UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

The asymptotic version of the Erdős–Sós conjecture and beyond

Akbar Davoodi, Diana Piguet,
Hanka Rada, Nicolás Sanhueza-Matamala

PREPRINT 2026-04

SERIE DE PRE-PUBLICACIONES

THE ASYMPTOTIC VERSION OF THE ERDŐS-SÓS CONJECTURE AND BEYOND

AKBAR DAVOODI, DIANA PIGUET, HANKA ŘADA, AND NICOLÁS SANHUEZA-MATAMALA

Abstract. Klimošová, Piguet, and Rozhoň conjectured that any graph with minimum degree $k/2$ and sufficiently many vertices of degree k should contain all trees with k edges. We prove an asymptotic version of this conjecture for dense host graphs. We obtain interesting corollaries: the first is an asymptotic version of the Erdős–Sós conjecture for dense host graphs, which works without any bounded-degree restriction on the guest trees. Secondly, by leveraging recent results by Pokrovsky, we can translate our results to sparse host graphs in the case of bounded-degree guest trees.

Contents

1. Introduction

One of the most classical questions in graph theory is to determine the number of edges in a host graph G that forces the existence of a copy of another guest graph H. Turán's theorem [Tur41] gives a complete answer whenever H is a clique, and the Erdős–Stone–Simonovits theorem [ES46; ES66] gives a satisfactory answer (up to lower order terms) for every graph with chromatic number at least 3. However, much less is known for bipartite H, and even the particular case of trees remains widely open. A seminal conjecture by Erdős and Sós [Erd64] says that graphs with average degree larger than $k-1$ should contain all k-edge trees.

Conjecture 1.1 (Erdős–Sós conjecture)**.** Every graph G with average degree $d(G) > k-1$ contains every tree with k edges.

The conjecture has been verified for specific families of trees (e.g. paths [EG59], trees with diameter at most four [McL05], spiders [FHL18], etc. See also [GKL16; Dav+18] for further variations in hypergraphs, and the survey by Stein [Ste20] for more results). Besomi, Stein and Pavez-Signé [BPS21] verified the Erdős–Soś conjecture for bounded-degree trees in dense host graphs, which then was extended to sparse host graphs by Pokrovskiy [Pok24b]. A solution of the Erdős–Sós conjecture for all large enough trees was announced in the early 1990s by Ajtai, Komlós, Simonovits and Szemerédi, though it has not yet been made available (a sketch of the proof can be found in [Ajt+15]).

For some classes of trees (such as stars, paths, trees with diameter at most three) even a weaker condition than that of Conjecture 1.1 on the host graph suffices: specifically, it is enough to assume that $\Delta(G) \geq k$ and $\delta(G) \geq k/2$ (this can be seen, e.g. by following the proofs by Erdős and Gallai [EG59]). This is indeed weaker, since if a graph satisfies $d(G) > k-1$ then a well-known argument implies that G contains a subgraph G' with $\Delta(G') \geq k$ and $\delta(G') \geq k/2$.

However, this new condition on the host graph is not enough to ensure the containment of all trees. It fails for trees of diameter four, as shown by examples of Havet, Reed, Stein and Wood [Hav+20, §1]. In those examples, the host graphs consist either of two cliques or a complete bipartite graph together with an extra universal vertex, which turns out to be the unique vertex of degree at least k. In view of this situation, it is natural to think that perhaps the following is true: if we have a substantial number of vertices of degree at least k in G (instead of just one), and we also have the condition $\delta(G) \geq k/2$, then we should find all k-edge trees as subgraphs of G. A conjecture along these lines was proposed by Klimošová, Piguet, and Rozhoň [Roz19, Conjecture 1.4].

Conjecture 1.2 (Klimošová, Piguet, Rozhoň)**.** Every n-vertex graph G with $\delta(G) \geq k/2$ and at least $n/(2\sqrt{k})$ vertices of degree at least k contains all k-edge trees.

Our main result is an approximate version of Conjecture 1.2 for dense graphs, i.e., with quadratically many edges.

Theorem 1.3 (Main result)**.** *For any $\eta, q > 0$, there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and all $k \geq qn$, any n-vertex graph G with minimal degree $\delta(G) \geq (1+\eta)k/2$ and with at least ηn vertices of degree at least $(1+\eta)k$ contains all k-edge trees.*

Theorem 1.3 implies an approximate dense version of the Erdős-Sós conjecture.

Corollary 1.4. *For any $\eta, q > 0$ there exists an $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and all $k \geq qn$ any n-vertex graph with average degree more than $(1+\eta)k$ contains any tree on at most k edges as its subgraph.*

Corollary 1.4 strengthens similar results by Rozhoň [Roz19] and Besomi, Pavez-Signé and Stein [BPS19; BPS21], which had the extra requirement that the trees T to be found as subgraphs satisfy $\Delta(T) = o(k)$. It also gives a proof independent of the one proposed by Ajtai, Komlós, Simonovits, and Szemerédi [Ajt+15] in the case the host graph is dense, under the very mild strengthening that its average degree is required to be slightly larger than k (see §11.1 for a comparison between their approach and ours). Recently, Reed and Stein [Ree25] announced a resolution of the Erdős–Sós conjecture for the case of 'dense trees', i.e. that one can replace $(1+\eta)k$ with $k-1$ in Corollary 1.4.

Both of our results mentioned above work only in the setting of dense host graphs and linear-sized trees. However, a recent key structural result by Pokrovskiy [Pok24a] proves that the dense case in fact encapsulates most of the difficulty of the general tree-embedding problem; his result reduces problems about embedding bounded-degree trees in host graphs (non-necessarily dense) to the case of dense host graphs. In our situation, these tools allow us to translate our Theorem 1.3 to the sparse setting, in the case of bounded-degree trees.

Corollary 1.5. *For $\Delta \in \mathbb{N}$, and $\eta > 0$, let $k_0 \in \mathbb{N}$ be sufficiently large. Then for any $k \geq k_0$ and any graph G with minimal degree $\delta(G) \geq (1+\eta)k/2$ and at least $\eta|V(G)|$ vertices of degree at least $(1+\eta)k$ contains as its subgraph any tree T on at most k edges with bounded maximal degree $\Delta(T) \leq \Delta$.*

There is a chance that our methods can be used to attack other tree-embedding conjectures which combine minimum and maximum degree conditions; see §11.2 for more discussion.

As a consequence of Corollary 1.4, we can quickly obtain bounds for the multicolour Ramsey numbers of trees. Given graphs $T_1, \ldots, T_r$, the *r-colour Ramsey number of* $T_1, \ldots, T_r$, denoted by $R_r(T_1, \ldots, T_r)$, is the least N so that every complete graph on N vertices which is edge-coloured with $\{1, \ldots, r\}$, contains a monochromatic copy of T_i in the ith colour, for some i. We write $R_r(T)$ if $T = T_1 = \cdots = T_r$. Erdős and Graham [Erd81] conjectured that for every $r \geq 2$ and every n-vertex tree T, the bound $R_r(T) \leq rn + O(1)$ should hold. This would follow from the validity of Conjecture 1.1. The case $r = 2$ was proven by Zhao [Zha11] for all large n. The following result gives an asymptotically tight bound for $R_r(T_1, \ldots, T_r)$ for arbitrary trees, generalising a result of Piguet and Stein [PS12] (for two colours) and Klimošová, Piguet, and Rozhoň [KPR20] (who assumed further 'skew' properties of the trees). Also if $T = T_1 = \cdots = T_r$ note that this gives an approximate version of the conjecture of Erdős and Graham.

Corollary 1.6. *For $r \geq 2$ and $\varepsilon > 0$, there exists n_0 such that for any trees $T_1, \dots, T_r$ with $\sum_{i=1}^{r} |V(T_i)| \geq n_0$, we have $R_r(T_1, \dots, T_r) \leq (1+\varepsilon) \sum_{i=1}^{r} |V(T_i)|$.*

1.1. **Rough sketch of the proof of our main result.** The proof follows the well-known regularity method. First, we prepare both the host graph and the tree for embedding (i.e., finding a copy of the tree within the graph). We prepare the host graph by applying Szemerédi's Regularity Lemma, which yields the so-called *reduced graph*. For the tree, we find a negligible set of cut vertices that partition the tree into smaller subtrees with specific properties.

Next, in the reduced graph, we identify a suitable structure to assist in embedding the tree. This involves selecting two adjacent clusters, each with a sufficiently large degree relative to a structure that matches the shape of the tree. One of these clusters will have a total (weighted) degree slightly greater than k (after re-scaling by the cluster size), a condition guaranteed by the degree-inheritance principle of the cluster graph. The second cluster will be a carefully chosen neighbour. These two adjacent clusters will host the cut vertices of the tree. The tree-specific structure is designed to accommodate the small subtrees. While many previous tree-embedding problems have used a simple matching for this structure, we have developed a significantly more sophisticated approach – a *skew-matching pair*. This structure builds on the idea of fractional matching but assigns different weights to each end-vertex. This refinement enables a much more precise replication of the global structure of the tree, offering a substantial improvement over previous methods and underscoring its key role in proving our result.

Finally, we embed the tree using the structure established in the cluster graph during the previous phase. The embedding process combines standard properties of regular pairs with probabilistic arguments.

1.2. **Organisation of the paper.** In §2, we set some very general notation. In §3, we derive from our main result (Theorem 1.3) a proof of the announced corollaries, Corollary 1.4 and Corollary 1.5. In §4, we first give some rough overview of the proof, and we prove our main result (Theorem 1.3) in §4.4, assuming the three main technical propositions: the Tree-Coating Lemma (Lemma 4.1), the Structural Proposition (Proposition 4.2), and the Tree-Embedding Lemma (Lemma 4.5). In §5 we prove the Tree-Coating Lemma.

The proof of the Structural Proposition (Proposition 4.2) spans the next four sections. In §6 we give some additional notation and definitions of the objects we will use in that proof, including the key 'skew-matchings'. In §7 we investigate the structure of skew-matchings in general graphs, based on known results for matchings in graphs. In §8 we prove various 'building blocks' associated with skew-matchings that serve as lemmas in our proof of Proposition 4.2 (Structural Proposition); then we finally give the proof in §9. Finally, in §10 we prove Lemma 4.5 (Tree Embedding Lemma). We conclude with closing remarks in §11.

We remark that the bulk of the length of the paper is the proof of the two main technical lemmas: the proof of the Structural Proposition (Proposition 4.2) (which is in §6–§9) and the proof of the Tree-Embedding Lemma (Lemma 4.5) (§10). The latter can be read independently of the previous sections (using the definition of skew-matchings pair as a black-box).

2. Notation

Basic notation. We sometimes use the hierarchy symbol $\ll$, where $a \ll b$ informally means *"a is much smaller than b"*, and formally translates to "there is a monotone increasing function $f : (0,1) \to (0,1)$ such that for any a, b satisfying $a \leq f(b)$, the following holds".

Graphs. Given a graph G and a subset $S \subseteq V(G)$, we write $G - S$ to refer to the graph obtained from G after removing S and any edges incident to S.

For two disjoint subsets X and Y of $V(G)$, we denote by $d(X,Y)$ the *bipartite density* of the pair (X,Y), defined by

$$d(X,Y) := \frac{|E(X,Y)|}{|X||Y|},$$

in which $|E(X,Y)|$ denotes the number of edges between X and Y. For a graph G, we denote by $d(G)$ the *average degree* of G, i.e. $d(G) := 2|E(G)|/|V(G)|$. For a vertex $v \in V(G)$, let $N_G(v)$ denote the set of neighbours of v in G. We will omit G from the notation if the graph is clear from context. A *vertex-cover* of a graph G (or just *cover* for short) is a set $C \subseteq V(G)$ such that for every edge $xy \in E(G)$ we have $\{x,y\} \cap C \neq \emptyset$.

A graph H *embeds* in a graph G if there is a subgraph $H' \subseteq G$ such that H' is isomorphic to H.

Digraphs. In *digraphs* every edge is oriented, meaning that it consists of an ordered pair of vertices, which we shall denote by a pair with an arrow on top (e.g. $\overrightarrow{uv}$ to mean the ordered pair (u,v)) to differentiate well with the non-oriented pairs. In the digraphs we will use during our proofs we admit cycles of length 2 (where the pairs of edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are both present), but no digraph will have parallel edges in the same direction, and we also forbid loops. If every time the pair $\overrightarrow{uv}$ is present we also have $\overrightarrow{vu}$, we say that the digraph is *symmetric*. Given a vertex $u \in V(G)$, the sets $N^+_G(u) = \{v \in V(G) : \overrightarrow{uv} \in E(G)\}$ and $N^-_G(u) = \{v \in V(G) : \overrightarrow{vu} \in E(G)\}$ are the *outneighbours* and *inneighbours* of u, respectively. Again, we will omit G from the notation if the digraph is clear from context.

Given a graph G, its *associated digraph* $G^{\leftrightarrow}$ is the digraph with the same vertex set as G and both $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are present for each undirected $uv \in E(G)$. Observe that this digraph is symmetric. Moreover, we use $V(G)$ instead of $V(G^{\leftrightarrow})$ and $N_G(u)$ instead of $N^+_G(u)$ or $N^-_G(u)$ when we deal with associated digraphs. (It is because in this case, we have $V(G) = V(G^{\leftrightarrow})$ and $N_G(u) = N^+_G(u) = N^-_G(u)$.)

Weighted graphs and digraphs. A *weighted graph* is a pair (G,w) where G is a graph and $w : E(G) \to \mathbb{R}^+$ is a function. We define the *weighted degree* of a vertex v as $\deg_w(v) := \sum_{u \in N(v)} w(vu)$. We can implicitly assume that $w(vu) = 0$ for any $uv \notin E(G)$, so it makes sense to evaluate w on non-edges of G. Given this assumption, we also have $\deg_w(v) = \sum_{u \in V(G)} w(vu)$. Given a set $S \subseteq V(G)$, we also define $\deg_w(v,S) := \sum_{u \in S} w(vu)$. Also, the *minimum* and *maximum weighted degree* are defined as $\delta_w(G) := \min\{\deg_w(v) : v \in V(G)\}$, and $\Delta_w(G) := \max\{\deg_w(v) : v \in V(G)\}$, respectively. Similarly, for $v \in V(G)$ we define $N_w(v) := \{u \in V(G) :\ w(vu) > 0\}$.

Similarly, we can define a weighted digraph putting a weight on each directed edge. We will mostly consider (weighted) *digraphs* $G^{\leftrightarrow}$ associated to weighted graphs (G,w) as well with their weights inherited from G, i.e. $w(\overrightarrow{uv}) = w(\overrightarrow{vu}) = w(uv)$. However, in general the weights in digraphs do not need to be symmetric, and we may have symmetric digraph with non-symmetric weights[1]. Similarly as above, we can define degrees in weighted directed graphs by $\deg_w(u) := \sum_{u \in V(G)} w(\overrightarrow{vu})$, $\deg_w(u,S) := \sum_{u \in S} w(\overrightarrow{vu})$, and $N_w(G) := \{u \in V(G) :\ w(\overrightarrow{vu}) > 0\}$.

3. Proof of the corollaries

3.1. **Proof of Corollary 1.4.** We prove that our main result (Theorem 1.3) implies the asymptotic version of the Erdős-Sós Conjecture in the setting of dense graphs (Corollary 1.4).

Proof of Corollary 1.4. Let $k = rn$ with $r \geq q > 0$, and let G be a graph on n vertices with average degree at least $(1+\eta)k$. It is well-known [Die17, Proposition 1.2.2] that G

[1]For example, in associated weighted digraphs $G^{\leftrightarrow}$ after altering its inherited weights; we may lose symmetry when using the concept of truncated weighted digraphs (Definition 6.12).

contains an induced subgraph H such that $\delta(H) \geq d(H)/2 \geq d(G)/2 \geq (1+\eta)k/2$. Let m be the number of vertices of H. We clearly have $(1+\eta)k/2 \leq \delta(H) < m \leq n$.

For a given $\lambda > 0$, let X_λ be the set of vertices of H whose degree in H is at least $(1+\lambda)k$. Then we have

$$(1+\eta)km \leq md(H) = \sum_{v \in V(H)} \deg_H(v) \leq |X_\lambda|m + (m - |X_\lambda|)(1+\lambda)k,$$

which, by rearranging, gives

$$|X_\lambda| \geq \frac{(\eta-\lambda)km}{m-(1+\lambda)k} \geq (\eta-\lambda)k.$$

From now on, fix the choice $\lambda := \eta k/(m+k)$. This choice satisfies $\eta \geq \lambda \geq \eta r/(1+r)$, and from the previous calculations we deduce that H satisfies $\delta(H) > (1+\lambda)k/2$, and has at least $(\eta-\lambda)k = \lambda m$ vertices of degree at least $(1+\lambda)k$. Thus, the statement follows by applying Theorem 1.3 to H, with λ playing the role of η. ■

3.2. **Proof of Corollary 1.5.** Now we prove Corollary 1.5, which is a sparse version of Theorem 1.3 for bounded-degree trees. We shall use the following very recent 'hyperstability' result of Pokrovskiy [Pok24a, Theorem 5.2].

Theorem 3.1. *For any $\Delta \geq 1$ and $\varepsilon > 0$, there exists $d > 0$ such that the following holds. Let T be a tree with d edges and $\Delta(T) \leq \Delta$. For any n-vertex graph G having no copies of T, it is possible to delete εdn edges to get a graph H each of whose components has a vertex-cover of order at most $(2+\varepsilon)d$.*

We shall also use the following result of Piguet and Stein [PS12, Theorem 2], which is a dense approximate version of the Loebl–Komlós–Sós conjecture.

Theorem 3.2. *Let $\eta, q \gg 1/n$. Let G be an n-vertex graph G and $k \geq qn$. If at least half of the vertices in G have degree at least $(1+\eta)k$, then G contains every tree with at most k edges.*

Proof of Corollary 1.5. The proof consists of several steps. First, we apply Theorem 3.1 to obtain a subgraph with nearly the same number of edges, but composed of a union of components, each with a small vertex-cover. In the second step, we select a suitable component to work with and consider two cases: either the component is sufficiently small, or it is still quite large. In the first case, we refine the component to obtain a subgraph H_1 which retains similar properties to the original graph. Exploiting the small size of the component, we reduce the problem to its dense instance and apply Theorem 1.3 (our main result) to H_1 to find a copy of the tree. In the second case, we carefully select a small subgraph H_2 of the component that includes the entire vertex-cover. This reduction transforms the problem into the dense instance of the Loebl-Komlós-Sós case, so we can apply Theorem 3.2 to H_2 to obtain a copy of the tree.

Step 1: Decomposition via hyperstability. Let $\Delta \in \mathbb{N}$ and $\eta > 0$ be given. Set

$$\varepsilon := \left(\frac{\eta}{8}\right)^4, \qquad \eta' := \frac{\eta}{6^2},$$

and let k_0 be large enough such that it satisfies the requirement on d in Theorem 3.1 with input Δ and ε, and ensures $k_0 \geq qn$ for any $n \geq n_0$, where $q = \max\left\{\varepsilon^{-1}, \Delta\right\}$ and n_0 is the largest threshold required by Theorems 3.2 and 1.3, so that the conditions of both theorems are satisfied.

Let T be a tree on at most k edges with $\Delta(T) \leq \Delta$. Assume that G does not contain any copy of T, as otherwise we are done. By Theorem 3.1, we can erase a set of edges $E' \subseteq E(G)$ with $|E'| \leq \varepsilon kn$ so that the resulting graph $H = (V(G), E(G) \setminus E')$ consists of components $C_1, \ldots, C_t$, each of which has a cover of order at most $3k$.

Step 2: Component selection. Let $L := \{v \in V(G) : \deg_G(v) \geq (1+\eta)k\}$ and $L_i = L \cap C_i$. Let $I^- := \{i \in [t] : |L_i| < \frac{\eta}{2}|C_i|\}$ and $I^+ := [t] \setminus I^-$. Let $L^+ := \bigcup_{i \in I^+} L_i$. We have

$$|L^+| = |L| - |L^-| \geq \eta n - \sum_{i \in I^-} |L_i| > \eta n - \frac{\eta}{2} \sum_{i \in I^-} |C_i| \geq \frac{\eta}{2} n \Big(2 - \frac{\sum_{i \in I^-} |C_i|}{n}\Big) \geq \frac{\eta}{2} n.$$

We shall discard $\bigcup_{i \in I^-} C_i$ and consider only $H^+ := \bigcup_{i \in I^+} C_i$, where each component contains a substantial portion of its vertices of large degree (in G).

Let $B_i := \{v \in C_i : \deg_H(v) < \deg_G(v) - \frac{\eta}{8}k\}$. In words, B_i corresponds to those vertices in C_i whose degree dropped substantially when erasing the edges E'. We have

$$\sum_{i \in I^+} |B_i| = \Bigg|\bigcup_{i \in I^+} B_i\Bigg| \leq \Bigg|\bigcup_{i \in [t]} B_i\Bigg| \leq \frac{2|E'|}{\eta k/8} \leq \frac{16\varepsilon k n}{\eta k} = \frac{16\varepsilon}{\eta} n,$$

which implies that

$$\sum_{i \in I^+} |C_i| \geq \sum_{i \in I^+} |L_i| = |L^+| \geq \frac{\eta n}{2} \geq \frac{\eta}{2} \cdot \frac{\eta}{16\varepsilon} \sum_{i \in I^+} |B_i|.$$

Hence, there is an index $i_0 \in I^+$ such that

$$|C_{i_0}| \geq \frac{\eta^2}{32\varepsilon} |B_{i_0}|. \tag{1}$$

We select this component and show there is a copy of T in C_{i_0}.

Step 3: Small component case - reduction to the dense case. First, assume that $|C_{i_0}| < (\eta')^{-1}k$. Then observe that, by (1),

$$|B_{i_0}| \leq \frac{32\varepsilon}{\eta^2} |C_{i_0}| < \frac{32\varepsilon}{\eta'\eta^2} k < \frac{\eta}{4} k.$$

Set $H_1 := H[C_{i_0} \setminus B_{i_0}]$. Trivially $|V(H_1)| \leq |C_{i_0}|$ holds. Then, as $i_0 \in I^+$, we have $|L_{i_0} \cap V(H_1)| \geq |L_{i_0}| - |B_{i_0}| \geq \frac{\eta}{2}|C_{i_0}| - \frac{32\varepsilon}{\eta^2}|C_{i_0}| \geq \frac{\eta}{4}|C_{i_0}| \geq \eta'|V(H_1)|$. Moreover, for every $v \in V(H_1)$, we have

$$\deg_{H_1}(v) \geq \deg_H(v) - |B_{i_0}| \geq \deg_G(v) - \frac{\eta}{8}k - \frac{\eta}{4}k.$$

Therefore, $\delta(H_1) \geq (1+\eta)k/2 - \frac{3\eta}{4}k/2 \geq (1+\eta')k/2$ and for any $v \in L_{i_0} \cap V(H_1)$, we obtain $\deg_{H_1}(v) \geq (1+\eta)k - \frac{3\eta}{8}k \geq (1+\eta')k$. We thus can apply Theorem 1.3 to H_1 (in which we have at least k vertices), with η' playing the role of η, $k/|V(H_1)|$ implicitly satisfying the role of q, and k corresponding to itself. This allows us to obtain a copy of T in $H_1 \subseteq G$.

Step 4: Large component case - reduction to the Loebl-Komlós-Sós dense case. We are left to consider the case when $|C_{i_0}| \geq (\eta')^{-1}k$. Let D_{i_0} be the vertex-cover of C_{i_0} of size at most $3k$. This means that every edge in C_{i_0} is incident to at least one vertex in D_{i_0}. Now, we note that

$$|L_{i_0}| \geq \frac{\eta}{2}|C_{i_0}| > \frac{32\varepsilon}{\eta^2}|C_{i_0}| + \frac{\eta}{4}|C_{i_0}| \geq |B_{i_0}| + \frac{\eta}{4\eta'}k \geq |B_{i_0}| + 3|D_{i_0}|,$$

where the first inequality holds since $i_0 \in I^+$, and the remaining inequalities follow from (1), the choices of parameters ε and η', and $|D_{i_0}| \leq 3k$. Hence, we can choose U_{i_0} as a subset of $L_{i_0} \setminus (D_{i_0} \cup B_{i_0})$ with size $2|D_{i_0}|$.

Now, we set $H_2 := H[U_{i_0} \cup D_{i_0}]$. Observe that as D_{i_0} is a vertex-cover and disjoint from U_{i_0}, so every vertex in U_{i_0} has all of its neighbors in H_2 in D_{i_0}. For any vertex $v \in U_{i_0} \cap V(H_2)$, we have

$$\deg_{H_2}(v) = \deg_H(v) \geq \deg_G(v) - \frac{\eta}{8}k \geq (1+\eta')k.$$

This is because $\deg_G(v) \geq (1+\eta)k$ for $v \in U_{i_0} \subseteq L_{i_0}$, and since $U_{i_0} \cap B_{i_0} = \emptyset$, the vertex v lost at most $\frac{\eta}{8}k$ of its degree in passing from G to H.

As ensuring a sufficiently high minimum degree in H_2 is challenging, our strategy is to apply Theorem 3.2 instead of Theorem 1.3 to H_2. To apply Theorem 3.2, we note that $|V(H_2)| = |U_{i_0}| + |D_{i_0}| = 3|D_{i_0}| \leq 9k$. Since $|U_{i_0}| = 2|D_{i_0}| > \frac{|V(H_2)|}{2}$, the conditions of Theorem 3.2 are satisfied. Thus, we apply Theorem 3.2 to H_2 (in which we have at least k vertices), with $k/|V(H_2)|$ playing the role of q, η' playing the role of η, and $|V(H_2)|$ playing the role of n. These choices ensure that the conditions of the theorem are satisfied, allowing us to find a copy of T in $H_2 \subseteq G$. This finishes this case, and since there are no more cases, this finishes the proof. ■

We remark that, using the stability result of Pokrovsky (Theorem 3.1) in conjunction with our dense approximate version of the Erdős-Sós conjecture (Corollary 1.4), it is possible to deduce an approximate version of the Erdős-Sós conjecture for bounded-degree trees in sparse graphs. However, this was already deduced by Pokrovsky [Pok24a, Theorem 1.6] from the results of Rozhoň [Roz19] and Besomi, Pavez-Signé and Stein [BPS19; BPS21] mentioned in the introduction. In fact, this strategy was used to obtain an *exact* version of the Erdős-Sós conjecture for large, bounded-degree trees [Pok24a, Theorem 1.16]. This strengthening additionally requires further stability analysis based on ideas of Besomi, Pavez-Signé and Stein [BPS21], see [Pok24b] for full details.

3.3. **Proof of Corollary 1.6.**

Proof of Corollary 1.6. We prove it with $2r\varepsilon$ in place of ε. Let $n := \sum_{i=1}^{r} |V(T_i)|$, which we assume to be sufficiently large with respect to r and ε; and let $N := (1 + 2r\varepsilon)n$. Let K_N be r-edge-coloured. By averaging, there must exist $i \in \{1, \dots, r\}$ such that the subgraph G_i consisting of the ith coloured edges has average degree at least $(1+\varepsilon)\max\{|V(T_i)|, \varepsilon n\}$. By Corollary 1.4, G_i must contain each tree with at most $\max\{|V(T_i)|, \varepsilon n\} \geq |E(T_i)|$ edges, so in particular there is an i-coloured copy of T_i in K_N, as desired. ■

4. Proof of the main result

This section focuses on proving Theorem 1.3. The proof relies on the regularity method, which consists of three standard steps: pre-processing the host graph G using Szemerédi's Regularity Lemma (Theorem 4.3) which yields a so-called *reduced graph* that captures the large-scale structure of G. Then, we find a suitable structure (Proposition 4.2) in the reduced graph. Finally, we use the structure in the reduced graph to embed the guest tree T in G (Lemma 4.5).

Before delving into the proof, we provide some insights and outline three key statements necessary for this proof. (Readers unfamiliar with the regularity method and its terminology can check §4.3 for the main concepts and definitions.) We will identify a specific structure in the reduced graph of the host graph, which we call a *skew-matching pair* (see Definition 6.4). This structure helps in embedding the tree. Similar to how a 'connected matching' in the reduced graph serves as a guide to embed a long path or a cycle in a host graph, a skew-matching pair in the reduced graph will correspond to a specific part of the host graph where there is enough space to embed the tree. However, the shape of the skew-matching pair is heavily dependent on the structure of the tree we want to embed. To understand what kind of shape we need, we will observe that any tree can be broken down into a negligible-sized set of cut vertices and two forests $\mathcal{F}_A, \mathcal{F}_B$ that consist of tiny trees. What is most relevant here is the colour classes (i.e. the natural bipartition) of the forests $\mathcal{F}_A$ and $\mathcal{F}_B$. Trees with forests with colour classes of the same sizes are grouped into the same tree-class $\mathcal{T}$. Then the skew-matching pair encodes and represents some tree-class $\mathcal{T}$.

With this structure in mind, we can expand our initial discussion of the three main necessary lemmas for our approach. We have a host graph G and a guest tree T which we need to embed into G. The first element of the proof is the Tree-Coating Lemma (Lemma 4.1), which guarantees that T belongs to a tree-class $\mathcal{T}$, according to the partition into cut vertices and forests which we mentioned before. This tree-class $\mathcal{T}$ defines the skew-matching pair we will look for in the reduced graph of G. The second element we need is the Structural Proposition (Proposition 4.2), that ensures that the required skew-matching pair exists in the reduced graph. Finally, we need the Tree-Embedding Lemma (Lemma 4.5). This lemma states that if the reduced graph contains a specific skew-matching pair, we can embed in the host graph G any tree belonging to the tree-class this pairs represents. This gives an embedding of T into G, as desired.

To keep this section from becoming too complex due to heavy notation and definitions, we are intentionally leaving out the precise definitions of the skew-matching pair and the corresponding tree-classes for now, treating these concepts as black-box structures. We will provide exact definitions when needed, in §6 and §5, respectively.

4.1. **Tree-classes and the Tree-Coating Lemma.** We begin by stating our Tree-Coating Lemma. This lemma ensures that any tree T belongs to a tree-class, depending on the outcome of some process which decomposes the tree into small parts. This decomposition depends on a parameter $\rho > 0$ which we are free to choose. As explained above, this gives a set of cut-vertices of negligible size and two forests $\mathcal{F}_A$ and $\mathcal{F}_B$ of tiny trees. Let a_1, a_2 be the size of the colour classes of $\mathcal{F}_A$, and b_1, b_2 the colour classes of $\mathcal{F}_B$. We then include T in the tree-class $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ (see Definition 5.3). For technical reasons, it is not actually T which is going to be decomposed but instead some slightly larger supertree $T' \supset T$, which is given by the following lemma.

Lemma 4.1 (Tree-Coating Lemma). *For any $1/16 \geq \eta \geq \rho > 0$ and any tree T of size $|V(T)| \geq 1000/\rho$, there are natural numbers $a_i, b_i \geq \eta|V(T)|$ for $i \in [2]$, and there is a tree $T' \supset T$ belonging to $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ with $|V(T')| \leq (1 + 4\eta)|V(T)|$.*

The proof of Lemma 4.1, together with the definition of $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$, are deferred to §5.

4.2. **Skew-matching pairs and the Structural Proposition.** A skew-matching σ is a substructure of a weighted graph, inspired by the concept of fractional matching. We have a fixed 'skew' value $\gamma > 0$ and for every edge uv we attribute some non-negative weight which is distributed unequally between u and v, in a ratio of γ. The weight $W(\sigma)$ of a skew-matching σ represents its size, i.e. the sum of all weights assigned to all edges. A (γ_A, γ_B)-skew-matching pair (σ_A, σ_B) consists of two skew-matchings σ_A and σ_B, each with respective skew parameters γ_A and γ_B. Additionally, these skew-matchings will be well-positioned relative to each other.

Proposition 4.2 (Structural Proposition). *Let $a_1, a_2, b_1, b_2 \in \mathbb{N}$ be such that $a_1 + a_2 + b_1 + b_2 = k$ and let (H, w) be a weighted graph with $w : E(H) \to (0, 1]$ such that $\delta_w(H) \geq \frac{k}{2}$ and $\Delta_w(H) \geq k$. Let $\gamma_A := \frac{a_2}{a_1}$ and $\gamma_B := \frac{b_2}{b_1}$. Then H admits a (γ_A, γ_B)-skew-matching pair (σ_A, σ_B) with weights $W(\sigma_A) = a_1 + a_2$ and $W(\sigma_B) = b_1 + b_2$.*

In §6, one can find the definition of a skew-matching pair, as well as all the notions needed to prove Proposition 4.2. Then the proof of Proposition 4.2 is given in §9.

4.3. **Reduced graph and the Embedding Lemma.** As already mentioned, we use the Szemerédi Regularity Lemma [Sze78] on the host graph to obtain a regular partition. The necessary definitions to work with graph regularity are as follows. Given a graph G and $\varepsilon > 0$, a pair (X, Y) with $X, Y \subseteq V(G)$ disjoint, is said to be *ε-regular*, if for any sets $X' \subseteq X$ and $Y' \subseteq Y$ with $|X'| \geq \varepsilon|X|$ and $|Y'| \geq \varepsilon|Y|$ it holds that $|d(G[X', Y']) - d(G[X, Y])| < \varepsilon$. We say that a partition $\{V_0, \ldots, V_t\}$ of $V(G)$ is an *ε-regular partition* if $|V_0| \leq \varepsilon|V(G)|$, and for every $1 \leq i \leq t$, all but at most εt values of $1 \leq j \leq t$ are such that the pair (V_i, V_j) is not ε-regular. We call an ε-regular partition *equitable* if $|V_i| = |V_j|$ for every $1 \leq i < j \leq t$. The following version of the Regularity Lemma follows by standard arguments from its 'degree form' [KS96, Theorem 1.10].

Theorem 4.3 (Szemerédi's Regularity Lemma)**.** *For every $\varepsilon > 0$ there is n_0 and M_0 such that every graph of size at least n_0 admits an ε-regular equitable partition $\{V_0, V_1, \ldots, V_t\}$ with $1/\varepsilon \leq t \leq M_0$.*

This regular partition is then captured in a *weighted reduced graph*, as defined below.

Definition 4.4 (Weighted $\mathfrak{d}$-reduced graph)**.** Given a graph G, an ε-regular equitable partition $\mathcal{P} = \{V_0, \ldots, V_t\}$ of $V(G)$ and $\mathfrak{d} > 0$, we define the $\mathfrak{d}$*-reduced graph* $\Gamma_{\mathfrak{d},\varepsilon}$ as follows. The vertex set of $\Gamma_{\mathfrak{d},\varepsilon}$ is $\{1, \ldots, t\}$, and there is an edge $ij \in E(\Gamma_{\mathfrak{d},\varepsilon})$ if and only if the pair (V_i, V_j) is ε-regular and $d(V_i, V_j) \geq \mathfrak{d}$. The *weighted* $\mathfrak{d}$*-reduced graph* consists of $\Gamma_{\mathfrak{d},\varepsilon}$ endowed with a natural weight function $w : E(\Gamma_{\mathfrak{d},\varepsilon}) \to [0, 1]$ defined by $w(ij) = d(V_i, V_j)$. Note that $\Gamma_{\mathfrak{d},\varepsilon}$ depends on the choice of the partition $\mathcal{P}$, but since it will always be clear from context which partition is being used, we omit it from the notation.

We shall apply the Structural Proposition (Proposition 4.2) in the weighted $\mathfrak{d}$-reduced graph $\Gamma_{\mathfrak{d},\varepsilon}$ to obtain a skew-matching pair there. That is precisely the input which the next lemma requires, and it provides an embedding of a tree in the original host graph.

Lemma 4.5 (Tree Embedding Lemma)**.** *For any $\eta, \mathfrak{d}, q > 0$, and $t \in \mathbb{N}$, there are $\varepsilon = \varepsilon(\eta, \mathfrak{d}, q), \rho = \rho(\eta, \mathfrak{d}, q, t) > 0$ and $n_0 = n_0(\eta, \mathfrak{d}, q, t) \in \mathbb{N}$ such that for $n \geq n_0$ the following holds. Suppose G is an n-vertex graph, and $\mathcal{P} = \{V_0, V_1, \ldots, V_N\}$ is an ε-regular equitable partition of G with $N \leq t$. Suppose $k \geq qn$ and that we have natural numbers $a_1, b_1, a_2, b_2 \geq \eta k$ such that*

$$k = a_1 + a_2 + b_1 + b_2\,.$$

Suppose the weighted $\mathfrak{d}$-reduced graph $\Gamma_{\mathfrak{d},\varepsilon}$ corresponding to G admits a $(a_2/a_1, b_2/b_1)$-skew-matching pair (σ_A, σ_B), with weights $W(\sigma_A) \geq (1 + \eta)(a_1 + a_2)N/n$ and $W(\sigma_B) \geq (1 + \eta)(b_1 + b_2)N/n$. Then G contains any tree $T \in \mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$.

Lemma 4.5 is proven in §10.

4.4. **The proof of Theorem 1.3.** Now, we give the proof of Theorem 1.3, assuming the validity of the main statements introduced before (Lemma 4.1, Proposition 4.2, Lemma 4.5).

Proof of Theorem 1.3. The proof splits naturally into four steps.

Step 1: Setting up the parameters. Suppose we are given input parameters $\eta > 0$ for the approximation factor, and $q > 0$ for the ratio of the size of the trees with respect to the size of the host graph. We may assume that $q, \eta \ll 1$, or we just replace them with smaller values. We set the following parameters to satisfy

$$0 < 1/n_0 \ll \rho \ll 1/M_0 \ll \varepsilon \ll \mathfrak{d} \ll \eta, q \ll 1, \tag{2}$$

as follows. We set $\mathfrak{d} := \frac{\eta q}{100}$, $\varepsilon := \min\{\frac{\mathfrak{d}}{2}, \varepsilon'\}$, where ε' is the output of the Tree Embedding Lemma (Lemma 4.5) given the input $\eta/40$, $\mathfrak{d}$, and q playing the roles of η, $\mathfrak{d}$, and q respectively. Let M_0, n_0' be the outputs of the Szemerédi Regularity Lemma (Theorem 4.3) with input ε. Let ρ, n_0'' be the output of the Tree Embedding Lemma (Lemma 4.5) with input $\eta/40, \mathfrak{d}, q, M_0$ in place of $\eta, \mathfrak{d}, q, t$. Finally, let $n_0 = \max\{n_0', n_0'', 1000/\rho\}$.

Step 2: Processing the tree. We apply the Tree-Coating Lemma (Lemma 4.1) with input $\eta/12$, ρ, and T, playing the roles of η, ρ, and T. We obtain $T' \in \mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ with $a_i, b_i \geq \frac{\eta k}{12} \geq \frac{\eta}{20}|V(T')|$ and $|V(T')| \leq (1 + \frac{\eta}{3})k$. Recall that, by assumption, the host graph G satisfies $\delta(G) \geq (1 + \eta)\frac{k}{2} \geq (1 + \frac{\eta}{20})\frac{|V(T')|}{2}$ and that at least $\frac{\eta n}{20}$ vertices of G have degree at least $(1 + \eta)k \geq (1 + \frac{\eta}{20})|V(T')|$.

Step 3: Preparing the host graph. We apply the Szemerédi Regularity Lemma (Theorem 4.3) on G with parameter ε. This application yields an ε-regular equitable partition

$\{V_0, V_1, \ldots, V_t\}$, with $1/\varepsilon \le t \le M_0$. Given this partition, we define the weighted $\mathfrak{d}$-reduced graph $\Gamma_{\mathfrak{d},\varepsilon}$ endowed with the weight function $w : E(\Gamma_{\mathfrak{d},\varepsilon}) \to [0,1]$, defined by $w(ij) := d(V_i, V_j)$. Recall that by the definition of $\Gamma_{\mathfrak{d},\varepsilon}$, if (V_i, V_j) is not an ε-regular pair, or if $d(V_i, V_j) < \mathfrak{d}$, then $ij \notin E(\Gamma_{\mathfrak{d},\varepsilon})$.

From now on, let $r = k/n$. Standard calculations show that $\Gamma_{\mathfrak{d},\varepsilon}$ inherits minimum and maximum degree conditions from G. Concretely, we claim that $(\Gamma_{\mathfrak{d},\varepsilon}, w)$ satisfies $\delta_w(\Gamma_{\mathfrak{d},\varepsilon}) \ge (1+\frac{\eta}{40})\frac{rt}{2}$ and $\Delta_w(\Gamma_{\mathfrak{d},\varepsilon}) \ge (1+\frac{\eta}{40})rt$. We give those arguments for completeness. Indeed, for $i \in V(\Gamma_{\mathfrak{d},\varepsilon})$ we have

$$\begin{aligned}
\deg_w(i) &= \sum_{j \in N_{\Gamma_{\mathfrak{d},\varepsilon}}(i)} w(ij) = \sum_{j \in N_{\Gamma_{\mathfrak{d},\varepsilon}}(i)} d(V_i, V_j) \\
&\ge \left(\sum_{j \in [t]\setminus\{i\}} d(V_i, V_j) \right) - (\varepsilon + \mathfrak{d})t = \left(\sum_{j \in [t]\setminus\{i\}} \sum_{v \in V_i} \frac{\deg_G(v, V_j)}{|V_i|^2} \right) - (\varepsilon + \mathfrak{d})t \\
&\ge \sum_{v \in V_i} \frac{\deg_G(v) - |V_i|}{|V_i|^2} - (\varepsilon + \mathfrak{d})t \ge \frac{\delta(G)}{|V_i|} - 1 - (\varepsilon + \mathfrak{d})t \\
&\ge \frac{(1+\eta)k/2}{|V_i|} - (2\varepsilon + \mathfrak{d})t = \frac{(1+\eta)rn/2}{|V_i|} - (2\varepsilon + \mathfrak{d})t \\
&\ge (1+\eta)\frac{rt}{2} - (2\varepsilon + \mathfrak{d})t \ge \left(1 + \frac{\eta}{40}\right)\frac{rt}{2},
\end{aligned}$$

where we used $|V_i| \le n/t$ in the second to last inequality, and $r \ge q$ and the choice of $\mathfrak{d}, \varepsilon$ in the last inequality.

To calculate $\Delta_w(\Gamma_{\mathfrak{d},\varepsilon})$, denote by L the set of $v \in V(G)$ such that $\deg_w(v) \ge (1+\eta)k = (1+\eta)rn$. By the pigeonhole principle, there is an index $i \in [t]$ with $|V_i \cap L| \ge \frac{\eta n - \varepsilon n}{t} \ge \eta(1-\varepsilon)n/t \ge \eta(1-\varepsilon)|V_i| > \varepsilon|V_i|$. Then

$$\begin{aligned}
\deg_w(i) &= \sum_{j \in N_{\Gamma_{\mathfrak{d},\varepsilon}}(i)} w(ij) = \sum_{j \in N_{\Gamma_{\mathfrak{d},\varepsilon}}(i)} d(V_i, V_j) \ge \left(\sum_{j \in [t]\setminus\{i\}} d(V_i, V_j) \right) - (\varepsilon + \mathfrak{d})t \\
&\ge \left(\sum_{j \in [t]\setminus\{i\}} d(V_i \cap L, V_j) - \varepsilon \right) - (\varepsilon + \mathfrak{d})t \\
&\ge \left(\sum_{v \in V_i \cap L} \frac{\deg_G(v) - |V_i|}{|V_i \cap L||V_i|} \right) - (2\varepsilon + \mathfrak{d})t \\
&\ge \left(\sum_{v \in V_i \cap L} \frac{(1+\eta)k}{|V_i \cap L||V_i|} \right) - 1 - (2\varepsilon + \mathfrak{d})t \ge (1+\eta)\frac{k}{|V_i|} - (3\varepsilon + \mathfrak{d})t \\
&= (1+\eta)\frac{rn}{|V_i|} - (3\varepsilon + \mathfrak{d})t \ge (1+\eta)rt - (3\varepsilon + \mathfrak{d})t \ge \left(1 + \frac{\eta}{40}\right) rt,
\end{aligned}$$

where we used $|V_i| \le n/t$ in the second to last inequality, and $r \ge q$ and the choice of $\mathfrak{d}, \varepsilon$ in the last inequality.

These estimates on $\delta_w(\Gamma_{\mathfrak{d},\varepsilon})$ and $\Delta_w(\Gamma_{\mathfrak{d},\varepsilon})$ allow us to apply the Structural Proposition (Proposition 4.2) with $(\Gamma_{\mathfrak{d},\varepsilon}, w)$, $(1+\frac{\eta}{40})rt$, $(1+\frac{\eta}{40})a_i t/n$ and $(1+\frac{\eta}{40})b_i t/n$ playing the roles of (H, w), k and a_i, b_i, for $i \in [2]$, respectively. We obtain an $(\frac{a_2}{a_1}, \frac{b_2}{b_1})$-skew-matching (σ_A, σ_B) with $W(\sigma_A) = (1+\frac{\eta}{40})(a_1+a_2)t/n$ and $W(\sigma_B) = (1+\frac{\eta}{40})(b_1+b_2)t/n$.

Step 4: Embedding the tree. Having found the skew-matching pair in the reduced graph, we finalise by applying the Tree Embedding Lemma (Lemma 4.5). First, we note that $\min\{a_1, a_2\} \ge \frac{\eta}{20}|V(T')| \ge \frac{\eta}{40}\left((1+\frac{\eta}{40})|V(T')|\right)$ and similarly for b_1, b_2, we have that $\min\{b_1, b_2\} \ge \frac{\eta}{40}((1+\frac{\eta}{40})|V(T')|)$. Thus, we can apply the Tree Embedding Lemma

(Lemma 4.5) with t and $\eta/40$, playing the roles of N and η, respectively. We obtain that G contains any tree in $\mathcal{T}^{\rho}_{a_1,b_1,a_2,b_2}$, so in particular $T' \subseteq G$. Since $T \subseteq T'$, this proves Theorem 1.3. ■

5. Coating the Tree

To prepare the embedding, we will first partition the trees into suitably small parts. We will use the following handy concept from Hladký, Komlós, Piguet, Simonovits, Stein, and Szemerédi [Hla+17, Definition 3.3]. It gives a partition of a tree into smaller trees that also satisfy several additional useful properties.

If T is a tree rooted at r, and $\widetilde{T} \subseteq T$ is a subtree with $r \notin V(\widetilde{T})$, the *seed of* $\widetilde{T}$ is the unique vertex $x \in V(T) \setminus V(\widetilde{T})$ which is farthest from r and also belongs to every (r, v)-path in T, for every $v \in V(\widetilde{T})$. We emphasize that the seed of $\widetilde{T}$ is *not* contained in $V(\widetilde{T})$, and that $\widetilde{T}$ does not necessarily contain all descendants of its seed.

Definition 5.1 (ℓ-fine partition)**.** Let T be a tree on k vertices rooted at a vertex r. An *ℓ-fine partition of* T is a quadruple $(W_A, W_B, \mathcal{F}_A, \mathcal{F}_B)$, where $W_A, W_B \subseteq V(T)$ and $\mathcal{F}_A, \mathcal{F}_B$ are families of subtrees of T such that

(A1) the three sets W_A, W_B, and $\{V(T^*)\}_{T^* \in \mathcal{F}_A \cup \mathcal{F}_B}$ partition $V(T)$ (in particular, the trees in $\mathcal{F}_A \cup \mathcal{F}_B$ are pairwise vertex-disjoint),

(A2) $r \in W_A \cup W_B$,

(A3) $\max\{|W_A|, |W_B|\} \leq 336k/\ell$,

(A4) for $w_1, w_2 \in W_A \cup W_B$, the distance between w_1 and w_2 in T is odd if and only if one of them lies in W_A and the other one in W_B,

(A5) $|V(T^*)| \leq \ell$ for every $T^* \in \mathcal{F}_A \cup \mathcal{F}_B$,

(A6) $V(T^*) \cap N(W_B) = \emptyset$ for every $T^* \in \mathcal{F}_A$, and $V(T^*) \cap N(W_A) = \emptyset$ for every $T^* \in \mathcal{F}_B$;

(A7) each tree of $\mathcal{F}_A \cup \mathcal{F}_B$ has its seeds in $W_A \cup W_B$,

(A8) $|N(V(T^*)) \cap (W_A \cup W_B)| \leq 2$ for each $T^* \in \mathcal{F}_A \cup \mathcal{F}_B$,

(A9) if $N(V(T^*)) \cap (W_A \cup W_B)$ contains two distinct vertices w_1, w_2 for some $T^* \in \mathcal{F}_A \cup \mathcal{F}_B$, then $\mathrm{dist}_T(w_1, w_2) \geq 6$.

The trees $T^* \in \mathcal{F}_A \cup \mathcal{F}_B$ will be called *shrubs*. We remark that ℓ-fine partitions can be defined differently so they satisfy even more properties, but we are only citing those we need. See Figure 1 for a visual representation of a tree together with an ℓ-fine partition.

The next lemma [Hla+17, Lemma 3.5] says that any tree has an ℓ-fine partition.

Lemma 5.2. *Let T be a tree on k vertices rooted at r, and let $1 \leq \ell \leq k$. Then T has an ℓ-fine partition.*

Definition 5.3 (Tree-class $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$)**.** We denote by $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ the set of trees T, such that there is a $(\rho|V(T)|)$-fine partition $(W_A, W_B, \mathcal{F}_A, \mathcal{F}_B)$ of T such that $|V_i(\mathcal{F}_A)| = a_i$, $|V_i(\mathcal{F}_B)| = b_i$, for $i \in \{1, 2\}$, where $V_1(\mathcal{F}_A)$ (resp. $V_2(\mathcal{F}_A)$) is the set of vertices of $\mathcal{F}_A$ that are at odd (resp. even) distance from W_A, and $V_i(\mathcal{F}_B)$ are defined analogously with respect to W_B.

As mentioned earlier, for technical reasons we shall work with a tree $T' \supset T$ which is slightly larger than T, but that belongs to the class $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ for some suitable values of a_1, a_2, b_1, b_2. In our case 'suitable' will mean that none of these values are very small; this fact will be useful during the embedding phase later. The Tree-Coating Lemma (Lemma 4.1) describes precisely the process of finding this supertree T'. We now have the tools and vocabulary to prove it. We restate the statement for the convenience of the reader.

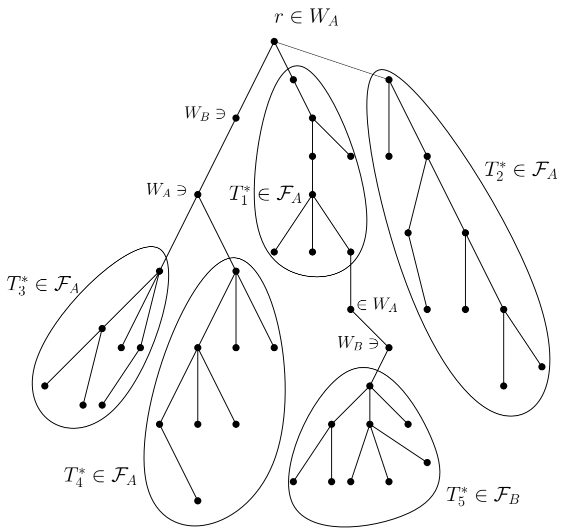


FIGURE 1. A schematic view of an ℓ-fine partition of a tree with 45 vertices rooted at r into $W_A, W_B, \mathcal{F}_A, \mathcal{F}_B$, satisfying (A1)–(A9).

Lemma 4.1 (Tree-Coating Lemma). *For any $1/16 \geq \eta \geq \rho > 0$ and any tree T of size $|V(T)| \geq 1000/\rho$, there are natural numbers $a_i, b_i \geq \eta|V(T)|$ for $i \in [2]$, and there is a tree $T' \supset T$ belonging to $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ with $|V(T')| \leq (1 + 4\eta)|V(T)|$.*

Proof of Lemma 4.1. Pick any vertex $r \in V(T)$ as the root of T and apply Lemma 5.2 with input T, $|V(T)|$, r, and $\rho|V(T)|$, playing the roles of T, k, r and ℓ to obtain a $\rho|V(T)|$-fine partition of T. First assume that both W_A and W_B are non-empty. Pick a vertex $x_A \in W_A$ and a vertex $x_B \in W_B$ and attach $\eta|V(T)|$ paths of length 2 to each of the two vertices x_A and x_B. This adds $\eta|V(T)|$ vertices to $\mathcal{F}_L \cap V_i(T)$ for $L \in \{A, B\}$ and $i \in [2]$, where $V_i(T), i \in [2]$ are the two colour classes of T, and increases the size of the tree by $4\eta|V(T)|$. As $2 \leq \rho|V(T)|$, all the newly formed shrubs in $\mathcal{F}_A \cup \mathcal{F}_B$ have size at most $\rho|V(T')|$ and thus, the newly obtained tree T' belongs to the class $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$, for suitable a_i and b_i, for $i \in [2]$.

If (w.l.o.g.) $W_A = \emptyset$, by Definition 5.1(A6), we have that $\mathcal{F}_A = \emptyset$. By (A3), we have $|W_B| \leq 336/\rho \leq |V(T)|/2$, and therefore $|\mathcal{F}_B \cap V_i(T)| \geq |V(T)|/4 \geq 4\eta|V(T)|$, for some $i \in [2]$. Add a vertex x_A to W_A, connecting it to any vertex in W_B, and as previously attach $\eta|V(T)|$ paths of length 2 to x_A. For the other side, if the vertices in $\mathcal{F}_B$ at odd distance to W_B are less than $\eta|V(T)|$, then place a star with $2\eta|V(T)|-1$ leaves centred on an arbitrary vertex $x_B \in W_B$. Otherwise attach $\eta|V(T)| - 2$ paths of length 2 to some vertex $x_B \in W_B$ and a star with 2 leaves and connect the centre to x_B as well. In this way, we ensure (as $3 \leq \rho|V(T)|$) that the newly created tree belongs to $\mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ for suitable $a_i, b_i \geq \eta|V(T)|$, $i \in [2]$, and that we have increased the tree by exactly $4\eta|V(T)|$ vertices. ∎

6. SKEW-MATCHINGS AND OTHER MATCHING STRUCTURES

In the next four sections, we prove the Structural Proposition (Proposition 4.2). In this section, we will introduce the basic definitions that will allow us to describe the structure we will seek in the reduced graph. The basic building block of our argument is a "skew" oriented fractional matching, defined in Section 6.2. Its definition is inspired by the standard fractional matching, which (we recall) is a weight function over the edges of a graph such that the sum of weights over the edges incident to any given vertex is at most one (Definition 6.1). As we shall see, a skew-matching satisfies a similar property, but instead is a weight function defined over *oriented* edges of the graph, which allows us to distribute the weight in every edge in an unbalanced way. We remark that essentially equivalent concepts were used before (e.g. [Bal+18, §2.4]) in the context of graph tilings.

After introducing the main definitions in §6.1 and §6.2, we introduce many extra auxiliary concepts. In §6.3 we introduce notation to compare different fractional matchings and skew-matchings; and in §6.4 we introduce the auxiliary notion of the *truncated weighted graph*.

6.1. **Fractional matchings.** The following standard definition generalises the notion of matchings to a fractional setting.

Definition 6.1 (Fractional matching)**.** A *fractional matching* is a weight function $\mu : E(G) \to [0,1]$ such that for any vertex $v \in V(G)$, we have $\sum_{u \in N(v)} \mu(uv) \leq 1$.

We shall abuse notation here and use the sign μ on vertices as well and we call

$$\mu(v) := \sum_{u \in N(v)} \mu(uv)$$

the *weight of* μ *on* v. The *weight* of a fractional matching μ is

$$W(\mu) := \sum_{e \in E(G)} \mu(e).$$

We say that two fractional matchings μ' and μ'' are *disjoint* if $\mu'(x) + \mu''(x) \leq 1$ for all $x \in V(G)$. We denote by $V(\mu)$ the set of vertices $v \in V(G)$ such that $\mu(v) > 0$.

6.2. **Skew-matchings.** Now we present the definition of skew-matchings. Note that, according to our convention on $G^{\leftrightarrow}$, both edges $\overrightarrow{uv}$ and $\overrightarrow{vu}$ are present in the following definition.

Definition 6.2 (γ-skew oriented fractional matching)**.** Let G be a graph, $G^{\leftrightarrow}$ its associated digraph, and $\gamma \geq 0$. A γ*-skew oriented fractional matching* (or just γ*-skew-matching*) is a function $\sigma : E(G^{\leftrightarrow}) \to [0,1]$ such that for any vertex $u \in V(G)$,

$$\sum_{v \in N(u)} \left(\frac{1}{1+\gamma} \sigma(\overrightarrow{uv}) + \frac{\gamma}{1+\gamma} \sigma(\overrightarrow{vu}) \right) \leq 1.$$

As explained before, γ-skew-matchings can be understood as fractional matchings (Definition 6.1) in graphs where the weight of the edge is distributed in an unbalanced way, meaning that one end of the edge gets γ times the weight of the other end, and the direction of this imbalance is given by the direction of the edge in the digraph $G^{\leftrightarrow}$.

With a slight abuse of notation, given a γ-skew-matching σ we shall use the symbol σ on vertices as well. We define

$$\sigma(u) := \sigma^1(u) + \sigma^2(u),$$

where

$$\sigma^1(u) := \frac{1}{1+\gamma} \sum_{v \in N(u)} \sigma(\overrightarrow{uv}) \qquad \text{and} \qquad \sigma^2(u) := \frac{\gamma}{1+\gamma} \sum_{v \in N(u)} \sigma(\overrightarrow{vu}).$$

Now we define two crucial concepts associated with a γ-skew-matching σ. The *anchor* of σ, denoted by $\mathcal{A}(\sigma)$, is the set

$$\mathcal{A}(\sigma) := \{u \in V(G) \ : \ \sigma^1(u) > 0\}.$$

The *weight* of σ is

$$W(\sigma) := \sum_{e \in E(G^{\leftrightarrow})} \sigma(e).$$

For our proof, we will usually need to work with several skew-matchings, possibly with different values of γ. If σ, σ' are γ-skew and γ'-skew-matchings respectively, we say σ, σ' are *disjoint* if, for every $u \in V(G)$,

$$\sigma(u) + \sigma'(u) \leq 1.$$

In what follows, we often work with a weighted graph (G, w) and its associated digraph $G^{\leftrightarrow}$ and we do not always explicitly distinguish between G and $G^{\leftrightarrow}$, as they are in one-to-one correspondence. The following definition is key: it essentially says that a skew-matching σ does not place too much weight in the neighbourhood of a vertex u, meaning that it respects the weight dictated by some edge weighting w. This will allow us to embed trees using the space, allowed by σ, that is also joined correctly with u.

Definition 6.3 (Fitting in the w-neighbourhood)**.** Let (G, w) be a weighted digraph, $\gamma \geq 0$, and $u \in V(G)$. Given a γ-skew oriented fractional matching σ, we say that its anchor *fits in the w-neighbourhood of u* if $\sigma^1(v) \leq w(\overrightarrow{uv})$, for every $v \in V(G)$.

Note that, in particular, if $\mathcal{A}(\sigma)$ fits in the w-neighbourhood of u, then $\mathcal{A}(\sigma) \subseteq N_w(u)$.

The following definition is also key. It corresponds to two skew-matchings, with different skews, which are disjoint from each other and they fit in the neighborhoods of two vertices forming an edge cd. This will be the structure we desire in the reduced graph. Referencing the ℓ-fine partitions of a tree T discussed before: the (few) vertices of the sets W_A, W_B will be mapped to the clusters corresponding to cd, while the shrubs in $\mathcal{F}_A, \mathcal{F}_B$ will be mapped in the space ensured by σ_A, σ_B, respectively.

Definition 6.4 (Edge-anchored skew-matching pair)**.** Let (G, w) be a weighted digraph, and $\gamma_A, \gamma_B \geq 0$. A (γ_A, γ_B)*-skew-matching pair anchored in* $\overrightarrow{cd} \in E(G)$ is a pair (σ_A, σ_B) such that

(B1) σ_A and σ_B are disjoint,

(B2) σ_A is a γ_A-skew-matching, whose anchor fits in the w-neighbourhood of c,

(B3) σ_B is a γ_B-skew-matching, whose anchor fits in the w-neighbourhood of d, and

(B4) for every $x \in N(c) \cap N(d)$, we have

$$\max\{w(\overrightarrow{cx}), w(\overrightarrow{dx})\} \geq \sigma_A^1(x) + \sigma_B^1(x).$$

Observe that there is no weight requirement on $\overrightarrow{cd}$.

We say that a weighted (unoriented) graph (G, w) *admits* a (γ_A, γ_B)-skew-matching pair (σ_A, σ_B), if (σ_A, σ_B) is defined on its associated weighted digraph $G^{\leftrightarrow}$ and is anchored in some edge $\overrightarrow{cd} \in E(G^{\leftrightarrow})$, and $w(cd) > 0$.

Remark 6.5. We can express (B4) equivalently by saying that there is a partition $\{X_c, X_d\}$ of $N(c) \cap N(d)$ such that for every $x \in X_c$, we have $w(\overrightarrow{cx}) \geq \sigma_A^1(x) + \sigma_B^1(x)$, and for every $x \in X_d$, we have $w(\overrightarrow{dx}) \geq \sigma_A^1(x) + \sigma_B^1(x)$.

We say that a vertex $v \in V(G)$ is *covered* by a fractional matching σ (or a skew oriented fractional matching) if $\sigma(v) = 1$. Analogously, we say that a set U is *covered*, if each vertex $v \in U$ is covered. Moreover, we define $V(\sigma) := \{v \in V(G) \ : \ \sigma(v) > 0\}$.

The value

$$\deg_w(u, \sigma) = \sum_{v \in V(G)} \min\{\sigma(v), w(\overrightarrow{uv})\}$$

is called the *saturation* of $N(u)$ by σ. We say that σ *saturates* $N(u) \cap U$ for some $U \subseteq V(G)$, if for every vertex $v \in U$, we have that $\sigma(v) \geq w(\overrightarrow{uv})$. If $U = V(G)$, we simply say that σ *saturates* the neighbourhood of u (or *saturates* $N(u)$ for short). If this happens, then $\deg_w(u) = \deg_w(u, \sigma)$.

6.3. **Comparing matchings.** Now we define a few relations between fractional matchings and skew-matchings, which will allow us to compare them to each other. Notice that both edges $\overrightarrow{xy}$ and $\overrightarrow{yx}$ are often used.

We begin by comparing fractional matchings between them. We write $\mu \leq \mu'$ for two fractional matchings μ and μ', whenever $\mu(xy) \leq \mu'(xy)$ for every $xy \in E(G)$. Moreover, $\mu = \mu'$ if and only if $\mu(xy) = \mu'(xy)$ for every $xy \in E(G)$.

Similarly, we can compare skew-matchings between them, even if they differ in their skews. A γ'-skew-matching σ' is a *skew sub-matching* of a γ-skew matching σ, if for every $\overrightarrow{uv} \in E(G^{\leftrightarrow})$, we have $\frac{\sigma'(\overrightarrow{uv})}{1+\gamma'} \leq \frac{\sigma(\overrightarrow{uv})}{1+\gamma}$ and $\frac{\gamma'}{1+\gamma'}\sigma'(\overrightarrow{uv}) \leq \frac{\gamma}{1+\gamma}\sigma(\overrightarrow{uv})$. This is denoted by $\sigma' \leq \sigma$.

In the next definition we introduce the language and notation to compare fractional matchings with skew-matchings, and vice versa.

Definition 6.6 ($\preceq, \trianglelefteq$)**.** Let G be a graph with associated digraph $G^{\leftrightarrow}$, let σ be a γ-skew oriented fractional matching in $G^{\leftrightarrow}$, and let μ be a fractional matching in G. We write $\mu \preceq \sigma$ if for every $\overrightarrow{xy} \in E(G^{\leftrightarrow})$ with $\mu(xy) > 0$, we have

$$\mu(xy) \leq \frac{1}{1+\gamma}\sigma(\overrightarrow{xy}) + \frac{\gamma}{1+\gamma}\sigma(\overrightarrow{yx}).$$

Similarly, we write $\sigma \trianglelefteq \mu$ if for every $\overrightarrow{xy} \in E(G^{\leftrightarrow})$ with $\sigma(\overrightarrow{xy}) + \sigma(\overrightarrow{yx}) > 0$, we have

$$\frac{\sigma(\overrightarrow{xy}) + \gamma\sigma(\overrightarrow{yx})}{1+\gamma} \leq \mu(xy).$$

Remark 6.7. We emphasize that in both definitions, that of $\mu \preceq \sigma$ and $\sigma \trianglelefteq \mu$, we require the inequality to hold for each oriented edge in $G^{\leftrightarrow}$, which means that for each (unoriented) edge $xy \in E(G)$ we need to verify the inequality both for $\overrightarrow{xy}$ and $\overrightarrow{yx}$.

For instance, in the definition of $\mu \preceq \sigma$ we need

$$\mu(xy) \leq \frac{1}{1+\gamma}\sigma(\overrightarrow{yx}) + \frac{\gamma}{1+\gamma}\sigma(\overrightarrow{xy})$$

to hold as well; and in the definition of $\sigma \trianglelefteq \mu$ we also need that

$$\frac{\sigma(\overrightarrow{yx}) + \gamma\sigma(\overrightarrow{xy})}{1+\gamma} \leq \mu(xy).$$

We also introduce notation to compare multiple skew-matchings with a fractional matching. Let σ_i be a γ_i-skew-matching for every $i \in [k]$. We write $\sum_{i=1}^k \sigma_i \trianglelefteq \mu$ if for every $\overrightarrow{xy} \in E(G^{\leftrightarrow})$ with $\sum_{i=1}^k \big(\sigma_i(\overrightarrow{xy}) + \sigma_i(\overrightarrow{yx})\big) > 0$, we have

$$\sum_{i=1}^k \frac{\sigma_i(\overrightarrow{xy}) + \gamma_i\sigma_i(\overrightarrow{yx})}{1+\gamma_i} \leq \mu(xy).$$

Remark 6.8. In the last definition, we compute a sum of skew-matchings with different skews. This sum does not define a new skew-matching. We only compare the weight of these skew-matchings on every oriented edge with the weight of the fractional matching μ on the corresponding unoriented edge (and we do it for both possible orientations). In this way we investigate if all these skew-matchings together "fit" in a fractional matching μ. In other words, if $\sum_{i=1}^k \sigma_i \trianglelefteq \mu$, we have $\sum_{i=1}^k \sigma_i(u) \leq \mu(u) \leq 1$ for all $u \in V(G)$.

Observation 6.9. Let G be a graph, $G^{\leftrightarrow}$ be its associated digraph, $\mu, \widehat{\mu}$ be fractional matchings in G and σ, $\widehat{\sigma}$ be skew fractional matchings in $G^{\leftrightarrow}$.

All the values $\mu(uv), \sigma(\overrightarrow{uv}), \mu(u), \widehat{\mu}(u)$ are non-negative real numbers. Therefore, expressions such as $\mu(uv) \leq \sigma(\overrightarrow{uv})$, $\mu(u) \leq \widehat{\mu}(u)$, $\sigma(\overrightarrow{uv}) + \mu(uv)$ are well-defined.

We can also gather some straightforward observations and consequences of our previous definitions, that will allow us to work more mechanically with different objects.

(i) Having $\mu(u) \leq \widehat{\mu}(u)$ for every $u \in V(G)$ does not imply that $\mu \leq \widehat{\mu}$.
(ii) Similarly, having $\sigma \trianglelefteq \mu$ does not imply that $\sigma(\overrightarrow{uv}) \leq \mu(uv)$ for every uv.
(iii) On the other hand, having $\sigma \trianglelefteq \mu$ implies that $\sigma(u) \leq \mu(u)$ for every $u \in V(G)$.
(iv) If $\mu \preceq \sigma$ and $\sigma \trianglelefteq \widehat{\mu}$, then $\mu \leq \widehat{\mu}$.
(v) Having $\sigma \trianglelefteq \mu$ and $\mu \preceq \widehat{\sigma}$ does not imply that $\sigma \leq \widehat{\sigma}$.
(vi) If $\sigma \trianglelefteq \mu$ and $W(\sigma) = 2W(\mu)$, then $\sigma(u) = \mu(u)$ for every $u \in V(G)$.

Note that, in general, we have $\sigma(u) \neq \sum_{v\in N(u)} \sigma(\overrightarrow{uv}) + \sigma(\overrightarrow{vu})$.

While fractional matchings are defined on a non-oriented graph G and skew-matchings are defined on $G^{\leftrightarrow}$, we can easily obtain a 1-skew-matching σ from a fractional matching μ. This can be done, for instance, by choosing an orientation for each edge $xy \in E(G)$ and setting $\sigma(\overrightarrow{xy}) = 2\mu(xy)$ and $\sigma(\overrightarrow{yx}) = 0$. We observe, however, that the definition of "weight" in both cases differs, because then

$$W(\sigma) = \sum_{\overrightarrow{xy} \in E(G^{\leftrightarrow})} \sigma(\overrightarrow{xy}) = \sum_{xy \in E(G)} (2\mu(xy) + 0) = 2W(\mu).$$

There are many different 1-skew-matchings σ that we can define using a fractional matching μ. For example $\sigma(\overrightarrow{xy}) = \sigma(\overrightarrow{yx}) = \mu(xy)$ is also a possibility. In this case, we again obtain that $W(\sigma) = 2W(\mu)$. Conversely, we can obtain a fractional matching μ from a 1-skew-matching σ just by "forgetting" the orientation of the edges. Then, we get

$$W(\mu) = \sum_{xy \in E(G)} \mu(xy) = \sum_{xy \in E(G)} \frac{\sigma(\overrightarrow{xy}) + \sigma(\overrightarrow{yx})}{2} = \frac{W(\sigma)}{2}.$$

The next lemma encapsulates the above discussion for future reference.

Lemma 6.10. *Let G be a graph and $G^{\leftrightarrow}$ its associated digraph. Let σ be a 1-skew-matching in $G^{\leftrightarrow}$. Then G has a fractional matching μ such that*
(i) *$W(\mu) = \frac{1}{2}W(\sigma)$,*
(ii) *for all $x \in V(G)$, $\sigma(x) = \mu(x)$, and*
(iii) *if σ' is a γ-skew-matching in $G^{\leftrightarrow}$ such that $\sigma \leq \sigma'$, then $\mu \preceq \sigma'$.*

In the following definition, we extend the concept of *covering* and *saturation* from fractional matchings or skew-matchings, to sum of skew-matchings, or sum of a fractional matching and skew-matchings. This is done in a natural way: the notions of covering and saturation refer to the weights that the given objects induce on the vertices and do not depend on the type of object.

Definition 6.11 (Saturation, Covering)**.** Let (G, w) be a weighted graph, $(G^{\leftrightarrow}, w)$ its associated weighted digraph, ν_1, ν_2 represent fractional matchings and/or skew-matchings[2] (including the option of everywhere 0-valued functions), and $u \in V(G)$. Then the value

$$\deg_w(u, \nu_1 + \nu_2) := \sum_{v \in N_w(u)} \min\{\nu_1(v) + \nu_2(v), w(\overrightarrow{uv})\}$$

is called the *saturation* of $N_w(u)$ by $\nu_1 + \nu_2$. We say that $\nu_1 + \nu_2$ *saturates the neighbourhood of* u (or *saturates* $N_w(u)$ for short) if $\deg_w(u)$ equals $\deg_w(u, \nu_1 + \nu_2)$.

Analogously, $\nu_1 + \nu_2$ *saturates* $N_w(u) \cap U$, for some $U \subseteq V(G)$, if for every vertex $v \in N_w(u) \cap U$ we have $\nu_1(v) + \nu_2(v) \geq w(\overrightarrow{uv})$. We say that $\nu_1 + \nu_2$ *covers* a vertex $v \in V(G)$ if $\nu_1(v) + \nu_2(v) = 1$. We say that it covers a set $S \subseteq V(G)$, if it covers every vertex $v \in S$.

6.4. **Truncated weighted graphs.** Now we shall define another important concept used in our proofs, that of a *truncated weighted graph.* To motivate this definition, consider the following scenario. Suppose we need to find a fractional matching in a weighted graph (G, w) of sufficiently large weight, but so far we have only found a fractional matching μ of insufficient weight. In our settings, we always need to make sure our fractional objects (matchings or skew-matchings) 'fit' in the w-neighbourhood of a given vertex, say c. Thus, if we want to build another matching $\bar{\mu}$, disjoint from μ, and we want to consider the sum $\mu + \bar{\mu}$, we need to keep in mind that we can only allocate weight in a vertex as determined by the weight function w. More precisely, for every vertex $x \in N(c)$ we need to ensure $\bar{\mu}(x) + \mu(x) \leq w(\overrightarrow{cx})$. To work with these kinds of restrictions, we will define a new weight function $\bar{w}$, which is essentially $w(\overrightarrow{cx}) - \mu(x)$

[2]The values of the skew are not relevant in this definition.

for every $x \in V(G)$, so the restriction we just discussed is expressed more succinctly as $\bar{\mu}(x) \leq \bar{w}(\overrightarrow{cx})$.

Definition 6.12 (Truncated weighted digraph)**.** Let (G, w) be a weighted symmetric digraph and μ be a fractional matching in G (forgetting the orientation of the edges). For every directed $ux \in E(G)$, let $\bar{w}(\overrightarrow{ux}) := \max\{0, w(\overrightarrow{ux}) - \mu(x)\}$. We call $(G, \bar{w})$ the *μ-truncated weighted digraph obtained from* (G, w).

Remark 6.13. In the previous definition we need G to be a symmetric digraph, to be able to define a fractional matching in a meaningful way; in fact we will use it essentially only for associated digraphs obtained from undirected graphs. But we do not require the weights themselves in (G, w) to be symmetric, and generally the truncated weighted digraphs $(G, \bar{w})$ will also not have symmetric weights.

The next proposition ensures that we can indeed correctly combine skew-matchings as long as they respect the weights from a truncated weighted digraph; so we achieved what we set out to do with the definition.

Proposition 6.14. *Let (G, w) be a weighted symmetric digraph, and $uv \in E(G)$. Let μ and $\bar{\mu}$ be disjoint fractional matchings in (unoriented) G and let $(G, \bar{w})$ be the μ-truncated weighted digraph obtained from (G, w). Suppose that*

(C1) *(σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair in (G, w) anchored in $\overrightarrow{uv}$ with $\sigma_A + \sigma_B \trianglelefteq \mu$; and*

(C2) *$(\bar{\sigma}_A, \bar{\sigma}_B)$ is a (γ_A, γ_B)-skew-matching pair in $(G, \bar{w})$ anchored in $\overrightarrow{uv}$ with $\bar{\sigma}_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$.*

Then $(\sigma_A + \bar{\sigma}_A, \sigma_B + \bar{\sigma}_B)$ is a (γ_A, γ_B)-skew-matching pair in (G, w), anchored in $\overrightarrow{uv}$, with $\sigma_A + \bar{\sigma}_A + \sigma_B + \bar{\sigma}_B \trianglelefteq \mu + \bar{\mu}$.

Proof. We need to verify the properties (B1)–(B4) for $(\sigma_A + \bar{\sigma}_A, \sigma_B + \bar{\sigma}_B)$. The disjointness of μ and $\bar{\mu}$, along with $\sigma_A + \sigma_B \trianglelefteq \mu$ and $\bar{\sigma}_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$, ensure that $\sigma_A + \bar{\sigma}_A + \sigma_B + \bar{\sigma}_B \trianglelefteq \mu + \bar{\mu}$. Thus, by Remark 6.8, we have that $\sigma_A + \bar{\sigma}_A$ is disjoint from $\sigma_B + \bar{\sigma}_B$, which gives (B1). This also gives that $\sigma_A + \bar{\sigma}_A$ is a γ_A-skew-matching and $\sigma_B + \bar{\sigma}_B$ is a γ_B-skew-matching.

Now we verify (B2)–(B3). For all $z \in N_{\bar{w}}(u)$, we have $\bar{w}(\overrightarrow{uz}) > 0$, and therefore $\bar{w}(\overrightarrow{uz}) = w(\overrightarrow{uz}) - \mu(z)$ by definition. Then $\sigma^1_A(z) + \bar{\sigma}^1_A(z) \leq \mu(z) + \bar{w}(\overrightarrow{uz}) = w(\overrightarrow{uz})$, where in the first inequality we used that $\sigma_A \trianglelefteq \mu$ and that $\bar{\sigma}_A$ fits in the $\bar{w}$-neighbourhood of u. On the other hand, if $\bar{w}(\overrightarrow{uz}) = 0$, then $\bar{\sigma}^1_A(z) = 0$. Thus, $\sigma^1_A(z) + \bar{\sigma}^1_A(z) = \sigma^1_A(z) \leq w(\overrightarrow{uz})$. Hence, we have that $\mathcal{A}(\sigma_A + \bar{\sigma}_A)$ fits in the w-neighbourhood of u, which gives (B2). A symmetric argument gives that $\mathcal{A}(\sigma_B + \bar{\sigma}_B)$ fits in the w-neighbourhood of v, thus giving (B3).

Finally, consider any $z \in N_w(u) \cap N_w(v)$. Using that $\mathcal{A}(\bar{\sigma}_A), \mathcal{A}(\bar{\sigma}_B)$ fit in the $\bar{w}$-neighbourhood of u and v respectively, we get

$$\bar{\sigma}^1_A(z) + \sigma^1_A(z) + \bar{\sigma}^1_B(z) + \sigma^1_B(z) \leq \max\{\bar{w}(\overrightarrow{uz}), \bar{w}(\overrightarrow{vz})\} + \sigma^1_A(z) + \sigma^1_B(z). \tag{3}$$

Now we consider three cases. If $z \in N_{\bar{w}}(u) \cap N_{\bar{w}}(v)$, then $\bar{w}(\overrightarrow{uz}) = w(\overrightarrow{uz}) - \mu(z)$ and $\bar{w}(\overrightarrow{vz}) = w(\overrightarrow{vz}) - \mu(z)$. Using this in (3) gives

$$\bar{\sigma}^1_A(z) + \sigma^1_A(z) + \bar{\sigma}^1_B(z) + \sigma^1_B(z) \leq \max\{w(\overrightarrow{uz}), w(\overrightarrow{vz})\} + \sigma^1_A(z) + \sigma^1_B(z) - \mu(z),$$

and the last term is at most $\max\{w(\overrightarrow{uz}), w(\overrightarrow{vz})\}$ since $\sigma_A + \sigma_B \trianglelefteq \mu$, thus giving (B4) in this case. Now assume that $z \in N_{\bar{w}}(u) \setminus N_{\bar{w}}(v)$. Then $0 < \bar{w}(\overrightarrow{uz}) = w(\overrightarrow{uz}) - \mu(z)$, but $\bar{w}(\overrightarrow{vz}) = 0$, so $\max\{\bar{w}(\overrightarrow{uz}), \bar{w}(\overrightarrow{vz})\} = w(\overrightarrow{uz}) - \mu(z)$. Using this in (3) gives

$$\bar{\sigma}^1_A(z) + \sigma^1_A(z) + \bar{\sigma}^1_B(z) + \sigma^1_B(z) \leq w(\overrightarrow{uz}) - \mu(z) + \sigma^1_A(z) + \sigma^1_B(z) \leq w(\overrightarrow{uz}),$$

where again we used $\sigma_A + \sigma_B \trianglelefteq \mu$ in the last inequality. The case $z \in N_{\bar{w}}(v) \setminus N_{\bar{w}}(u)$ follows by a symmetric argument, so it only remains to check the case where $z \notin N_{\bar{w}}(u) \cup$

$N_{\bar{w}}(v)$. In this case, $\bar{w}(\overrightarrow{uz}) = \bar{w}(\overrightarrow{vz}) = 0$, so $\bar{\sigma}^1_A(z) + \bar{\sigma}^1_B(z) = 0$. Hence, left-hand side of (3) becomes

$$\sigma^1_A(z) + \sigma^1_B(z) \le \max\{w(\overrightarrow{uz}), w(\overrightarrow{vz})\},$$

where we used that (B4) holds for (σ_A, σ_B). This gives (B4), and we are done. ■

Remark 6.15. Proposition 6.14 is meaningful in the case where we take some of the skew-matchings to be identically equal to zero. We denote such a zero-valued skew-matching by $\sigma_\emptyset$. Indeed, any γ-skew-matching σ with its anchor fitting in the w-neighbourhood of a vertex v can be paired-up with an empty skew-matching to form a (γ, γ')-skew pair $(\sigma, \sigma_\emptyset)$ in (G, w) for any $\gamma' \ge 0$. This will allow us to apply Proposition 6.14 replacing the input of a skew pair with just a single skew-matching.

The next proposition will be useful to estimate degrees of a vertex in different combinations of matchings and weights obtained after truncations.

Proposition 6.16. *Let (G, w) be a weighted symmetric digraph. Let $\mu' \le \mu$ be fractional matchings in (unoriented) G and let (G, w') be the μ'-truncated weighted digraph obtained from (G, w). Then*

$$\deg_w(v, \mu') + \deg_{w'}(v, \mu - \mu') = \deg_w(v, \mu).$$

Proof. We have

$$\deg_w(v, \mu') + \deg_{w'}(v, \mu - \mu') = \sum_{x \in V(G)} \left(\min\{w(\overrightarrow{vx}), \mu'(x)\} + \min\{w'(\overrightarrow{vx}), \mu(x) - \mu'(x)\}\right),$$

so it suffices to show that, for every $x \in V(G)$, we have

$$\min\{w(\overrightarrow{vx}), \mu'(x)\} + \min\{w'(\overrightarrow{vx}), \mu(x) - \mu'(x)\} = \min\{w(\overrightarrow{vx}), \mu(x)\}.$$

Suppose first that $\mu'(x) < w(\overrightarrow{vx})$ holds. Then we have $w'(\overrightarrow{vx}) = w(\overrightarrow{vx}) - \mu'(x)$, and the left-hand side of the desired equality becomes $\mu'(x) + \min\{w(\overrightarrow{vx}) - \mu'(x), \mu(x) - \mu'(x)\} = \min\{w(\overrightarrow{vx}), \mu(x)\}$, as we wanted to show. Hence, we can suppose that $w(\overrightarrow{vx}) \le \mu'(x)$. In this case, we have $w'(\overrightarrow{vx}) = 0$. Then the left-hand side of the desired equality becomes $w(\overrightarrow{vx})$, which is also equal to $\min\{w(\overrightarrow{vx}), \mu(x)\}$ since $w(\overrightarrow{vx}) \le \mu'(x) \le \mu(x)$. ■

7. Fractional structure of Gallai–Edmonds decompositions

In this section, we investigate the structure of skew-matchings in general graphs, relying on known structural results about matchings in graphs. Our starting point is the classical Gallai–Edmonds decomposition theorem [Gal64; Edm65], which provides a vertex-partition which exhibits crucial information about maximal matchings in a graph. Recall that a graph G is said to be *factor-critical* if for any vertex $v \in V(G)$ there is a perfect matching $M \subseteq E(G)$ covering $G - \{v\}$. We say that a component of a graph is *factor-critical* if the graph induced by this component is factor-critical. The following statement appears in [Die17, Theorem 2.2.3].

Theorem 7.1 (Gallai–Edmonds theorem)**.** *For any graph G, there is a set $S \subseteq V(G)$, called a* separator*, such that each component of $G - S$ is factor-critical and there is a matching $M_S \subseteq E(G)$ between S and $V(G) \setminus S$, that covers S and matches the elements of S into different components of $G - S$; i.e., for any component K of $G - S$, we have $|V(K) \cap V(M_S)| \le 1$.*

Given a graph G and $S \subseteq V(G)$, $M_S \subseteq E(G)$ as in Theorem 7.1, we say (G, S, M_S) is a *Gallai–Edmonds triple*. It will be important to us to consider the components in $G - S$ and to distinguish whether they correspond to single vertices or not. We let $\mathcal{K}_S$ be the set of components of $G - S$, and we let $\mathcal{K}^*_S \subseteq \mathcal{K}_S$ correspond to the non-singleton components of $G - S$. We also let $U_S = \{v \in V(G) : \{v\} \in \mathcal{K}_S \setminus \mathcal{K}^*_S\}$ be the set of vertices corresponding to singleton components in $\mathcal{K}_S$. See Figure 2 for an illustration of all of these objects.

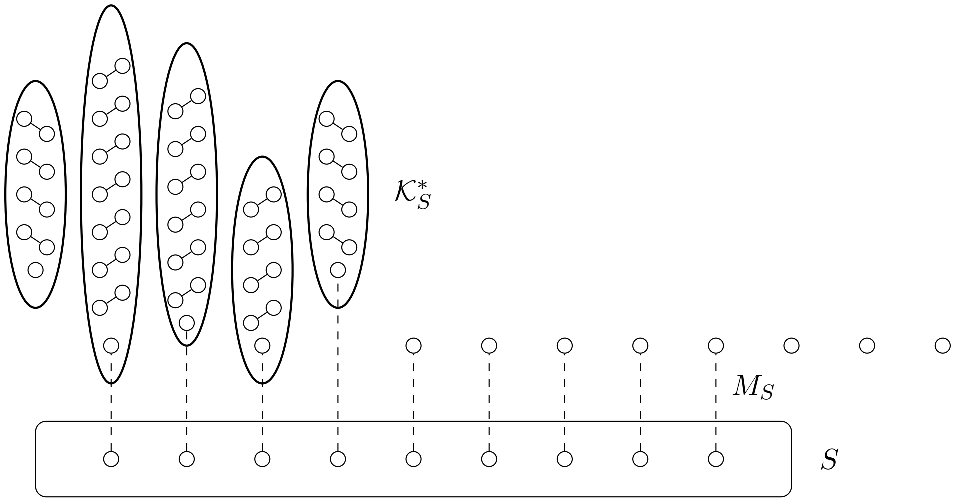


FIGURE 2. A view of a (G, S, M) Gallai–Edmonds triple. Note that however there might be more edges, here we just draw the matching M_S between S and components of $G - S$ by dashed edges. We also illustrate the notion of a factor-critical component by completing M_S with a matching (full edges) in each non-singleton component of $\mathcal{K}^* - V(M_S)$.

In our proofs we will work with fractional and skew-fractional matchings which are supported on edges beyond those of M_S. The following definition captures the set of edges that we will use.

Definition 7.2 (Gallai–Edmonds support)**.** Let (G, S, M) be a Gallai–Edmonds triple. We say that $E_{S,M} \subseteq E(G)$ is the *Gallai–Edmonds support* of (G, S, M) if

$$E_{S,M} := M \cup E(G[S, U_S]) \cup \bigcup_{K \in \mathcal{K}_S^*} E(G[K]).$$

Remark 7.3. Observe that $E_{S,M}$ does not include any edge with two endpoints in S.

7.1. **Fractional matchings and c-optimal fractional matchings.** We now consider fractional matchings associated with a Gallai–Edmonds triple (G, S, M). The fractional matchings we consider will be supported by the Gallai–Edmonds support, and cover both the separator S and the non-singleton components of $G - S$.

Definition 7.4 (Fractional Gallai–Edmonds triple)**.** Let μ be a fractional matching in G. We say that (G, S, μ) is a *fractional Gallai–Edmonds triple* if there is a matching $M \subseteq E(G)$, such that (G, S, M) is a Gallai–Edmonds triple, and

(D1) μ is supported in the Gallai–Edmonds support $E_{S,M}$,

(D2) $S \cup \bigcup_{K \in \mathcal{K}_S^*} V(K)$ is covered by μ.

Among all fractional Gallai–Edmond triples (G, S, μ), we will work with those that are in some sense "optimal" with respect to a vertex c: they put as much weight as possible in the neighbourhood of c; with respect to a weight function w on the edges of G. To formalise this definition we will need a fractional notion of an *alternating path.*

Definition 7.5 (Alternating path)**.** For a fractional matching μ in a graph G and a vertex $u \in V(G)$, a *μ-alternating path starting at u* is a path $P = (u, v_1, v_2, \dots, v_\ell)$ in G starting at u, such that $\mu(v_{2i-1}v_{2i}) > 0$ for all $i \in \{1, \dots, \lfloor \ell/2 \rfloor\}$. The *thickness* of P is $\theta(P) := \min\{\mu(v_{2i-1}v_{2i}) \,:\, i \in \{1, \dots, \lfloor \ell/2 \rfloor\}\}$.

Given this, we can precisely define what are the fractional matchings we want to use in our proofs.

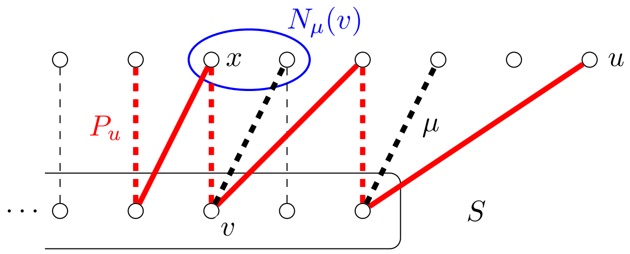


FIGURE 3. An alternating path P_u in bold red edges starting at u in a (G, S, M) Gallai–Edmonds triple structure. The support of the fractional matching μ is illustrated in dashed edges. For any vertex v in $V(P_u) \cap S$, the set $N_\mu(v)$ (shown in blue) consists of the neighbours of v connected by support edges of μ.

Definition 7.6 (c-optimal fractional matchings)**.** Let (G, w) be a weighted graph, $G^{\leftrightarrow}$ be its associated digraph (with inherited weights), (G, S, M) be a Gallai–Edmonds triple, and $c \in S$. We say that a fractional matching μ is a *c-optimal fractional matching* if (G, S, μ) is a fractional Gallai–Edmonds triple, and moreover the following holds. For each singleton component $\{u\}$ of $G - S$ with $\mu(u) < w(\overrightarrow{cu})$, for each μ-alternating path P_u starting at u, and for each $v \in V(P_u) \cap S$, let

$$N_\mu(v) = \{x \in N(v) : \mu(vx) > 0\}.$$

Then we have

(E1) $N_\mu(v)$ consists of singleton components of $G - S$,

(E2) $\mu(v) = 1$,

(E3) for each $x \in N_\mu(v)$ we have $\mu(x) \leq w(\overrightarrow{cx})$

Remark 7.7. If $\{u\}$ is a singleton component of $G - S$ with $\mu(u) < w(\overrightarrow{cu})$, then we have $N_w(u) \subseteq S$. Also, for each $v \in N_w(u)$ the edge uv forms a μ-alternating path of length 1. Then in particular properties (E1)–(E3) apply to all $v \in N_w(u)$.

Remark 7.8. If v is as above, and $x \in N_\mu(v)$, property (E3) implies that $w(\overrightarrow{cx}) \geq \mu(x) > 0$, so in particular $x \in N_w(c)$. In short, $N_\mu(v) \subseteq N_w(c)$.

The following property is key, and shows that indeed c-optimal fractional matchings exist given a Gallai–Edmonds triple (G, S, M), as long as we pick c in the separator S.

Proposition 7.9. *Let (G, w) be a weighted graph, (G, S, M) be a Gallai–Edmonds triple, and $c \in S$. Then there exists a fractional Gallai–Edmonds triple (G, S, μ), such that μ is a c-optimal fractional matching.*

To prove Proposition 7.9 we will need the following simple result.

Proposition 7.10. *Every factor-critical graph on more than one vertex has a perfect fractional matching.*

Proof. Since G is factor-critical, for each vertex $v \in V(G)$ there exists a matching M_v which covers $G - \{v\}$. Let $\mu_v : E(G) \to \{0, 1\}$ be such that $\mu_v(M_v) \equiv 1$ and $\mu_v(E(G) \setminus M_v) \equiv 0$. Then $\mu = \sum_{v \in V(G)} \mu_v / (|V(G)| - 1)$ is a perfect fractional matching of G. ■

Proof of Proposition 7.9. Among all possible Gallai–Edmonds triples (G, S, M), choose one with matching M_S that minimizes the number of vertices in $N(c)$ which lie in singleton components of $G - S$ and are not contained in M_S. Let E_{S,M_S} be the corresponding Gallai–Edmonds support of (G, S, M_S). The matching M_S can be understood as a fractional matching μ_S with weights 1 on M_S and 0 on other edges. This fractional matching can be extended to cover all non-trivial components K of $G - S$; either by using Proposition 7.10 if $V(K) \cap M_S = \emptyset$, or using the factor-criticality of K to find a perfect matching in $K - V(M_S)$. By doing this, we have obtained a fractional matching μ' such that

(i) μ' covers S,
(ii) the only uncovered vertices by μ' form isolated components in $G - S$,
(iii) μ' is supported in E_{S,M_S}, and
(iv) for each isolated component $\{u\}$ of $G - S$, we have $\mu'(u) \in \{0, 1\}$, and $\mu'(u) = 1$ if and only if u was covered by M_S.

Claim 7.11. *μ' satisfies (E2) and (E1).*

Proof of the claim. Let $\{u\}$ be any singleton component of $G - S$ with $\mu'(u) < w(\overrightarrow{cu})$, let P_u be any μ'-alternating path P_u starting at u, and let $v \in V(P_u) \cap S$ be arbitrary. Since μ' covers S, indeed (E2) holds.

It only remains to see that (E1) holds. Note that $N_{\mu'}(v) \cap S = \emptyset$ because μ' is supported in E_{S,M_S}. Moreover, it follows from $0 \leq \mu'(u) < w(\overrightarrow{cu}) \leq 1$ that $u \in N(c)$. By (iv), we have that $\mu'(u) = 0$, and that u was not covered by M_S. For a contradiction, suppose that $x \in N_{\mu'}(v)$ is not a singleton component of $G - S$. By definition of $N_{\mu'}(v)$, we have that $\mu'(vx) > 0$. We also know that $x \notin S$. Since μ' is supported in E_{S,M_S} in fact we deduce that $vx \in M_S$. Since $v \in V(P_u) \cap S$, we deduce that there exists $i \geq 1$ and a μ'-alternating path $uv_1v_2 \cdots v_{2i-1}v_{2i}$ with $\mu'(v_{2j-1}v_{2j}) = 1$ for all $1 \leq j \leq i$, and $v_{2i-1} = v$ and $v_{2i} = x$. This implies that $v_1v_2, \ldots, v_{2i-1}v_{2i} \subseteq M_S$. Define $M'_S := (M_S \setminus \{v_1v_2, \ldots, v_{2i-1}v_{2i}\}) \cup \{uv_1, v_2v_3, \ldots, v_{2i-2}v_{2i-1}\}$. In summary, by passing from M_S to M'_S we have uncovered x but covered u.

But (G, S, M'_S) is a Gallai–Edmonds triple, and since $u \in N(c)$ and x does not form a singleton component in $G - S$, by passing from M_S to M'_S we have decreased the number of vertices in $N(c)$ which lie in singleton components and are not covered. This contradicts the minimality of M_S. Thus, (E1) indeed holds. □

Let μ be the fractional matching that maximizes $\deg_w(c, \mu)$ among all fractional matchings μ' whose support is a subset of E_{S,M_S} and which satisfy (E1), and (E2). We claim that μ satisfies (E3). Aiming at a contradiction, suppose this is not the case. Thus, there exist a singleton component $\{u\}$ of $G-S$ with $\mu(u) < w(\overrightarrow{cu})$, a μ-alternating path P_{uv} starting at u and finishing at $v \in S$, and $x \in N_\mu(v)$ such that $\mu(x) > w(\overrightarrow{cx})$. We can assume that among all the possible such paths we choose one of shortest length. Using (E1) we obtain that $x \in G - S$. Note that $x \notin V(P_{uv})$, as otherwise we can pass to a shorter path. Then the path P consisting of P_{uv}, extended with vx at the end is a μ-alternating path starting from u. Indeed, $x \in N_\mu(v)$ and therefore $\mu(vx) > 0$, with $x \notin S$.

Suppose $P = v_0v_1 \cdots v_{2\ell}$ with $v_0 = u$ and $v_{2\ell} = x$. By definition of P and $N_\mu(v)$, we have that the thickness $\theta(P)$ of P (see Definition 7.5) satisfies $\theta(P) > 0$. By (E1), we have that $E(P) \subseteq E_{S,M_S}$.

Let $\delta := \min\{\mu(x) - w(\overrightarrow{cx}), w(\overrightarrow{cu}) - \mu(u), \theta(P)\} > 0$. We define a new fractional matching μ_P such that $\mu_P(e) = \mu(e)$ if $e \notin E(P)$; and $\mu_P(v_{2i+1}v_{2i+2}) := \mu(v_{2i+1}v_{2i+2}) - \delta$ and $\mu_P(v_{2i}v_{2i+1}) := \mu(v_{2i}v_{2i+1}) + \delta$ for every $i \in \{0, \ldots, \ell - 1\}$. Then $\mu_P(z) = \mu(z)$ for every $z \in V(G) \setminus \{u, x\}$, and $\mu_P(x) \geq w(\overrightarrow{cx})$. We also have $w(\overrightarrow{cu}) \geq \mu_P(u) > \mu(u)$. Since $u \in N(c)$, this implies $\deg_w(c, \mu_P) > \deg_w(c, \mu)$, contradicting the maximality of $\deg_w(c, \mu)$ (or equivalently, our assumption that μ fails to satisfy (E3)). ■

7.2. **Reachable vertices.** Now we take a closer look at the structure of c-optimal fractional matchings. Of particular interest are the vertices outside the separator S that belong to alternating paths, which we call *reachable*.

Definition 7.12 (Reachable vertices)**.** Let (G, S, M) be a Gallai–Edmonds triple, $c \in S$ and $w : E(G) \to [0, 1]$ be a weight function. Let (G, S, μ) be a fractional Gallai–Edmonds triple such that μ is c-optimal. We define the set $\mathcal{R}$ of *reachable vertices with respect to* c as the set of vertices in $V(G) \setminus S$ contained in some μ-alternating path P_u starting at a singleton component u of $G - S$ with $\mu(u) < w(\overrightarrow{cu})$. Also, let $S_{\mathcal{R}} = \bigcup_{x \in \mathcal{R}} N(x)$.

If $x \in \mathcal{R}$, then either x is a singleton component in $G-S$ for which $w(\overrightarrow{cx}) > 0$, or there exists a path P_u as in the definition such that $x \in V(P_u)$. By taking the neighbour of x in P_u that is closer to u, we obtain some $v \in S$ such that $x \in N_\mu(v)$, so by Remark 7.8 we have that $w(\overrightarrow{cx}) > 0$ as well. Hence, in any case, we have verified the following three consequences.

Observation 7.13. We have $\mathcal{R} \subseteq U_S$, i.e. $\mathcal{R}$ consists only of singleton components of $G - S$. In particular, $S_\mathcal{R} \subseteq S$.

Observation 7.14. If $x \in \mathcal{R}$, then $w(\overrightarrow{cx}) > 0$. In particular, $\mathcal{R} \subseteq N_w(c)$.

Observation 7.15. For every $y \in \mathcal{R}$, we have that $w(\overrightarrow{cy}) \geq \mu(y)$.

The following observation allows us to compare the sizes of $\mathcal{R}$ and $S_\mathcal{R}$.

Observation 7.16. For every $y \in S_\mathcal{R}$ and $z \in V(H)$ such that $\mu(yz) > 0$, we have $z \in \mathcal{R}$. In particular, we have

$$\sum_{x\in\mathcal{R}} \mu(x) = \sum_{y\in S_\mathcal{R}} \mu(y),$$

and $|S_\mathcal{R}| \leq |\mathcal{R}|$.

Proof. Let $y \in S_\mathcal{R}$ and $z \in V(H)$ such that $\mu(yz) > 0$. We have yz is in the Gallai–Edmonds support, hence $z \notin S$. Since $S_\mathcal{R}$ is the set of neighbourhoods of vertices in $\mathcal{R}$, there exists $r \in \mathcal{R}$ such that $ry \in E(H)$. By the definition of $\mathcal{R}$, y is the endpoint of a suitable μ-alternating path which we can extend by adding z; this implies that $z \in \mathcal{R}$, as desired.

This implies that for every edge such that $\mu(xy) > 0$, $x \in \mathcal{R}$ if and only if $y \in S_\mathcal{R}$. This yields the claimed equality. Since μ covers S (by property (D2)), the right-hand side equals $|S_\mathcal{R}|$. On the other hand, the left-hand side is at most $|\mathcal{R}|$ because μ is a fractional matching. This proves the claimed inequality. ∎

7.3. **GE pairs.** In the proof of Proposition 4.2 we shall work with the concept of *GE pair*. This will be a pair $(\tilde{\sigma}, \tilde{\mu})$ consisting of a γ-skew-matching $\tilde{\sigma}$ and a fractional matching $\tilde{\mu}$, which are disjoint. Essentially, we want to generalise the concept of a fractional Gallai–Edmonds triple by allowing some of the weight to be provided by a skew-matching, not only by a fractional matching.

Definition 7.17. Let (G, w) be a weighted digraph, (G, S, M) be a Gallai–Edmonds triple, $c \in S$, and $\gamma > 1$. Let μ be a c-optimal fractional matching and let $\mathcal{R}$ be the set of reachable vertices with respect to c and μ. We say that a pair $(\tilde{\sigma}, \tilde{\mu})$ is a *γ-GE pair for c with respect to w and μ*, if

(F1) $\tilde{\sigma}$ is a γ-skew-matching,

(F2) $\tilde{\mu}$ is a fractional matching disjoint from $\tilde{\sigma}$,

(F3) $\mathcal{A}(\tilde{\sigma})$ is contained in $S_\mathcal{R}$,

(F4) $\mathcal{A}(\tilde{\sigma})$ fits in the w-neighbourhood of c,

(F5) $V(\tilde{\sigma}) \setminus \mathcal{A}(\tilde{\sigma}) \subseteq \mathcal{R}$.

(F6) for all $y \in \mathcal{R}$ we have $\tilde{\mu}(y) + \tilde{\sigma}(y) \leq w(\overrightarrow{cy})$,

(F7) for all $y \in V(G) \setminus \mathcal{R}$ we have $\tilde{\mu}(y) + \tilde{\sigma}(y) \geq w(\overrightarrow{cy})$,

(F8) $\tilde{\mu} + \tilde{\sigma}$ covers S, and

(F9) $\tilde{\mu}$ equals to μ when restricted to the graph $G - (\mathcal{R} \cup S_\mathcal{R})$ and any supporting edge of $\tilde{\mu}$ intersecting the set $\mathcal{R} \cup S_\mathcal{R}$ lies in the bipartite graph $G[\mathcal{R}, S_\mathcal{R}]$.

For brevity, we will say that $(\tilde{\sigma}, \tilde{\mu})$ is a *$(G, w, S, M, c, \mu, \gamma)$-GE pair*.

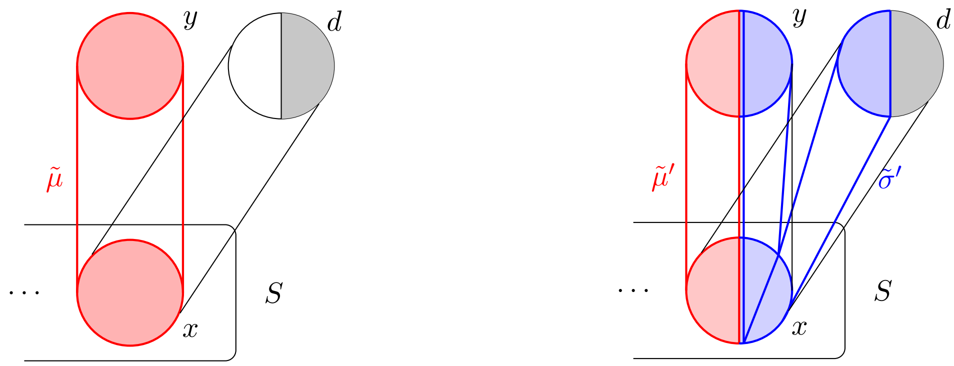
Remark 7.18. Note that in our definition we only consider the case $\gamma > 1$. While it is in principle possible to define γ-GE pairs with $\gamma \leq 1$, this will never be used in our proofs. The main reason are properties (F3) and (F5). For any the directed edges $\overrightarrow{uv}$ with $\tilde{\sigma}(\overrightarrow{uv}) > 0$ must satisfy $u \in S_{\mathcal{R}}$ and $v \in \mathcal{R}$. Hence, if $\gamma \leq 1$ then in such a situation the contribution of $\overrightarrow{uv}$ to the weight of u is at least the contribution of that edge in v. This means that we could replace the weight in $\tilde{\sigma}$ with some weight in μ (which covers u and v equally) without decreasing the total weight. Thus the definition only represents a real gain over fractional Gallai–Edmonds triples in the case $\gamma > 1$.

Remark 7.19. A γ-GE pair always exists as, for any c-optimal fractional matching μ, the pair $(\mu, \emptyset)$ satisfies Definition 7.17. Formally, this follows from the fact that (G, S, μ) is a fractional Gallai–Edmonds triple, together with the definition of $\mathcal{R}$ and Observations 7.15 and 7.16.

7.4. **Separation with GE pairs.** Let $(\tilde{\sigma}, \tilde{\mu})$ be a $(G, w, S, M, c, \mu, \gamma)$-GE pair. Among all such pairs, we will work with pairs which saturate the neighbourhood of c as much as possible, that is, we want that $\deg_w(c, \tilde{\mu} + \tilde{\sigma})$ is maximum over all choices of $(\tilde{\sigma}, \tilde{\mu})$. In such a case, we say that $(\tilde{\sigma}, \tilde{\mu})$ is *optimal.*

We adapt a generalization of an alternating path argument to obtain some structural information about optimal GE pairs; that is the content of the next two lemmas.

Lemma 7.20 (Separating Lemma I)**.** *Let $(\tilde{\sigma}, \tilde{\mu})$ be an optimal $(G, w, S, M, c, \mu, \gamma)$-GE pair. Suppose there exists $d \in \mathcal{R}$ such that $\tilde{\sigma}(d) + \tilde{\mu}(d) < w(\overrightarrow{cd})$. Then we have $\tilde{\sigma}(x) = w(\overrightarrow{cx})$ for all $x \in N_w(c) \cap N_w(d)$.*



(A) For simplicity, we illustrate here the easy case, when $w(\overrightarrow{cd}) = 1/2$, $w(\overrightarrow{cx}) = w(\overrightarrow{cy}) = 1$, $\tilde{\mu}(xy) = 1$ thus $\tilde{\sigma}(x) = 0$, and $\tilde{\sigma}(d) + \tilde{\mu}(d) = 0$. This situation contradicts the conclusion of Lemma 7.20.

(B) Assuming $\gamma = 2$, we obtain $\delta = 1/4$, and after modification, we have a new pair $(\tilde{\mu}', \tilde{\sigma}')$ with $\tilde{\mu}'(xy) = 1/2$, $\tilde{\sigma}'(\overrightarrow{xy}) = 3/4$, $\tilde{\sigma}'(\overrightarrow{xd}) = 3/4$. Observe that we have $\tilde{\sigma}'(d) + \tilde{\mu}'(d) = \tilde{\sigma}'(d) = 1/2 = w(\overrightarrow{cd})$.

Figure 4. Situations in the proof of Lemma 7.20. In the second figure $\tilde{\mu}' + \tilde{\sigma}'$ covers x and y as in the first figure, but additionally it covers also half of d. Hence $\deg_w(c, \tilde{\mu}' + \tilde{\sigma}') > \deg_w(c, \tilde{\mu} + \tilde{\sigma})$. Therefore, $(\tilde{\mu}, \tilde{\sigma})$ was not an optimal GE pair, as assumed.

Proof. Let $d \in \mathcal{R}$ as in the statement, and let $x \in N_w(c) \cap N_w(d)$ be arbitrary. Since $d \in \mathcal{R}$, we have $x \in S_{\mathcal{R}}$ by Observation 7.13. We also have $d \in N_w(c)$ by Observation 7.14. Now, we leverage the fact that if there is any possibility to increase the total saturation (w.r.t. c); then clearly the GE pair $(\tilde{\sigma}, \tilde{\mu})$ is not maximizing the saturation of $N_w(c)$, which will be a contradiction.

By (F4), we have $\tilde{\sigma}(x) \leq w(\overrightarrow{cx})$. Aiming at a contradiction, suppose that $\tilde{\sigma}(x) < w(\overrightarrow{cx})$. By (F8), we have that $\tilde{\sigma}(x) + \tilde{\mu}(x) = 1$, so we must have $\tilde{\mu}(x) > 0$. Hence, there exists $y \in N_{\tilde{\mu}}(x)$ such that $\tilde{\mu}(xy) > 0$. See Picture (A) of Figure 4 for an example. Note that it could happen that $y = d$. Define

$$\delta := \min\left\{\frac{\gamma-1}{\gamma}(w(\overrightarrow{cx}) - \tilde{\sigma}(x)), \frac{1}{\gamma}(w(\overrightarrow{cd}) - \tilde{\sigma}(d) - \tilde{\mu}(d)), \frac{\gamma-1}{\gamma}\tilde{\mu}(xy)\right\}. \tag{4}$$

The first and second terms of this minimum are strictly positive by assumption, and the last term is positive by the choice of y. Hence, $\delta > 0$.

We define $\tilde{\mu}'$ by

$$\begin{aligned}
\tilde{\mu}'(xy) &= \tilde{\mu}(xy) - \frac{\gamma}{\gamma-1}\delta \overset{(4)}{\geq} 0, \text{ and}\\
\tilde{\mu}'(uv) &= \tilde{\mu}(uv), \text{ in any other case.}
\end{aligned}$$

Now, we suppose first that $y \neq d$. We define $\tilde{\sigma}'$ by

$$\begin{aligned}
\tilde{\sigma}'(\overrightarrow{xd}) &= \tilde{\sigma}(\overrightarrow{xd}) + (1+\gamma)\delta,\\
\tilde{\sigma}'(\overrightarrow{xy}) &= \tilde{\sigma}(\overrightarrow{xy}) + \frac{1+\gamma}{\gamma-1}\delta, \text{ and}\\
\tilde{\sigma}'(\overrightarrow{uv}) &= \tilde{\sigma}(\overrightarrow{uv}), \text{ in any other case.}
\end{aligned}$$

See Picture (B) of Figure 4 for an illustration. Now, we have

$$\tilde{\mu}'(d) + \tilde{\sigma}'(d) = \tilde{\mu}(d) + \tilde{\sigma}(d) + \gamma\delta \overset{(4)}{\leq} w(\overrightarrow{cd}) \leq 1, \tag{5}$$

$$\tilde{\mu}'(y) + \tilde{\sigma}'(y) = \tilde{\mu}(y) - \frac{\gamma}{\gamma-1}\delta + \tilde{\sigma}(y) + \frac{\gamma}{\gamma-1}\delta = \tilde{\mu}(y) + \tilde{\sigma}(y) \overset{(F6)}{\leq} w(\overrightarrow{cy}) \leq 1, \tag{6}$$

$$\tilde{\sigma}'(x) + \tilde{\mu}'(x) = \tilde{\sigma}(x) + \delta + \frac{\delta}{\gamma-1} + \tilde{\mu}(x) - \frac{\gamma}{\gamma-1}\delta = \tilde{\sigma}(x) + \tilde{\mu}(x) \overset{(F2)}{=} 1, \tag{7}$$

$$\tilde{\sigma}'(x) = \tilde{\sigma}(x) + \delta + \frac{\delta}{\gamma-1} \overset{(4)}{\leq} w(\overrightarrow{cx}). \tag{8}$$

In the case when $y = d$, we define σ' as before for every edge except $\overrightarrow{xd}$, where we set

$$\tilde{\sigma}'(\overrightarrow{xd}) = \tilde{\sigma}(\overrightarrow{xd}) + (1+\gamma)\delta + \frac{1+\gamma}{\gamma-1}\delta.$$

Observe that (7)–(8) still hold with this choice; and instead of (5) and (6) we have

$$\tilde{\mu}'(d) + \tilde{\sigma}'(d) = \tilde{\mu}(d) - \frac{\gamma}{\gamma-1}\delta + \tilde{\sigma}(d) + \gamma\delta + \frac{\gamma}{\gamma-1}\delta \overset{(4)}{\leq} w(\overrightarrow{cd}). \tag{9}$$

We claim that, in any case, $(\tilde{\sigma}', \tilde{\mu}')$ is a $(G, w, S, M, c, \mu, \gamma)$-GE pair. First, observe that by the assumption that $d \in \mathcal{R}$, we get that $x \in S_{\mathcal{R}}$. Together with (F9) on the pair $(\tilde{\sigma}, \tilde{\mu})$, we deduce that $y \in \mathcal{R}$. Property (F1) follows directly from the definition of $\tilde{\sigma}'$ and from (5)-(6) and (8)-(9); and (F2) follows from (5)-(7) and (9). Properties (F3) and (F5) are trivial, since $x \in S_{\mathcal{R}}$ and $d, y \in \mathcal{R}$. Property (F4) follows from (8); and (F6) follows from (5)-(6) and (9). We also have that $\tilde{\sigma}'(z) + \tilde{\mu}'(z) = \tilde{\sigma}(z) + \tilde{\mu}(z)$ for any $z \notin \{d, x, y\}$, so to get (F7) and (F8), it suffices to check it for x, and this follows from (7). The last property (F9) follows from the fact that the pair $(\tilde{\sigma}', \tilde{\mu}')$ satisfies (F9), together with the fact that $x \in S_{\mathcal{R}}$ and $d, y \in \mathcal{R}$.

Finally, we note that the change in $\deg_w(c, \tilde{\sigma}' + \tilde{\mu}')$ with respect to $\deg_w(c, \tilde{\sigma} + \tilde{\mu})$ depends only (possibly) on the vertices d, y and x, and we have already checked that the change in x and in y is zero. From (5) or (9), we get

$$\deg_w(c, \tilde{\sigma}' + \tilde{\mu}') - \deg_w(c, \tilde{\sigma} + \tilde{\mu}) = \gamma\delta > 0$$

where we used $\delta > 0$ in the last step. This contradicts the optimality of $(\tilde{\sigma}, \tilde{\mu})$. ∎

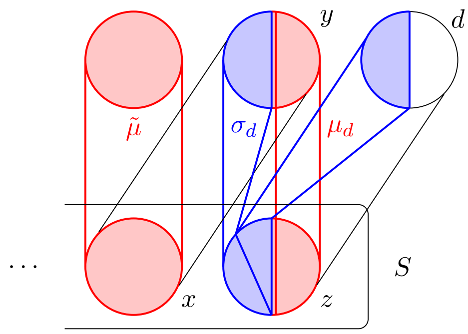
Lemma 7.21 (Separating Lemma II)**.** *Let $(\tilde{\sigma}, \tilde{\mu})$ be an optimal $(G, w, S, M, c, \mu, \gamma)$-GE pair. Suppose there exists $d \in \mathcal{R}$ such that $\tilde{\sigma}(d) + \tilde{\mu}(d) < w(\overrightarrow{cd})$.*

Let $\sigma_d \leq \tilde{\sigma}$ and $\mu_d \leq \tilde{\mu}$ be the γ-skew-matching and fractional matching, respectively, obtained from $\tilde{\sigma}$ and $\tilde{\mu}$, respectively, by considering only the edges of their support that intersect $N_w(d)$. Then there is no edge between the sets

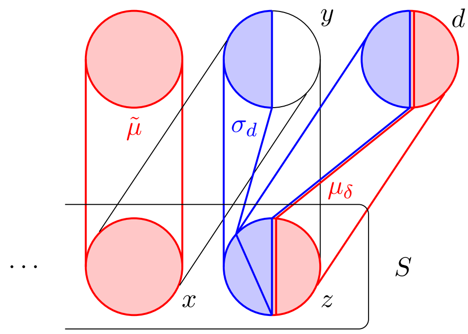
$$\{x \in N_w(c) \cap S \,:\, \tilde{\sigma}(x) < w(\overrightarrow{cx})\}$$

and

$$\{y \in N_w(c) \setminus S \,:\, \sigma_d(y) + \mu_d(y) > 0\}.$$



(A) For simplicity, we illustrate here the trivial case, when $w(\overrightarrow{cd}) = w(\overrightarrow{cx}) = w(\overrightarrow{cy}) = 1$, $\tilde{\mu}(x) = 1$ thus $\tilde{\sigma}(x) = 0$, and $\tilde{\sigma}(d) + \tilde{\mu}(d) = 1/2 < w(\overrightarrow{cd})$. The vertices x and y are connected by an edge, contradicting the conclusion of Lemma 7.21.



(B) The parameter $\delta = 1/2$ in this situation. This amount of weight in μ_d is transferred from zy to zd. The skew matching $\tilde{\sigma}$ doesn't change. Now $\tilde{\sigma}(y) + \mu_\delta(y) = 1/2 < w(\overrightarrow{cy})$.

Figure 5. The new GE pair created in the right picture saturates the neighbourhood of c by the same amount as the one on the left. As xy is an edge the GE pair $(\tilde{\sigma}, \mu_\delta)$ cannot be optimal, and thus neither is $(\tilde{\sigma}, \tilde{\mu})$, contradicting the assumption of Lemma 7.21.

Proof. Arguing by contradiction, we suppose that there is a vertex $x \in N_w(c) \cap S$ with $w(\overrightarrow{cx}) > \tilde{\sigma}(x)$ that is adjacent to a vertex $y \in N_w(c) \setminus S$ with $\sigma_d(y) + \mu_d(y) > 0$. By Separating Lemma I (Lemma 7.20), we know that $x \notin N_w(d)$. This implies, in particular, that $y \neq d$. It also implies that $\mu_d(x) = 0$. Also observe that, because $\sigma_d(y) + \mu_d(y) > 0$, and the way σ_d and μ_d is defined, we have that $y \in \mathcal{R}$ by (F5) and (F9). This straightforwardly leads to $x \in S_{\mathcal{R}}$.

We now split the proof into two cases.

Case 1: $y \in V(\mu_d)$. First consider the case when $y \in V(\mu_d)$, i.e. that $\mu_d(y) > 0$. Then there must exist some $z \in N_w(d) \cap S_{\mathcal{R}}$ with $0 < \mu_d(zy)$, and thus $\tilde{\mu}(zy) > 0$. Let

$$\delta = \min\left\{\tilde{\mu}(zy), w(\overrightarrow{cd}) - \tilde{\sigma}(d) - \tilde{\mu}(d)\right\}. \tag{10}$$

The first term is non-zero by the choice of z; and the second term is non-zero by our assumption on d. Hence, $\delta > 0$.

We define a skew matching σ_δ and a fractional matching μ_δ by setting $\sigma_\delta \equiv \tilde{\sigma}$ on all edges, and

$$\mu_\delta(zd) = \tilde{\mu}(zd) + \delta,$$
$$\mu_\delta(zy) := \tilde{\mu}(zy) - \delta,$$

and $\mu_\delta = \tilde{\mu}$ on any other edge. See Figure 5 for illustration.

Observe that the construction also implies that we have not modified the weights of vertices outside of $\{z, d, y\}$. Also observe that

$$\sigma_\delta(z) + \mu_\delta(z) = \tilde{\sigma}(z) + \tilde{\mu}(z) + \delta - \delta = \tilde{\sigma}(z) + \tilde{\mu}(z) \overset{\text{(F8)}}{=} 1, \tag{11}$$

$$\sigma_\delta(y) + \mu_\delta(y) = \tilde{\sigma}(y) + \tilde{\mu}(y) - \delta \overset{\text{(F6)}}{<} w(\overrightarrow{cy}) \leq 1, \tag{12}$$

$$\sigma_\delta(d) + \mu_\delta(d) = \tilde{\sigma}(d) + \tilde{\mu}(d) + \delta \overset{(10)}{\leq} w(\overrightarrow{cd}) \leq 1. \tag{13}$$

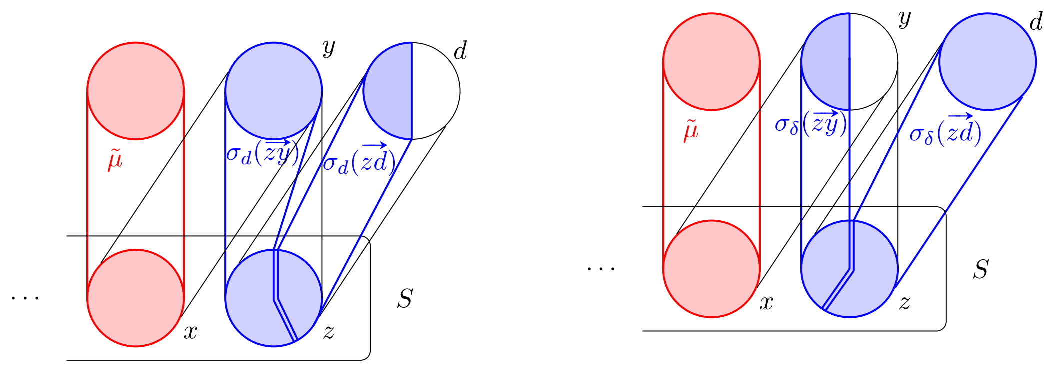
We will show that $(\sigma_\delta, \mu_\delta)$ is a $(G, w, S, M, c, \mu, \gamma)$-GE pair. Properties (F1) and (F3)–(F5) are automatic as $\sigma_\delta \equiv \tilde{\sigma}$. Property (F2) follows from (11)–(13). Property (F6) follows from (12)–(13). As $d, y \in \mathcal{R}$, we have that (F7)–(F8) follow from (11). Finally, property (F9) follows from the fact that $zd \in G[S_{\mathcal{R}}, \mathcal{R}]$.

Now, the fact that there is an edge between x and y and from (12), we deduce from Lemma 7.20 that $(\sigma_\delta, \mu_\delta)$ is not an optimal $(G, w, S, M, c, \mu, \gamma)$-GE pair. As

$$\deg_w(c, \sigma_\delta + \mu_\delta) = \deg_w(c, \tilde{\sigma} + \tilde{\mu}) + \delta - \delta = \deg_w(c, \tilde{\sigma} + \tilde{\mu}),$$

we deduce that $(\tilde{\sigma}, \tilde{\mu})$ is not an optimal $(G, w, S, M, c, \mu, \gamma)$-GE pair, either. This is a contradiction.

Case 2: $y \notin V(\mu_d)$. Since $\sigma_d(y) + \mu_d(y) > 0$, this implies that $y \in V(\sigma_d) \setminus V(\mu_d)$, i.e., there is a $z \in N_w(d)$ such that $\tilde{\sigma}(\overrightarrow{zy}) > 0$ and $\tilde{\mu}(zy) = 0$. By (F3)–(F4), we have that in fact $z \in N_w(c) \cap N_w(d)$.



(A) The vertices x and y are connected, contradicting the conclusion of Lemma 7.21. In this picture $\sigma_d(y) > 0$. However, $y \notin V(\mu_d)$ (i.e., we are in Case 2)

(B) We obtain a new skew-matching σ_δ by increasing the weight of σ_d on the edge zd and decreasing it by the same amount on the edge zy. This "transfer the empty space" from d to y, which is directly connected to x.

FIGURE 6. In both pictures above, we have a GE pair with the same saturation of $N_w(c)$. If the fist one is optimal, so is the second one. However, this is a contradiction with Lemma 7.20.

Let

$$\delta := \min\left\{\frac{\sigma(\overrightarrow{zy})}{\gamma}, \frac{w(\overrightarrow{cd}) - \tilde{\sigma}(d) - \tilde{\mu}(d)}{\gamma}\right\}, \tag{14}$$

From the choice of z and our assumption, we have that $\delta > 0$.

Define a skew-matching σ_δ by setting

$$\sigma_\delta(\overrightarrow{zd}) := \tilde{\sigma}(\overrightarrow{zd}) + (1+\gamma)\delta,$$

$$\sigma_\delta(\overrightarrow{zy}) := \tilde{\sigma}(\overrightarrow{zy}) - (1+\gamma)\delta,$$

and $\sigma_\delta \equiv \tilde{\sigma}$ on all other edges. Also let $\mu_\delta \equiv \tilde{\mu}$. See Figure 6 for illustration.

Observe that the construction also implies that we do not modify the weights of vertices outside $\{z, d, y\}$. Also note that we have

$$\sigma_\delta(d) + \mu_\delta(d) = \tilde{\sigma}(d) + \tilde{\mu}(d) + \gamma\delta \overset{(14)}{\leq} w(\overrightarrow{cd}) \leq 1, \tag{15}$$

$$\sigma_\delta(y) + \mu_\delta(y) = \tilde{\sigma}(y) + \tilde{\mu}(y) - \gamma\delta \overset{(F6)}{<} w(\overrightarrow{cy}) \leq 1, \tag{16}$$

$$\sigma_\delta(z) + \mu_\delta(z) = \tilde{\sigma}(z) + \tilde{\mu}(z) + \delta - \delta = \tilde{\sigma}(z) + \tilde{\mu}(z) \overset{(F8)}{=} 1, \tag{17}$$

$$\sigma_\delta(z) = \tilde{\sigma}(z) + \delta - \delta = \tilde{\sigma}(z) \leq w(\overrightarrow{cz}) \leq 1. \tag{18}$$

We will show that $(\sigma_\delta, \mu_\delta)$ is a $(G, w, S, M, c, \mu, \gamma)$-GE pair. Properties (F1) and (F2) follow from (15)–(17). Properties (F3) and (F5) follow from the definition of σ_δ and that $d \in \mathcal{R}$. Property (F4) follows from (18). Property (F6) follows from (15) and (16). Properties (F7) and (F8) follow from (17). Finally, (F9) is trivial since $\mu_\delta \equiv \tilde{\mu}$.

Since $d, y \in N_w(c)$, we have that

$$\deg_w(c, \sigma_\delta + \tilde{\mu}) = \deg_w(c, \tilde{\sigma} + \tilde{\mu}) + \gamma\delta - \gamma\delta = \deg_w(c, \tilde{\sigma} + \tilde{\mu}). \tag{19}$$

By (16), using that xy is an edge, Separating Lemma I (Lemma 7.20) implies that $(\sigma_\delta, \mu_\delta)$ is not an optimal$(G, w, S, M, c, \mu, \gamma)$-GE pair; but then (19) implies that neither is $(\tilde{\sigma}, \tilde{\mu})$. This is contradiction, and it finishes the proof. ■

Remark 7.22. After a careful moment of reflection, one can observe that the statements of Separating Lemma I (Lemma 7.20) and Separating Lemma II (Lemma 7.21) are about the same underlying property, just one step further in an "$(\tilde{\sigma}+\tilde{\mu})$-alternating path" from d in Lemma 7.21. Similar properties hold for longer "alternating paths". However, for the sake of simplification, here we choose to focus on just what we actually need in the proof.

8. The Matching Lemmas

To streamline the proof of Proposition 4.2, we have extracted and formalized some recurring arguments into stand-alone lemmas. These lemmas not only simplify the current proof but also offer versatile, black-box tools that can be applied in future work involving skew-matching pairs.

This section is organised as follows. In §8.1 we introduce basic lemmas that serve as foundational building blocks. These lemmas are used directly in the proof of Proposition 4.2 or as components of the more intricate 'advanced' matching lemmas presented in §8.2. There are four advanced matching lemmas: the Improved Balancing Lemma, the Completion Lemma, the Greedy Lemma, and the $(k, k/2)$ Lemma. The proof of the basic lemmas are presented in §8.3, and the next subsections present each a proof of an advanced matching lemma.

8.1. Basic Matching Lemmas.

The first lemma determines how large a skew-matching can be fitted inside a given fractional matching. The set U represents the neighbourhood of a vertex where we want the anchor to fit in.

Lemma 8.1 (Extending-out)**.** *Let H be a graph and $U, V \subseteq V(H)$ be disjoint sets. Suppose there is a fractional matching μ between U and V. Then there is a γ-skew-matching σ in $H^{\leftrightarrow}$ such that*

(G1) $\sigma \trianglelefteq \mu$,

(G2) $W(\sigma) = (1 + \min\{\gamma, \gamma^{-1}\})W(\mu)$, *and*

(G3) $\mathcal{A}(\sigma) \subseteq U$.

The second lemma will be used in situations where we have "access" to the fractional matching from both sides (i.e., the fractional matching lies within the neighbourhood of a vertex). In this situation, we can pack the skew-matching more efficiently.

Lemma 8.2 (Balancing-out)**.** *Let H be a graph, and let $U \subseteq V(H)$, such that there is a fractional matching μ in $H[U]$. Then there is a γ-skew-matching $\sigma \in H^{\leftrightarrow}$ such that*

(H1) $\sigma \trianglelefteq \mu$,

(H2) $W(\sigma) = 2W(\mu)$, *and*

(H3) $\mathcal{A}(\sigma) \subseteq U$.

The next lemma is a more specific version of the previous two, stated directly for weighted graphs. It will provide a lower bound on the weight of a skew-matching that can be built while respecting the saturation of the neighbourhood of a vertex by a fractional matching μ.

Lemma 8.3 (Combination)**.** *Let (H, w) be a weighted graph, $v \in V(H)$, μ a fractional matching in H and $\gamma \geq 0$. Then there is a γ-skew-matching σ in $H^{\leftrightarrow}$ such that*

(I1) $\sigma \trianglelefteq \mu$,

(I2) $W(\sigma) \geq \deg_w(v, \mu)$, *and*

(I3) $\mathcal{A}(\sigma)$ *fits in the w-neighbourhood of v.*

In contrast with the previous lemmas, the next lemma will be used to build a fractional matching from the existence of a skew-matching. This lemma is presented in a more general form, where one γ_A-skew-matching is found inside another γ_B-skew-matching. Applying this with $\gamma_A = 1$, this can be combined with Lemma 6.10 to obtain fractional matchings.

Lemma 8.4 (Extending-out skew-matching)**.** *Let (H, w) be a weighted graph, $\gamma_A > 0$ and $\gamma_B \geq 1$. Let $u \in V(H)$, and σ_B be a γ_B-skew-matching in $H^{\leftrightarrow}$ such that $\mathcal{A}(\sigma_B)$ fits in the w-neighbourhood of u. Then there is a γ_A-skew-matching in $H^{\leftrightarrow}$ such that*

(J1) $\sigma_A \leq \sigma_B$,

(J2) $W(\sigma_A) = \frac{1+\min\{\gamma_A, \gamma_A^{-1}\}}{1+\gamma_B} W(\sigma_B)$, *and*

(J3) $\mathcal{A}(\sigma_A)$ *fits in the w-neighbourhood of u.*

8.2. **Advanced Matching Lemmas.** The first of the more complex lemmas gives a condition under which we can completely fill up a fractional matching with a skew-matching pair.

Lemma 8.5 (Improved balancing)**.** *Let H be a graph and $U, V \subseteq V(H)$ be disjoint sets. Suppose there is a fractional matching μ running between U and V. Let $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ be such that*

(K1) $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 = 2W(\mu)$,

(K2) $\max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\} \leq W(\mu)$.

Set $\gamma_A := \alpha_2/\alpha_1$ and $\gamma_B := \beta_2/\beta_1$. Then there is a γ_A-skew-matching σ_A and a γ_B-skew-matching σ_B in $H^{\leftrightarrow}$, such that

(K3) $\sigma_A + \sigma_B \trianglelefteq \mu$,

(K4) $W(\sigma_A) = \alpha_1 + \alpha_2$, $W(\sigma_B) = \beta_1 + \beta_2$,

(K5) $\mathcal{A}(\sigma_A) \subseteq U$, *and* $\mathcal{A}(\sigma_B) \subseteq U \cup V$.

The following lemma is the most complex one. As in the previous lemma, we combine two different skew-matchings and place them disjointly within a fractional matching. However, in this case, we impose significantly weaker conditions for the graph to satisfy, leading to a much more challenging scenario.

Lemma 8.6 (Completion)**.** *Let (H, w) be a weighted graph, $U, V \subseteq V(H)$ be disjoint sets, $u \in V(H)$, and μ be a fractional matching running between U and V. Let $\alpha_1, \beta_1 > 0$ and $\alpha_2, \beta_2 \geq 0$ be such that*

(L1) *for all $y \in V$, we have $\mu(y) \leq w(\overrightarrow{uy})$,*

(L2) *$\max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\} \leq W(\mu)$, and*

(L3) *$\min\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\} \geq W(\mu)$.*

Set $\gamma_A := \alpha_2/\alpha_1$ and $\gamma_B := \beta_2/\beta_1$. Then $H^{\leftrightarrow}$ admits a γ_A-skew-matching σ_A and a γ_B-skew-matching σ_B such that

(L4) *$\sigma_A + \sigma_B \trianglelefteq \mu$,*

(L5) *$W(\sigma_A) = \alpha_1 + \alpha_2$,*

(L6) *$W(\sigma_B) \geq \max\{0, \deg_w(u, \mu) - W(\sigma_A)\}$,*

(L7) *$\mathcal{A}(\sigma_A) \subseteq U$,*

(L8) *$\mathcal{A}(\sigma_B)$ fits in the w-neighbourhood of u, and moreover,*

(L9) *if $\gamma_B \leq 1$, we can also ensure that $\sigma_A(y) + \sigma_B(y) = \mu(y)$ for all $y \in V$.*

During the proof of Proposition 4.2, we repeatedly encounter situations where we rely on a minimum degree condition to place our skew-matching. We present three lemmas, which have similar, but slightly different, scenarios and outcomes. We recall the notation $\deg_w(v, S) := \sum_{u \in S} w(\overrightarrow{vu})$ for $v \in V(H)$ and $S \subseteq V(H)$.

Lemma 8.7 (First Greedy Lemma)**.** *Let (H, w) be a weighted graph, $u, v \in V(H)$, $\kappa \geq 0$, (σ_A, σ_B) be a (γ_A, γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{uv} \in E(H^{\leftrightarrow})$, and let $U, V \subseteq V(H)$ be disjoint sets. Suppose that*

(M1) *$\deg_w(u, V) \geq \kappa + \sum_{x \in V}(\sigma_A(x) + \sigma_B(x))$, and*

(M2) *$|N_w(x) \cap U| \geq \gamma_A\kappa + \sum_{y \in U}(\sigma_A(y) + \sigma_B(y))$, for every $x \in V$.*

Then there is a γ_A-skew oriented fractional matching σ'_A in $H^{\leftrightarrow}$ with

(M3) *$\mathcal{A}(\sigma'_A) \subseteq V$,*

(M4) *$W(\sigma'_A) \geq (1 + \gamma_A)\kappa$,*

(M5) *σ'_A is supported in $H[V, U]$, and such that*

(M6) *$(\sigma_A + \sigma'_A, \sigma_B)$ is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{uv}$.*

Lemma 8.8 (Second Greedy Lemma)**.** *Let (H, w) be a weighted graph, $u, v \in V(H)$, $\kappa \geq 0$, (σ_A, σ_B) be a (γ_A, γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{uv} \in E(H^{\leftrightarrow})$, and let $U, V \subseteq V(H)$ be disjoint sets. Suppose that*

(N1) *$\deg_w(u, V) \geq \kappa + \sum_{x \in V}(\sigma_A(x) + \sigma_B(x))$,*

(N2) *$|N_w(x) \cap (U \cup V)| \geq (1 + \gamma_A)\kappa + \sum_{y \in U \cup V}(\sigma_A(y) + \sigma_B(y))$, for every $x \in V$,*

Then there is a γ_A-skew oriented fractional matching σ'_A in $H^{\leftrightarrow}$ with

(N3) *$\mathcal{A}(\sigma'_A) \subseteq V$,*

(N4) *$W(\sigma'_A) \geq (1 + \gamma_A)\kappa$,*

(N5) *σ'_A is supported in $\{xy \in E(H) : x \in V, y \in V \cup U\}$, and such that*

(N6) *$(\sigma_A + \sigma'_A, \sigma_B)$ is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{uv}$.*

Lemma 8.9 (Third Greedy Lemma)**.** *Let (H,w) be a weighted graph, $u,v \in V(H)$, $\kappa \geq 0$, (σ_A,σ_B) be a (γ_A,γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{uv} \in E(H^{\leftrightarrow})$, and let $U,V \subseteq V(H)$ be disjoint sets. Suppose that*

(O1) $|U| \geq \gamma_A\kappa + \sum_{y\in U}(\sigma_A(y)+\sigma_B(y))$, *and*

(O2) $\deg_w(u, N_w(y)\cap V) \geq \kappa + \sum_{x\in V}(\sigma_A(x)+\sigma_B(x))$, *for every $y \in U$.*

Then there is a γ_A-skew oriented fractional matching σ'_A in $H^{\leftrightarrow}$ with

(O3) $\mathcal{A}(\sigma'_A) \subseteq V$,

(O4) $W(\sigma'_A) \geq (1+\gamma_A)\kappa$,

(O5) σ'_A *is supported in $H[V,U]$, and such that*

(O6) $(\sigma_A+\sigma'_A,\sigma_B)$ *is a (γ_A,γ_B)-skew-matching anchored in $\overrightarrow{uv}$.*

The following proposition will allow us to conclude if there is an edge cd and a skew-matching which saturates sufficiently each of the neighbourhoods of c and d. This lemma (in a less general version, using only matchings) appeared in the work of Ajtai, Komlós, Simonovits, and Szemerédi [Ajt+15]; we adapt their strategy to our situation.

Lemma 8.10 (The $(k,k/2)$-lemma)**.** *Let $k \geq 2$. Let (H,w) be a weighted graph, $cd \in E(H)$, and μ a fractional matching in H, such that*

(P1) $\deg_w(c,\mu) \geq k$, *and*

(P2) $\deg_w(d,\mu) \geq k/2$.

Let $\alpha_1,\beta_1 > 0$ and $\alpha_2,\beta_2 \geq 0$ be such that $\alpha_1+\alpha_2+\beta_1+\beta_2 = k$. Set $\gamma_A := \alpha_2/\alpha_1$ and $\gamma_B := \beta_2/\beta_1$. Then $H^{\leftrightarrow}$ admits a (γ_A,γ_B)-skew-matching pair (σ_A,σ_B), anchored in $\overrightarrow{cd}$ or in $\overrightarrow{dc}$, such that

(P3) $W(\sigma_A) = \alpha_1+\alpha_2$,

(P4) $W(\sigma_B) = \beta_1+\beta_2$, *and*

(P5) $\sigma_A+\sigma_B \trianglelefteq \mu$.

8.3. **The proofs of the Basic Matching Lemmas.** We give the proof of the Basic Matching Lemmas (Lemmas 8.1–8.4).

Proof of Lemma 8.1 (Extending-out). For every xy with $x \in U$ and $y \in V$, define

$$\sigma(\overrightarrow{xy}) := (1+\gamma)\frac{\mu(xy)}{\max\{1,\gamma\}},$$

and set σ to 0 on all other edges. It is straightforward to verify that σ is a γ-skew fractional matching. Indeed, for $x \in U$ we have $\sum_y\big(\sigma(\overrightarrow{xy})/(1+\gamma)\big) = \sum_y \mu(xy)/\max\{1,\gamma\} \leq \mu(x) \leq 1$, and for $y \in V$ we have $\sum_x\big(\gamma\,\sigma(\overrightarrow{xy})/(1+\gamma)\big) = \gamma\sum_x \mu(xy)/\max\{1,\gamma\} \leq \mu(y) \leq 1$.

Now we verify the required properties. Since the only edges $\overrightarrow{xy}$ with non-zero weight have $x \in U$, this readily implies that $\mathcal{A}(\sigma) \subseteq U$, which gives (G3). Also, we have

$$W(\sigma) = \sum_{x\in U, y\in V}\sigma(\overrightarrow{xy}) = \frac{1+\gamma}{\max\{1,\gamma\}}\sum_{x\in U,y\in V}\mu(xy) = (1+\min\{\gamma,\gamma^{-1}\})W(\mu),$$

which gives (G2). Finally, for $x \in U$ and $y \in V$, we have

$$\frac{\sigma(\overrightarrow{xy})+\gamma\sigma(\overrightarrow{yx})}{1+\gamma} = \frac{\sigma(\overrightarrow{xy})}{1+\gamma} = \frac{\mu(xy)}{\max\{1,\gamma\}} \leq \mu(xy),$$

and

$$\frac{\sigma(\overrightarrow{yx})+\gamma\sigma(\overrightarrow{xy})}{1+\gamma}=\frac{\gamma\sigma(\overrightarrow{xy})}{1+\gamma}=\frac{\gamma\mu(xy)}{\max\{1,\gamma\}}\leq\mu(xy).$$

This means that $\sigma \trianglelefteq \mu$, so (G1) holds. ■

Proof of Lemma 8.2 (Balancing-out). For every xy with $x, y \in U$, we define $\sigma(\overrightarrow{xy}) = \sigma(\overrightarrow{yx}) := \mu(xy)$; and we set σ to 0 on all other edges. We have

$$W(\sigma)=\sum_{\{xy\in E(G)\,:\,x,y\in U\}}\sigma(\overrightarrow{xy})+\sigma(\overrightarrow{yx})=\sum_{\{xy\in E(G)\,:\,x,y\in U\}}2\mu(xy)=2W(\mu).$$

Therefore, for $x, y \in U$ we obtain

$$\frac{\sigma(\overrightarrow{xy})+\gamma\sigma(\overrightarrow{yx})}{1+\gamma}=\frac{(1+\gamma)\mu(xy)}{1+\gamma}=\mu(xy),$$

and

$$\frac{\sigma(\overrightarrow{yx})+\gamma\sigma(\overrightarrow{xy})}{1+\gamma}=\frac{(1+\gamma)\mu(xy)}{1+\gamma}=\mu(xy).$$

Hence, $\sigma \trianglelefteq \mu$. Finally, $\mathcal{A}(\sigma) \subseteq U$ holds by construction (because no directed edge with tail outside U receives weight). We thus have (H1)–(H3). ■

Proof of Lemma 8.3 (Combination). Let $U := V(\mu)$, and let $\tilde{\mu} \leq \mu$ be a maximal fractional matching so that $\tilde{\mu}(x) \leq w(\overrightarrow{vx})$ for all $x \in U$. By Lemma 8.2 there is a γ-skew-matching $\tilde{\sigma} \trianglelefteq \tilde{\mu}$ of weight $W(\tilde{\sigma}) = 2W(\tilde{\mu})$ with its anchor $\mathcal{A}(\tilde{\sigma})$ contained in U. By the choice of $\tilde{\mu}$ and $\tilde{\sigma} \trianglelefteq \tilde{\mu}$, we have that the anchor $\mathcal{A}(\tilde{\sigma})$ fits in the w-neighbourhood of v.

Let (H, w') be the $\tilde{\mu}$-truncated weighted graph obtained from (H, w). We define $U' := N_{w'}(v) = \{x \in V(H) : w'(\overrightarrow{vx}) > 0\}$ and $V' := U \setminus U'$. Let $\mu' \leq \mu - \tilde{\mu}$ be maximal so that $\mu'(x) \leq w'(\overrightarrow{vx})$ for all $x \in U'$ and has only support edges intersecting U'.

Observe that we have $w'(\overrightarrow{vx}) = 0$ or $w'(\overrightarrow{vy}) = 0$ for every xy with $\mu'(xy) > 0$. Indeed, if not, then we could increase $\tilde{\mu}(xy)$ by a small $\varepsilon > 0$, staying within the edge-wise bounds $\tilde{\mu} \leq \mu$ and the vertex bounds $\tilde{\mu}(x) \leq w(\overrightarrow{vx})$ and $\tilde{\mu}(y) \leq w(\overrightarrow{vy})$, contradicting the maximality of $\tilde{\mu}$. Hence, μ' runs between U' and V'.

Because of this, we have $W(\mu') = \sum_{x\in U'} \mu'(x)$, and since $\mu'(x) \leq w'(\overrightarrow{vx})$ for all $x \in U'$ then we also have $W(\mu') = \deg_{w'}(v, \mu')$.

We apply Lemma 8.1 with U', V', H, μ' playing the roles of U, V, H, μ, respectively. By doing so, we obtain a γ-skew-matching $\sigma' \in H^{\leftrightarrow}$ such that $\sigma' \trianglelefteq \mu'$, with weight $W(\sigma') = (1 + \min\{\gamma, \gamma^{-1}\})W(\mu') \geq \deg_{w'}(v, \mu')$, and such that its anchor $\mathcal{A}(\sigma')$ is contained in U'. By the definition of U' and μ', this implies that $\mathcal{A}(\sigma')$ fits in the w'-neighbourhood of v.

We apply Proposition 6.14 (with $(H, w), vx, \tilde{\mu}, \mu', w', \tilde{\sigma}, \sigma_\emptyset, \sigma'$, and $\sigma_\emptyset$, with an arbitrary x, playing the role of $(G, w), uv, \mu, \bar{\mu}, \sigma_A, \sigma_B, \bar{\sigma}_A$, and $\bar{\sigma}_B$ respectively) to consider the sum of $\tilde{\sigma}$ and σ' Let $\sigma := \tilde{\sigma} + \sigma'$. Thanks to the definition of $\tilde{\mu}$, we have $\sigma \trianglelefteq \mu$. Moreover, σ has weight $W(\sigma) \geq \deg_w(v, \tilde{\mu}) + \deg_{w'}(v, \mu - \tilde{\mu}) = \deg_w(v, \mu)$, where we used $\deg_w(v, \tilde{\mu}) = 2W(\tilde{\mu})$ and Proposition 6.16 with $H, w, \tilde{\mu}, \mu$, and w' paying the role of G, w, μ', μ and w', respectively. Also, its anchor $\mathcal{A}(\sigma)$ fits in the w-neighbourhood of v. This terminates the proof of Lemma 8.3. ■

Proof of Lemma 8.4 (Extending-out Skew-matching). Define a γ_A-skew-matching σ_A so that for each $\overrightarrow{xy} \in E(H^{\leftrightarrow})$ with $\sigma_B(\overrightarrow{xy}) > 0$, we have

$$\sigma_A(\overrightarrow{xy}):=\frac{1+\min\{\gamma_A,\gamma_A^{-1}\}}{1+\gamma_B}\sigma_B(\overrightarrow{xy}),$$

and $\sigma_A(\overrightarrow{xy}) := 0$ for all other $\overrightarrow{xy} \in E(H^{\leftrightarrow})$. Note that since $\gamma_B \geq 1$, we have $1 + \min\{\gamma_A, \gamma_A^{-1}\} \leq 2 \leq 1 + \gamma_B$, which ensures this indeed defines a γ_A-skew-matching.

Then we have

$$W(\sigma_A) = \frac{1+\min\{\gamma_A, \gamma_A^{-1}\}}{1+\gamma_B} W(\sigma_B),$$

which gives (J2). For every $\overrightarrow{xy} \in E(H^{\leftrightarrow})$ we have

$$\frac{\sigma_A(\overrightarrow{xy})}{1+\gamma_A} = \frac{1+\min\{\gamma_A, \gamma_A^{-1}\}}{1+\gamma_A} \cdot \frac{\sigma_B(\overrightarrow{xy})}{1+\gamma_B} \leq \frac{\sigma_B(\overrightarrow{xy})}{1+\gamma_B}.$$

Thus, $\sigma_A^1(x) \leq \sigma_B^1(x)$ for all $x \in V(H)$. Further, for all $\overrightarrow{xy} \in E(H^{\leftrightarrow})$, we have

$$\frac{\gamma_A \sigma_A(\overrightarrow{xy})}{1+\gamma_A} = \frac{\gamma_A + \min\{(\gamma_A)^2, 1\}}{1+\gamma_A} \cdot \frac{\sigma_B(\overrightarrow{xy})}{1+\gamma_B} \leq \frac{\gamma_B \sigma_B(\overrightarrow{xy})}{1+\gamma_B},$$

since $\frac{\gamma_A + \min\{\gamma_A^2, 1\}}{1+\gamma_A} \leq 1$ (it equals 1 if $\gamma_A \geq 1$ and equals γ_A if $\gamma_A \leq 1$), and $\gamma_B \geq 1$. This means that $\sigma_A \leq \sigma_B$, so (J1) holds. Finally, from $\sigma_A^1(x) \leq \sigma_B^1(x)$ we see that $\mathcal{A}(\sigma_A)$ fits in the w-neighbourhood of u, because $\mathcal{A}(\sigma_B)$ also fits in the w-neighbourhood of u, so we have (J3). ∎

8.4. **Proof of the Improved Balancing Lemma.** We shall need the following auxiliary 'allocation' lemma.

Lemma 8.11. *Let $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma \geq 0$ such that*

(Q1) $\alpha_1 + \alpha_2 + \beta_1 + \beta_2 \leq 2\gamma$, *and*

(Q2) $\min\{\beta_1, \beta_2\} + \max\{\alpha_1, \alpha_2\} \leq \gamma$,

then there exist $\hat{\beta}_1 \leq \beta_1$ and $\hat{\beta}_2 \leq \beta_2$ with $\hat{\beta}_1 \cdot \beta_2 = \hat{\beta}_2 \cdot \beta_1$ such that

(Q3) $\hat{\beta}_1 + (\beta_2 - \hat{\beta}_2) + \alpha_1 \leq \gamma$ *and*

(Q4) $\hat{\beta}_2 + (\beta_1 - \hat{\beta}_1) + \alpha_2 \leq \gamma$.

Proof. Without loss of generality we can assume that $\beta_2 \geq \beta_1$. We will also assume that $\alpha_2 \geq \alpha_1$, and we explain how to remove this assumption later. Those two assumptions imply that $\beta_1 + \alpha_2 \leq \beta_1 + \alpha_1 \leq \gamma$. If $\beta_2 + \alpha_2 \leq \gamma$, then we set $\hat{\beta}_1 = \beta_1$ and $\hat{\beta}_2 = \beta_2$, from which it is straightforward to verify (Q3)–(Q4). Hence, we may assume that $\beta_2 + \alpha_2 > \gamma$. Together with $\gamma \geq \alpha_2 + \beta_1$ we get

$$\beta_2 > \gamma - \alpha_2 \geq \beta_1.$$

This implies that there exists $\lambda \in [0, 1]$ such that

$$\lambda\beta_2 + (1-\lambda)\beta_1 = \gamma - \alpha_2.$$

Set $\hat{\beta}_2 := \lambda\beta_2$ and $\hat{\beta}_1 = \lambda\beta_1$. This choice gives $\hat{\beta}_2 + (\beta_1 - \hat{\beta}_1) + \alpha_2 = \gamma$, so (Q4) holds. To see (Q3), we note that

$$\begin{aligned}\hat{\beta}_1 + (\beta_2 - \hat{\beta}_2) + \alpha_1 &= \alpha_1 - (\lambda\beta_2 + (1-\lambda)\beta_1) + \beta_2 + \beta_1 \\ &= \alpha_1 + \alpha_2 + \beta_2 + \beta_1 - \gamma \leq \gamma,\end{aligned}$$

where in the last inequality we used (Q1). Finally, note that by defining instead $\hat{\beta}_2 = (1-\lambda)\beta_2$ and $\hat{\beta}_1 = (1-\lambda)\beta_1$ we get (Q3)–(Q4) with the roles of α_1, α_2 swapped, which removes the assumption on α_1, α_2. ∎

Proof of Lemma 8.5 (Improved balancing). Our assumptions (K1) and (K2) imply that we can apply Lemma 8.11 with $\alpha_1, \alpha_2, \beta_1, \beta_2, W(\mu)$ playing the roles of $\alpha_1, \alpha_2, \beta_1, \beta_2, \gamma$. We obtain $\hat{\beta}_1 \leq \beta_1$ and $\hat{\beta}_2 \leq \beta_2$ such that $\hat{\beta}_2/\hat{\beta}_1 = \beta_2/\beta_1 = \gamma_B$ and such that

$$\hat{\beta}_1 + (\beta_2 - \hat{\beta}_2) + \alpha_1 \leq W(\mu), \tag{20}$$

and

$$\hat{\beta}_2 + (\beta_1 - \hat{\beta}_1) + \alpha_2 \leq W(\mu). \tag{21}$$

For every $x \in U, y \in V$, we set

$$\sigma_A(\overrightarrow{xy}) := \frac{\alpha_1 + \alpha_2}{W(\mu)}\mu(xy), \qquad \sigma_B(\overrightarrow{xy}) := \frac{\hat{\beta}_1 + \hat{\beta}_2}{W(\mu)}\mu(xy),$$

and

$$\sigma_B(\overrightarrow{yx}) := \frac{(\beta_1 + \beta_2) - (\hat{\beta}_1 + \hat{\beta}_2)}{W(\mu)}\mu(xy),$$

and we put σ_A and σ_B equal to 0 on all other edges. Note that $\mathcal{A}(\sigma_A) \subseteq U$ and $\mathcal{A}(\sigma_B) \subseteq U \cup V$, so (K5) holds. The weights are

$$W(\sigma_A) = \sum_{x\in U, y\in V} \sigma_A(\overrightarrow{xy}) = \frac{\alpha_1 + \alpha_2}{W(\mu)} \sum_{x\in U, y\in V} \mu(xy) = \alpha_1 + \alpha_2$$

and

$$W(\sigma_B) = \sum_{x\in U, y\in V} \Big(\sigma_B(\overrightarrow{xy}) + \sigma_B(\overrightarrow{yx})\Big) = \frac{\beta_1 + \beta_2}{W(\mu)} \sum_{x\in U, y\in V} \mu(xy) = \beta_1 + \beta_2,$$

so (K4) holds. Finally, we observe that $\hat{\beta}_1 = t\beta_1$ and $\hat{\beta}_2 = t\beta_2$, for some $t \in [0, 1]$. From this, it follows that for every $x \in U, y \in V$, we have

$$\frac{\sigma_A(\overrightarrow{xy}) + \gamma_A \sigma_A(\overrightarrow{yx})}{1+\gamma_A} + \frac{\sigma_B(\overrightarrow{xy}) + \gamma_B \sigma_B(\overrightarrow{yx})}{1+\gamma_B} = \frac{\alpha_1 + \hat{\beta}_1 + (\beta_2 - \hat{\beta}_2)}{W(\mu)}\mu(xy) \overset{(20)}{\leq} \mu(xy),$$

and

$$\frac{\gamma_A\sigma_A(\overrightarrow{xy}) + \sigma_A(\overrightarrow{yx})}{1+\gamma_A} + \frac{\gamma_B\sigma_B(\overrightarrow{xy}) + \sigma_B(\overrightarrow{yx})}{1+\gamma_B} = \frac{\alpha_2 + \hat{\beta}_2 + (\beta_1 - \hat{\beta}_1)}{W(\mu)}\mu(xy) \overset{(21)}{\leq} \mu(xy),$$

so $\sigma_A + \sigma_B \trianglelefteq \mu$ holds, giving (K3). ■

8.5. **Proof of the Completion Lemma.** The proof is technical, so we divide it into six main steps.

(i) *Step 1: Defining σ_A.* The fractional matching μ is sufficiently large to accommodate the entire γ_A-skew-matching σ_A. Since the anchor must be contained in U, there is no flexibility in the choice of orientation. We define σ_A by distributing its weight proportionally according to the weight of the fractional matching μ.

(ii) *Step 2: Partitioning μ and σ_A.* We partition μ into μ' and $\widehat{\mu}$ such that all of μ' can be "reached from u from both sides" (i.e., both endpoints are in the w-neighborhood of u). This partition makes μ' easier to handle, as we can choose on which side to place the anchor. Similarly, we partition σ_A into σ'_A and $\widehat{\sigma_A}$ according to the proportions of μ' and $\widehat{\mu}$ on each edge.

(iii) *Step 3: Perfectly filling μ'.* We complete σ'_A by adding σ'_B to perfectly fill μ'. This is achieved in two steps, formulated in Claim 8.12 and Claim 8.13. In Claim 8.12, we identify a minimum $\bar{\sigma}_B$ to compensate for the skew in σ'_A. This requires carefully selecting the orientation of $\bar{\sigma}_B$ on each support edge of μ'. After compensating for the skew in σ'_A, we proceed to Claim 8.13, where $\bar{\sigma}_B$ is complemented by a γ_B-skew-matching to obtain σ'_B, balancing its orientations within the remainder of μ'.

(iv) *Step 4: Fitting the remainder into $\widehat{\mu}$.* The fractional matching $\widehat{\mu}$ can only be accessed from u on one side, meaning there is no choice regarding the placement of the anchor for a γ_B-skew-matching. After removing a fractional sub-matching $\tilde{\mu}$ with $\widehat{\sigma}_A \trianglelefteq \tilde{\mu}$ (i.e., where $\widehat{\sigma}_A$ "lives"), we fit a γ_B-skew-matching into the remaining portion $\widehat{\mu} - \tilde{\mu}$. After these four steps, we will have proven properties (L1)–(L8).

(v) *Steps 5 and 6: The case $\gamma_B \le 1$.* We are left to refine our approach to prove property (L9) under the additional assumption that $\gamma_B \le 1$. This is treated in two additional steps, depending on the case if $\gamma_A \ge 1$ and on the case when $\gamma_A < 1$. When $\gamma_A \ge 1$, the skews γ_A, γ_B work in our favour and the proof of property (L9) is straightforward using the objects we have defined so far. However, if $\gamma_A < 1$, we have to define things differently. Instead of placing a γ_B-skew-matching in the fractional matching $\widehat{\mu} - \tilde{\mu}$ as previously, we first need to complement $\widehat{\sigma}_A$ with a γ_B-skew-matching σ that together with $\widehat{\sigma}_A$ uses the fractional matching $\widehat{\mu}$ equally on both of its sides. Only then we can complete it with a γ_B-skew-matching σ' to ensure the set V is fully covered.

Proof of Lemma 8.6 (Completion). We follow the six steps outlined above.

Step 1: Defining σ_A. For all $x \in U$ and $y \in V$, we set

$$\sigma_A(\overrightarrow{xy}) := \frac{\alpha_1 + \alpha_2}{W(\mu)} \mu(xy), \tag{22}$$

and we put σ_A equal to 0 on all other edges. Observe that

$$W(\sigma_A) = \sum_{x \in U, y \in V} \sigma_A(\overrightarrow{xy}) = \frac{\alpha_1 + \alpha_2}{W(\mu)} \sum_{x \in U, y \in V} \mu(xy) = \alpha_1 + \alpha_2. \tag{23}$$

By construction, we have that

$$\mathcal{A}(\sigma_A) \subseteq U, \tag{24}$$

i.e. the anchor of σ_A is contained in U. Using (L2), we also have

$$\frac{\max\{1, \gamma_A\}}{1 + \gamma_A} \sigma_A(\overrightarrow{xy}) = \frac{\max\{\alpha_1, \alpha_2\}}{\alpha_1 + \alpha_2} \sigma_A(\overrightarrow{xy}) \le \frac{W(\mu)}{\alpha_1 + \alpha_2} \sigma_A(\overrightarrow{xy}) = \mu(xy),$$

and therefore $\sigma_A \trianglelefteq \mu$.

Step 2: Partitioning μ and σ_A. Let $\mu' \le \mu$ be a fractional matching such that for all $x \in U$ and $y \in V$ we have

$$\mu'(xy) := \frac{\min\{w(\overrightarrow{ux}), \mu(x)\}}{\mu(x)} \mu(xy). \tag{25}$$

By (L1), we have

$$\mu'(y) \le \mu(y) \le w(\overrightarrow{uy}), \text{ for all } y \in V. \tag{26}$$

Directly from the definition, we also have

$$\mu'(x) = \min\{w(\overrightarrow{ux}), \mu(x)\}, \text{ for all } x \in U. \tag{27}$$

and thus

$$\begin{aligned} \deg_w(u, \mu') &= \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu'(x)\} + \sum_{y \in V} \min\{w(\overrightarrow{uy}), \mu'(y)\} \\ &\overset{(26),(27)}{\le} \sum_{x \in U} \mu'(x) + \sum_{y \in V} \mu'(y) = 2W(\mu'). \end{aligned} \tag{28}$$

Now set

$$\widehat{\mu} := \mu - \mu'. \tag{29}$$

We partition σ_A into the part that will fit in μ' and the part that will fit in $\widehat{\mu}$. For all $x \in U, y \in V$, set

$$\sigma'_A(\overrightarrow{xy}) := \frac{\mu'(xy)}{\mu(xy)} \sigma_A(\overrightarrow{xy}) = \frac{\alpha_1 + \alpha_2}{W(\mu)} \cdot \mu'(xy), \tag{30}$$

and let σ'_A be equal to 0 on all other edges. Let

$$\widehat{\sigma}_A := \sigma_A - \sigma'_A. \tag{31}$$

Concretely, we have for $x \in U, y \in V$,

$$\widehat{\sigma}_A(\overrightarrow{xy}) = \frac{\widehat{\mu}(xy)}{\mu(xy)}\sigma_A(\overrightarrow{xy}) = \frac{\alpha_1 + \alpha_2}{W(\mu)} \cdot \widehat{\mu}(xy), \tag{32}$$

and $\widehat{\sigma}_A$ equal to 0 on all other edges. Observe that, by our construction and the fact that $\sigma_A \trianglelefteq \mu$, we have that $\sigma'_A \trianglelefteq \mu'$ and

$$\widehat{\sigma}_A \trianglelefteq \widehat{\mu}. \tag{33}$$

Step 3: Filling μ' perfectly. In the first claim, our goal is to counterbalance the skew of σ'_A by carefully positioning a minimal γ_B-skew-matching, ensuring that each support edge of the fractional matching carries equal weight at both endpoints.

Claim 8.12. *There are a fractional matching $\bar{\mu} \leq \mu'$ and a γ_B-skew-matching $\bar{\sigma}_B$ such that $\sigma'_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$, and $W(\sigma'_A) + W(\bar{\sigma}_B) = 2W(\bar{\mu})$.*

Before proving the claim, we gather two useful facts for the rest of the proof. First, we claim that

$$\text{if } \max\{1, \gamma_A\} > \min\{1, \gamma_A\}, \text{ then } \max\{1, \gamma_B\} > \min\{1, \gamma_B\}. \tag{34}$$

Indeed, suppose otherwise. Then $\gamma_A \neq 1$ and $\gamma_B = 1$, so $\alpha_1 \neq \alpha_2$ and $\beta_1 = \beta_2$. We have

$$\begin{aligned}\max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\} &= \max\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\} \\ &> \min\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\} \\ &\overset{\text{(L3)}}{\geq} W(\mu) \overset{\text{(L2)}}{\geq} \max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\},\end{aligned}$$

a contradiction. This proves (34).

Next, we gather a useful inequality. Observe that

$$\begin{aligned}&\frac{\max\{1, \gamma_A\}}{1 + \gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\min\{\beta_1, \beta_2\}}{W(\mu)}\mu'(xy) \\ &\overset{(30)}{=} \frac{\max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\}}{W(\mu)}\mu'(xy) \\ &\overset{\text{(L3)-(L2)}}{\leq} \frac{\min\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\}}{W(\mu)}\mu'(xy) \\ &= \frac{\min\{1, \gamma_A\}}{1 + \gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\max\{\beta_1, \beta_2\}}{W(\mu)}\mu'(xy).\end{aligned}$$

and therefore

$$\frac{\max\{1, \gamma_A\} - \min\{1, \gamma_A\}}{1 + \gamma_A}\sigma'_A(\overrightarrow{xy}) \leq \frac{\max\{\beta_1, \beta_2\} - \min\{\beta_1, \beta_2\}}{W(\mu)}\mu'(xy). \tag{35}$$

Proof of Claim 8.12. We will separate the proof into two cases, depending if $(1-\gamma_A)(1-\gamma_B) \leq 0$, or $(1 - \gamma_A)(1 - \gamma_B) > 0$.

Case 1: $(1 - \gamma_A)(1 - \gamma_B) \leq 0$. We define a γ_B-skew matching $\bar{\sigma}_B$ as follows. If $\gamma_A = 1$, then $\bar{\sigma}_B$ is identically zero. Otherwise, if $\gamma_A \neq 1$, for every $x \in U$ and every $y \in V$ we choose $\bar{\sigma}_B(\overrightarrow{xy})$ so that

$$\bar{\sigma}_B(\overrightarrow{xy}) := \frac{1 + \gamma_B}{1 + \gamma_A} \cdot \frac{\max\{1, \gamma_A\} - \min\{1, \gamma_A\}}{\max\{1, \gamma_B\} - \min\{1, \gamma_B\}} \cdot \sigma'_A(\overrightarrow{xy}), \tag{36}$$

and $\bar{\sigma}_B(\overrightarrow{xy}) = 0$ in every other edge. Observe that if $\gamma_A \neq 1$, by (34), we also have $\gamma_B \neq 1$. Together with $\gamma_B \leq 1$, this yields $\gamma_B < 1$, and in particular, $\max\{1, \gamma_B\} > \min\{1, \gamma_B\}$ and thus $\bar{\sigma}_B(\overrightarrow{xy})$ is correctly defined.

We claim that

$$\sigma'_A + \bar{\sigma}_B \trianglelefteq \mu' \tag{37}$$

holds (note this also verifies that $\bar{\sigma}_B$ is indeed a γ_B-skew matching).

If $\gamma_A = 1$ then (37) is immediate, because $\sigma'_A \trianglelefteq \mu'$ is true. Hence, we assume $\gamma_A \neq 1$. We need to verify, for every $x \in U$, $y \in V$, that

$$\frac{\sigma'_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\bar{\sigma}_B(\overrightarrow{xy})}{1+\gamma_B} \leq \mu'(xy), \tag{38}$$

and

$$\frac{\gamma_A\sigma'_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\gamma_B\bar{\sigma}_B(\overrightarrow{xy})}{1+\gamma_B} \leq \mu'(xy), \tag{39}$$

hold, as required by the desired condition (37).

Inequalities (35) and (36) together, imply, for each $x \in U$, $y \in V$, that

$$\bar{\sigma}_B(\overrightarrow{xy}) \leq \frac{\beta_1+\beta_2}{W(\mu)}\mu'(xy),$$

and thus

$$\begin{aligned}\frac{\max\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\min\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{xy}) &\overset{(30)}{\leq} \frac{\max\{\alpha_1,\alpha_2\}+\min\{\beta_1,\beta_2\}}{W(\mu)}\mu'(xy) \\ &\overset{(L2)}{\leq} \mu'(xy).\end{aligned} \tag{40}$$

Now, from (36), we obtain

$$\begin{aligned}\frac{\min\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\max\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{xy}) &= \frac{\max\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\min\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{xy}) \\ &\overset{(40)}{\leq} \mu'(xy).\end{aligned} \tag{41}$$

Inequalities (40) and (41) imply (38) and (39), as the condition $(1-\gamma_A)(1-\gamma_B) \leq 0$ and (34) imply that $\gamma_A > 1$ if and only if $\gamma_B < 1$. Thus indeed (37) holds.

Finally, for every $x \in U$ and every $y \in N_{\mu'}(x)$, set

$$\begin{aligned}\bar{\mu}(xy) :&= \frac{\min\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\max\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{xy}) \\ &= \frac{\max\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\min\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{xy}),\end{aligned}$$

where the equality is due to (36). From the definition we obtain $W(\sigma'_A) + W(\bar{\sigma}_B) = 2W(\bar{\mu})$, and that $\sigma'_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$, as desired.

Case 2: $(1-\gamma_A)(1-\gamma_B) > 0$ In this case, the proof goes similarly as above, but the support edges of $\bar{\sigma}_B$ will go in the opposite direction as the support edges of σ'_A. This means that for every $x \in U$ and every $y \in V$ we set

$$\bar{\sigma}_B(\overrightarrow{yx}) := \frac{1+\gamma_B}{1+\gamma_A} \cdot \frac{\max\{1,\gamma_A\} - \min\{1,\gamma_A\}}{\max\{1,\gamma_B\} - \min\{1,\gamma_B\}} \cdot \sigma'_A(\overrightarrow{xy}), \tag{42}$$

and $\bar{\sigma}_B = 0$ on all other edges.

Analogously as above, we claim that (37) holds. We need to verify for every $x \in U$ and every $y \in V$ that

$$\frac{\sigma'_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\gamma_B\bar{\sigma}_B(\overrightarrow{yx})}{1+\gamma_B} \leq \mu'(xy), \tag{43}$$

and

$$\frac{\gamma_A\sigma'_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\bar{\sigma}_B(\overrightarrow{yx})}{1+\gamma_B} \leq \mu'(xy). \tag{44}$$

The calculations go verbatim, simply by switching the orientation of the support edges of $\bar{\sigma}_B$ and realising that $(1-\gamma_A)(1-\gamma_B) > 0$ implies that $\gamma_A = \max\{1,\gamma_A\}$ if and only if $1 = \min\{1,\gamma_B\}$.

Then, we set

$$\begin{aligned}\bar{\mu}(xy) :=& \frac{\min\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\max\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{yx})\\ =& \frac{\max\{1,\gamma_A\}}{1+\gamma_A}\sigma'_A(\overrightarrow{xy}) + \frac{\min\{1,\gamma_B\}}{1+\gamma_B}\bar{\sigma}_B(\overrightarrow{yx})\end{aligned}$$

for every $x \in U$ and every $y \in N_\mu(x)$. We obtain $W(\sigma'_A) + W(\bar{\sigma}_B) = 2W(\bar{\mu})$, and that $\sigma'_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$, as desired. □

In this second claim, we complete the remaining unmatched portion of the fractional matching using a γ_B-skew-matching, alternating its orientation to ensure that the weight on each support edge is equally balanced between its endpoints.

Claim 8.13. *There is a γ_B-skew-matching σ'_B with*

$$W(\sigma'_A) + W(\sigma'_B) = 2W(\mu') \tag{45}$$

and

$$\sigma'_A + \sigma'_B \trianglelefteq \mu'. \tag{46}$$

Moreover, $\sigma'_B \geq \bar{\sigma}_B$, and the anchor of σ'_B fits in the w-neighbourhood of u.

Proof of the claim. Let $(H, \bar{w})$ be the $\bar{\mu}$-truncated weighted graph obtained from (H, w). Recall that $\mu' \leq \mu$ and that $V(\mu) \subseteq U \cup V$. We apply Lemma 8.2 (Balancing-out) with $\mu' - \bar{\mu}$ in place of μ. We obtain a γ_B-skew-matching $\tilde{\sigma}_B \trianglelefteq \mu' - \bar{\mu}$ in $H^{\leftrightarrow}$ of weight $W(\tilde{\sigma}_B) = 2W(\mu' - \bar{\mu})$ and with its anchor $\mathcal{A}(\tilde{\sigma}_B)$ contained in $U \cup V$.

Set $\sigma'_B := \bar{\sigma}_B + \tilde{\sigma}_B$. As $\tilde{\sigma}_B \trianglelefteq \mu' - \bar{\mu}$ and $\sigma'_A + \bar{\sigma}_B \trianglelefteq \bar{\mu}$, we have $\sigma'_A + \sigma'_B \trianglelefteq \mu'$, as required, and this also shows that σ'_B is well-defined. Observe that $\sigma'_B \geq \bar{\sigma}_B$ holds immediately. From Claim 8.12, we have $W(\sigma'_A) + W(\bar{\sigma}_B) = 2W(\bar{\mu})$. Thus we obtain $W(\sigma'_A) + W(\sigma'_B) = 2W(\mu')$, as required.

Finally, using $\sigma'_B \trianglelefteq \mu'$ and both (26) and (27), we obtain that the anchor $\mathcal{A}(\sigma'_B)$ fits in the w-neighbourhood of u. □

Step 4: Fitting the remainder into $\widehat{\mu}$. Recall that $\widehat{\mu} := \mu - \mu'$ and $\widehat{\sigma}_A := \sigma_A - \sigma'_A$. By (33), we can let $\tilde{\mu} \leq \widehat{\mu}$ be a minimal fractional matching such that $\widehat{\sigma}_A \trianglelefteq \tilde{\mu}$. Recall that σ_A (and therefore, $\widehat{\sigma}_A$) is supported only in edges $\overrightarrow{xy}$ with $x \in U$ and $y \in V$. This implies that, for every such $\overrightarrow{xy}$, we have

$$\tilde{\mu}(xy) = \frac{\max\{1,\gamma_A\}}{1+\gamma_A}\sigma_A(\overrightarrow{xy}),$$

and we have $\tilde{\mu}(xy) = 0$ for any other edge. Hence, we have

$$W(\tilde{\mu}) = \max\{1,\gamma_A\}\frac{W(\widehat{\sigma}_A)}{1+\gamma_A}. \tag{47}$$

Now we obtain a γ_B-skew-matching $\widehat{\sigma}_B$ such that $\widehat{\sigma}_B \trianglelefteq \widehat{\mu} - \tilde{\mu}$.

Claim 8.14. *There is a γ_B-skew-matching $\widehat{\sigma}_B \trianglelefteq \widehat{\mu} - \tilde{\mu}$ in $H^{\leftrightarrow}$ of weight*

$$W(\widehat{\sigma}_B) = (1 + \min\{\gamma_B, \gamma_B^{-1}\})W(\widehat{\mu} - \tilde{\mu}), \tag{48}$$

and such that $\mathcal{A}(\widehat{\sigma}_B) \subseteq V$.

Proof of the claim. Recalling the definitions, we have $\widehat{\mu} - \tilde{\mu} \leq \widehat{\mu} = \mu - \mu' \leq \mu$. Since μ is a matching running between U and V by assumption, the same is true for $\widehat{\mu} - \tilde{\mu}$. Hence, the claim follows by applying Lemma 8.1 with $\widehat{\mu} - \tilde{\mu}, \gamma_B, V, U$ playing the roles of μ, γ, U, V. □

Let $\sigma_B := \widehat{\sigma}_B + \sigma'_B$. Since $\widehat{\sigma}_B \trianglelefteq \hat{\mu} - \tilde{\mu} \leq \mu - \mu'$ and $\sigma'_B \trianglelefteq \mu'$, it quickly follows that σ_B is a γ_B-skew-matching. Recall that σ_A was defined at the beginning of the proof. We shall prove that σ_A, σ_B satisfy the required (L4)–(L8).

To see (L4), note that

$$\sigma_B + \sigma_A = \widehat{\sigma}_B + \sigma'_B + \sigma'_A + \widehat{\sigma}_A \trianglelefteq (\widehat{\mu} - \widetilde{\mu}) + \mu' + \widetilde{\mu} = \mu,$$

where we used the definition of σ_B and (31) in the first equality, and then we used the choice of $\widehat{\sigma}_B$, (46) and (33). Point (L5) follows from (23).

To see (L6), we shall estimate $W(\sigma_B)$ using all our work so far. We have

$$\begin{aligned}
W(\sigma_B) &= W(\widehat{\sigma}_B) + W(\sigma'_B) \\
&\overset{(48)}{\geq} W(\widehat{\mu} - \tilde{\mu}) + W(\sigma'_B) \\
&\overset{(45)}{=} 2W(\mu') - W(\sigma'_A) + W(\widehat{\mu} - \tilde{\mu}) \\
&\overset{(47)}{\geq} 2W(\mu') - W(\sigma'_A) + W(\widehat{\mu}) - W(\widehat{\sigma}_A) \\
&\overset{(28)}{\geq} \deg_w(u, \mu') - W(\sigma'_A) + W(\widehat{\mu}) - W(\widehat{\sigma}_A) \\
&\overset{(31)}{=} \deg_w(u, \mu') + W(\widehat{\mu}) - W(\sigma_A) \\
&= \sum_{x \in V(H)} \min\{w(\overrightarrow{ux}), \mu'(x)\} + W(\widehat{\mu}) - W(\sigma_A) \\
&= \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu'(x)\} + \sum_{x \in V} \min\{w(\overrightarrow{ux}), \mu'(x)\} + W(\widehat{\mu}) - W(\sigma_A) \\
&\overset{(26)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu'(x)\} + \sum_{x \in V} \mu'(x) + W(\widehat{\mu}) - W(\sigma_A) \\
&= \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu'(x)\} + \sum_{x \in V} \mu'(x) + \sum_{x \in V} \hat{\mu}(x) - W(\sigma_A) \\
&\overset{(29)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu'(x)\} + \sum_{x \in V} \mu(x) - W(\sigma_A) \\
&\overset{(27)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu(x)\} + \sum_{x \in V} \mu(x) - W(\sigma_A) \\
&\overset{(26)}{=} \sum_{x \in U \cup V} \min\{w(\overrightarrow{ux}), \mu(x)\} - W(\sigma_A) \\
&= \deg_w(u, \mu) - W(\sigma_A),
\end{aligned}$$

which, together with the obvious $W(\sigma_B) \geq 0$, gives (L6).

Point (L7) follows from (24). Finally, to verify (L8), we need to check that $\sigma^1_B(x) \leq w(\overrightarrow{ux})$ for all $x \in V(H)$. It suffices to check it for $x \in U \cup V$. Suppose first that $x \in U$. Note that $\mathcal{A}(\widehat{\sigma}_B) \subseteq V$ follows from our choice in Claim 8.14, so we have $\widehat{\sigma}^1_B(x) = 0$, and $\mathcal{A}(\sigma'_B)$ fits in the w-neighbourhood of u by our choice in Claim 8.13. Hence we have $\sigma^1_B(x) = \sigma'_B(x) \leq w(\overrightarrow{ux})$, as required. Now, suppose that $x \in V$. Here, we have $\sigma^1_B(x) \leq \mu(x) \leq w(\overrightarrow{ux})$, where in the first inequality we used (L4), and the second inequality follows from (26). Thus (L8) holds.

We have thus proven (L1)–(L8) and it only remains to prove (L9). So we may assume that $\gamma_B \leq 1$, as otherwise there is nothing to show.

Let us explain our strategy here. We will need to divide the proof in two cases, depending if $\gamma_A \geq 1$, or not. In any case, we shall keep the definition of σ_A, σ'_A, and $\widehat{\sigma}_A$ as in (22), (30), and (32), as well as we shall keep the definition of σ'_B from Claim 8.13 and $\bar{\sigma}_B$ from Claim 8.12. We keep $\widehat{\sigma}_B$ from Claim 8.14 as well, but we shall use it to determine σ_B only in some cases. This is the part where the value of γ_A becomes relevant. Indeed, while verifying (L9) is natural in the case when $\gamma_A \geq 1$ and $\gamma_B \leq 1$, one has to define things differently when $\gamma_A < 1$. In this case, we shall not use $\widehat{\sigma}_B$ in the definition of σ_B as above. The main problem is that $\widehat{\sigma}_A$ does not saturate $\tilde{\mu}$ in V.

Therefore, it has to be suitably completed by some γ_B-skew-matching. To achieve this, we shall proceed similarly as in Claim 8.12. This γ_B-skew-matching is then completed with a γ_B-skew-matching in a standard way to fill the left-over of $\widehat{\mu}$ in V. This is possible, as $\gamma_B \leq 1$. After that, we glue the respective γ_A and γ_B-skew-matchings together and check that indeed fulfill the required properties. Now we turn to the details.

Step 5: Saturating V, case $\gamma_A \geq 1$. We consider first the easiest case when $\gamma_A \geq 1$. Observe that $\mathcal{A}(\widehat{\sigma}_B) \subseteq V$ implies that $\widehat{\sigma}_B(\overrightarrow{xy}) = 0$ for all $x \in U, y \in V$. From $\widehat{\sigma}_B \trianglelefteq \widehat{\mu} - \widetilde{\mu}$ we have $\widehat{\sigma}_B(y) \leq \widehat{\mu}(y) - \widetilde{\mu}(y)$ for all $y \in V$. From $\gamma_B \leq 1$ and (48) we get $W(\widehat{\sigma}_B) = (1+\gamma_B)W(\widehat{\mu} - \widetilde{\mu})$, and therefore

$$\sum_{y\in V} \widehat{\sigma}_B(y) = \frac{W(\widehat{\sigma}_B)}{1+\gamma_B} = W(\widehat{\mu} - \widetilde{\mu}) = \sum_{y\in V} (\widehat{\mu}(y) - \widetilde{\mu}(y)),$$

which together with the above, implies that for all $y \in V$, we have

$$\widehat{\sigma}_B(y) = \widehat{\mu}(y) - \tilde{\mu}(y).$$

Now note that from $\widehat{\sigma}_A \trianglelefteq \tilde{\mu}$ we have $\widehat{\sigma}_A(y) \leq \tilde{\mu}(y)$ for all $y \in V$. Note that from $\gamma_A \geq 1$ and (47), we obtain $\gamma_A W(\widehat{\sigma}_A) = (1+\gamma_A)W(\tilde{\mu})$. Together with $\mathcal{A}(\widehat{\sigma}_A) \subseteq \mathcal{A}(\sigma_A) \subseteq U$, a similar argument as above gives, for all $y \in V$,

$$\widehat{\sigma}_A(y) = \tilde{\mu}(y).$$

Next, by Claim 8.13 we have $\sigma'_B + \sigma'_A \trianglelefteq \mu'$ and $W(\sigma'_B) + W(\sigma'_A) = 2W(\mu')$. By a similar reasoning as above, we have, for all $y \in V$,

$$\sigma'_B(y) + \sigma'_A(y) = \mu'(y). \tag{49}$$

Using the three equalities above, together with $\sigma_B := \widehat{\sigma}_B + \sigma'_B$ and (31), (29), we obtain

$$\sigma_A(y) + \sigma_B(y) = \widehat{\sigma}_B(y) + \sigma'_B(y) + \widehat{\sigma}_A(y) + \sigma'_A(y) = \widehat{\mu}(y) + \mu'(y) = \mu(y)$$

for all $y \in V$, thus giving (L9) in the case $\gamma_A \geq 1$.

Step 6: Saturating V, case $\gamma_A < 1$. From now on, we can assume, in addition to $\gamma_B \leq 1$, that $\gamma_A < 1$. By (34), in fact we have that $\gamma_B < 1$. Observe now that $(1-\gamma_A)(1-\gamma_B) > 0$, as in the second case of Claim 8.12. We shall define a new γ_B-skew-matching in place of σ_B to conclude.

Let σ be the maximal γ_B-skew-matching with $\sigma + \widehat{\sigma}_A \trianglelefteq \widehat{\mu}$ such that for every $x \in U$ and every $y \in V$ we have

$$\frac{1-\gamma_A}{1+\gamma_A}\widehat{\sigma}_A(\overrightarrow{xy}) \geq \frac{1-\gamma_B}{1+\gamma_B}\sigma(\overrightarrow{yx}), \tag{50}$$

and $\sigma = 0$ on all other edges. Analogously as in the second case of Claim 8.12, the definition implies that we obtain equality in (50). The calculations go verbatim, replacing $\bar{\sigma}_B$ by σ, σ'_A by $\widehat{\sigma}_A$, and μ' by $\widehat{\mu}$.

Next, let $\tilde{\mu}' \leq \widehat{\mu}$ be the fractional matching such that

$$\tilde{\mu}'(xy) := \frac{\widehat{\sigma}_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\gamma_B\sigma(\overrightarrow{yx})}{1+\gamma_B} = \frac{\sigma(\overrightarrow{yx})}{1+\gamma_B} + \frac{\gamma_A\widehat{\sigma}_A(\overrightarrow{xy})}{1+\gamma_A}, \tag{51}$$

where the equality follows from the fact that we have equality in (50) for each $\overrightarrow{xy}$. From (51), we have

$$W(\widehat{\sigma}_A) + W(\sigma) = 2W(\tilde{\mu}') \tag{52}$$

and, similarly as before, we also obtain

$$\sigma(y) + \widehat{\sigma}_A(y) = \sum_{x\in U} \frac{\sigma(\overrightarrow{yx})}{1+\gamma_B} + \frac{\gamma_A\widehat{\sigma}_A(\overrightarrow{xy})}{1+\gamma_A} = \tilde{\mu}'(y) \tag{53}$$

for all $y \in V$. From (51) we also have that

$$\sigma + \widehat{\sigma}_A \trianglelefteq \tilde{\mu}'. \tag{54}$$

Recall that $\gamma_B < 1$. Apply Lemma 8.1 with $V, U, \widehat{\mu} - \tilde{\mu}', \gamma_B$ playing the roles of U, V, μ, γ, respectively. We obtain a γ_B-skew-matching $\sigma' \trianglelefteq \widehat{\mu} - \tilde{\mu}'$ in $H^{\leftrightarrow}$ of weight

$$W(\sigma') = (1 + \gamma_B) W(\widehat{\mu} - \tilde{\mu}'), \tag{55}$$

with $\mathcal{A}(\sigma') \subseteq V$. In particular, this means that $\sigma'^2(y) = 0$ for all $y \in V$. Using $\gamma_B < 1$ and (55) we obtain, for all $y \in V$, that

$$\sigma'(y) = \widehat{\mu}(y) - \tilde{\mu}'(y). \tag{56}$$

Now we shall 'glue' the skew-matchings together and check we indeed have the required properties. Set $\sigma''_B := \sigma'_B + \sigma + \sigma'$. We now claim that σ_A, σ''_B are the required skew-matchings, i.e. they satisfy (L4)–(L9).

To see (L4), from (46) we have $\sigma'_A + \sigma'_B \trianglelefteq \mu'$, from (54) we have $\sigma + \widehat{\sigma}_A \trianglelefteq \tilde{\mu}'$, and from the definition of σ' we have $\sigma' \trianglelefteq \widehat{\mu} - \tilde{\mu}'$. Therefore, by (29), we have

$$\sigma_A + \sigma''_B = (\sigma'_A + \widehat{\sigma}_A) + (\sigma'_B + \sigma + \sigma') \trianglelefteq \mu' + \tilde{\mu}' + (\widehat{\mu} - \tilde{\mu}') = \mu$$

so (L4) holds. Item (L5) follows from (23).

To see (L6), we estimate

$$\begin{aligned}
W(\sigma''_B) &= W(\sigma'_B) + W(\sigma) + W(\sigma') \\
&\overset{(45)}{=} 2W(\mu') - W(\sigma'_A) + W(\sigma) + W(\sigma') \\
&\overset{(52)}{=} 2W(\mu') - W(\sigma'_A) + 2W(\tilde{\mu}') - W(\widehat{\sigma}_A) + W(\sigma') \\
&\overset{(55)}{=} 2W(\mu') - W(\sigma'_A) + 2W(\tilde{\mu}') - W(\widehat{\sigma}_A) + (1 + \gamma_B) W(\hat{\mu} - \tilde{\mu}') \\
&\overset{(31)}{\geq} 2W(\mu') + W(\widehat{\mu}) - W(\sigma_A) \\
&= \sum_{x \in U} \mu'(x) + \sum_{y \in V} (\mu'(y) + \widehat{\mu}(y)) - W(\sigma_A) \\
&\overset{(27)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu(x)\} + \sum_{y \in V} (\mu'(y) + \widehat{\mu}(y)) - W(\sigma_A) \\
&\overset{(29)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu(x)\} + \sum_{y \in V} \mu(y) - W(\sigma_A) \\
&\overset{(26)}{=} \sum_{x \in U} \min\{w(\overrightarrow{ux}), \mu(x)\} + \sum_{y \in V} \min\{w(\overrightarrow{uy}), \mu(y)\} - W(\sigma_A) \\
&= \deg_w(u, \mu) - W(\sigma_A),
\end{aligned}$$

which together with the trivial $W(\sigma''_B) \geq 0$ gives (L6). Property (L7) follows from (24).

Now we verify (L8). Our definitions of σ and σ' imply that $\mathcal{A}(\sigma), \mathcal{A}(\sigma') \subseteq V$. Hence, for $x \in U$, we have that $(\sigma''_B)^1(x) = (\sigma'_B)^1(x) \leq w(\overrightarrow{ux})$, where the inequality follows because $\mathcal{A}(\sigma'_B)$ fits in the w-neighbourhood of u. For $y \in V$, from (L4) and (26) we have $(\sigma''_B)^1(y) \leq \mu(y) \leq w(\overrightarrow{uy})$. Thus (L8) holds.

Finally, we verify (L9). For any $y \in V$, we have

$$\begin{aligned}
\sigma''_B(y) + \sigma_A(y) &= \sigma'_A(y) + \sigma'_B(y) + \widehat{\sigma}_A(y) + \sigma(y) + \sigma'(y) \\
&\overset{(49),(53),(56)}{=} \mu'(y) + \tilde{\mu}'(y) + \widehat{\mu}(y) - \tilde{\mu}'(y) = \mu'(y) + \widehat{\mu}(y) \overset{(29)}{=} \mu(y),
\end{aligned}$$

which implies (L9). This concludes the proof of Lemma 8.6. ■

8.6. **Proof of the Greedy Lemmas.** Now we shall prove the 'Greedy Lemmas', i.e. Lemmas 8.7–8.9.

Proof of Lemma 8.7. We shall define the required skew-matching σ'_A by a greedy iterative process. We will define a sequence $\sigma'_0, \ldots, \sigma'_t$ of γ_A-skew-matchings, where σ'_i will be obtained from σ'_{i-1} by increasing the value in exactly one edge. Then σ'_A will be the final skew-matching obtained at the end of this procedure.

The following claim ensures we can carry one iterative step of the process.

Claim 8.15. *Let σ' be a γ_A-skew-matching supported in $H[V,U]$ with $\mathcal{A}(\sigma') \subseteq V$. If $W(\sigma') < (1+\gamma_A)\kappa$, then there exists $x \in N_w(u) \cap V$ such that $w(\overrightarrow{ux}) > \sigma_A(x) + \sigma_B(x) + \sigma'(x)$, and $y \in N_w(x) \cap U$ that is not covered by $\sigma_A + \sigma_B + \sigma'_A$, i.e. that $\sigma_A(y) + \sigma_B(y) + \sigma'_A(y) < 1$.*

Proof of the claim. Suppose the desired x does not exist. Then, for all $x \in V$ we have $w(\overrightarrow{ux}) \leq \sigma_A(x) + \sigma_B(x) + \sigma'(x)$. Taking the sum over all $x \in V$ and using (M1) we get $\kappa \leq \sum_{x\in V} \sigma'(x) = W(\sigma')/(1+\gamma_A)$, a contradiction. Now suppose the desired y does not exist. Then for all $y \in N_w(x) \cap U$ we have $\sigma_A(y) + \sigma_B(y) + \sigma'(y) \geq 1$. Taking the sum over all $y \in N_w(x) \cap U$, we obtain $|N_w(x) \cap U| \leq \sum_{y \in N_w(x)\cap U}(\sigma_A(y) + \sigma_B(y) + \sigma'(y)) \leq \sum_{y\in U}(\sigma_A(y) + \sigma_B(y) + \sigma'(y))$. Together with (M2) we obtain $\kappa\gamma_A \leq \sum_{y\in U} \sigma'(y) = \gamma_A W(\sigma')/(1+\gamma_A)$, a contradiction. □

Initially, let σ'_0 be the identically zero γ_A-skew-matching. Next, given $i \geq 0$, suppose that we are given σ'_i, which is supported on $H[V,U]$, with $\mathcal{A}(\sigma'_i) \subseteq V$, which is disjoint from $\sigma_A + \sigma_B$, and such that $\sigma_A(x) + \sigma_B(x) + \sigma'_i(x) \leq w(\overrightarrow{ux})$ for all $x \in V$. We run the following algorithm:

(i) If $W(\sigma'_i) \geq (1+\gamma_A)\kappa$, then we set $\sigma'_A := \sigma'_i$ and we finalise the construction.
(ii) Otherwise, we have $W(\sigma'_i) < (1+\gamma_A)\kappa$. By the claim, there exists $x \in V$ and $y \in N_w(x) \cap U$ such that $w(\overrightarrow{ux}) > \sigma_A(x) + \sigma_B(x) + \sigma'_i(x)$ and $\sigma_A(y) + \sigma_B(y) + \sigma'_i(y) < 1$. We define σ'_{i+1} from σ'_i by increasing the weight maximally on $\overrightarrow{xy}$ such that σ'_{i+1} is still a γ_A-skew-matching disjoint from $\sigma_A + \sigma_B$, and such that $\sigma_A(x) + \sigma_B(x) + \sigma'_{i+1}(x) \leq w(\overrightarrow{ux})$.

Observe that this process strictly increases the weight of the current matching in each iteration, and further, it chooses each edge $\overrightarrow{xy}$, for $x \in V$ and $y \in U$, at most once. Hence the process must stop in at most $|U||V|$ steps.

The γ_A-skew-matching σ'_A, by construction, satisfies (M3)–(M5). It remains to verify (M6), for which we need to verify that (B1)–(B4) hold for $(\sigma_A+\sigma'_A, \sigma_B)$ and $\overrightarrow{uv}$. Property (B1) follows directly from our construction, since σ'_A is disjoint from $\sigma_A + \sigma_B$. Property (B3) is true by assumption, because (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{uv}$. To see (B2), we need to prove that $\sigma_A + \sigma'_A$ fits in the w-neighbourhood of u. Note that for every $x \in V(H)$ for which $\sigma'^1_A(x) > 0$, by construction we have

$$w(\overrightarrow{ux}) \geq \sigma'_A(x) + \sigma_A(x) + \sigma_B(x).$$

Together with the fact that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{uv}$, this gives (B2). Property (B4) also follows from this inequality and the fact that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{uv}$. This finishes the proof. ■

Proof of Lemma 8.8. We first shall reserve some space for the anchor of σ'_A. We do this by following a greedy process that defines a sequence of functions $A_0, A_1, \ldots$, each from $V(H)$ to $[0,1]$, and supported on V (i.e. we will have $A_i(x) = 0$ for all $x \notin V$). The next claim will guarantee that we can carry one step of the process.

Claim 8.16. *Let $A : V(H) \to [0,1]$ be a function supported on V. Suppose that $\sum_{x\in V} A(x) < \kappa$. Then there exists $x \in V$ such that $w(\overrightarrow{ux}) > \sigma_A(x) + \sigma_B(x) + A(x)$.*

Proof of the claim. Suppose otherwise. Then we have $w(\overrightarrow{ux}) \leq \sigma_A(x) + \sigma_B(x) + A(x)$ for all $x \in V$. Summing over all $x \in V$, we obtain $\deg_w(u,V) \leq \sum_{x\in V}(\sigma_A(x)+\sigma_B(x)) + \sum_{x\in V} A(x)$, which together with $\sum_{x\in V} A(x) < \kappa$ contradicts (N1). □

Let $A_0 : V(H) \to [0,1]$ be the identically-zero function. Next, given $i \geq 0$ and a function $A_i : V(H) \to [0,1]$ supported on V, we execute the following process:

(i) If $\sum_{x\in V} A_i(x) = \kappa$; then set $A := A_i$, and halt the process.

(ii) Otherwise, by the claim, there exists $x \in V$ such that $w(\overrightarrow{ux}) > \sigma_A(x) + \sigma_B(x) + A(x)$. Define $A_{i+1} : V(H) \to [0,1]$ from A_i by increasing the value maximally on x, subject to $\sigma_A(x) + \sigma_B(x) + A_{i+1}(x) \leq w(\overrightarrow{ux})$ and $\sum_{x\in V} A_{i+1}(x) \leq \kappa$.

Since each $x \in V$ can be chosen at most once during the process, we must reach $\sum_{x\in V} A_i(x) = \kappa$ after at most $|V|$ steps. Thus the process ends up defining $A : V(H) \to [0,1]$ supported on V, such that $\sum_{x\in V} A(x) = \kappa$.

Now we shall use A to define our desired skew-matching σ'_A. To do this, we will also use an iterative process, constructing skew-matchings $\sigma'_0, \sigma'_1, \dots$ and functions $A'_0, A'_1, \dots$, and we will maintain the invariants $\sum_{x\in V}(A'_i(x) + \sigma'^1_i(x)) = \kappa$ and $\mathcal{A}(\sigma'_i) \subseteq V$ during the process. Again, we state a claim ensuring we can carry with one iteration of the process.

Claim 8.17. *Suppose $A : V(H) \to [0,1]$ is supported on V and such that $\sum_{z\in V} A(z) > 0$. Suppose σ' is a γ_A-skew-matching such that $\sum_{z\in V}(A(z)+\sigma'^1(z)) = \kappa$ and $\mathcal{A}(\sigma') \subseteq V$. Then there exists $x \in V$ and $y \in N_w(x) \cap (U \cup V)$ with $A(x) > 0$ and $\sigma_A(y) + \sigma_B(y) + A(y) + \sigma'(y) < 1$.*

Proof of the claim. Select any $x \in V$ with $A(x) > 0$. Aiming at a contradiction, suppose for all $y \in N_w(x) \cap (U \cup V)$ we have $\sigma_A(y) + \sigma_B(y) + A(y) + \sigma'(y) \geq 1$. Taking the sum of $\sigma_A(y) + \sigma_B(y) + A(y) + \sigma'(y)$ over all $y \in N_w(x) \cap (U \cup V)$, we get

$$\sum_{y\in N_w(x)\cap(U\cup V)} (\sigma_A(y) + \sigma_B(y) + A(y) + \sigma'(y)) \geq |N_w(x) \cap (U \cup V)|. \tag{57}$$

On the other hand, from (N2) we get

$$\begin{aligned}
|N_w(x) \cap (U \cup V)| &\geq (1 + \gamma_A)\kappa + \sum_{y\in U\cup V} (\sigma_A(y) + \sigma_B(y)) \\
&= \gamma_A\kappa + \sum_{z\in V}(A(z) + \sigma'(z)) + \sum_{z\in U\cup V} (\sigma_A(z) + \sigma_B(z)) \\
&= \gamma_A\kappa + \sum_{z\in V} \sigma'^1(z) + \sum_{z\in U\cup V} (\sigma_A(z) + \sigma_B(z) + A(z)) \\
&> (1 + \gamma_A)\sum_{z\in V} \sigma'^1(z) + \sum_{z\in U\cup V} (\sigma_A(z) + \sigma_B(z) + A(z)),
\end{aligned}$$

where in the last step we used $\sum_{z\in V} \sigma'^1(z) < \kappa$. From $\mathcal{A}(\sigma') \subseteq V$, we get that

$$(1 + \gamma_A)\sum_{z\in V} \sigma'^1(z) = \sum_{z\in V}\sum_{x\in N(z)} \sigma'(\overrightarrow{xz}) = \sum_{z\in U\cup V} \sigma'(z).$$

Combining the last two inequalities we obtain the desired contradiction with (57), which proves the claim. □

The process is described as follows. Initially, let σ'_0 be the identically-zero skew-matching, and let $A'_0 := A$. This clearly satisfies $\sum_{x\in V}(A'_0(x)+\sigma'^1_0(x)) = \sum_{x\in V} A(x) = \kappa$ and $\mathcal{A}(\sigma'_0) \subseteq V$, as required.

Next, given $i \geq 0$ and A'_i, σ'_i with $\sum_{x\in V}(A'_i(x) + \sigma'^1_i(x)) = \kappa$ and $\mathcal{A}(\sigma'_i) \subseteq V$, we do the following.

(i) If $\sum_{x\in V} A'_i(x) = 0$, then we let $A' := A'_i$ and $\sigma'_A := \sigma'_i$, and halt the process.

(ii) Otherwise, by the previous claim there exists $x \in V$ and $y \in N_w(x) \cap (U \cup V)$ with $A_i(x) > 0$ and $\sigma_A(y) + \sigma_B(y) + A_i(y) + \sigma'_i(y) < 1$.

Let δ be maximum such that $\delta \leq A_i(x)$ and $\gamma_A\delta \leq 1-(\sigma_A(y)+\sigma_B(y)+A_i(y)+\sigma'_i(y))$. We define A_{i+1} from A_i by decreasing the value of $A_i(x)$ in δ, and we define σ'_{i+1} from σ'_i by adding $(1 + \gamma_A)\delta$ to the weight of $\overrightarrow{xy}$. This ensures that $\sigma'^1_i(x)$ increases by δ and $\sigma'^2_i(y)$ increases by $\gamma_A\delta$; so $\sum_{x\in V}(A'_{i+1}(x)+\sigma'^1_{i+1}(x)) = \kappa$ holds.

Note that each $\overrightarrow{xy}$ with $x \in V, y \in U \cup V$ is chosen at most once during the iterations, so the process finishes after at most $|V||V \cup U|$ steps. So the process halts, and at the end of the process we have obtained a γ_A-skew-matching σ'_A, with $\mathcal{A}(\sigma'_A) \subseteq V$ and $\sum_{x \in V} \sigma'^1_A(x) = \kappa$.

We verify the required properties (N3)–(N6). Properties (N3) and (N5) follow straight from the construction; and property (N4) follows from $\mathcal{A}(\sigma'_A) \subseteq V$ together with $\sum_{x \in V} \sigma'^1_A(x) = \kappa$. To see (N6), we need to verify that (B1)–(B4) hold for $(\sigma_A + \sigma'_A, \sigma_B)$ and $\overrightarrow{uv}$. The construction of σ'_A implies that $\sigma_A + \sigma'_A$ is a γ_A-skew-matching and for all $x \in V(H)$ with $\sigma'_A(x) > 0$, we have $\sigma^1_A(x) + \sigma_B(x) + \sigma'^1_A(x) \leq w(\overrightarrow{ux})$. This, together with the fact that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{uv}$, readily gives (B1)–(B4), and finishes the proof. ■

Proof of Lemma 8.9. The proof proceeds analogously to the proof of Lemma 8.7, with the difference that as long as $W(\sigma'_A) < (1 + \gamma_A)\kappa$, we first find a vertex $y \in U$ such that $\sigma_A(y) + \sigma_B(y) + \sigma'_A(y) < 1$ and a vertex $x \in N_w(y) \cap V$ with $w(\overrightarrow{ux}) > \sigma_A(x) + \sigma_B(x) + \sigma'_A(x)$. We increment σ'_A by putting a maximal weight on $\overrightarrow{xy}$, so that σ'_A stays a γ_A-skew-matching, σ'_A is disjoint from $\sigma_A + \sigma_B$ and $\sigma'_A(x) + \sigma_A(x) + \sigma_B(x) \leq w(\overrightarrow{ux})$.

Hence, $\mathcal{A}(\sigma'_A) \subseteq V$, the support edges of σ'_A run from V to U, the anchor of σ'_A fits in the w-neighbourhood of u, we have $W(\sigma'_A) = (1 + \gamma_A)\kappa$, and $(\sigma_A + \sigma'_A, \sigma_B)$ is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{uv}$. ■

8.7. **Proof of the $(k, k/2)$-Lemma.** Now we prove the $(k, k/2)$-Lemma, Lemma 8.10. We shall need the following auxiliary lemma. In some easy cases, the fractional matching can be partitioned into two disjoint matchings that will host each one of the skew-matchings from our desired skew-matching pair; the next lemma finds the required skew-matchings in this situation.

Lemma 8.18 (Filling disjoint matchings). *Let (H, w) be a weighted graph and $uv \in E(H)$. Let μ_1, μ_2, μ be fractional matchings satisfying $\mu_1 + \mu_2 \leq \mu$, and let $(H, \bar{w})$ be the μ_1-truncated weighted graph obtained from (H, w). Then, for any $\gamma_1, \gamma_2 > 0$, there is a (γ_1, γ_2)-skew-matching pair (σ_1, σ_2) in (H, w) anchored in $\overrightarrow{uv}$ with*

(R1) $W(\sigma_1) = \deg_w(u, \mu_1)$,

(R2) $W(\sigma_2) = \deg_{\bar{w}}(v, \mu_2)$, *and*

(R3) $\sigma_1 + \sigma_2 \trianglelefteq \mu$.

Proof. First, we construct σ_1. To do so, we use Lemma 8.3 (Combination) with

object	(H, w)	u	μ_1	γ_1
in place of	(H, w)	v	μ	γ

We obtain a γ_1-skew-matching σ_1 in $H^{\leftrightarrow}$ with $\sigma \trianglelefteq \mu_1$ and such that the anchor $\mathcal{A}(\sigma_1)$ fits in the w-neighbourhood of u. By decreasing the weight if necessary, we can also assume that $W(\sigma_1) = \deg_w(u, \mu_1)$.

Secondly, we build σ_2. We use Lemma 8.3 (Combination) a second time, now with

object	$(H, \bar{w})$	v	μ_2	γ_2
in place of	(H, w)	v	μ	γ

We obtain a γ_2-skew-matching σ_2 in $H^{\leftrightarrow}$ such that $\sigma_2 \trianglelefteq \mu_2$ and $\mathcal{A}(\sigma_2)$ fits in the $\bar{w}$-neighbourhood of v. Again, decreasing the weight if necessary we can assume that $W(\sigma_2) = \deg_{\bar{w}}(v, \mu_2)$.

Now we combine both skew-matchings to form a skew-matching pair. Let $\sigma_\emptyset$ be the empty skew-matching. Applying Proposition 6.14 (recall Remark 6.15) with

object	(H, w)	uv	μ_1	μ_2	$\bar{w}$	$(\sigma_1, \sigma_\emptyset)$	$(\sigma_\emptyset, \sigma_2)$	γ_1	γ_2
in place of	(G, w)	uv	μ	$\bar{\mu}$	$\bar{w}$	(σ_A, σ_B)	$(\bar{\sigma}_A, \bar{\sigma}_B)$	γ_A	γ_B

we obtain that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair in (H, w) anchored in $\overrightarrow{uv}$ with $\sigma_1 + \sigma_2 \trianglelefteq \mu$. By construction, it satisfies (R1)–(R3). ■

Proof of Lemma 8.10. Without loss of generality, we may assume that

$$\deg_w(c, \mu) = k \qquad \text{and} \qquad \deg_w(d, \mu) = k/2, \tag{58}$$

as we can decrease the weight function w so that it is satisfied. A second assumption we may do without loss of generality is

$$\alpha_1 + \alpha_2 \leq \frac{k}{2}, \tag{59}$$

as otherwise we can switch the roles of α_1, α_2 and β_1, β_2.

The proof consists of five phases. In the first phase, we partition the fractional matching into meaningful parts, where we will apply some of the matching lemmas from this section. You can refer to Figure 7 for illustration of how the fractional matchings are defined. In the second phase, we can construct the required matchings quickly in some 'easy' cases, mostly relying on applications of Lemma 8.18. After this is done, we get some extra assumptions about the weights of our auxiliary matchings. Now it will not be immediately clear whether the (γ_A, γ_B)-skew-matching pair will be anchored at $\overrightarrow{cd}$ or $\overrightarrow{dc}$; and this decision is made in the third phase. In the fourth phase, we construct partial skew-matchings inside the meaningful parts. Finally, in the fifth phase, we combine these partial skew-matchings to form the desired skew-matching pair.

Step 1: Partitioning the fractional matching into meaningful bits. Each part of the fractional matching is connected to vertices c and d in different ways. To handle them systematically, we divide μ into 'homogeneous bits' –groups with similar properties–, so that they can be treated in the same way. The fractional matching μ_d will represent the portion that is strongly connected to vertex d (from both sides) and can be efficiently packed using a skew-matching anchored at d. On the other hand, the fractional matching $\bar{\mu}_d$ will be connected to d from only one side and is further split into μ'_d, which will also be connected to vertex c, and μ^*, which will be exclusive or 'private' to d. Finally, μ_c will be the part of the fractional matching that is private to c, and it will include sections that are connected to c either from both sides or from only one side.

Let $\mu_d \leq \mu$ be a fractional matching of maximal weight such that $\mu_d(x) \leq w(\overrightarrow{dx})$ for all $x \in V(H)$. Note that this implies that if $\mu_d(xy) > 0$, then $x, y \in N_w(d)$. We have

$$\deg_w(d, \mu_d) = 2W(\mu_d) \geq \deg_w(c, \mu_d). \tag{60}$$

Next, let (H, w') be the μ_d-truncated weighted graph obtained from (H, w) (Definition 6.12). Set $U := N_{w'}(d)$ and $V := V(H) \setminus U$. We have

$$\mu_d(x) = w(\overrightarrow{dx}) \tag{61}$$

for all $x \in V$. Also observe that for all $x \in U$, from the definition of w' we have

$$w'(\overrightarrow{dx}) + \mu_d(x) = w(\overrightarrow{dx}). \tag{62}$$

Now, let $\bar{\mu}_d \leq \mu - \mu_d$ be a maximal fractional matching supported only on edges intersecting the set U, and such that

$$\bar{\mu}_d(x) \leq w'(\overrightarrow{dx}) \tag{63}$$

for all $x \in U$. We claim that $\bar{\mu}_d$ is supported in $H[U, V]$. Indeed, otherwise there is an edge xy, completely contained in U, with $\bar{\mu}_d(xy) > 0$. This implies that $w'(\overrightarrow{dx}) > 0$ and $w'(\overrightarrow{dy}) > 0$; and therefore $w(\overrightarrow{dx}) > \mu_d(x)$ and $w(\overrightarrow{dy}) > \mu_d(y)$; but this contradicts the maximality of μ_d. Now, the fact that $\bar{\mu}_d$ runs between U and V implies that

$$\deg_{w'}(d, \bar{\mu}_d) = W(\bar{\mu}_d). \tag{64}$$

It also gives

$$\bar{\mu}_d(x) + \mu_d(x) = \min\{w(\overrightarrow{dx}), \mu(x)\}$$

for all $x \in U$, and

$$w(\overrightarrow{dx}) \overset{(61)}{=} \mu_d(x) \leq \bar{\mu}_d(x) + \mu_d(x) \leq \mu(x)$$

for all $x \in V$. This yields

$$\deg_w(d, \mu_d + \bar{\mu}_d) = \deg_w(d, \mu) = \frac{k}{2}. \tag{65}$$

Moreover, observe that by construction we have $\mu_d(x) = \min\{\mu_d(x), w(\overrightarrow{dx})\}$ for all $x \in V(H)$. Hence, we have

$$\begin{aligned} 2W(\mu_d) + W(\bar{\mu}_d) &\overset{(60),(64)}{=} \deg_w(d, \mu_d) + \deg_{w'}(d, \bar{\mu}_d) \\ &= \sum_{x \in V(H)} \mu_d(x) + \sum_{x \in U} \min\{w'(\overrightarrow{dx}), \bar{\mu}_d(x)\} \\ &= \sum_{x \in V} \mu_d(x) + \sum_{x \in U} \Big(\mu_d(x) + \min\{w'(\overrightarrow{dx}), \bar{\mu}_d(x)\}\Big) \\ &= \sum_{x \in V} \mu_d(x) + \sum_{x \in U} \min\{w'(\overrightarrow{dx}) + \mu_d(x), \bar{\mu}_d(x) + \mu_d(x)\} \\ &\overset{(62)}{=} \sum_{x \in V} \mu_d(x) + \sum_{x \in U} \min\{w(\overrightarrow{dx}), \bar{\mu}_d(x) + \mu_d(x)\} \\ &\overset{(61)}{=} \sum_{x \in V} \min\{w(\overrightarrow{dx}), \mu_d(x) + \bar{\mu}_d(x)\} + \sum_{x \in U} \min\{w(\overrightarrow{dx}), \bar{\mu}_d(x) + \mu_d(x)\} \\ &= \deg_w(d, \mu_d + \bar{\mu}_d). \end{aligned} \tag{66}$$

Let $\mu'_d \leq \bar{\mu}_d$ be a fractional matching such that, for all $y \in V$,

$$\mu'_d(y) = \min\{w'(\overrightarrow{cy}), \bar{\mu}_d(y)\}. \tag{67}$$

Let $\mu^* := \bar{\mu}_d - \mu'_d$. Since $\mu^*, \mu'_d \leq \bar{\mu}_d$ and $\bar{\mu}_d$ is supported in $H[U, V]$, we have that μ'_d and μ^* run between U and V as well. Straight from the definition, we have

$$W(\mu^*) = W(\bar{\mu}_d) - W(\mu'_d). \tag{68}$$

Moreover, from $\bar{\mu}_d = \mu^* + \mu'_d$ we have

$$\deg_w(c, \mu_d + \bar{\mu}_d) \leq \deg_w(c, \mu_d + \mu'_d) + W(\mu^*). \tag{69}$$

We finalise the construction of objects by setting $\mu_c := \mu - (\mu_d + \bar{\mu}_d)$. Let $(H, \bar{w})$ be the $\bar{\mu}_d$-truncated graph obtained from (H, w'). Observe that $(H, \bar{w})$ also corresponds to the $(\mu_d + \bar{\mu}_d)$-truncated graph obtained from (H, w).

Observe that

$$\begin{aligned} \deg_{\bar{w}}(c, \mu_c) = \deg_w(c, \mu) - \deg_w(c, \mu_d + \bar{\mu}_d) &\overset{(58),(69)}{\geq} k - \deg_w(c, \mu_d + \mu'_d) - W(\mu^*) \\ &\overset{(58)}{\geq} 2\deg_w(d, \mu) - 2\Big(W(\mu_d) + W(\mu'_d)\Big) - W(\mu^*) \\ &= 2\Big(\deg_w(d, \mu_d) + \deg_{w'}(d, \bar{\mu}_d)\Big) - 2W(\mu_d) - 2W(\mu'_d) - W(\mu^*) \\ &\overset{(60),(64)}{=} 4W(\mu_d) + 2W(\bar{\mu}_d) - 2W(\mu_d) - 2W(\bar{\mu}_d) + W(\mu^*) \\ &= 2W(\mu_d) + W(\mu^*). \end{aligned} \tag{70}$$

Step 2: The simple cases. The most delicate part of the proof involves handling the fractional matching μ'_d, as its space might be shared by both skew-matchings: one anchored at d and the other at c. Before diving into this difficulty, let us first get rid of some easier cases, when the fractional matching μ'_d does not need to be shared by the skew-matchings. The main outcome after this step is that we will get some numerical information about the weights of the auxiliary matchings, encapsulated in inequalities (71) and (79).

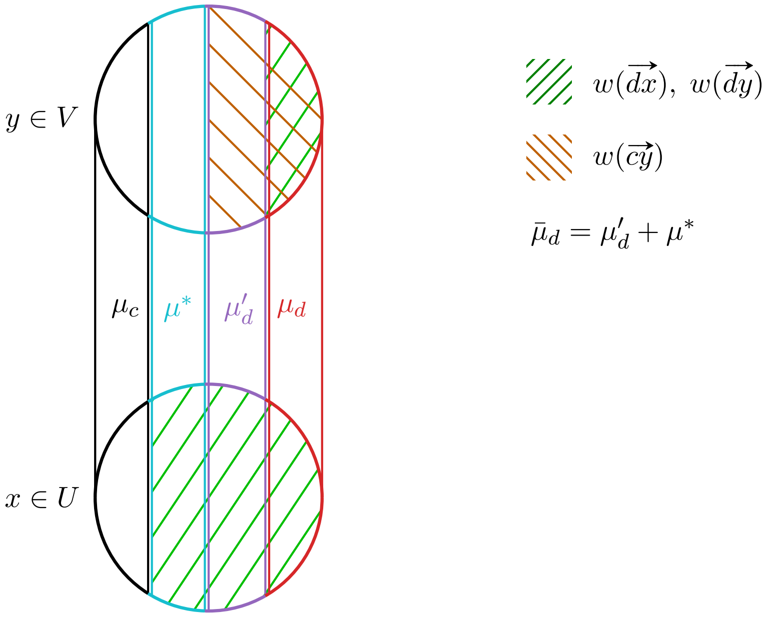


FIGURE 7. A representation of one edge xy and how it is partitioned in different fractional matchings. Depending on $w(\overrightarrow{dx}), w(\overrightarrow{dy}), w(\overrightarrow{cy})$ the fractional matchings $\mu_d, \mu'_d, \mu^*, \mu_c$ will look differently. Note that $w(\overrightarrow{cx})$ has not been represented, as it is irrelevant for the definition of the different fractional matchings. Also $\mu_d(y)$ may not necessarily be "covered" by $N_w(c)$ (i.e. $\mu_d(y) < w(\overrightarrow{cy})$). However, in this case $\mu'_d(xy) = 0$. Also note that $\mu_c(y)$ may be partially "covered" by $N_w(c)$. However, if this is the case, than $\mu^*(xy) = 0$.

The first case we consider is when $2W(\mu_d) + W(\mu^*) \geq \beta_1 + \beta_2$ holds. If this holds, then by (70) we obtain that $\deg_{\bar{w}}(c, \mu_c) \geq \beta_1 + \beta_2$. On the other hand, (65) and (59) together imply that $\deg_w(d, \mu_d + \bar{\mu}_d) \geq \alpha_1 + \alpha_2$. We recall that, by definition of μ_c, we have $\mu_c + \mu_d + \bar{\mu}_d \trianglelefteq \mu$; and also that $(H, \bar{w})$ is the $(\mu_d + \bar{\mu}_d)$-truncated graph obtained from (H, w). Thus we can apply Lemma 8.18 (Filling disjoint matchings) with

object	(H,w)	dc	$\mu_d + \bar{\mu}_d$	μ_c	μ	$\bar{w}$	γ_A	γ_B
in place of	(H,w)	uv	μ_1	μ_2	μ	$\bar{w}$	γ_1	γ_2

to obtain the required (γ_A, γ_B)-skew-matching pair (σ_A, σ_B) with $\sigma_A + \sigma_B \trianglelefteq \mu_c + \mu_d + \bar{\mu}_d = \mu$, and we are done in this case. So, from now on, we may assume that

$$2W(\mu_d) + W(\mu^*) < \beta_1 + \beta_2. \tag{71}$$

The second simple case we consider is when $2W(\mu_d) \geq \alpha_1 + \alpha_2$. If this holds, then we recall (60) and we use it to select $\mu''_d \leq \mu_d$ to be such that $\deg_w(d, \mu''_d) = 2W(\mu''_d) = \alpha_1 + \alpha_2$. Also, let $\mu''_c = \mu - \mu''_d$. Let (H, w'') be the μ''_d-truncated graph obtained from (H, w). By applying Proposition 6.16 with $(H, w), \mu''_d, \mu, w''$ playing the role of $(G, w), \mu', \mu, w'$, we see that

$$\deg_{w''}(c, \mu''_c) = \deg_w(c, \mu) - \deg_w(c, \mu''_d) \geq k - 2W(\mu''_d) = \beta_1 + \beta_2.$$

Hence, we can apply Lemma 8.18 with

object	(H,w)	dc	μ''_d	μ''_c	μ	w''	γ_A	γ_B
in place of	(H,w)	uv	μ_1	μ_2	μ	$\bar{w}$	γ_1	γ_2

to obtain the required (γ_A, γ_B)-skew-matching pair (σ_A, σ_B) with $\sigma_A + \sigma_B \trianglelefteq \mu''_c + \mu''_d = \mu$. Thus, from now on, we can assume that $2W(\mu_d) < \alpha_1 + \alpha_2$ holds.

Finally, the last simple case we will consider is when $\alpha_1 + \alpha_2 \leq 2W(\mu_d) + W(\mu^*)$. Since $2W(\mu_d) < \alpha_1 + \alpha_2$, we can select $\hat{\mu}_d \leq \mu^*$ to be such that

$$2W(\mu_d) + W(\hat{\mu}_d) = \alpha_1 + \alpha_2. \tag{72}$$

We recall that (H, w') is the μ_d-truncated weighted graph obtained from (H, w). Then, we can apply Lemma 8.18 with

object	(H,w)	dc	μ_d	μ'_d	$\mu_d + \mu'_d$	w'	γ_A	γ_B
in place of	(H,w)	uv	μ_1	μ_2	μ	$\bar{w}$	γ_1	γ_2

to obtain a (γ_A, γ_B)-skew-matching pair (σ'_A, σ'_B) that is anchored in $\overrightarrow{dc}$ with

$$W(\sigma'_A) = \deg_w(d, \mu_d) = 2W(\mu_d) \tag{73}$$

and

$$W(\sigma'_B) = \deg_{w'}(c, \mu'_d) \geq \deg_w(c, \mu_d + \mu'_d) - 2W(\mu_d) = \deg_w(c, \mu_d + \mu'_d) - W(\sigma'_A). \tag{74}$$

Moreover, we have

$$\sigma'_A + \sigma'_B \trianglelefteq \mu_d + \mu'_d. \tag{75}$$

Next, let $(H, \hat{w})$ be the μ'_d-truncated weighted graph obtained from (H, w'). We recall that $U = N_{w'}(d)$, and since $\hat{w} \leq w'$ we get $N_{\hat{w}}(d) \subseteq U$.

We claim that

$$V(\mu^*) \cap N_{\hat{w}}(c, V) = \emptyset. \tag{76}$$

Indeed, from (67) we see that for any $y \in V$ with $\mu^*(y) = \bar{\mu}_d(y) - \mu'_d(y) > 0$, we get $\hat{w}(\overrightarrow{cy}) \leq w'(\overrightarrow{cy}) - \mu'_d(y) = \mu'_d(y) - \mu'_d(y) = 0$.

We also recall that, by the definition of $\hat{\mu}_d$ and $\bar{\mu}_d$, we have $\hat{\mu}_d(x) \leq \bar{\mu}_d(x) - \mu'_d(x) \leq w'(\overrightarrow{dx}) - \mu'_d(x) \leq \hat{w}(\overrightarrow{dx})$ for all $x \in U$. Therefore, we have

$$\deg_{\hat{w}}(d, \hat{\mu}_d) = \sum_{x \in U} \min\{\hat{w}(\overrightarrow{dx}), \hat{\mu}_d(x)\} = \sum_{x \in U} \hat{\mu}_d(x) = W(\hat{\mu}_d), \tag{77}$$

where in the last equality we used that $\hat{\mu}_d \leq \bar{\mu}_d$, and that $\bar{\mu}_d$ is supported in $H[U, V]$.

We apply Lemma 8.3 (Combination) with

object	$(H,\hat{w})$	d	$\hat{\mu}_d$	γ_A
in place of	(H,w)	v	μ	γ

The outcome is an γ_A-skew-matching σ''_A in $(H, \hat{w})$, such that $\sigma''_A \trianglelefteq \hat{\mu}_d$, and with weight $W(\sigma''_A) \geq \deg_{\hat{w}}(d, \hat{\mu}_d)$. Using (77), (72) and (73), we obtain

$$W(\sigma''_A) \geq \deg_{\hat{w}}(d, \hat{\mu}_d) = W(\hat{\mu}_d) = \alpha_1 + \alpha_2 - 2W(\mu_d) = \alpha_1 + \alpha_2 - W(\sigma'_A), \tag{78}$$

so we can assume, decreasing the weight of σ''_A if necessary, that in fact $W(\sigma''_A) = \alpha_1 + \alpha_2 - W(\sigma'_A)$ holds.

Now we wish to apply Proposition 6.14 (recall Remark 6.15) with

object	(H,w)	$(H,\hat{w})$	dc	$\mu_d + \mu'_d$	$\hat{\mu}_d$	$\hat{w}$	(σ'_A, σ'_B)	$(\sigma''_A, \sigma_\emptyset)$	γ_A	γ_B
in place of	(G,w)	$(G,\bar{w})$	uv	μ	$\bar{\mu}$	$\bar{w}$	(σ_A, σ_B)	$(\bar{\sigma}_A, \bar{\sigma}_B)$	γ_A	γ_B

Let us quickly verify that our choice of objects satisfies the required conditions. We have that $\mu_d + \mu'_d$ and $\hat{\mu}_d$ are disjoint, because $\hat{\mu}_d \leq \mu^* = \bar{\mu}_d - \mu'_d$ and $\bar{\mu}_d \leq \mu - \mu_d$. We observe that the choice $\hat{w}$ (as the μ'_d-truncated weighted graph from (H, w')) also implies that $(H, \bar{w})$ is the $(\mu_d + \mu'_d)$-truncated weighted graph obtained from (H, w), as required. Given this, then (C1) follows from (75), and (C2) follows from the choice of σ''_A.

From the application of Proposition 6.14 we obtain a (γ_A, γ_B)-skew-matching pair (σ_A, σ'_B) in (H, w) anchored in $\overrightarrow{dc}$, where $\sigma_A := \sigma'_A + \sigma''_A$.

Now, we observe that by applying Proposition 6.16 twice (first with $(H, w), \mu'_d, \mu_d + \mu'_d + \hat{\mu}_d$, and w' playing the role of $(G, w), \mu', \mu,$, and w', and then with $(H, w'), \mu'_d, \mu'_d + \hat{\mu}_d$, and $\hat{w}$ playing the role of $(G, w), \mu', \mu$, and w') we obtain

$$\begin{aligned}
\deg_w(c, \mu_d + \mu'_d + \hat{\mu}_d) &= \deg_{w'}(c, \mu'_d + \hat{\mu}_d) + \deg_w(c, \mu_d) \\
&= \deg_{\hat{w}}(c, \hat{\mu}_d) + \deg_{w'}(c, \mu'_d) + \deg_w(c, \mu_d) \\
&\leq \deg_{\hat{w}}(c, \hat{\mu}_d) + \deg_{w'}(c, \mu'_d) + 2W(\mu_d),
\end{aligned}$$

where in the last inequality we used $\deg_w(c, \mu_d) \le 2W(\mu_d)$. Next, from (76) and $\hat{\mu}_d \le \mu^* \le \bar{\mu}_d \subseteq H[U, V]$, we get that for every edge $uv \in E(H)$ with $u \in U$, $v \in V$ and $\hat{\mu}_d(uv) > 0$, it must hold that $\hat{w}(\overrightarrow{cv}) = 0$. Hence, we obtain that $\deg_{\hat{w}}(c, \hat{\mu}_d) \le W(\hat{\mu}_d)$. Combining this with the last displayed inequality, we obtain

$$\deg_w(c, \mu_d + \mu'_d + \hat{\mu}_d) \le W(\hat{\mu}_d) + \deg_{w'}(c, \mu'_d) + 2W(\mu_d) = W(\sigma'_B) + \alpha_1 + \alpha_2,$$

where the last equality is now direct by using (78), (74) and (73).

Last, let (H, w'') be the $(\mu_d + \mu'_d + \hat{\mu}_d)$-truncated graph obtained from (H, w). Using our upper bound for $\deg_w(c, \mu_d + \mu'_d + \hat{\mu}_d)$, we see that

$$\deg_{w''}(c, \mu - \hat{\mu}_d - \mu'_d - \mu_d) = \deg_w(c, \mu) - \deg_w(c, \mu_d + \mu'_d + \hat{\mu}_d) \ge k - W(\sigma'_B) - \alpha_1 - \alpha_2.$$

We apply Lemma 8.3 (Combination) with

object	(H, w'')	c	$\mu - (\mu_d + \mu'_d + \bar{\mu}_d)$	γ_B
in place of	(H, w)	v	μ	γ

to obtain a γ_B-skew-matching σ''_B satisfying $\sigma''_B \trianglelefteq \mu - (\mu_d + \mu'_d + \bar{\mu}_d)$, and that has weight $W(\sigma''_B) = k - W(\sigma'_B) - \alpha_1 - \alpha_2 = \beta_1 + \beta_2 - W(\sigma'_B)$. Combined with Proposition 6.14 with

object	(H, w)	dc	$\mu_d + \mu'_d + \hat{\mu}_d$	$\mu - (\mu_d + \mu'_d + \hat{\mu}_d)$	w''	(σ_A, σ'_B)	$(\emptyset, \sigma'_B)$
in place of	(G, w)	uv	μ	$\bar{\mu}$	$\bar{w}$	(σ_A, σ_B)	$(\bar{\sigma}_A, \bar{\sigma}_B)$

we get the desired (γ_A, γ_B)-skew-matching pair (σ_A, σ_B), with $\sigma_B := \sigma'_B + \sigma''_B$ and $\sigma_A + \sigma_B \trianglelefteq \mu$.

Hence, from now on we may assume that

$$2W(\mu_d) + W(\mu^*) < \alpha_1 + \alpha_2. \tag{79}$$

Step 3: Deciding how to anchor the skew-matching pair. To manage this effectively, we will use the sophisticated Lemma 8.6 (Completion) in the next step. At this point, we determine which configuration satisfies the conditions, which, in turn, dictates which skew-matching should be anchored to each vertex. The fact that μ'_d governs how the skew-matching pair is anchored, while the other fractional matchings are irrelevant, follows from the property established below in (82), which states that we can fit the same amount of skew-matching in μ_c as in μ_d.

Let

$$\begin{aligned} a &:= \alpha_1 + \alpha_2 - 2W(\mu_d) - W(\mu^*), \\ b &:= \beta_1 + \beta_2 - 2W(\mu_d) - W(\mu^*). \end{aligned}$$

By (71) and (79), we obtain that $a > 0$ and $b > 0$. Set

$$a_1 := \frac{a}{1 + \gamma_A}, \qquad a_2 := \gamma_A a_1, \qquad b_1 := \frac{b}{1 + \gamma_B}, \qquad b_2 := \gamma_B b_1.$$

Clearly, $a_1, a_2, b_1, b_2 \ge 0$. We have

$$\begin{aligned} a_1 + a_2 + b_1 + b_2 &= \alpha_1 + \alpha_2 + \beta_1 + \beta_2 - 4W(\mu_d) - 2W(\mu^*) \\ &= k - 4W(\mu_d) - 2W(\mu^*) \\ &\overset{(58)}{=} 2\deg_w(d, \mu) - 4W(\mu_d) - 2W(\mu^*) \\ &\overset{(65),(66)}{=} 2W(\bar{\mu}_d) - 2W(\mu^*) \\ &\overset{(68)}{=} 2W(\mu'_d). \end{aligned} \tag{80}$$

Now, assume that

$$\max\{a_1, a_2\} + \min\{b_1, b_2\} \le W(\mu'_d). \tag{81}$$

Then, we shall find below a suitable (γ_A,γ_B)-skew-matching anchored in $\overrightarrow{dc}$. In the case (81) does not hold, we find a suitable (γ_A,γ_B)-skew-matching anchored in $\overrightarrow{cd}$ using the same argumentation as below, just interchanging $\alpha,\alpha_i,\gamma_A,a,a_i$ with $\beta,\beta_i,\gamma_B,b,b_i$.

Step 4: Building the skew-matchings. In this phase, we apply the relevant matching lemmas from this section, based on the specific properties of the fractional matchings we are completing.

By Lemma 8.2 (Balancing-out) (with $\mu_d,V(H)$ in place of μ,U) there is a γ_A-skew-matching $\tilde\sigma_A \trianglelefteq \mu_d$ in $H^{\leftrightarrow}$ of weight $W(\tilde\sigma_A)=2W(\mu_d)$. By the definition of μ_d the anchor of $\tilde\sigma_A$ fits in the w-neighbourhood of d.

Now, we wish to apply Lemma 8.6 (Completion) with

object	(H,w')	U	V	a_1	a_2	b_1	b_2	μ'_d	c
in place of	(H,w)	U	V	α_1	α_2	β_1	β_2	μ	u

To be able to do so, we need to verify the corresponding (L1)–(L3). Properties (L1) and (L2) follow from (67) and (81), respectively; and (L3) follows by combining (81) with (80). The application of the Completion Lemma gives a γ_A-skew-matching σ'_A and a γ_B-skew-matching σ'_B satisfying (L4)–(L8). In particular, we have $\sigma'_A+\sigma'_B \trianglelefteq \mu'_d$; we have $W(\sigma'_A)=a_1+a_2=a$ and $W(\sigma'_B)\ge \deg_{w'}(c,\mu'_d)-W(\sigma'_A)$. Moreover, we have $\mathcal{A}(\sigma'_A)\subseteq U$, and the anchor of σ'_B fits in the w'-neighbourhood of c.

We claim that (σ'_A,σ'_B) is a (γ_A,γ_B)-skew-matching pair anchored in $\overrightarrow{dc}$, with respect to w'. For this, we need to verify the required (B1)–(B4). Property (B1) follows from $\sigma'_A+\sigma'_B \trianglelefteq \mu'_d$; and we already verified (B3). Using (63), we see that for every $x\in U$, $\sigma'^1_A(x)+\sigma'^1_B(x)\le \mu'_d(x)\le \bar\mu_d(x)\le w'(\overrightarrow{dx})$. Since $\mathcal{A}(\sigma'_A)\subseteq U$ and $N_{w'}(d)=U$, this gives both (B2) and (B4).

Since (H,w') is the μ_d-truncated weighted graph obtained from (H,w), and $\tilde\sigma_A \trianglelefteq \mu_d$, we can apply Proposition 6.14 (with $(H,w),dc\mu_d,\mu'_d,w',(\tilde\sigma_A,\sigma_\emptyset)$, and (σ'_A,σ'_B) playing the role of $(G,w),\mu,\bar\mu,\bar w,(\sigma_A,\sigma_B)$, and $(\bar\sigma_A,\bar\sigma_B)$) to see that the pair $(\tilde\sigma_A+\sigma'_A,\sigma'_B)$ is a (γ_A,γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$ with $\tilde\sigma_A+\sigma'_A+\sigma'_B \trianglelefteq \mu_d+\mu'_d$.

Let (H,w^*) be the $(\mu_d+\mu'_d)$-truncated graph obtained from (H,w). We wish to apply Lemma 8.1 (Extending-out Lemma) with U,V,H,μ^* playing the roles of U,V,H,μ; we can do this because μ^* is supported in $H[U,V]$. From the lemma, we obtain a γ_A-skew-matching σ^*_A such that $\sigma^*_A \trianglelefteq \mu^*$ and its anchor is contained in U. We can also assume that

$$W(\sigma^*_A)=W(\mu^*)$$

(the lemma gives a skew-matching with larger weight, which we can scale down, and this scaling down does not break the property $\sigma^*_A \trianglelefteq \mu^*$).

Now we claim that the anchor of σ^*_A fits in the w^*-neighbourhood of d. Indeed, since $\mathcal{A}(\sigma^*_A)\subseteq A$ and $\sigma^*_A \trianglelefteq \mu^*$, it suffices to verify that $\mu^*(x)\le w^*(x)$ holds for $x\in U$. Since $\mu^*=\bar\mu_d-\mu'_d$, from (67) and (62) we obtain $\mu^*(x)\le w(\overrightarrow{dx})-(\mu_d(x)+\mu'_d(x))$, from which the desired inequality follows by the definition of w^*. This allows us to apply Proposition 6.14 (with $(H,w),dc,\mu_d+\mu'_d,\mu^*,w^*,(\tilde\sigma_A+\sigma'_A,\sigma'_B)$, and $(\sigma^*_A,\sigma_\emptyset)$ playing the role of $(G,w),\mu,\bar\mu,\bar w,(\sigma_A,\sigma_B)$, and $(\bar\sigma_A,\bar\sigma_B)$) again, to obtain that the pair $(\tilde\sigma_A+\sigma'_A+\sigma^*_A,\sigma'_B)$ is a (γ_A,γ_B)-skew-matching pair in (H,w) anchored in $\overrightarrow{dc}$ with $\tilde\sigma_A+\sigma'_A+\sigma^*_A+\sigma'_B \trianglelefteq \mu_d+\bar\mu_d$.

Now, let $(H,\bar w)$ be the $(\mu_d+\bar\mu_d)$-truncated weighted graph obtained from (H,w). Recall that $\mu_c=\mu-(\mu_d+\bar\mu_d)$. We have the following:

$$\begin{aligned}\deg_{\bar w}(c,\mu_c) &= \sum_{x\in V(H)} \min\{\bar w(\overrightarrow{cx}),\mu_c(x)\}\\ &= \sum_{x\in V(H)} \max\{0,\min\{w(\overrightarrow{cx}),\mu(x)\}-(\mu_d(x)+\bar\mu_d(x))\}\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{x\in V(H)} \min\{w(\overrightarrow{cx}),\mu(x)\} - 2W(\mu_d+\bar{\mu}_d)\\
&= \deg_w(c,\mu) - 2W(\mu_d+\bar{\mu}_d) \overset{(58)}{=} 2\deg_w(d,\mu) - 2W(\mu_d+\bar{\mu}_d)\\
&\overset{(66),(65)}{=} 2W(\mu_d) \overset{(60)}{=} \deg_w(d,\mu_d). \qquad (82)
\end{aligned}$$

By Lemma 8.3 (Combination) (with $H,\bar{w},\mu_c,c$ playing the roles of H,w,μ,v) there exists a γ_B-skew-matching $\tilde{\sigma}_B \trianglelefteq \mu_c$ in $\bar{H}^{\leftrightarrow}$ with weight $W(\tilde{\sigma}_B) \geq \deg_{\bar{w}}(c,\mu_c)$, and its anchor $\mathcal{A}(\tilde{\sigma}_B)$ fits in the $\bar{w}$-neighbourhood of c.

Step 5: Gluing skew-matchings together. The final step is to combine all the skew-matchings and verify that the resulting skew-matching pair satisfies the required properties.

We check a quick inequality before proceeding. Note that, since w' is the μ_d-truncated weight obtained from w, we have $w'(\overrightarrow{cx}) \geq w(\overrightarrow{cx}) - \mu_d(x)$ for each $x \in V(H)$, and therefore

$$\deg_{w'}(c,\mu'_d) + 2W(\mu_d) \geq \sum_{x\in V(H)} \big(\min\{w'(\overrightarrow{cx}),\mu'_d(x)\} + \mu_d(x)\big) \geq \deg_w(c,\mu_d+\mu'_d) \quad (83)$$

By Proposition 6.14 (with $(H,w), dc, \mu_d+\bar{\mu}_d, \mu_c, \bar{w}, (\tilde{\sigma}_A+\sigma'_A+\sigma^*_A, \sigma'_B)$, and $(\sigma_\emptyset,\tilde{\sigma}_B)$ playing the role of $(G,w),\mu,\bar{\mu},\bar{w},(\sigma_A,\sigma_B)$, and $(\bar{\sigma}_A,\bar{\sigma}_B)$) the pair $(\tilde{\sigma}_A+\sigma'_A+\sigma^*_A, \tilde{\sigma}_B+\sigma'_B)$ is a (γ_A,γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$ with $\tilde{\sigma}_A+\sigma'_A+\sigma^*_A+\tilde{\sigma}_B+\sigma'_B \trianglelefteq \mu$. Set $\sigma_A := \tilde{\sigma}_A+\sigma'_A+\sigma^*_A$ and $\sigma_B := \tilde{\sigma}_B+\sigma'_B$. Then $\sigma_A+\sigma_B \trianglelefteq \mu$. We have that

$$W(\sigma_A) = W(\tilde{\sigma}_A) + W(\sigma'_A) + W(\sigma^*_A) = 2W(\mu_d) + W(\mu^*) + a = \alpha_1+\alpha_2,$$

and

$$\begin{aligned}
W(\sigma_B) &= W(\tilde{\sigma}_B) + W(\sigma'_B) \geq \deg_{\bar{w}}(c,\mu_c) + \deg_{w'}(c,\mu'_d) - W(\sigma'_A)\\
&= \deg_{\bar{w}}(c,\mu_c) + \deg_{w'}(c,\mu'_d) - (\alpha_1+\alpha_2-2W(\mu_d)-W(\mu^*))\\
&\overset{(83)}{\geq} \deg_{\bar{w}}(c,\mu_c) + \deg_w(c,\mu_d+\mu'_d) + W(\mu^*) - (\alpha_1+\alpha_2)\\
&\overset{(69)}{\geq} \deg_{\bar{w}}(c,\mu_c) + \deg_w(c,\mu_d+\bar{\mu}_d) - (\alpha_1+\alpha_2)\\
&\overset{\text{Prop. 6.16}}{\geq} \deg_w(c,\mu) - (\alpha_1+\alpha_2) \overset{(58)}{\geq} k - (\alpha_1+\alpha_2) = \beta_1+\beta_2.
\end{aligned}$$

If $W(\sigma_B) > \beta_1+\beta_2$, we can scale it down. This finishes the proof of Lemma 8.10. ∎

9. The Structural Proposition

The goal of this section is to prove the Structural Proposition (Proposition 4.2), which we restate here for convenience.

Proposition 4.2 (Structural Proposition). *Let $a_1,a_2,b_1,b_2 \in \mathbb{N}$ be such that $a_1+a_2+b_1+b_2 = k$ and let (H,w) be a weighted graph with $w : E(H) \to (0,1]$ such that $\delta_w(H) \geq \frac{k}{2}$ and $\Delta_w(H) \geq k$. Let $\gamma_A := \frac{a_2}{a_1}$ and $\gamma_B := \frac{b_2}{b_1}$. Then H admits a (γ_A,γ_B)-skew-matching pair (σ_A,σ_B) with weights $W(\sigma_A) = a_1+a_2$ and $W(\sigma_B) = b_1+b_2$.*

The proof is split across this section and spans several subsections. We begin by giving a sketch of the proof, highlighting the various cases that will arise during our analysis. Then we proceed with the main proof in the remaining subsections.

9.1. **Sketch of the proof.** We aim to find appropriate neighbouring vertices c,d in $V(H)$ and to construct a sufficiently large skew-matching pair anchored at those two vertices. In the course of the proof, we will choose c to have total degree at least k, and vertex d to have total degree at least $k/2$. The proof splits into several cases, considering different configurations of the graph H, as well as the different possible structures of the pair (σ_A,σ_B) of skew-matchings.

(i) *The fractional matching cover case.* In the first two claims (Claims 9.1 and 9.2), we consider the situation where a fractional matching obtained from a Gallai-Edmonds triple is sufficient to build the entire skew-matching pair. This arises when the fractional matching covers well the neighbourhood of vertex c.

The case is divided into two claims, depending on the position of c. Claim 9.1 considers the configuration where c lies in the separator S. Then finding a suitable vertex d, whose neighbourhood is also completely covered by the fractional matching, is easy. The skew-matching pair is obtained by applying the $(k, k/2)$-Lemma (Lemma 8.10).

On the other hand, Claim 9.2 considers the configuration where c does not lie in the separator S. Then, having chosen a neighbouring vertex d, it may be necessary to slightly alter the given fractional matching to ensure it covers well its neighbourhood, while still keeping a good coverage of the neighbourhood of c. After building a suitable fractional matching, we again apply the $(k, k/2)$-Lemma (Lemma 8.10). Assuming that we are not in the fractional matching case provides important structural information about the graph, in particular on the existence of the set $\mathcal{R}$ and $S_{\mathcal{R}}$ from Definition 7.12.

(ii) *The easy skew case.* In Claim 9.3, we consider the case where the skew-matching structure has favorable parameters, making it easy to build. Then most of the skew-matching pair is built using the fractional matching from the GE triple, and the left-over can be handled with a greedy argument. This allows us to make some basic assumptions on the structure of the skew-matching pair we aim to embed for the rest of the proof.

(iii) *The skew-matching cover case.* In the next two claims (Claim 9.4 and Claim 9.5) we aim to replace part of the (unsatisfactory) fractional matching by "blowing it" into a γ_B-skew-matching in order to cover the neighbourhood of c as much as possible. For this purpose, we will use a GE pair (Definition 7.17). Here, Claim 9.4 will take care of the case where the "blowed part" can actually accommodate the whole skew-matching σ_B. The skew-matching σ_A will be built within the remaining fractional matching. Secondly, Claim 9.5 covers the complementary situation where the obtained skew-matching is perhaps not large enough to accommodate the whole one part of the pair, but similarly as in the *fractional matching case*, it covers well the neighbourhood of c.

After this step, we will have found an optimal GE pair, which in particular allow us to use the Separating Lemmas (Lemma 7.20 and Lemma 7.21), to obtain two important "Separating Claims" in our situation (Claim 9.6 and Claim 9.7).

(iv) *The balanced case.* Harnessing the assumptions we made on the configuration of the graph H, in Claim 9.8 we manage to build the required pair of skew-matchings, under the condition that it is reasonably balanced, i.e., none of the four parts a_1, a_2, b_1, b_2 exceeds half of the total weight $a_1 + a_2 + b_1 + b_2$. This allows us to assume from now on that the pair of skew-matchings has a huge b_2.

(v) *The large $S_{\mathcal{R}}$-case.* This auxiliary case allows us to extract further assumptions on the configuration of the graph H. In Claim 9.9 we assume that the neighbourhoods of two elements of $\mathcal{R}$ do not intersect much, leading to the existence of a large $S_{\mathcal{R}}$. The large size of $S_{\mathcal{R}}$ allows us to build the required pair of skew-matching. This enables us to make a crucial assumption for the following case, i.e., that there is a 'flabellum structure' emanating from $S_{\mathcal{R}}$ and spreading to $\mathcal{R}$.

(vi) *The flabellum case.* In Claim 9.10, we assume the graph H contains a large 'flabellum structure'. Intuitively, a flabellum should have two parts: a smaller one, called the base, from which the structure 'expands' to the second (larger) part. In our graph H, this is represented by a bipartite graph: the smaller colour class fits in $S_{\mathcal{R}}$ and forms the base of the flabellum; the larger colour class contains $\mathcal{R}$ and has the property that the neighbourhood of each of its vertices intersects the base substantially. We shall use the flabellum to host the (very large) σ_B and then build σ_A somehow in the leftover of the graph.

(vii) *The avoiding case.* Finally, we treat the last possible configuration of the graph H. We assume that the 'flabellum structure' emanating from $S_{\mathcal{R}}$ is too small to accommodate the whole σ_B. We proceed as follows: We build a large part of σ_B within the fractional matching and skew-matching covering the base of the flabellum structure. Then we build the rest of σ_B and the whole σ_A working greedily from the part of the neighbourhood of c that is not covered by the flabellum structure. We manage to do this by using the fact that we start from something that is not in the flabellum structure and thus we can avoid its base. We also heavily exploit the structural information obtained in the "Separating Claims" (Claim 9.6 and Claim 9.7) to avoid the whole fractional matching and skew-matching covering the base of the flabellum.

Now we begin our proof of Proposition 4.2, which will take up the rest of this section.

Proof of Proposition 4.2. For brevity, say that a *good matching* is a (γ_A, γ_B)-skew-matching pair (σ_A, σ_B) anchored in some edge $\overrightarrow{xy} \in E(H^{\leftrightarrow})$ with $w(xy) > 0$ such that $W(\sigma_A) = a_1 + a_2$ and $W(\sigma_B) = b_1 + b_2$. Our goal is to show that $H^{\leftrightarrow}$ has a good matching.

Note that if $uv \in E(H)$ is such that $w(uv) = 0$, then we can remove uv from H and this does not affect any of our assumptions. Thus we can assume that $w(uv) > 0$ for each $uv \in E(H)$. In particular, for every $v \in V(H)$ we have $N(v) = N_H(v) = N_w(v)$. We will use this during the whole proof. Let $(H^{\leftrightarrow}, w)$ is a weighted directed graph associated with (H, w), with its weight function inherited from (H, w), i.e., $w(\overrightarrow{uv}) = w(\overrightarrow{vu}) = w(uv)$.

9.2. **Proof of Proposition 4.2: The fractional matching cover case.** We begin by applying the Gallai–Edmonds theorem (Theorem 7.1) on H, which provides us with a Gallai–Edmonds triple (H, S, M_S). Recall that $\mathcal{K}_S^*$ is the set of vertices in non-singleton components in $H - S$, and U_S is the set of vertices which correspond to singleton components in $H - S$.

By assumption, there exists a vertex c with $\deg_w(c) \geq k$. Depending on the location of c and its neighbours within the set S, there are two situations where we can quickly find good matchings in (H, w), as the next two claims show. Recall the meaning of a fractional Gallai–Edmonds triple (Definition 7.4).

Claim 9.1. *Suppose there exists $c \in S$ and a fractional Gallai–Edmonds triple (H, S, μ) such that $\deg_w(c, \mu) \geq k$. Then $H^{\leftrightarrow}$ has a good matching.*

Proof of Claim 9.1. Let $d \in N(c) \setminus S$ (this exists, because in the Gallai–Edmonds triple, M_S matches c with some vertex not in S). Let K be the component of $H - S$ that contains d, so that $N_H(d) \subseteq S \cup (V(K) \setminus \{d\})$. The definition of the fractional Gallai–Edmonds triple implies that μ covers $N_H(d)$, so we have $\deg_w(d, \mu) = \deg_w(d) \geq \delta_w(H) \geq k/2$. We obtain the required good matching immediately from an application of Lemma 8.10 (the $(k, k/2)$-Lemma). $\square$

Claim 9.2. *Suppose there exists $c \in V(H) \setminus S$ with $\deg_w(c) \geq k$. Then $H^{\leftrightarrow}$ has a good matching.*

Proof of Claim 9.2. Suppose first that there exists $d \in N(c) \setminus S$. In this case, both c and d are not in S. Let μ be any fractional matching such that (H, S, μ) is a fractional Gallai–Edmonds triple (at least one must exist, e.g., by applying Proposition 7.9 with an arbitrary $c' \in S$ in place of c). By definition, μ covers all of S and each non-singleton component of $H - S$, and therefore it must cover $N(c) \cup N(d)$. Hence, we have $\deg_w(c, \mu) \geq \deg_w(c) \geq k$ and $\deg_w(d, \mu) \geq \deg_w(d) \geq \delta_w(H) \geq \frac{k}{2}$. Then, an application of the $(k, k/2)$-Lemma (Lemma 8.10) yields the existence of the required good matching. From now on, we let $M = M_S$ for simplicity of notation.

Hence, we may assume from now on that $N(c) \subseteq S$ holds. Pick $d \in N_H(c) \subseteq S$ arbitrarily. Our objective now is to construct a fractional matching μ_d that allows us to apply the $(k, k/2)$-Lemma once more with input edge cd. The construction of μ_d will need several steps, the first of which is to define an auxiliary weighted graph (H, w') where w' is obtained from w by decreasing the weight in some of its edges, as follows. Recall that, by assumption, $\deg_w(d) \geq \frac{k}{2}$. We obtain w' from w by decreasing its value on the edges incident to d so that $\deg_{w'}(d) = \frac{k}{2}$, and such that $N_w(d) \cap U_S = N_{w'}(d) \cap U_S$. The only purpose of w' is to define a suitable μ_d as an input of the $(k, k/2)$-Lemma.

Our second step to find μ_d is to construct an initial fractional matching μ_1 with the help of the First Greedy Lemma (Lemma 8.7), as follows. Let $V = U_S \cap N(d)$, i.e. it consists of all vertices forming singleton components in $H - S$ that are incident to d.

Let $\sigma_\emptyset : E(H^{\leftrightarrow}) \to [0, 1]$ be the function which is identically zero everywhere, and note that it is (trivially) a 1-skew oriented fractional matching. Also, define $\kappa = \deg_{w'}(d, V)$. We wish to apply Lemma 8.7 with

object	(H, w')	(d, c)	S	V	$\sigma_\emptyset$	$\sigma_\emptyset$	1	1	κ
in place of	(H, w)	(u, v)	U	V	σ_A	σ_B	γ_A	γ_B	κ

We can indeed do so: condition (M1) reduces to $\deg_{w'}(d, V) \geq \kappa$, which trivially holds. To check (M2), we note that for any $x \in V$, by our choice of w', we have $N_{w'}(x) = N_w(x) = N_H(x)$. Since $N_H(x) \subseteq S$, we have

$$\begin{aligned}|N_{w'}(x) \cap S| = |N_H(x)| \geq \deg_w(x) \geq \delta_w(H) \geq \frac{k}{2}\\ = \deg_{w'}(d) \geq \deg_{w'}(d, V) = \kappa,\end{aligned}$$

as required.

Hence, from Lemma 8.7 we obtain a 1-skew oriented fractional matching σ of weight $W(\sigma) \geq 2\kappa$, whose support is contained in $H[V, S]$, and such that $(\sigma, \sigma_\emptyset)$ is anchored in $\overrightarrow{dc}$ (with respect to w'), and moreover the anchor of σ is contained in V. Then we have $\kappa = \sum_{u \in V} w'(\overrightarrow{du}) \geq \sum_{u \in V} \sigma^1(u) = \frac{1}{2} W(\sigma) = \kappa$. This means that σ saturates $N_{w'}(d) \cap V$. Forgetting the orientation (formally, by Lemma 6.10) we obtain from σ a fractional matching μ_1 in H of weight $\deg_{w'}(d, V)$ that saturates $N_{w'}(d) \cap V$.

Our next step is to obtain a new fractional matching μ_2 to cover S. Recall that $M_S \subseteq H$ is a matching such that (H, S, M_S) is a Gallai–Edmonds triple. Let μ_2 be a fractional matching supported on M_S, disjoint from μ_1, and of maximum possible weight. We have that $\mu_1 + \mu_2$ fully covers S.

Next, we obtain a new fractional matching μ_3 to cover the non-singleton components in $H - S$. For any component K of $H - S$ that is not a singleton, there is at most one vertex $v \in V(K)$ for which $\mu_2(v) > 0$, and every other vertex in K receives zero weight under μ_2. As K is factor-critical, there is a perfect matching M_K in $K - v$. By Proposition 7.10 there is a fractional matching μ'_K that completely covers K. We shall build a fractional matching μ_K by

$$\mu_K := \mu_2(v) \cdot \mathbf{1}_{M_K} + (1 - \mu_2(v)) \cdot \mathbf{1}_{\mu'_K},$$

so μ_K is disjoint from $\mu_1 + \mu_2$ and supported completely in $E(K)$. Let $\mu_3 = \sum_K \mu_K$, where the sum ranges over all non-singleton components of $H - S$. Then μ_3 is also disjoint from $\mu_1 + \mu_2$.

We can finally define our desired matching as $\mu_d := \mu_1 + \mu_2 + \mu_3$. By construction, μ_d covers S and also covers all non-singleton components of $H - S$. Moreover μ_d saturates $N_{w'}(d) \cap V$ and thus $\deg_w(d, \mu_d) \geq \deg_{w'}(d, \mu_d) = \deg_{w'}(d) = k/2$. As μ_d covers S, we have $\deg_w(c, \mu_d) = \deg_w(c) \geq k$. We can now use the $(k, k/2)$-Lemma (Lemma 8.10), with

object	(H,w)	c	d	μ_d	k	a_1	a_2	b_1	b_2
in place of	(H,w)	c	d	μ	k	α_1	α_2	β_1	β_2

to obtain the required good matching and conclude the proof. □

Now, we fix important objects (vertices and fractional matchings) for the remainder of the proof and record key properties of those objects. Fix an arbitrary vertex c with $\deg_w(c) \geq k$. If $c \notin S$, we are done by Claim 9.2. Hence, we can assume that

(S1) $c \in S$.

Recall that U_S is the set of vertices that correspond to singleton components in $H - S$. Now we apply Proposition 7.9 to (H,w), (H,S,M) and c. We obtain a c-optimal fractional matching μ, which means the following: for each $u \in U_S$ such that $\mu(u) < w(\overrightarrow{cu})$, for each μ-alternating path P_u starting at u, and for each $v \in V(P_u) \cap S$, the set $N_\mu(v) = \{x \in N(v) : \mu(vx) > 0\}$ satisfies

(S2) $N_\mu(v) \subseteq U_S$;

(S3) $\mu(v) = 1$; and

(S4) for every $y \in N_\mu(v)$, $0 < \mu(y) \leq w(\overrightarrow{cy})$.

From Remark 7.8, we get

(S5) $N_\mu(v) \subseteq N(c)$.

Moreover, since (H,S,μ) is a fractional Gallai–Edmonds triple (Definition 7.4) we also have

(S6) μ is supported in the Gallai–Edmonds support $E_{S,M}$, and

(S7) $S \cup \bigcup_{K \in \mathcal{K}_S^*} V(K)$ is covered by μ. In particular, $S \cup \bigcup_{K \in \mathcal{K}_S^*} V(K) \subseteq V(\mu)$.

Let $\mathcal{R}$ be the set of reachable vertices (Definition 7.12) with respect to (H,S,μ), also let $S_{\mathcal{R}}$ as in that definition, i.e. the neighbourhood of the set of reachable vertices. By Observations 7.13 and 7.14, we have that

(S8) $\mathcal{R} \subseteq U_S \cap N_w(c)$, and

(S9) $S_{\mathcal{R}} \subseteq S$.

If $\deg_w(c,\mu) \geq k$, we would be done by Claim 9.1. Hence, we can assume that

(S10) $\deg_w(c,\mu) < k$.

Note that in particular, (S10) implies that μ does not saturate $N_w(c)$, that is, there exists $u \in N_w(c)$ such that $\mu(u) < w(cu)$. By (S7), we have that $u \in U_S$, i.e., $\{u\}$ is a singleton component of $H - S$. This implies that cu is an $(E(H),\mu)$-alternating path, so $u \in \mathcal{R}$. In summary,

(S11) $N_w(c) \cap \mathcal{R} \cap \{u : \mu(u) < w(\overrightarrow{cu})\} \neq \emptyset$.

Since $a_1 + a_2 + b_1 + b_2 = k$, without loss of generality we can suppose that

$$a_2 + b_1 \leq \frac{k}{2}, \tag{84}$$

as otherwise we can just swap the roles of a_1, b_1 with a_2, b_2. Set $\gamma_A := \frac{a_2}{a_1}$ and $\gamma_B := \frac{b_2}{b_1}$.

9.3. **Proof of Proposition 4.2: The easy skew case.** Now we can find a good matching if the skew of the tree satisfies a favourable condition.

Claim 9.3. *If $a_1 + b_1 \geq k/2$, then $H^{\leftrightarrow}$ has a good matching.*

Proof of Claim 9.3. The proof has three parts. First we will define auxiliary objects: a weighted graph (H, w_d) and a fractional matching μ_d, obtained from (H, w) and μ, respectively; and then gather properties of those objects. The second part of the proof is to use the auxiliary objects as an input for an application of the Completion Lemma, which gives a skew-matching as an output. In the third part we apply the Combination and Greedy matching lemmas to obtain two further skew-matchings. The combination of all the skew-matchings we have constructed yields the result.

Step 1: Setting the auxiliary objects. Let $d \in N(c) \cap \mathcal{R}$ be an arbitrary vertex with $\mu(d) < w(\overrightarrow{cd})$, which exists by (S11). Since $d \in \mathcal{R}$, we have in fact that $N(d) \subseteq S$.

We obtain a new weighted graph (H, w_d), where w_d is obtained from w by decreasing the weight function on edges incident to d (other than cd) in such a way that $\deg_{w_d}(d, \mu) = k/2$ holds. The goal of w_d is to define a fractional matching μ_d and filling it completely with some skew-matching σ'_A and σ_B.

Next, we define a new fractional matching $\mu_d \leq \mu$ using the following procedure. We recall that by (S6) we have that μ is not supported in any edge with two endpoints in S; and that $N_{w_d}(d) \subseteq N_w(d) \subseteq S$. Initially, we obtain μ_d from μ by setting its value on every edge which does not touch $N_{w_d}(d)$ to 0. At this point, we have that μ_d is supported only in edges with one endpoint in $N_{w_d}(d) \subseteq S$ and the other in $V(H) \setminus S$. Next, we process every $x \in N_{w_d}(d)$ in turn. If $\mu_d(x) \leq w_d(\overrightarrow{dx})$ we do nothing, otherwise we decrease the value of μ_d on the edges incident to x so that $\mu_d(x) = w_d(\overrightarrow{dx})$ holds. This finishes the construction of μ_d.

Note that by our construction, $\min\{\mu_d(x), w_d(\overrightarrow{dx})\} = \mu_d(x)$ if $x \in N_{w_d}(d)$, and $\mu_d(x) = 0$ for $x \in S \setminus N_{w_d}(d)$. Also, if $\mu_d(x) < \mu(x)$, then $\mu_d(x) = w_d(\overrightarrow{dx})$. This implies that

$$\min\{\mu(x), w_d(\overrightarrow{dx})\} = \min\{\mu_d(x), w_d(\overrightarrow{dx})\} = \mu_d(x) \text{ for every } x \in N_{w_d}(d). \tag{85}$$

Using all of this, we get

$$\begin{aligned} W(\mu_d) &= \sum_{xy \in E(H)} \mu_d(xy) = \sum_{x \in N_{w_d}(d)} \mu_d(x) = \sum_{x \in N_{w_d}(d)} \min\{\mu(x), w_d(\overrightarrow{dx})\} \\ &= \deg_{w_d}(d, \mu) = \frac{k}{2}. \end{aligned} \tag{86}$$

Before continuing, we gather some inequalities. From (84) and the assumption $a_1 + b_1 \geq k/2$, we deduce

$$\gamma_A = \frac{a_2}{a_1} \leq 1. \tag{87}$$

Since $a_1 + b_1 \geq k/2$, we get $a_2 + b_2 \leq k/2$; similarly from (84) we also have $a_1 + b_2 \geq k/2$. Using all of this and (86), we get

$$\max\{b_1, b_2\} + a_2 \leq \frac{k}{2} = \deg_{w_d}(d, \mu) = W(\mu_d) \leq \min\{b_1, b_2\} + a_1. \tag{88}$$

Now we want to use (S2)–(S5) with d playing the role of u, for which we verify the necessary conditions. Since $d \in \mathcal{R}$, we have $d \in U_S$ by (S8); and by the choice of d we also have $\mu(d) < w(\overrightarrow{cd}) = w_d(\overrightarrow{cd})$, so indeed d can play the role of u in that property. For any $x \in N_{w_d}(d)$, by Remark 7.7 we have that it can play the role of v in (S2)–(S5). We deduce that for any $x \in N_{w_d}(d)$ and any $y \in N_\mu(x)$, by (S4) we have

$$\mu_d(y) \leq \mu(y) \leq w(\overrightarrow{cy}) = w_d(\overrightarrow{cy}). \tag{89}$$

Step 2: Applying the Completion Lemma. Recall that $V(\mu_d)$ denotes the set of vertices v such that $\mu_d(v) > 0$. Now we want to apply Lemma 8.6 (Completion) with

object	(H, w_d)	$N_{w_d}(d)$	$V(\mu_d) \setminus S$	b_1	b_2	a_1	a_2	μ_d	c
in place of	(H, w)	U	V	α_1	α_2	β_1	β_2	μ	u

Let us verify the required conditions (L1)–(L3). Since $N_{w_d}(d) \subseteq S$, we indeed have that $N_{w_d}(d) \cap (V(\mu_d) \setminus S) = \emptyset$. By construction, μ_d is supported only in edges with one endpoint in $N_{w_d}(d)$ and the other in $V(\mu_d) \setminus S$, as required. If $y \in V(\mu_d) \setminus S$, there exists $x \in N_{w_d}(d)$ such that $y \in N_\mu(x)$, so from (89) we get that

$$\mu_d(y) \leq w_d(\overrightarrow{cy}) \text{ for all } y \in V(\mu_d) \setminus S, \tag{90}$$

which gives (L1). Inequality (88) implies (L2) and (L3). Finally, (87) also allows us to use part (L9).

Hence, $H^{\leftrightarrow}$ admits a γ_A-skew-matching σ'_A and a γ_B-skew-matching σ_B satisfying (L4)–(L9). In particular, we have that $\sigma'_A + \sigma_B \trianglelefteq \mu_d$ and $W(\sigma_B) = b_1 + b_2$. We also have that

$$W(\sigma'_A) \geq \deg_{w_d}(c, \mu_d) - W(\sigma_B). \tag{91}$$

Moreover, we have that the anchor $\mathcal{A}(\sigma_B)$ is contained in $N_{w_d}(d)$, the anchor $\mathcal{A}(\sigma'_A)$ fits in the w_d-neighbourhood of c, and

$$\text{for all } y \in V(\mu_d) \setminus S, \text{ we have } \sigma'_A(y) + \sigma_B(y) = \mu_d(y). \tag{92}$$

Note that for each $y \in V(\mu_d) \setminus S$, we have that there exists $x \in S \cap N_H(d)$ with $\mu(xy) \geq \mu_d(xy) > 0$. We have $d \in U_S$ and $\mu_d(d) \leq \mu(d) < w(\overrightarrow{cd})$. Hence dxy is a μ-alternating path and y is reachable, so $V(\mu_d) \setminus S \subseteq \mathcal{R}$. Therefore, together with (92) and (86), we have

$$\sum_{y \in \mathcal{R}} \left(\sigma'_A(y) + \sigma_B(y)\right) = \sum_{y \in V(\mu_d) \setminus S} \mu_d(y) = W(\mu_d) = \frac{k}{2}. \tag{93}$$

Using $\sigma'_A + \sigma_B \trianglelefteq \mu_d$ and the properties of σ'_A, σ_B, we have that (σ'_A, σ_B) is a (γ_A, γ_B)-skew pair in (H, w_d), anchored in $\overrightarrow{cd}$ (we use (85) to check (B3) and (B4)).

Step 3: Two more skew-matchings. Let (H, w') be the μ_d-truncated weighted graph obtained from (H, w_d). Let $\mu_c = \mu - \mu_d$.

By Lemma 8.3 (Combination) there is a γ_A-skew-matching $\sigma^*_A \trianglelefteq \mu_c$ with its anchor $\mathcal{A}(\sigma^*_A)$ fitting in the w'-neighbourhood of c and of weight

$$W(\sigma^*_A) \geq \deg_{w'}(c, \mu_c). \tag{94}$$

By Proposition 6.14 (with $(H, w_d), cd, \mu_d, \mu_c, w', \sigma_A, \sigma_B, \sigma^*_A$, and $\sigma_\emptyset$ playing the role of $(G, w), uv, \mu, \bar{\mu}, \bar{w}, \sigma_A, \sigma_B, \bar{\sigma}_A$, and $\bar{\sigma}_B$, respectively) we have that $(\sigma'_A + \sigma^*_A, \sigma_B)$ is a (γ_A, γ_B)-skew pair anchored in $\overrightarrow{cd}$ with $\sigma'_A + \sigma^*_A + \sigma_B \trianglelefteq \mu$. Moreover, we have (with explanations to follow)

$$\begin{aligned} W(\sigma^*_A + \sigma'_A + \sigma_B) \geq W(\sigma^*_A) + W(\sigma'_A) + W(\sigma_B) &\overset{(91),(94)}{\geq} \deg_{w'}(c, \mu_c) + \deg_{w_d}(c, \mu_d) \\ &= \deg_{w_d}(c, \mu) = \deg_w(c, \mu). \end{aligned}$$

Here, the last equality follows from the construction of w_d, from which we infer $w_d(\overrightarrow{cx}) = w(\overrightarrow{cx})$ for all $x \in V(H)$. The penultimate inequality follows from Proposition 6.16 (with $(H, w_d), \mu_d, \mu$, and w' playing the role of $(G, w), \mu', \mu$, and w', respectively). In summary, we obtain

$$W(\sigma^*_A + \sigma'_A + \sigma_B) \geq \deg_w(c, \mu). \tag{95}$$

Let $\kappa' = (k - W(\sigma_B + \sigma'_A + \sigma^*_A))(1 + \gamma_A)^{-1}$. We want to use the First Greedy Lemma (Lemma 8.7) with

object	(H, w)	$\overrightarrow{cd}$	$(\sigma'_A + \sigma^*_A, \sigma_B)$	$V(H) \setminus \mathcal{R}$	$\mathcal{R}$	κ'
in place of	(H, w)	$\overrightarrow{uv}$	(σ_A, σ_B)	U	V	κ

To verify we can do so, we check the required inequalities (M1) and (M2). We note first that, all support edges of μ_d intersect $N_{w_d}(d)$, by construction. Hence, if $y \in \mathcal{R}$

and $\mu_d(y) > 0$, by (89) we have $\mu(y) \leq w(\overrightarrow{cy})$. Using this, together with $\sigma_A^* \trianglelefteq \mu_c$ and $\sigma_A' + \sigma_B \trianglelefteq \mu_d$, we obtain

$$\begin{aligned}\sum_{x\in\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\} &= \sum_{x\in\mathcal{R}} \mu(x) = \sum_{x\in\mathcal{R}} \mu_c(x) + \sum_{x\in\mathcal{R}} \mu_d(x)\\ &\geq \sum_{x\in\mathcal{R}} \left(\sigma_A^*(x) + \sigma_A'(x) + \sigma_B(x)\right). \end{aligned} \tag{96}$$

Also note that if $y \in V(H)$ is such that $0 < \mu(y) < w(\overrightarrow{cy})$, then $y \notin S$ (by (S7)) and hence $y \in \mathcal{R}$ by definition. Hence,

$$\sum_{x\in V(H)\setminus\mathcal{R}} w(\overrightarrow{cx}) \leq \sum_{x\in V(\mu)\setminus\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\}. \tag{97}$$

Using this, we can verify (M1). Indeed,

$$\begin{aligned}\deg_w(c,\mathcal{R}) &\geq \deg_w(c) - \sum_{x\in V(H)\setminus\mathcal{R}} w(\overrightarrow{cx})\\ &\overset{(97)}{\geq} \deg_w(c) - \sum_{x\in V(\mu)\setminus\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\}\\ &\geq k - \sum_{x\in V(\mu)\setminus\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\}\\ &\geq k - \deg_w(c,\mu) + \sum_{x\in\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\}\\ &\overset{(95)}{\geq} k - W(\sigma_A' + \sigma_A^* + \sigma_B) + \sum_{x\in\mathcal{R}} \min\{w(\overrightarrow{cx}), \mu(x)\}\\ &\overset{(96)}{\geq} k - W(\sigma_A' + \sigma_A^* + \sigma_B) + \sum_{x\in\mathcal{R}} (\sigma_A'(x) + \sigma_A^*(x) + \sigma_B(x))\\ &\geq \kappa' + \sum_{x\in\mathcal{R}} (\sigma_A'(x) + \sigma_A^*(x) + \sigma_B(x)),\end{aligned}$$

Next, for any $x \in \mathcal{R}$, we have $N_w(x) \subseteq S \setminus \mathcal{R}$, so

$$\begin{aligned}|N_w(x) \cap (V(H)\setminus\mathcal{R})| \geq \deg_w(x) \geq \frac{k}{2} &\overset{(93)}{\geq} k - \sum_{x\in\mathcal{R}} (\sigma_B(x) + \sigma_A'(x) + \sigma_A^*(x))\\ &\geq k - W(\sigma_B + \sigma_A' + \sigma_A^*) + \sum_{y\in V(H)\setminus\mathcal{R}} (\sigma_B(y) + \sigma_A'(y) + \sigma_A^*(y))\\ &\geq (1+\gamma_A)\kappa' + \sum_{y\in V(H)\setminus\mathcal{R}} (\sigma_B(y) + \sigma_A'(y) + \sigma_A^*(y)),\end{aligned}$$

which is (M2), as required.

The outcome of Lemma 8.7 is a γ_A-skew-matching $\tilde{\sigma}_A$ of weight $W(\tilde{\sigma}_A) \geq (1+\gamma_A)\kappa' = k - W(\sigma_B + \sigma_A' + \sigma_A^*)$ such that $\sigma_A := \sigma_A' + \sigma_A^* + \tilde{\sigma}_A$ is such that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{cd}$. We have that $W(\sigma_B) = b_1 + b_2$ and $W(\sigma_A) \geq k - (b_1 + b_2) = a_1 + a_2$, so we have found the required good matching, proving the claim. □

In the rest of the proof we may assume that

$$\max\{a_1, a_2\} + b_1 < \frac{k}{2}. \tag{98}$$

This inequality, together with $a_1 + a_2 + b_1 + b_2 = k$, implies $b_2 + \min\{a_1, a_2\} > k/2 > b_1 + \max\{a_1, a_2\} \geq b_1 + \min\{a_1, a_2\}$, which in turn implies $b_2 > b_1$. Hence, we have

(S12) $\gamma_B = b_2/b_1 > 1$.

9.4. **Proof of Proposition 4.2: The skew-matching cover case.** In the remainder of the proof we will need the concept of a *GE pair* (Definition 7.17). Recall that we have already defined the fractional matching μ, which is c-optimal. We will let $(\tilde{\sigma}, \tilde{\mu})$ be a $(H, w, S, M, c, \gamma_B)$-GE pair, which means that

(S13) $\tilde{\sigma}$ is a γ_B-skew-matching,

(S14) $\tilde{\mu}$ is a fractional matching disjoint from $\tilde{\sigma}$,

(S15) the anchor $\mathcal{A}(\tilde{\sigma})$ of $\tilde{\sigma}$ is contained in $S_{\mathcal{R}}$,

(S16) $\mathcal{A}(\tilde{\sigma})$ fits in the w-neighbourhood of c,

(S17) $V(\tilde{\sigma}) \setminus \mathcal{A}(\tilde{\sigma}) \subseteq \mathcal{R}$,

(S18) for every $y \in \mathcal{R}$ we have $\tilde{\mu}(y) + \tilde{\sigma}(y) \leq w(\overrightarrow{cy})$,

(S19) for every $y \in V(H) \setminus \mathcal{R}$ we have $\tilde{\mu}(y) + \tilde{\sigma}(y) \geq w(\overrightarrow{cy})$,

(S20) $\tilde{\mu} + \tilde{\sigma}$ covers S, and

(S21) $\tilde{\mu}$ equals μ when restricted to the graph $H - (\mathcal{R} \cup S_{\mathcal{R}})$ and any supporting edge of $\tilde{\mu}$ intersecting the set $\mathcal{R} \cup S_{\mathcal{R}}$ lies in the bipartite graph $H[\mathcal{R}, S_{\mathcal{R}}]$.

Such objects exist by Remark 7.19. Over all possible $(H, w, S, M, c, \gamma_B)$-GE pairs $(\tilde{\sigma}, \tilde{\mu})$, we can assume we choose $(\tilde{\sigma}, \tilde{\mu})$ to be optimal, meaning that, over the possible choices,

(S22) $\tilde{\mu} + \tilde{\sigma}$ maximises the saturation $\deg_w(c, \tilde{\mu} + \tilde{\sigma})$ of $N(c)$.

As we shall see now, we can conclude if $\tilde{\sigma}$ has enough weight.

Claim 9.4. *Suppose that $W(\tilde{\sigma}) \geq b_1 + b_2$. Then $H^{\leftrightarrow}$ has a good matching.*

Proof of Claim 9.4. Let $d \in \mathcal{R}$ be arbitrary. By (S8) we have that $d \in N(c)$. Next, let σ_B be a γ_B-skew-matching obtained by scaling down $\tilde{\sigma}$, so that $\sigma_B \leq \tilde{\sigma}$ and $W(\sigma_B) = b_1 + b_2$. Let $\sigma'_B := \tilde{\sigma} - \sigma_B$, so σ'_B is a γ_B-skew-matching of weight $W(\tilde{\sigma}) - (b_1 + b_2)$.

We wish to apply Lemma 8.4 (Extending-out skew-matching) with

object	(H, w)	c	σ'_B	1	γ_B
in place of	(H, w)	u	σ_B	γ_A	γ_B

Indeed we can do so: we have $\gamma_B \geq 1$ by (S12); and as $\sigma'_B \leq \tilde{\sigma}$, by (S16), we have that the anchor $\mathcal{A}(\sigma'_B)$ fits in the w-neighbourhood of c. From this application we obtain a 1-skew-matching σ'_A of weight $2W(\sigma'_B)/(1 + \gamma_B)$, and such that $\sigma'_A \leq \sigma'_B$. Using Lemma 6.10, we obtain from σ'_A a fractional matching $\mu_{\tilde{\sigma}}$ such that

$$W(\mu_{\tilde{\sigma}}) = \frac{1}{2} W(\sigma'_A) = \frac{W(\sigma'_B)}{1 + \gamma_B} = \frac{W(\tilde{\sigma})}{1 + \gamma_B} - b_1, \tag{99}$$

and, for each $x \in V(H)$,

$$\mu_{\tilde{\sigma}}(x) = \sigma'_A(x), \tag{100}$$

and

$$\mu_{\tilde{\sigma}} \preceq \tilde{\sigma} - \sigma_B. \tag{101}$$

Note that, since $\sigma'_A \leq \sigma'_B \leq \tilde{\sigma}$, from (S15) we have that the anchor of σ'_A is in $S_{\mathcal{R}}$. Then (S17) implies that $W(\sigma'_A) = \sum_{x \in S} \sum_{y \in N(x)} \sigma'_A(xy)$, and the analogous equation for $W(\sigma'_B)$ also holds. Using $\sigma'_A \leq \sigma'_B$, we get that

$$W(\sigma'_B) = \sum_{x \in S} \sum_{y \in N(x)} \sigma'_B(xy) \geq \frac{1 + \gamma_B}{2} \sum_{x \in S} \sum_{y \in N(x)} \sigma'_A(xy) = \frac{1 + \gamma_B}{2} W(\sigma'_A) = W(\sigma'_B),$$

so $\sigma'_B(xy) = (1 + \gamma_B)\sigma'_A(xy)/2$ holds for every edge. This implies that, for each $x \in S$,

$$\sigma'_A(x) = \sigma'_B(x). \tag{102}$$

Similarly, the same argument we used above shows that for each $x \in S$ we have $\sigma_B(x) = \sigma^1_B(x)$. Moreover, since σ_B is γ_B-skew anchored in S, we have $(1+\gamma_B)\sum_{x\in S}\sigma^1_B(x) = W(\sigma_B)$. Hence (since $N(d) \subseteq S$), we have

$$\deg_w(d, \sigma_B) \leq \sum_{x\in S}\sigma^1_B(x) = \frac{1}{1+\gamma_B}W(\sigma_B) = b_1. \tag{103}$$

Now we claim that for each $x \in S$, we have

$$\mu_{\tilde{\sigma}}(x) + \tilde{\mu}(x) \geq w(\overrightarrow{dx}) - \sigma_B(x). \tag{104}$$

Indeed, let $x \in S$. We have

$$\begin{aligned}\mu_{\tilde{\sigma}}(x) + \tilde{\mu}(x) &\overset{(100)}{=} \sigma'_A(x) + \tilde{\mu}(x) \overset{(102)}{=} \sigma'_B(x) + \tilde{\mu}(x) = (\sigma'_B(x) + \tilde{\mu}(x) + \sigma_B(x)) - \sigma_B(x)\\ &= 1 - \sigma_B(x) \geq w(\overrightarrow{dx}) - \sigma_B(x),\end{aligned}$$

in the last line we used that $\tilde{\sigma} + \tilde{\mu} = \sigma'_B + \sigma_B + \tilde{\mu}$ covers S by (S20). This proves (104).

Next, consider $\mu_d \leq \mu_{\tilde{\sigma}} + \tilde{\mu}$ to be a maximal fractional matching such that for each $x \in N(d)$ we have $\mu_d(x) \leq \max\{0, w(\overrightarrow{dx}) - \sigma_B(x)\}$, and $\mu_d(xy) = 0$ if xy does not intersect $N_w(d)$. Since $\mu_{\tilde{\sigma}} \preceq \sigma'_B$ and $\sigma'_B \leq \tilde{\sigma}$, from (S15) and further since $N(d) \subseteq S$, we get that μ_d is only supported in edges with exactly one endpoint in $S_{\mathcal{R}}$.

We claim that, in fact, for each $x \in N(d)$, we have

$$\mu_d(x) = \max\{0, w(\overrightarrow{dx}) - \sigma_B(x)\}, \tag{105}$$

By the maximal choice of μ_d, it is enough to check that for all such x, it holds that $\mu_{\tilde{\sigma}}(x) + \tilde{\mu}(x) \geq w(\overrightarrow{dx}) - \sigma_B(x)$, and this is true by (104). This gives (105).

We thus have

$$\begin{aligned}W(\mu_d) &= \sum_{x\in N_w(d)} \mu_d(x) \overset{(105)}{=} \sum_{x\in N_w(d)} \max\{0, w(\overrightarrow{dx}) - \sigma_B(x)\}\\ &= \sum_{x\in N_w(d)} \Big(w(\overrightarrow{dx}) - \min\{w(\overrightarrow{dx}), \sigma_B(x)\}\Big) = \deg_w(d) - \deg_w(d, \sigma_B)\\ &\geq \frac{k}{2} - \deg_w(d, \sigma_B) \overset{(103)}{\geq} \frac{k}{2} - b_1 \overset{(98)}{\geq} \max\{a_1, a_2\}.\end{aligned}$$

We apply Lemma 8.1 (Extending-out) with

object	H	$N_w(d)$	$V(H)\setminus S$	μ_d	γ_A
in place of	H	U	V	μ	γ

to obtain a γ_A-skew-matching σ_A such that $\sigma_A \trianglelefteq \mu_d$, which has weight

$$W(\sigma_A) \geq (1 + \min\{\gamma_A, \gamma_A^{-1}\})W(\mu_d) \geq a_1 + a_2,$$

and its anchor $\mathcal{A}(\sigma_A)$ is contained in $N_w(d)$.

We claim that (σ_A, σ_B) is the desired (γ_A, γ_B)-skew-matching, anchored in $\overrightarrow{dc}$. To do so, let us verify (B1)–(B4). Indeed, by construction, σ_A is disjoint from σ_B. We have that the anchor $\mathcal{A}(\sigma_A)$ is contained in $N_w(d)$. By (S16) and $\sigma_B \leq \tilde{\sigma}$, we get that $\mathcal{A}(\sigma_B)$ fits in the w-neighbourhood of c. Also, for each $x \in N_w(d) \cap N_w(c)$ for which $\mu_d(x) > 0$ we have that

$$\sigma_A(x) \leq \mu_d(x) \leq w(\overrightarrow{dx}) - \sigma_B(x).$$

In particular, for this we can deduce that the anchor $\mathcal{A}(\sigma_A)$ fits in the w-neighbourhood of d; and moreover, that (B4) holds. This proves the claim. □

Owing to Claim 9.4, we may assume from now on that

$$W(\tilde{\sigma}) < b_1 + b_2. \tag{106}$$

Now we show that we are also done if $\tilde{\sigma} + \tilde{\mu}$ has sufficient weight in $N_w(c)$, i.e., if $\deg_w(c, \tilde{\sigma} + \tilde{\mu}) \geq k$ holds.

Claim 9.5. *If* $\deg_w(c, \tilde{\sigma} + \tilde{\mu}) \geq k$, *then* $H^{\leftrightarrow}$ *has a good matching.*

Proof of Claim 9.5. Let $d \in \mathcal{R}$ be arbitrary. We shall build a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{dc}$. We summarise the proof strategy now. The neighbourhood of c receives enough weight by $\tilde{\mu} + \tilde{\sigma}$, so we can build the whole pair within those objects. We shall use the full $\tilde{\sigma}$ to start building a γ_B-skew-matching with a fraction of the total required weight. In order to build the left-over of the skew pair in $\tilde{\mu}$, we shall first define a weight w', somewhat similarly to a "truncated weight" (Definition 6.12), but with the skew-matching $\tilde{\sigma}$ playing the role of the fractional matching μ in that definition. Next, we shall partition $\tilde{\mu}$ into three auxiliary fractional matchings $\mu_d, \bar{\mu}, \mu_c$, and μ', so that they are homogeneous with respect to the weights from c and d. The fractional matching μ_d represents the part of $\tilde{\mu}$ that is anchored in d and where we can place the γ_A-skew matching, and $\bar{\mu}$ its complement, where only the γ_B-skew matching can be placed. We further divide μ_d into two matchings μ_c and μ'. The matching μ_c represents the part of μ_d that is not only accessible from d, but as well accessible from c on both its side, allowing a lot of flexibility to build the γ_B-skew matching (using Lemma 8.6). Before doing that however, we aim to build as much of the γ_A-skew matching in what remains of μ_d and μ' represents how much of $\mu_d - \mu_c$ we need for that. That is, if we manage to find the whole γ_A-skew matching in $\mu_d - \mu_c$, then μ' might be slightly smaller that $\mu_d - \mu_c$. This is a special (easier case) we treat separately in Step 4. If we do not manage to find the whole γ_A-skew matching in $\mu_d - \mu_c$, then the rest will have to share the space with the γ_B-skew matching in μ_c (and $\mu' = \mu_d - \mu_c$). The latter situation is the core of the proof and is in Step 5 further divided in two cases (Cases 1 and Case 2) according to how much of the γ_B-skew matching we manage to build in μ_c beside the leftover of the γ_A-skew matching.

Some readers may be slightly confused during the proof with the order in which the skew matchings and in which order the truncated weights are defined. The philosophy behind it is the following. The fractional matching μ_c is defined first, w.r.t. w'. It is then cut away (reserved for later) and $\mu_d - \mu_c$ is filled using the truncated weight $\tilde{w}$, obtained from w'. Next, the reserved fractional matching μ_c is used up using the weight w'. Last, if needed, we cut away the whole μ_d, which is already completely used up, and fit whatever we can in $\bar{\mu}$ using the truncated weight $\bar{w}$. At the end we glue the different bits together.

Step 1: Defining the "truncated weight". By (S8) we have that $d \in N_w(c)$. We define an auxiliary weight w' by setting

$$w'(\overrightarrow{ux}) = \begin{cases} \max\{0, w(\overrightarrow{ux}) - \tilde{\sigma}(x)\} & \text{if } u \in \{c, d\} \text{ and } x \in V(H) \setminus \{c, d\}, \\ w(\overrightarrow{ux}) & \text{otherwise.} \end{cases}$$

As mentioned before, w' is analogous to a truncated weight (Definition 6.12), but taken instead with respect to a skew-matching $\tilde{\sigma}$ and somewhat simplified. Observe that

$$\begin{aligned} \deg_{w'}(c, \tilde{\mu}) &= \sum_{v \in V(H)} \big(\min\{w'(\overrightarrow{cx}) + \tilde{\sigma}(x), \tilde{\mu}(x) + \tilde{\sigma}(x)\} - \tilde{\sigma}(x)\big) \\ &\geq \sum_{x \in V(H)} \min\{w(\overrightarrow{cx}), \tilde{\mu}(x) + \tilde{\sigma}(x)\} - W(\tilde{\sigma}) \\ &= \deg_w(c, \tilde{\mu} + \tilde{\sigma}) - W(\tilde{\sigma}). \end{aligned} \tag{107}$$

By (S20) we have, for each $x \in S$, that $\tilde{\mu}(x) = 1 - \tilde{\sigma}(x) \geq w(\overrightarrow{dx}) - \tilde{\sigma}(x)$; hence $w'(\overrightarrow{dx}) \leq \tilde{\mu}(x)$ holds for each $x \in S$. This implies that

$$\deg_{w'}(d, \tilde{\mu}) = \sum_{x \in S} \min\{w'(\overrightarrow{dx}), \tilde{\mu}(x)\} = \sum_{x \in S} w'(\overrightarrow{dx}) \geq \sum_{x \in S} (w(\overrightarrow{dx}) - \tilde{\sigma}(x))$$

$$= \deg_w(d) - \sum_{x \in S_{\mathcal{R}}} \tilde{\sigma}(x) \geq \frac{k}{2} - \frac{W(\tilde{\sigma})}{1+\gamma_B}, \tag{108}$$

where in the last inequality we used (S15), (S17) and $\deg_w(d) \geq k/2$.

Step 2: Defining the auxiliary fractional matchings. Let $\mu_d \leq \tilde{\mu}$ be a maximal fractional matching such that for each $x \in N_{w'}(d)$, we have $\mu_d(x) \leq w'(\overrightarrow{dx})$ and its support edges intersect $N_{w'}(d)$. We have $\mu_d(x) = \min\{w'(\overrightarrow{dx}), \tilde{\mu}(x)\}$ for each $x \in S$, and therefore

$$\deg_{w'}(d, \mu_d) = \deg_{w'}(d, \tilde{\mu}). \tag{109}$$

Now, let $\mu_c \leq \mu_d$ be a maximal fractional matching so that $\mu_c(x) \leq w'(\overrightarrow{cx})$ for all $x \in N_{w'}(d)$. Similarly, by construction we have that $\mu_c(x) = \min\{w'(\overrightarrow{cx}), \mu_d(x)\} = \min\{w'(\overrightarrow{cx}), w'(\overrightarrow{dx}), \tilde{\mu}(x)\}$ for all $x \in S$. Let $\mu' \leq \mu_d - \mu_c$ be maximal such that

$$W(\mu') \leq \frac{a_1 + a_2}{1 + \min\{\gamma_A, \gamma_A^{-1}\}}. \tag{110}$$

Last, let $(H, \tilde{w})$ be the μ_c-truncated graph obtained from (H, w'), let $(H, \bar{w})$ be the μ_d-truncated weighted graph obtained from (H, w'), and set $\bar{\mu} = \tilde{\mu} - \mu_d$.

We now claim the following property, that we will need repeatedly later:

$$\text{for all } v \notin S,\ \mu_c(v) \leq w'(\overrightarrow{cv}). \tag{111}$$

Indeed, if $\mu_c(v) = 0$ there is nothing to check, so assume $\mu_c(v) > 0$. We have $0 < \mu_c(v) \leq \mu_d(v) \leq \tilde{\mu}(v)$. In particular, there exists $u \in S$ such that $\mu_d(uv) > 0$. By definition of μ_d, we must have $u \in N_{w'}(d) \subseteq S_{\mathcal{R}}$ and also $0 < \mu_d(uv) \leq \tilde{\mu}(uv)$. By (S21), we get $v \in \mathcal{R}$. Hence by (S18) we have $\tilde{\mu}(v) + \tilde{\sigma}(v) \leq w(\overrightarrow{cv})$, and thus $\mu_c(v) \leq \tilde{\mu}(v) \leq w'(\overrightarrow{cv})$. This shows that (111) holds.

Step 3: Starting building the γ_A-skew-matching in μ'. We apply Lemma 8.1 (Extending-out) with

object	H	S	$V(H) \setminus S$	μ'	γ_A
in place of	H	U	V	μ	γ

By doing so, we obtain a γ_A-skew-matching $\tilde{\sigma}_A$ in $H^{\leftrightarrow}$ such that

$$\tilde{\sigma}_A \trianglelefteq \mu' \leq \mu_d - \mu_c, \tag{112}$$

of weight

$$W(\tilde{\sigma}_A) = (1 + \min\{\gamma_A, \gamma_A^{-1}\}) W(\mu') \overset{(110)}{\leq} a_1 + a_2, \tag{113}$$

and whose anchor $\mathcal{A}(\tilde{\sigma}_A)$ is contained in S. We claim that

$$\text{the anchor } \mathcal{A}(\tilde{\sigma}_A) \text{ fits in the } \tilde{w}\text{-neighbourhood of } d. \tag{114}$$

Indeed, for all $x \in N_{w'}(d) \subseteq S$, using (112) we have

$$\begin{aligned}
\tilde{\sigma}_A^1(x) \leq \mu'(x) \leq \mu_d(x) - \mu_c(x) &\overset{(111)}{=} \mu_d(x) - \min\{w'(\overrightarrow{cx}), \mu_d(x)\} \\
&= \max\{0, \mu_d(x) - w'(\overrightarrow{cx})\} \leq \max\{0, w'(\overrightarrow{dx}) - \mu_c(x)\} = \tilde{w}(\overrightarrow{dx}),
\end{aligned}$$

where the last inequality follows from $\mu_d(x) \leq w'(\overrightarrow{dx})$ and $\mu_c(x) \leq w'(\overrightarrow{cx})$.

Now we claim that

$$W(\tilde{\sigma}_A) \geq \deg_{\tilde{w}}(c, \mu'). \tag{115}$$

Indeed, $\deg_{\tilde{w}}(c, \mu')$ is the sum of the terms $\min\{\tilde{w}(\overrightarrow{cx}), \mu'(x)\}$, and those are all zero for $x \in S$. Hence the sum incorporates the weight μ' of each edge at most once, and thus is at most $\deg_{\tilde{w}}(c, \mu') \leq W(\mu')$. Then we get the desired inequality by (113).

Step 4: The special (easy) case when $W(\mu_d - \mu_c)$ is large. Before continuing, we will show that we can conclude in the case where the inequality $W(\mu_d - \mu_c) > W(\mu')$ holds. If the fractional matching μ' does not spend the whole weight of $\mu_d - \mu_c$, it means it is big enough to accommodate a whole γ_A-skew matching of weight $a_1 + a_2$. Indeed, we have in this case equality in (110), leading to equality in (113), i.e. we have that $W(\tilde{\sigma}_A) = a_1 + a_2$. This means that all is left to do is to build the rest of the γ_B-skew fractional matching and glue everything together.

First, we apply Lemma 8.2 (Balancing-Out) with

object	H	$V(\mu_c)$	μ_c	γ_B
in place of	H	U	μ	γ

to obtain a γ_B-skew matching $\widehat{\sigma}_B$ such that $\widehat{\sigma}_B \trianglelefteq \mu_c$ and $W(\widehat{\sigma}_B) = 2W(\mu_c)$. Since $\widehat{\sigma}_B \trianglelefteq \mu_c$, together with the fact that $\mu_c(x) \le w'(\overrightarrow{cx})$ for $x \in S$ and (111), implies that in fact $\widehat{\sigma}_B$ fits in the w'-neighbourhood of c.

Secondly, we let $(H, \hat{w})$ be the μ'-truncated graph obtained from $(H, \tilde{w})$. We claim that

$$\deg_{\widehat{w}}(c, \tilde{\mu} - \mu' - \mu_c) \ge b_1 + b_2 - W(\widehat{\sigma}_B) - W(\tilde{\sigma}). \tag{116}$$

Indeed, using Proposition 6.16 twice in the first two equalities

object	$(H, \tilde{w})$	μ'	$\tilde{\mu} - \mu_c$	$\hat{w}$
object	(H, w')	μ_c	$\tilde{\mu}$	$\tilde{w}$
in place of	(G, w)	μ'	μ	w'

we have

$$\begin{aligned}
\deg_{\widehat{w}}(c, \tilde{\mu} - \mu' - \mu_c) &= \deg_{\tilde{w}}(c, \tilde{\mu} - \mu_c) - \deg_{\tilde{w}}(c, \mu') \\
&= \deg_{w'}(c, \tilde{\mu}) - \deg_{w'}(c, \mu_c) - \deg_{\tilde{w}}(c, \mu') \\
&\ge \deg_{w'}(c, \tilde{\mu}) - 2W(\mu_c) - \deg_{\tilde{w}}(c, \mu') \\
&= \deg_{w'}(c, \tilde{\mu}) - W(\hat{\sigma}_B) - \deg_{\tilde{w}}(c, \mu') \\
&\overset{(107)}{\ge} \deg_{w'}(c, \tilde{\mu} + \tilde{\sigma}) - W(\tilde{\sigma}) - W(\hat{\sigma}_B) - \deg_{\tilde{w}}(c, \mu') \\
&\overset{(115)}{\ge} \deg_{w'}(c, \tilde{\mu} + \tilde{\sigma}) - W(\tilde{\sigma}) - W(\hat{\sigma}_B) - W(\tilde{\sigma}_A) \\
&\overset{(113)}{\ge} \deg_{w'}(c, \tilde{\mu} + \tilde{\sigma}) - W(\tilde{\sigma}) - W(\hat{\sigma}_B) - (a_1 + a_2) \\
&= \deg_{w'}(c, \tilde{\mu} + \tilde{\sigma}) - k - W(\tilde{\sigma}) - W(\hat{\sigma}_B) + (b_1 + b_2),
\end{aligned}$$

which gives (116) by recalling that $\deg_{w'}(c, \tilde{\mu} + \tilde{\sigma}) \ge k$ holds by the claim assumption. Thanks to (116), we can use Lemma 8.3 (Combination) with

object	$(H, \hat{w})$	c	$\tilde{\mu} - \mu_c - \mu'$	γ_B
in place of	(H, w)	v	μ	γ

to obtain a γ_B-skew matching $\bar{\sigma}_B$ such that $\bar{\sigma}_B \trianglelefteq \tilde{\mu} - \mu_c - \mu'$, $W(\bar{\sigma}_B) \ge b_1 + b_2 - W(\widehat{\sigma}_B) - W(\tilde{\sigma})$, and such that $\mathcal{A}(\bar{\sigma}_B)$ fits in the $\hat{w}$-neighbourhood of c.

We have finalised the construction of the skew matchings, and we need to 'glue' them; namely, we need to argue that $\tilde{\sigma} + \widehat{\sigma}_B + \bar{\sigma}_B$ forms a skew-matching pair together with $\tilde{\sigma}_A$. Recall that $\sigma_\emptyset$ is the empty skew-matching. We apply Proposition 6.14 with

object	$(H, \tilde{w})$	(d, c)	μ'	$\tilde{\mu} - \mu_c - \mu'$	$(H, \hat{w})$	$\tilde{\sigma}_A$	$\sigma_\emptyset$	$\sigma_\emptyset$	$\bar{\sigma}_B$
in place of	(H, w)	(u, v)	μ	$\bar{\mu}$	$(H, \bar{w})$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

to obtain that $(\tilde{\sigma}_A, \bar{\sigma}_B)$ is a (γ_A, γ_B)-skew matching pair in $(H, \tilde{w})$, anchored in $\overrightarrow{dc}$, with $\tilde{\sigma}_A + \bar{\sigma}_B \trianglelefteq \tilde{\mu} - \mu_c$. Next, we apply Proposition 6.14 again, now with

object	(H, w')	(d, c)	μ_c	$\tilde{\mu} - \mu_c$	$(H, \tilde{w})$	$\sigma_\emptyset$	$\widehat{\sigma}_B$	$\tilde{\sigma}_A$	$\bar{\sigma}_B$
in place of	(H, w)	(u, v)	μ	$\bar{\mu}$	$(H, \bar{w})$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

to obtain that $(\tilde{\sigma}_A, \widehat{\sigma}_B + \bar{\sigma}_B)$ is a (γ_A, γ_B)-skew matching pair in (H, w'), anchored in $\overrightarrow{dc}$, with $\tilde{\sigma}_A + \bar{\sigma}_B + \widehat{\sigma}_B \trianglelefteq \tilde{\mu}$.

We would like to apply Proposition 6.14 again to incorporate $\tilde{\sigma}$, but formally this does not work because w' is not a truncated weight obtained from w. Nevertheless, the argument is morally the same and we sketch it for completeness. We need to show that $(\bar{\sigma}_A, \widehat{\sigma}_B + \bar{\sigma}_B + \tilde{\sigma})$ is a (γ_A, γ_B)-skew matching pair in (H, w), anchored in $\overrightarrow{dc}$. We shall verify (B1)–(B4). Property (B1) follows because $\tilde{\sigma}_A + \bar{\sigma}_B + \widehat{\sigma}_B \trianglelefteq \tilde{\mu}$ and (S14). We already have (B2). To see (B3), let $x \in N_w(c)$. Using property (B3) for $(\bar{\sigma}_A, \widehat{\sigma}_B + \bar{\sigma}_B)$ and w', we see that $\widehat{\sigma}^2_B(x) + \bar{\sigma}^1_B(x) + \tilde{\sigma}^1(x) \leq w'(\overrightarrow{cx}) + \tilde{\sigma}(x) = w(\overrightarrow{cx})$, where the last inequality is because the anchor of $\tilde{\sigma}$ fits in the w-neighbourhood of c. Lastly, (B4) follows as in the proof of Proposition 6.14, considering three cases depending if $x \in N_{w'}(c)$ and $x \in N_{w'}(d)$, or not; we omit further details.

To finish, we argued at the beginning of this step that the assumption $W(\mu_d - \mu_c) > W(\mu')$ implies that $W(\tilde{\sigma}_A) = a_1 + a_2$, and by construction we have $W(\tilde{\sigma}) + W(\widehat{\sigma}_B) + W(\bar{\sigma}_B) = b_1 + b_2$. Hence, we are done with the construction in this case.

Step 5: The main case distinction. Hence, from now on, we can assume that $W(\mu_d - \mu_c) \leq W(\mu')$. Since $\mu' \leq \mu_d - \mu_c$, this implies that

$$\mu' = \mu_d - \mu_c. \tag{117}$$

Now, we set the auxiliary parameters

$$\alpha_1 := a_1 - \frac{W(\tilde{\sigma}_A)}{1+\gamma_A}, \qquad \alpha_2 := \gamma_A \alpha_1,$$
$$\beta_1 := b_1 - \frac{W(\tilde{\sigma})}{1+\gamma_B}, \qquad \beta_2 := \gamma_B \beta_1.$$

Note that $\alpha_1, \alpha_2 \geq 0$ thanks to (113), and $\beta_1, \beta_2 \geq 0$ thanks to (106). We will use the parameters $\alpha_1, \alpha_2, \beta_1, \beta_2$ to build auxiliary skew-matchings in the cases that follow.

We do some preliminary calculations that will be useful. Using that $\mu_c(x) \leq \mu_d(x) \leq w'(\overrightarrow{dx})$ for all $x \in S$, we get that $W(\mu_c) = \deg_{w'}(d, \mu_c)$ and that $W(\mu_d) = \deg_{w'}(d, \mu_d)$. From this, we can observe that

$$\begin{aligned}
W(\mu_c) &= \deg_{w'}(d, \tilde{\mu}) + W(\mu_c) - \deg_{w'}(d, \tilde{\mu}) \\
&\overset{(109)}{=} \deg_{w'}(d, \tilde{\mu}) + W(\mu_c) - \deg_{w'}(d, \mu_d) \\
&= \deg_{w'}(d, \tilde{\mu}) - W(\mu_d - \mu_c) \\
&\overset{(117)}{=} \deg_{w'}(d, \tilde{\mu}) - W(\mu') \\
&\overset{(108)}{\geq} \frac{k}{2} - \frac{W(\tilde{\sigma})}{1+\gamma_B} - W(\mu') \\
&\overset{(113)}{=} \frac{k}{2} - \frac{W(\tilde{\sigma})}{1+\gamma_B} - \max\{1, \gamma_A\}\frac{W(\tilde{\sigma}_A)}{1+\gamma_A} \\
&\overset{(98)}{>} \max\{a_1, a_2\} + b_1 - \frac{W(\tilde{\sigma})}{1+\gamma_B} - \max\{1, \gamma_A\}\frac{W(\tilde{\sigma}_A)}{1+\gamma_A} \\
&= \max\{1, \gamma_A\}\left(a_1 - \frac{W(\tilde{\sigma}_A)}{1+\gamma_A}\right) + \beta_1 \\
&= \max\{\alpha_1, \alpha_2\} + \beta_1 \\
&\geq \max\{\alpha_1, \alpha_2\} + \min\{\beta_1, \beta_2\}.
\end{aligned} \tag{118}$$

From now on, we separate the proof of the claim into two cases, depending if $W(\mu_c)$ is sufficiently large with respect to $\alpha_1, \alpha_2, \beta_1, \beta_2$ or not.

Case 1: $W(\mu_c)$ is large. In this case, we will assume that

$$\min\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\} < W(\mu_c). \tag{119}$$

Since μ_c is large enough, we can accommodate the left-over of the skew-matching pair in it and will not need to use $\bar{\mu}$ at all. However, the setting is such that we cannot guarantee the use of Lemma 8.6 (Completion), and therefore, we shall have to build the skew-matching pair in μ_c "by hand".

Case 1, Step I: Building a skew-matching pair in μ_c. We will define two auxiliary skew-matchings, $\widehat{\sigma}_A$ and $\widehat{\sigma}_B$, which are γ_A-skew and γ_B-skew, respectively. For every $xy \in E(H)$ with $x \in S$, $y \notin S$, we let

$$\widehat{\sigma}_A(\overrightarrow{xy}) := \frac{\alpha_1+\alpha_2}{W(\mu_c)}\mu_c(xy),$$

and $\widehat{\sigma}_A$ takes the value 0 in any other case. Note that

$$W(\widehat{\sigma}_A) = \sum_{x\in S}\sum_{y\notin S}\frac{\alpha_1+\alpha_2}{W(\mu_c)}\mu_c(xy) = \alpha_1+\alpha_2. \tag{120}$$

The definition of $\widehat{\sigma}_B$ depends if $\gamma_A < 1$ or not. For brevity, we use the Iverson bracket notation: for a logical statement P the value $[P]$ equals 1 if P is true, and 0 otherwise. Now consider any $xy \in E(H)$ with $x \in S$ and $y \notin S$. We define

$$\widehat{\sigma}_B(\overrightarrow{xy}) := [\gamma_A < 1]\frac{\beta_1+\beta_2}{W(\mu_c)}\mu_c(xy),$$

as well as

$$\widehat{\sigma}_B(\overrightarrow{yx}) := [\gamma_A \geq 1]\frac{\beta_1+\beta_2}{W(\mu_c)}\mu_c(xy);$$

and $\widehat{\sigma}_B$ takes the value 0 in every other ordered edge. Similarly as before, we can observe that

$$W(\widehat{\sigma}_B) = \beta_1+\beta_2. \tag{121}$$

Now, we claim that

$$\widehat{\sigma}_A + \widehat{\sigma}_B \trianglelefteq \mu_c.$$

To do this, let $xy \in E(H)$ be arbitrary such that $x \in S$ and $y \notin S$ (as otherwise there is nothing to check). Observe that

$$\begin{aligned}\frac{\widehat{\sigma}_A(\overrightarrow{xy})+\gamma_A\widehat{\sigma}_A(\overrightarrow{yx})}{1+\gamma_A} &+ \frac{\widehat{\sigma}_B(\overrightarrow{xy})+\gamma_B\widehat{\sigma}_B(\overrightarrow{yx})}{1+\gamma_B}\\ &= \left(\frac{\alpha_1+\alpha_2}{1+\gamma_A} + ([\gamma_A<1]+\gamma_B[\gamma_A\geq 1])\frac{\beta_1+\beta_2}{1+\gamma_B}\right)\frac{\mu_c(xy)}{W(\mu_c)}\\ &= (\alpha_1+\beta_1([\gamma_A<1]+\gamma_B[\gamma_A\geq 1]))\frac{\mu_c(xy)}{W(\mu_c)}.\end{aligned}$$

By (S12), we have that $\beta_2 = \gamma_B\beta_1 \geq \beta_1$. A brief moment of reflection (considering the cases $\gamma_A \geq 1$ and $\gamma_A < 1$ separately) then implies that $\alpha_1+\beta_1([\gamma_A<1]+\gamma_B[\gamma_A\geq 1])$ is equal either to $\max\{\alpha_1,\alpha_2\}+\min\{\beta_1,\beta_2\}$ or $\min\{\alpha_1,\alpha_2\}+\max\{\beta_1,\beta_2\}$. In any case, either by (118) or (119), we can conclude this term is at most $W(\mu_c)$. This implies that we have

$$\frac{\widehat{\sigma}_A(\overrightarrow{xy})+\gamma_A\widehat{\sigma}_A(\overrightarrow{yx})}{1+\gamma_A} + \frac{\widehat{\sigma}_B(\overrightarrow{xy})+\gamma_B\widehat{\sigma}_B(\overrightarrow{yx})}{1+\gamma_B} \leq \mu_c(xy),$$

as desired. An analogous calculation implies that

$$\frac{\widehat{\sigma}_A(\overrightarrow{yx})+\gamma_A\widehat{\sigma}_A(\overrightarrow{xy})}{1+\gamma_A} + \frac{\widehat{\sigma}_B(\overrightarrow{yx})+\gamma_B\widehat{\sigma}_B(\overrightarrow{xy})}{1+\gamma_B} \leq \mu_c(xy),$$

thus indeed $\widehat{\sigma}_A + \widehat{\sigma}_B \trianglelefteq \mu_c$ holds.

Case 1, Step II: Gluing the skew-matchings together. Now, we claim that $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ is a (γ_A,γ_B)-skew-matching pair in (H, w') anchored in $\overrightarrow{dc}$, with respect to w'. We check the required properties (B1)–(B4). The disjointness of $\widehat{\sigma}_A$ and $\widehat{\sigma}_B$ follows from $\widehat{\sigma}_A+\widehat{\sigma}_B \trianglelefteq \mu_c$, so (B1) holds. We check that $\widehat{\sigma}_A$ fits in the w'-neighbourhood of d. Since $\widehat{\sigma}_A$ is supported only in directed edges xy with $x \in S$ and $y \notin S$, we have $\widehat{\sigma}_A^1(v) = 0$ for each $v \notin S$. For each $v \in S$, again $\widehat{\sigma}_A + \widehat{\sigma}_B \trianglelefteq \mu_c$ implies that $\widehat{\sigma}_A^1(v) \leq \mu_c(v) \leq \mu_d(v) \leq w'(\overrightarrow{dv})$ (where we used $\mu_c \leq \mu_d$, the definition of μ_d, and $v \in S$ in the last inequalities); this implies that $\widehat{\sigma}_A$ fits in the w'-neighbourhood of d. Now, we check that $\widehat{\sigma}_B$ fits in the w'-neighbourhood of c. If $v \in S$, a similar argument as before (using the definition of μ_c) gives that $\widehat{\sigma}_B^1(v) \leq \mu_c(v) \leq w'(\overrightarrow{cv})$. Hence, we can suppose that $v \notin S$. We have $0 < \widehat{\sigma}_B^1(v) \leq \mu_c(v) \leq w'(\overrightarrow{cv})$, where the last inequality is from (111). Thus indeed $\widehat{\sigma}_B$ fits in the w'-neighbourhood of c; hence (B2)–(B3) hold. Finally, for each $v \in N_{w'}(c) \cap N_{w'}(d) \subseteq S$ we have $\widehat{\sigma}_A^1(v) + \widehat{\sigma}_B^1(v) \leq \mu_c(v) \leq w'(\overrightarrow{cv})$ which gives (B4). We have then checked that $(\widehat{\sigma}_A, \widehat{\sigma}_B)$ is a (γ_A, γ_B)-skew-matching pair in (H, w') anchored in $\overrightarrow{dc}$.

Set $\sigma_A := \widehat{\sigma}_A + \tilde{\sigma}_A$, and let $\sigma_\emptyset$ be the identically-zero skew-matching. Recalling (112) and (114) reveals that we can use Proposition 6.14 with

object	(H, w')	(d, c)	μ_c	$\mu_d - \mu_c$	$(H, \tilde{w})$	$\widehat{\sigma}_A$	$\widehat{\sigma}_B$	$\tilde{\sigma}_A$	$\sigma_\emptyset$
in place of	(H, w)	(u, v)	μ	$\bar{\mu}$	$(H, \bar{w})$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

and we get that $(\sigma_A, \widehat{\sigma}_B)$ is a (γ_A, γ_B)-skew-matching in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$ (with respect to w') with $\sigma_A + \widehat{\sigma}_B \trianglelefteq \mu_d$.

We set $\sigma_B := \tilde{\sigma} + \widehat{\sigma}_B$. Using (120) and the definition of α_1, α_2, we have

$$W(\sigma_A) = W(\tilde{\sigma}_A) + \alpha_1 + \alpha_2 = W(\tilde{\sigma}_A) + (1 + \gamma_A)\left(a_1 - \frac{W(\tilde{\sigma}_A)}{1 + \gamma_A}\right) = a_1 + a_2,$$

and similarly, using (121) and the definition of β_1, β_2, we have

$$W(\sigma_B) = W(\tilde{\sigma}) + \beta_1 + \beta_2 = b_1 + b_2.$$

To conclude in this case, we just need to check that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$. Recall that $\mu_d \leq \tilde{\mu}$. Since $\tilde{\mu}$ and $\tilde{\sigma}$ are disjoint, so are σ_A and σ_B, which gives (B1). Point (B2) follows from the fact (that we have already checked) that σ_A is a γ_A-skew-matching which fits in the w'-neighbourhood of d; and $w' \leq w$. To see (B3) we need to check that σ_B is a γ_B-skew-matching, whose anchor fits in the w-neighbourhood of c. We already know that $\widehat{\sigma}_B$ is a γ_B-skew-matching whose anchor fits in the w'-neighbourhood of c. Because of (S15) and (S17), we just need to check the property for $x \in S_{\mathcal{R}}$. In such a case, we have

$$\sigma_B^1(x) = \widehat{\sigma}_B^1(x) + \tilde{\sigma}^1(x) = \widehat{\sigma}_B^1(x) + \tilde{\sigma}(x) \leq w'(\overrightarrow{cx}) + \tilde{\sigma}(x). \tag{122}$$

We check two cases depending on the definition of w'. If $w'(\overrightarrow{cx}) = 0$, then (122) gives $\sigma_B^1(x) \leq \tilde{\sigma}(x) \leq w(\overrightarrow{cx})$, as desired. Otherwise, we have $w'(\overrightarrow{cx}) = w(\overrightarrow{cx}) - \tilde{\sigma}(x)$, so $\sigma_B^1(x) \leq w(\overrightarrow{cx})$, in which case again we are done. This gives (B3). Finally, to see (B4) let $x \in N(c) \cap N(d) \subseteq S$. Note that $\sigma_A^1(x) + \sigma_B^1(x) \leq \mu_d(x) + \tilde{\sigma}(x) \leq w'(\overrightarrow{dx}) + \tilde{\sigma}(x) \leq \max\{w(\overrightarrow{dx}), \tilde{\sigma}(x)\} \leq \max\{w(\overrightarrow{dx}), w(\overrightarrow{cx})\}$, where in the second to last inequality we used the definition of w', and in the last inequality we used (S16). This finishes the proof in Case 1.

Case 2: $W(\mu_c)$ is small. We can assume that Case 1 does not hold. Hence (119) fails to hold, and we can assume that

$$\min\{\alpha_1, \alpha_2\} + \max\{\beta_1, \beta_2\} \geq W(\mu_c). \tag{123}$$

In this case, since $W(\mu_c)$ is small, we shall need possibly to complement whatever we build in μ_d by a γ_B-skew-matching in $\bar{\mu}$. On the other hand, the setting is such that for building a skew-matching pair in μ_c, we may use Lemma 8.6 (Completion).

Case 2, Step I: Building a skew-matching pair in μ_c. We wish to apply Lemma 8.6 (Completion) with

object	(H, w')	$N_{w'}(d)$	$V(H) \setminus S$	c	μ_c	α_1	α_2	β_1	β_2
in place of	(H, w)	U	V	u	μ	α_1	α_2	β_1	β_2

We check the required hypothesis. To check (L1), we need that for all $y \notin S$, $\mu_c(y) \leq w'(\overrightarrow{cy})$, and this follows from (111). The other required inequalities (L2) and (L3) follow from (118) and (123), respectively.

The application of Lemma 8.6 gives us a γ_A-skew-matching σ'_A and a γ_B-skew-matching σ'_B with $\sigma'_A + \sigma'_B \trianglelefteq \mu_c$ such that

$$W(\sigma'_A) = a_1 + a_2 - W(\tilde{\sigma}_A) \tag{124}$$

and

$$W(\sigma'_B) \geq \deg_{w'}(c, \mu_c) - W(\sigma'_A), \tag{125}$$

the anchor $\mathcal{A}(\sigma'_B)$ fits in the w'-neighbourhood of c, and $\mathcal{A}(\sigma'_A)$ is contained in $N_{w'}(d)$.

We claim that (σ'_A, σ'_B) forms a (γ_A, γ_B)-skew pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$, with respect to w'. Indeed, (B1) follows from $\sigma'_A + \sigma'_B \trianglelefteq \mu_c$; we have that $\mathcal{A}(\sigma'_B)$ fits in the w'-neighbourhood of c, implying (B3). To see (B2), we need to check that σ'_A is a γ_A-skew-matching whose anchor fits in the w'-neighbourhood of d. This follows because of $\sigma'_A + \sigma'_B \trianglelefteq \mu_c$ and the fact that $\mu_c(x) \leq \mu_d(x) \leq w'(\overrightarrow{xd})$ holds for all $x \in S$, by the construction of μ_c, μ_d. Finally, to see (B4) we can argue similarly using $\sigma'_A + \sigma'_B \trianglelefteq \mu_c$.

Case 2, Step II: Completing the skew-matching pair in $\bar{\mu}$. By Lemma 8.3 with

object	$(H, \bar{w})$	c	$\bar{\mu}$	γ_B
in place of	(H, w)	v	μ	γ

there is a γ_B-skew-matching $\sigma^*_B \trianglelefteq \bar{\mu}$ of weight

$$W(\sigma^*_B) \geq \deg_{\bar{w}}(c, \bar{\mu}) \tag{126}$$

with its anchor $\mathcal{A}(\sigma^*_B)$ fitting in the $\bar{w}$-neighbourhood of c.

Case 2, Step III: Gluing the skew-matching pairs together. Recall that $\sigma_\emptyset$ is the zero skew-matching. Together with (112) and (114), we can use Proposition 6.14 with

object	(H, w')	(d, c)	μ_c	μ'	$(H, \tilde{w})$	σ'_A	σ'_B	$\tilde{\sigma}_A$	$\sigma_\emptyset$
in place of	(H, w)	(u, v)	μ	$\bar{\mu}$	$(H, \bar{w})$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

and obtain that the pair $(\sigma'_A + \tilde{\sigma}_A, \sigma'_B)$ forms a (γ_A, γ_B)-skew-matching in H anchored in $\overrightarrow{dc}$ (w.r.t. w') with $\sigma'_A + \tilde{\sigma}_A + \sigma'_B \trianglelefteq \mu_d$.

We apply Proposition 6.14 once again, this time with

object	(H, w')	(d, c)	μ_d	$\bar{\mu}$	$(H, \bar{w})$	$\sigma'_A + \tilde{\sigma}_A$	σ'_B	$\sigma_\emptyset$	σ^*_B
in place of	(H, w)	(u, v)	μ	$\bar{\mu}$	$(H, \bar{w})$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

and by doing so we obtain that $(\sigma'_A + \tilde{\sigma}_A, \sigma'_B + \sigma^*_B)$ is a (γ_A, γ_B)-skew pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{dc}$ (w.r.t. w') with $\sigma'_A + \tilde{\sigma}_A + \sigma'_B + \sigma^*_B \leq \tilde{\mu}$. Note that we have

$$W(\sigma'_A + \tilde{\sigma}_A) \stackrel{(124)}{=} a_1 + a_2 - W(\tilde{\sigma}_A) + W(\tilde{\sigma}_A) = a_1 + a_2. \tag{127}$$

Recalling that $(H, \tilde{w})$ is the μ_c-truncated graph obtained from (H, w'), and that $(H, \bar{w})$ is the μ_d-truncated weighted graph obtained from (H, w'); we also have

$$\begin{aligned} W(\sigma'_B + \sigma^*_B) &\stackrel{(125),(126)}{\geq} \deg_{\bar{w}}(c, \bar{\mu}) + \deg_{w'}(c, \mu_c) - W(\sigma'_A) \\ &\stackrel{(124)}{=} \deg_{\bar{w}}(c, \tilde{\mu} - \mu_d) + \deg_{w'}(c, \mu_c) - (a_1 + a_2) + W(\tilde{\sigma}_A) \\ &\stackrel{(115)}{\geq} \deg_{\bar{w}}(c, \tilde{\mu} - \mu_d) + \deg_{w'}(c, \mu_c) + \deg_{\tilde{w}}(c, \mu_d - \mu_c) - (a_1 + a_2) \end{aligned}$$

Now we apply Proposition 6.16 twice, to deduce

$$\begin{aligned} W(\sigma'_B + \sigma^*_B) &\geq \deg_{\bar{w}}(c, \tilde{\mu} - \mu_d) + \deg_{w'}(c, \mu_d) - (a_1 + a_2). \\ &\geq \deg_{w'}(c, \tilde{\mu}) - (a_1 + a_2) \stackrel{(107)}{\geq} \deg_w(c, \tilde{\mu} + \tilde{\sigma}) - W(\tilde{\sigma}) - (a_1 + a_2) \\ &\geq k - W(\tilde{\sigma}) - (a_1 + a_2), \end{aligned} \tag{128}$$

where the last inequality holds by our assumption in the statement of the claim.

We can finally define our final skew-matchings that will allow us to conclude. Let $\sigma_B := \tilde{\sigma} + \sigma'_B + \sigma^*_B$ and $\sigma_A = \sigma'_A + \tilde{\sigma}_A$. By (127) and (128), we have $W(\sigma_A) \geq a_1 + a_2$ and $W(\sigma_B) \geq k - W(\tilde{\sigma}) - (a_1 + a_2) + W(\tilde{\sigma}) = b_1 + b_2$. We claim that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{dc}$; this gives the existence of the desired good matching in $H^{\leftrightarrow}$.

We check the required properties (B1)–(B4). To see (B1), note that since $\tilde{\mu}$ and $\tilde{\sigma}$ are disjoint and $\sigma_A + \sigma'_B + \sigma^*_B \trianglelefteq \tilde{\mu}$, we have that σ_A and σ_B are disjoint. As $(\sigma'_A + \tilde{\sigma}_A, \sigma'_B + \sigma^*_B)$ is a (γ_A, γ_B)-skew pair anchored in $\overrightarrow{dc}$ w.r.t. w', the anchor of $\sigma_A = \sigma'_A + \tilde{\sigma}_A$ fits in the w-neighbourhood of d, so (B2) holds.

Now we check that (B3) holds. We have already checked that the anchor of $\sigma'_B + \sigma^*_B$ fits in the w'-neighbourhood of c. This means that we only need to check that for any $x \in V(H)$ with $\tilde{\sigma}^1(x) > 0$, we have that

$$w(\overrightarrow{cx}) \geq \sigma^1_B(x). \tag{129}$$

To see this, let x be a vertex with $\tilde{\sigma}^1(x) > 0$. By (S15)-(S16), we have that $x \in N_w(c) \cap S_{\mathcal{R}}$. Suppose first that $w'(\overrightarrow{cx}) = 0$. In this case, $\mu_c(x) = 0$ and thus $\sigma'_B(x) = 0$. Also we have then that $\bar{w}(\overrightarrow{cx}) = 0$, and therefore $\sigma^{*1}_B(x) = 0$. Hence, we have that $\sigma^1_B(x) = \tilde{\sigma}^1_B(x)$. By (S16), we have that $\tilde{\sigma}^1(x) \leq w(\overrightarrow{cx})$, and therefore we obtain

$$w(\overrightarrow{cx}) \geq \tilde{\sigma}(x) = \tilde{\sigma}^1(x) + \sigma^{*1}_B(x) + \sigma'^1_B(x) = \sigma^1_B(x),$$

so (129) holds in this case. Hence, we can assume that $w'(\overrightarrow{cx}) > 0$. In this case (by the definition of w') we have $w'(\overrightarrow{cx}) = w(\overrightarrow{cx}) - \tilde{\sigma}(x)$. Hence, using that $\sigma'_B + \sigma^*_B$ fits in the w'-neighbourhood of c, we get

$$\sigma^1_B(x) = \tilde{\sigma}^1(x) + \sigma'^1_B(x) + \sigma^{*1}_B(x) \leq \tilde{\sigma}(x) + w'(\overrightarrow{cx}) = w(\overrightarrow{cx}).$$

Hence (129) holds in all cases, and therefore, (B3) holds.

It remains to check that (B4) holds. Let $x \in N_w(c) \cap N_w(d)$ be arbitrary. Recall that we know already that $(\sigma'_A + \tilde{\sigma}_A, \sigma'_B + \sigma^*_B)$ is a (γ_A, γ_B)-skew pair with respect to w'. This implies that

$$\sigma'^1_A(x) + \tilde{\sigma}^1_A(x) + \sigma'^1_B(x) + \sigma^{*1}_B(x) \leq \max\{w'(\overrightarrow{cx}), w'(\overrightarrow{dx})\}.$$

Hence, we have

$$\begin{aligned}
\sigma^1_A(x) + \sigma^1_B(x) &= \sigma'^1_A(x) + \tilde{\sigma}^1_A(x) + \sigma'^1_B(x) + \sigma^{*1}_B(x) + \tilde{\sigma}^1(x) \\
&\leq \max\{w'(\overrightarrow{cx}), w'(\overrightarrow{dx})\} + \tilde{\sigma}(x) \\
&= \max\{w(\overrightarrow{cx}) - \tilde{\sigma}(x), w(\overrightarrow{dx}) - \tilde{\sigma}(x)\} + \tilde{\sigma}(x) \\
&= \max\{w(\overrightarrow{cx}), w(\overrightarrow{dx})\},
\end{aligned}$$

where in the second-to-last step we have used the definition of w', and the fact that $\tilde{\sigma}$ fits in the w-neighbourhood of c (so $w(\overrightarrow{cx}) \geq \tilde{\sigma}^1(x) = \tilde{\sigma}(x)$ holds). Since x was arbitrary, this implies (B4) holds. This finishes the proof of the claim that (σ_A, σ_B) is a good matching; which in turn finishes the proof in Case 2, and the proof of Claim 9.5. □

Let

$$\tilde{\mathcal{R}} := \{d \in V(H) \,:\, \tilde{\sigma}(d) + \tilde{\mu}(d) < w(\overrightarrow{cd})\}.$$

If $\tilde{\mathcal{R}}$ is empty, then we would have $\deg_w(c, \tilde{\sigma} + \tilde{\mu}) = \sum_{x \in V} w(\overrightarrow{cx}) = \deg_w(c) \geq k$, so we would be done by Claim 9.5. Hence, from now on, we can and will assume that

(S23) $\tilde{\mathcal{R}} \neq \emptyset$,

Observe that by (S19) we have $\tilde{\mathcal{R}} \subseteq \mathcal{R}$. Since $(\tilde{\sigma}, \tilde{\mu})$ is an optimal $(H, w, S, M, c, \mu, \gamma_B)$-GE pair by (S22), we can apply the Separating Lemmas (Lemma 7.20 and Lemma 7.21) to it with any choice of $d \in \tilde{\mathcal{R}}$. From those applications we directly obtain the following claims.

Claim 9.6 (First Separating Claim)**.** *For any $d \in \tilde{\mathcal{R}}$ we have $\tilde{\sigma}(x) = w(\overrightarrow{cx})$ for all $x \in N_w(c) \cap N_w(d)$.*

Claim 9.7 (Second Separating Claim)**.** *Let $d \in \tilde{\mathcal{R}}$, and let $\sigma_d \leq \tilde{\sigma}$ and $\mu_d \leq \tilde{\mu}$ be the γ_B-skew-matching and fractional matching, respectively, obtained from $\tilde{\sigma}$ and $\tilde{\mu}$, respectively, by considering only the edges of their support that intersect $N_w(d)$. Then there is no edge between the sets*

$$\{x \in N_w(c) \cap S \,:\, \tilde{\sigma}(x) < w(\overrightarrow{cx})\}$$

and

$$\{y \in N_w(c) \setminus S \,:\, \sigma_d(y) + \mu_d(y) > 0\}.$$

9.5. **Proof of Proposition 4.2: The balanced case.** Now we have the tools to conclude in the case where $b_2 \leq k/2$. We discuss a bit about our strategy here. This case is still somewhat manageable, in the sense that the γ_B-skew-matching would fit in the fractional matching built on top of the neighbourhood of d, for some $d \in \mathcal{R}$. However, we do not have the condition that $\min\{b_1, b_2\} + \max\{a_1, a_2\} \leq k/2$ in order to use directly the Completion Lemma (Lemma 8.6). To overcome this problem and to ensure that we can actually embed in the mentioned fractional matching over $N_w(d)$ at least as much as the degree of c into it, we shall leverage the fact we can fill things more efficiently by using the skew-matching $\tilde{\sigma}$.

Claim 9.8. *Suppose that $b_2 \leq k/2$. Then $H^{\leftrightarrow}$ admits a good matching.*

Proof of Claim 9.8. The proof has three steps. We begin by picking any $d \in \tilde{\mathcal{R}}$ (this exists, by (S23)).

Step 1: Building the γ_B-skew-matching σ_B. Let $\sigma'_B \leq \tilde{\sigma}$ be a maximal γ_B-skew-matching such that for every $x \in N_w(d) \cap N_w(c)$ we have $\sigma'_B(x) = \min\{\tilde{\sigma}(x), w(\overrightarrow{dx})\}$. We recall that by (S16), we have $\sigma'_B(x) \leq \tilde{\sigma}(x) = 0$ for $x \notin N_w(c)$. Hence, by construction, the anchor $\mathcal{A}(\sigma'_B)$ fits in the w-neighbourhood of d.

Let w' be the weight function defined as

$$w'(\overrightarrow{uv}) = \begin{cases} w(\overrightarrow{dx}) - \sigma'_B(x) & \text{if } \{u,v\} = \{d,x\} \text{ with } x \in S, \\ \max\{0,\, w(\overrightarrow{cx}) - \sigma'_B(x)\} & \text{if } \{u,v\} = \{c,x\} \text{ with } x \in V(H) \setminus \{c\}, \\ w(\overrightarrow{uv}) & \text{otherwise.} \end{cases}$$

The definition of w' and σ'_B, and a quick case analysis, shows that for every $x \in S$ we have $w'(\overrightarrow{dx}) = w(\overrightarrow{dx}) - \sigma'_B(x) \leq 1 - \tilde{\sigma}(x)$. For every $x \in S$ we have $1 - \tilde{\sigma}(x) = \tilde{\mu}(x)$, by (S20), which combined with the previous bound gives, for all $x \in S$,

$$w'(\overrightarrow{dx}) \leq \tilde{\mu}(x). \tag{130}$$

Now, let $\tilde{\mu}_d \leq \tilde{\mu}$ be a maximal fractional matching whose support edges intersect $N_{w'}(d)$, and such that for all $x \in N_{w'}(d) \subseteq S$, we have $\tilde{\mu}_d(x) \leq w'(\overrightarrow{dx})$ and

$$W(\tilde{\mu}_d) \leq \gamma_B \left(b_1 - \frac{W(\sigma'_B)}{1+\gamma_B} \right). \tag{131}$$

By assumption, we have $b_2 \leq k/2$, and therefore

$$\begin{aligned} \gamma_B \left(b_1 - \frac{W(\sigma'_B)}{1+\gamma_B} \right) &= b_2 - \frac{\gamma_B}{1+\gamma_B} W(\sigma'_B) \leq \frac{k}{2} - \frac{\gamma_B}{1+\gamma_B} W(\sigma'_B) \overset{\text{(S12)}}{\leq} \frac{k}{2} - \frac{W(\sigma'_B)}{1+\gamma_B} \\ &\leq \sum_{x \in S} \left(w(\overrightarrow{dx}) - \sigma'_B(x) \right) = \sum_{x \in S} w'(\overrightarrow{dx}) \overset{(130)}{\leq} \deg_{w'}(d, \tilde{\mu}). \end{aligned}$$

Hence, by construction, in fact we attain equality in (131). We thus have

$$W(\tilde{\mu}_d) = \gamma_B \left(b_1 - \frac{W(\sigma'_B)}{1+\gamma_B} \right). \tag{132}$$

We apply Lemma 8.1 (Extending-out), with

object	H	$N_{w'}(d)$	$V(H)\setminus S$	$\tilde\mu_d$
in place of	H	U	V	μ

The outcome of this application is an γ_B-skew-matching $\tilde\sigma_B$ in $H^{\leftrightarrow}$ with

$$\tilde\sigma_B \trianglelefteq \tilde\mu_d, \tag{133}$$

and of weight

$$\begin{aligned} W(\tilde\sigma_B) &= (1+\min\{\gamma_B,\gamma_B^{-1}\})W(\tilde\mu_d) \overset{(S12)}{=} (1+\gamma_B^{-1})W(\tilde\mu_d) \\ &\overset{(132)}{=} b_1+b_2-W(\sigma'_B), \end{aligned} \tag{134}$$

and whose anchor satisfies $\mathcal{A}(\tilde\sigma_B)\subseteq N_{w'}(d)\subseteq S$. By the way w' and $\tilde\mu_d$ were defined and (133) we have that

$$\text{the anchor } \mathcal{A}(\tilde\sigma_B) \text{ fits in the } w'\text{-neighbourhood of } d. \tag{135}$$

We finish the construction of σ_B by setting

$$\sigma_B := \sigma'_B + \tilde\sigma_B. \tag{136}$$

By (134), we have

$$W(\sigma_B) = W(\sigma'_B) + W(\tilde\sigma_B) = b_1+b_2. \tag{137}$$

Step 2: Starting to build σ_A. In this step, we try to build a γ_A-skew-matching, which is as large as possible, and is contained in the leftover of $\tilde\mu$ and in the leftover of $\tilde\sigma$.

By construction, the edges supporting $\tilde\mu_d$ intersect $N_{w'}(d)$. By Claim 9.6 we have that $\tilde\sigma(x)=w(\overrightarrow{cx})$ for all $x\in N_w(c)\cap N_{w'}(d)$, and by (S16) we have $\tilde\sigma(x)=w(\overrightarrow{cx})=0$ for all $x\notin N_w(c)$. Then (in particular) the equality $\tilde\sigma(x)=w(\overrightarrow{cx})$ holds for all $x\in N_{w'}(d)\cap V(\tilde\mu_d)$. For $x\in S\cap N_w(c)$, by the definition of $\tilde\mu_d$ we have that if $\tilde\mu_d(x)>0$, then $w'(\overrightarrow{dx})>0$. This, by definition of w', means that $w(\overrightarrow{dx})>\sigma'_B(x)$ for such x. By the definition of σ'_B, then we have $\sigma'_B(x)=\tilde\sigma(x)=w(\overrightarrow{cx})$, which then means that $w'(\overrightarrow{cx})=0$. Hence, we have that

$$\deg_{w'}(c, V(\tilde\mu_d)\cap S)=0. \tag{138}$$

Now, let $y\in V(\tilde\mu_d)\setminus S$. For such y, there exists $v\in S\cap N_w(d)$ such that $0<\tilde\mu_d(vy)$. Because $\tilde\mu_d\le\tilde\mu$, by (S21) $\tilde\mu_d$ is supported in the edges between $S_{\mathcal{R}}$ and $\mathcal{R}$, and thus $y\in\mathcal{R}$ and thus (S18) implies $\tilde\mu_d(y)+\tilde\sigma(y)\le\tilde\mu(y)+\tilde\sigma(y)\le w(\overrightarrow{cy})$, implying $\tilde\mu_d(y)\le w(\overrightarrow{cy})-\tilde\sigma(y)\le w'(\overrightarrow{cy})$. Thus we have

$$\begin{aligned} \deg_{w'}(c,\tilde\mu_d) &= \sum_{x\in N_{w'}(c)} \min\{w'(\overrightarrow{cx}),\tilde\mu_d(x)\} \overset{(138)}{=} \sum_{x\in N_{w'}(c)\setminus S} \min\{w'(\overrightarrow{cx}),\tilde\mu_d(x)\} \\ &= \sum_{x\in\mathcal{R}} \tilde\mu_d(x) \le W(\tilde\mu_d) \overset{(134)}{\le} W(\tilde\sigma_B), \end{aligned} \tag{139}$$

using that $\tilde\mu_d$ is supported in the edges between $S_{\mathcal{R}}$ and $\mathcal{R}$.

Let $(H,\bar w)$ be the $\tilde\mu_d$-truncated weighted graph obtained from (H,w'). We want to apply Lemma 8.4 (Extending-out skew-matching) with

object	$(H,\bar w)$	1	γ_B	c	$\tilde\sigma-\sigma'_B$
in place of	(H,w)	γ_A	γ_B	u	σ_B

To do so, we recall that $\gamma_B>1$, by (S12). We also need to check that the anchor of $\tilde\sigma-\sigma'_B$ fits in the $\bar w$-neighbourhood of c. Indeed, we have $\mathcal{A}(\tilde\sigma-\sigma'_B)\subseteq S_{\mathcal{R}}$ by (S15) and $\sigma'_B\le\tilde\sigma$, by definition. For any $x\in S$, if $\tilde\sigma(x)\le w(\overrightarrow{dx})$, we have $\sigma'_B(x)=\tilde\sigma(x)$, so there is nothing to check. So we can assume $\tilde\sigma(x)>w(\overrightarrow{dx})$, so $\sigma'_B(x)=w(\overrightarrow{dx})$. Then we have

$$\begin{aligned} \tilde\sigma^1(x)-(\sigma'_B)^1(x) &= \tilde\sigma(x)-\sigma'_B(x) = \tilde\sigma(x)-w(\overrightarrow{dx}) \le w(\overrightarrow{cx})-w(\overrightarrow{dx}) \\ &= w(\overrightarrow{cx})-\sigma'_B(x)+\sigma'_B(x)-w(\overrightarrow{dx}) \end{aligned}$$

$$\leq w'(\overrightarrow{cx}) + \sigma'_B(x) - w(\overrightarrow{dx}) = w'(\overrightarrow{cx}) - w'(\overrightarrow{dx})$$
$$\leq w'(\overrightarrow{cx}) - \tilde{\mu}_d(x) = \bar{w}(\overrightarrow{cx}),$$

as required. The application of Lemma 8.4 then gives a 1-skew-matching σ_c of weight $W(\sigma_c) = 2W(\tilde{\sigma} - \sigma'_B)/(1+\gamma_B)$ whose anchor fits in the $\bar{w}$-neighbourhood of c, and $\sigma_c \leq \tilde{\sigma} - \sigma'_B$. By Lemma 6.10 with

object	H	σ_c	$\tilde{\sigma} - \sigma'_B$	γ_B
in place of	G	σ	σ'	γ

we get a fractional matching μ_c such that

$$\mu_c \preceq \tilde{\sigma} - \sigma'_B, \tag{140}$$

and

$$W(\mu_c) = \frac{W(\tilde{\sigma} - \sigma'_B)}{1+\gamma_B}.$$

The last equality and (140) imply that, in fact, for each $x \in S$, we have

$$\mu_c(x) = \tilde{\sigma}(x) - \sigma'_B(x),$$

and therefore, for each $x \in S$,

$$\mu_c(x) + \tilde{\mu}(x) + \sigma'_B(x) = \tilde{\sigma}(x) + \tilde{\mu}(x) = 1 \geq w(\overrightarrow{cx}).$$

Observe that by (S17) (and the fact that $\mu_c \preceq \tilde{\sigma} - \sigma'_B$ and $\sigma'_B \leq \tilde{\sigma}$), we have that $\mu_c(x) = \sigma'_B(x) = 0$ for all $x \in V(H) \setminus (S \cup \mathcal{R})$. Together with (S19), we get that for each $x \in V(H) \setminus (S_{\mathcal{R}} \cup \mathcal{R})$, we have $\mu_c(x) + \tilde{\mu}(x) + \sigma'_B(x) \geq w(\overrightarrow{cx})$. Together with the last displayed inequality, we in fact have deduced that, for all $x \in V(H) \setminus \mathcal{R}$, we have

$$\mu_c(x) + \tilde{\mu}(x) + \sigma'_B(x) \geq w(\overrightarrow{cx}). \tag{141}$$

As $\mu_c \preceq \tilde{\sigma} - \sigma'_B$, using (S18) we can deduce that for each $x \in \mathcal{R}$,

$$\mu_c(x) + \tilde{\mu}(x) + \sigma'_B(x) \leq \tilde{\sigma}(x) + \tilde{\mu}(x) \leq w(\overrightarrow{cx}). \tag{142}$$

By $\tilde{\mu}_d \leq \tilde{\mu}$, (140) and (S14), we have that μ_c and $\tilde{\mu} - \tilde{\mu}_d$ are disjoint and thus $\mu_c + \tilde{\mu} - \tilde{\mu}_d$ is a fractional matching. We apply Lemma 8.3 (Combination)

object	$(H, \bar{w})$	c	$\mu_c + \tilde{\mu} - \tilde{\mu}_d$	γ_A
in place of	(H, w)	v	μ	γ

to obtain a γ_A-skew-matching σ'_A such that

$$\sigma'_A \trianglelefteq \mu_c + \tilde{\mu} - \tilde{\mu}_d, \tag{143}$$

with weight

$$W(\sigma'_A) \geq \deg_{\bar{w}}(c, \mu_c + \tilde{\mu} - \tilde{\mu}_d), \tag{144}$$

and

$$\text{the anchor } \mathcal{A}(\sigma'_A) \text{ fits in the } \bar{w}\text{-neighbourhood of } c. \tag{145}$$

Recall that $\sigma_\emptyset$ is the identically-zero matching. Owing to (133), (135), (143), (145), and the fact that $\bar{w}$ is the $\tilde{\mu}_d$-truncated weight obtained from w', we can apply Proposition 6.14 with

object	(H, w')	cd	$\tilde{\mu}_d$	$\mu_c + \tilde{\mu} - \tilde{\mu}_d$	$\bar{w}$	$\sigma_\emptyset$	$\tilde{\sigma}_B$	σ'_A	$\sigma_\emptyset$
in place of	(G, w)	uv	μ	$\bar{\mu}$	$\bar{w}$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

to obtain that $(\sigma'_A, \tilde{\sigma}_B)$ is a (γ_A, γ_B)-skew-matching pair in $H^{\leftrightarrow}$, anchored in $\overrightarrow{cd}$ (with respect to w') with $\sigma'_A + \tilde{\sigma}_B \trianglelefteq \mu_c + \tilde{\mu}$.

We now give further estimates on $W(\sigma'_A)$. We have

$$W(\sigma'_A) \overset{(144)}{\geq} \deg_{\bar{w}}(c, \mu_c + \tilde{\mu} - \tilde{\mu}_d) = \sum_{x \in N_{\bar{w}}(c)} \min\{\bar{w}(\overrightarrow{cx}), \mu_c(x) + \tilde{\mu}(x) - \tilde{\mu}_d(x)\}$$

$$\begin{aligned}
&= \sum_{x \in N_{w'}(c)} \min\{\overline{w}(\overrightarrow{cx}), \mu_c(x) + \tilde{\mu}(x) - \tilde{\mu}_d(x)\} \\
&\geq \sum_{x \in N_{w'}(c)} (\min\{w'(\overrightarrow{cx}), \mu_c(x) + \tilde{\mu}(x)\} - \tilde{\mu}_d(x)) \\
&= \deg_{w'}(c, \mu_c + \tilde{\mu}) - \sum_{x \in N_{w'}(c)} \tilde{\mu}_d(x) \overset{(138)}{\geq} \deg_{w'}(c, \mu_c + \tilde{\mu}) - W(\tilde{\mu}_d) \\
&\overset{(139)}{\geq} \deg_{w'}(c, \mu_c + \tilde{\mu}) - W(\tilde{\sigma}_B), \qquad (146)
\end{aligned}$$

here in the second line we used that $N_{\overline{w}}(c) \subseteq N_{w'}(c)$, and then that $\overline{w}(\overrightarrow{cx}) \geq w'(\overrightarrow{cx}) - \tilde{\mu}_d(x)$, both things holding by definition of $\overline{w}$.

Now we want to show that (σ'_A, σ_B) is a (γ_A, γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{cd}$ (with respect to w). For this, we need to verify properties (B1)–(B4). By (143), (140), (133), we can appeal to the definition of μ_c and σ'_B to deduce that σ'_A is disjoint from $\sigma'_B + \tilde{\sigma}_B = \sigma_B$, which gives (B1). As $(\sigma'_A, \tilde{\sigma}_B)$ is a (γ_A, γ_B)-skew pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{cd}$ (with respect to w'), we automatically have that the anchor $\mathcal{A}(\sigma'_A)$ fits in the w-neighbourhood of c, which gives (B2). To see (B3), we use $\sigma'_B + \tilde{\sigma}_B = \sigma_B$ together with (135) to get $\sigma^1_B(x) = (\sigma'_B)^1(x) + \tilde{\sigma}^1_B(x) \leq (\sigma'_B)^1(x) + w'(\overrightarrow{dx})$ for all $x \in V$. From the definition of w' we get that the last term is at most $w(\overrightarrow{dx})$, which gives (B3).

It remains to check (B4). Let $x \in N_w(d) \cap N_w(c)$ be arbitrary. We note first that

$$\sigma'^1_A(x) + \tilde{\sigma}^1_B(x) \leq \bar{w}(\overrightarrow{cx}) + \tilde{\mu}_d(x) \leq \max\{w'(\overrightarrow{cx}), \tilde{\mu}_d(x)\} \leq \max\{w'(\overrightarrow{cx}), w'(\overrightarrow{dx})\},$$

where in the first inequality we used that the anchor of σ'_A fits in the $\bar{w}$-neighbourhood of c together with (133), in the second we used that $\bar{w}$ is the $\bar{\mu}_d$-truncated weight obtained from w', and finally, we used the definition of $\tilde{\mu}_d$. Using this, we get

$$\sigma'^1_A(x) + \sigma^1_B(x) \leq \max\{w'(\overrightarrow{cx}), w'(\overrightarrow{dx})\} + \sigma'_B(x) \leq \max\{w(\overrightarrow{cx}), w(\overrightarrow{dx}), \sigma'_B(x)\},$$

where we used $\sigma'_B + \tilde{\sigma}_B = \sigma_B$ and the definition of w'. The last term is at most $\max\{w(\overrightarrow{cx}), w(\overrightarrow{dx})\}$, because $\sigma'_B(x) \leq \tilde{\sigma}(x) \leq w(\overrightarrow{cx})$ by (S15)–(S16). This implies (B4).

Step 3: Complementing σ_A. Summarising our efforts so far, we have obtained that (σ'_A, σ_B) is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{cd}$, with respect to w. By (137), we only need to focus on σ'_A. In this step, we use a greedy argument to complement the already-obtained γ_A-skew-matching σ'_A. By doing so, we will obtain a γ_A-skew-matching $\widehat{\sigma_A}$, so we can conclude our construction by setting $\sigma_A = \sigma'_A + \widehat{\sigma_A}$.

We proceed as follows. If $W(\sigma'_A) \geq a_1 + a_2$, then we are actually done (since then (σ'_A, σ_B) is the desired good matching), so we can finish by setting $\widehat{\sigma}_A \equiv 0$. Hence, we can assume that $W(\sigma'_A) < a_1 + a_2$.

We also note that from (133), (136), and (143), we get

$$\sigma'_A + \sigma_B - \sigma'_B \trianglelefteq \tilde{\mu} + \mu_c. \qquad (147)$$

Let

$$\kappa' := a_1 - \frac{W(\sigma'_A)}{1 + \gamma_A}.$$

We wish to apply Lemma 8.7 (First Greedy Lemma), with

object	(H, w)	σ'_A	σ_B	c	d	$\mathcal{R}$	S	κ'
in place of	(H, w)	σ_A	σ_B	u	v	V	U	κ

To do so, we verify the required properties (M1)–(M2). Indeed, we have (using $\sigma'_B \leq \tilde{\sigma}$ in the fourth line)

$$\deg_w(c, \mathcal{R}) = \deg_w(c) - \deg_w(c, V(H) \setminus \mathcal{R}) = \deg_w(c) - \sum_{y \notin \mathcal{R}} w(\overrightarrow{cy})$$

$$
\begin{aligned}
&\overset{(141)}{=} \deg_w(c) - \sum_{y\notin\mathcal{R}} \min\{w(\overrightarrow{cy}), \tilde{\mu}(y) + \sigma'_B(y) + \mu_c(y)\}\\
&= \deg_w(c) - \deg_w(c, \tilde{\mu} + \sigma'_B + \mu_c) + \sum_{y\in\mathcal{R}} \min\{w(\overrightarrow{cy}), \tilde{\mu}(y) + \sigma'_B(y) + \mu_c(y)\}\\
&\geq k - \deg_w(c, \tilde{\mu} + \tilde{\sigma} + \mu_c) + \sum_{y\in\mathcal{R}} \min\{w(\overrightarrow{cy}), \tilde{\mu}(y) + \sigma'_B(y) + \mu_c(y)\}\\
&\overset{(142)}{=} k - \deg_w(c, \tilde{\mu} + \tilde{\sigma} + \mu_c) + \sum_{y\in\mathcal{R}} (\tilde{\mu}(y) + \sigma'_B(y) + \mu_c(y))\\
&\overset{(144)}{\geq} (a_1 + a_2) - W(\sigma'_A) + \sum_{y\in\mathcal{R}} (\tilde{\mu}(y) + \sigma'_B(y) + \mu_c(y))\\
&\overset{(147)}{\geq} a_1 - \frac{W(\sigma'_A)}{1+\gamma_A} + \sum_{y\in\mathcal{R}} \left(\sigma'_A(y) + \sigma_B(y)\right),
\end{aligned}
$$

which verifies (M1). To see (M2), we need to verify, for any $x \in \mathcal{R}$, that we have

$$
\deg_w(x, S) \geq \sum_{y\in S} \left(\sigma_B(y) + \sigma'_A(y)\right) + \left(a_2 - \frac{\gamma_A W(\sigma'_A)}{1+\gamma_A}\right). \tag{148}
$$

Indeed, let $x \in \mathcal{R}$ be arbitrary. We have

$$
\begin{aligned}
\deg_w(x, S) = \deg_w(x) \geq \frac{k}{2} &\overset{(98)}{>} b_1 + \max\{a_1, a_2\} \overset{(137)}{=} \frac{1}{1+\gamma_B} W(\sigma_B) + \max\{a_1, a_2\}\\
&\geq \sum_{y\in S} \sigma_B(y) + \max\{a_1, a_2\} = \sum_{y\in S} \sigma_B(y) + (\max\{a_1, a_2\} - a_2) + a_2.
\end{aligned}
$$

We also note that

$$
\begin{aligned}
\sum_{y\in S} \sigma'_A(y) &\leq \frac{\gamma_A}{1+\gamma_A} \sum_{(u,y),y\in S} \sigma'_A(\overrightarrow{uy}) + \frac{1}{1+\gamma_A} \sum_{(y,u),y\in S} \sigma'_A(\overrightarrow{yu})\\
&\leq \frac{\gamma_A}{1+\gamma_A} W(\sigma'_A) + \frac{1-\gamma_A}{1+\gamma_A} \sum_{(y,u),y\in S} \sigma'_A(\overrightarrow{yu}) \leq \frac{1}{1+\gamma_A} W(\sigma'_A),
\end{aligned}
$$

so to reach (148) it suffices to show that

$$
\frac{1}{1+\gamma_A} W(\sigma'_A) \leq \frac{\gamma_A}{1+\gamma_A} W(\sigma'_A) + (\max\{a_1, a_2\} - a_2) \tag{149}
$$

We do this as follows. Suppose first that $a_2 \geq a_1$, so $\gamma_A \geq 1$ and $\max\{a_1, a_2\} - a_2 = 0$. In this case, (149) follows immediately from $\gamma_A \geq 1$. Thus we can assume that $a_2 < a_1$, so $\gamma_A < 1$, and $\max\{a_1, a_2\} - a_2 = a_1 - a_2$. Recall that we assume $W(\sigma'_A) < a_1 + a_2$, so we have

$$
\frac{1}{1+\gamma_A} W(\sigma'_A) \leq \frac{\gamma_A}{1+\gamma_A} W(\sigma'_A) + \frac{1-\gamma_A}{1+\gamma_A} (a_1 + a_2) = \frac{\gamma_A}{1+\gamma_A} W(\sigma'_A) + (a_2 - a_1),
$$

which again gives (149). Thus indeed (148) holds, and we can apply Lemma 8.7 with the above-mentioned desired parameters.

The outcome of Lemma 8.7 is a γ_A-skew-matching $\widehat{\sigma}_A$ in $H^{\leftrightarrow}$ of weight $W(\widehat{\sigma}_A) \geq a_1 + a_2 - W(\sigma'_A)$ such that $(\sigma'_A + \widehat{\sigma}_A, \sigma_B)$ is a (γ_A, γ_B)-skew-matching anchored in $\overrightarrow{cd}$. Set $\sigma_A := \sigma'_A + \widehat{\sigma}_A$. We have

$$
W(\sigma_A) = W(\sigma'_A) + W(\widehat{\sigma}_A) \geq W(\sigma'_A) + a_1 + a_2 - W(\sigma'_A) = a_1 + a_2,
$$

which, together with (137), implies we have found our desired good matching. □

In the following, we may assume that

$$b_2 > k/2, \tag{150}$$

and thus

$$a_1 + a_2 + b_1 < k/2. \tag{151}$$

9.6. **Proof of Proposition 4.2: The large $S_{\mathcal{R}}$ case.** In this case we will argue that if the neighbourhoods of two elements of $\mathcal{R}$ do not intersect much, the set $S_{\mathcal{R}}$ will be unusually large, allowing us to conclude.

Claim 9.9. *If there are $d_1, d_2 \in \mathcal{R}$ such that*

$$\sum_{u \in N_w(d_2)} w(\overrightarrow{d_1 u}) \leq b_1, \tag{152}$$

then $H^{\leftrightarrow}$ has a good matching.

In the proof of Claim 9.9 we will not use the GE pair $(\tilde{\mu}, \tilde{\sigma})$, but only the c-optimal fractional matching μ.

Proof of Claim 9.9. Suppose d_1, d_2 are as in the statement. The strategy here is to first build σ_A, which is anchored in S, and its anchor w-fits in the neighbourhood of $d_1 \in \mathcal{R}$. Then we will set up two auxiliary matchings μ_A and μ'. The third step is to use the condition (152) to deduce that there is enough space in $S_{\mathcal{R}}$ to build σ_B, anchored in $\mathcal{R}$, that also w-fits in the neighbourhood of c. Both matchings will be found by invoking Lemma 8.1 (Extending-out).

Step 1: Building σ_A. Let $d_1, d_2 \in \mathcal{R}$ be such that they satisfy (152). By Observation 7.13, we have $N_w(d_1) \subseteq S_{\mathcal{R}} \subseteq S$. By (S7), each $x \in N_w(d_i)$ is covered by μ. In particular, $1 = \mu(x) \geq w(\overrightarrow{d_i x})$ holds. Now, let $\mu_{d_1} \leq \mu$ be a maximal fractional matching such that for any $u \in N_w(d_1)$, we have $\mu_{d_1}(u) \leq w(\overrightarrow{d_1 u})$. This implies that for any such u, in fact we have $\mu_{d_1}(u) = w(\overrightarrow{d_1 u})$, so it follows that

$$W(\mu_{d_1}) = \deg_w(d_1, \mu_{d_1}) = \deg_w(d_1) \geq \frac{k}{2}. \tag{153}$$

Now we apply Lemma 8.1 (Extending-out), with

object	H	$N_w(d_1)$	$V(H) \setminus S$	μ_{d_1}	γ_A
in place of	H	U	V	μ	γ

and deduce the existence of a γ_A-skew-matching $\sigma_A \trianglelefteq \mu_{d_1}$ in $H^{\leftrightarrow}$ of weight

$$W(\sigma_A) = (1 + \min\{\gamma_A, \gamma_A^{-1}\}) W(\mu_{d_1}) \geq W(\mu_{d_1}) \overset{(153)}{\geq} \frac{k}{2} \overset{(151)}{\geq} a_1 + a_2,$$

and such that its anchor $\mathcal{A}(\sigma_A)$ is contained in $N_w(d_1) \subseteq S$. By decreasing the weight in the edges of σ_A if necessary, we can assume that in fact

$$W(\sigma_A) = a_1 + a_2. \tag{154}$$

By the construction of μ_{d_1} and the fact that $\sigma_A \trianglelefteq \mu_{d_1}$, the anchor $\mathcal{A}(\sigma_A)$ fits in the w-neighbourhood of d_1.

Step 2: Constructing the auxiliary fractional matchings. Now we will set up several auxiliary fractional matchings. Define μ_A as a fractional matching such that $\mu_A \leq \mu_{d_1}$ and minimum such that $\sigma_A \trianglelefteq \mu_A$. Note that, by the fact that the anchor of σ_A is contained in S, then the only directed edges with non-zero weight in σ_A are of the form (x, y) with $x \in S$ and $y \notin S$. Then the minimality of μ_A implies that for such a pair $xy \in E(H)$,

$$\mu_A(xy) = \frac{1}{1 + \gamma_A} \max\{\sigma_A(\overrightarrow{xy}) + \gamma_A \sigma_A(\overrightarrow{yx}), \sigma_A(\overrightarrow{yx}) + \gamma_A \sigma_A(\overrightarrow{xy})\}$$

$$= \frac{1}{1+\gamma_A} \max\{\sigma_A(\overrightarrow{xy}), \gamma_A \sigma_A(\overrightarrow{xy})\}$$
$$= \sigma_A(\overrightarrow{xy}) \frac{\max\{1,\gamma_A\}}{1+\gamma_A} = \sigma_A(\overrightarrow{xy}) \frac{\max\{a_1,a_2\}}{a_1+a_2},$$

which implies that

$$W(\mu_A) = W(\sigma_A) \frac{\max\{a_1,a_2\}}{a_1+a_2} = \max\{a_1,a_2\}.$$

Therefore, we conclude that

$$\deg_w(d_1, \mu_A) \leq W(\mu_A) = \max\{a_1, a_2\} \leq a_1 + a_2. \tag{155}$$

Let (H, w') be the μ_A-truncated weighted graph obtained from (H, w). Now we will define another fractional matching μ', by

$$\mu'(xy) = \begin{cases} \mu(xy) - \mu_A(xy) & xy \in H[\mathcal{R}, S_{\mathcal{R}}] \\ 0 & \text{otherwise.} \end{cases}$$

In words, μ' is equal to $\mu - \mu_A$ whenever the corresponding edge touches $\mathcal{R}$, and zero otherwise. Note that $\mu' \leq \mu - \mu_A \leq \mu$.

To understand μ' we recall a few consequences of the definition of reachable vertices (Definition 7.12). By Observation 7.15, together with the definition of w', we get that

$$w'(\overrightarrow{cy}) \geq w(\overrightarrow{cy}) - \mu_A(y) \geq \mu'(y) \text{ for all } y \in \mathcal{R}, \tag{156}$$

and recall from Observation 7.16 that we have $\sum_{x \in S_{\mathcal{R}}} \mu(x) = \sum_{x \in \mathcal{R}} \mu(x)$.

Using this, we obtain

$$\begin{aligned}
\deg_{w'}(c, \mu') &\geq \sum_{y \in \mathcal{R}} \min\{w'(\overrightarrow{cy}), \mu'(y)\} \overset{(156)}{\geq} \sum_{y \in \mathcal{R}} \mu'(y) \\
&= \sum_{y \in \mathcal{R}} (\mu(y) - \mu_A(y)) \\
&\geq \sum_{y \in \mathcal{R}} \mu(y) - W(\mu_A) = \sum_{x \in S_{\mathcal{R}}} \mu(x) - W(\mu_A) \\
&\overset{(S3)}{=} |S_{\mathcal{R}}| - W(\mu_A) \overset{(155)}{\geq} |S_{\mathcal{R}}| - (a_1 + a_2) \\
&\geq |N_w(d_1) \cup N_w(d_2)| - (a_1 + a_2) \\
&= |N_w(d_2)| + |N_w(d_1) \setminus N_w(d_2)| - (a_1 + a_2) \\
&\geq \deg_w(d_2) + \deg_w(d_1, S_{\mathcal{R}} \setminus N_w(d_2)) - (a_1 + a_2) \\
&= \deg_w(d_2) + \deg_w(d_1) - \deg_w(d_1, N_w(d_2)) - (a_1 + a_2) \\
&\overset{(152)}{\geq} \deg_w(d_1) + \deg_w(d_2) - (a_1 + a_2) - b_1 \\
&\geq \frac{k}{2} + \frac{k}{2} - (a_1 + a_2) - b_1 = b_2 = \max\{b_1, b_2\}.
\end{aligned}$$

Step 3: Building σ_B. Now we will use Lemma 8.1 (Extending out), with

object	H	$\mathcal{R}$	$S_{\mathcal{R}}$	μ'	γ_B
in place of	H	U	V	μ	γ

which implies the existence of a γ_B-skew-matching σ_B such that $\sigma_B \trianglelefteq \mu'$, of weight

$$W(\sigma_B) = b_1 + b_2 \leq (1 + \min\{\gamma_B, \gamma_B^{-1}\}) \deg_{w'}(c, \mu'),$$

and with its anchor $\mathcal{A}(\sigma_B)$ contained in $\mathcal{R}$. Thanks to this, together with the fact that $\sigma_B \trianglelefteq \mu'$ and the definition of (H, w'), we can infer that the anchor $\mathcal{A}(\sigma_B)$ fits in the w'-neighbourhood of c. By Proposition 6.14

object	(H,w)	d_1c	μ_A	μ'	w'	σ_A	$\sigma_\emptyset$	$\sigma_\emptyset$	σ_B
in place of	(G,w)	uv	μ	$\bar{\mu}$	$\bar{w}$	σ_A	σ_B	$\bar{\sigma}_A$	$\bar{\sigma}_B$

we obtain that (σ_A, σ_B) is a (γ_A, γ_B)-skew pair anchored in $\overrightarrow{d_1c}$. □

In the rest of the proof, we may assume that for any $d, d' \in \mathcal{R}$ we have that

$$\sum_{u \in N_w(d')} w(du) > b_1, \tag{157}$$

as otherwise we would be done by Claim 9.9.

9.7. **Proof of Proposition 4.2: The flabellum case.** Before proceeding with the proof, let us summarise our strategy here. As b_2 is so large (by inequality (150)), the biggest challenge is to find enough space for a γ_B-skew-matching σ_B of weight $b_1 + b_2$. For any $d \in \mathcal{R}$, we can easily accommodate part (at least half) of such a γ_B-skew-matching using edges between $\mathcal{R}$ and $N_H(d)$. This is because inequality (157) makes it actually very easy to find space for its anchor in $S_{\mathcal{R}} \subseteq S$.

Unfortunately, $\mathcal{R}$ might be too small to host the whole *tail* (i.e., the non-anchor part) of σ_B. To solve this problem, we shall try to extend the space where to place the *tail* of σ_B. To this aim, we define below an auxiliary set X_d. Here we specifically use the fact that we aim to use it only for at most half of the *tail* of σ_B. Claim 9.10 then says that if $X_d \cup \mathcal{R}$ is large enough to fit the whole tail of σ_B, then we can find the required (γ_A, γ_B)-skew-matching in H.

Referencing the 'flabellum structure' described informally at the beginning of the section when we sketched the proof; the small 'base' of the flabellum will be contained in $S_{\mathcal{R}}$, and the large part will be $X_d \cup \mathcal{R}$. These two structures will be used to host σ_B; and σ_A will be built after that, in the leftover.

Here is the key definition for this step. For any $d \in \mathcal{R}$, set

$$X_d := \left\{ u \in N_w(c) \setminus (\mathcal{R} \cup N_w(d)) \ : \ \sum_{x \in N_w(u)} w(\overrightarrow{dx}) \geq \frac{b_1}{2} \right\}.$$

Claim 9.10. *Suppose there is $d \in \tilde{\mathcal{R}}$ such that $|\mathcal{R} \cup X_d| \geq b_2$. Then $H^{\leftrightarrow}$ admits a good matching.*

Proof of Claim 9.10. The proof has three steps. Choosing d as in the statement, in the first step, we build a γ_B-skew-matching σ_B^* completely outside $\mathcal{R}$, with anchor in $N_w(d)$ and tail in X_d. This matching will contain only a fraction of the weight of the required γ_B-skew-matching. In the second step, we build a γ_B-skew-matching σ'_B which completes the required weight, and σ'_B will be such that its tail is in $\mathcal{R}$. In the last step, we build the γ_A-skew-matching σ_A.

Recall that $(\tilde{\mu}, \tilde{\sigma})$ is the $(H, w, S, M, c, \mu, \gamma_B)$-GE pair, chosen on §9.4 ('The skew-matching cover case'), and it satisfies (S15)–(S22). Let $\mu_d \leq \tilde{\mu}$ and $\sigma_d \leq \tilde{\sigma}$ be defined as in Claim 9.7, i.e. discarding from $\tilde{\mu}$ and $\tilde{\sigma}$ all support edges not intersecting $N_w(d)$. We define $\mathcal{R}_{\text{cov}} = \mathcal{R} \cap V(\mu_d + \sigma_d)$. Observe that

$$V(\mu_d + \sigma_d) = N_w(d) \cup \mathcal{R}_{\text{cov}} \tag{158}$$

Indeed, by (S20) we clearly have $V(\mu_d + \sigma_d) \cap S = N_w(d)$. Since $\sigma_d \leq \tilde{\sigma}$, by (S17), we have $V(\sigma_d) \setminus \mathcal{A}(\sigma_d) \subseteq \mathcal{R}$; and since $\mu_d \leq \tilde{\mu}$ by Observation 7.16 we have $V(\mu_d) \setminus S \subseteq \mathcal{R}$, which implies the equality above.

Step 1: Starting building γ_B-skew-matching using $X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\text{cov}})$ for the tail. Let d be as in the statement. The goal of this step is to build a γ_B-skew-matching σ_B^*. If $b_2 < |\mathcal{R}_{\text{cov}}|$, we simply set $\sigma_B^* \equiv \sigma_\emptyset$ to be the empty skew-matching, and finalise the construction. Otherwise, we have that $b_2 \geq |\mathcal{R}_{\text{cov}}|$. Let

$$\kappa' := \frac{b_2 - |\mathcal{R}_{\text{cov}}|}{\gamma_B}.$$

We wish to apply Lemma 8.9 (Third Greedy Lemma) with

object	$X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})$	$N_w(d)$	d	c	γ_B	γ_A	$(\sigma_\emptyset, \sigma_\emptyset)$	κ'
in place of	U	V	u	v	γ_A	γ_B	(σ_A, σ_B)	κ

Observe that $N_w(d) \subseteq S$, so $\mathcal{R}$ and $N_w(d)$ are disjoint; also X_d is disjoint from $N_w(d)$ by definition. Hence $X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})$ is disjoint from $N_w(d)$. We need to check (O1)–(O2). With our choice of parameters, (O1) translates to

$$|X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})| = |X_d \cup \mathcal{R}| - |\mathcal{R}_{\mathrm{cov}}| \geq b_2 - |\mathcal{R}_{\mathrm{cov}}|,$$

which follows from (158), the fact that X_d is disjoint from $\mathcal{R} \cup N_w(d)$, and our main assumption $|\mathcal{R} \cup X_d| \geq b_2$.

To see (O2), we gather one extra inequality. From (158) we can deduce that σ_d is supported between $N_w(d)$ and $\mathcal{R}$, with the anchor in $N_w(d)$. Since $\gamma_B > 1$ by (S12), we must have that $\sum_{x \in N(d)} \sigma_d(x) \leq \sum_{x \in \mathcal{R}} \sigma_d(x)$. Similarly, using $\mu_d \leq \tilde{\mu}$ and Observation 7.16 we have $\sum_{x \in N(d)} \mu_d(x) \leq \sum_{x \in \mathcal{R}} \mu_d(x)$. Combining these two bounds, we obtain

$$\frac{k}{2} \leq |N(d)| \leq \sum_{x \in N(d)} (\sigma_d(x) + \mu_d(x)) \leq \sum_{x \in \mathcal{R}} (\sigma_d(x) + \mu_d(x)) \leq |\mathcal{R}_{\mathrm{cov}}| \tag{159}$$

Next, note that for any $y \in X_d$ we have (using the definition of X_d in the first inequality),

$$\begin{aligned} \deg_w(d, N_w(y) \cap N_w(d)) = \sum_{x \in N_w(y) \cap N_w(d)} w(\overrightarrow{dx}) = \sum_{x \in N_w(y)} w(\overrightarrow{dx}) \geq \frac{b_1}{2} = \frac{b_2}{2\gamma_B} \\ \geq \frac{b_2 - k/2}{\gamma_B} \overset{(159)}{\geq} \frac{b_2 - |\mathcal{R}_{\mathrm{cov}}|}{\gamma_B}, \end{aligned}$$

giving the required inequality in this case. Finally, for $y \in \mathcal{R}$, we have

$$\deg_w(d, N_w(y) \cap N_w(d)) = \sum_{x \in N_w(y) \cap N_w(d)} w(\overrightarrow{dx}) = \sum_{x \in N_w(y)} w(\overrightarrow{dx}) \overset{(157)}{\geq} b_1,$$

which allows us to conclude as before. This gives the desired inequality for all $y \in X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})$, so we obtain (O2).

As a consequence of the application of Lemma 8.9, we deduce the existence of a γ_B-skew-matching σ_B^*, of weight

$$W(\sigma_B^*) = (1 + \gamma_B)\frac{b_2 - |\mathcal{R}_{\mathrm{cov}}|}{\gamma_B} \leq (1 + \gamma_B)\frac{b_1}{2},$$

with its anchor $\mathcal{A}(\sigma_B^*)$ fitting in the w-neighbourhood of d and the support edges of σ_B^* lying in $H[N_w(d), X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})]$. Therefore, only the anchor $\mathcal{A}(\sigma_B^*)$ intersects $N_w(d)$, and σ_B^* does not intersect $\mathcal{R}_{\mathrm{cov}}$.

Note that, no matter if $b_2 < |\mathcal{R}_{\mathrm{cov}}|$ or not, in both cases our construction of σ_B^* ensures that we have

$$W(\sigma_B^*) = (1 + \gamma_B)\frac{\max\{0, b_2 - |\mathcal{R}_{\mathrm{cov}}|\}}{\gamma_B}, \tag{160}$$

and ensures that all vertices from $\mathcal{R}_{\mathrm{cov}}$ do not receive any weight from σ_B^*.

Step 2: Completing the γ_B-skew-matching using $\mathcal{R}_{\mathrm{cov}}$ for the tail. Let $\mathcal{R}_d \subseteq \mathcal{R}_{\mathrm{cov}}$ be of size $|\mathcal{R}_d| = \min\{|\mathcal{R}_{\mathrm{cov}}|, b_2\}$. We wish to apply Lemma 8.9 (Third Greedy Lemma) again, this time with

object	$\mathcal{R}_d$	$N_w(d)$	d	c	γ_B	γ_A	$(\sigma_B^*, \sigma_\emptyset)$	$|\mathcal{R}_d|/\gamma_B$
in place of	U	V	u	v	γ_A	γ_B	(σ_A, σ_B)	κ

Again, we need to check (O1)–(O2). With our choice of parameters, (O1) translates to

$$|\mathcal{R}_d| \geq |\mathcal{R}_d| + \sum_{y\in\mathcal{R}_d} \sigma_B^*(y),$$

which indeed holds, since in any case our construction ensures that σ_B^* has zero weight in all vertices of $\mathcal{R}_d \subseteq \mathcal{R}_{\mathrm{cov}}$, so the second sum is identically zero.

To verify (O2), we note that for every $y \in \mathcal{R}_d$ we have

$$\begin{aligned}\deg_w(d, N_w(y)\cap N_w(d)) &= \sum_{x\in N_w(y)} w(\overrightarrow{dx}) \overset{(157)}{>} b_1 = \frac{b_2}{\gamma_B} = \frac{|\mathcal{R}_d| + (b_2 - |\mathcal{R}_d|)}{\gamma_B}\\ &= \frac{|\mathcal{R}_d|}{\gamma_B} + \frac{\max\{0, b_2 - |\mathcal{R}_{\mathrm{cov}}|\}}{\gamma_B}\\ &\overset{(160)}{=} \frac{|\mathcal{R}_d|}{\gamma_B} + \frac{W(\sigma_B^*)}{1+\gamma_B} = \frac{|\mathcal{R}_d|}{\gamma_B} + \sum_{x\in N_w(d)} \sigma_B^*(x),\end{aligned}$$

where the last equality follows from the fact the anchor of σ_B^* is contained in $N_w(d)$.

Thanks to Lemma 8.9, we obtain a γ_B-skew-matching σ_B' of weight

$$W(\sigma_B') = (1+\gamma_B)\frac{|\mathcal{R}_d|}{\gamma_B} \leq \frac{1+\gamma_B}{\gamma_B}|\mathcal{R}_{\mathrm{cov}}| \tag{161}$$

such that $\sigma_B := \sigma_B^* + \sigma_B'$ is a γ_B-skew-matching in $H^{\leftrightarrow}$, with its anchor $\mathcal{A}(\sigma_B)$ fitting in the w-neighbourhood of d. Note that we have

$$W(\sigma_B) = W(\sigma_B^*) + W(\sigma_B') = (1+\gamma_B)\frac{b_2}{\gamma_B} = b_1 + b_2. \tag{162}$$

Step 3: Building the γ_A-skew-matching. In order to build the γ_A-skew-matching, we shall heavily leverage the 'separation' properties obtained in Claim 9.6 and Claim 9.7. Indeed, we shall use the fact that there is a "separation" between the (fractional, and skew) matching covering the neighbourhood $N_w(d)$ and the matching built on top of the rest of $S_{\mathcal{R}}$. In this way we can guarantee that the γ_A-skew-matching we build greedily from its anchor can avoid the already-built γ_B-skew-matching.

Let $S_{\tilde\sigma} := \{x \in S : \tilde\sigma(x) = w(\overrightarrow{cx})\}$. Observe that $N_w(d) \subseteq S_{\tilde\sigma}$, thanks to Claim 9.6 and $d \in \tilde{\mathcal{R}}$.

Define

$$\begin{aligned}V' &= N_w(c) \setminus (\mathcal{R}_d \cup S_{\tilde\sigma}),\\ U' &= V(H) \setminus (\mathcal{R}_{\mathrm{cov}} \cup V').\end{aligned}$$

Clearly, U' and V' are disjoint. Since $\mathcal{R}_d \subseteq \mathcal{R}_{\mathrm{cov}} \subseteq \mathcal{R} \subseteq N_w(c)$, we have

$$U' \cup V' = V(H) \setminus \mathcal{R}_d. \tag{163}$$

Our goal now is to apply Lemma 8.8 (Second Greedy Lemma) with

object	V'	U'	$\overrightarrow{cd}$	$(\sigma_\emptyset, \sigma_B)$	a_1
in place of	V	U	$\overrightarrow{uv}$	(σ_A, σ_B)	κ

Let us verify the required (N1)–(N2) conditions hold. By definition, we have for all $x \in S_{\tilde\sigma}$ that $w(\overrightarrow{cx}) = \tilde\sigma(x)$. Using (S15) we also recall that the anchor of $\tilde\sigma$ is in $S_{\mathcal{R}}$, which contains $S_{\tilde\sigma}$. Therefore,

$$\deg_w(c, S_{\tilde\sigma}) = \sum_{x\in S_{\tilde\sigma}} w(\overrightarrow{cx}) = \sum_{x\in S_{\tilde\sigma}} \tilde\sigma(x) \leq \frac{W(\tilde\sigma)}{1+\gamma_B} \overset{(106)}{<} b_1. \tag{164}$$

Using this, we have

$$\begin{aligned}\deg_w(c, N_w(c)\setminus(\mathcal{R}_d \cup S_{\tilde\sigma})) &\geq \deg_w(c) - \deg_w(c, S_{\tilde\sigma}) - |\mathcal{R}_d|\\ &\overset{(164)}{>} \deg_w(c) - b_1 - |\mathcal{R}_d|\end{aligned}$$

$$
\begin{aligned}
&\overset{(161)}{\geq} k - b_1 - \frac{\gamma_B W(\sigma'_B)}{1+\gamma_B}\\
&\geq a_1 + b_2 - \frac{\gamma_B W(\sigma'_B)}{1+\gamma_B}\\
&\overset{(162)}{=} a_1 + b_2 - \frac{\gamma_B}{1+\gamma_B}\left(b_1 + b_2 - W(\sigma^*_B)\right)\\
&= a_1 + \frac{\gamma_B}{1+\gamma_B} W(\sigma^*_B)\\
&= a_1 + \sum_{x \in X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}})} \sigma^*_B(x)\\
&= a_1 + \sum_{x \in N_w(c) \setminus (\mathcal{R}_d \cup S_{\tilde{\sigma}})} \sigma^*_B(x)\\
&= a_1 + \sum_{x \in N_w(c) \setminus (\mathcal{R}_d \cup S_{\tilde{\sigma}})} \sigma_B(x)
\end{aligned}
$$

where the third to last line follows since, by construction, σ^*_B only puts weight from the tail in X_d, i.e. $\sigma^*_B(x) = (\sigma^*_B)^2(x)$ for $x \in X_d$; the second to last line follows from the fact that $V(\sigma^*_B) \cap \mathcal{R}_d = \emptyset$ and $X_d \cup (\mathcal{R} \setminus \mathcal{R}_{\mathrm{cov}}) \subseteq N_w(c) \setminus S_{\tilde{\sigma}}$; and the last line follows from the fact that $\sigma_B - \sigma^*_B = \sigma'_B$ does not have any weight outside $\mathcal{R}_d \cup S_{\tilde{\sigma}}$. This gives (N1).

To see that (N2) holds, we proceed as follows. First, we have

$$
\sum_{y \in \mathcal{R}_{\mathrm{cov}}} \sigma_B(y) = \sum_{y \in \mathcal{R}} \sigma'_B(y) = \frac{\gamma_B}{1+\gamma_B} W(\sigma'_B) \overset{(161)}{=} |\mathcal{R}_d| = \min\{|\mathcal{R}_{\mathrm{cov}}|, b_2\} \geq \frac{k}{2}, \tag{165}
$$

where the last inequality holds by (159) and (150). Next, we note that, since $\mathcal{R}$ consists of singletons and $\mathcal{R}_d \subseteq \mathcal{R}$, every neighbour of $\mathcal{R}_d$ must be in S. Note that Claim 9.7 forbids edges from $\mathcal{R}_{\mathrm{cov}}$ to $S \setminus S_{\tilde{\sigma}}$, and $\mathcal{R}_d \subseteq \mathcal{R}_{\mathrm{cov}}$. We conclude that there is no edge between the set $N_w(c) \setminus (\mathcal{R}_d \cup S_{\tilde{\sigma}})$ and the set $\mathcal{R}_d$. Therefore, for any $x \in N_w(c) \setminus (\mathcal{R}_d \cup S_{\tilde{\sigma}})$, we have

$$
\begin{aligned}
|N_w(x) \cap (V(H) \setminus \mathcal{R}_d)| &\geq \deg_w(x, V(H) \setminus \mathcal{R}_d)\\
&= \deg_w(x) \geq \frac{k}{2} \overset{(165)}{>} k - \sum_{y \in \mathcal{R}_{\mathrm{cov}}} \sigma_B(y)\\
&= (a_1 + a_2) + W(\sigma_B) - \sum_{y \in \mathcal{R}_{\mathrm{cov}}} \sigma_B(y)\\
&= (a_1 + a_2) + \sum_{y \in V(H) \setminus \mathcal{R}_{\mathrm{cov}}} \sigma_B(y)\\
&= (a_1 + a_2) + \sum_{y \in V(H) \setminus \mathcal{R}_d} \sigma_B(y),
\end{aligned}
$$

where the last line follows from the fact that $V(\sigma_B) \cap \mathcal{R}_{\mathrm{cov}} \subseteq \mathcal{R}_d$. Together with (163), this gives (N2).

As a consequence of the application of Lemma 8.8, we obtain a γ_A-skew-matching σ_A of weight $W(\sigma_A) = a_1 + a_2$ such that the pair (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair in $H^{\leftrightarrow}$ anchored in $\overrightarrow{cd}$. Together with (162), this finishes the proof of Claim 9.10. □

9.8. **Proof of Proposition 4.2: The avoiding case.** For the rest of the proof, we may assume that for every $d \in \tilde{\mathcal{R}}$, we have

$$
|\mathcal{R} \cup X_d| < b_2, \tag{166}
$$

as otherwise we would be done by Claim 9.10.

Now we will finalise the proof of Proposition 4.2 by constructing a good matching. The construction will have four main steps. First, we select a suitable region in $S_{\mathcal{R}} \cup \mathcal{R}$

that is “large enough”. After picking this special region, we pick the anchor d for the γ_A-skew-matching to be a vertex “avoiding” this region. In the second step, we shall construct a partial γ_B-skew-matching using this region and estimate its size w.r.t. the degree of d into this region. In the third step, we shall construct the γ_A-skew-matching by leveraging the Separation Claims (Claim 9.6 and Claim 9.7) to avoid the large, already built, part of the γ_B-skew-matching. In the last step, we complement the already built γ_B-skew-matching to obtain the full pair. This last step relies on the same avoiding strategy as in Step 3.

Step 1: Defining a separating region and finding d. Slightly counter-intuitively, we first pick an auxiliary vertex d'. This vertex and, in particular, its large neighbourhood in $S_{\mathcal{R}}$, will allow us to build a pair $(\mu_{d'}, \sigma_{d'})$ which we shall fill by a γ_B-skew-matching in the second step.

Pick any $d' \in \tilde{\mathcal{R}}$. Let $\mu_{d'} \le \tilde{\mu}$ and $\sigma_{d'} \le \tilde{\sigma}$ be as in Claim 9.7, i.e. they are obtained from μ and σ by setting the weight to zero in all edges not touching $N_w(d')$, and keeping every other edge intact.

We recall, from (S15)–(S17), that $\tilde{\sigma}$ is supported only in directed edges (u, v) with $u \in N_w(c) \cap S$ and $v \in \mathcal{R}$. Together with (106), we have

$$b_1 > \frac{W(\tilde{\sigma})}{1+\gamma_B} = \sum_{x \in N_w(c) \cap S} \tilde{\sigma}(x). \tag{167}$$

Also, observe that by Claim 9.6, and by the definition of $\sigma_{d'}$, we have

$$\deg_w(c, N_w(d')) = \sum_{x \in N_w(d')} w(\overrightarrow{cx}) = \sum_{x \in N_w(d') \cap N_w(c)} \tilde{\sigma}(x) = \frac{W(\sigma_{d'})}{1+\gamma_B}. \tag{168}$$

Thus,

$$\begin{aligned}
\sum_{x \in N_w(c) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d'))} w(\overrightarrow{cx}) &= \deg_w(c, V(H) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d'))) \\
&\ge \deg_w(c) - |\mathcal{R} \cup X_{d'}| - \deg_w(c, N_w(d')) \\
&\ge k - |\mathcal{R} \cup X_{d'}| - \deg_w(c, N_w(d')) \\
&\overset{(166)}{>} k - b_2 - \deg_w(c, N_w(d')) \\
&\overset{(168)}{=} k - b_2 - \sum_{x \in N_w(d') \cap N_w(c)} \tilde{\sigma}(x) \\
&\ge b_1 - \sum_{x \in N_w(d') \cap N_w(c)} \tilde{\sigma}(x) \\
&\overset{(167)}{>} \sum_{x \in N_w(c) \cap S} \tilde{\sigma}(x) - \sum_{x \in N_w(d') \cap N_w(c)} \tilde{\sigma}(x) \\
&= \sum_{x \in (N_w(c) \setminus N_w(d')) \cap S} \tilde{\sigma}(x) \\
&\ge \sum_{x \in N_w(c) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d'))} \tilde{\sigma}(x),
\end{aligned}$$

where the last inequality follows because every x counted in the last sum and not in the previous sum must belong to $N_w(c) \setminus (S \cup \mathcal{R})$, and for those vertices we have $\tilde{\sigma}(x) = 0$. This chain of inequalities implies that there is at least one vertex $d \in N_w(c) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d'))$ such that $w(\overrightarrow{cd}) > \tilde{\sigma}(d)$. Pick such a d such that $\deg_w(d, N_w(d'))$ is maximum.

Now, since $d \in N_w(c) \setminus (\mathcal{R} \cup N_w(d') \cup X_{d'})$, from the definition of $X_{d'}$ we have

$$\deg_w(d', N_w(d)) = \sum_{x \in N_w(d)} w(\overrightarrow{d'x}) < \frac{b_1}{2}.$$

Then,

$$\begin{aligned} W(\mu_{d'}) + \frac{W(\sigma'_d)}{1+\gamma_B} &= \sum_{x \in N_w(d')} (\mu_{d'}(x) + \sigma_{d'}(x)) = \sum_{x \in N_w(d')} (\tilde{\mu}(x) + \tilde{\sigma}(x)) \\ &\overset{(S20)}{=} |N_w(d')| \qquad (169) \\ &\geq \deg_w(d, N(d')) + \deg_w(d') - \deg_w(d', N(d)) \\ &> \deg_w(d, N(d')) + \frac{k}{2} - \frac{b_1}{2}. \qquad (170) \end{aligned}$$

Step 2: Starting to build the γ_B-skew-matching. As explained earlier, we build a γ_B-skew-matching σ_B^* in $V(\mu_{d'})$, which is then completed by $\sigma_{d'}$.

We note first that

$$V(\mu_{d'}) \setminus S \subseteq \mathcal{R}, \qquad (171)$$

that is, $\mu_{d'}$ is supported between $\mathcal{R}$ and $S_{\mathcal{R}}$. Indeed, by construction, $V(\mu_{d'}) \cap S \subseteq N_w(d') \subseteq S_{\mathcal{R}}$, since $d' \in \mathcal{R}$. Since $\mu_{d'} \leq \tilde{\mu}$, by (S21) for each $v \in V(\mu_{d'}) \cap S$, each edge with non-zero weight in $\mu_{d'}$ joined to v must have its other endpoint in $\mathcal{R}$.

This allows us to apply Lemma 8.1 (Extending out) with

object	H	$\mathcal{R}$	S	$\mu_{d'}$	γ_B
in place of	H	U	V	μ	γ

This application yields a γ_B-skew-matching $\sigma_B^* \trianglelefteq \mu_{d'}$ of weight

$$W(\sigma_B^*) = (1 + \gamma_B^{-1}) W(\mu_{d'})$$

with its anchor $\mathcal{A}(\sigma_B^*)$ contained in $\mathcal{R}$.

Let $x \in \mathcal{A}(\sigma_B^*) \subseteq \mathcal{R}$ be arbitrary, and suppose $\sigma_B^*(x) > 0$. Since $\sigma_B^* \trianglelefteq \mu_{d'} \leq \tilde{\mu}$, by (S18) we obtain that $(\sigma_B^*)^1(x) \leq \tilde{\mu}(x) \leq w(\overrightarrow{cx})$. We deduce that the anchor $\mathcal{A}(\sigma_B^*)$ fits in the w-neighbourhood of c.

We have

$$\begin{aligned} W(\sigma_B^* + \sigma_{d'}) &= (1+\gamma_B^{-1}) W(\mu_{d'}) + (1+\gamma_B) \frac{W(\sigma_{d'})}{1+\gamma_B} \\ &= W(\mu'_d) + \frac{W(\sigma_{d'})}{1+\gamma_B} + \gamma_B^{-1} W(\mu_{d'}) + \frac{\gamma_B}{1+\gamma_B} W(\sigma_{d'}) \\ &\overset{(170)}{>} \deg_w(d, N_w(d')) + \frac{k-b_1}{2} + \gamma_B^{-1} \left(W(\mu'_d) + \frac{W(\sigma_{d'})}{1+\gamma_B} \right) \\ &\overset{(169)}{\geq} \deg_w(d, N_w(d')) + \frac{k-b_1}{2} + \gamma_B^{-1} |N_w(d')| \\ &\geq \deg_w(d, N_w(d')) + \frac{k - b_1 + \gamma_B^{-1} b_2}{2} \\ &= \deg_w(d, N_w(d')) + \frac{k}{2}. \end{aligned}$$

Hence, by the maximal choice of d, we have that

$$W(\sigma_B^* + \sigma_{d'}) \geq \deg_w(x, N_w(d')) + \frac{k}{2} \qquad (172)$$

holds for any $x \in N_w(c) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d'))$ that also satisfies $w(\overrightarrow{cx}) > \tilde{\sigma}(x)$.

Step 3: Building the γ_A-skew-matching σ_A. We find a fractional matching $\widehat{\mu}$, completely avoiding the already build γ_B-skew-matching, and we exploit the unique property of the anchor vertex d to deduce that d has large degree into this matching, allowing us to build a sufficiently large γ_A-skew-matching σ_A.

Using that $\sigma_B^* \trianglelefteq \mu_{d'}$, we check that $\frac{\gamma_B}{1+\gamma_B} W(\sigma_B^*) \leq W(\mu_{d'})$. Observe that

$$\begin{aligned}\frac{\gamma_B}{1+\gamma_B} W(\sigma_B^* + \tilde{\sigma}) = \frac{\gamma_B}{1+\gamma_B} W(\sigma_B^*) + \frac{\gamma_B}{1+\gamma_B} W(\tilde{\sigma}) \quad &\leq W(\mu_{d'}) + \frac{\gamma_B}{1+\gamma_B} W(\tilde{\sigma}) \\ \leq |\mathcal{R}| < b_2,&\end{aligned}$$

where the last line follows because $\mu_{d'}$ and $\tilde{\sigma}$ are disjoint, and supported between $\mathcal{R}$ and $S_{\mathcal{R}}$, by (171) and (S17); and the last inequality follows from (166). Hence, we deduce

$$W(\tilde{\sigma} + \sigma_B^*) < b_1 + b_2.$$

Now, note that we have $V(\mu_{d'} + \sigma_{d'}) \cap S = N_w(d')$ and $V(\mu_{d'} + \sigma_{d'}) \setminus S \subseteq \mathcal{R}$. Since $\mathcal{R}$ consists of singletons, all of its neighbours are in S. Hence, since $w(\overrightarrow{cd}) > \tilde{\sigma}(d)$, by Claim 9.7, we have that $N_w(d) \cap V(\mu_{d'} + \sigma_{d'}) = N_w(d')$. Therefore, we get

$$\begin{aligned}\deg_w(d, V(H) \setminus V(\mu_{d'} + \sigma_{d'})) &= \deg_w(d) - \deg_w(d, N_w(d')) \\ &\overset{(172)}{>} \frac{k}{2} - \left(W(\sigma_B^* + \sigma_{d'}) - \frac{k}{2}\right) \\ &= k - W(\sigma_B^* + \sigma_{d'}) \\ &= a_1 + a_2 + b_1 + b_2 - W(\sigma_B^* + \tilde{\sigma}) + W(\tilde{\sigma} - \sigma_{d'}) \\ &> a_1 + a_2 + W(\tilde{\sigma} - \sigma_{d'}). \end{aligned} \tag{173}$$

Let $\widehat{\mu} = \tilde{\mu} - \mu_{d'}$. Observe that $V(\widehat{\mu})$ does not intersect $V(\mu_{d'} + \sigma_{d'}) \cap S$. Indeed, if $0 < \sigma_{d'}(x) < 1$, then $x \in N_w(d')$, and thus $\tilde{\mu}(x) = \mu_{d'}(x)$. Hence, $\widehat{\mu}(x) = 0$. Define a weight function $\widehat{w}$ by

$$\widehat{w}(\overrightarrow{uv}) = \begin{cases} \max\{0, w(\overrightarrow{ux}) - \tilde{\sigma}(x) - \mu_{d'}(x)\} & \text{for } u \in \{c, d\}, v = x \in V(H), \\ w(\overrightarrow{uv}) & \text{otherwise.} \end{cases}$$

We claim that we have

$$\deg_{\widehat{w}}(d, \widehat{\mu}) > a_1 + a_2. \tag{174}$$

Indeed,

$$\begin{aligned}\deg_{\widehat{w}}(d, \widehat{\mu}) &= \sum_{x \in V(H)} \min\{\widehat{w}(\overrightarrow{dx}), \widehat{\mu}(x)\} \\ &\geq \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \min\{\widehat{w}(\overrightarrow{dx}), \widehat{\mu}(x)\} \\ &\geq \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \min\{w(\overrightarrow{dx}) - \tilde{\sigma}(x) - \mu_{d'}(x),\ \tilde{\mu}(x) - \mu_{d'}(x)\} \\ &= \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \min\{w(\overrightarrow{dx}) - \tilde{\sigma}(x),\ \tilde{\mu}(x)\} \\ &= \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \big(w(\overrightarrow{dx}) - \tilde{\sigma}(x)\big) \\ &= \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \big(w(\overrightarrow{dx}) - \tilde{\sigma}(x) + \sigma_{d'}(x)\big) \\ &\geq \deg_w\big(d, V(H) \setminus V(\mu_{d'} + \sigma_{d'})\big) - \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \big(\tilde{\sigma}(x) - \sigma_{d'}(x)\big) \\ &\overset{(173)}{>} a_1 + a_2 + W(\tilde{\sigma} - \sigma_{d'}) - \sum_{x \in V(H) \setminus V(\mu_{d'} + \sigma_{d'})} \big(\tilde{\sigma}(x) - \sigma_{d'}(x)\big) \\ &\geq a_1 + a_2. \end{aligned}$$

where in the fifth line we used $N_w(d) \subseteq S_{\mathcal{R}}$ and (S19). In the sixth line we used that $x \notin V(\mu_{d'} + \sigma_{d'})$.

We apply Lemma 8.3 (Combination) with

object	$(H, \widehat{w})$	d	$\widehat{\mu}$	γ_A
in place of	(H, w)	v	μ	γ

and we obtain as a consequence a γ_A-skew-matching σ_A, which satisfies $\sigma_A \trianglelefteq \widehat{\mu}$, and whose weight (scaling down if necessary, using (174)) is

$$W(\sigma_A) = a_1 + a_2.$$

Moreover, its anchor $\mathcal{A}(\sigma_A)$ fits in the $\widehat{w}$-neighbourhood of d.

We shall argue now that $(\sigma_A, \tilde{\sigma}+\sigma_B^*)$ is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{dc}$. We need to check properties (B1)–(B4). The disjointedness of σ_A with $\tilde{\sigma} + \sigma_B^*$ follows from the fact that $\sigma_A \trianglelefteq \widehat{\mu} = \tilde{\mu} - \mu_{d'}$ and $\sigma_B^* \trianglelefteq \mu_{d'}$, together with the fact that $\tilde{\mu}$ is disjoint from $\tilde{\sigma}$. This gives (B1). We have already stated above that $\mathcal{A}(\sigma_A)$ fits in the $\widehat{w}$-neighbourhood of d, which gives (B2) of Definition 6.4. We have also said that the anchor $\mathcal{A}(\sigma_B^*)$ fits in the w-neighbourhood of c, and also we have $\mathcal{A}(\sigma_B^*) \subseteq \mathcal{R}$. By (S15)-(S16), we have that the anchor $\mathcal{A}(\tilde{\sigma})$ is contained in S and fits in the w-neighbourhood of c. As $\mathcal{R}$ and $S_{\mathcal{R}}$ are disjoint, we have that $\mathcal{A}(\tilde{\sigma} + \sigma_B^*)$ fits in the w-neighbourhood of c, so we have (B3).

Finally, to see (B4), we note first that $N(c) \cap N(d) \subseteq N(d) \subseteq S_{\mathcal{R}}$. Since σ_B^* is anchored in $\mathcal{R}$, which is disjoint from $S_{\mathcal{R}}$, for each $x \in N(c) \cap N(d)$ we have $(\sigma_B^*)^1(x) = 0$. Hence, to see (B4) it suffices to show, for every $x \in N(c) \cap N(d)$, that

$$\max\{w(\overrightarrow{cx}), w(\overrightarrow{dx})\} \geq \sigma_A^1(x) + \tilde{\sigma}^1(x)$$

holds. Suppose first that $\sigma_A^1(x) = 0$. Then to get the desired inequality it suffices to check that $w(\overrightarrow{cx}) \geq \tilde{\sigma}^1(x)$ holds, and this is true because the anchor of $\tilde{\sigma}$ fits in the w-neighbourhood of c. Hence, we can assume that $\sigma_A^1(x) > 0$. Since $\sigma_A \trianglelefteq \widehat{\mu} = \tilde{\mu} - \mu_{d'}$, we have that $\tilde{\mu}(x) \neq \mu_{d'}(x)$, meaning that $x \notin N(d')$, and therefore, $\mu_{d'}(x) = 0$. On the other hand, since the anchor of σ_A fits in the $\widehat{w}$-neighbourhood of d, from $\sigma_A^1(x) > 0$ we also have that $\widehat{w}(\overrightarrow{dx}) \geq \sigma_A^1(x) > 0$. From the definition of $\widehat{w}$, this implies that $\widehat{w}(\overrightarrow{dx}) = w(\overrightarrow{dx}) - \tilde{\sigma}(x)$ (since $\mu_{d'}(x) = 0$). Putting all together, we have

$$w(\overrightarrow{dx}) = \widehat{w}(\overrightarrow{dx}) + \tilde{\sigma}(x) \geq \sigma_A^1(x) + \tilde{\sigma}^1(x),$$

as desired. This finishes the verification of (B4).

Step 4: Completing the γ_B-skew-matching. Having found σ_A, we need to extend $\tilde{\sigma} + \sigma_B^*$ in order to obtain σ_B. We use a similar trick as in the previous step, but instead of picking an anchor d that avoids $\sigma_B^* + \sigma_{d'}$, we shall select "avoiding vertices" to place the anchor of $\widehat{\sigma}_B$, the missing part of the γ_B-skew-matching. Using their avoiding properties, we can greedily build then the tail of $\widehat{\sigma}_B$.

Define

$$V := N_{\widehat{w}}(c) \setminus (\mathcal{R} \cup X_{d'} \cup N_w(d')), \text{ and}$$
$$U := V(H) \setminus (V(\sigma_{d'} + \mu_{d'}) \cup V).$$

We gather first some useful observations. We note first that, since $V(\sigma_{d'} + \mu_{d'}) \subseteq N_w(d') \cup \mathcal{R}$, we have that $V \cap V(\sigma_{d'} + \mu_{d'}) = \emptyset$. This implies that

$$U \cup V = V(H) \setminus V(\sigma_{d'} + \mu_{d'}). \tag{175}$$

Suppose that $x \notin \mathcal{R} \cup X_{d'}$. The claim is that $x \in N(d') \cup V$, or $w(\overrightarrow{cx}) = 0$. Indeed, suppose that $x \notin \mathcal{R} \cup X_{d'} \cup N(d') \cup V$. In particular we have $x \notin N_{\widehat{w}}(c)$, so $\widehat{w}(\overrightarrow{cx}) = 0$. If $x \in S$, we have $\mu_{d'}(x) = 0$ (because $x \notin N(d')$) and $w(\overrightarrow{cx}) \geq \tilde{\sigma}(x)$ (since $\tilde{\sigma}$ fits in the w-neigbourhood of c), then by definition of $\hat{w}$ we have $0 = \widehat{w}(\overrightarrow{cx}) = w(\overrightarrow{cx}) - \tilde{\sigma}(x) \geq 0$, hence $w(\overrightarrow{cx}) = 0$, as desired. On the other hand, if $x \notin S$, then we have that $\mu_{d'}(x) = \tilde{\sigma}(x) = 0$ (by (S17), (171) and $x \notin \mathcal{R}$). Then again by definition of $\hat{w}$ we have $0 = \widehat{w}(\overrightarrow{cx}) = w(\overrightarrow{cx}) =$

0, as desired. We conclude that $V(H) \setminus (\mathcal{R} \cup X_{d'}) \subseteq N(d') \cup V \cup \{x : w(\vec{cx}) = 0\}$, so in particular

$$\sum_{x \in V(H) \setminus (\mathcal{R} \cup X_{d'})} w(\vec{cx}) \leq \sum_{x \in N_w(d')} w(\vec{cx}) + \sum_{x \in V} w(\vec{cx}). \tag{176}$$

Similarly, again using (S17) we deduce that for each $x \in (V \cup N_w(d)) \setminus S$, we have $\tilde{\sigma}(x) = 0$. In particular, since V and $N_w(d')$ are disjoint, we have

$$\sum_{x \in V} \tilde{\sigma}(x) + \sum_{x \in N(d')} \tilde{\sigma}(x) \leq \sum_{x \in S} \tilde{\sigma}(x). \tag{177}$$

Finally, we observe the following. Suppose $x \in N_{\widehat{w}}(c) \setminus (\mathcal{R} \cup N(d'))$; and consider $Y = \mathcal{R} \cap (V(\mu_{d'} + \sigma_{d'}))$. We claim that

$$N(x) \cap Y = \emptyset. \tag{178}$$

Indeed, since $N(Y) \subseteq N(\mathcal{R}) = S_{\mathcal{R}} \subseteq S$, it suffices to analyse the case where $x \in S$. Since $x \in N_{\widehat{w}}(c)$, we have $\widehat{w}(\vec{cx}) > 0$. By definition of $\widehat{w}$, this implies that $w(\vec{cx}) > \tilde{\sigma}(x) + \mu_{d'}(x) \geq \tilde{\sigma}(x)$. Hence, by Claim 9.7, we have that x has no neighbours in Y, as desired.

We wish to apply Lemma 8.8 (Second Greedy Lemma) with

object	V	U	$\vec{cd}$	$(\tilde{\sigma} + \sigma_B^*, \sigma_A)$	γ_B	γ_A	$((b_1 + b_2) - W(\sigma_B^* + \tilde{\sigma}))/(1 + \gamma_B)$
in place of	V	U	$\vec{uv}$	(σ_A, σ_B)	γ_A	γ_B	κ

Note that U and V are disjoint. We verify the required conditions (N1)–(N2). We have

$$\begin{aligned}
\deg_w(c, V) = \sum_{x \in V} w(\vec{cx}) &\overset{(176)}{\geq} \sum_{x \in V(H) \setminus (\mathcal{R} \cup X_{d'})} w(\vec{cx}) - \sum_{x \in N_w(d')} w(\vec{cx}) \\
&\geq \sum_{x \in V(H)} w(\vec{cx}) - |\mathcal{R} \cup X_{d'}| - \sum_{x \in N_w(d')} w(\vec{cx}) \\
&\overset{\text{Claim 9.6}}{=} \sum_{x \in V(H)} w(\vec{cx}) - |\mathcal{R} \cup X_{d'}| - \sum_{x \in N_w(d')} \tilde{\sigma}(x) \\
&= \deg_w(c) - |\mathcal{R} \cup X_{d'}| - \sum_{x \in N_w(d')} \tilde{\sigma}(x) \\
&\overset{(177)}{\geq} \deg_w(c) - |\mathcal{R} \cup X_{d'}| - \sum_{x \in S} \tilde{\sigma}(x) + \sum_{x \in V} \tilde{\sigma}(x) \\
&\overset{(166)}{\geq} k - b_2 - \sum_{x \in S_{\mathcal{R}}} \tilde{\sigma}(x) + \sum_{x \in V} \tilde{\sigma}(x) \\
&\overset{\text{(S15)\&(S17)}}{\geq} k - b_2 - \frac{W(\tilde{\sigma})}{1 + \gamma_B} + \sum_{x \in V} \tilde{\sigma}(x) \\
&= b_1 - \frac{W(\tilde{\sigma})}{1 + \gamma_B} + (a_1 + a_2) + \sum_{x \in V} \tilde{\sigma}(x) \\
&\geq b_1 - \frac{W(\tilde{\sigma})}{1 + \gamma_B} + \sum_{x \in V} \sigma_A(x) + \sum_{x \in V} \tilde{\sigma}(x) \\
&\geq b_1 - \frac{W(\tilde{\sigma} + \sigma_B^*)}{1 + \gamma_B} + \sum_{x \in V} (\sigma_A(x) + \tilde{\sigma}(x) + \sigma_B^*(x)),
\end{aligned}$$

where in the last line we used that $\sigma_B^* \trianglelefteq \mu_{d'}$, and for every $x \in V$ we have $\mu_{d'}(x) = 0$, since $\mu_{d'}$ is supported between $N_w(d')$ and $\mathcal{R}$. This implies that $\sigma_B^*(x) = 0$ for all $x \in V$; and so indeed we have (N1).

Now we check (N2). Recall from (175) that $U \cup V = V(H) \setminus V(\sigma_{d'} + \mu_{d'})$, and that $\mu_{d'}$ and $\sigma_{d'}$ are supported in $N(d') \cup \mathcal{R}$. Also notice that if $x \in V$, then $\widehat{w}(\overrightarrow{cx}) > 0$, i.e., $w(\overrightarrow{cx}) > \tilde{\sigma}(x)$ and thus (172) applies to all $x \in V$. Using this, we have, for all $x \in V$,

$$
\begin{aligned}
|N(x) \cap (U \cup V)| &\geq \deg_w(x, V \cup U) \\
&\geq \deg_w(x, V(H) \setminus V(\sigma_{d'} + \mu_{d'})) \\
&= \deg_w(x) - \deg_w(x, \mathcal{R} \cap V(\sigma_{d'} + \mu_{d'})) - \deg_w(x, N_w(d')) \\
&\geq \frac{k}{2} - \deg_w(x, \mathcal{R} \cap V(\sigma_{d'} + \mu_{d'})) - \deg_w(x, N_w(d')) \\
&\overset{(178)}{=} \frac{k}{2} - \deg_w(x, N_w(d')) \\
&\overset{(172)}{\geq} \frac{k}{2} - \left(W(\sigma_B^* + \sigma_{d'}) - \frac{k}{2} \right) = k - W(\sigma_B^* + \sigma_{d'}) \\
&= b_1 + b_2 - W(\sigma_B^* + \tilde{\sigma}) + W(\tilde{\sigma} - \sigma_{d'}) + (a_1 + a_2) \\
&\geq b_1 + b_2 - W(\sigma_B^* + \tilde{\sigma}) + W(\tilde{\sigma} - \sigma_{d'}) + \sum_{x \in V \cup U} \sigma_A(x) \\
&\geq b_1 + b_2 - W(\sigma_B^* + \tilde{\sigma}) + \sum_{x \in V \cup U} (\sigma_A(x) + \tilde{\sigma}(x) - \sigma_{d'}(x)) \\
&= (b_1 + b_2 - W(\sigma_B^* + \tilde{\sigma})) + \sum_{x \in V \cup U} (\sigma_A(x) + \tilde{\sigma}(x) + \sigma_B^*(x)) \,,
\end{aligned}
$$

where in the last inequality we use the fact that $\sigma_{d'}(x) = \sigma_B^*(x) = 0$ for $x \in U \cup V$ (this follows from $\sigma_B^* \trianglelefteq \mu_{d'}$ together with (175)).

The outcome of the application of Lemma 8.8 is a γ_B-skew-matching $\widehat{\sigma}_B$ in $H^{\leftrightarrow}$ of weight $W(\widehat{\sigma}_B) = (b_1 + b_2) - W(\sigma_B^* + \tilde{\sigma})$ such that $(\widehat{\sigma}_B + \sigma_B^* + \tilde{\sigma}, \sigma_A)$ is a (γ_B, γ_A)-skew-matching anchored in $\overrightarrow{cd}$. We set $\sigma_B := \widehat{\sigma}_B + \sigma_B^* + \tilde{\sigma}$ and have

$$W(\sigma_B) = b_1 + b_2.$$

Hence, (σ_A, σ_B) is a good matching. This finishes the proof of Proposition 4.2. ∎

10. Embedding the Tree

In this section, we prove the Tree Embedding Lemma (Lemma 4.5), which we restate for convenience.

Lemma 4.5 (Tree Embedding Lemma)**.** *For any $\eta, \mathfrak{d}, q > 0$, and $t \in \mathbb{N}$, there are $\varepsilon = \varepsilon(\eta, \mathfrak{d}, q), \rho = \rho(\eta, \mathfrak{d}, q, t) > 0$ and $n_0 = n_0(\eta, \mathfrak{d}, q, t) \in \mathbb{N}$ such that for $n \geq n_0$ the following holds. Suppose G is an n-vertex graph, and $\mathcal{P} = \{V_0, V_1, \dots, V_N\}$ is an ε-regular equitable partition of G with $N \leq t$. Suppose $k \geq qn$ and that we have natural numbers $a_1, b_1, a_2, b_2 \geq \eta k$ such that*

$$k = a_1 + a_2 + b_1 + b_2 \,.$$

Suppose the weighted $\mathfrak{d}$-reduced graph $\Gamma_{\mathfrak{d},\varepsilon}$ corresponding to G admits a $(a_2/a_1, b_2/b_1)$-skew-matching pair (σ_A, σ_B), with weights $W(\sigma_A) \geq (1+\eta)(a_1 + a_2)N/n$ and $W(\sigma_B) \geq (1+\eta)(b_1 + b_2)N/n$. Then G contains any tree $T \in \mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$.

The section is organised as follows. We gather useful results about probability and the regularity method in §10.1. In §10.2, we prove a lemma that embeds a 'shrub' in a regular pair and will be used repeatedly during the proof. We give a sketch of the proof in §10.3. Then we proceed with the main proof, which is split in several steps, and takes the remainder of the section.

10.1. **Preliminaries.** We will need a bounded-differences inequality [McD89].

Lemma 10.1. *Let $X_1, \dots, X_N$ be independent random variables, with $X_i \in \Lambda_i$. Let $f : \prod_{i=1}^N \Lambda_i \to \mathbb{R}$ be a function such that for any $z, z' \in \prod_{i=1}^N \Lambda_i$ which differ only in the kth coordinate, we have $|f(z) - f(z')| \leq c_k$. Then, the random variable $X = f(X_1, \dots, X_N)$ satisfies, for any $t \geq 0$,*

$$\Pr[X \geq \mathbf{E}[X] + t] \leq \exp\left(-\frac{2t^2}{\sum_{i=1}^N c_i^2}\right).$$

We now collect a few results about graph regularity and regular pairs. Let (X, Y) be an ε-regular pair with density $\mathfrak{d}$, and $Y' \subseteq Y$. Say that $x \in X$ is *typical to* Y' if x has at least $(\mathfrak{d} - \varepsilon)|Y'|$ neighbours in Y'. The following fact is well-known (cf. [Die17, Lemma 7.5.1]).

Lemma 10.2. *Let (X, Y) be an ε-regular pair, and let $Y' \subseteq Y$ with $|Y'| \geq \varepsilon|Y|$. Then all but at most $\varepsilon|X|$ vertices in X are typical to Y'.*

We generalise this to deduce that many vertices are typical to many sets.

Lemma 10.3. *Let $X, Y_1, \dots, Y_N$ be such that (X, Y_i) is an ε-regular pair for each $1 \leq i \leq N$, and for each i let $Y_i' \subseteq Y_i$ with $|Y_i'| \geq \varepsilon|Y_i|$. Then all but at most $\sqrt{\varepsilon}|X|$ vertices in X are typical to all but at most $\sqrt{\varepsilon}N$ of the sets $Y_1', \dots, Y_N'$.*

Proof. Let B be an auxiliary bipartite graph with classes X and $[N] = \{1, \dots, N\}$, where $x \in X$ and $i \in [N]$ are joined by an edge if x is *not* typical to Y_i'. By Lemma 10.2, each i is adjacent to at most $\varepsilon|X|$ edges in B, so B has at most $\varepsilon|X|N$ edges. A double-counting argument shows that the number of vertices of X which have degree at least $\sqrt{\varepsilon}N$ in B is at most $\sqrt{\varepsilon}|X|$, as required. ■

10.2. **Embedding shrubs.** Now, we state and prove an auxiliary lemma, which concerns the embedding of a 'shrub' in a regular pair. The proof follows from very standard arguments about embedding in regular pairs, so the reader familiar with graph regularity may safely skip its proof.

Definition 10.4. We say that (F, r, x) is a *rooted shrub* if F is a tree, $r \in V(F)$, and either $x = \emptyset$ or $x \in V(F)$ and the distance of x to r is even and at least 4. We say that r is the *root* of F, and if $x \neq \emptyset$ we call x the *adventitious root* of F (and say that F has an adventitious root).

Lemma 10.5 (Embedding a shrub)**.** *Suppose that $2\varepsilon \leq \mathfrak{d} \leq 1/3$ and $\varepsilon < \frac{\mathfrak{d}^2\tilde{\eta}}{8}$. Let (F, r, x) be a rooted shrub. Let (X, Y) be an ε-regular pair in a graph G with $d(X, Y) \geq \mathfrak{d}$ and $|X| = |Y|$. Let $U \subseteq X \cup Y$ and $P \subset X$ be disjoint sets, and $v \in X \setminus P$, and suppose*

(T1) $|X \cap U|, |Y \cap U| \geq \tilde{\eta}|X|$,

(T2) $|P| \geq 2\varepsilon|X|$,

(T3) $\deg_G(v, Y \cap U) \geq 2\varepsilon|Y|$,

(T4) *F has at most $\varepsilon|X|$ vertices.*

Then there is an embedding φ of F in G with $\varphi(V(F) \setminus \{x, r\}) \subseteq U$, $\varphi(r) = v$, and $\varphi(x) \in P$.

Proof. Let $m = |X| = |Y|$. We suppose first that $x \neq \emptyset$, i.e. that F contains an adventitious root x, and we define $\varphi(x)$ first. Since $|Y \cap U| \geq \tilde{\eta}m \geq \varepsilon m$, then Lemma 10.2 implies that all but at most εm vertices in X are typical to $Y \cap U$. In particular, we may select one such typical vertex w which lies in P, and we set $\varphi(x) = w \in P$. By the choice of w, we have $\deg(w, Y \cap U) \geq (\mathfrak{d} - \varepsilon)|Y \cap U| \geq \frac{\mathfrak{d}\tilde{\eta}}{2}m$. Set $N_w := N(w) \cap Y \cap U$. If there is no adventitious root in F, we simply set $N_w := Y \cap U$. In any case, $N_w \subseteq Y \cap U$ is defined and it holds that $|N_w| \geq \frac{\mathfrak{d}\tilde{\eta}}{2}m$.

Let X' be the set of vertices in X that are typical to N_w, and let Y' be the set of vertices in Y that are typical to $X \cap U$. Again, Lemma 10.2 implies that $|X \setminus X'| \leq \varepsilon m$ and $|Y \setminus Y'| \leq \varepsilon m$. Observe that for every $u \in X'$, we have

$$\deg(u, N_w \cap Y') \geq (\mathfrak{d} - \varepsilon)|N_w| - |Y \setminus Y'| \geq \frac{\mathfrak{d}^2 \tilde{\eta}}{4} m - \varepsilon m \geq \varepsilon m \geq |V(F)|;$$

for every $u \in Y'$, we have

$$\deg(u, X' \cap U) \geq (\mathfrak{d} - \varepsilon)|X \cap U| - |X \setminus X'| \geq \frac{\mathfrak{d}\tilde{\eta}}{2} m - \varepsilon m \geq \varepsilon m \geq |V(F)|,$$

and

$$\deg(v, Y' \cap U) \geq \deg(v, Y \cap U) - |Y \setminus Y'| \geq 2\varepsilon m - \varepsilon m \geq |V(F)|.$$

Therefore we can greedily embed $V(F) \setminus \{x\}$ in $U \cup \{v\}$ so that r is mapped to v, the neighbours of r are mapped to Y', the vertices of even distance from r in X' and the vertices of odd distance at least 3 from r to $N_w \cap Y'$. ■

10.3. **Sketch of the proof.** We will find an embedding of a tree T in a graph G, provided G has a regular partition whose weighted $\mathfrak{d}$-reduced graph has a suitable skew-matching pair. The proof will consist of seven steps.

(i) *Setting the stage.* First, we will set the stage and constants. By assumption, the tree T to be embedded has a corresponding fine partition $(W_A, W_B, \mathcal{F}_A, \mathcal{F}_B)$ which decomposes T into 'seeds' and 'shrubs'. The reduced host graph has a skew-matching pair (σ_A, σ_B) which is anchored to an edge (V_c, V_d). We select buffer and reservoir sets U and Q in $V(G) \setminus (V_c \cup V_d)$.
(ii) *Embedding the seeds.* Next, we will embed the 'seeds' of T, meaning we embed $W_A \cup W_B$ into $V_c \cup V_d$.
(iii) *Allocating the shrubs.* Then we will allocate (but not embed yet) the shrubs from $\mathcal{F}_A \cup \mathcal{F}_B$ in a suitable way, according to the weights indicated by (σ_A, σ_B). This is done using a randomised procedure. At the end of this step, for each shrub F, we will have selected a cluster V_i where one colour class of the shrub will be embedded.
(iv) *Allocating the roots.* We reserve, for each shrub F, enough space in the selected cluster V_i. We will select pairwise disjoint sets $R_F \subseteq V_i$, one for each $F \in \mathcal{F}_A \cup \mathcal{F}_B$. This R_F will be a 'private' set for the shrub F, with enough space for the embedding of F.
(v) *Finding suitable clusters.* Having chosen V_i and R_F for all shrubs F, we will argue that there is always (even after having embedded some shrubs) a way to choose another 'suitable' cluster V_ℓ for each F, in such a way that there is enough space in $G[V_i, V_\ell]$ to embed F.
(vi) *First Embedding Phase.* In the last two steps, we will carry the actual embedding of the shrubs. Here, we will try to embed the shrubs in their respective chosen bipartite graphs $G[V_i, V_j]$, avoiding (nearly completely) the reservoir set Q. However, this embedding can fail for some (few) shrubs.
(vii) *Second Embedding Phase.* In this phase, the remaining few shrubs, which failed to be embedded in the previous phase, are finally embedded in Q. For this purpose, we use Lemma 10.5.

The rest of this section is dedicated to the proof of Lemma 4.5, and spans several subsections.

10.4. **Proof of Lemma 4.5: Setting the stage.** Suppose $\mathfrak{d}, \eta, q > 0$ and $t \in \mathbb{N}$ are given. Set

$$\eta' := q\eta^3.$$

We assume $0 < \mathfrak{d} \ll \eta' \ll \eta \ll 1$; otherwise we decrease their values to satisfy this hierarchy. Set

$$\varepsilon := \frac{\mathfrak{d}^2 \cdot (\eta')^2}{100}, \qquad \rho := \min\left\{\frac{(\eta')^4\eta^2}{10^3 t^3}, \frac{\mathfrak{d}^2\eta}{100}, \frac{\varepsilon}{2t}\right\}, \qquad n_0 := \frac{4 \cdot 10^3 \cdot t}{\rho^2}.$$

The parameters satisfy

$$0 < 1/n_0 \ll \rho \ll \varepsilon \ll \mathfrak{d} \ll \eta' \ll \eta \ll 1. \tag{179}$$

Let G be a given graph on n vertices, and let $\Gamma := \Gamma_{\mathfrak{d},\varepsilon}$ be the reduced graph with $|V(\Gamma_{\mathfrak{d},\varepsilon})| = N \leq t$, and let $\mathcal{P} = \{V_0, V_1, \ldots, V_N\}$ be the ε-regular equitable partition which yields Γ, so $i \in V(\Gamma)$ corresponds to the cluster $V_i \subseteq V(G)$. Let $w : E(\Gamma) \to \{0\} \cup [\mathfrak{d}, 1]$ be the weight function defined by $w(ij) := d(V_i, V_j)$, if $d(V_i, V_j) \geq \mathfrak{d}$ and $w(ij) = 0$, otherwise. Let $\overrightarrow{cd}$ be the edge in $E(\Gamma^{\leftrightarrow})$ where (σ_A, σ_B) is anchored.

Let $m := |V_1|$ be the common size of all clusters $V_1, \ldots, V_N$. Note that since $\mathcal{P}$ is an ε-regular equitable partition, we have

$$(1 - \varepsilon)\frac{n}{N} \leq m \leq \frac{n}{N}. \tag{180}$$

Let $T \in \mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$ be an arbitrary tree, which we need to embed into G. Let $(W_A, W_B, \mathcal{F}_A, \mathcal{F}_B)$ be the $(\rho|V(T)|)$-fine partition of T that witnesses $T \in \mathcal{T}^{\rho}_{a_1,a_2,b_1,b_2}$. By assumption, the values a_1, a_2, b_1, b_2, k must satisfy

$$a_1 + a_2 + b_1 + b_2 = k \geq qn \tag{181}$$

$$a_1, a_2, b_1, b_2 \geq \eta k. \tag{182}$$

Let $\gamma_A := a_2/a_1$ and $\gamma_B := b_2/b_1$. From (182), we easily get

$$\gamma_A, \gamma_B \geq \eta. \tag{183}$$

We recall that the definition of σ^1 (Definition 6.2) implies that, for every $i \in V(\Gamma)$,

$$\sum_{j \in N_\Gamma(i)} \sigma_A(\overrightarrow{ij}) = (1 + \gamma_A)\sigma^1_A(i), \tag{184}$$

and similarly,

$$\sum_{j \in N_\Gamma(i)} \sigma_B(\overrightarrow{ij}) = (1 + \gamma_B)\sigma^1_B(i). \tag{185}$$

By assumption, we have that (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{cd}$ with weights $W(\sigma_A), W(\sigma_B)$. In particular, from (B2) we have that if $\sigma^1_A(i) > 0$, then $i \in N_\Gamma(c)$, and therefore $\sigma_A(\overrightarrow{ij}) > 0$ implies that $i \in N_\Gamma(c)$ and $j \in N_\Gamma(i)$. This implies that

$$W(\sigma_A) = \sum_{e \in E(\Gamma^{\leftrightarrow})} \sigma_A(e) = \sum_{i \in N_\Gamma(c)} \sum_{j \in N_\Gamma(i)} \sigma_A(\overrightarrow{ij}) = (1 + \gamma_A) \sum_{i \in N_\Gamma(c)} \sigma^1_A(i),$$

where we used (184) in the last step. We also have as assumption that $W(\sigma_A)n \geq (1 + \eta)(a_1 + a_2)N = (1 + \eta)a_1(1 + \gamma_A)N$. Combined with the previous calculations, we can write

$$\sum_{i \in N_\Gamma(c)} \sigma^1_A(i) \geq (1 + \eta)a_1\frac{N}{n}. \tag{186}$$

The analogous reasoning also yields

$$\sum_{i \in N_\Gamma(d)} \sigma^1_B(i) \geq (1 + \eta)b_1\frac{N}{n}. \tag{187}$$

Let $1 \leq i \leq N$ be arbitrary. By (A3), we have

$$|W_A \cup W_B| \leq \frac{672|V(T)|}{\rho|V(T)|} < \frac{10^3}{\rho} \leq \frac{\rho n}{4N} < \frac{\rho|V_i|}{2}, \tag{188}$$

where the second to last inequality follows from the choice of n_0 and the inequalities $n \geq n_0$ and $N \leq t$; and the last inequality follows from (180).

For any $\mathcal{F} \subseteq \mathcal{F}_A$ denote by $V_1(\mathcal{F})$ and $V_2(\mathcal{F})$ the set of vertices in $\mathcal{F}$ at an odd distance, and respectively even distance, from W_A. Similarly, for $\mathcal{F} \subseteq \mathcal{F}_B$, we define $V_1(\mathcal{F})$ and $V_2(\mathcal{F})$ to be the vertices of $\mathcal{F}$ at odd, and respectively even, distance from W_B.

Choose $Q \subseteq V(G) \setminus (V_c \cup V_d)$ arbitrarily under the condition that $|Q \cap V_i| = \frac{\eta}{4}|V_i|$ for every $V_i \in \mathcal{P} \setminus \{V_c, V_d\}$. We shall use this set in a second phase of the embedding to deal with a small proportion of the shrubs we fail to embed in a first attempt. Let $U \subseteq V(G) \setminus (Q \cup V_c \cup V_d)$ be an arbitrary set such that $|U \cap V_i| = \frac{\eta}{4}|V_i|$ for every $V_i \in \mathcal{P} \setminus \{V_c, V_d\}$. When we embed the small shrubs of $\mathcal{F}_A \cup \mathcal{F}_B$, we shall use the set U as a buffer that will guarantee us we have enough free neighbours to choose a typical vertex from. Thus, by construction, we have, for each $i \in [N] \setminus \{c, d\}$,

$$|V_i \cap U| = |V_i \cap Q| = \frac{\eta}{4}|V_i| = \frac{\eta}{4}m. \tag{189}$$

10.5. **Proof of Lemma 4.5: Embedding the seeds.** In the following, we find a suitable embedding of the seeds of T. More precisely, we shall find an embedding φ_0 of $T[W_A \cup W_B]$ in $G[V_c, V_d]$, and subsets $I_x \subseteq V(\Gamma)$, one for each $x \in W_A \cup W_B$, that satisfy

(U1) $\varphi_0(W_A) \subseteq V_c$, and $\varphi_0(W_B) \subseteq V_d$;

(U2) for each $x \in W_A$, $I_x \subseteq N_\Gamma(c)$, and for each $x \in W_B$, $I_x \subseteq N_\Gamma(d)$;

(U3) $|N(\varphi_0(x)) \cap V_i \setminus (Q \cup U)| \geq (w(ci) - \varepsilon)|V_i \setminus (Q \cup U)|$ and $|N(\varphi_0(x)) \cap V_i \cap Q| \geq 7\varepsilon|V_i|$ for each $x \in W_A$ and for each $i \in I_x$,

(U4) $|N(\varphi_0(x)) \cap V_i \setminus (Q \cup U)| \geq (w(di) - \varepsilon)|V_i \setminus (Q \cup U)|$ and $|N(\varphi_0(x)) \cap V_i \cap Q| \geq 7\varepsilon|V_i|$ for each $x \in W_B$ and for each $i \in I_x$,

(U5) $\sum_{i \notin I_x \cap I_{x'}} \sigma_A^1(i)|V_i| \leq 5\sqrt{\varepsilon}n$, for every $x, x' \in W_A$,

(U6) $\sum_{i \notin I_x \cap I_{x'}} \sigma_B^1(i)|V_i| \leq 5\sqrt{\varepsilon}n$, for every $x, x' \in W_B$.

Let us digest the meaning of some of these properties. Intuitively, (U5) says that for every $x, x' \in W_A$, both of the sets I_x and I'_x (and therefore also their intersection) contains most of $N_\Gamma(c)$; and (U3) says that every $x \in W_A$ will be embedded in a vertex $\varphi_0(x) \in V_c$ such that for each $i \in I_x$, $\varphi_0(x)$ has large degree both to $V_i \setminus (Q \cup U)$ and $V_i \cap Q$. (U6) and (U4) state the analogous properties for W_B.

To find the required embedding, we will embed the seeds $W_A \cup W_B$ in $G[V_c, V_d]$ one after the other. Recall that by Definition 4.4, for any $j \in [N]$, $N_\Gamma(j) \subseteq V(\Gamma)$ denotes the set of indices i such that the cluster V_i forms an ε-regular pair together with V_j of density at least $\mathfrak{d}$.

Let $V'_c \subseteq V_c$ be the set of vertices that are typical to all but at most $\sqrt{\varepsilon}|N_\Gamma(c)|$ of the sets $V_i \setminus (Q \cup U)$ with $i \in N_\Gamma(c) \setminus \{d\}$, typical to all but at most $\sqrt{\varepsilon}|N_\Gamma(c)|$ sets $Q \cap V_i$ with $i \in N_\Gamma(c) \setminus \{d\}$, and typical to V_d. Define $V'_d \subseteq V_d$ analogously.

We estimate the size of V'_c. By Lemma 10.3, we see that at most $\sqrt{\varepsilon}|V_c|$ vertices in V_c are not included in V'_c because of the first requirement; and similarly at most $\sqrt{\varepsilon}|V_c|$ vertices fail to be included in V'_c because of the second requirement. By Lemma 10.2, at most $\varepsilon|V_c| \leq \sqrt{\varepsilon}|V_c|$ are lost in V'_c because of the third requirement. The analysis for V'_d is the same, and thus, we get that

$$|V'_c|, |V'_d| \geq (1 - 3\sqrt{\varepsilon})|V_c| = (1 - 3\sqrt{\varepsilon})|V_d|.$$

From the definition of V'_c and the previous bound, we have that every $v \in V'_c$ satisfies

$$\deg_G(v, V'_d) \geq (\mathfrak{d} - \varepsilon)|V_d| - |V_d \setminus V'_d| \geq \frac{\mathfrak{d}}{2}|V_d| \overset{(188)}{>} |W_A \cup W_B|,$$

where in the last step, we also used $\mathfrak{d} \geq \rho$. Similarly, for any vertex $v \in V'_d$, we have $\deg(v, V'_c) \geq |W_A \cup W_B|$. We can thus define an embedding φ_0 of $T[W_A \cup W_B]$ in $G[V'_c, V'_d]$ vertex by vertex, in a greedy fashion, in such a way that $\varphi_0(W_A) \subseteq V'_c$ and $\varphi_0(W_B) \subseteq V'_d$. Thus, (U1) holds.

For each $x \in W_A \cup W_B$, let $j_x \in \{c, d\}$ be such that $\varphi_0(x) \in V_{j_x}$. We define $I_x \subseteq N_\Gamma(j_x) \setminus \{c, d\}$ as the set of indices i of the clusters V_i for which $\varphi_0(x)$ is typical to $V_i \setminus (Q \cup U)$ and typical to $V_i \cap Q$. By construction, (U2) holds.

Now, we check that the sets I_x satisfy (U3) and (U4). Indeed, for each $i \in I_x$, the vertex $\varphi_0(x)$ is typical to the set $V_i \setminus (Q \cup U)$, and thus

$$\deg_G(\varphi_0(x), V_i \setminus (Q \cup U)) \geq (d(V_{j_x}, V_i) - \varepsilon)|V_i \setminus (Q \cup U)| = (w(j_x i) - \varepsilon)|V_i \setminus (Q \cup U)|.$$

Similarly, $\varphi_0(x)$ is typical to $V_i \cap Q$, and thus

$$\begin{aligned}\deg_G(\varphi_0(x), V_i \cap Q) &\geq (d(V_{j_x}, V_i) - \varepsilon)|V_i \cap Q| = (w(j_x i) - \varepsilon)|V_i \cap Q| \\ &= (w(j_x i) - \varepsilon)\frac{\eta}{4}|V_i| \geq \frac{\mathfrak{d}}{2}\frac{\eta}{4}|V_i| \overset{(179)}{>} 7\varepsilon|V_i|,\end{aligned}$$

where in the second to last inequality, we used that $i \in N_\Gamma(j_x)$, and therefore $w(j_x i) = d(V_{j_x}, V_i) \geq \mathfrak{d}$; together with $\mathfrak{d} \gg \varepsilon$. Thus we have obtained (U3) and (U4).

Now, we show that the sets I_x satisfy (U5) and (U6). We focus on showing (U5), as the proof of (U6) follows mutatis mutandis. Let $x, x' \in W_A$ be arbitrary. By the definition of V'_c, there are at most $\sqrt{\varepsilon}|N_\Gamma(c)| \leq \sqrt{\varepsilon}N$ indices $i \in N_\Gamma(c) \setminus \{d\}$ such that $\varphi_0(x)$ is not typical to $V_i \setminus (Q \cup U)$, and at most $\sqrt{\varepsilon}|N_\Gamma(c)| \leq \sqrt{\varepsilon}N$ indices $i \in N_\Gamma(c) \setminus \{d\}$ for which $\varphi_0(x)$ is not typical with respect to $V_i \cap Q$. The same can be said of $\varphi_0(x')$.

Therefore, $|N_\Gamma(c) \setminus (I_x \cup \{d\})| \leq 2\sqrt{\varepsilon}N$ and $|N_\Gamma(c) \setminus (I_{x'} \cup \{d\})| \leq 2\sqrt{\varepsilon}N$ hold, from which we can deduce that

$$|N_\Gamma(c) \setminus (I_x \cap I_{x'})| \leq 4\sqrt{\varepsilon}N + 1.$$

Next, note that since (σ_A, σ_B) is a (γ_A, γ_B)-skew-matching pair anchored in $\overrightarrow{cd}$, then σ_A is a γ_A-skew-matching anchored in $N_\Gamma(c)$. In particular, we have that $\sigma^1_A(i) \leq w(ci) \leq 1$ for all $i \in N_\Gamma(c)$, and $\sigma^1_A(i) = 0$ for every $i \notin N_\Gamma(c)$. Therefore, we have

$$\sum_{i \notin I_x \cap I_{x'}} \sigma^1_A(i) \leq \sum_{i \in (N_\Gamma(c) \setminus I_x) \cup (N_\Gamma(c) \setminus I_{x'}) \cup \{d\}} \sigma^1_A(i) \leq |N_\Gamma(c) \setminus (I_x \cap I_{x'})| \leq 5\sqrt{\varepsilon}N.$$

The bounds in (180) imply that $|V_i| = m \leq n/N$ holds for each $i \in [N]$, so the previous inequality yields (U5).

10.6. **Proof of Lemma 4.5: Allocating the shrubs.** To decide where we shall embed each shrub F of $\mathcal{F}_A \cup \mathcal{F}_B$, we first decide where the colour class containing the root of F should go. We will use a probabilistic argument to assign a cluster i_F for each shrub $F \in \mathcal{F}_A \cup \mathcal{F}_B$. This will be done in such a way that the neighbours of F (in T), which must be in $W_A \cup W_B$ and are already embedded via φ_0, can be appropriately joined whenever we embed the shrub F in V_{i_F}.

Now, we turn to the details. Recall that $\eta' := q\eta^3$. We will construct partitions $\mathcal{F}_A = \mathcal{F}^1_A \cup \cdots \cup \mathcal{F}^N_A$ and $\mathcal{F}_B = \mathcal{F}^1_B \cup \cdots \cup \mathcal{F}^N_B$ such that for each $i \in [N]$, and each $L \in \{A, B\}$, we have

- (V1) $|V_1(\mathcal{F}^i_L)| \leq (1 - \eta')\sigma^1_L(i)|V_i \setminus (Q \cup U)|$,
- (V2) $|V_2(\mathcal{F}^i_L)| \leq (1 - \eta')\sum_{j \in [N]} \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{ij})|V_j \setminus (Q \cup U)|$,
- (V3) if $\mathcal{F}^i_L \neq \emptyset$, then $\sigma^1_L(i) \geq \eta'$,
- (V4) for each $F \in \mathcal{F}^i_A \cup \mathcal{F}^i_B$, we have that $i \in \bigcap_{x \in X_F} I_x$, where we set $X_F := \varphi_0(N_T(V(F)) \cap (W_A \cup W_B))$.

We show the existence of a suitable partition of $\mathcal{F}_A$, since the argument for $\mathcal{F}_B$ can be done in the same way. Let

$$\Xi_c := \{i \in N_\Gamma(c) : \sigma^1_A(i) \geq \eta'\} \subseteq V(\Gamma).$$

In words, Ξ_c are the indices of the clusters which we can use to allocate trees from $\mathcal{F}_A$ while complying with (V3). For any $i \notin \Xi_c$, we will set $\mathcal{F}^i_A := \emptyset$.

Let $F \in \mathcal{F}_A$, and recall the definition of X_F from Item (V4). By (A6)–(A7), we have $N_T(V(F)) \cap W_B = \emptyset$, and therefore $X_F = \varphi_0(N_T(V(F)) \cap (W_A \cup W_B)) \subseteq \varphi_0(W_A) \subseteq V_c$. Observe that by (A8), we have $|X_F| \leq 2$. For each $F \in \mathcal{F}_A$, define

$$I_F := \Xi_c \cap \bigcap_{x \in X_F} I_x, \tag{190}$$

so $I_F \subseteq \Xi_c$ are the indices corresponding to the clusters which are permissible for the allocation of F. We will ultimately allocate F to $\mathcal{F}^i_A$ while ensuring that $i \in I_F$ holds, and this choice will ensure that (V4) holds.

Before continuing with the proof, we need the following crucial estimate.

Claim 10.6. *For any $F \in \mathcal{F}_A$, we have*

$$\tilde{a}_1 := \left(1 + \frac{\eta}{3}\right) a_1 \leq \sum_{i \in I_F} \sigma^1_A(i) |V_i \setminus (Q \cup U)| \tag{191}$$

Proof of the claim. Let S_F be the term in the right-hand side of (191). Using (189),

$$\begin{aligned} S_F &\geq \left(1 - \frac{\eta}{2}\right) m \sum_{i \in I_F} \sigma^1_A(i) \\ &\overset{(190)}{\geq} \left(1 - \frac{\eta}{2}\right) m \left(\sum_{i \in N_\Gamma(c)} \sigma^1_A(i) - \sum_{i \notin \Xi_c} \sigma^1_A(i) - \sum_{i \notin \bigcap_{x \in X_F} I_x} \sigma^1_A(i) \right). \end{aligned}$$

We estimate the last two sums in the last expression. From the definition of Ξ_c and (180), we have that $\sum_{i \notin \Xi_c} \sigma^1_A(i) m \leq \eta' m(N - |\Xi_c|) \leq \eta' N m \leq \eta' n$; and from (U5), we get that $m \sum_{i \notin \bigcap_{x \in X_F} I_x} \sigma^1_A(i)$ is at most $5\sqrt{\varepsilon} n$. Using this, we obtain

$$S_F \geq \left(1 - \frac{\eta}{2}\right) \left(m \sum_{i \in N_\Gamma(c)} \sigma^1_A(i) - \eta' n - 5\sqrt{\varepsilon} n \right) \geq \left(1 - \frac{\eta}{2}\right) \left(m \sum_{i \in N_\Gamma(c)} \sigma^1_A(i) - 2\eta' n \right),$$

where in the last inequality, we used the definition of ε to deduce $5\sqrt{\varepsilon} \leq \eta'$. Next, using (186) and (180), we get

$$S_F \geq \left(1 - \frac{\eta}{2}\right) \left(m(1+\eta) a_1 \frac{N}{n} - 2\eta' n \right) \geq \left(1 - \frac{\eta}{2}\right) \left((1-\varepsilon)(1+\eta) a_1 - 2\eta' n \right).$$

Using (181) and (182), we get that $a_1 \geq \eta k \geq \eta q n$, and together with the choice of $\eta' = q\eta^3$, we deduce that $2\eta' n \leq 2\eta^2 a_1$. This gives

$$S_F \geq \left(1 - \frac{\eta}{2}\right) \left((1-\varepsilon)(1+\eta) - 2\eta^2 \right) a_1 \geq \left(1 + \frac{\eta}{3}\right) a_1 = \tilde{a}_1,$$

where in the last line, we used that $\varepsilon \ll \eta \ll 1$, according to (179). This proves the claim. □

We will assign each shrub $F \in \mathcal{F}_A$ to $\mathcal{F}^i_A$, by choosing some $i \in I_F$, independently at random, with probability

$$p^i_F := \Pr[F \in \mathcal{F}^i_A] = \frac{\sigma^1_A(i)}{\sum_{\ell \in I_F} \sigma^1_A(\ell)}.$$

We note that, in particular, (191) implies $I_F \neq \emptyset$, so the probabilities p^i_F are well-defined. Note that by construction, any such assignment will satisfy (V3)–(V4). We shall show that with non-zero probability also (V1)–(V2) are satisfied. This suffices to prove the existence of the desired partition.

We will estimate the expectation of $|V_1(\mathcal{F}^i_A)|$ and $|V_2(\mathcal{F}^i_A)|$ for each $i \in \Xi_c$, then we will show that those values are concentrated around its expectation. Note that $|V_1(\mathcal{F}^i_A)|$ can be written as the sum of the independent random variables $\{X^i_F : F \in \mathcal{F}_A\}$, each of which takes the value $|V_1(F)|$ with probability p^i_F, and takes the value 0 otherwise. Thus, using the linearity of expectation, we have that

$$\begin{aligned}\mathbf{E}[|V_1(\mathcal{F}^i_A)|] &= \sum_{F \in \mathcal{F}_A} p^i_F |V_1(F)| = \sum_{F \in \mathcal{F}_A} \frac{\sigma^1_A(i)}{\sum_{\ell \in I_F} \sigma^1_A(\ell)} |V_1(F)| \\ &\leq \sum_{F \in \mathcal{F}_A} \frac{\sigma^1_A(i)|V_i \setminus (Q \cup U)|}{\tilde{a}_1} |V_1(F)|,\end{aligned}$$

where in the last inequality, we used (191), and also the fact that $|V_i \setminus (Q \cup U)|$ is equal for each i (which follows from (189) and from the fact that $Q \cap U = \emptyset$). We have that $\sum_{F \in \mathcal{F}_A} |V_1(F)| = a_1$, and using this, we get

$$\begin{aligned}\mathbf{E}[|V_1(\mathcal{F}^i_A)|] &\leq \frac{a_1}{\tilde{a}_1} \sigma^1_A(i)|V_i \setminus (Q \cup U)| = \frac{\sigma^1_A(i)|V_i \setminus (Q \cup U)|}{1 + \frac{\eta}{3}} \\ &\leq \left(1 - \frac{\eta}{4}\right) \sigma^1_A(i)|V_i \setminus (Q \cup U)|.\end{aligned}$$

Using these estimates, we have

$$(1 - \eta')\sigma^1_A(i)|V_i \setminus (Q \cup U)| - \mathbf{E}[|V_1(\mathcal{F}^i_A)|] \geq \left(\frac{\eta}{4} - \eta'\right) \sigma^1_A(i)|V_i \setminus (Q \cup U)| \geq \frac{\eta\eta'}{16N} n,$$

where we used $\sigma^1_A(i) \geq \eta'$ (as $i \in \Xi_c$) and $|V_i \setminus (Q \cup U)| \geq |V_i|/2$ (which follows from (189)) in the last inequality.

Now, note that changing the allocation of a single shrub F changes the value of $|V_1(\mathcal{F}^i_A)|$ by at most $c_F := |V_1(F)|$. Since the partition is a $(\rho|V(T)|)$-fine-partition, we have by (A5) that $c_F \leq \rho|V(T)| = \rho k$ for every $F \in \mathcal{F}_A$. Moreover, we have $\sum_{F \in \mathcal{F}_A} c_F \leq k$. Therefore,

$$\sum_{F \in \mathcal{F}_A} c_F^2 \leq (\rho k) \sum_{F \in \mathcal{F}_A} c_F \leq \rho k^2.$$

Putting all together, using Lemma 10.1 and recalling $\eta' = q\eta^3$ and $k \leq n$, we have

$$\begin{aligned}\Pr\left[|V_1(\mathcal{F}^i_A)| > (1 - \eta')\sigma^1_A(i)|V_i \setminus (Q \cup U)|\right] &\leq \Pr\left[|V_1(\mathcal{F}^i_A)| > \mathbf{E}[|V_1(\mathcal{F}^i_A)|] + \frac{\eta\eta'}{16N} n\right] \\ &\leq \exp\left(-\frac{2\left(\frac{\eta\eta' n}{16N}\right)^2}{\sum_F c_F^2}\right) = \exp\left(-\frac{2(\frac{q\eta^4}{16N} n)^2}{\sum_F c_F^2}\right) \\ &\leq \exp\left(-\frac{2(\frac{q\eta^4}{16N} n)^2}{\rho k^2}\right) \leq \exp\left(-\frac{q^2\eta^8}{128N^2\rho}\right) < \frac{1}{2N},\end{aligned}$$

where in the last inequality, we used $\rho < q^2\eta^8/(128N^2 \ln(2N))$, which indeed is valid by the choice of ρ.

Similarly, to control $|V_2(\mathcal{F}^i_A)|$, we need an upper bound on its expectation. The main point here is that by (184), we have that $\sum_{j \in [N]} \frac{\gamma_A}{1+\gamma_A} \sigma_A(\overrightarrow{ij}) = \gamma_A \sigma^1_A(i)$, and we can use that $|V_j \setminus (Q \cup U)|$ have the same size for each choice of $j \in [N]$ again. Thus, (V2) will hold if we ensure that $|V_2(\mathcal{F}^i_A)| \leq (1 - \eta')\gamma_A \sigma^1_A(i)|V_i \setminus (Q \cup U)|$ holds for every $i \in [N]$. To get this, we argue in the same way as in the calculation of $\mathbf{E}[|V_1(\mathcal{F}^i_A)|]$, but now, we use that $a_2 = \sum_{F\in\mathcal{F}_A} |V_2(F)|$ and also $a_2 = \gamma_A a_1$. We have that

$$\begin{aligned}\mathbf{E}[|V_2(\mathcal{F}_A^i)|] &= \sum_{F\in\mathcal{F}_A} p_F^i |V_2(F)| \le \frac{a_2}{\tilde{a}_1}\sigma_A^1(i)|V_i\setminus(Q\cup U)| \\ &\le \left(1-\frac{\eta}{4}\right)\gamma_A\sigma_A^1(i)|V_i\setminus(Q\cup U)|.\end{aligned}$$

From this, essentially the same argument (using Lemma 10.1, and our choice of ρ) as before gives that

$$\Pr\left[|V_2(\mathcal{F}_A^i)| > (1-\eta')\gamma_A\sigma_A^1(i)|V_i\setminus(Q\cup U)|\right] < \frac{1}{2N}.$$

Thus, using a union bound over the at most N possible values of i, we see that the random allocation satisfies (V1)–(V2) simultaneously for every i with positive probability: this implies that the desired partition exists.

10.7. **Proof of Lemma 4.5: Allocating the roots.** Now, we will determine suitable sets to 'root' the shrubs. Given a shrub $F \in \mathcal{F}_A \cup \mathcal{F}_B$, recall that by (A7), F has a *seed* $s_F \in W_A \cup W_B$. In T, there must be a unique neighbour r_F of s_F inside $V(F)$; we will say that r_F is the *root* of F. Alternatively, r_F is the unique vertex in $V(F)$, which is closest to the root of T. Recall that the seed s_F is already embedded in $V_c \cup V_d$ via φ_0.

This step aims to define sets R_F, one for each $F \in \mathcal{F}_A \cup \mathcal{F}_B$, that satisfy the following properties.

(W1) if $i \in [N]$ is such that $F \in \mathcal{F}_A^i \cup \mathcal{F}_B^i$, then $R_F \subseteq V_i \setminus (Q\cup U)$,

(W2) $|R_F| = |V_1(F)|$,

(W3) $R_F \subseteq N_G(\varphi_0(s_F))$, where $s_F \in W_A \cup W_B$ is the seed of F,

(W4) all the sets R_F are pairwise-disjoint.

We proceed as follows. Let $i \in [N]$ for which $\mathcal{F}_A^i \cup \mathcal{F}_B^i$ is non-empty. Suppose first that $i \in N_\Gamma(c) \setminus N_\Gamma(d)$. In this case, we have $\sigma_B^1(i) = 0 < \eta'$ (the anchor of σ_B fits in the w-neighbourhood of d) and thus $\mathcal{F}_B^i = \emptyset$ by (V3). Hence, in this case, we have $\mathcal{F}_A^i \cup \mathcal{F}_B^i \subseteq \mathcal{F}_A$. By (V3) again, we have that $\sigma_A^1(i) \ge \eta'$, and therefore $\sigma_A^1(i) - \varepsilon \ge (1-\eta')\sigma_A^1(i)$ (where we used $\varepsilon \le (\eta')^2$). Let $F \in \mathcal{F}_A^i$, and let $s_F \in W_A$ be the seed of F. As the anchor of σ_A fits in the w-neighbourhood of c by (B2), we have that $\sigma_A^1(i) \le w(ci) = d(V_c, V_i)$. By (V4), (U3) and (V1), we get

$$\begin{aligned}|N_G(\varphi_0(s_F)) \cap V_i \setminus (Q\cup U)| &\ge (w(ci)-\varepsilon)|V_i\setminus(Q\cup U)| \ge (\sigma_A^1(i)-\varepsilon)|V_i\setminus(Q\cup U)| \\ &\ge (1-\eta')\sigma_A^1(i)|V_i\setminus(Q\cup U)| \ge |V_1(\mathcal{F}_A^i)|.\end{aligned}$$

This means that we can find a set $R_F \subseteq \big(N_G(\varphi_0(s_F)) \cap V_i\big) \setminus (Q\cup U)$ of size precisely $|R_F| = |V_1(F)|$. Moreover, we can do it in such a way that all the sets $R_{F'} \subseteq V_i$, for $F' \in \mathcal{F}_A^i$, are pairwise-disjoint. (Indeed, for this fixed i we have $|N_G(\varphi_0(s_F)) \cap V_i \setminus (Q\cup U)| \ge |V_1(\mathcal{F}_A^i)|$, so choosing the R_F one by one uses at most the remaining total demand and preserves enough capacity in $V_i \setminus (Q\cup U)$ for the shrubs still to be assigned.) The argument is analogous if $i \in N_\Gamma(d) \setminus N_\Gamma(c)$, and in that case we can also find pairwise-disjoint sets $R_F \subseteq V_i$, for all shrubs $F \in \mathcal{F}_B^i$.

It is only left to consider the case where $i \in N_\Gamma(c) \cap N_\Gamma(d)$. By (B4), we have that one of $w(ci)$ or $w(di)$ is at least $\sigma_A^1(i) + \sigma_B^1(i)$. Without loss of generality, we can assume that the former inequality holds, i.e. $w(ci) \ge \sigma_A^1(i) + \sigma_B^1(i)$ (the proof in the complementary case is analogous). In this case, we first define R_F for all shrubs $F \in \mathcal{F}_B^i$ as above, which we can do since σ_B is anchored in $N_\Gamma(d)$. Next, for the remaining shrubs $F \in \mathcal{F}_A^i$ with seeds s_F, we note that

$$\begin{aligned}|N_G(\varphi_0(s_F)) \cap V_i \setminus (Q\cup U)| &\ge (w(ci)-\varepsilon)|V_i\setminus(Q\cup U)| \\ &\ge (\sigma_A^1(i)+\sigma_B^1(i)-\varepsilon)|V_i\setminus(Q\cup U)| \ge |V_1(\mathcal{F}_A^i)| + |V_1(\mathcal{F}_B^i)|.\end{aligned}$$

So we can again find a set $R_F \subseteq N(\varphi_0(s_F)) \cap V_i \setminus (Q \cup U)$ of size $|R_F| = |V_1(F)|$. Moreover, we can find such set that is also disjoint from every previously chosen $R_{F'} \subseteq V_i$. (Here $(w(ci) - \varepsilon)|V_i \setminus (Q \cup U)| \geq |V_1(\mathcal{F}^i_A)| + |V_1(\mathcal{F}^i_B)|$, so the same greedy capacity vs demand argument ensures pairwise disjointness.) We have thus ensured (W1)–(W4) hold.

10.8. **Proof of Lemma 4.5: Finding suitable clusters.** Summarising what we have done so far: we have already embedded $W_A \cup W_B$ in $G[V_c \cup V_d]$ via φ_0, we have already allocated each shrub $F \in \mathcal{F}_A \cup \mathcal{F}_B$ to a cluster $i \in V(\Gamma)$ where $V_1(F)$ will be embedded. Moreover, for each such shrub, we have also reserved a 'private' set $R_F \subseteq V_i$ of the right size $|V_1(F)|$.

Now, for each shrub $F \in \mathcal{F}^i_A \cup \mathcal{F}^i_B$, we would like to show the existence of another index $\ell \in V(\Gamma)$, such that F can be embedded in $G[V_i, V_\ell]$. In fact, we will show that we can find so many such ℓ that some of the graphs $G[V_i, V_\ell]$ can be used to embed F even if we have previously embedded other shrubs and we need to avoid using the space used by these shrubs.

We introduce some notation that we use to work with partial functions from one set to another. Let $Z, X' \subseteq X$, Y be an arbitrary set and $\varphi : X' \to Y$ be a function. We say that φ is a *partial function* from X to Y, and by $\varphi(Z)$, we mean $\varphi(Z \cap X')$, i.e. the set of images of the elements in Z which do have their image defined by φ.

Now, recall that φ_0 is the embedding we have already defined on $W_A \cup W_B$. We say that a partial function $\varphi : V(T) \to V(G)$ is a *partial embedding of T* if

(X1) φ extends φ_0,

(X2) φ is defined precisely on $W_A \cup W_B$ and $\bigcup_{F \in \mathcal{F}} V(F)$, where $\mathcal{F} \subseteq \mathcal{F}_A \cup \mathcal{F}_B$; and

(X3) for each $i \in [N]$, we have $\varphi(V_1(\mathcal{F}^i_A) \cup V_1(\mathcal{F}^i_B)) \subseteq V_i$.

(X4) for each $i \in [N]$, we have $\varphi(V_2(\mathcal{F}^i_A) \cup V_2(\mathcal{F}^i_B)) \cap V_i = \emptyset$

Thus, a partial embedding is an embedding defined on the seeds $W_A \cup W_B$ and a subset of the shrubs in $\mathcal{F}_A \cup \mathcal{F}_B$. The last property means that if a shrub satisfies $F \in \mathcal{F}^i_A \cup \mathcal{F}^i_B$ —that is, F is a shrub allocated to the ith cluster— and its image is defined by φ, then we naturally must have $\varphi(V_1(F)) \subseteq V_i$. In the next step, we will try to extend partial embeddings by including one more shrub at a time.

Given a partial embedding φ, $L \in \{A, B\}$ and $1 \leq i \leq N$, we will say that $\ell \in [N]$ is a *target index for* (φ, L, i) if

(Y1) $|\varphi(\mathcal{F}^i_L) \cap V_\ell| < (1 + \eta'/2)(1 - \eta')\frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)|$, and

(Y2) $\frac{(\eta')^2}{2}\frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)| \geq \rho k$.

Intuitively, if ℓ is a target index for (φ, L, i); then we can try to extend φ by embedding a shrub $F \in \mathcal{F}^i_L$ in $G[V_i, V_\ell]$. Now, we prove that target indexes always exist.

Claim 10.7 (Finding a target index)**.** *Let φ be a partial embedding, and let $L \in \{A, B\}$ and $1 \leq i \leq N$ be such that $\mathcal{F}^i_L \neq \emptyset$. Then a target index exists for (φ, L, i).*

Proof of the claim. First, we shall observe that actually most of the clusters V_ℓ satisfy (Y2). Intuitively, if (Y2) fails for a given $1 \leq \ell \leq N$, the skew-matching σ_L will have very little weight on the pair $\overrightarrow{i\ell}$. More precisely, we will show the following. Denote by $\mathcal{Y} \subseteq V(\Gamma)$ the set of all ℓ which do not satisfy (Y2). We claim that

$$\sum_{\ell \in \mathcal{Y}} \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)| \leq \frac{\eta'}{200}\sum_{\ell \in [N]} \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)|. \tag{192}$$

Let $X_{\mathcal{Y}}$ be the left-hand side of (192). By definition, we have

$$\frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)| < \frac{2\rho k}{(\eta')^2}$$

for any $\ell \in \mathcal{Y}$, and therefore, we have

$$X_{\mathcal{Y}} < \frac{2\rho k N}{(\eta')^2} < \frac{\eta(\eta')^2}{400}\frac{n}{N},$$

where we have used that $k \leq n$ and $\rho < (\eta')^4\eta/(800N^2)$ in the second inequality. From (180), we can deduce $n/(2N) \leq |V_i \setminus (Q \cup U)|$, and therefore

$$X_{\mathcal{Y}} \leq \frac{\eta(\eta')^2}{200}|V_i \setminus (Q \cup U)| \leq \frac{\eta\eta'}{200}\sigma_L^1(i)|V_i \setminus (Q \cup U)|,$$

where in the last inequality, we used that $\mathcal{F}_L^i \neq \emptyset$ (by assumption) together with (V3). Next, using (183) first and then (184)–(185), we arrive at

$$X_{\mathcal{Y}} \leq \frac{\eta'}{200}\gamma_L\sigma_L^1(i)|V_i \setminus (Q \cup U)| \leq \frac{\eta'}{200}\sum_{\ell\in[N]}\frac{\gamma_L}{1+\gamma_L}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)|,$$

so indeed (192) holds.

Now, we argue that there must exist $\ell \notin \mathcal{Y}$ which satisfies (Y1), i.e. such that the inequality $|\varphi(\mathcal{F}_L^i) \cap V_\ell| < (1+\eta'/2)(1-\eta')\frac{\gamma_L}{1+\gamma_L}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)|$ holds. For this, we will find a lower bound for the sum

$$S := \sum_{\ell\notin\mathcal{Y}}\left(1+\frac{\eta'}{2}\right)(1-\eta')\frac{\gamma_L}{1+\gamma_L}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)|.$$

Indeed, we have

$$\begin{aligned}
S &= \left(1+\frac{\eta'}{2}\right)(1-\eta')\frac{\gamma_L}{1+\gamma_L}\sum_{\ell\notin\mathcal{Y}}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)| \\
&= \left(1+\frac{\eta'}{2}\right)(1-\eta')\frac{\gamma_L}{1+\gamma_L}\left(\sum_{\ell\in[N]}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)| - \sum_{\ell\in\mathcal{Y}}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)|\right) \\
&\overset{(192)}{\geq} \left(1+\frac{\eta'}{2}\right)(1-\eta')\left(1-\frac{\eta'}{200}\right)\sum_{\ell\in[N]}\frac{\gamma_L}{1+\gamma_L}\sigma_L(\vec{i\ell})|V_\ell \setminus (Q \cup U)| \\
&\overset{(V2)}{\geq} \left(1+\frac{\eta'}{2}\right)\left(1-\frac{\eta'}{200}\right)|V_2(\mathcal{F}_L^i)| \geq \left(1+\frac{\eta'}{4}\right)|V_2(\mathcal{F}_L^i)| > |V_2(\mathcal{F}_L^i)|.
\end{aligned}$$

Thus, $S > |V_2(\mathcal{F}_L^i)|$. To finish, we use that the vertices in $V_1(\mathcal{F}_L^i)$ are mapped only to V_i via φ (because φ is a partial embedding), and thus

$$\sum_{\ell\notin\mathcal{Y}}|\varphi(\mathcal{F}_L^i) \cap V_\ell| \leq \sum_{\ell\in N_\Gamma(i)}|\varphi(\mathcal{F}_L^i) \cap V_\ell| = |\varphi(\mathcal{F}_L^i) \setminus V_i| \leq |V_2(\mathcal{F}_L^i)| < S.$$

Hence, there is an ℓ satisfying (Y1) and (Y2), as required. □

Given a partial embedding φ, $L \in \{A, B\}$ and $1 \leq i \leq N$, let $\mathcal{L}_{\varphi,L,i}$ be the set of all target indexes for (φ, L, i). In this notation, Claim 10.7 ensures that $\mathcal{L}_{\varphi,L,i}$ is non-empty for φ, L and i whenever $\mathcal{F}_L^i$ is nonempty. For technical reasons, we will need to compare the sets $\mathcal{L}_{\varphi,L,i}$ and $\mathcal{L}_{\varphi',L,i}$ for two different partial embeddings φ, φ'. What we need is the following easy fact, which says that the set of target indexes is "decreasing".

Claim 10.8. *Let $L \in \{A, B\}$ and $1 \leq i \leq N$, and let φ, φ' be two partial embeddings. If φ' extends φ, then $\mathcal{L}_{\varphi',L,i} \subseteq \mathcal{L}_{\varphi,L,i}$.*

Proof of the claim. Given $\ell \in \mathcal{L}_{\varphi',L,i}$, we need to show that $\ell \in \mathcal{L}_{\varphi,L,i}$. We need to check that (Y1)–(Y2) hold for ℓ and φ. Since (Y2) depends only on i, L, ℓ and not on φ, φ'; we only need to check that (Y1) holds. Since φ' extends φ, we have that $|\varphi(\mathcal{F}_L^i) \cap V_\ell| \leq |\varphi'(\mathcal{F}_L^i) \cap V_\ell|$. Combined with the fact that (Y1) holds for ℓ, φ', L, i, this shows that (Y1) holds for ℓ, φ, L, i, as required. □

10.9. **Proof of Lemma 4.5: First Embedding Phase.** After all this preparation, it is finally the turn to embed the shrubs in $\mathcal{F}_A \cup \mathcal{F}_B$. We will fix an enumeration of all the shrubs and define our embedding iteratively, incorporating one shrub at a time, respecting this order. In a first attempt, we shall try to use Lemma 10.5 for each shrub while avoiding the set Q for all but a constant number of vertices (which will correspond to the adventitious roots). It turns out that this procedure can fail for some shrubs; those will not be embedded in this step but instead deferred to the Second Embedding Phase (as we will show later, the number of vertices of the shrub which will fail to be embedded in the First Embedding Phase will be tiny).

From now on, we will work with a fixed enumeration of the shrubs in $\mathcal{F}_A \cup \mathcal{F}_B$. Let $J := |\mathcal{F}_A| + |\mathcal{F}_B|$ be the total number of shrubs. Let $F_1, F_2, \dots, F_J$ be an enumeration of the shrubs in $\mathcal{F}_A \cup \mathcal{F}_B$, chosen so that each tree in $\mathcal{F}_A^1 \cup \mathcal{F}_B^1$ appears before each tree in $\mathcal{F}_A^2 \cup \mathcal{F}_B^2$ in the ordering, and so on. Formally, we will have that for every $1 \leq i_1 < i_2 \leq N$; if $F_{j_1} \in \mathcal{F}_A^{i_1} \cup \mathcal{F}_B^{i_1}$ and $F_{j_2} \in \mathcal{F}_A^{i_2} \cup \mathcal{F}_B^{i_2}$, then $j_1 < j_2$.

Recall that for each shrub $F \in \mathcal{F}_A \cup \mathcal{F}_B$, we have defined sets X_F (in Step 3) and R_F (in Step 4), and we have identified a root $r_F \in V(F)$ and a seed $s_F \in W_A \cup W_B$ (in Step 4). If $1 \leq j \leq J$ is such that $F = F_j$, from now on, we will call these objects X_j, R_j, r_j, s_j, respectively.

Before continuing with the embedding, we will formally obtain "rooted shrubs" from the shrubs $F_1, \dots, F_J$ to be able to apply Lemma 10.5. To do so, we just need to specify the roots and adventitious roots in each F_j, which we do as follows. We have already (at the beginning of Step 4) identified the seed $s_j \in W_A \cup W_B$ and the root $r_j \in V(F_j)$ of F_j. T is a tree and therefore every vertex from $W_A \cup W_B$ can have at most one neighbour in F_j. By (A8), there are at most two neighbours of $W_A \cup W_B$ in $V(F_j)$. If there is no neighbour of $W_A \cup W_B$ in $V(F_j)$ apart from r_j, we set $x_j = \emptyset$, i.e. F_j has no adventitious root. Otherwise, there exists $x_j \neq r_j$ in $V(F_j)$ which is a neighbour of $W_A \cup W_B$. By (A9), x_j and r_j have distance at least 4 in F_j. Thus, we can set x_j as the adventitious root of F_j. In all cases, (F_j, r_j, x_j) defines a valid rooted shrub, and thus $\{(F_j, r_j, x_j)\}_{j=1}^J$ is a family of vertex-disjoint rooted shrubs.

Now, we describe our embedding process in detail. We begin by describing two certain invariants that will be useful to track during the construction of the embeddings. We will say that a partial embedding φ is *reasonable* if

(Z1) for all distinct $i, \ell \in [N]$, and all $L \in \{A, B\}$,
$$|\varphi(\mathcal{F}_L^i) \cap V_\ell| < \frac{\gamma_L}{1+\gamma_L} \sigma_L(\overrightarrow{i\ell}) |V_\ell \setminus (Q \cup U)|.$$

In what follows, we will work with reasonable embeddings only.

The second invariant we need concerns the sets of vertices we would like to avoid using when embedding F_j. For $1 \leq j \leq J$, we will say that φ is a *j-partial embedding* if it is defined only for (a subset of) the trees in $F_1, \dots, F_j$ (and none of the trees $F_{j+1}, \dots, F_J$). In our setting, a $(j-1)$-partial embedding φ is given, and we want to find space to embed F_j. We do not want to use vertices in Q (those are reserved for the adventitious roots and the Second Embedding Phase), we do not want to use vertices in R_k for $k > j$, and obviously, we do not want to use vertices already used by φ. This defines the set of *forbidden vertices for (φ, j) as*
$$Z_{\varphi,j}^{\text{forb}} := Q \cup \text{im}(\varphi) \cup \bigcup_{j+1 \leq k \leq J} R_k.$$
We also set $Z_{\varphi,J+1}^{\text{forb}} := Q \cup \text{im}(\varphi)$.

The importance of reasonable $(j-1)$-partial embeddings φ is that they leave sufficient space in each cluster while avoiding the forbidden vertices $Z_{\varphi,j}^{\text{forb}}$. We express this as a claim.

Claim 10.9. *Let $1 \leq j \leq J+1$ and let φ be a reasonable $(j-1)$-partial embedding. Then, for any $1 \leq i \leq N$, we have $|V_i \setminus Z_{\varphi,j}^{\text{forb}}| \geq |V_i \cap U|$.*

Proof of the claim. We first estimate the vertices of V_i already used by the V_1-parts of embedded shrubs. Since for each $i \in [N]$ we have $\varphi(V_1(\mathcal{F}_A^i) \cup V_1(\mathcal{F}_B^i)) \subseteq V_i$, it follows that

$$|V_i \cap \varphi(V_1(\mathcal{F}_A \cup \mathcal{F}_B))| = \sum_{L\in\{A,B\}} \sum_{k\in[N]} |V_i \cap \varphi(V_1(\mathcal{F}_L^k))| = \sum_{L\in\{A,B\}} |V_i \cap \varphi(V_1(\mathcal{F}_L^i))|. \tag{193}$$

Moreover, since φ is a $(j-1)$-partial embedding, only shrubs among $F_1, \dots, F_{j-1}$ contribute to the left-hand side above; while for $k \geq j+1$, the set R_k contributes $|V_1(F_k)|$ whenever $F_k \in \mathcal{F}_A^i \cup \mathcal{F}_B^i$. Therefore,

$$|V_i \cap \varphi(V_1(\mathcal{F}_A \cup \mathcal{F}_B))| + \sum_{j+1\leq k\leq J} |V_i \cap R_k| \leq |V_1(\mathcal{F}_A^i)| + |V_1(\mathcal{F}_B^i)|. \tag{194}$$

On the other hand, we have

$$\begin{aligned}
|V_i \cap \varphi(V_2(\mathcal{F}_A \cup \mathcal{F}_B))| &= \sum_{L\in\{A,B\}} \sum_{k\in[N]\setminus\{i\}} |V_i \cap \varphi(V_2(\mathcal{F}_L^k))| \\
&\leq \sum_{L\in\{A,B\}} \sum_{k\in[N]\setminus\{i\}} |V_i \cap \varphi(\mathcal{F}_L^k)| \\
&\overset{(Z1)}{\leq} \sum_{L\in\{A,B\}} \sum_{k\in[N]\setminus\{i\}} \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{ki})|V_i \setminus (Q \cup U)| \\
&\leq |V_i \setminus (Q \cup U)|(\sigma_A^2(i) + \sigma_B^2(i)),
\end{aligned} \tag{195}$$

where we used the definition of σ_A^2 and σ_B^2 in the last step.

Putting it all together, we have

$$\begin{aligned}
&|V_i \setminus Z_{\varphi,j}^{\text{forb}}| = \left| V_i \setminus \left(Q \cup \operatorname{im}(\varphi) \cup \bigcup_{j+1\leq k\leq J} R_k \right) \right| \\
&\geq |V_i \setminus Q| - |V_i \cap \varphi(V_1(\mathcal{F}_A \cup \mathcal{F}_B))| - |V_i \cap \varphi(V_2(\mathcal{F}_A \cup \mathcal{F}_B))| - \sum_{j+1\leq k\leq J} |V_i \cap R_k| \\
&\overset{(194)}{\geq} |V_i \setminus Q| - |V_i \cap \varphi(V_2(\mathcal{F}_A \cup \mathcal{F}_B))| - (|V_1(\mathcal{F}_A^i)| + |V_1(\mathcal{F}_B^i)|) \\
&\overset{(V1)}{\geq} |V_i \setminus Q| - |V_i \cap \varphi(V_2(\mathcal{F}_A \cup \mathcal{F}_B))| - |V_i \setminus (Q \cup U)|(\sigma_A^1(i) + \sigma_B^1(i)) \\
&\overset{(195)}{\geq} |V_i \setminus Q| - |V_i \setminus (Q \cup U)|(\sigma_A^1(i) + \sigma_B^1(i) + \sigma_A^2(i) + \sigma_B^2(i)) \\
&\geq |V_i \setminus Q| - |V_i \setminus (Q \cup U)| \geq |V_i \cap U|,
\end{aligned}$$

where in the second to last we used that σ_A, σ_B are disjoint skew-matchings, and in the last inequality we used that U and Q are disjoint. This proves the claim. $\square$

The importance of target indices is that they will allow us to extend reasonable embeddings to reasonable embeddings, as shown in the following claim.

Claim 10.10. *Let φ be a reasonable partial embedding that does not embed the shrub F_j. Let i, L and ℓ be such that $F_j \in \mathcal{F}_L^i$, and that ℓ is a target index for (φ, L, i). Suppose φ' extends φ by embedding $V_1(F_j)$ in V_i and $V_2(F_j)$ in V_ℓ. Then φ' is a reasonable partial embedding.*

Proof of the claim. To see that (Z1) holds for φ', it is enough to check the property for i, ℓ, and L, since φ is reasonable and the embedding only changed because of F_j. Thus, we have

$$|\varphi'(\mathcal{F}_L^i) \cap V_\ell| \leq |\varphi(\mathcal{F}_L^i) \cap V_\ell| + |V_2(F_j)| \leq |\varphi(\mathcal{F}_L^i) \cap V_\ell| + |V(F_j)| \overset{(A5)}{\leq} |\varphi(\mathcal{F}_L^i) \cap V_\ell| + \rho k$$

$$\leq \left((1+\eta'/2)(1-\eta') + \frac{\eta'^2}{2}\right) \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)|,$$
$$< \frac{\gamma_L}{1+\gamma_L}\sigma_L(\overrightarrow{i\ell})|V_\ell \setminus (Q \cup U)|,$$

where we used both (Y1)–(Y2) of the definition of the target index in the second to last inequality. This proves that φ' is reasonable. $\square$

As we said before, an embedding of a considered shrub can fail. Now, we can precisely describe the condition that ensures that the embedding of a shrub is successful or not. Suppose $1 \leq j \leq J$ is given, and let φ be a $(j-1)$-partial embedding. Let $1 \leq i \leq N$ and $L \in \{A, B\}$ be such that $F_j \in \mathcal{F}^i_L$. We say that a target index $\ell \in \mathcal{L}_{\varphi,L,i}$ is *successful for* F_j if there exists $v \in R_j$ such that

$$\deg_G\left(v, V_\ell \setminus Z^{\mathrm{forb}}_{\varphi,j}\right) \geq 2\varepsilon|V_\ell|, \tag{196}$$

We can now state the following crucial claim. This claim ensures that if a successful target index exists, then a reasonable $(j-1)$-partial embedding can be extended to a reasonable j-partial embedding by including F_j.

Claim 10.11. *Suppose $1 \leq j \leq J$ is given, that $1 \leq i \leq N$ and $L \in \{A, B\}$ are such that $F_j \in \mathcal{F}^i_L$. Suppose φ is a reasonable $(j-1)$-partial embedding such that $R_j \cap \mathrm{im}(\varphi) = \emptyset$, only adventitious roots are embedded in Q, and that ℓ is a successful target index for (φ, L, i). Then, there exists a reasonable j-partial embedding φ', that extends φ by embedding F_j in $G[V_i, V_\ell]$. Moreover, $\varphi'(r_j) \in R_j$, $\varphi'(x_j) \in Q \cap V_i$, and $V(F_j) \setminus \{r_j, x_j\}$ is mapped to $(V_i \cup V_\ell) \setminus Z^{\mathrm{forb}}_{\varphi,j}$.*

To be clear, if $x_j = \emptyset$, the 'moreover' part just claims that $\varphi'(r_j) \in R_j$ and $V(F_j)\setminus\{r_j\}$ is mapped to $(V_i \cup V_\ell) \setminus Z^{\mathrm{forb}}_{\varphi,j}$.

Proof of the claim. If $x_j = \emptyset$, define $P_j := (Q \cap V_i) \setminus \mathrm{im}(\varphi)$; otherwise x_j is an adventitious root of (F_j, r_j, x_j). Therefore, by (A8), x_j has a unique neighbour, say y_j, in $W_A \cup W_B$. We define $P_j := (Q \cap V_i \cap N(\varphi_0(y_j))) \setminus \mathrm{im}(\varphi)$. Define $\tilde{U} := (V_i \cup V_\ell) \setminus Z^{\mathrm{forb}}_{\varphi,j}$. Since ℓ is successful, we can choose a vertex $v_j \in R_j$ that satisfies (196), by the assumption that $R_j \cap \mathrm{im}(\varphi) = \emptyset$, we know that v_j is available to use in the embedding. We wish to apply Lemma 10.5 with

object	V_i	V_ℓ	$\tilde{U}$	P_j	v_j	(F_j, r_j, x_j)	$\eta/4$
in place of	X	Y	U	P	v	(T, r, x)	$\tilde{\eta}$

We will check that the required (T1)–(T4) hold.

We begin by checking (T1), which means that we need to prove that $|V_i \cap \tilde{U}| \geq \eta m/4$ and $|V_\ell \cap \tilde{U}| \geq \eta m/4$ hold. We have that $|V_i \cap \tilde{U}| = |V_i \setminus Z^{\mathrm{forb}}_{\varphi,j}|$, so from Claim 10.9, we deduce that $|V_i \cap \tilde{U}| \geq |V_i \cap U|$. From (189), we have that $|V_i \cap U| = \eta m/4$, so we conclude that $|V_i \cap \tilde{U}| \geq \eta m/4$. The proof of $|V_\ell \cap \tilde{U}| \geq \eta m/4$ is identical.

To check (T2), we must show that $|P_j| \geq 2\varepsilon m$ holds. We might suppose we are in the case where the adventitious root x_j exists and we have set $P_j = (Q \cap V_i \cap N(\varphi_0(y_j))) \setminus \mathrm{im}(\varphi)$, the other case is more straightforward. By assumption, only adventitious roots are contained in $\mathrm{im}(\varphi) \cap Q$. Since only predecessors of $W_A \cup W_B$ can be adventitious roots, we deduce that $|\mathrm{im}(\varphi) \cap Q| \leq |W_A \cup W_B| \leq 5\varepsilon|V_i|$, we used (188) and $\rho \leq \varepsilon$ in the last step. On the other hand, we have that $y_j \in N_T(V(F_j)) \cap (W_A \cup W_B)$ thus $i \in I_{y_j}$ by (V4), and therefore $|Q \cap V_i \cap N(\varphi_0(y_j))| \geq 7\varepsilon|V_i|$ follows from (U3) or (U4). Hence, we have $|P_j| \geq |P \cap V_i \cap N(\varphi_0(y_j))| - |\mathrm{im}(\varphi) \cap Q| \geq 7\varepsilon|V_i| - 5\varepsilon|V_i| = 2\varepsilon|V_i|$, so (T2) holds.

Finally, (T3) holds because v_j was chosen to satisfy (196); and (T4) follows from the fact that we are working with a $\rho|V(T)|$-fine partition Thus, we can use Lemma 10.5 as intended. Define φ' as φ extended by the inclusion of F_j. By construction, it is simple to check that indeed $\varphi'(r_j) \in R_j$, $\varphi'(x_j) \in Q \cap V_i$, and $V(F_j) \setminus \{r_j, x_j\}$ is mapped to $(V_i \cup V_\ell) \setminus Z^{\text{forb}}_{\varphi,j}$, as required. Finally, φ' is a reasonable j-partial embedding by Claim 10.10. □

Now, we can iteratively define j-partial embeddings for all $0 \leq j \leq J$. We will also define an increasing family $\mathcal{T}_j$ of trees for all $0 \leq j \leq J$, $\mathcal{T}_j$ will correspond to the set of *postponed shrubs* among $F_1, \dots, F_j$ whose embedding failed and have been postponed to the Second Embedding Phase.

Recall that φ_0 corresponds to the already-defined embedding of $T[W_A \cup W_B]$ done in Step 2. Clearly, φ_0 is reasonable and $(Q \cup \bigcup_{k=1}^J R_k) \cap \operatorname{im}(\varphi_0) = \emptyset$. Also, let $\mathcal{T}_0 = \emptyset$.

Next, suppose we are given $1 \leq j \leq J$, the set $\mathcal{T}_{j-1}$ and a reasonable $(j-1)$-partial embedding φ_{j-1} such that $\bigcup_{k=j}^J R_k \cap \operatorname{im}(\varphi_{j-1}) = \emptyset$ and $Q \cap \operatorname{im}(\varphi_{j-1})$ contains only adventitious roots. In the j-th round of the embedding process we will define a reasonable j-partial embedding φ_j which extends φ_{j-1}, and $\mathcal{T}_j \supseteq \mathcal{T}_{j-1}$, as follows. Let $L \in \{A, B\}$ and $1 \leq i \leq N$ such that $F_j \in \mathcal{F}^i_L$. We consider the set $\mathcal{L}_{\varphi_{j-1},L,i}$ of target indexes for (φ_{j-1}, L, i). We know by Claim 10.7 that $\mathcal{L}_{\varphi_{j-1},L,i}$ is non-empty. There are two possibilities: either there exists a successful index $\ell \in \mathcal{L}_{\varphi_{j-1},L,i}$ or not. If there exists a successful ℓ, then by Claim 10.11 there exists a reasonable j-partial embedding φ_j which extends φ_{j-1}, and embeds F_j. Moreover, $\varphi_j(F_j) \cap Z^{\text{forb}}_{\varphi_{j-1},j} \subseteq \{\varphi_j(x_j), \varphi_j(r_j)\}$, $\varphi_j(x_j) \subseteq Q$, $\varphi_j(r_j) \subseteq R_j$, $R_j \subseteq Z^{\text{forb}}_{\varphi_{j-1},j} \setminus Q$ and $R_k \cap (Q \cup R_j) = \emptyset$ for all $k \geq j+1$.

Therefore, we have that $\bigcup_{k=j+1}^J R_k \cap \operatorname{im}(\varphi_j) = \emptyset$, and $Q \cap \operatorname{im}(\varphi_j)$ contains only adventitious roots. In this case, F_j was not postponed and we set $\mathcal{T}_j := \mathcal{T}_{j-1}$. Otherwise, if there is no successful $\ell \in \mathcal{L}_{\varphi_{j-1},L,i}$, then, we set $\varphi_j := \varphi_{j-1}$ and $\mathcal{T}_j := \mathcal{T}_{j-1} \cup \{F_j\}$. This process finishes with a final J-partial embedding $\varphi^\star := \varphi_J$ and a final set of postponed shrubs $\mathcal{T}^\star := \mathcal{T}_J$, by construction each shrub not in $\mathcal{T}^\star$ is embedded by $\varphi^\star$.

Here, we record a crucial property of how we defined the embeddings. Every shrub $F_j \in \mathcal{T}^\star$ was postponed only if there was no successful index ℓ among all its target indexes. Thus, we have

(Z2) for every $F_j \in \mathcal{T}^\star$ such that $F_j \in \mathcal{F}^i_L$, every target index $\ell \in \mathcal{L}_{\varphi_{j-1},L,i}$, and for every $v \in R_j$, we have

$$\deg_G\left(v, V_\ell \setminus Z^{\text{forb}}_{\varphi_{j-1},j}\right) < 2\varepsilon|V_\ell|.$$

10.10. **Proof of Lemma 4.5: Second Embedding Phase.** To conclude the embedding of the whole tree T, it only remains to extend the embedding $\varphi^\star$ by defining the embedding of every postponed shrub in $\mathcal{T}^\star$. For each $1 \leq i \leq N$, let $\mathcal{T}^\star_{i,A} = \mathcal{T}^\star \cap \mathcal{F}^i_A$, $\mathcal{T}^\star_{i,B} = \mathcal{T}^\star \cap \mathcal{F}^i_B$, and $\mathcal{T}^\star_i = \mathcal{T}^\star_{i,A} \cup \mathcal{T}^\star_{i,B}$. We will proceed in rounds, extending $\varphi^\star$ one shrub at a time. We do this by incorporating each $F_j \in \mathcal{T}^\star$ in increasing order of j.

The following claim implies that each set $\mathcal{T}^\star_i$ of postponed trees is tiny.

Claim 10.12. *For each $1 \leq i \leq N$ and $L \in \{A, B\}$, we have $|V_1(\mathcal{T}^\star_{i,L})| \leq \varepsilon|V_i|$.*

Proof of the claim. To see this, we need to carefully track what happens when a shrub F is postponed and ends up belonging in $\mathcal{T}^\star_{i,L}$.

Let $j_{\max}$ be the maximum value of j such that $F_j \in \mathcal{T}^\star_{i,L}$, and let $\ell_{\max}$ be any target index in $\mathcal{L}_{\varphi_{j_{\max}-1},L,i}$. We clearly have $\ell_{\max} \neq i$. Now, let j be arbitrary such that $F_j \in \mathcal{T}^\star_{i,L}$. By the choice of $j_{\max}$, we have that $j \leq j_{\max}$ and that $\varphi_{j_{\max}}$ extends φ_j, therefore by Claim 10.8, we have that $\mathcal{L}_{\varphi_{j_{\max}-1},L,i} \subseteq \mathcal{L}_{\varphi_{j-1},L,i}$. This implies that $\ell_{\max}$ is a target index for (φ_{j-1}, L, i). Therefore, by (Z2), we deduce that for any $v \in R_j$,

$$\deg_G\left(v, V_{\ell_{\max}} \setminus Z^{\text{forb}}_{\varphi_{j-1},j}\right) < 2\varepsilon|V_{\ell_{\max}}|. \tag{197}$$

Now, we make the following crucial observation: Because of the way we ordered the shrubs, during the definition of the partial embeddings $\varphi_j, \varphi_{j+1}, \dots, \varphi_{j_{\max}}$, we have only considered trees which are in $\mathcal{F}^i_A \cup \mathcal{F}^i_B$. This means, by (W1), that $R_j \cup R_{j+1} \cup \dots \cup R_{j_{\max}}$

are all contained in V_i, and thus are disjoint from $V_{\ell_{\max}}$. Together with $\mathrm{im}(\varphi_{j-1}) \subseteq \mathrm{im}(\varphi_{j_{\max}})$, this implies that

$$V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j_{\max}-1},j_{\max}} \subseteq V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j-1},j}.$$

Combined with (197), we obtain that for each $v \in R_j$,

$$\deg_G\left(v, V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j_{\max}-1},j_{\max}}\right) < 2\varepsilon|V_{\ell_{\max}}|. \tag{198}$$

Let $R_{i,L} := \bigcup_j R_j$, where the union is taken over all j such that the shrub F_j belongs in $\mathcal{T}^\star_{i,L}$. Since j was arbitrary in the argument above, (198) holds for each $v \in R_{i,L}$.

Claim 10.9 implies that $|V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j_{\max}-1},j_{\max}}| \geq |V_i \cap U| = \eta|V_i|/4$. Since $\varepsilon \ll \mathfrak{d} \ll \eta$, we deduce that

$$2\varepsilon|V_{\ell_{\max}}| < (\mathfrak{d} - \varepsilon)\left|V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j_{\max}-1},j_{\max}}\right|.$$

Using the terminology of Lemma 10.2, this and (198) implies that every vertex in $R_{i,L}$ is 'atypical' with respect to $V_{\ell_{\max}} \setminus Z^{\mathrm{forb}}_{\varphi_{j_{\max}-1},j_{\max}}$. Thus, Lemma 10.2 implies that $|R_{i,L}| \leq \varepsilon|V_i|$, and we can conclude that

$$|V_1(\mathcal{T}^\star_{i,L})| = \sum_{j:F_j\in\mathcal{T}^\star_{i,L}} |V_1(F_j)| = \sum_{j:F_j\in\mathcal{T}^\star_{i,L}} |R_j| = |R_{i,L}|,$$

where we used (W2) in the second to last equality. □

Now, using this claim, we can define the embedding of the shrubs, as we explained before. The following claim takes care of one round of this process by embedding a single extra shrub.

Claim 10.13. *Let i, L be such that $F_j \in \mathcal{T}^\star_{i,L}$, and suppose φ is a reasonable partial embedding of T which extends $\varphi^\star$ and does not embed F_j. Moreover, suppose that* $\mathrm{im}(\varphi) \cap Q \cap V_i$ *contains only adventitious roots and roots of trees in $\mathcal{T}^\star_i$; and that ℓ is a target index for (φ, i, L). Then there is a reasonable partial embedding φ' of T which extends φ by embedding F_j. Moreover, r_j, x_j are mapped to $Q \cap V_i$, and $V(F_j) \setminus \{r_j, x_j\}$ is mapped to $(V_i \cup V_\ell) \setminus (Q \cup \mathrm{im}(\varphi))$.*

Proof of the claim. Define $\tilde{U} := (V_i \cup V_\ell) \setminus (Q \cup \mathrm{im}(\varphi))$. Note that φ is a reasonable J-partial embedding and therefore $Z^{\mathrm{forb}}_{\varphi,J+1} = Q \cup \mathrm{im}(\varphi)$ and $\tilde{U} = (V_i \cup V_\ell) \setminus Z^{\mathrm{forb}}_{\varphi,J+1}$. Then Claim 10.9 implies that $|V_i \cap \tilde{U}| \geq |V_i \cap U|$ and $|V_\ell \cap \tilde{U}| \geq |V_\ell \cap U|$.

Recall that $s_j \in W_A \cup W_B$ is the seed of F_j and is already embedded by φ (since it extends $\varphi^\star$). We define $P'_j := (N(\varphi(s_j)) \cap Q \cap V_i) \setminus \mathrm{im}(\varphi)$ as the set of vertices where we will look for a vertex to embed r_j. By assumption, $\mathrm{im}(\varphi) \cap Q \cap V_i$ contains only adventitious roots and roots of trees in $\mathcal{T}^\star_i$ and adventitious roots embedded in the First Embedding Phase, which amounts to at most $|W_A \cup W_B|$. Then Claim 10.12 implies that

$$|\,\mathrm{im}(\varphi) \cap Q \cap V_i| \leq |V_1(\mathcal{T}^\star_{i,A})| + |V_1(\mathcal{T}^\star_{i,B})| + |W_A \cup W_B| \leq 3\varepsilon|V_i|.$$

On the other hand, we have that $s_j \in N_T(V(F_j)) \cap (W_A \cup W_B)$, thus $i \in I_{s_j}$ by (V4), and therefore $|Q \cap V_i \cap N(\varphi(s_j))| \geq 7\varepsilon|V_i|$ follows from (U3) or (U4). Putting all together, we have that $|P'_j| \geq 7\varepsilon|V_i| - 3\varepsilon|V_i| \geq 4\varepsilon|V_i|$. Since $|V_\ell \cap \tilde{U}| \geq |V_\ell \cap U| = \frac{\eta}{4}|V_i| \geq \varepsilon|V_i|$, Lemma 10.2 implies that all but at most $\varepsilon|V_i|$ vertices in V_i have degree at least $(\mathfrak{d} - \varepsilon)|V_\ell \cap \tilde{U}| \geq 2\varepsilon|V_i|$ into $V_\ell \cap \tilde{U}$, where the inequality follows from (179). In particular, we can select a vertex $v_j \in P'_j$ such that $\deg_G(v_j, V_\ell \cap \tilde{U}) \geq 2\varepsilon|V_i|$.

Now, we define the set P_j where the adventitious root, if it exists, will be embedded. The only extra care needed here is that we cannot use v_j. If $x_j = \emptyset$, define $P_j := (Q \cap V_i) \setminus (\mathrm{im}(\varphi) \cup \{v_j\})$; otherwise x_j is an adventitious root of (F_j, r_j, x_j). Therefore, by (A8), x_j has a unique neighbour, say y_j, in $W_A \cup W_B$. We define $P_j := (Q \cap V_i \cap N_G(\varphi(y_j))) \setminus (\mathrm{im}(\varphi) \cup \{v_j\})$.

We will apply Lemma 10.5 with

object	V_i	V_ℓ	$\tilde{U}$	P_j	v_j	(F_j, r_j, x_j)	$\eta/4$
in place of	X	Y	U	P	v	(T, r, x)	$\tilde{\eta}$

We will quickly check that the required (T1)–(T4) hold. We have already checked (T1). The proof of (T2) follows along the same lines we have used to check the lower bound on $|P'_j|$, so we omit it. The choice of v_j was done precisely to ensure that (T3) holds, and (T4) follows by (A5) since we are working with a $\rho|V(T)|$-fine partition. Thus, we can use Lemma 10.5 as intended.

Define φ' as φ extended by the inclusion of F_j. By construction, we have that $\varphi'(r_j) = v_j \in Q \cap V_i$, if $x_j \neq \emptyset$ then $\varphi'(x_j) \in Q \cap V_i$, and $V(F_j) \setminus \{r_j, x_j\}$ is mapped to $(V_i \cup V_\ell) \setminus (Q \cup \mathrm{im}(\varphi))$, as required. Finally, φ' is reasonable by Claim 10.10. □

Given the last claim, we can conclude as follows. Let $J^\star := |\mathcal{T}^\star|$, i.e. the number of postponed trees. We set $\varphi^\star_0 := \varphi^\star$, which, of course, is a partial embedding of T that extends $\varphi^\star$. By construction, $\varphi^\star$ only embeds adventitious roots in Q.

Next, assume that $1 \leq k \leq J^\star$ is given and that we have constructed a reasonable partial embedding $\varphi^\star_{k-1}$ which extends $\varphi^\star$ and embeds precisely the first $k-1$ shrubs in $\mathcal{T}^\star$. Assume also that, for each $1 \leq i \leq N$, $\mathrm{im}(\varphi^\star_{k-1}) \cap Q \cap V_i$ contains only adventitious roots and roots of trees in $\mathcal{T}^\star_i$. Let F_j be the kth remaining shrub in $\mathcal{T}^\star$. Let i, L be such that $F_j \in \mathcal{T}^\star_{i,L}$. Let ℓ be a target index for $(\varphi^\star_{k-1}, L, i)$, this exists by Claim 10.7. We apply Claim 10.13 to find an reasonable partial embedding $\varphi^\star_k$ which extends $\varphi^\star_{k-1}$ by embedding F_j, and moreover $\varphi^\star_k(V(F_j)) \cap (Q \cap V_i) \subseteq \{\varphi^\star_k(r_j), \varphi^\star_k(x_j)\}$. This implies that $\varphi^\star_k$ embeds precisely the first k shrubs of $\mathcal{T}^\star$ and also, for each $1 \leq i \leq N$, $\mathrm{im}(\varphi^\star_k) \cap Q \cap V_i$ contains only adventitious roots and roots of trees in $\mathcal{T}^\star_i$; so we obtained a reasonable partial embedding which allows us to continue this process.

At the end of this process, we obtain a final embedding $\varphi^\star_{J^\star}$. By construction, $\varphi^\star_{J^\star}$ extends $\varphi^\star$ and also embeds all shrubs in $\mathcal{T}^\star$. Thus, it is an embedding of the whole tree T. This (finally!) finishes the proof of Lemma 4.5. ■

11. Concluding remarks

11.1. Comparison with the proof proposed by Ajtai, Komlós, Simonovits, and Szemerédi. This comparison is based on personal communication between the second author and Miklós Simonovits and we focus here only on their proof when specialised to the setting of Corollary 1.4, i.e., to the so-called *approximate dense case*. Their full result is much stronger than Corollary 1.4.

The major difference comes from the basic properties we may assume about the cluster graph. While in the present paper we may only assume that the cluster graph has minimum degree[3] slightly more than $k/2$ and maximum degree slightly larger than k, in the proof by Ajtai, Komlós, Simonovits, and Szemerédi, they may assume in addition an average degree slightly above k.

In some parts, they actually exploit only the existence of one cluster of degree slightly more than k together with the minimal degree condition, i.e., they use the same properties of the cluster graph as we do. This leads to some similarities between the two proofs (in the part treated in §9.2 and §9.3). However, then they do use the additional property of the average degree of the cluster graph. Missing this additional property in our setting, we need to fight harder to obtain a suitable structure, treated by more case distinctions (represented by the cases analysed in §9.4–§9.8).

11.2. Further variations on Erdős–Sós. We believe that our approach, combined with Simonovits' Stability Method and ad hoc analysis of the close-to-extremal graphs, should provide a proof of the Erdős–Sós conjecture (Conjecture 1.1) in the context of dense graphs. We recall that the best-known exact results for Conjecture 1.1 are in

[3]For simplicity of explanation, in this subsection all degrees in the cluster graph are scaled by a factor corresponding to the size of a cluster.

the dense setting by Besomi, Pavez-Signé and Stein [BPS21], then extended to sparse setting by Pokrovsky [Pok24b]; both of those results assume the tree is sufficiently large and that $\Delta(T) = O(1)$. A recent result by Reed and Stein [RS25] proves the conjecture for large trees whose number of vertices is very close to the number of vertices of the host graph, without maximum degree restrictions on the trees; an extension was recently announced in [Ree25].

Regarding steps towards proving Conjecture 1.2, replacing the minimum-degree requirement by $k/2$ and the large-degree requirement by k could probably also be dealt with using Simonovits' Stability Method, as well. However, requiring only a sublinear number of vertices of high degree (instead of $\Omega(n)$ such vertices) would require a novel approach.

It is possible that our techniques would also be of help in studying related tree-embedding conjectures which combine minimum and maximum degree conditions; this is in contrast to Conjecture 1.2 which requires many vertices of large degree. We summarise some of those conjectures, starting with the following conjecture by Besomi, Pavez-Signé and Stein [BPS19, Conjecture 1.1]

Conjecture 11.1 (Besomi, Pavez-Signé, Stein)**.** *Let $k \in \mathbb{N}$, let $\alpha \in \left(0, \frac{1}{3}\right)$, and let G be a graph with $\delta(G) \geq (1+\alpha)\frac{k}{2}$ and $\Delta(G) \geq 2(1-\alpha)k$. Then G contains each tree with k edges.*

Hyde and Reed [HR23] proved a relaxation of the $\alpha = 0$ case, where the maximum degree condition is replaced with $\Delta(G) \geq f(k)$, for some larger function of k. Such a relaxation is in some sense necessary, since if $\delta(G) = \lfloor (k-1)/2 \rfloor$ then one in fact needs $\Delta(G)$ to be at least quadratic in k, as shown by some examples [HR23, §3].

The maximum degree condition in the previous conjecture is not suspected to be best-possible for $\alpha = 1/3$. In that case, there is a previous conjecture by Havet, Reed, Stein and Wood [Hav+20, Conjecture 1.1].

Conjecture 11.2 (Havet, Reed, Stein, Wood)**.** *Let $k \in \mathbb{N}$, let G be a graph of minimum degree at least $\lfloor 2k/3 \rfloor$ and maximum degree at least k. Then every tree with k edges is a subgraph of G.*

Partial results by replacing the minimum or maximum degree conditions can be found in [Hav+20]; and a proof in the particular case of spanning trees (i.e. $k = n-1$) was obtained by Reed and Stein [RS23a; RS23b]. Finally, Pokrovskiy, Versteegen and Williams [PVW25] recently made progress both on Conjecture 11.1 and Conjecture 11.2 for large, bounded-degree trees.

Acknowledgment. We would like to thank Jan Hladký for drawing our attention on the fact that Theorem 1.3 implies the approximate Erdős-Sós Conjecture for dense graphs (Corollary 1.4). The fourth author thanks Giovanne Santos for discussions.

An extended abstract of this work appeared in the proceedings of EUROCOMB '23 [Dav+23]. Part of the work leading to this result was done while all the authors were affiliated with The Czech Academy of Sciences, Institute of Computer Science and were supported by the long-term strategic development financing of the Institute of Computer Science (RVO: 67985807) and by the Czech Science Foundation, grant number 19-08740S. N. Sanhueza-Matamala was supported by ANID-FONDECYT Iniciación Nº11220269 grant and ANID-FONDECYT Regular Nº1251121 grant.

References

[Ajt+15] Miklos Ajtai, János Komlós, Miklós Simonovits, and Endre Szemerédi. Exact solution of the Erdős-Sós conjecture (2015). Talk at GT2015, Nyborg, Denmark.

[Bal+18] József Balogh, Andrew McDowell, Theodore Molla, and Richard Mycroft. Triangle-tilings in graphs without large independent sets. *Combin. Probab. Comput.* **27**(4) (2018), 449–474.

[BPS19] Guido Besomi, Matías Pavez-Signé, and Maya Stein. Degree conditions for embedding trees. *SIAM J. Discrete Math.* **33**(3) (2019), 1521–1555.

[BPS21] Guido Besomi, Matías Pavez-Signé, and Maya Stein. On the Erdős-Sós conjecture for trees with bounded degree. *Combin. Probab. Comput.* **30**(5) (2021), 741–761.

[Dav+18] Akbar Davoodi, Ervin Győri, Abhishek Methuku, and Casey Tompkins. An Erdős–Gallai type theorem for uniform hypergraphs. *European Journal of Combinatorics* **69** (2018), 159–162.

[Dav+23] Akbar Davoodi, Diana Piguet, Hanka Řada, and Nicolás Sanhueza-Matamala. Beyond the Erdős–Sós conjecture. *Proceedings of the 12th European Conference on Combinatorics, Graph Theory and Applications EUROCOMB'23.* Masaryk University Press, 2023, 328–335.

[Die17] Reinhard Diestel. Graph theory. Fifth Edition. Vol. 173. Graduate Texts in Mathematics. Springer, Berlin, 2017, xviii+428.

[Edm65] Jack Edmonds. Paths, trees, and flowers. *Canadian J. Math.* **17** (1965), 449–467.

[EG59] P. Erdős and T. Gallai. On maximal paths and circuits of graphs. *Acta Math. Acad. Sci. Hungar.* **10** (1959), 337–356 (unbound insert).

[Erd64] P. Erdős. Extremal problems in graph theory. *Theory of Graphs and its Applications (Proc. Sympos. Smolenice, 1963).* Publ. House Czech. Acad. Sci., Prague, 1964, 29–36.

[Erd81] Paul Erdős. Some new problems and results in graph theory and other branches of combinatorial mathematics. *Combinatorics and graph theory (Calcutta, 1980).* Vol. 885. Lecture Notes in Math. Springer, Berlin-New York, 1981, 9–17.

[ES46] P. Erdös and A. H. Stone. On the structure of linear graphs. *Bull. Amer. Math. Soc.* **52** (1946), 1087–1091.

[ES66] P. Erdős and M. Simonovits. A limit theorem in graph theory. *Studia Sci. Math. Hungar.* **1** (1966), 51–57.

[FHL18] Genghua Fan, Yanmei Hong, and Qinghai Liu. The Erdös-Sós conjecture for spiders. Preprint (2018).

[Gal64] T. Gallai. Maximale systeme unabhängiger Kanten. *Magyar Tud. Akad. Mat. Kutató Int. Közl.* **9** (1964), 401–413 (1965).

[GKL16] Ervin Győri, Gyula Y. Katona, and Nathan Lemons. Hypergraph extensions of the Erdős-Gallai Theorem. *European Journal of Combinatorics* **58** (2016), 238–246.

[Hav+20] Frédéric Havet, Bruce Reed, Maya Stein, and David R. Wood. A variant of the Erdös-Sós conjecture. *J. Graph Theory* **94**(1) (2020), 131–158.

[Hla+17] Jan Hladký, János Komlós, Diana Piguet, Miklós Simonovits, Maya Stein, and Endre Szemerédi. The approximate Loebl-Komlós-Sós conjecture IV: Embedding techniques and the proof of the main result. *SIAM J. Discrete Math.* **31**(2) (2017), 1072–1148.

[HR23] Joseph Hyde and Bruce Reed. Graphs of minimum degree at least $\lfloor \frac{d}{2} \rfloor$ and large enough maximum degree embed every tree with d vertices. *Procedia Comput. Sci.* **223** (2023), 217–222.

[KPR20] Tereza Klimošová, Diana Piguet, and Václav Rozhoň. A version of the Loebl-Komlós-Sós conjecture for skew trees. *European J. Combin.* **88** (2020), 103106, 28.

[KS96] J. Komlós and M. Simonovits. Szemerédi's regularity lemma and its applications in graph theory. *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993).* Vol. 2. Bolyai Soc. Math. Stud. János Bolyai Math. Soc., Budapest, 1996, 295–352.

[McD89] Colin McDiarmid. On the method of bounded differences. *Surveys in Combinatorics, 1989 (Norwich, 1989).* Vol. 141. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 1989, 148–188.

[McL05] Andrew McLennan. The Erdős-Sós conjecture for trees of diameter four. *J. Graph Theory* **49**(4) (2005), 291–301.

[Pok24a] Alexey Pokrovskiy. Hyperstability in the Erdős-Sós Conjecture. Preprint (2024).

[Pok24b] Alexey Pokrovskiy. Notes on embedding trees in graphs with $O(|T|)$-sized covers. Preprint (2024).

[PS12] Diana Piguet and Maya Jakobine Stein. An approximate version of the Loebl–Komlós–Sós conjecture. *Journal of Combinatorial Theory, Series B* **102**(1) (2012), 102–125.

[PVW25] Alexey Pokrovskiy, Leo Versteegen, and Ella Williams. Embedding trees using minimum and maximum degree conditions. Preprint (2025).

[Ree25] Bruce Reed. The Erdős-Sós conjecture and some (minor) variants (2025). Talk at LAGOS 2025, Buenos Aires, Argentina.

[Roz19] Václav Rozhoň. A local approach to the Erdős-Sós conjecture. *SIAM J. Discrete Math.* **33**(2) (2019), 643–664.

[RS23a] Bruce Reed and Maya Stein. Spanning trees in graphs of high minimum degree with a universal vertex I: An asymptotic result. *J. Graph Theory* **102**(4) (2023), 737–783.

[RS23b] Bruce Reed and Maya Stein. Spanning trees in graphs of high minimum degree with a universal vertex II: A tight result. *J. Graph Theory* **102**(4) (2023), 797–821.

[RS25] Bruce Reed and Maya Stein. Embedding nearly spanning trees. *Combin. Probab. Comput.* **34**(6) (2025), 927–931.

[Ste20] Maya Stein. Tree containment and degree conditions. *Discrete Mathematics and Applications*. Vol. 165. Springer Optim. Appl. Springer, Cham, 2020, 459–486.

[Sze78] Endre Szemerédi. Regular partitions of graphs. *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*. Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, 399–401.

[Tur41] Pál Turán. Egy gráfelméleti szélsőértékfeladatról. *Mat. Fiz. Lapok* **48** (1941), 436–452.

[Zha11] Yi Zhao. Proof of the $(n/2 - n/2 - n/2)$ conjecture for large n. *Electron. J. Combin.* **18**(1) (2011), Paper 27, 61.

The Czech Academy of Sciences, Institute of Computer Science, Pod Vodárenskou věží 2, Prague,182 07, Czech Republic
Email address: `davoodi@cs.cas.cz, piguet@cs.cas.cz`

Czech Technical University in Prague, Faculty of Information Technology, Department of Applied Mathematics, Thákurova 9, 160 00 Prague 6, Czech Republic
Email address: `hanka.rada@fit.cvut.cz`

Departamento de Ingeniería Matemática and CI^{2}MA, Universidad de Concepción, Chile
Email address: `nsanhuezam@udec.cl`

Centro de Investigación en Ingeniería Matemática (CI^2MA)

PRE-PUBLICACIONES 2025 - 2026

- 2025-22 FERNANDO ARTAZA-COVARRUBIAS, TONATIUH SANCHEZ-VIZUET, MANUEL SOLANO: *A coupled HDG discretization for the interaction between acoustic and elastic waves*
- 2025-23 EIDER ALDANA, RICARDO OYARZÚA, JESÚS VELLOJÍN: *Numerical analysis of a three-field formulation for a reverse osmosis model*
- 2025-24 JULIO ARACENA, FLORIAN BRIDOUX, MAXIMILIEN GADOULEAU, PIERRE GUILLON, KEVIN PERROT, ADRIEN RICHARD, GUILLAUME THEYSSIER: *On the Dynamics of Bounded-Degree Automata Networks*
- 2025-25 DANIEL CAJAS GUIJARRO, JOHN CAJAS GUIJARRO: *Resonant and non-resonant double Hopf bifurcation in a 4D Goodwin model with wage inequality*
- 2025-26 ALONSO J. BUSTOS, SERGIO CAUCAO: *A Banach space mixed formulation for the unsteady Brinkman problem with spatially varying porosity*
- 2025-27 JULIO ARACENA, KATERIN DE LA HOZ, ALEXIS POINDRON, LILIAN SALINAS: *An exploratory approach to the compatibility and inference of unate functions*
- 2025-28 JESSIKA CAMAÑO, SERGIO CAUCAO, CRISTIAN INZUNZA: *A five-field mixed formulation for the fully dynamic Biot–Brinkman model*
- 2025-29 JUAN BARAJAS-CALONGE, MAURICIO SEPÚLVEDA, NICOLÁS TORRES, LUIS M. VILLADA: *Dynamically consistent finite volume scheme for a bimonomeric simplified model with inflammation processes for Alzheimer's disease*
- 2026-01 ALONSO J. BUSTOS, GABRIEL N. GATICA, RICARDO RUIZ-BAIER, BENJAMÍN N. VENEGAS: *A perturbed threefold saddle-point formulation yielding new mixed finite element methods for poroelasticity with reduced symmetry*
- 2026-02 JORGE AGUAYO, RODOLFO ARAYA: *A Nitsche-type finite element method for the linear elasticity equation with discontinuity jumps*
- 2026-03 SERGIO CAUCAO, GABRIEL N. GATICA, ADRIAN SUAREZ, IVAN YOTOV: *A skew-symmetry-based mixed formulation for an Oseen-type Kelvin–Voigt–Brinkman–Forchheimer model*
- 2026-04 AKBAR DAVOODI, DIANA PIGUET, HANKA RADA, NICOLÁS SANHUEZA-MATAMALA: *The asymptotic version of the Erdős–Sós conjecture and beyond*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: DIRECTOR, CENTRO DE INVESTIGACIÓN EN INGENIERÍA MATEMÁTICA, UNIVERSIDAD DE CONCEPCIÓN, CASILLA 160-C, CONCEPCIÓN, CHILE, TEL.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl

Centro de Investigación en
Ingeniería Matemática (CI^{2}MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl