\Omega) \hookrightarrow L^{p+l}(\Omega)$.

The energy of solutions for system (2.4) is defined by

$$\begin{aligned}
E(t) &= \frac{1}{2}\sum_{i=1}^{3}(\mu-\gamma_i)\int_\Omega |\nabla w_i|^2dx + \frac{1}{2}\sum_{i=1}^{3}\int_\Omega |w_i'|^2dx + \frac{\lambda+\mu}{2}\sum_{i=1}^{3}\int_\Omega |\, div w_i|^2dx \\
&+ \frac{b}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(\sigma,t)|^2d\sigma dx + \frac{1}{p^2}\sum_{i=1}^{3}\int_\Omega |w_i(t)|^p dx \\
&- \frac{1}{p}\sum_{i=1}^{3}\int_\Omega |w_i|^p \ln|w_i|dx + \frac{1}{2}r\circ\nabla\nu, \qquad (2.9)
\end{aligned}$$

where

$$r\circ\nabla\nu = \sum_{i=1}^{3}\int_0^{+\infty} r_i(s)\int_\Omega |\nabla\nu_i(x,s)|^2dxds. \qquad (2.10)$$

It is easy to verify that

$$E'(t) = \frac{1}{2}r'\circ\nabla\nu - b\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\varrho)|\phi_i(\sigma,t)|^2d\sigma dx \le 0, \qquad (2.11)$$

where

$$r'\circ\nabla\nu = \sum_{i=1}^{3}\int_0^{+\infty} r_i'(s)\int_\Omega |\nabla\nu_i(x,s)|^2dxds.$$

Theorem 2.5. *(Exponential stability result)[[28]] Assume that (H1)-(H3) hold true and (2.7). Then there exist positive constants k and K such that*

$$E(t) \le Ke^{-kt}. \qquad (2.12)$$

3 Blow-Up Result

This section is devoted to the blow up of the system. Let us define the following functionals:

$$\begin{aligned}
J(w(t)) = \frac{1}{2}\sum_{i=1}^{3}(\mu-\gamma_i)\int_\Omega |\nabla w_i|^2\,dx + \frac{1}{p^2}\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx \\
- \frac{1}{p}\sum_{i=1}^{3}\int_\Omega |w_i|^p\ln|w_i|\,dx + \frac{\mu+\lambda}{2}\sum_{i=1}^{3}\int_\Omega |\mathrm{div}\, w_i|^2\,dx \qquad (3.1)
\end{aligned}$$

and

$$\begin{aligned}
I(w(t)) = \sum_{i=1}^{3}(\mu-\gamma_i)\int_\Omega |\nabla w_i|^2\,dx - \sum_{i=1}^{3}\int_\Omega |w_i|^p\ln|w_i|\,dx \\
+ (\mu+\lambda)\sum_{i=1}^{3}\int_\Omega |\mathrm{div}\, w_i|^2\,dx \qquad (3.2)
\end{aligned}$$

It is clear that

$$J(w(t)) = \frac{1}{p}I(u(t)) + \frac{1}{p^2}\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx + \left(\frac{p-2}{2p}\right)\sum_{i=1}^{3}(\mu-\gamma_i)\int_\Omega |\nabla w_i|^2\,dx$$
$$+\frac{p-2}{2p}\sum_{i=1}^{3}(\mu+\lambda)\int_\Omega |\mathrm{div}\, w_i|^2\,dx \tag{3.3}$$

and

$$\begin{aligned} E(t) = E(w(t)) &= \frac{1}{2}\sum_{i=1}^{3}\|w_i'(t)\|_2^2 + \frac{1}{2}r\circ\nabla\nu + \frac{b}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i|^2\,ds\,dx + J(w(t)) \\ &= \frac{1}{2}\sum_{i=1}^{3}\|w_i'(t)\|_2^2 + \frac{1}{2}(\mu+\lambda)\sum_{i=1}^{3}\int_\Omega |\mathrm{div}\ w_i|^2\,dx + \frac{1}{2}r\circ\nabla\nu \\ &+ \frac{b}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i|^2\,ds\,dx + \frac{1}{p}I(u(t)) + \frac{1}{p^2}\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx \\ &+ \frac{p-2}{2p}\sum_{i=1}^{3}(\mu-\gamma_i)\int_\Omega |\nabla w_i|^2\,dx \end{aligned} \tag{3.4}$$

Remark 3.1. *The functional $I(w(t))$ differs from that defined in [28]. If $\mathbb{I}(0) > 0$ in (2.7) is replaced by $I(0) > 0$, the results on global existence and exponential stability remain valid, since $\mathbb{I}(0) \geq I(0)$.*

Let us define the potential depth as

$$0 < d = \inf_{w\in H_0^1(\Omega))^3\setminus\{0\}}\ \sup_{l\geq 0} J(lw), \tag{3.5}$$

As demonstrated in [10, 21, 20, 11], it is therefore satisfactory

$$0 < d = \inf_{w\in\mathcal{N}} J(w(t)), \tag{3.6}$$

where $\mathcal{N}$ is the well-known Nehari manifold and is defined as,(see[13, 12])

$$\mathcal{N} = \left\{w \in H_0^1(\Omega))^3 \setminus \{0\} \mid I(w) = 0\right\}.$$

Lemma 3.2. *Let $w \in (H_0^1)^3 \setminus \{0\}$. Then, we have:*

(a) $\lim_{l\to 0^+} J(lw) = 0, \quad \lim_{l\to+\infty} J(lw) = -\infty.$

(b) There exists a unique $l^* > 0$ such that $\frac{d}{dl}J(lw)\big|_{l=l^*} = 0$, and $J(lw)$ is increasing on $(0, l^*)$, decreasing on $(l^*, +\infty)$ and attains its maximum at l^*. Moreover $l^* < 1$ if $I(w) < 0$.

(c) $I(lw) > 0$ for $0 < l < l^*$, $I(lw) < 0$ for $l^* < l < +\infty$, and $I(l^*w) = 0$.

Proof. We will just prove the existence of l^* that satisfies $l^* < 1$ if $I(w) < 0$. The proof of the other points is classical so we omit it. We start by proving that the equation $g(l) = a_1 l^{p-2} + a_2 l^{p-2} \ln l = a_3$, has a unique solution, where a_1, a_2 and a_3 are positive constants. It is clear that g is continuous and differentiable on $(0, +\infty)$ and we have

$$g'(l) = a_1(p-2)l^{p-3} + a_2(p-2)l^{p-3}lnl + a_2 l^{p-3} = l^{p-3}\left(a_1(p-2) + a_2(p-2)lnl + a_2\right).$$

Besides, it is easy to see that

$$g'(l) = 0 \Longleftrightarrow l = l_1 = e^{-\frac{(p-2)a_1+a_2}{(p-2)a_2}},$$

and g is decreasing on $(0, l_1)$ and increasing on $(l_1, +\infty)$. Since $g(l_1) < 0$ and $\lim\limits_{l\to+\infty} g(l) = +\infty$, then the equation $g(l) = a_3$ has a unique solution l^* and we have

$$l^* < 1 \Longleftrightarrow a_3 < a_1.$$

Let

$$\begin{aligned} h(l) = J(lw) = \frac{1}{2}\sum_{i=1}^{3}(\mu-\gamma_i)l^2\left\|\nabla w_i\right\|_2^2 + \frac{l^p}{p^2}\sum_{i=1}^{3}\int_\Omega |w_i|^p dx - \frac{l^p}{p}\sum_{i=1}^{3}\ln l \\ - \frac{l^p}{p}\int_\Omega |w_i|^p \ln|w_i|\, dx + \frac{l^2(\mu+\lambda)}{2}\sum_{i=1}^{3}\|\text{div } w_i\|_2^2. \end{aligned}$$

Differentiate $h(l)$ with respect to l, we get

$$\begin{aligned} h'(l) &= l\sum_{i=1}^{3}\left((\mu-\gamma_i)\|\nabla w_i\|_2^2 + (\lambda+\mu)\|\text{div } w_i\|_2^2\right) \\ &\quad - l^{p-1}\sum_{i=1}^{3}\left(\int_\Omega |w_i|^p \ln|w_i|\, dx + \ln l\, \|w_i\|_p^p\right) \\ &= l\Bigg(\sum_{i=1}^{3}\left((\mu-\gamma_i)\|\nabla w_i\|_2^2 + (\lambda+\mu)\|\text{div } w_i\|_2^2\right) \\ &\qquad - l^{p-2}\sum_{i=1}^{3}\int_\Omega |w_i|^p\ln|w_i|\, dx - l^{p-2}\ln l\sum_{i=1}^{3}\|w_i\|_p^p\Bigg). \end{aligned} \tag{3.7}$$

Putting

$$a_1 = \sum_{i=1}^{3}\int_\Omega |w_i|^p \ln|w_i|\, dx,\; a_2 = \sum_{i=1}^{3}\|w_i\|_p^p,\; a_3 = \sum_{i=1}^{3}\left((\mu-\gamma_i)\|\nabla w_i\|_2^2 + (\lambda+\mu)\|\text{div } w_i\|_2^2\right),$$

we see that $h'(l) = 0$ has a unique solution l^* which satisfies

$$l^* < 1 \Longleftrightarrow \sum_{i=1}^{3}\left((\mu-\gamma_i)\|\nabla w_i\|_2^2 + (\lambda+\mu)\|\text{div } w_i\|_2^2\right) < \sum_{i=1}^{3}\int_\Omega |w_i|^p\ln|w_i|\, dx \Longleftrightarrow I(w) < 0.$$

□

Now (2.11) can be expressed as

$$E'(t) + b\sum_{i=1}^{3}\int_{\Omega}\int_{-\infty}^{+\infty}(\sigma^2+\varrho)|\phi_i(\sigma,t)|^2 d\sigma dx \le 0, \tag{3.8}$$

and hence

$$E(t) + b\int_0^t \sum_{i=1}^{3}\left(\int_{\Omega}\int_{-\infty}^{+\infty}(\sigma^2+\varrho)|\phi_i(\sigma,t)|^2 d\sigma dx\right) ds \le E(0),\ \forall\, 0\le t\le T_{max}. \tag{3.9}$$

Lemma 3.3. *If $E(0) < d$ and $I(w_0) < 0$, then the solution w of the problem (2.4) satisfies*

$$I(w) < 0 \quad \textit{and} \quad E(t) < d \quad \textit{for } t \in [0, T_{\max}). \tag{3.10}$$

Proof. From (3.9), it is clear that $E(t) < d$. Given that $I(w_0) < 0$ and w is continuous on $[0, T_{\max})$

$$I(w(t)) < 0 \quad \text{for some interval } [0, t_1) \subset [0, T_{\max}). \tag{3.11}$$

Let t_0 be the maximum time that satisfies (3.11). If $t_0 < T_{max}$, then $I(w(t_0)) = 0$ and $w(t_0) \ne 0$, that is $w(t_0) \in \mathcal{N}$.
Thus, we obtain from (3.6) that

$$J(w(t_0)) \ge \inf_{w\in\mathcal{N}} J(w(t)) = d.$$

However, this contradicts the fact

$$J(w(t_0)) \le E(t_0) \le E(0) < d.$$

The proof is now complete. □

Remark 3.4. *If $I(w_0) < 0$, then according to Lemma 3.2 and Lemma 3.3, we have, for any small positive constant E_1 such that $0 < E_1 \le d$,*

$$\begin{aligned} E_1 \le d \le J(l_* w_i(t)) &= \frac{(l^*)^p}{p^2}\sum_{i=1}^{3}\int_{\Omega}|w_i|^p\,dx + (l^*)^2\left(\frac{p-2}{2p}\right)\sum_{i=1}^{3}(\mu-\gamma_i)\int_{\Omega}|\nabla w_i|^2\,dx \\ &\quad + (l^*)^2\left(\frac{p-2}{2p}\right)\sum_{i=1}^{3}(\mu+\lambda)\int_{\Omega}|div\, w_i|^2\,dx \\ &< \frac{1}{p^2}\sum_{i=1}^{3}\int_{\Omega}|w_i|^p\,dx + \left(\frac{p-2}{2p}\right)\sum_{i=1}^{3}(\mu-\gamma_i)\int_{\Omega}|\nabla w_i|^2\,dx \\ &\quad + \left(\frac{p-2}{2p}\right)\sum_{i=1}^{3}(\mu+\lambda)\int_{\Omega}|div\, w_i|^2\,dx. \end{aligned} \tag{3.12}$$

Notations: Throughout the remaining of the paper, we will adopt the following notations:

$$\gamma^* = \max_{1\le i\le 3}\gamma_i,\ \gamma_* = \min_{1\le i\le 3}\gamma_i,\ 0 < \alpha = \frac{1}{2}\frac{\mu-\gamma^*}{\mu-\gamma_*} < 1,\ \delta = \frac{\mu(p-2)}{8}. \tag{3.13}$$

Theorem 3.5. *Let the conditions $(H1)-(H3)$ holds and $E(0)<\alpha E_1$ and $I(w_0)<0$, where E_1 is as in Remark 3.4. Moreover, suppose that*

$$\gamma^* < \min\Big\{\frac{\mu}{2}, \frac{\mu p(p-2)}{4}\Big\}. \tag{3.14}$$

Then, the solution to problem (2.4) blows up in finite time.

Proof. Let's define the function $\mathcal{G}(t)$ as

$$\mathcal{G}(t) = \alpha E_1 - E(t). \tag{3.15}$$

From (2.11) in (3.15), we have

$$\mathcal{G}'(t) = -E'(t) \geq b\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}(\sigma^2+\varrho)|\phi_i(\sigma,t)|^2 d\sigma\, dx \geq 0 \tag{3.16}$$

By using (2.9) and (3.12) in (3.15), we get

$$\begin{aligned} 0<\mathcal{G}(0)\leq \mathcal{G}(t) &= \alpha d - E(t)\\ &\leq \frac{1}{p}\sum_{i=1}^{3}\int_\Omega |w_i|^p\ln|w_i|\,dx := K(w(t)). \end{aligned}\tag{3.17}$$

Now, let us define

$$\mathscr{D}(t) = \mathcal{G}(t)^{1-\beta} + \varepsilon\sum_{i=1}^{3}\int_\Omega w_i(t)w_i'(t)\,dx := \mathcal{G}(t)^{1-\beta}+\varepsilon N(t) \tag{3.18}$$

for some $0<\beta<1$ to be determined and differentiating $\mathscr{D}(t)$, then using (2.4), we obtain

$$\begin{aligned} \mathscr{D}'(t) =&(1-\beta)\mathcal{G}(t)^{-\beta}\mathcal{G}'(t) + \varepsilon\sum_{i=1}^{3}\Big\{\|w_i'(t)\|_2^2 - (\mu-\gamma_i)\|\nabla w_i(t)\|_2^2 - (\mu+\lambda)\|div w_i(t)\|_2^2\Big\}\\ &\underbrace{-\varepsilon\sum_{i=1}^{3}\int_\Omega\nabla w_i\int_0^{+\infty}r_i(s)\nabla\nu_i\,dsdx}_{J_1} + \varepsilon\sum_{i=1}^{3}\int_\Omega|w_i(x,t)|^p\ln|w_i|\,dx\\ &\underbrace{-\varepsilon\, b\sum_{i=1}^{3}\int_\Omega w_i\int_{-\infty}^{+\infty}\phi_i(\sigma,t)\varpi(\sigma)d\sigma dx}_{J_2} \end{aligned}\tag{3.19}$$

Therefore, by using (2.9), we get that, for any $0 < \delta_1 < 1 - \frac{2}{p}$,

$$\begin{aligned}\mathscr{D}'(t) &= \varepsilon\left(\frac{p(1-\delta_1)+2}{2}\right)\sum_{i=1}^{3}\|w_i'(t)\|_2^2 + \varepsilon\left(\frac{p(1-\delta_1)-2}{2}\right)\sum_{i=1}^{3}(\mu-\gamma_i)\|\nabla w_i(t)\|_2^2 \\ &+ \varepsilon(\mu+\lambda)\left(\frac{p(1-\delta_1)-2}{2}\right)\sum_{i=1}^{3}\|\mathrm{div} w_i(t)\|_2^2 - \varepsilon p(1-\delta_1)E(t) \\ &+ \varepsilon\delta_1 K(w(t)) + \frac{\varepsilon p(1-\delta_1)}{2} r\circ\nabla\nu - \varepsilon J_1 - \varepsilon J_2 + \frac{\varepsilon(1-\delta_1)}{p}\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx \\ &+ \varepsilon\frac{bp(1-\delta_1)}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(\sigma,t)|^2\,d\sigma dx + (1-\beta)\mathcal{G}(t)^{-\beta}\mathcal{G}'(t).\end{aligned} \tag{3.20}$$

Using Young's inequality, we get that

$$\begin{aligned}|J_1| &\le \delta\sum_{i=1}^{3}\int_\Omega|\nabla w_i|^2\,\mathrm{d}x + \frac{1}{4\delta}\sum_{i=1}^{3}\gamma_i\int_0^{+\infty} r_i(s)\int_\Omega|\nabla\nu_i|^2\,\mathrm{d}x ds \\ &\le \delta\sum_{i=1}^{3}\int_\Omega|\nabla w_i|^2\,\mathrm{d}x + \frac{\gamma^*}{4\delta} r\circ\nabla\nu,\end{aligned} \tag{3.21}$$

Applying the inequality

$$\int_\Omega |u|^p\,dx \le C\left(c_0(\Omega,p) + \int_\Omega |u|^p\ln|u|\,dx\right),$$

one has

$$\begin{aligned}|J_2| &\le bA_0^{\frac{1}{2}}\sum_{i=1}^{3}\|w_i\|_{L^2}\left(\int_\Omega\int_{-\infty}^{\infty}\phi_i(\sigma,t)\left(\varrho+\sigma^2\right)d\sigma dx\right)^{\frac{1}{2}} \\ &\le bA_0^{\frac{1}{2}}\sum_{i=1}^{3}C(\Omega,p)\left(\int_\Omega|w_i|^p\ln|w_i|\,dx + c_0(\Omega,p)\right)^{\frac{1}{p}}\left(\int_\Omega\int_{-\infty}^{\infty}\phi_i(\sigma,t)\left(\varrho+\sigma^2\right)d\sigma dx\right)^{\frac{1}{2}} \\ &\le bA_0^{\frac{1}{2}}\tilde{C}(\Omega,p)\left(K(w(t))^{\frac{1}{p}} + (3c_0)^{\frac{1}{p}}\right)\left(\int_\Omega\int_{-\infty}^{\infty}\phi_i(\sigma,t)\left(\varrho+\sigma^2\right)d\sigma dx\right)^{\frac{1}{2}}.\end{aligned} \tag{3.22}$$

By selecting $0 < \beta < 1 - \frac{2}{p}$ and combining with (3.16) and (3.17), (3.22) can be reduced to

$$\begin{aligned}|J_2| &\le bA_0^{\frac{1}{2}}\tilde{C}(\Omega,p)\left(K(w(t))^{\frac{1}{p}-\frac{1}{2}+\frac{\beta}{2}} + (3c_0)^{\frac{1}{p}}K(w(t))^{-\frac{1}{2}+\frac{\beta}{2}}\right)K(w(t))^{\frac{1}{2}}K(w(t))^{-\frac{\beta}{2}}\mathcal{G}'(t)^{\frac{1}{2}} \\ &\le bA_0^{\frac{1}{2}}\tilde{C}(\Omega,p)\left(\mathcal{G}(0)^{\frac{1}{p}-\frac{1}{2}+\frac{\beta}{2}} + (3c_0)^{\frac{1}{p}}\mathcal{G}(0)^{-\frac{1}{2}+\frac{\beta}{2}}\right)\left(K(w(t)) + \mathcal{G}(t)^{-\beta}\mathcal{G}'(t)\right) \\ &\le \frac{\delta_1}{2}K(w(t)) + C_{\delta_1}\mathcal{G}(t)^{-\beta}\mathcal{G}'(t),\end{aligned} \tag{3.23}$$

for any $\delta_1 > 0$. By substituting (3.21), (3.23) and (3.15) into (3.20), we obtain

$$\begin{aligned}
\mathscr{D}'(t) \geq{}& (1-\beta-\varepsilon C_{\delta_1})\,\mathcal{G}(t)^{-\beta}\mathcal{G}'(t) + \varepsilon\left(\frac{p(1-\delta_1)+2}{2}\right)\sum_{i=1}^{3}\|w_i'(t)\|_2^2 \\
&+ \varepsilon\left(\frac{(p(1-\delta_1)-2)(\mu-\gamma^*)}{2} - \delta\right)\sum_{i=1}^{3}\|\nabla w_i(t)\|_2^2 + \frac{\delta_1}{2}K(w(t)) \\
&+ \varepsilon\left(\frac{(p(1-\delta_1)-2)\,(\mu+\lambda)}{2}\right)\sum_{i=1}^{3}\|\mathrm{div} w_i(t)\|_2^2 - \varepsilon p(1-\delta_1)\alpha E_1 \\
&+ \varepsilon\left(\frac{p(1-\delta_1)}{2} - \frac{\gamma^*}{4\delta}\right) r\circ\nabla\nu + \frac{\varepsilon(1-\delta_1)}{p}\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx \\
&+ \frac{\varepsilon b p(1-\delta_1)}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(\sigma,t)|^2\,d\sigma dx + \varepsilon p(1-\delta_1)\mathcal{G}(t). \qquad (3.24)
\end{aligned}$$

Then by using (3.12) in (3.24), we find that

$$\begin{aligned}
\mathscr{D}'(t) \geq{}& (1-\beta-\varepsilon C_{\delta_1})\,\mathcal{G}(t)^{-\beta}\mathcal{G}'(t) + \varepsilon\left(\frac{p+2}{2}\right)\sum_{i=1}^{3}\|w_i'(t)\|_2^2 \\
&+ \varepsilon\left(\frac{(p(1-\delta_1)-2)\,(\mu-\gamma^*) - \alpha(p-2)(1-\delta_1)(\mu-\gamma_*)}{2} - \delta\right)\sum_{i=1}^{3}\|\nabla w_i(t)\|_2^2 \\
&+ \varepsilon\left(\frac{(p(1-\delta_1)-2)(\mu+\lambda)}{2} - \frac{\alpha(p-2)(1-\delta_1)}{2}\right)\sum_{i=1}^{3}\|\mathrm{div} w_i(t)\|_2^2 \\
&+ \varepsilon\left(\frac{p}{2} - \frac{\gamma^*}{4\delta}\right) r\circ\nabla\nu + \frac{\varepsilon}{p}(1-\alpha)(1-\delta_1)\sum_{i=1}^{3}\int_\Omega |w_i|^p\,dx \\
&+ \frac{\varepsilon b p}{2}\sum_{i=1}^{3}\int_\Omega\int_{-\infty}^{+\infty}|\phi_i(\sigma,t)|^2 d\sigma dx + \varepsilon p(1-\delta_1)\mathcal{G}(t) + \frac{\delta_1}{2}K(w(t)). \qquad (3.25)
\end{aligned}$$

By the help of (3.13) and (3.14) we have

$$\frac{(p-2)\,(\mu-\gamma^*-\alpha(\mu-\gamma_*))}{2} - \delta = \frac{(p-2)(\mu-\gamma^*)}{4} - \frac{\mu(p-2)}{8} > \frac{\mu(p-2)}{8} - \frac{\mu(p-2)}{8} = 0,$$

and

$$\frac{p}{2} - \frac{\gamma^*}{4\delta} = \frac{p}{2} - \frac{2\gamma^*}{\mu(p-2)} > \frac{p}{2} - \frac{p}{2} = 0.$$

Then we can choose δ_1 small enough such that

$$\frac{(p(1-\delta_1)-2)\,(\mu-\gamma^*) - \alpha(p-2)(1-\delta_1)(\mu-\gamma_*)}{2} - \delta > 0$$

and

$$\frac{(p(1-\delta_1)-2)(\mu+\lambda)}{2} - \frac{\alpha(p-2)(1-\delta_1)}{2} > 0.$$

Now, we fix $\varepsilon > 0$ sufficiently small so that $1 - \beta - \varepsilon C_{\delta_1} > 0$. Then (3.25) implies that

$$\mathscr{D}'(t) \geq c_1 \left\{ \mathcal{G}(t) + K(w(t)) + \sum_{i=1}^{3} \left[\|w_i'(t)\|_2^2 + \|\nabla w_i(t)\|_2^2 + \| \operatorname{div} w_i(t)\|_2^2 \right] \right\} \tag{3.26}$$

Once more, let's assume that $\varepsilon > 0$ is small enough to obtain

$$\mathscr{D}(0) = \mathcal{G}(0) + \varepsilon \sum_{i=1}^{3} \int_{\Omega} w_0(x,0) \partial_t w_i(x,0)\, dx > 0 \tag{3.27}$$

After that, we derive from (3.26) and (3.27).

$$\mathscr{D}(t) \geq \mathscr{D}(0) > 0, \ \forall\, t \geq 0.$$

Moreover, we also impose that $0 < \beta < \frac{1}{2} - \frac{1}{p}$. Then we have

$$\mathscr{D}(t)^{\frac{1}{1-\beta}} \leq 2^{\frac{1}{1-\beta}} \left(\mathcal{G}(t) + |N(t)|^{\frac{1}{1-\beta}} \right) \tag{3.28}$$

and

$$\begin{aligned} |N(t)|^{\frac{1}{1-\beta}} \leq & C \sum_{i=1}^{3} \|\partial_t w_i\|_2^{\frac{1}{1-\beta}} \|w_i\|_2^{\frac{1}{1-\beta}} \\ \leq & C \sum_{i=1}^{3} \|\partial_t w_i\|_2^{\frac{1}{1-\beta}} \|w_i\|_p^{\frac{1}{1-\beta}} \\ \leq & C \sum_{i=1}^{3} \left(\|\partial_t w_i\|_2^2 + \|w_i\|_p^{\frac{2}{1-2\beta}} \right). \end{aligned} \tag{3.29}$$

Note that we can obtain a more precise estimate than that in (3.17):

$$0 < \mathcal{G}(t) = \alpha E_1 - E(t) \leq K(w(t)) - \frac{1-\alpha}{p^2} \sum_{i=1}^{3} \|w_i\|_p^p,$$

which entails that

$$\left(\sum_{i=1}^{3} \|w_i\|_p^{\frac{2}{1-2\beta}} \right)^{\frac{(1-2\beta)p}{2}} \leq C \sum_{i=1}^{3} \|w_i\|_p^p \leq C \frac{p^2}{1-\alpha} K(w(t)).$$

Then the above estimate together with (3.17) and (3.29) yields that

$$\begin{aligned} |N(t)|^{\frac{1}{1-\beta}} \leq & C \sum_{i=1}^{3} \|\partial_t w_i\|_2^2 + C K(w(t))^{\frac{2}{(1-2\beta)p}} \\ \leq & C \sum_{i=1}^{3} \left(\|w_i'\|_2^2 + K(w(t))^{\frac{2}{(1-2\beta)p}-1} K(w(t)) \right) \\ \leq & C \sum_{i=1}^{3} \left(\|w_i'\|_2^2 + \mathcal{G}(0)^{\frac{2}{(1-2\beta)p}-1} K(w(t)) \right) \end{aligned}$$

Substituting the above inequality into (3.28) and combining with (3.26) implies that

$$\mathscr{D}(t)^{\frac{1}{1-\beta}} \leq C\mathscr{D}'(t), \quad t \in (0, T_{max}).$$

It implies that

$$T_{max} < C\frac{\beta}{1-\beta}\mathscr{D}(0)^{\frac{\beta}{\beta-1}}.$$

Consequently, the solution blows up at finite time. □

4 Numerical Approximation

In this section, we illustrate numerically the qualitative behavior of the system, including the decay of energy in the stable regime (as established in [28] and recalled in Section 2) and the finite-time blow-up proved in Section 3. For simplicity, we consider numerical examples in dimension $n = 2$. The real case $n = 3$ is merely a general physical description that, due to symmetry, can be reduced to two dimensions $\Omega = (0,1)^2$.

4.1 Finite Volume Approximation

We consider the finite volume method (FVM) for spatial discretization of $w = (w_1, w_2)^T$, with $w_k = w_k(\mathbf{x}, t)$, $k = 1, 2$ and $\boldsymbol{x} = (x, y)$, based on a discretization of finite differences of flux [32]. In this sense, let $\mathcal{T} = \bigcup_{i,j} K_{ij}$ be a rectangular nonuniform structured mesh for the domain $\Omega = (0, l_1) \times (0, l_2)$ in small $N_1 \times N_2$ control volumes $K_{ij} = (x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}) \times (y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}})$, with $x_{i+\frac{1}{2}} = x_{i-\frac{1}{2}} + \delta x_i$, $i = 0, \ldots, N_1$, $y_{j+\frac{1}{2}} = y_{j-\frac{1}{2}} + \delta y_j$, $j = 0, \ldots, N_2$ and $\sum_{i-1}^{N_1} \delta x_i = \sum_{j-1}^{N_2} \delta y_j = 1$. The unknown $w(x, y, t)$ is approximated by $\mathbf{w} = w_{ij}(t)$ in the control volume K_{ij}. When integrating the Laplace operator in a control volume, there are 4 fluxes on the edges of the rectangle of the form $\int_\sigma \nabla w \cdot \mathbf{n} ds(\mathbf{x})$ which is approximated by finite differences in each edge $x = x_{i+\frac{1}{2}}$ and $y = y_{j+\frac{1}{2}}$. In summary, the Laplace operator is approximated by (see [32]):

$$\begin{aligned}\Delta w(x_i, y_j) &\approx \left(\mathbf{D}^2\mathbf{w}\right)_{ij} \\ &= \frac{1}{\delta x_i}\left(\frac{w_{i+1,j} - w_{i,j}}{\delta x_{i+\frac{1}{2}}} - \frac{w_{i,j} - w_{i-1,j}}{\delta x_{i-\frac{1}{2}}}\right) + \frac{1}{\delta y_j}\left(\frac{w_{i,j+1} - w_{i,j}}{\delta y_{j+\frac{1}{2}}} - \frac{w_{i,j} - w_{i,j-1}}{\delta y_{j-\frac{1}{2}}}\right) \quad (4.1)\end{aligned}$$

with $\delta x_{i+\frac{1}{2}} = x_{i+1} - x_i$, $\delta y_{j+\frac{1}{2}} = y_{j+1} - y_j$ and $u_{0,j} = u_{i,0} = u_{i,N_2+1} = u_{N_1+1,j} = u_{0,0} = u_{N_1+1,0} = u_{0,N_2+1} = u_{N_1+1,N_2+1} = 0$, for $i = 1, \ldots, N_1$, $j = 1, \ldots, N_2$. On the other hand, the approximation of the grad-div operator is also obtained from the approximation of the flux that are deduced from the integration over a control volume K_{ij}:

$$\int_{K_{ij}} \nabla(\mathrm{div}(w))\mathrm{d}\mathbf{x} = \begin{pmatrix} \displaystyle\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathrm{div}(w)(x_{i+\frac{1}{2}}, y)\mathrm{d}y - \int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathrm{div}(w)(x_{i-\frac{1}{2}}, y)\mathrm{d}y \\ \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathrm{div}(w)(x, y_{j+\frac{1}{2}})\mathrm{d}x - \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathrm{div}(w)(x, y_{j-\frac{1}{2}})\mathrm{d}x \end{pmatrix}$$

in which each term can be approximated by the following numerical fluxes:

$$\int_{y_{j-\frac{1}{2}}}^{y_{j+\frac{1}{2}}} \mathrm{div}(w)(x_{i+\frac{1}{2}}, y)\mathrm{d}y \approx H_{i+\frac{1}{2},j}$$

$$= \delta y_j \left(\frac{w^1_{i+1,j} - w^1_{i,j}}{\delta x_{i+\frac{1}{2}}} + \frac{w^2_{i+1,j+1} - w^2_{i,j-1} - w^2_{i+1,j} + w^2_{i,j}}{\delta y_{j+\frac{1}{2}} + \delta y_{j-\frac{1}{2}}} \right)$$

$$\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \mathrm{div}(w)(x, y_{j+\frac{1}{2}})\mathrm{d}x \approx H_{i,j+\frac{1}{2}}$$

$$= \delta x_i \left(\frac{w^1_{i+1,j+1} - w^1_{i,j+1} - w^1_{i-1,j} + w^1_{i,j}}{\delta x_{i+\frac{1}{2}} + \delta x_{i-\frac{1}{2}}} + \frac{w^2_{i,j+1} - w^2_{i,j}}{\delta y_{j+\frac{1}{2}}} \right)$$

where $\mathbf{w} = (\mathbf{w}^1, \mathbf{w}^2)^T$ and $\mathbf{w}^k = (w^k_{ij})_{i,j,k}$ for $k = 1, 2$. In summary, the approximation of the grad-div operator is as

$$\nabla(\mathrm{div}(w)) \approx \begin{pmatrix} (\mathbf{D}_{xx}\mathbf{w}^1)_{ij} & (\mathbf{D}_{xy}\mathbf{w}^2)_{ij} \\ (\mathbf{D}_{yx}\mathbf{w}^1)_{ij} & (\mathbf{D}_{yy}\mathbf{w}^2)_{ij} \end{pmatrix}$$

where

$$(\mathbf{D}_{xx}\mathbf{w})_{ij} = \frac{1}{\delta x_i} \left(\frac{w_{i+1,j} - w_{i,j}}{\delta x_{i+\frac{1}{2}}} - \frac{w_{i,j} - w_{i-1,j}}{\delta x_{i-\frac{1}{2}}} \right)$$

$$(\mathbf{D}_{xy}\mathbf{w})_{ij} = \frac{w_{i+1,j+1} + 2w_{i,j} + w_{i-1,j-1} - w_{i,j+1} - w_{i+1,j} - w_{i-1,j} - w_{i,j-1}}{\delta x_i \left(\delta y_{j+\frac{1}{2}} + \delta y_{j-\frac{1}{2}} \right)}$$

$$(\mathbf{D}_{yx}\mathbf{w})_{ij} = \frac{w_{i+1,j+1} + 2w_{i,j} + w_{i-1,j-1} - w_{i,j+1} - w_{i+1,j} - w_{i-1,j} - w_{i,j-1}}{\delta y_j \left(\delta x_{i+\frac{1}{2}} + \delta x_{i-\frac{1}{2}} \right)}$$

$$(\mathbf{D}_{yy}\mathbf{w})_{ij} = \frac{1}{\delta y_j} \left(\frac{w_{i,j+1} - w_{i,j}}{\delta y_{j+\frac{1}{2}}} - \frac{w_{i,j} - w_{i,j-1}}{\delta y_{j-\frac{1}{2}}} \right)$$

4.2 Linear equations of Motion

Let the vector $\mathbf{w}(t) = (\mathbf{w}^1(t), \mathbf{w}^2(t))^T = [w_1(t), \ldots, w^1_J(t), w^2_1(t), \ldots, w^2_J(t)]^\top$, an approximation of $w(x, t)$ in $\mathbb{R}^{2J}$ with $J = N_1 + N_2$. Taking into account the approximations (4.1) and (4.1) in the system (1.1), the following equation of motion is obtained

$$\mathbf{M}\ddot{\mathbf{w}}(t) + \mathbf{K}\mathbf{w}(t) + \mathbf{C}\dot{\mathbf{w}}(t) = \mathbf{J}(\mathbf{w}) \tag{4.2}$$

where $\mathbf{M} = \mathbf{I}_{2J\times 2J}$ is the identity matrix of size $2J \times 2J$,

$$\mathbf{K} \begin{pmatrix} \mathbf{w}^1 \\ \mathbf{w}^2 \end{pmatrix} = \begin{pmatrix} -\mu\mathbf{D}^2\mathbf{w}^1 - (\mu + \lambda) \left(\mathbf{D}_{xx}\mathbf{w}^1 + \mathbf{D}_{xy}\mathbf{w}^2 \right) \\ -\mu\mathbf{D}^2\mathbf{w}^2 - (\mu + \lambda) \left(\mathbf{D}_{yx}\mathbf{w}^1 + \mathbf{D}_{yy}\mathbf{w}^2 \right) \end{pmatrix}$$

is the stiffness matrix, and

$$\mathbf{C} = \mathbf{C}_{memo} + \mathbf{C}_{frac}$$

is the dissipation matrix given by the sum of the two matrices taking part in the approximation of the dissipative terms of the equation $(1.1)_1$:

- $\mathbf{C}_{memo}$ which characterizes the infinite memory dissipative term $\int_0^\infty q(s)\Delta w(t-s)\,ds$;
- $\mathbf{C}_{frac}$ which characterizes the fractional derivative dissipative term $\partial_t^{\varpi,\varsigma}\omega(t)$;

4.3 Time discretization

In order to preserve the energy with a second-order scheme in time, we choose a β-Newmark scheme for w. The method consists of updating the displacement, velocity and acceleration vectors at the current time $t^n = n\delta t$ to the time $t^{n+1} = (n+1)\delta t$, a small time interval δt later. The Newmark algorithm [34] is based on a set of two relations expressing the forward displacement $\mathbf{w}^{n+1}$ and velocity $\dot{\mathbf{w}}^{n+1}$ in terms of their current values and the forward and current values of the acceleration:

$$\dot{\mathbf{w}}^{n+1} = \dot{\mathbf{w}}^{n} + (1-\gamma)\delta t\,\ddot{\mathbf{w}}^{n} + \gamma\delta t\,\ddot{\mathbf{w}}^{n+1}, \tag{4.3}$$

$$\mathbf{w}^{n+1} = \mathbf{w}^{n} + \delta t\dot{\mathbf{w}}^{n} + \left(\frac{1}{2}-\beta\right)\delta t^2\,\ddot{\mathbf{w}}^{n} + \beta\delta t^2\,\ddot{\mathbf{w}}^{n+1}, \tag{4.4}$$

where β and γ are parameters of the methods that will be fixed later. Returning now to the description of nonlocal dissipative matrices, we have the following.

4.4 Infinite memory term.

The infinite memory term $\int_0^\infty r(s)\Delta w(t-s)\,ds$ taking part in the equation $(1.1)_1$, can be approximated by $\mathbf{C}_{memo}\mathbf{w}$. Before specifying the matrix associated with this decay, we must note that this type of infinite memory terms have already been treated numerically in multiple works, for example in [36] in which reasonable results are obtained, however the energy is not conserved and spurious oscillations occur in the decay. In order to avoid these unwanted oscillations, we consider another approximation in which we use the modified model (2.4), and we discretize the variable o^t introduced in (2.2). This allows us to obtain a conservative scheme whose energy will be numerically decreasing. To do this, we approximate $o^t(x,s)$ by $o_j^{m,n}$, for $j=1,\dots,J$, $n=1,\dots,N$ and $m=1,\dots,M$ in the equation (2.2), and defining

$$o_j^{m,n} := \omega_j^n - \omega_j^{n-m} \tag{4.5}$$

Replacing (4.3)-(4.4) in (4.5), we obtain

$$\begin{aligned} o_j^{m,n+1} - o_j^{m-1,n} &= \omega_j^{n+1} - \omega_j^n \\ &= \delta t\left(\frac{\beta}{\gamma}\dot{\omega}_j^{n+1} + \left(1-\frac{\beta}{\gamma}\right)\dot{\omega}_j^{n+1}\right) - \frac{2\beta-\gamma}{2\gamma}\delta t^2\ddot{\omega}_j^n \end{aligned}$$

Then $\gamma = \frac{1}{2}$, $\beta = \frac{1}{2}\gamma$ is chosen, in order to obtain the following conservative scheme

$$o_j^{m,n+1} = o_j^{m-1,n} + \delta t\dot{\omega}_j^{n+\frac{1}{2}} \tag{4.6}$$

with $\dot{\omega}_j^{n+\frac{1}{2}} = \dfrac{\dot{\omega}_j^n + \dot{\omega}_j^{n+1}}{2}$. Then from (2.4), the approximation of the infinite-memory term at time $t = t_{n+1}$ can be written as

$$
\begin{aligned}
-\int_0^{+\infty} q(s)\Delta o^{t_{n+1}}(x,s)ds &\approx -\delta t \sum_{m=1}^{M} q_m \mathbf{D}^2 \mathbf{o}^{m,n+1} \\
&= -\delta t \sum_{m=1}^{M} q_m \mathbf{D}^2 \mathbf{o}^{m-1,n} - \delta t^2 \left(\sum_{m=1}^{M} q_m\right) \mathbf{D}^2 \dot{\mathbf{w}}^{n+\frac{1}{2}} \qquad (4.7)
\end{aligned}
$$

4.5 Fractional derivative term.

In order to numerically simulate the improper integral $(2.1)_3$, we consider $R > 0$ sufficiently large, so that

$$
\partial_t^{\varpi,\varsigma}\omega(t) \approx 2b \int_0^R \theta(\xi,t)\zeta(\xi)d\xi
$$

(we note the parity of the function $\theta\zeta$ with respect to ξ from (2.1)). Let $\xi_\ell := \ell\delta\xi$ $\ell = 1,\ldots,L$, $\delta\xi = L/R$. From (2.1), we define

$$
\zeta_\ell = |\xi_\ell|^{(2\varpi-1)/2}, \quad \ell = 1,\ldots,L, \quad 0 < \varpi < 1.
$$

In the case of this dissipative term, we will simply be inspired by the work of [31], where considering the augmented model of [1] results in a conservative scheme and decreasing numerical energy. Thus, an approximation of the fractional derivative term, is given by

$$
\partial_t^{\varpi,\varsigma}\omega(t) \approx 2b\delta\xi \sum_{\ell=1}^{L} \zeta_\ell \theta_\ell^n. \qquad (4.8)
$$

On the other hand, the system (2.1) can be discretized using the Crank–Nicolson method [35], in order to maintain the conservation of energy, or its nondecrease in case of dissipation. Then, we obtain the following conservative numerical scheme:

$$
\theta_\ell^{n+1} = \theta_\ell^n - \delta t\left(\xi_\ell^2 + \varsigma\right)\theta_\ell^{n+\frac{1}{2}} + \delta t \zeta_\ell \dot{\mathbf{w}}^{n+\frac{1}{2}} \qquad (4.9)
$$

Combining then (4.7) and (4.9) with (4.3) and (4.4), and replacing these expressions in (4.2) for $t = t_{n+1}$ gives the following system of nonlinear equations describing the first part of the conservative scheme:

$$
\begin{aligned}
&\left(\mathbf{M} + \gamma\delta t\,\mathbf{C} + \beta\delta t^2\,\mathbf{K}\right)\ddot{\mathbf{w}}^{n+1} - \mathbf{J}(\mathbf{w}^{n+1}) = \\
&\quad -\mathbf{C}\left(2\dot{\mathbf{w}}^n + (1-\gamma)\,\delta t\ddot{\mathbf{w}}^n\right) - \mathbf{K}\left(\mathbf{w}^n + \delta t\dot{\mathbf{w}}^n + \left(\frac{1}{2} - \beta\right)\delta t^2\ddot{\mathbf{w}}^n\right), \\
&\quad -\delta t \sum_{m=1}^{M} q_k \mathbf{D}^2 \mathbf{o}^{m-1,n} - \frac{\delta t^2}{2}\left(\sum_{m=1}^{M} q_k\right)\mathbf{D}^2\dot{\mathbf{w}}^n - 2b\delta\xi \sum_{\ell=1}^{L} \widetilde{\zeta}_\ell \theta_\ell^n \qquad (4.10)
\end{aligned}
$$

with $\mathbf{C} = \mathbf{C}_{memo} + \mathbf{C}_{frac}$ and

$$\begin{cases} \mathbf{C}_{memo} = -\frac{\delta t}{2}\left(\sum_{m=1}^{M} q_k\right)\mathbf{D}^2 \\ \mathbf{C}_{frac} = \delta t b\left(\sum_{\ell=1}^{L} \frac{2\zeta_\ell^2\delta\xi}{2+\delta t\,(\xi_\ell^2+\varsigma)}\right)\mathbf{I}_J, \qquad \text{with} \quad \widetilde{\zeta}_\ell = \frac{2-\delta t\,(\xi_\ell^2+\varsigma)}{2+\delta t\,(\xi_\ell^2+\varsigma)}\zeta_\ell \end{cases}$$

4.6 Source term $J(\mathbf{w})$

In the first instance we propose a discretizations of the nonlinear term $\mathbf{J}(\mathbf{w})$ for the scheme (4.10), in a quite natural and naive way as

$$\mathbf{J}(\mathbf{w}^{n+1})_j = \mathbf{J}(\omega_j^{n+1}) = \omega_j^{n+1}|\omega_j^{n+1}|^\varrho \ln|\omega_j^{n+1}| \tag{4.11}$$

This choice is reasonable; however, it does not preserve the system's energy when the dissipative terms in equation (1.1) and the corresponding (4.10) scheme are not taken into account. Specifically, when analyzing the (4.10) scheme combined with (4.11), a numerical dissipation of energy is observed, as shown in Figure **??**, along with persistent oscillations around this decay. While dissipation is desirable, we aim for a more accurate scheme free of spurious numerical effects, enabling a clearer assessment of the performance of the approximate dissipative terms. For this reason, we subsequently propose the following conservative scheme:

$$\begin{aligned} \left(\mathbf{M}+\gamma\delta t\,\mathbf{C}+\beta\delta t^2\,\mathbf{K}\right)\ddot{\mathbf{w}}^{n+1} - \mathcal{J}(\mathbf{w}^n,\mathbf{w}^{n+1}) &= \mathcal{L}(\mathbf{w}^n,\dot{\mathbf{w}}^n,\ddot{\mathbf{w}}^n,\mathbf{o}^{\cdot,n},\theta^n) \\ &- \mathbf{C}\left(\dot{\mathbf{w}}^n + (1-\gamma)\,\delta t\ddot{\mathbf{w}}^n\right) - \mathbf{K}\left(\mathbf{w}^n+\delta t\dot{\mathbf{w}}^n+\left(\frac{1}{2}-\beta\right)\delta t^2\ddot{\mathbf{w}}^n\right), \\ &- \delta t\sum_{m=1}^{M} q_m\mathbf{D}^2\mathbf{o}^{m-1,n} - \frac{\delta t^2}{2}\left(\sum_{m=1}^{M} q_m\right)\mathbf{D}^2\dot{\mathbf{w}}^n - b\delta\xi\sum_{\ell=1}^{L}\widetilde{\zeta}_\ell\theta_\ell^n \end{aligned} \tag{4.12}$$

where

$$\mathcal{J}(\mathbf{w}^n,\mathbf{w}^{n+1})_j = \begin{cases} \dfrac{\mathbf{F}(\omega_j^{n+1})-\mathbf{F}(\omega_j^n)}{\omega_j^{n+1}-\omega_j^n} & \text{if } \omega_j^n \neq \omega_j^{n+1} \\ 0 & \text{otherwise,} \end{cases} \qquad j=1,\ldots,J, \tag{4.13}$$

with $\mathbf{F}(\mathbf{w})_j = \dfrac{1}{\varrho^2}|\omega_j|^\varrho(\ln|\omega_j|^\varrho - 1)$ and

$$\begin{aligned} \mathcal{L}(\mathbf{w}^n,\dot{\mathbf{w}}^n,\ddot{\mathbf{w}}^n,\mathbf{o}^{\cdot,n},\theta^n) &= -\left(\mathbf{M}\ddot{\mathbf{w}}^n+\mathbf{K}\mathbf{w}^n+\mathbf{K}\dot{\mathbf{w}}^n\right) \\ &- \delta t\sum_{m=1}^{M} q_m\mathbf{D}^2\mathbf{o}^{m,n} - b\delta\xi\sum_{\ell=1}^{L}\zeta_\ell\theta_\ell^n \end{aligned} \tag{4.14}$$

4.7 Discrete energy and stability

In order to analyze the stability properties of the proposed scheme, we consider a discrete energy functional naturally induced by the algebraic structure of the fully discrete system (4.10)–(4.9). This construction follows the approach introduced in [37], where auxiliary variables are employed to reformulate both the fractional damping and the memory term within a unified framework.

The discrete energy is defined consistently with the numerical implementation and includes contributions from the kinetic and elastic terms, as well as from the auxiliary variables associated with the fractional damping and the memory effect, together with the logarithmic potential term. More precisely, it is computed from the discrete solution $(\mathbf{w}^n, \dot{\mathbf{w}}^n, \boldsymbol{\theta}^n, \mathbf{o}^n)$ in a form that mirrors the continuous energy while accounting for the quadrature and discretization procedures involved.

Although this discrete energy does not coincide exactly with the continuous functional due to spatial discretization and numerical integration of the nonlocal terms, it preserves its qualitative structure and, in particular, its dissipative character.

Discrete energy decay. Under the choice $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$ in the Newmark scheme, and using Crank–Nicolson discretizations for the auxiliary variables, the discrete energy satisfies

$$E^{n+1} \leq E^n. \tag{4.15}$$

This monotonicity property follows from the discrete balance obtained by testing the scheme against the midpoint velocity $\dot{\mathbf{w}}^{n+\frac{1}{2}}$, and from the dissipative contributions induced by the fractional and memory terms. The resulting behavior is fully consistent with the continuous energy estimate derived in Section 2 and with the structure of the augmented formulations considered in [37].

4.8 Numerical examples

We illustrate the behavior of the proposed numerical scheme on the square domain $\Omega = (0,1)^2$, using a uniform mesh with $n = 200$ interior nodes in each direction, so that $\Delta x = \Delta y = 1/(n+1)$. The physical parameters are fixed as $\rho_1 = \rho_2 = 1$, $\mu = 2$, $\lambda = -1$, and $p = 3$. The Newmark parameters are chosen as $\gamma = \frac{1}{2}$ and $\beta = \frac{1}{4}$.

4.8.1 Energy decay test

We first consider the regime corresponding to Theorem 2.5, where the solution remains globally bounded and the energy decays over time.

We take initial data of the form

$$w_1(x, y, 0) = A\,\Phi(x, y), \qquad w_2(x, y, 0) = -A\,\Phi(x, y), \qquad \partial_t w_1 = \partial_t w_2 = 0,$$

with $\Phi(x, y) = \sin(\pi x)\sin(\pi y)$.

The amplitude A is chosen sufficiently small so that the initial data lie in the stable set of the potential well, namely

$$I(w_0) > 0, \qquad E(0) < E_1.$$

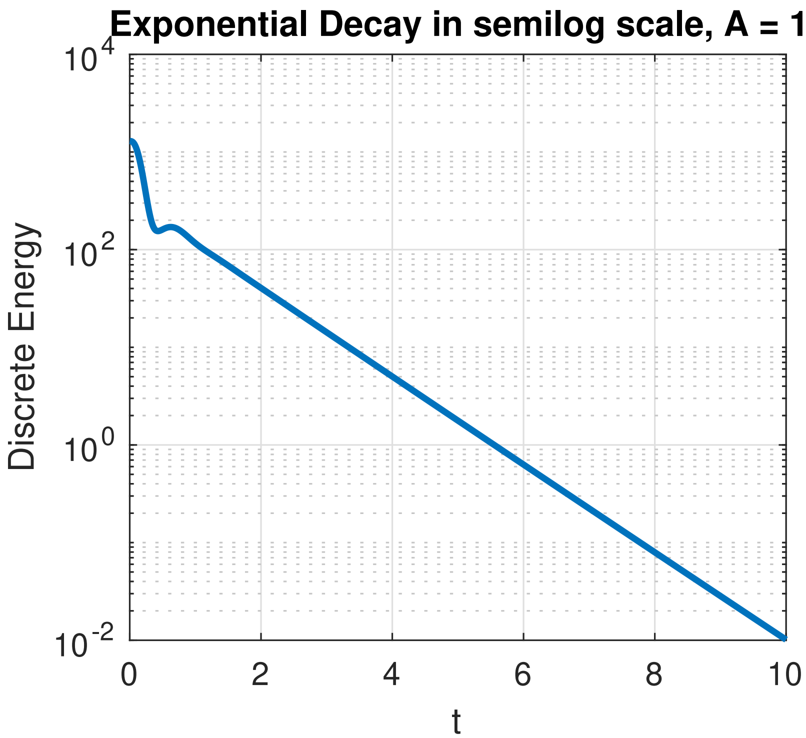
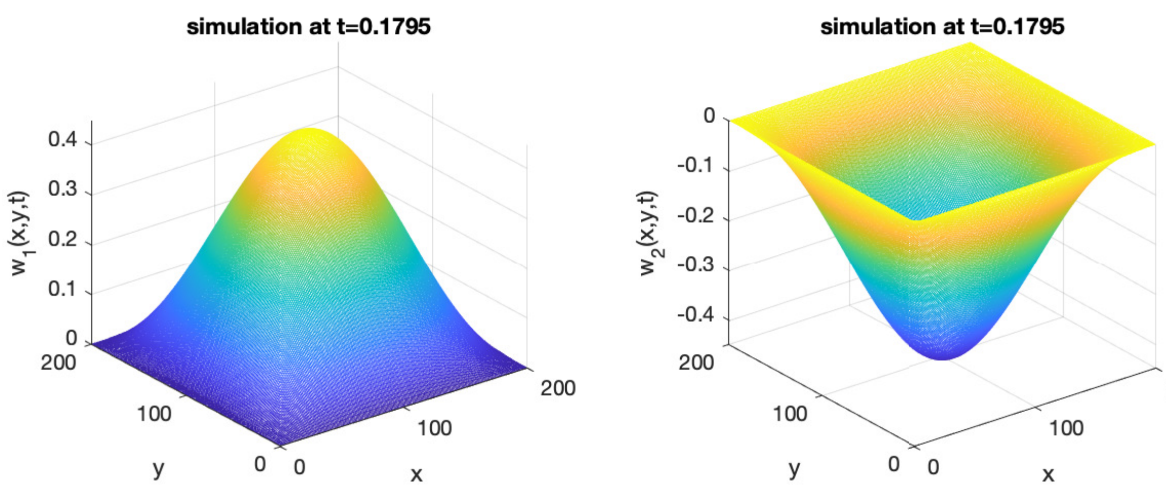


Figure 1: Time evolution of the discrete energy for the decay test with $A = 1$, $T = 10$, and $N_t = 1000$ (hence $\Delta t = 10^{-2}$). The energy decreases monotonically and exhibits an exponential decay, in agreement with the theoretical prediction.



(a) Profile of w_1 at $t = 0.18$.

(b) Profile of w_2 at $t = 0.18$.

Figure 2: Spatial profiles of the displacement components w_1 and w_2 in the stable regime for $A = 1$. The solution is displayed at $t = 0.18$, a time comparable to the blow-up time observed in the unstable case, highlighting the contrast between both behaviors.

For small A, the quadratic part of the energy, which scales as A^2, dominates the nonlinear logarithmic term, which behaves like $A^p \log A$ and becomes negligible as $A \to 0$.

In the present simulation, we take

$$A = 1,$$

which satisfies the above conditions.

The numerical scheme is run with $T = 10$ and $N_t = 1000$, corresponding to $\Delta t = 10^{-2}$. The evolution of the discrete energy is shown in Figure 1, where a clear monotone decay is observed. For comparison with the blow-up case, the solution profiles are displayed at time $t = 0.1795$ in Figure 2.

These results confirm that, when the initial data belong to the stable regime, the solution remains bounded and the energy decays over time, in agreement with the theoretical predictions.

4.8.2 Numerical blow-up test

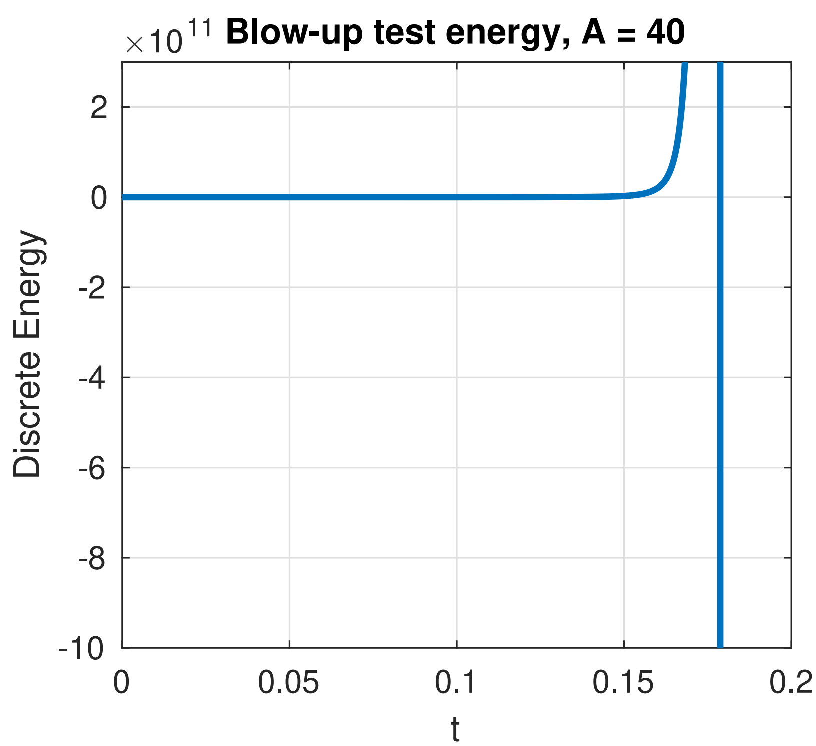
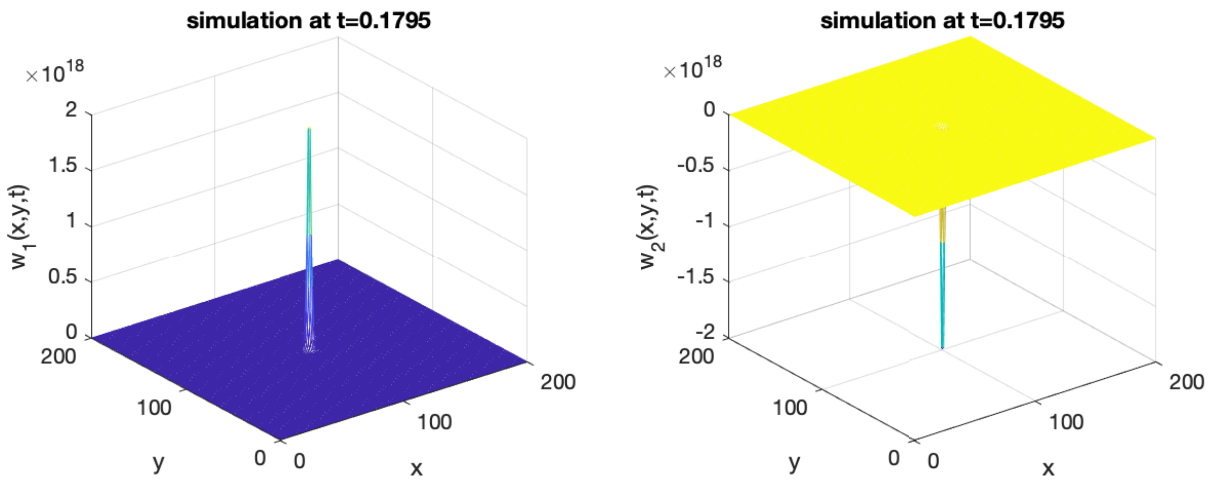


Figure 3: Time evolution of the discrete energy for the blow-up test with $A = 40$, $T = 0.25$, and $N_t = 1000$ (hence $\Delta t = 2.5 \times 10^{-4}$). The energy grows rapidly and becomes non-finite at approximately $t = 0.1795$, indicating the occurrence of blow-up.

We now illustrate the finite-time blow-up behavior predicted by Theorem 3.5. In order to satisfy the structural condition required in that theorem, we choose the memory kernel

$$r(s) = 0.5e^{-s},$$



(a) Profile of w_1.

(b) Profile of w_2.

Figure 4: Spatial profiles of the displacement components w_1 and w_2 in the blow-up regime for the initial amplitude $A = 40$. These profiles illustrate the rapid growth of the solution before the numerical blow-up time $t \approx 0.1795$.

so that

$$\gamma^* = \int_0^\infty r(s)\, ds = 0.5.$$

For the parameters used in the simulation, namely $\mu = 2$, $\lambda = -1$ and $p = 3$, condition (3.16) becomes

$$\gamma^* < \min\left\{\frac{\mu}{2}, \frac{\mu p(p-2)}{4}\right\} = 1,$$

which is satisfied. We consider the initial data

$$w_1(x, y, 0) = A\Phi(x, y), \qquad w_2(x, y, 0) = -A\Phi(x, y),$$

with zero initial velocity and

$$\Phi(x, y) = \sin(\pi x)\sin(\pi y).$$

The amplitude is chosen as $A = 40$. For this choice, the discrete counterparts of the functionals appearing in the blow-up analysis give

$$I(w_0) = -4.60 \times 10^6, \qquad J(w_0) = -7.25 \times 10^5, \qquad E(0) = -7.25 \times 10^5.$$

Thus $I(w_0) < 0$. Moreover, since $E(0) < 0$ and $\alpha = 0.5$, the condition

$$E(0) < \alpha E_1$$

is satisfied for any admissible $E_1 > 0$ such that $0 < E_1 \leq d$. The simulation is performed with $T = 0.25$ and $N_t = 1000$, corresponding to $\Delta t = 2.5 \times 10^{-4}$. The numerical blow-up is detected when the computed energy becomes non-finite. In the present experiment, this occurs at approximately

$$t \approx 0.1795.$$

The corresponding energy evolution and the profiles of w_1 and w_2 are displayed in Figures 3 and 4. These results provide numerical evidence of finite-time blow-up for large initial data and are fully consistent with the unstable dynamics predicted by the analytical study.

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