A five-field mixed formulation for the fully dynamic Biot–Brinkman model

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A five-field mixed formulation for the fully dynamic Biot–Brinkman model *

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Abstract

We propose and analyze a mixed formulation for the fully dynamic Biot–Brinkman system, which models the coupled effects of internal viscous diffusion and deformation of the porous skeleton. The classical formulation is expressed in terms of displacement, fluid velocity, and pore pressure. In this work, we reformulate the problem in terms of the fluid velocity, pore pressure, and poroelastic stress tensor, together with two additional variables: the structural velocity and the rotation rate. In particular, we eliminate the solid displacement and the rotation tensor from the set of primary variables. Both quantities are subsequently recovered by a simple post-processing step. This yields a five-field mixed formulation. We establish existence and uniqueness of a weak solution and derive stability bounds based on the Babuška–Brezzi theory for perturbed saddle-point problems, as well as on the theoretical framework developed by Showalter for semilinear degenerate evolution equations. We then develop a semidiscrete continuous-in-time approximation based on stable mixed finite elements. Moreover, employing a backward Euler time discretization, we introduce a fully discrete finite element scheme. We prove that both schemes are well-posed, derive stability bounds, and establish the corresponding error estimates. Finally, we numerically verify the theoretical convergence rates and perform classical benchmark tests to illustrate the robustness and versatility of the method across different domains and parameter regimes.

Key words: Biot model, Brinkman equations, mixed finite element methods

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

For decades, Darcy's law [27] has been the canonical model for describing fluid flow in porous media under laminar, low velocity conditions. It postulates a linear relation between the fluid pressure gradient and the filtration (Darcy) velocity relative to the porous medium, and has proven effective in diverse applications, including groundwater hydrology, reservoir engineering, and mass transport in biological media [6]. However, Darcy's law does not account for internal viscous diffusion nor

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for inertial corrections that may emerge at moderate pore-scale Reynolds numbers. To capture such regimes, Brinkman proposed an extension of Darcy's model that augments the drag term with a viscous diffusion contribution, formally analogous to the Stokes operator, thereby accounting for internal viscous dissipation within the porous structure [13]. This Brinkman description has proved especially useful for large-pore materials, fibrous media, biological tissues, and configurations with significant hydrodynamic interactions [45, 2, 31, 34]. It provides a natural intermediate description between Darcy and Stokes flow when microstructural effects and shear layers are non-negligible.

In the macroscopic poroelastic setting, coupling Darcy flow with elastic deformation yields the classical Biot model [7, 42, 8], in which the saturating fluid moves through the pore network according to Darcy's law. Variational formulations, existence and uniqueness of weak solutions, and stable, convergent mixed finite element schemes have been established [37, 9, 43], clarifying key issues such as mass conservation, robustness under extreme parameter regimes, and the mitigation of spurious pore-pressure oscillations. Nevertheless, because the internal flow in the Biot framework is governed by Darcy's law, the aforementioned limitations persist when inertial and shear effects become significant. To overcome these drawbacks, several extensions replace Darcy's law in Biot's theory with more general momentum balances. A prominent example is the Biot–Brinkman coupling, which augments the pore-scale momentum equation with a Stokes-type viscous diffusion term to represent internal shear. Recent theoretical and computational studies on Biot–Brinkman couplings include vorticity-based mixed formulations and robust discretizations via conforming finite elements, discontinuous Galerkin, and virtual element methods; see for instance [16, 30, 15, 25, 33]. In [16], a two length scale model is developed for coupled flow-solid mechanics and validated against data for chemically driven deformation in viscoplastic/elastic solids, yielding an error-function dependence of permeability on the microporous clay fraction. In [30], generalized Biot–Brinkman equations for multi-network poroelasticity are analyzed, and three-field finite elements are introduced alongside norms and preconditioners uniformly robust across relevant parameter regimes, with corroborating simulations. In [15], a steady coupled formulation with divergence-conforming filtration fluxes and parameter-weighted spaces is proposed; solvability is proved together with optimal, parameter-robust error estimates (robust for large Lamé parameters and small permeability/storativity) and effective block preconditioning is demonstrated. In [25], locking is addressed via a four-field formulation introducing the total pressure; parameter-independent well-posedness is established and mixed interior-penalty DG schemes are designed with optimal energy and L^2 error bounds and documented robustness. Finally, [33] formulates, analyzes, and implements a four-field discrete scheme based on Nitsche's technique within the virtual element framework; using a suitably weighted norm, a robust a priori analysis and error estimates are derived and verified numerically, showing robustness with respect to the physical parameters.

In this work we consider the fully dynamic Biot–Brinkman model. This system couples second-order poroelasticity with a Brinkman-type momentum balance for the pore fluid, leading to a strongly coupled, time-dependent problem. The interplay between internal viscous diffusion and skeleton deformation gives rise to a genuinely multiphysics regime, characterized by the interaction of spatial and temporal scales. To the best of our knowledge, a unified framework that establishes well-posedness while also providing stable and convergent discretizations for this fully dynamic coupling is still lacking. The purpose of the present work is to develop and analyze a five-field mixed formulation of the Biot–Brinkman problem and to study a suitable conforming numerical discretization. To this end, and motivated by [18, 19, 39], we reformulate the model by introducing the structural velocity and the rotation rate tensor as additional unknowns, alongside the poroelastic stress tensor, fluid velocity, and pore pressure. In doing so, the displacement and the rotation tensor are eliminated from the formulation, although they may be subsequently recovered through post-processing. We establish the existence of a solution to the continuous weak formulation by combining results from [38] and [11].

Uniqueness is obtained by showing that the homogeneous problem admits only the trivial solution. Stability of the weak solution is then derived through an energy estimate. We also develop a semidiscrete continuous-in-time finite element approximation, where arguments analogous to those used in the continuous setting are applied. In this case, the functional framework naturally leads to the use of stable mixed finite element families for elasticity and Stokes such as Arnold–Falk–Winther and Taylor–Hood elements, respectively. Also, we carry out a detailed error analysis for the semidiscrete scheme and establish optimal convergence rates. In addition, using the backward Euler method, we derive the fully discrete scheme and establish its well-posedness and error analysis. We further obtain optimal-order error estimates for the post-processed variables, explicitly incorporating the composite trapezoidal quadrature rule used in their reconstruction, which represents a novel feature of the present work. Finally, we present numerical experiments that confirm the theoretical convergence rates, together with applied examples that illustrate the robustness and strong performance of the scheme across different parameter regimes. In particular, the applied tests indicate that the method remains numerically locking-free with respect to the storage coefficient, which we regard as a key practical advantage of the proposed approach.

We have organized the contents of this paper as follows. The remainder of this section introduces the standard notation and functional spaces used throughout. In Section 2, we present the model problem and develop the five-field mixed variational formulation. Section 3 establishes well posedness of the weak formulation. In Section 4, we introduce and analyze the semidiscrete continuous-in-time scheme, deriving error estimates and convergence rates. Section 5 is devoted to the fully discrete approximation, where we establish its well-posedness and derive the corresponding error estimates, including those for the post-processed variables. In Section 6, we report numerical experiments that confirm the theoretical convergence rates and illustrate the robustness of the proposed five-field mixed formulation; we also include classical benchmarks, namely a cantilever configuration and the Mandel–Cryer problem, carried out in two and three dimensions, respectively. Finally, the paper ends with conclusions in Section 7.

Preliminary notations. Let $\Omega \subset \mathrm{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $\mathrm{s} \geq 0$ and $\mathrm{p} \in [1,+\infty]$, we denote by $\mathrm{L}^\mathrm{p}(\Omega)$ and $\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)$ the usual Lebesgue and Sobolev spaces endowed with the norms $\|\cdot\|_{\mathrm{L}^\mathrm{p}(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{\mathrm{s},\mathrm{p}}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,\mathrm{p}}(\Omega) = \mathrm{L}^\mathrm{p}(\Omega)$. If $\mathrm{p} = 2$, we write $\mathrm{H}^\mathrm{s}(\Omega)$ in place of $\mathrm{W}^{\mathrm{s},2}(\Omega)$, and denote the corresponding norm by $\|\cdot\|_{\mathrm{H}^\mathrm{s}(\Omega)}$. By $\mathbf{H}$ and $\mathbb{H}$ we will denote the corresponding vectorial and tensorial counterparts of a generic scalar functional space H. The $\mathrm{L}^2(\Omega)$ inner product for scalar, vector, or tensor valued functions is denoted by $(\cdot,\cdot)_\Omega$. The $\mathrm{L}^2(\Gamma)$ inner product or duality pairing is denoted by $\langle\cdot,\cdot\rangle_\Gamma$. Moreover, let V be a separable Banach space endowed with the norm $\|\cdot\|_\mathrm{V}$. We recall the Bochner spaces $\mathrm{L}^2(0,T;\mathrm{V})$, $\mathrm{H}^s(0,T;\mathrm{V})$ (for integer $s \geq 1$), $\mathrm{L}^\infty(0,T;\mathrm{V})$, and $\mathrm{W}^{1,\infty}(0,T;\mathrm{V})$, endowed with the norms

$$\|f\|^2_{\mathrm{L}^2(0,T;\mathrm{V})} := \int_0^T \|f(t)\|^2_\mathrm{V}\,dt\,, \quad \|f\|^2_{\mathrm{H}^s(0,T;\mathrm{V})} := \int_0^T \sum_{i=0}^{s} \|\partial_t^i f(t)\|^2_\mathrm{V}\,dt,$$

$$\|f\|_{\mathrm{L}^\infty(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \|f(t)\|_\mathrm{V}, \quad \text{and} \quad \|f\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{V})} := \operatorname*{ess\,sup}_{t\in[0,T]} \big\{\|f(t)\|_\mathrm{V} + \|\partial_t f(t)\|_\mathrm{V}\big\}.$$

In turn, for any vector field $\mathbf{v} := (v_i)_{i=1,d}$, we set the gradient and divergence operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial\, v_i}{\partial\, x_j}\right)_{i,j=1,d} \quad \text{and} \quad \mathrm{div}(\mathbf{v}) := \sum_{j=1}^{d} \frac{\partial\, v_j}{\partial\, x_j}.$$

On the other hand, we introduce the Hilbert spaces

$$\mathbb{H}(\mathbf{div};\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^2(\Omega)\Big\}\,,$$

equipped with the natural norm

$$\|\boldsymbol{\tau}\|^2_{\mathbb{H}(\mathbf{div};\Omega)} := \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{\mathbf{L}^2(\Omega)} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\,.$$

Furthermore, to prepare the forthcoming analysis, we introduce the following closed subspaces of $\mathbb{L}^2(\Omega)$ and $\mathrm{L}^2(\Omega)$:

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) := \Big\{ \boldsymbol{\chi} \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\chi} = -\boldsymbol{\chi}^{\mathtt{t}} \Big\} \quad \text{and} \quad \mathrm{L}^2_0(\Omega) := \Big\{ q \in \mathrm{L}^2(\Omega) : \quad (q,1)_\Omega = 0 \Big\}. \tag{1.1}$$

In what follows, when no confusion arises, $|\cdot|$ denotes the Euclidean norm in R^d or $\mathrm{R}^{d\times d}$. In addition, in the sequel we will make use of the well-known Cauchy–Schwarz inequality given by

$$\int_\Omega |f\,g| \leq \|f\|_{\mathrm{L}^2(\Omega)}\,\|g\|_{\mathrm{L}^2(\Omega)} \quad \forall\, f,\, g \in \mathrm{L}^2(\Omega)\,, \tag{1.2}$$

and Young's inequality, for $a, b \geq 0$, and $\delta > 0$,

$$a\,b \leq \frac{\delta^{\mathrm{p}/2}}{\mathrm{p}}\,a^{\mathrm{p}} + \frac{1}{\mathrm{q}\,\delta^{\mathrm{q}/2}}\,b^{\mathrm{q}}\,. \tag{1.3}$$

2 The model problem and its five-field mixed formulation

In this section we present the Biot–Brinkman model, reformulated in a form suitable for the forthcoming analysis, and derive the associated five-field mixed variational formulation.

2.1 The model problem

Let $\Omega \subset \mathrm{R}^d$ be a homogeneous porous domain. We consider the fully dynamic Biot–Brinkman system (see, e.g., [16, 30, 15, 25]), which models the interaction between a deformable porous skeleton and a viscous fluid. Let $\rho_p > 0$ denote the solid density, $\nu > 0$ the fluid viscosity, $\mathtt{D} > 0$ the Darcy coefficient, $\alpha \in (0,1]$ the Biot–Willis coefficient, and $s_0 \geq 0$ the storage coefficient. Given body forces $\mathbf{f}_p$, $\mathbf{g} : [0,T] \to \mathrm{R}^d$, a scalar source $g : [0,T] \to \mathrm{R}$, and suitable initial data $\boldsymbol{\eta}_0$, $\boldsymbol{\eta}_{t,0}$, $\mathbf{u}_0 : \Omega \to \mathrm{R}^d$ and $p_0 : \Omega \to \mathrm{R}$, the governing equations read

$$\rho_p \frac{\partial^2 \boldsymbol{\eta}}{\partial t^2} - \mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p\,, \quad \frac{\partial \mathbf{u}}{\partial t} - \nu\,\Delta\mathbf{u} + \mathtt{D}\,\mathbf{u} + \nabla p = \mathbf{g} \quad \text{in } \Omega \times (0,T]\,, \tag{2.1a}$$

$$\frac{\partial}{\partial t}(s_0\,p + \alpha\,\mathrm{div}(\boldsymbol{\eta})) + \mathrm{div}(\mathbf{u}) = g \quad \text{in } \Omega \times (0,T]\,, \tag{2.1b}$$

$$\boldsymbol{\eta} = \mathbf{0}, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma \times (0,T]\,, \quad (p,1)_\Omega = 0 \quad \text{in } (0,T]\,, \tag{2.1c}$$

$$\boldsymbol{\eta}(0) = \boldsymbol{\eta}_0\,, \quad \frac{\partial \boldsymbol{\eta}}{\partial t}(0) = \boldsymbol{\eta}_{t,0}\,, \quad \mathbf{u}(0) = \mathbf{u}_0\,, \quad p(0) = p_0 \quad \text{in } \Omega\,, \tag{2.1d}$$

where $\boldsymbol{\eta}$ is the solid displacement, $\mathbf{u}$ the fluid velocity, p the pore pressure, and $\boldsymbol{\sigma}_p$ the poroelastic stress tensor defined by

$$\boldsymbol{\sigma}_p = \boldsymbol{\sigma}_e - \alpha\,p\,\mathbb{I}, \tag{2.2}$$

with $\mathbb{I}$ the $d \times d$ identity tensor. The elastic stress $\boldsymbol{\sigma}_e$ satisfies the constitutive relation

$$A(\boldsymbol{\sigma}_e) = \mathbf{e}(\boldsymbol{\eta}), \quad \text{where} \quad \mathbf{e}(\boldsymbol{\eta}) := \frac{1}{2}\big(\nabla\boldsymbol{\eta} + (\nabla\boldsymbol{\eta})^{\mathtt{t}}\big), \tag{2.3}$$

and A denotes the compliance tensor. In the isotropic case, A is symmetric and positive definite, and takes the explicit form

$$A(\boldsymbol{\tau}) := \frac{1}{2\mu}\left(\boldsymbol{\tau} - \frac{\lambda}{2\mu + d\lambda}\mathrm{tr}(\boldsymbol{\tau})\mathbb{I}\right), \quad \text{and} \quad A^{-1}(\boldsymbol{\tau}) = 2\mu\,\boldsymbol{\tau} + \lambda\,\mathrm{tr}(\boldsymbol{\tau})\mathbb{I} \qquad \forall\,\boldsymbol{\tau} \in \mathrm{R}^{d\times d},$$

where $\mu > 0$ and $\lambda \geq 0$ are the Lamé parameters. Moreover, A satisfies the coercivity and boundedness properties

$$\frac{1}{2\mu + d\lambda}\boldsymbol{\tau} : \boldsymbol{\tau} \;\leq A(\boldsymbol{\tau}) : \boldsymbol{\tau} \leq \frac{1}{2\,\mu}\boldsymbol{\tau} : \boldsymbol{\tau} \qquad \forall\,\boldsymbol{\tau} \in \mathrm{R}^{d\times d}. \tag{2.4}$$

To facilitate the reformulation of the model, we introduce the rotation tensor $\boldsymbol{\rho} := \frac{1}{2}\left(\nabla\boldsymbol{\eta} - (\nabla\boldsymbol{\eta})^{\mathrm{t}}\right)$, and observe from (2.2) and (2.3) that

$$A(\boldsymbol{\sigma}_p + \alpha\, p\mathbb{I}) = \mathbf{e}(\boldsymbol{\eta}) = \nabla\boldsymbol{\eta} - \boldsymbol{\rho} \quad \text{in} \quad \Omega \times (0,T]. \tag{2.5}$$

As a consequence, by applying the trace operator in (2.5), we deduce

$$\mathrm{div}(\boldsymbol{\eta}) = \mathrm{tr}(\mathbf{e}(\boldsymbol{\eta})) = \mathrm{tr}(A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})) \quad \text{in} \quad \Omega \times (0,T]. \tag{2.6}$$

Notice that, by using the identity (2.6), the boundary condition $\boldsymbol{\eta} = \mathbf{0}$ on $\Gamma \times (0,T]$, and applying the divergence theorem, we obtain that

$$(\mathrm{tr}(A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})), 1)_\Omega = (\mathrm{div}(\boldsymbol{\eta}), 1)_\Omega = \langle \boldsymbol{\eta}\cdot\boldsymbol{\nu}, 1\rangle_\Gamma = 0 \quad \text{in} \quad (0,T]. \tag{2.7}$$

We next follow the approach from [39] (see also [18, 19]), and introduce, respectively, the structural velocity and rotation rate as

$$\mathbf{u}_s := \frac{\partial\boldsymbol{\eta}}{\partial t} \quad \text{and} \quad \boldsymbol{\gamma} := \frac{\partial\boldsymbol{\rho}}{\partial t} \quad \text{in} \quad \Omega \times (0,T]. \tag{2.8}$$

In addition, by combining (2.1b) with (2.6), differentiating (2.5) with respect to time, and considering (2.7), we may reformulate the original system (2.1) in terms of the variables $\boldsymbol{\sigma}_p$, $\mathbf{u}$, p, $\mathbf{u}_s$, and $\boldsymbol{\gamma}$ as follows:

$$\frac{\partial}{\partial t}A(\boldsymbol{\sigma}_p + \alpha\, p\mathbb{I}) - \nabla\mathbf{u}_s + \boldsymbol{\gamma} = 0 \quad \text{in } \Omega \times (0,T], \tag{2.9a}$$

$$\rho_p\frac{\partial\mathbf{u}_s}{\partial t} - \mathbf{div}(\boldsymbol{\sigma}_p) = \mathbf{f}_p \quad \text{in } \Omega \times (0,T], \tag{2.9b}$$

$$\frac{\partial\mathbf{u}}{\partial t} - \nu\Delta\mathbf{u} + \mathtt{D}\mathbf{u} + \nabla p = \mathbf{g} \quad \text{in } \Omega \times (0,T], \tag{2.9c}$$

$$\frac{\partial}{\partial t}\big(s_0\, p + \alpha\,\mathrm{tr}\,(A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I}))\big) + \mathrm{div}(\mathbf{u}) = g \quad \text{in } \Omega \times (0,T], \tag{2.9d}$$

$$(p,1)_\Omega = 0 \quad \text{and} \quad (\mathrm{tr}\,(A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})), 1)_\Omega = 0 \quad \text{in } (0,T], \tag{2.9e}$$

$$\mathbf{u}_s = \mathbf{0}, \quad \mathbf{u} = \mathbf{0} \quad \text{on } \Gamma \times (0,T], \tag{2.9f}$$

$$\mathbf{u}_s(0) = \mathbf{u}_{s,0}, \quad \mathbf{u}(0) = \mathbf{u}_0, \quad p(0) = p_0 \quad \text{in } \Omega, \tag{2.9g}$$

where $\mathbf{u}_{s,0} := \boldsymbol{\eta}_{t,0}$. Observe that, in view of (2.9a) and (2.9d), an initial condition for $\boldsymbol{\sigma}_p$ is required. Accordingly, starting from the initial datum $\mathbf{u}_{s,0}$, we will construct compatible initial conditions for $\boldsymbol{\sigma}_p$ and $\boldsymbol{\gamma}$ consistent with the system (see Lemma 3.5 in Section 3.2).

We also note that the original unknowns $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ were eliminated from the system, but may be recovered by time integration of $\mathbf{u}_s$ and $\boldsymbol{\gamma}$, respectively. Indeed, for all $t \in [0,T]$, we have

$$\boldsymbol{\eta}(t) = \boldsymbol{\eta}_0 + \int_0^t \mathbf{u}_s(s)\,ds \quad \text{and} \quad \boldsymbol{\rho}(t) = \boldsymbol{\rho}_0 + \int_0^t \boldsymbol{\gamma}(s)\,ds\,, \tag{2.10}$$

where $\boldsymbol{\rho}_0 := \frac{1}{2}(\nabla\boldsymbol{\eta}_0 - \nabla\boldsymbol{\eta}_0^{\mathtt{t}})$.

2.2 The mixed variational formulation

In this section, we derive the variational formulation of the system (2.9). We begin by testing equation (2.9a) against a tensor-valued function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, integrating by parts, and incorporating the boundary condition $\mathbf{u}_s = \mathbf{0}$ on $\Gamma \times (0,T]$ (cf. (2.9f)), which yields

$$(\partial_t A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I}), \boldsymbol{\tau})_\Omega + (\mathbf{u}_s, \mathbf{div}(\boldsymbol{\tau}))_\Omega + (\boldsymbol{\gamma}, \boldsymbol{\tau})_\Omega \;=\; 0 \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)\,, \tag{2.11}$$

where we used the notation $\partial_t := \frac{\partial}{\partial t}$. All terms in (2.11) are well-defined provided that $\mathbf{u}_s \in \mathbf{L}^2(\Omega)$, $\boldsymbol{\sigma}_p, \boldsymbol{\gamma} \in \mathbb{L}^2(\Omega)$, and $p \in \mathrm{L}^2(\Omega)$. Moreover, recalling (2.9e), we obtain that $p \in \mathrm{L}^2_0(\Omega)$ (cf. (1.1)). In turn, from the definition of $\boldsymbol{\gamma}$ (cf. (2.8)), we see that $\boldsymbol{\gamma} \in \mathbb{L}^2_{\mathrm{skew}}(\Omega)$ (cf. (1.1)).

We now consider the weak formulation of equation (2.9b), given by

$$\rho_p(\partial_t \mathbf{u}_s, \mathbf{v}_s)_\Omega - (\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{v}_s)_\Omega \;=\; (\mathbf{f}_p, \mathbf{v}_s)_\Omega \qquad \forall\, \mathbf{v}_s \in \mathbf{L}^2(\Omega)\,, \tag{2.12}$$

which second term reveals that the appropriate function space for $\boldsymbol{\sigma}_p$ is $\mathbb{H}(\mathbf{div};\Omega)$. In addition, to ensure the symmetry of $\boldsymbol{\sigma}_p$, we impose the condition

$$(\boldsymbol{\sigma}_p, \boldsymbol{\chi})_\Omega \;=\; 0 \qquad \forall\, \boldsymbol{\chi} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,. \tag{2.13}$$

In turn, as suggested by the Dirichlet boundary condition satisfied by the velocity $\mathbf{u}$ (cf. (2.9f)), we notice that the appropriate trial and test spaces reduces in this case to $\mathbf{H}^1_0(\Omega)$. Thus, performing the usual integration by parts procedure to (2.9c), we get

$$(\partial_t \mathbf{u}, \mathbf{v})_\Omega + \nu(\nabla\mathbf{u}, \nabla\mathbf{v})_\Omega + \mathtt{D}(\mathbf{u},\mathbf{v})_\Omega - (p, \operatorname{div}(\mathbf{v}))_\Omega = (\mathbf{g}, \mathbf{v})_\Omega \quad \forall\, \mathbf{v} \in \mathbf{H}^1_0(\Omega)\,. \tag{2.14}$$

Next, we impose the weak form of (2.9d), which after a simple algebraic manipulation may be written as

$$s_0(\partial_t p, q)_\Omega + (\partial_t A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I}), \alpha\, q\,\mathbb{I})_\Omega + (\operatorname{div}(\mathbf{u}), q)_\Omega = (g, q)_\Omega \quad \forall\, q \in \mathrm{L}^2(\Omega)\,. \tag{2.15}$$

We observe that the variational identity (2.15) remains valid when restricted to test functions $q \in \mathrm{L}^2_0(\Omega)$ (cf. (1.1)). In fact, assuming that $g \in \mathrm{L}^2_0(\Omega)$, we note that all three terms on the left-hand side of (2.15) vanish when tested against constant functions: the first term vanishes since $p \in \mathrm{L}^2_0(\Omega)$, the second term vanishes due to identity (2.9e), and the third term vanishes as a consequence of the divergence theorem and the homogeneous Dirichlet condition imposed on $\mathbf{u}$. Accordingly, we may equivalently formulate (2.15) by taking $q \in \mathrm{L}^2_0(\Omega)$ as the test space.

For the subsequent analysis, we introduce the product spaces

$$\mathbb{H} \;:=\; \mathbb{H}(\mathbf{div};\Omega) \times \mathbf{H}^1_0(\Omega) \times \mathrm{L}^2_0(\Omega) \quad \text{and} \quad \mathbb{Q} \;:=\; \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega),$$

together with the notation

$$\underline{\boldsymbol{\sigma}} := (\boldsymbol{\sigma}_p, \mathbf{u}, p), \quad \underline{\boldsymbol{\tau}} := (\boldsymbol{\tau}, \mathbf{v}, q) \in \mathbb{H}, \quad \text{and} \quad \underline{\mathbf{u}} := (\mathbf{u}_s, \boldsymbol{\gamma}), \quad \underline{\mathbf{v}} := (\mathbf{v}_s, \boldsymbol{\chi}) \in \mathbb{Q},$$

and we equip them with the product norms

$$\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}^2 := \|\boldsymbol{\tau}\|_{\mathbb{H}(\mathbf{div};\Omega)}^2 + \|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}^2 + \|q\|_{\mathrm{L}^2(\Omega)}^2 \quad \text{and} \quad \|\underline{\mathbf{v}}\|_{\mathbb{Q}}^2 := \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega)}^2 + \|\boldsymbol{\chi}\|_{\mathbb{L}^2(\Omega)}^2 .$$

Hence, by combining equations (2.11)–(2.15), we obtain the mixed variational formulation of (2.9), that is, given

$$\mathbf{g} : [0,T] \to \mathbf{L}^2(\Omega)\,, \quad g : [0,T] \to \mathrm{L}_0^2(\Omega)\,, \quad \mathbf{f}_p : [0,T] \to \mathbf{L}^2(\Omega)\,,$$
$$\mathbf{u}_0 \in \mathbf{H}_0^1(\Omega)\,, \quad p_0 \in \mathrm{L}_0^2(\Omega)\,, \quad \text{and} \quad \mathbf{u}_{s,0} \in \mathbf{L}^2(\Omega)\,,$$

find $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{H} \times \mathbb{Q}$ such that $\mathbf{u}(0) = \mathbf{u}_0$, $p(0) = p_0$, $\mathbf{u}_s(0) = \mathbf{u}_{s,0}$, and for a.e. $t \in (0,T)$,

$$\begin{aligned} \frac{\partial}{\partial t}[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}(t)), \underline{\boldsymbol{\tau}}] + [\mathcal{A}(\underline{\boldsymbol{\sigma}}(t)), \underline{\boldsymbol{\tau}}] + [\mathcal{B}'(\underline{\mathbf{u}}(t)), \underline{\boldsymbol{\tau}}] &= [\mathbf{F}(t), \underline{\boldsymbol{\tau}}] \qquad \forall\, \underline{\boldsymbol{\tau}} \in \mathbb{H}\,, \\ \frac{\partial}{\partial t}[\mathcal{E}_2(\underline{\mathbf{u}}(t)), \underline{\mathbf{v}}] - [\mathcal{B}(\underline{\boldsymbol{\sigma}}(t)), \underline{\mathbf{v}}] &= [\mathbf{G}(t), \underline{\mathbf{v}}] \qquad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}\,, \end{aligned} \tag{2.16}$$

where the operators $\mathcal{E}_1, \mathcal{A} : \mathbb{H} \to \mathbb{H}'$, $\mathcal{B} : \mathbb{H} \to \mathbb{Q}'$ and $\mathcal{E}_2 : \mathbb{Q} \to \mathbb{Q}'$ are defined by

$$[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] := (A(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I}), \boldsymbol{\tau} + \alpha\, q\, \mathbb{I})_\Omega + (\mathbf{u}, \mathbf{v})_\Omega + s_0 (p, q)_\Omega\,, \tag{2.17}$$

$$[\mathcal{A}(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] := \nu (\nabla \mathbf{u}, \nabla \mathbf{v})_\Omega + \mathtt{D}(\mathbf{u}, \mathbf{v})_\Omega - (p, \mathrm{div}(\mathbf{v}))_\Omega + (\mathrm{div}(\mathbf{u}), q)_\Omega\,, \tag{2.18}$$

$$[\mathcal{B}(\underline{\boldsymbol{\tau}}), \underline{\mathbf{v}}] := (\mathbf{v}_s, \mathbf{div}(\boldsymbol{\tau}))_\Omega + (\boldsymbol{\chi}, \boldsymbol{\tau})_\Omega\,, \tag{2.19}$$

$$[\mathcal{E}_2(\underline{\mathbf{u}}), \underline{\mathbf{v}}] := \rho_p (\mathbf{u}_s, \mathbf{v}_s)_\Omega\,. \tag{2.20}$$

The linear functionals $\mathbf{F} \in \mathbb{H}'$ and $\mathbf{G} \in \mathbb{Q}'$ are given by

$$[\mathbf{F}, \underline{\boldsymbol{\tau}}] \;:=\; (\mathbf{g}, \mathbf{v})_\Omega + (g, q)_\Omega \quad \text{and} \quad [\mathbf{G}, \underline{\mathbf{v}}] \;:=\; (\mathbf{f}_p, \mathbf{v}_s)_\Omega\,. \tag{2.21}$$

Throughout, the notation $[\cdot,\cdot]$ denotes the duality pairing associated with the corresponding operator. Finally, we define $\mathcal{B}' : \mathbb{Q} \to \mathbb{H}'$ as the adjoint of $\mathcal{B}$, satisfying $[\mathcal{B}'(\underline{\mathbf{v}}), \underline{\boldsymbol{\tau}}] = [\mathcal{B}(\underline{\boldsymbol{\tau}}), \underline{\mathbf{v}}]$ for all $(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) \in \mathbb{H} \times \mathbb{Q}$.

3 The well-posedness of the model

In this section, we establish the solvability of the variational problem (2.16). To this end, we begin by presenting a set of auxiliary results that will be instrumental in the analysis.

3.1 Preliminaries

We first discuss the stability properties of the operators involved. It is straightforward to observe that the operators $\mathcal{E}_1$, $\mathcal{E}_2$, $\mathcal{A}$, and $\mathcal{B}$ (cf. (2.17)–(2.20)), as well as the functionals $\mathbf{F}$ and $\mathbf{G}$ (cf. (2.21)), are linear. In addition, applying the Cauchy–Schwarz inequality (cf. (1.2)) and simple computations yield the following bounds:

$$\big|[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}]\big| \le \|\mathcal{E}_1\| \|\underline{\boldsymbol{\sigma}}\|_{\mathbb{H}} \, \|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}\,, \quad \big|[\mathcal{A}(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}]\big| \le \|\mathcal{A}\| \|\underline{\boldsymbol{\sigma}}\|_{\mathbb{H}} \, \|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}\,, \quad \big|[\mathcal{E}_2(\underline{\mathbf{u}}), \underline{\mathbf{v}}]\big| \le \|\mathcal{E}_2\| \|\underline{\mathbf{u}}\|_{\mathbb{Q}} \, \|\underline{\mathbf{v}}\|_{\mathbb{Q}}\,,$$
$$\big|[\mathcal{B}(\underline{\boldsymbol{\tau}}), \underline{\mathbf{v}}]\big| \le \|\mathcal{B}\| \|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}} \, \|\underline{\mathbf{v}}\|_{\mathbb{Q}}\,, \quad |[\mathbf{F}, \underline{\boldsymbol{\tau}}]| \le \|\mathbf{F}\| \|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}, \quad |[\mathbf{G}, \underline{\mathbf{v}}]| \le \|\mathbf{G}\| \|\underline{\mathbf{v}}\|_{\mathbb{Q}}\,,$$

with operator norms given by

$$\|\mathcal{E}_1\| \;:=\; \max\left\{\frac{1}{\mu}, \frac{\alpha^2 d}{\mu} + s_0, 1\right\}, \quad \|\mathcal{A}\| \;:=\; 2\max\left\{\sqrt{d},\, \nu,\, \mathtt{D}\right\}\,, \quad \|\mathcal{E}_2\| \;:=\; \rho_p\,, \quad \|\mathcal{B}\| \;:=\; 1\,,$$
$$\|\mathbf{F}\| \;:=\; \left\{\|\mathbf{g}\|_{\mathbf{L}^2(\Omega)}^2 + \|g\|_{\mathrm{L}^2(\Omega)}^2\right\}^{1/2} \quad \text{and} \quad \|\mathbf{G}\| \;:=\; \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega)}\,.$$

These estimates ensure that the involved operators and functionals are bounded and continuous. Moreover, the operators $\mathcal{E}_1$, $\mathcal{A}$, and $\mathcal{E}_2$ are symmetric, and non-negative. In particular, for every $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \mathbf{v}, q) \in \mathbb{H}$ and $\underline{\mathbf{v}} = (\mathbf{v}_s, \boldsymbol{\chi}) \in \mathbb{Q}$, the following bounds hold

$$\begin{aligned} [\mathcal{E}_1(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] &= \|A^{1/2}(\boldsymbol{\tau} + \alpha q \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|q\|^2_{\mathrm{L}^2(\Omega)}\,, \\ [\mathcal{A}(\underline{\boldsymbol{\tau}}), \underline{\boldsymbol{\tau}}] &\geq \min\{\nu, \mathtt{D}\}\, \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)}\,, \quad [\mathcal{E}_2(\underline{\mathbf{v}}), \underline{\mathbf{v}}] = \rho_p \|\mathbf{v}_s\|^2_{\mathbf{L}^2(\Omega)}\,. \end{aligned} \tag{3.1}$$

We now state two inf-sup conditions that play a key role in the analysis of the problem.

Lemma 3.1 *There exist positive constants β_1 and β_2 such that*

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbb{H}} \frac{[\mathcal{B}(\underline{\boldsymbol{\tau}}), \underline{\mathbf{v}}]}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}} \geq \beta_1 \|\underline{\mathbf{v}}\|_{\mathbb{Q}} \qquad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}\,, \tag{3.2}$$

and

$$\sup_{\mathbf{0} \neq \mathbf{v} \in \mathbf{H}^1_0(\Omega)} \frac{(q, \operatorname{div}(\mathbf{v}))_\Omega}{\|\mathbf{v}\|_{\mathbf{H}^1(\Omega)}} \geq \beta_2 \|q\|_{\mathrm{L}^2(\Omega)} \qquad \forall\, q \in \mathrm{L}^2_0(\Omega)\,. \tag{3.3}$$

Proof. The inf-sup condition (3.2) follows from the the framework developed in [29, Section 2.4.3.1], where the linear elasticity problem with homogeneous Dirichlet boundary conditions is analyzed. There it is shown that there exists $\beta_1 > 0$ such that

$$\sup_{\mathbf{0} \neq \underline{\boldsymbol{\tau}} \in \mathbb{H}_0} \frac{[\mathcal{B}(\underline{\boldsymbol{\tau}}), \underline{\mathbf{v}}]}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}} \geq \beta_1 \|\underline{\mathbf{v}}\|_{\mathbb{Q}} \qquad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}\,,$$

with

$$\mathbb{H}_0 := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}; \Omega) : \quad (\operatorname{tr}(\boldsymbol{\tau}), 1)_\Omega = 0 \right\}.$$

Then, since $\mathbb{H}_0 \subset \mathbb{H} := \mathbb{H}(\mathbf{div}; \Omega)$ it follows immediately (3.2). In turn, for (3.3) we resort to [28, Corollary B.71]. □

Remark 3.1 *We emphasize that the inf-sup condition (3.3) is not mandatory for the well-posedness of (2.16). However, it will be useful for deriving stability and error bounds for the pore pressure p that are independent of the storage coefficient s_0.*

The following norm equivalence will also be required in the analysis.

Lemma 3.2 *There exist constants $\mathcal{C}_{\mathtt{p}}$, $c_{\mathtt{p}} > 0$, depending on μ, λ, α, s_0, and the space dimension d, such that*

$$c_{\mathtt{p}} \left(\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|q\|^2_{\mathrm{L}^2(\Omega)} \right) \leq \|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\, \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + s_0 \|q\|^2_{\mathrm{L}^2(\Omega)} \leq \mathcal{C}_{\mathtt{p}} \left(\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|q\|^2_{\mathrm{L}^2(\Omega)} \right) \tag{3.4}$$

for all $(\boldsymbol{\tau}, q) \in \mathbb{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$.

Proof. Let $(\boldsymbol{\tau}, q) \in \mathbb{L}^2(\Omega) \times \mathrm{L}^2(\Omega)$, beginning with the upper bound of (3.4), the triangle inequality and the upper bound for A (cf. (2.4)), yield

$$\|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\, \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} \leq \frac{1}{\mu} \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha^2 d}{\mu} \|q\|^2_{\mathrm{L}^2(\Omega)}\,,$$

which implies

$$\|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + s_0\|q\|^2_{\mathrm{L}^2(\Omega)} \le \frac{1}{\mu}\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \left(\frac{\alpha^2 d}{\mu} + s_0\right)\|q\|^2_{\mathrm{L}^2(\Omega)},$$

and yields the upper bound of (3.4) with $\mathcal{C}_{\mathtt{p}} := \max\left\{\frac{1}{\mu},\, \frac{\alpha^2 d}{\mu} + s_0\right\}$.

For the lower bound in (3.4), using the coercivity bound for A (cf. (2.4)) the triangle inequality, and after a simple algebraic manipulation, we get

$$\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} \le 2(2\mu + d\lambda)\left(\|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \frac{\alpha^2\, d}{2\mu}\|q\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{3.5}$$

Hence, upon adding $\|q\|^2_{\mathrm{L}^2(\Omega)}$ to both sides of (3.5) and factoring accordingly, we obtain

$$\begin{aligned}
&\|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|q\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad\le 2(2\mu + d\lambda)\left(\|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{s_0}\left(\frac{\alpha^2\, d}{2\mu} + \frac{1}{2(2\mu + d\lambda)}\right) s_0\|q\|^2_{\mathrm{L}^2(\Omega)}\right) \\
&\quad\le 2(2\mu + d\lambda)\max\left\{1, \frac{1}{s_0}\left(\frac{\alpha^2\, d}{2\mu} + \frac{1}{(2\mu + d\lambda)}\right)\right\}\left(\|A^{1/2}(\boldsymbol{\tau} + \alpha\, q\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + s_0\|q\|^2_{\mathrm{L}^2(\Omega)}\right).
\end{aligned}$$

Therefore, defining $c_{\mathtt{p}} := \left(2(2\mu + d\lambda)\max\left\{1, \frac{1}{s_0}\left(\frac{\alpha^2\, d}{2\mu} + \frac{1}{(2\mu + d\lambda)}\right)\right\}\right)^{-1}$, we conclude the proof. □

We continue by introducing some definitions and notations. In what follows, a linear operator $\mathcal{A}$ from a real vector space E to its algebraic dual $E^\star$ is said to be symmetric and monotone if, respectively,

$$[\mathcal{A}(x), y] = [\mathcal{A}(y), x] \quad \forall\, x,\, y \in E\,, \quad \text{and} \quad [\mathcal{A}(x), x] \ge 0 \quad \forall\, x \in E\,.$$

In addition, let us denote by $Rg(\mathcal{A})$ the range of $\mathcal{A}$. We also recall that the dual of a seminormed space is the space of all linear functionals that are continuous with respect to the seminorm.

The following result is a slight simplification of [38, Theorem IV.6.1(b)], which will be used to establish the existence of a solution to (2.16).

Theorem 3.3 *Let the linear, symmetric and monotone operator $\mathcal{N}$ be given from the real vector space E to its algebraic dual $E^\star$, and let E_b' be the Hilbert space which is the topological dual of the seminormed space $(E, |\cdot|_b)$, where*

$$|x|_b = [\mathcal{N}(x), x]^{1/2} \quad x \in E\,.$$

Let $\mathcal{M} : E \to E_b'$ be an operator with domain $\mathcal{D} = \left\{x \in E \,:\, \mathcal{M}(x) \in E_b'\right\}$. Assume that $\mathcal{M}$ is monotone and $R(\mathcal{N} + \mathcal{M}) = E_b'$. Then, for each $f \in \mathrm{W}^{1,1}(0,T;E_b')$ and for each $u_0 \in \mathcal{D}$, there is a solution $u : [0,T] \to E$ of

$$\frac{\partial}{\partial\, t}\big(\mathcal{N}(u(t))\big) + \mathcal{M}\big(u(t)\big) \;=\; f(t) \quad \text{for a.e.} \quad 0 < t < T\,, \tag{3.6}$$

with

$$\mathcal{N}(u) \in \mathrm{W}^{1,\infty}(0,T;E_b')\,, \quad u(t) \in \mathcal{D} \quad \text{for all } 0 \le t \le T\,, \quad \text{and} \quad \mathcal{N}(u(0)) = \mathcal{N}(u_0)\,.$$

To apply Theorem 3.3 to our context, we begin by rewriting the variational formulation (2.16) in the abstract form (3.6). This is accomplished by setting

$$E := \mathbb{H}\times\mathbb{Q}\,,\quad \underline{u} := \begin{pmatrix}\underline{\boldsymbol{\sigma}}\\ \underline{\mathbf{u}}\end{pmatrix}\,,\quad \mathcal{N} := \begin{pmatrix}\mathcal{E}_1 & \mathbf{0}\\ \mathbf{0} & \mathcal{E}_2\end{pmatrix}\,,\quad \mathcal{M} := \begin{pmatrix}\mathcal{A} & \mathcal{B}'\\ -\mathcal{B} & \mathbf{0}\end{pmatrix}\,,\quad \text{and}\quad f := \begin{pmatrix}\mathbf{F}\\ \mathbf{G}\end{pmatrix}\,. \tag{3.7}$$

Furthermore, the seminorm induced by the operator $\mathcal{N}$ (cf. (3.7), (2.17), (2.20)) is defined by

$$|(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})|^2_{\mathcal{N}} := \|A^{1/2}(\boldsymbol{\tau}+\alpha\, q\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + s_0\|q\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{v}_s\|^2_{\mathbf{L}^2(\Omega)} \qquad \forall\,(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}\,.$$

Then, considering $s_0,\rho_p>0$ and invoking the norm equivalence established in Lemma 3.2, we conclude that

$$|(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})|^2_{\mathcal{N}} \;\equiv\; \|\boldsymbol{\tau}\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{v}\|^2_{\mathbf{L}^2(\Omega)} + \|q\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{v}_s\|^2_{\mathbf{L}^2(\Omega)} \qquad \forall\,(\underline{\boldsymbol{\tau}},\underline{\mathbf{v}})\in\mathbb{H}\times\mathbb{Q}\,.$$

In order to specify the dual space E'_b and the domain $\mathcal{D}$, we introduce the product spaces

$$\mathbb{H}'_2 := \mathbb{L}^2(\Omega)\times\mathbf{L}^2(\Omega)\times\mathrm{L}^2(\Omega)\,,\quad \mathbb{Q}'_2 := \mathbf{L}^2(\Omega)\times\{\mathbf{0}\}\,,$$

and define

$$E'_b := \mathbb{H}'_2\times\mathbb{Q}'_2 \quad\text{and}\quad \mathcal{D} := \Big\{(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\in\mathbb{H}\times\mathbb{Q}:\quad \mathcal{M}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\in E'_b\Big\}\,. \tag{3.8}$$

In the next section, we verify that the assumptions of Theorem 3.3 are satisfied. This verification will then allow us to conclude the existence of a solution to problem (2.16).

3.2 The resolvent system and compatible initial data

We now proceed to verify the range condition required by Theorem 3.3. To this end, we consider the corresponding resolvent system: given data functionals $\widehat{\mathbf{F}}\in\mathbb{H}'_2$ and $\widehat{\mathbf{G}}\in\mathbb{Q}'_2$, we seek $(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\in\mathbb{H}\times\mathbb{Q}$ such that

$$\begin{aligned}[(\mathcal{E}_1+\mathcal{A})(\underline{\boldsymbol{\sigma}}),\underline{\boldsymbol{\tau}}] + [\mathcal{B}'(\underline{\mathbf{u}}),\underline{\boldsymbol{\tau}}] &= [\widehat{\mathbf{F}},\underline{\boldsymbol{\tau}}] \qquad \forall\,\underline{\boldsymbol{\tau}}\in\mathbb{H}\,,\\ [\mathcal{B}(\underline{\boldsymbol{\sigma}}),\underline{\mathbf{v}}] - [\mathcal{E}_2(\underline{\mathbf{u}}),\underline{\mathbf{v}}] &= -[\widehat{\mathbf{G}},\underline{\mathbf{v}}] \quad \forall\,\underline{\mathbf{v}}\in\mathbb{Q}\,.\end{aligned} \tag{3.9}$$

Here, the linear functionals $\widehat{\mathbf{F}}$ and $\widehat{\mathbf{G}}$ are defined for all $\underline{\boldsymbol{\tau}} := (\boldsymbol{\tau},\mathbf{v},q)\in\mathbb{H}$ and $\underline{\mathbf{v}} := (\mathbf{v}_s,\boldsymbol{\chi})\in\mathbb{Q}$ by

$$\begin{aligned}[\widehat{\mathbf{F}},\underline{\boldsymbol{\tau}}] &:= (\widehat{\mathbf{f}},\boldsymbol{\tau})_\Omega + (\widehat{\mathbf{g}},\mathbf{v})_\Omega + (\widehat{g},q)_\Omega\,,\\ [\widehat{\mathbf{G}},\underline{\mathbf{v}}] &:= (\widehat{\mathbf{f}}_p,\mathbf{v}_s)_\Omega\,,\end{aligned} \tag{3.10}$$

with given data $\widehat{\mathbf{f}}\in\mathbb{L}^2(\Omega)$, $\widehat{\mathbf{g}}\in\mathbf{L}^2(\Omega)$, $\widehat{g}\in\mathrm{L}^2_0(\Omega)$, and $\widehat{\mathbf{f}}_p\in\mathbf{L}^2(\Omega)$.

To establish the existence of a solution to (3.9), we invoke [11, Theorem 4.3.1] (see also [23, Theorem 3.1] for its extension to the Banach setting), which guarantees the well-posedness of perturbed saddle-point problems. We first note that the kernel of the operator $\mathcal{B}$ (cf. (2.19)) is the product space

$$\mathbf{V} := \mathbf{K}\times\mathbf{H}^1_0(\Omega)\times\mathrm{L}^2_0(\Omega),\quad \text{where}\quad \mathbf{K} := \Big\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div};\Omega):\quad \mathbf{div}(\boldsymbol{\tau})=\mathbf{0}\quad\text{and}\quad \boldsymbol{\tau}=\boldsymbol{\tau}^{\mathrm{t}}\Big\}.$$

Furthermore, the properties satisfied by the operators $\mathcal{E}_1$, $\mathcal{A}$, and $\mathcal{E}_2$ (cf. (3.1)) result in two key facts. First, the operator $\mathcal{E}_1+\mathcal{A}$ is non-negative and $\mathbb{H}$-coercive on $\mathbf{V}$. Indeed, for every $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau},\mathbf{v},p)\in\mathbf{V}$, from (3.1) and the equivalence of norms established in Lemma 3.2, we obtain

$$[(\mathcal{E}_1+\mathcal{A})(\underline{\boldsymbol{\tau}}),\underline{\boldsymbol{\tau}}] \;\geq\; \min\{1,c_{\mathtt{p}},\nu,\mathtt{D}\}\Big(\|\boldsymbol{\tau}\|^2_{\mathbb{H}(\mathbf{div};\Omega)} + \|\mathbf{v}\|^2_{\mathbf{H}^1(\Omega)} + \|q\|^2_{\mathrm{L}^2(\Omega)}\Big)\,.$$

Second, the operator $\mathcal{E}_2$ (cf. (2.20)) is symmetric and non-negative. Then, in combination with the inf-sup condition satisfied by $\mathcal{B}$ (cf. (3.2) in Lemma 3.1), these properties allow us to establish the well-posedness of the resolvent problem.

Lemma 3.4 *Given $\widehat{\mathbf{F}} \in \mathbb{H}_2'$ and $\widehat{\mathbf{G}} \in \mathbb{Q}_2'$ (cf. (3.10)), there exists a unique solution $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) \in \mathbb{H} \times \mathbb{Q}$ of the resolvent system (3.9).*

Proof. The result is a direct consequence of the fact that $(\widehat{\mathbf{F}}, \widehat{\mathbf{G}}) \in \mathbb{H}_2' \times \mathbb{Q}_2'$, in combination with the properties established above for the operators $\mathcal{E}_1 + \mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}_2$, and an application of the well-posedness result in [11, Theorem 4.3.1]. □

We conclude this section by deriving an appropriate initial condition result, which is essential for applying Theorem 3.3 in our setting.

Lemma 3.5 *Assume initial conditions $\mathbf{u}_0 \in \mathbf{H}_{\mathbf{u}}$, $p_0 \in \mathrm{H}_p$, and $\mathbf{u}_{s,0} \in \mathbf{H}_{\mathbf{u}_s}$ (cf. (2.9g)), where*

$$\mathbf{H}_{\mathbf{u}} := \Big\{ \mathbf{v} \in \mathbf{H}_0^1(\Omega) : \quad \Delta \mathbf{v} \in \mathbf{L}^2(\Omega) \Big\}, \quad \mathrm{H}_p := \Big\{ q \in \mathrm{H}^1(\Omega) : \quad (q,1)_\Omega = 0 \Big\}, \\ \text{and} \quad \mathbf{H}_{\mathbf{u}_s} := \Big\{ \mathbf{v}_s \in \mathbf{H}_0^1(\Omega) : \quad \mathbf{div}(\mathbf{e}(\mathbf{v}_s)) \in \mathbf{L}^2(\Omega) \Big\}. \tag{3.11}$$

Then, there exists $(\boldsymbol{\sigma}_{p,0}, \gamma_0) \in \mathbb{H}(\mathbf{div};\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ such that, if we set $\underline{\boldsymbol{\sigma}}_0 := (\boldsymbol{\sigma}_{p,0}, \mathbf{u}_0, p_0) \in \mathbb{H}$ and $\underline{\mathbf{u}}_0 := (\mathbf{u}_{s,0}, \gamma_0) \in \mathbb{Q}$, there holds

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\sigma}}_0 \\ \underline{\mathbf{u}}_0 \end{pmatrix} \in \mathbb{H}_2' \times \mathbb{Q}_2'.$$

Proof. Let $\mathbf{u}_0 \in \mathbf{H}_{\mathbf{u}}$ and $p_0 \in \mathrm{H}_p$, recalling the definition of $\mathcal{A}$ (cf. (2.18)), we integrate by parts in the reverse direction, i.e., transferring derivatives from $\mathbf{v}$ to $(\mathbf{u}_0, p_0)$, to obtain

$$[\mathcal{A}(\underline{\boldsymbol{\sigma}}_0), \underline{\boldsymbol{\tau}}] = -\nu(\Delta \mathbf{u}_0, \mathbf{v})_\Omega + \mathtt{D}(\mathbf{u}_0, \mathbf{v})_\Omega + (\nabla p_0, \mathbf{v})_\Omega + (q, \mathrm{div}(\mathbf{u}_0))_\Omega \tag{3.12}$$

for all $(\mathbf{v}, q) \in \mathbf{H}_0^1(\Omega) \times \mathrm{L}_0^2(\Omega)$. Next, given $\mathbf{u}_{s,0} \in \mathbf{H}_{\mathbf{u}_s}$, we define

$$\boldsymbol{\sigma}_{p,0} := \mathbf{e}(\mathbf{u}_{s,0}) \quad \text{and} \quad \gamma_0 := \frac{1}{2}\big(\nabla \mathbf{u}_{s,0} - (\nabla \mathbf{u}_{s,0})^{\mathtt{t}}\big). \tag{3.13}$$

Since $\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})) \in \mathbf{L}^2(\Omega)$ by assumption (cf. (3.11)), it follows that $\boldsymbol{\sigma}_{p,0} \in \mathbb{H}(\mathbf{div};\Omega)$. Moreover, γ_0 is skew-symmetric by construction, hence $\gamma_0 \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$ (cf. (1.1)). In addition, testing $\mathbf{div}(\boldsymbol{\sigma}_{p,0}) = \mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0}))$ in Ω and the identities in (3.13) against arbitrary functions yields

$$(\mathbf{div}(\boldsymbol{\sigma}_{p,0}), \mathbf{v}_s)_\Omega = (\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})), \mathbf{v}_s)_\Omega, \quad (\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi})_\Omega = 0, \\ \text{and} \quad (\gamma_0, \boldsymbol{\tau})_\Omega = \frac{1}{2}(\nabla \mathbf{u}_{s,0} - (\nabla \mathbf{u}_{s,0})^{\mathtt{t}}, \boldsymbol{\tau})_\Omega, \tag{3.14}$$

for all $(\mathbf{v}_s, \boldsymbol{\chi}, \boldsymbol{\tau}) \in \mathbf{L}^2(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \times \mathbb{H}(\mathbf{div};\Omega)$. We notice that the second identity of (3.14) follows from the symmetry of $\mathbf{e}(\mathbf{u}_{s,0})$, which implies $\mathbf{e}(\mathbf{u}_{s,0}) : \boldsymbol{\chi} = 0$ for all $\boldsymbol{\chi} \in \mathbb{L}^2_{\mathtt{skew}}(\Omega)$. Then, recalling the definition of $\mathcal{B}$ (cf. (2.19)), the first row of (3.14) implies

$$[\mathcal{B}(\underline{\boldsymbol{\sigma}}_0), \underline{\mathbf{v}}] = (\mathbf{div}(\boldsymbol{\sigma}_{p,0}), \mathbf{v}_s)_\Omega + (\boldsymbol{\sigma}_{p,0}, \boldsymbol{\chi})_\Omega = (\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})), \mathbf{v}_s)_\Omega \quad \forall\, \underline{\mathbf{v}} \in \mathbb{Q}.$$

In turn, by integration by parts and using that $\mathbf{u}_{s,0} = \mathbf{0}$ on Γ, since $\mathbf{u}_{s,0} \in \mathbf{H}_{\mathbf{u}_s}$, we obtain

$$(\mathbf{u}_{s,0}, \mathbf{div}(\boldsymbol{\tau}))_\Omega = -(\nabla \mathbf{u}_{s,0}, \boldsymbol{\tau})_\Omega \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega). \tag{3.15}$$

Thus, combining (3.15) with the last identity in (3.14), we obtain

$$[\mathcal{B}'(\underline{\mathbf{u}}_0), \underline{\boldsymbol{\tau}}] = -(\nabla \mathbf{u}_{s,0}, \boldsymbol{\tau})_\Omega + \frac{1}{2}\big(\nabla \mathbf{u}_{s,0} - (\nabla \mathbf{u}_{s,0})^{\mathtt{t}}, \boldsymbol{\tau}\big)_\Omega = -\big(\mathbf{e}(\mathbf{u}_{s,0}), \boldsymbol{\tau}\big)_\Omega \qquad \forall\, \underline{\boldsymbol{\tau}} \in \mathbb{H}.$$

Hence, we have constructed a pair $(\underline{\boldsymbol{\sigma}}_0, \underline{\mathbf{u}}_0)$ satisfying

$$\begin{pmatrix} \mathcal{A} & \mathcal{B}' \\ -\mathcal{B} & 0 \end{pmatrix} \begin{pmatrix} \underline{\boldsymbol{\sigma}}_0 \\ \underline{\mathbf{u}}_0 \end{pmatrix} = \begin{pmatrix} \mathbf{F}_0 \\ \mathbf{G}_0 \end{pmatrix}, \tag{3.16}$$

where the right-hand side functionals are defined, for all $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau}, \mathbf{v}, q) \in \mathbb{H}$ and $\underline{\mathbf{v}} = (\mathbf{v}_s, \boldsymbol{\chi}) \in \mathbb{Q}$, by

$$\begin{aligned} [\mathbf{F}_0, \underline{\boldsymbol{\tau}}] &:= -\big(\mathbf{e}(\mathbf{u}_{s,0}), \boldsymbol{\tau}\big)_\Omega - \big(\nu\, \Delta \mathbf{u}_0 - \mathtt{D}\, \mathbf{u}_0 - \nabla p_0, \mathbf{v}\big)_\Omega + \big(q, \operatorname{div}(\mathbf{u}_0)\big)_\Omega, \\ [\mathbf{G}_0, \underline{\mathbf{v}}] &:= -\big(\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})), \mathbf{v}_s\big)_\Omega. \end{aligned} \tag{3.17}$$

Finally, using the additional regularity of $\mathbf{u}_0$, p_0, and $\mathbf{u}_{s,0}$ (cf. (3.11)), together with the Cauchy–Schwarz inequality (cf. (1.2)), we obtain

$$\begin{aligned} |[\mathbf{F}_0, \underline{\boldsymbol{\tau}}]| \leq\ & \|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{L}^2(\Omega)} \|\boldsymbol{\tau}\|_{\mathbb{L}^2(\Omega)} + \big(\nu\, \|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \mathtt{D}\, \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\nabla p_0\|_{\mathbf{L}^2(\Omega)}\big)\, \|\mathbf{v}\|_{\mathbf{L}^2(\Omega)} \\ & + \|\operatorname{div}(\mathbf{u}_0)\|_{\mathrm{L}^2(\Omega)} \|q\|_{\mathrm{L}^2(\Omega)}, \end{aligned} \tag{3.18}$$

and

$$|[\mathbf{G}_0, \underline{\mathbf{v}}]| \leq \|\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0}))\|_{\mathbf{L}^2(\Omega)}\, \|\mathbf{v}_s\|_{\mathbf{L}^2(\Omega)}, \tag{3.19}$$

which implies that $\mathbf{F}_0 \in \mathbb{H}_2'$ and $\mathbf{G}_0 \in \mathbb{Q}_2'$ and completes the proof. □

3.3 The main result

We now establish the well-posedness of problem (2.16).

Theorem 3.6 *Let $\mathbf{u}_0 \in \mathbf{H}_{\mathbf{u}}$, $p_0 \in \mathrm{H}_p$, and $\mathbf{u}_{s,0} \in \mathbf{H}_{\mathbf{u}_s}$ (cf. (3.11)), and let $(\underline{\boldsymbol{\sigma}}_0, \underline{\mathbf{u}}_0)$ be the compatible initial data constructed in Lemma 3.5, where $\underline{\boldsymbol{\sigma}}_0 := (\boldsymbol{\sigma}_{p,0}, \mathbf{u}_0, p_0)$ and $\underline{\mathbf{u}}_0 := (\mathbf{u}_{s,0}, \gamma_0)$. Then, for each*

$$\mathbf{g} \in \mathrm{W}^{1,1}(0,T; \mathbf{L}^2(\Omega)), \quad g \in \mathrm{W}^{1,1}(0,T; \mathrm{L}_0^2(\Omega)), \quad \text{and} \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T; \mathbf{L}^2(\Omega)),$$

there exists a unique solution $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{H} \times \mathbb{Q}$ to (2.16), such that

$$\underline{\boldsymbol{\sigma}} \in \mathrm{W}^{1,\infty}(0,T; \mathbb{H}_2'), \quad \mathbf{u}_s \in \mathrm{W}^{1,\infty}(0,T; \mathbf{L}^2(\Omega)), \quad \text{and} \quad (\underline{\boldsymbol{\sigma}}(0), \mathbf{u}_s(0)) = (\underline{\boldsymbol{\sigma}}_0, \mathbf{u}_{s,0}).$$

Proof. We first recall that the variational problem (2.16) falls within the abstract setting of Theorem 3.3, via the identifications specified in (3.7) and (3.8). Since $\mathcal{E}_1$ and $\mathcal{E}_2$ (cf. (2.17) and (2.20), respectively) are linear, symmetric, and monotone, it follows that $\mathcal{N}$ also satisfies these properties. Moreover, recalling the definition of $\mathcal{M}$ (cf. (3.7)), and using that the monotonicity of $\mathcal{A}$ follows from (3.1), we note that this property is inherited by $\mathcal{M}$. In addition, Lemma 3.4 guarantees that $\mathrm{Rg}(\mathcal{N} + \mathcal{M}) = E_b'$, and Lemma 3.5 ensures that, for any admissible initial data $(\mathbf{u}_0, p_0, \mathbf{u}_{s,0})$, we may construct $(\underline{\boldsymbol{\sigma}}_0, \underline{\mathbf{u}}_0) \in \mathcal{D}$ satisfying the required compatibility condition. Therefore, by invoking Theorem 3.3, we conclude that there exists at least one solution $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{H} \times \mathbb{Q}$ to (2.16) such that

$$\underline{\boldsymbol{\sigma}} \in \mathrm{W}^{1,\infty}(0,T; \mathbb{H}_2') \quad \text{and} \quad \mathbf{u}_s \in \mathrm{W}^{1,\infty}(0,T; \mathbf{L}^2(\Omega)),$$

with initial condition $(\underline{\boldsymbol{\sigma}}(0), \mathbf{u}_s(0)) = (\underline{\boldsymbol{\sigma}}_0, \mathbf{u}_{s,0})$.

We next establish uniqueness. Since the problem is linear, it suffices to prove that the homogeneous problem, i.e., with zero data, admits only the trivial solution. In fact, taking $\mathbf{F} = \mathbf{0}$ and $\mathbf{G} = \mathbf{0}$ in (2.16), $(\underline{\boldsymbol{\sigma}}_0, \mathbf{u}_{s,0}) = (\mathbf{0}, \mathbf{0})$, and testing it with the solution $((\boldsymbol{\sigma}_p, \mathbf{u}, p), (\mathbf{u}_s, \boldsymbol{\gamma}))$, we obtain

$$\frac{1}{2}\partial_t \left(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha p \mathbb{I})\|_{\mathbb{L}^2(\Omega)}^2 + \|\mathbf{u}\|_{\mathbf{L}^2(\Omega)}^2 + s_0 \|p\|_{\mathrm{L}^2(\Omega)}^2 + \rho_p \|\mathbf{u}_s\|_{\mathbf{L}^2(\Omega)}^2 \right) + \min\{\nu, \mathtt{D}\} \|\mathbf{u}\|_{\mathbf{H}^1(\Omega)}^2 \leq 0.$$

Integrating from 0 to $t \in (0,T]$, and using the homogeneous initial data, we deduce

$$\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha p\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{u}(s)\|^2_{\mathbf{H}^1(\Omega)}\, ds \le 0\,,$$

which, thanks to Lemma 3.2, immediately implies

$$\boldsymbol{\sigma}_p(t) = \mathbf{0}, \quad \mathbf{u}(t) = \mathbf{0}, \quad p(t) = 0, \quad \mathbf{u}_s(t) = \mathbf{0}, \quad \text{for all } t \in (0,T].$$

Finally, invoking the inf-sup condition (3.2), applied to $\underline{\mathbf{u}} = (\mathbf{u}_s, \boldsymbol{\gamma})$, and the first equation of (2.16), we obtain

$$\beta_1\|\boldsymbol{\gamma}(t)\|_{\mathbb{L}^2(\Omega)} \le \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbb{H}} \frac{[\mathcal{B}'(\underline{\mathbf{u}}(t)), \underline{\boldsymbol{\tau}}]}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}} = \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbb{H}} \frac{-\partial_t[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}(t)), \underline{\boldsymbol{\tau}}] - [\mathcal{A}(\underline{\boldsymbol{\sigma}}(t)), \underline{\boldsymbol{\tau}}]}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}} = 0,$$

which yields $\boldsymbol{\gamma}(t) = \mathbf{0}$ for all $t \in (0,T]$. Hence, the solution is unique. □

Before establishing a stability bound for the solution of (2.16), we first derive a bound at $t = 0$. For the sake of the stability analysis developed below in Theorem 3.9, we additionally assume that the source term g is time-independent.

Lemma 3.7 *Under the assumptions of Theorem 3.6, further assuming that g is time-independent and that the data are sufficiently regular, there exist constants $\mathcal{C}_{1,\mathrm{ic}}$, $\mathcal{C}_{2,\mathrm{ic}} > 0$, independent of s_0, such that*

$$\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(0)\|_{\mathbb{L}^2(\Omega)} \le \mathcal{C}_{1,\mathtt{ic}}\Big(\|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{L}^2(\Omega)} + \|p_0\|_{\mathrm{L}^2(\Omega)}\Big)\,, \tag{3.20}$$

and

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(0)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{u}(0)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t p(0)\|_{\mathrm{L}^2(\Omega)} + \sqrt{\rho_p}\,\|\partial_t\mathbf{u}_s(0)\|_{\mathbf{L}^2(\Omega)} \\
&\quad \le\; \mathcal{C}_{2,\mathtt{ic}}\Big(\|\mathbf{f}_p(0)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{g}(0)\|_{\mathbf{L}^2(\Omega)} + \|\Delta\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\nabla p_0\|_{\mathbf{L}^2(\Omega)} \\
&\qquad + \|\nabla\mathbf{u}_{s,0}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0}))\|_{\mathbf{L}^2(\Omega)} + \frac{1}{\sqrt{s_0}}\Big(\|g\|_{\mathrm{L}^2(\Omega)} + \|\mathrm{div}(\mathbf{u}_0)\|_{\mathrm{L}^2(\Omega)} + \|\nabla\mathbf{u}_{s,0}\|_{\mathbb{L}^2(\Omega)}\Big)\Big)\,.
\end{aligned} \tag{3.21}$$

Proof. We begin by recalling that the initial conditions were constructed in Lemma 3.5, and according to Theorem 3.6, we have that $((\boldsymbol{\sigma}_p(0), \mathbf{u}(0), p(0)), \mathbf{u}_s(0)) = ((\boldsymbol{\sigma}_{p,0}, \mathbf{u}_0, p_0), \mathbf{u}_{s,0})$. Then, using the property (2.4), we obtain the estimate

$$\|A^{1/2}(\boldsymbol{\sigma}_p(0) + \alpha\, p(0)\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} \le \frac{1}{\mu}\Big(\|\boldsymbol{\sigma}_p(0)\|^2_{\mathbb{L}^2(\Omega)} + \alpha^2 d\,\|p(0)\|^2_{\mathrm{L}^2(\Omega)}\Big)\,,$$

which together with (3.13) concludes (3.20) with $\mathcal{C}_{1,\mathtt{ic}}$ a positive constant depending on μ, α and d.

Next, we evaluate (2.16) at $t = 0$ and test it against $\underline{\boldsymbol{\tau}} = \partial_t\underline{\boldsymbol{\sigma}}(0) = (\partial_t\boldsymbol{\sigma}_p(0), \partial_t\mathbf{u}(0), \partial_t p(0))$, and we also differentiate the second row of (2.16) with respect to time, evaluate at $t = 0$, and test it against $\underline{\mathbf{v}} = (\mathbf{0}, \boldsymbol{\gamma}(0))$, which altogether yields

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\,\|\partial_t\mathbf{u}_s(0)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad = \big(\mathbf{g}(0), \partial_t\mathbf{u}(0)\big)_\Omega + \big(g, \partial_t p(0)\big)_\Omega + \big(\mathbf{f}_p(0), \partial_t\mathbf{u}_s(0)\big)_\Omega \\
&\quad - \big[\mathcal{A}(\underline{\boldsymbol{\sigma}}_0), \partial_t\underline{\boldsymbol{\sigma}}(0)\big] - \big(\mathbf{u}_{s,0}, \mathbf{div}(\partial_t\boldsymbol{\sigma}_p(0))\big)_\Omega + \big(\mathbf{div}(\boldsymbol{\sigma}_p(0)), \partial_t\mathbf{u}_s(0)\big)_\Omega\,.
\end{aligned} \tag{3.22}$$

Then, bearing in mind (3.12), (3.15), and the first identity of (3.14), with $\boldsymbol{\sigma}_p(0) = \boldsymbol{\sigma}_{p,0}$, we observe that the last three terms in (3.22) may be rewritten as follows:

$$\begin{aligned}
&\big[\mathcal{A}(\underline{\boldsymbol{\sigma}}_0), \partial_t \underline{\boldsymbol{\sigma}}(0)\big] + \big(\mathbf{u}_{s,0}, \mathbf{div}(\partial_t \boldsymbol{\sigma}_p(0))\big)_\Omega - \big(\mathbf{div}(\boldsymbol{\sigma}_p(0)), \partial_t \mathbf{u}_s(0)\big)_\Omega \\
&\quad = -\nu(\Delta \mathbf{u}_0, \partial_t \mathbf{u}(0))_\Omega + \mathtt{D}(\mathbf{u}_0, \partial_t \mathbf{u}(0))_\Omega + (\nabla p_0, \partial_t \mathbf{u}(0))_\Omega + (\partial_t p(0), \operatorname{div}(\mathbf{u}_0))_\Omega \\
&\qquad -(\nabla \mathbf{u}_{s,0}, \partial_t \boldsymbol{\sigma}_p(0))_\Omega - (\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})), \partial_t \mathbf{u}_s(0))_\Omega \,.
\end{aligned} \tag{3.23}$$

Hence, from (3.22) and (3.23), in combination with the Cauchy–Schwarz inequality, we get

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{u}_s(0)\|^2_{\mathbf{L}^2(\Omega)} \\
&\quad \le \|\mathbf{g}(0)\|_{\mathbf{L}^2(\Omega)}\|\partial_t \mathbf{u}(0)\|_{\mathbf{L}^2(\Omega)} + \|g\|_{\mathrm{L}^2(\Omega)}\|\partial_t p(0)\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{f}_p(0)\|_{\mathbf{L}^2(\Omega)}\|\partial_t \mathbf{u}_s(0)\|_{\mathbf{L}^2(\Omega)} \\
&\quad + \big(\nu\,\|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \mathtt{D}\,\|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\nabla p_0\|_{\mathbf{L}^2(\Omega)}\big)\,\|\partial_t \mathbf{u}(0)\|_{\mathbf{L}^2(\Omega)} + \|\operatorname{div}(\mathbf{u}_0)\|_{\mathrm{L}^2(\Omega)}\|\partial_t p(0)\|_{\mathrm{L}^2(\Omega)} \\
&\quad + \|\nabla \mathbf{u}_{s,0}\|_{\mathbb{L}^2(\Omega)}\|\partial_t \boldsymbol{\sigma}_p(0)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0}))\|_{\mathbf{L}^2(\Omega)}\|\partial_t \mathbf{u}_s(0)\|_{\mathbf{L}^2(\Omega)}\,.
\end{aligned} \tag{3.24}$$

Then, observing, in analogy with (3.5), that

$$\|\partial_t \boldsymbol{\sigma}_p(0)\|^2_{\mathbb{L}^2(\Omega)} \le C_1\left(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t p(0)\|^2_{\mathrm{L}^2(\Omega)}\right), \tag{3.25}$$

where C_1 is a positive constant depending on α, d, λ, and μ. By joining (3.24) and (3.25), together with the Young inequality (cf. (1.3)), we deduce (3.21), where $\mathcal{C}_{2,\mathrm{ic}}$ is a positive constant depending only on ν, $\mathtt{D}$, μ, α, d, and λ. □

We next recall from [21, Lemma 3.3] a result that will be employed to derive the stability bound for the solution of (2.16) without relying on Grönwall's inequality.

Lemma 3.8 *Suppose that for all $t \in (0,T]$,*

$$\chi^2(t) + R(t) \;\le\; A(t) + 2\int_0^t B(s)\,\chi(s)\,ds\,,$$

where χ, R, A, and B are non-negative functions. Then, for all $t \in (0,T]$,

$$\sqrt{\chi^2(t) + R(t)} \;\le\; \sup_{0\le s\le T}\sqrt{A(s)} + 2\int_0^t B(s)\,ds\,.$$

We conclude with the aforementioned stability bound.

Theorem 3.9 *Let the solution of (2.16) be the one established in Theorem 3.6, and suppose sufficient regularity of the data. Assume, in addition, that the source term g is time-independent. Then there exists a constant $\mathcal{C}_{\mathtt{st}} > 0$, independent of s_0, such that*

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})\|_{\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}\|_{\mathrm{H}^1(0,T;\mathbf{H}^1(\Omega))} \\
&\quad + \sqrt{s_0}\|p\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega))} + \|p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{u}_s\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} + \|\boldsymbol{\gamma}\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \\
&\quad \le \mathcal{C}_{\mathtt{st}}\left\{\|\mathbf{g}\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\Omega))} + \left(1 + \frac{1}{\sqrt{s_0}}\right)\|g\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{f}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}_p\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}\right. \\
&\qquad + \|d_t\mathbf{f}_p\|_{\mathrm{L}^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{g}(0)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|p_0\|_{\mathrm{H}^1(\Omega)} + \sqrt{s_0}\|p_0\|_{\mathrm{L}^2(\Omega)} \\
&\qquad \left. + \sqrt{\rho_p}\|\mathbf{u}_{s,0}\|_{\mathbf{L}^2(\Omega)} + \|\nabla \mathbf{u}_{s,0}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{H}(\mathbf{div};\Omega)} + \frac{1}{\sqrt{s_0}}\Big(\|\operatorname{div}(\mathbf{u}_0)\|_{\mathrm{L}^2(\Omega)} + \|\nabla \mathbf{u}_{s,0}\|_{\mathbb{L}^2(\Omega)}\Big)\right\}.
\end{aligned} \tag{3.26}$$

Proof. We proceed as in [18, Theorem 4.12] and [19, Theorem 4.13]. In fact, we begin by testing equation (2.16) with the pair $(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) = (\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$, which yields

$$\frac{1}{2}\partial_t\Big(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s\|^2_{\mathbf{L}^2(\Omega)}\Big)$$
$$+\,\nu\|\nabla\mathbf{u}\|^2_{\mathbb{L}^2(\Omega)} + \mathtt{D}\|\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)} = (\mathbf{g},\mathbf{u})_\Omega + (g,p)_\Omega + (\mathbf{f}_p,\mathbf{u}_s)_\Omega\,.$$

Next, integrate over $(0,t)$ with $t\in(0,T]$, and apply the Cauchy–Schwarz and Young inequalities to obtain

$$\begin{aligned}
&\frac{1}{2}\Big(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)}\Big)\\
&+\min\left\{\nu,\frac{\mathtt{D}}{2}\right\}\int_0^t\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)}ds \le C_1\int_0^t\Big(\|\mathbf{g}\|^2_{\mathbf{L}^2(\Omega)} + \|g\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega)}\Big)ds\\
&+\frac{1}{2}\Big(\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s(0)\|^2_{\mathbf{L}^2(\Omega)}\Big)\\
&+\max\left\{\frac{\delta_1}{2},\frac{\delta_2}{2}\right\}\int_0^t\Big(\|p\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{u}_s\|^2_{\mathbf{L}^2(\Omega)}\Big)ds\,,
\end{aligned}\tag{3.27}$$

where C_1 is a positive constant depending on $\mathtt{D}$, δ_1 and δ_2. Then, in order to bound the last term of (3.27), we resort to the inf-sup conditions especified in Lemma 3.1. Indeed, employing the Cauchy–Schwarz inequality in combination with (3.2), the first row of (2.16) with $\underline{\boldsymbol{\tau}} = (\boldsymbol{\tau},\mathbf{0},0)$, and (2.4), we obtain

$$\begin{aligned}
\beta_1\Big(\|\mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\gamma}(t)\|^2_{\mathbb{L}^2(\Omega)}\Big)^{1/2} &\le \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbb{H}}\frac{(\mathbf{u}_s(t),\mathbf{div}(\boldsymbol{\tau}))_\Omega + (\boldsymbol{\gamma}(t),\boldsymbol{\tau})_\Omega}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}}\\
&= \sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}\in\mathbb{H}}\frac{-(\partial_t A(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(t),\boldsymbol{\tau})_\Omega}{\|\underline{\boldsymbol{\tau}}\|_{\mathbb{H}}} \le \frac{1}{\sqrt{2\mu}}\,\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)}\,,
\end{aligned}$$

whence

$$\|\mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\gamma}(t)\|^2_{\mathbb{L}^2(\Omega)} \le \frac{1}{2\,\mu\,\beta_1^2}\,\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)}\,. \tag{3.28}$$

Similarly, but employing (3.3) and the first row of (2.16) with $\underline{\boldsymbol{\tau}} = (\mathbf{0},\mathbf{v},0)$, we may deduce a bound for p independent of s_0, such that

$$\|p(t)\|^2_{\mathrm{L}^2(\Omega)} \le C_2\left\{\|\partial_t\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}(t)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{g}(t)\|^2_{\mathbf{L}^2(\Omega)}\right\}, \tag{3.29}$$

with C_2 depending on $\nu,\mathtt{D}$ and β_2. Thus, integrating (3.28) and (3.29) over $(0,t)$ with $t\in(0,T]$, combining them with (3.27), properly choosing δ_1 and δ_2, and bearing in mind that g is assumed to be time-independent, we obtain

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)}\\
&+\int_0^t\Big(\|\mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} + \|p\|^2_{\mathrm{L}^2(\Omega)}\Big)ds \quad \le C_3\left\{\int_0^t\Big(\|\mathbf{g}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}_p\|^2_{\mathbf{L}^2(\Omega)}\Big)ds + T\|g\|^2_{\mathrm{L}^2(\Omega)}\right.\\
&+\|A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\,\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{u}_s(0)\|^2_{\mathbf{L}^2(\Omega)}\\
&\left.+\int_0^t\Big(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{u}\|^2_{\mathbf{L}^2(\Omega)}\Big)ds\right\},
\end{aligned}\tag{3.30}$$

where C_3 is a positive constant depending on $\mathtt{D}$, ν, β_1, β_2, and μ. In addition, testing the second row of (2.16) with $\underline{\mathbf{v}} = (\mathbf{div}(\boldsymbol{\sigma}_p), \mathbf{0})$, we obtain

$$\|\mathbf{div}(\boldsymbol{\sigma}_p)\|_{\mathbf{L}^2(\Omega)} \leq \|\mathbf{f}_p\|_{\mathbf{L}^2(\Omega)} + \rho_p \|\partial_t \mathbf{u}_s\|_{\mathbf{L}^2(\Omega)} \,. \tag{3.31}$$

Next, in order to derive bounds for the terms involving time derivatives on the right-hand sides of (3.30) and (3.31), we differentiate in time the equations in (2.16) and test the resulting system with $(\underline{\boldsymbol{\tau}}, \underline{\mathbf{v}}) := ((\partial_t \boldsymbol{\sigma}_p, \partial_t \mathbf{u}, \partial_t p), (\partial_t \mathbf{u}_s, \partial_t \gamma))$. Then, similarly to (3.27), integrating the resulting identity over $(0, t)$ for any $t \in (0, T]$ and applying the Cauchy–Schwarz and Young inequalities, we derive

$$\begin{aligned}
&\frac{1}{2}\Big(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|\partial_t p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\partial_t \mathbf{u}_s(t)\|^2_{\mathbf{L}^2(\Omega)}\Big) \\
&\quad + \min\left\{\nu, \frac{\mathtt{D}}{2}\right\} \int_0^t \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} ds \leq \frac{1}{2\mathtt{D}} \int_0^t \|\partial_t \mathbf{g}\|^2_{\mathbf{L}^2(\Omega)} ds + \int_0^t \|\partial_t \mathbf{f}_p\|_{\mathbf{L}^2(\Omega)} \|\partial_t \mathbf{u}_s\|_{\mathbf{L}^2(\Omega)} ds \\
&\quad + \frac{1}{2}\Big(\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|\partial_t p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\partial_t \mathbf{u}_s(0)\|^2_{\mathbf{L}^2(\Omega)}\Big) \,,
\end{aligned} \tag{3.32}$$

where we notice that the term depending on g vanish since it is assumed independent of time. Next, employing Lemma 3.8 with $\chi := \sqrt{\rho_p}\|\partial_t \mathbf{u}_s\|_{\mathbf{L}^2(\Omega)}$, $B = \dfrac{1}{\sqrt{\rho_p}}\|\partial_t \mathbf{f}_p\|_{\mathbf{L}^2(\Omega)}$, and R and A representing the remaining terms, we conclude that there exists a positive constant C_4 depending on ρ_p, $\mathtt{D}$ and ν such that

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}(t)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t p(t)\|_{\mathrm{L}^2(\Omega)} + \sqrt{\rho_p}\|\partial_t \mathbf{u}_s(t)\|_{\mathbf{L}^2(\Omega)} \\
&\quad + \left(\int_0^t \|\partial_t \mathbf{u}\|^2_{\mathbf{H}^1(\Omega)} ds\right)^{1/2} \leq C_4 \Bigg\{ \left(\int_0^t \|\partial_t \mathbf{g}\|^2_{\mathbf{L}^2(\Omega)} ds\right)^{1/2} + \int_0^t \|\partial_t \mathbf{f}_p\|_{\mathbf{L}^2(\Omega)} ds \\
&\quad + \|\partial_t A^{1/2}(\boldsymbol{\sigma}_p + \alpha\, p\, \mathbb{I})(0)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}(0)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t p(0)\|_{\mathrm{L}^2(\Omega)} + \sqrt{\rho_p}\|\partial_t \mathbf{u}_s(0)\|_{\mathbf{L}^2(\Omega)} \Bigg\} \,.
\end{aligned} \tag{3.33}$$

Hence, the bound (3.26) follows from (3.28), (3.30), (3.31), and (3.33), together with the bounds for the initial conditions established in Lemma 3.7, and some algebraic manipulations. □

Remark 3.2 *We stress the importance of assuming g to be time-independent in order to derive (3.33). Otherwise, the term $\int_0^t \|\partial_t\, g\|_{\mathrm{L}^2(\Omega)} \|\partial_t\, p\|_{\mathrm{L}^2(\Omega)}\, ds$ would appear on the right-hand side of (3.32), which requires a bound similar to (3.29). However, carrying this out introduces the additional term $\|\partial_{tt}\, \mathbf{u}\|_{\mathbf{L}^2(\Omega)}$, making it impossible to close the proof unless a Grönwall inequality or adaptation of Lemma 3.8 is applied with a constant that depends on s_0.*

4 Semidiscrete continuous-in-time approximation

In this section we introduce the semidiscrete continuous-in-time approximation of (2.16). We begin by establishing its well-posedness, following the abstract framework already developed in Section 3. Once existence and uniqueness of a solution have been ensured, we turn to the error analysis, from which we derive error estimates and the corresponding convergence rates.

4.1 Existence and uniqueness of a solution

Let $\mathcal{T}_h$ be a shape-regular triangulation of Ω made up of triangles K (when $d = 2$) or tetrahedra K (when $d = 3$). For each $K \in \mathcal{T}_h$, let h_K denote its diameter, and define the mesh size as $h := \max\{h_K : K \in \mathcal{T}_h\}$. Given an integer $k \geq 0$ and an element $K \in \mathcal{T}_h$, we denote by $\mathrm{P}_k(K)$ the space of polynomials of total degree less than or equal to k on K. The corresponding vector, and tensor valued versions are defined by $\mathbf{P}_k(K) := [\mathrm{P}_k(K)]^d$ and $\mathbb{P}_k(K) := [\mathrm{P}_k(K)]^{d\times d}$, respectively. In turn, the associated global finite element spaces are given by

$$\begin{aligned}
\mathrm{P}_k(\Omega) &:= \Big\{ q_h \in \mathrm{L}^2(\Omega) : \quad q_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbf{P}_k(\Omega) &:= \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}, \\
\mathbb{P}_k(\Omega) &:= \Big\{ \boldsymbol{\zeta}_h \in \mathbb{L}^2(\Omega) : \quad \boldsymbol{\zeta}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}.
\end{aligned}$$

For the discretization, we employ the following finite element subspaces. The spaces

$$\mathbb{H}_h^{\boldsymbol{\sigma}_p} := \mathbb{P}_{k+1}(\Omega) \cap \mathbb{H}(\mathbf{div};\Omega)\,, \qquad \mathbf{Q}_h^{\mathbf{u}_s} := \mathbf{P}_k(\Omega)\,, \qquad \mathbb{Q}_h^{\gamma} := \mathbb{P}_k(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)\,, \tag{4.1}$$

are used for the poroelastic stress tensor $\boldsymbol{\sigma}_p$, the structural velocity $\mathbf{u}_s$, and the rotation rate $\boldsymbol{\gamma}$, respectively. In turn, for the fluid velocity $\mathbf{u}$ and pore pressure p, we adopt the pair:

$$\mathbf{H}_h^{\mathbf{u}} := \mathbf{C}(\bar{\Omega}) \cap \mathbf{P}_{k+2}(\Omega) \cap \mathbf{H}^1_0(\Omega)\,, \qquad \mathrm{H}_h^{p} := \mathrm{C}(\bar{\Omega}) \cap \mathrm{P}_{k+1}(\Omega) \cap \mathrm{L}^2_0(\Omega)\,. \tag{4.2}$$

Then, similarly to the continuous setting and in preparation for the forthcoming analysis, defining

$$\begin{aligned}
\underline{\boldsymbol{\sigma}}_h := (\boldsymbol{\sigma}_{ph}, \mathbf{u}_h, p_h), \quad \underline{\boldsymbol{\tau}}_h := (\boldsymbol{\tau}_h, \mathbf{v}_h, q_h) \in \mathbb{H}_h := \mathbb{H}_h^{\boldsymbol{\sigma}_p} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{p}\,, \\
\underline{\mathbf{u}}_h := (\mathbf{u}_{sh}, \boldsymbol{\gamma}_h), \quad \underline{\mathbf{v}}_h := (\mathbf{v}_{sh}, \boldsymbol{\chi}_h) \in \mathbb{Q}_h := \mathbf{Q}_h^{\mathbf{u}_s} \times \mathbb{Q}_h^{\gamma}\,,
\end{aligned}$$

the semidiscrete continuous-in-time approximation to (2.16) reads: Find $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{H}_h \times \mathbb{Q}_h$ such that, for a.e. $t \in (0,T)$

$$\begin{aligned}
\frac{\partial}{\partial t}[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}_h(t)), \underline{\boldsymbol{\tau}}_h] + [\mathcal{A}(\underline{\boldsymbol{\sigma}}_h(t)), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\underline{\mathbf{u}}_h(t)), \underline{\boldsymbol{\tau}}_h] &= [\mathbf{F}(t), \underline{\boldsymbol{\tau}}_h] \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{H}_h\,, \\
\frac{\partial}{\partial t}[\mathcal{E}_2(\underline{\mathbf{u}}_h(t)), \underline{\mathbf{v}}_h] - [\mathcal{B}(\underline{\boldsymbol{\sigma}}_h(t)), \underline{\mathbf{v}}_h] &= [\mathbf{G}(t), \underline{\mathbf{v}}_h] \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h\,.
\end{aligned} \tag{4.3}$$

We recall that the finite element subspaces defined in (4.1) correspond to the Arnold–Falk–Winther (AFW) element of order $k \geq 0$, whose stability for the Hilbertian mixed formulation of linear elasticity was proved in [5]. In particular, these spaces satisfy the following discrete inf-sup condition: there exists a constant $\beta_{1,\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\underline{\boldsymbol{\tau}}_h\in\mathbb{H}_h} \frac{[\mathcal{B}(\underline{\boldsymbol{\tau}}_h), \underline{\mathbf{v}}_h]}{\|\underline{\boldsymbol{\tau}}_h\|_{\mathbb{H}}} \geq \beta_{1,\mathtt{d}}\, \|\underline{\mathbf{v}}_h\|_{\mathbb{Q}} \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h\,. \tag{4.4}$$

In turn, the finite element spaces defined in (4.2) correspond to the classical Taylor–Hood elements [41]. It is well known that this pair satisfies the discrete inf-sup condition (see, e.g., [10]): there exists a constant $\beta_{2,\mathtt{d}} > 0$, independent of h, such that

$$\sup_{\mathbf{0}\neq\mathbf{v}_h\in\mathbf{H}_h^{\mathbf{u}}} \frac{(q_h, \mathrm{div}(\mathbf{v}_h))_\Omega}{\|\mathbf{v}_h\|_{\mathbf{H}^1(\Omega)}} \geq \beta_{2,\mathtt{d}}\, \|q_h\|_{\mathrm{L}^2(\Omega)} \qquad \forall\, q_h \in \mathrm{H}_h^{p}\,. \tag{4.5}$$

Remark 4.1 *Besides (4.1), other elasticity-stable mixed finite element families satisfying the discrete inf-sup condition (4.4) may also be employed. For instance, the PEERS [4], Amara–Thomas [3], Stenberg [40], or Cockburn–Gopalakrishnan–Guzmán [22] families of mixed finite element spaces. On the other hand, instead of the Taylor–Hood pair (4.2), any other inf-sup stable Stokes finite element pair could be employed. To mention, we may consider the MINI element [11, Sections 8.4.2, 8.6, and 8.7] or the Crouzeix–Raviart element with tangential jump penalization (see [26] for the lowest-order discrete inf-sup condition and, e.g., [17] for the cubic-order case).*

Now, to construct a suitable approximation of the initial conditions $(\underline{\boldsymbol{\sigma}}_0, \underline{\mathbf{u}}_0)$ established in Lemma 3.5, we consider a slight modification of (3.16), that is, we chose $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\mathbf{u}}_h^0) = ((\boldsymbol{\sigma}_{ph}^0, \mathbf{u}_h^0, p_h^0), (\mathbf{u}_{sh}^0, \gamma_h^0)) \in \mathbb{H}_h \times \mathbb{Q}_h$ which solves

$$\begin{aligned} [\widetilde{\mathcal{A}}(\underline{\boldsymbol{\sigma}}_h^0), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\underline{\mathbf{u}}_h^0), \underline{\boldsymbol{\tau}}_h] &= [\mathbf{F}_0, \underline{\boldsymbol{\tau}}_h] + (\boldsymbol{\sigma}_{p,0}, \boldsymbol{\tau}_h)_\Omega + (p_0, q_h)_\Omega \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{H}_h \\ -[\mathcal{B}(\underline{\boldsymbol{\sigma}}_h^0), \underline{\mathbf{v}}_h] &= [\mathbf{G}_0, \underline{\mathbf{v}}_h] \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h\,, \end{aligned} \tag{4.6}$$

where $\mathbf{F}_0$ and $\mathbf{G}_0$ are given by (3.17), and the operator $\widetilde{\mathcal{A}} : \mathbb{H} \to \mathbb{H}'$ is defined as

$$[\widetilde{\mathcal{A}}(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] := (\boldsymbol{\sigma}_p, \boldsymbol{\tau})_\Omega + (p, q)_\Omega + [\mathcal{A}(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}] \quad \forall\, \underline{\boldsymbol{\sigma}}, \underline{\boldsymbol{\tau}} \in \mathbb{H}\,.$$

We first notice that the discrete kernel associated to $\mathcal{B}$ is given by $\mathbf{V}_h := \mathbf{K}_h \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^p$, where

$$\mathbf{K}_h := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}_p} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = \mathbf{0} \quad \text{and} \quad (\boldsymbol{\tau}_h, \boldsymbol{\chi}_h)_\Omega = 0 \quad \forall\, \boldsymbol{\chi}_h \in \mathbb{Q}_h^\gamma \Big\}\,. \tag{4.7}$$

Then, $\widetilde{A}$ is clearly linear, bounded, moreover, by (3.1), it is $\mathbb{H}$-coercive on $\mathbf{V}_h$, and since $\mathcal{B}$ satisfies the discrete inf-sup condition (4.4), a direct application of the Babuška–Brezzi theorem (cf. [29, Theorem 2.3]) implies the well-posedness of (4.6). Moreover, thanks to the estimates obtained for the functionals $\mathbf{F}_0$ and $\mathbf{G}_0$ (cf. (3.18) and (3.19), respectively) and the definition of $\boldsymbol{\sigma}_{p,0}$ (cf. (3.13)), we conclude that there exists a positive constant $\widetilde{\mathcal{C}}_{0,\mathtt{d}}$, depending only on ν, $\mathtt{D}$, and $\beta_{1,\mathtt{d}}$, such that

$$\begin{aligned} &\|\underline{\boldsymbol{\sigma}}_h^0\|_{\mathbb{H}}^2 + \|\underline{\mathbf{u}}_h^0\|_{\mathbb{Q}}^2 = \|\boldsymbol{\sigma}_{ph}^0\|_{\mathbb{H}(\mathbf{div};\Omega)}^2 + \|\mathbf{u}_h^0\|_{\mathbf{H}^1(\Omega)}^2 + \|p_h^0\|_{\mathrm{L}^2(\Omega)}^2 + \|\mathbf{u}_{sh}^0\|_{\mathbf{L}^2(\Omega)}^2 + \|\gamma_h^0\|_{\mathbb{L}^2(\Omega)}^2 \\ &\leq \widetilde{\mathcal{C}}_{0,\mathtt{d}} \Big(\|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}_0\|_{\mathbf{H}(\mathrm{div};\Omega)}^2 + \|p_0\|_{\mathrm{H}^1(\Omega)}^2 + \|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{H}(\mathbf{div};\Omega)}^2 \Big). \end{aligned} \tag{4.8}$$

Thus, from (4.6) we deduce a initial condition for (4.3), that is $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\mathbf{u}}_h^0)$, solution of

$$\begin{aligned} [\mathcal{A}(\underline{\boldsymbol{\sigma}}_h^0), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\underline{\mathbf{u}}_h^0), \underline{\boldsymbol{\tau}}_h] &= [\mathbf{F}_{h,0}, \underline{\boldsymbol{\tau}}_h]\,, \\ -[\mathcal{B}(\underline{\boldsymbol{\sigma}}_h^0), \underline{\mathbf{v}}_h] &= [\mathbf{G}_0, \underline{\mathbf{v}}_h]\,, \end{aligned} \tag{4.9}$$

with $[\mathbf{F}_{h,0}, \underline{\boldsymbol{\tau}}_h] := [\mathbf{F}_0, \underline{\boldsymbol{\tau}}_h] + (\boldsymbol{\sigma}_{p,0} - \boldsymbol{\sigma}_{ph}^0, \boldsymbol{\tau}_h)_\Omega + (p_0 - p_h^0, q_h)_\Omega$, which, thanks to (3.18) and (4.8), yields

$$\begin{aligned} |[\mathbf{F}_{h,0}, \underline{\boldsymbol{\tau}}_h]| \leq C_{0,\mathtt{d}} \Big(&\|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)}^2 + \|\mathbf{u}_0\|_{\mathbf{H}(\mathrm{div};\Omega)}^2 + \|p_0\|_{\mathrm{H}^1(\Omega)}^2 \\ &+ \|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{H}(\mathbf{div};\Omega)}^2 \Big)^{1/2} \|(\boldsymbol{\tau}_h, \mathbf{v}_h, q_h)\|_{\mathbb{L}^2(\Omega)\times\mathbf{L}^2(\Omega)\times\mathrm{L}^2(\Omega)}\,, \end{aligned} \tag{4.10}$$

with $C_{0,\mathtt{d}} > 0$ depending only on ν, $\mathtt{D}$, and $\beta_{1,\mathtt{d}}$. Thus $\mathbf{F}_{h,0} \in \mathbb{H}_2'$. We note that this choice is necessary to guarantee that the discrete initial data is compatible in the sense of Lemma 3.5, which is needed for the application of Theorem 3.3.

In this way, the well-posedness of (4.3), follows analogously to its continuous counterpart provided in Theorem 3.6. More precisely, to verify the corresponding range condition, we recall the definition of $\mathbf{V}_h$ (cf. (4.7)), and observe that $\mathcal{E}_1 + \mathcal{A}$, $\mathcal{B}$ and $\mathcal{E}_2$ satisfy the required hypothesis of [11, Theorem 4.3.1] in the discrete setting. Hence, the discrete counterpart of Lemma 3.4 follows immediately.

Now, we can establish the following well-posedness result.

Theorem 4.1 *For each compatible initial data $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\mathbf{u}}_h^0)$ satisfying (4.9) and*

$$\mathbf{g} \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega)), \quad g \in \mathrm{W}^{1,1}(0,T;\mathrm{L}^2_0(\Omega)), \quad \text{and} \quad \mathbf{f}_p \in \mathrm{W}^{1,1}(0,T;\mathbf{L}^2(\Omega)),$$

there exists a unique solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{H}_h \times \mathbb{Q}_h$ to (4.3), such that

$$\underline{\boldsymbol{\sigma}}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbb{H}_h), \quad \mathbf{u}_{sh} \in \mathrm{W}^{1,\infty}(0,T;\mathbf{Q}_h^{\mathbf{u}_s}), \quad \text{and} \quad (\underline{\boldsymbol{\sigma}}_h(0), \mathbf{u}_{sh}(0)) = (\underline{\boldsymbol{\sigma}}_h^0, \mathbf{u}_{sh}^0).$$

Moreover, assuming that the source term g is independent of time, there exists a positive constant $\mathcal{C}_{\mathtt{st,d}}$, independent of h and s_0, such that

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha\, p_h\, \mathbb{I})\|_{\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_h\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\mathbf{u}_h\|_{\mathrm{H}^1(0,T;\mathbf{H}^1(\Omega))} + \sqrt{s_0}\|p_h\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega))} + \|p_h\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{u}_{sh}\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\boldsymbol{\gamma}_h\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \leq \mathcal{C}_{\mathtt{st,d}} \Bigg\{ \|\mathbf{g}\|_{\mathrm{H}^1(0,T;\mathbf{L}^2(\Omega))} + \left(1 + \frac{1}{\sqrt{s_0}}\right) \|g\|_{\mathrm{L}^2(\Omega)} \\
&\quad + \|\mathbf{f}_p\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}_p\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))} + \|d_t \mathbf{f}_p\|_{\mathrm{L}^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{g}(0)\|_{\mathbf{L}^2(\Omega)} \\
&\quad + \left(1 + \sqrt{s_0} + \frac{1}{\sqrt{s_0}}\right) \Big(\|\Delta \mathbf{u}_0\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|_{\mathbf{H}(\mathrm{div};\Omega)} + \|p_0\|_{\mathrm{H}^1(\Omega)} + \|\mathbf{e}(\mathbf{u}_{s,0})\|_{\mathbb{H}(\mathbf{div};\Omega)} \Big) \Bigg\}.
\end{aligned} \tag{4.11}$$

Proof. Since $\mathbb{H}_h \subset \mathbb{H}$ and $\mathbb{Q}_h \subset \mathbb{Q}$, considering $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\mathbf{u}}_h^0)$ satisfying (4.9), and employing the continuity and monotonicity properties of the operators $\mathcal{N}$ and $\mathcal{M}$, together with the discrete inf-sup conditions (4.4) and (4.5), the existence and uniqueness of the solution $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{H}_h \times \mathbb{Q}_h$ to (4.3) follow along the same lines as in the proof of Theorem 3.6. Moreover, the discrete version of the stability bound (3.26) is derived by following the proof of Theorem 3.9, but with the corresponding discrete initial data on the right-hand side, bounding

$$\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha\, p_h\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|p_h(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\mathbf{u}_{sh}(0)\|_{\mathbf{L}^2(\Omega)} \tag{4.12}$$

and

$$\|\partial_t A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha\, p_h\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|\partial_t p_h(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\partial_t \mathbf{u}_{sh}(0)\|^2_{\mathbf{L}^2(\Omega)}, \tag{4.13}$$

in terms of the corresponding continuous initial data. In particular, the bound for (4.12) follows directly from (4.8). On the other hand, to bound (4.13), we proceed as follows: we test the first row of (4.3) with $\partial_t \underline{\boldsymbol{\sigma}}_h^0 := \big(\partial_t \boldsymbol{\sigma}_{ph}(0), \partial_t \mathbf{u}_h(0), \partial_t p_h(0)\big)$, differentiate the second row of (4.3) with respect to time and test it with $\underline{\mathbf{v}}_h = (\mathbf{0}, \boldsymbol{\gamma}_h(0))$, and finally evaluate at $t = 0$ to obtain

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha\, p_h\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|\partial_t p_h(0)\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad = -\big[\mathcal{A}(\underline{\boldsymbol{\sigma}}_h(0)), \partial_t \underline{\boldsymbol{\sigma}}_h^0\big] - (\mathbf{div}(\partial_t \boldsymbol{\sigma}_{ph})(0), \mathbf{u}_{sh}(0))_\Omega + [\mathbf{F}(0), \partial_t \underline{\boldsymbol{\sigma}}_h^0].
\end{aligned} \tag{4.14}$$

From the first row of (4.9) with $\underline{\boldsymbol{\tau}}_h = \partial_t\, \underline{\boldsymbol{\sigma}}_h^0$, and using the estimates (4.8) and (4.10), we have that

$$\begin{aligned}
&\big[\mathcal{A}(\underline{\boldsymbol{\sigma}}_h(0)), \partial_t \underline{\boldsymbol{\sigma}}_h^0\big] + (\mathbf{div}(\partial_t \boldsymbol{\sigma}_{ph})(0), \mathbf{u}_{sh}(0))_\Omega = [\mathbf{F}_{h,0}, \partial_t \underline{\boldsymbol{\sigma}}_h^0] - (\boldsymbol{\gamma}_h^0, \partial_t \boldsymbol{\sigma}_{ph}(0))_\Omega \\
&\quad \leq C_1 \Big(\|\Delta \mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}(\mathrm{div};\Omega)} + \|p_0\|^2_{\mathrm{H}^1(\Omega)} + \|\mathbf{e}(\mathbf{u}_{s,0})\|^2_{\mathbb{H}(\mathbf{div};\Omega)} \Big)^{1/2} \|\partial_t \underline{\boldsymbol{\sigma}}_h^0\|_{\mathbb{H}}\,,
\end{aligned} \tag{4.15}$$

with C_1 a positive constant depending on ν, $\mathtt{D}$, and $\beta_{1,\mathtt{d}}$. Then, using (4.15) the discrete version of (3.25), bounding $[\mathbf{F}(0), \partial_t \underline{\boldsymbol{\sigma}}_h^0]$ by $\|\mathbf{g}(0)\|_{\mathbf{L}^2(\Omega)}\, \|\partial_t \mathbf{u}_h(0)\|_{\mathbf{L}^2(\Omega)} + \dfrac{1}{s_0} \|g\|_{\mathrm{L}^2(\Omega)}\, s_0\, \|\partial_t p(0)\|_{\mathrm{L}^2(\Omega)}$, and applying the Cauchy–Schwarz and Young inequalities, we may conclude that (4.14) yields

$$\begin{aligned}
&\|\partial_t A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha\, p_h\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{u}_h(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|\partial_t p_h(0)\|^2_{\mathrm{L}^2(\Omega)} \\
&\quad \leq C_2 \bigg\{ \|\mathbf{g}(0)\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{s_0} \|g\|^2_{\mathrm{L}^2(\Omega)} + \bigg(1 + \frac{1}{s_0}\bigg) \Big(\|\Delta \mathbf{u}_0\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{u}_0\|^2_{\mathbf{H}(\mathrm{div};\Omega)} \\
&\qquad + \|p_0\|^2_{\mathrm{H}^1(\Omega)} + \|\mathbf{e}(\mathbf{u}_{s,0})\|^2_{\mathbb{H}(\mathbf{div};\Omega)} \Big) \bigg\}\,,
\end{aligned} \tag{4.16}$$

with C_2 a positive constant depending on ν, $\mathtt{D}$, and $\beta_{1,\mathtt{d}}$. In turn, testing the second row of (4.3) against $\underline{\mathbf{v}}_h = \partial_t \underline{\mathbf{u}}_h^0 := \big(\partial_t \mathbf{u}_{sh}(0), \mathbf{0}\big)$, and taking $t = 0$, we obtain

$$\rho_p \|\partial_t \mathbf{u}_{sh}(0)\|^2_{\mathbf{L}^2(\Omega)} = (\mathbf{div}(\boldsymbol{\sigma}_{ph})(0), \partial_t \mathbf{u}_s(0))_\Omega + (\mathbf{f}_p(0), \partial_t \mathbf{u}_{sh}(0))_\Omega\,. \tag{4.17}$$

Thus, testing the second row of (4.9) against $\underline{\mathbf{v}}_h = \partial_t \underline{\mathbf{u}}_h^0 := \big(\partial_t \mathbf{u}_{sh}(0), \mathbf{0}\big)$, and bearing in mind the first identity of (3.14), with $\boldsymbol{\sigma}_{ph}(0) = \boldsymbol{\sigma}_{ph}^0$, we deduce that

$$(\mathbf{div}(\boldsymbol{\sigma}_{ph})(0), \partial_t \mathbf{u}_s(0))_\Omega = (\mathbf{div}(\boldsymbol{\sigma}_{ph}^0), \partial_t \mathbf{u}_s(0))_\Omega = (\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0})), \partial_t \mathbf{u}_{sh}(0))_\Omega\,,$$

which in combination with (4.17), and the Cauchy–Schwarz and Young inequalities, yields

$$\rho_p \|\partial_t \mathbf{u}_{sh}(0)\|^2_{\mathbf{L}^2(\Omega)} \leq C_3 \Big(\|\mathbf{f}_p(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{div}(\mathbf{e}(\mathbf{u}_{s,0}))\|^2_{\mathbf{L}^2(\Omega)} \Big)\,, \tag{4.18}$$

with C_3 a positive constant depending on ρ_p. Hence, joining (4.16) and (4.18), completes the proof of the discrete stability bound (4.11). □

4.2 Error analysis

We now proceed to establish convergence rates for the semidiscrete scheme. To this end, for the velocity and pressure variables, we consider the orthogonal projection operators (cf. [28, Section 1.6.3]):

$$\boldsymbol{\Pi}_{c,h}^{1,k} : \mathbf{H}_0^1(\Omega) \to \mathbf{H}_h^{\mathbf{u}}\,, \quad \text{and} \quad \Pi_{c,h}^{0,k} : \mathrm{L}_0^2(\Omega) \to \mathrm{H}_h^p\,,$$

defined with respect to the natural scalar products. In particular, for each $q \in \mathrm{L}_0^2(\Omega)$, the operator satisfies

$$(q - \Pi_{c,h}^{0,k}(q), q_h)_\Omega = 0 \quad \forall\, q_h \in \mathrm{H}_h^p\,. \tag{4.19}$$

For the poroelastic stress tensor, we consider the Brezzi–Douglas–Marini projector $\Pi_h^{\mathrm{BDM}} : \mathbb{H}(\mathbf{div};\Omega) \cap \mathbb{H}^1(\Omega) \to \mathbb{H}_h^{\boldsymbol{\sigma}_p}$ (cf. [12]), which, for each $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div};\Omega) \cap \mathbb{H}^1(\Omega)$, satisfies

$$(\mathbf{div}(\Pi_h^{\mathrm{BDM}}(\boldsymbol{\sigma})), \mathbf{v}_{sh})_\Omega = (\mathbf{div}(\boldsymbol{\sigma}), \mathbf{v}_{sh})_\Omega \quad \forall\, \mathbf{v}_{sh} \in \mathbf{Q}_h^{\mathbf{u}_s}\,. \tag{4.20}$$

Finally, for the structural velocity and the rate of rotation, we introduce the L^2-orthogonal projection operators

$$\mathbf{P}_{d,h}^{0,k} : \mathbf{L}^2(\Omega) \to \mathbf{Q}_h^{\mathbf{u}_s}, \qquad \mathbb{P}_{d,h}^{0,k} : \mathbb{L}^2_{\mathtt{skew}}(\Omega) \to \mathbf{Q}_h^{\gamma},$$

where, for each $\mathbf{v}_s \in \mathbf{L}^2(\Omega)$, the operator $\mathbf{P}_{d,h}^{0,k}$ satisfy

$$(\mathbf{v}_s - \mathbf{P}_{d,h}^{0,k}(\mathbf{v}_s), \mathbf{v}_{sh})_\Omega = 0 \qquad \forall\, \mathbf{v}_{sh} \in \mathbf{Q}_h^{\mathbf{u}_s}. \tag{4.21}$$

We conclude this discussion by recalling the standard approximation properties of these operators, which are established in [28, Proposition 1.134] and [12, Section III.3.3]:

$(\mathbf{AP}_h^{\mathbf{u}})$ There exists a positive constant C, independent of h, such that for each $s \in [1, k+1]$, and for each $\mathbf{v} \in \mathbf{H}^{s+2}(\Omega)$, there holds

$$\|\mathbf{v} - \boldsymbol{\Pi}_{c,h}^{1,k}(\mathbf{v})\|_{\mathbf{H}^1(\Omega)} \le C\, h^{s+1} |\mathbf{v}|_{\mathbf{H}^{s+2}(\Omega)}.$$

$(\mathbf{AP}_h^{p})$ There exists a positive constant C, independent of h, such that for each $s \in [1, k+1]$ and for each $q \in \mathrm{H}^{s+1}(\Omega)$, there holds

$$\|q - \Pi_{c,h}^{0,k}(q)\|_{\mathrm{L}^2(\Omega)} \le c\, h^{s+1} |q|_{\mathrm{H}^{s+1}(\Omega)}.$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}_p})$ There exists a positive constant C, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{H}^l(\Omega)$, there holds

$$\|\boldsymbol{\tau} - \Pi_h^{\mathrm{BDM}}(\boldsymbol{\tau})\|_{\mathbb{H}(\mathbf{div};\Omega)} \;\le\; C\, h^l \Big(\|\boldsymbol{\tau}\|_{\mathbb{H}^l(\Omega)} + \|\mathbf{div}(\boldsymbol{\tau})\|_{\mathbf{H}^l(\Omega)} \Big).$$

$(\mathbf{AP}_h^{\mathbf{u}_s})$ There exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v}_s \in \mathbf{H}^l(\Omega)$, there holds

$$\|\mathbf{v}_s - \mathbf{P}_{d,h}^{0,k}(\mathbf{v}_s)\|_{\mathbf{L}^2(\Omega)} \le C h^l \|\mathbf{v}_s\|_{\mathbf{H}^l(\Omega)}.$$

$(\mathbf{AP}_h^{\gamma})$ There exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{\eta} \in \mathbb{H}^l(\Omega)$, there holds

$$\|\boldsymbol{\chi} - \mathbb{P}_{d,h}^{0,k}(\boldsymbol{\chi})\|_{\mathbb{L}^2(\Omega)} \le C h^l \|\boldsymbol{\chi}\|_{\mathbb{H}^l(\Omega)}.$$

Now, let $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) = ((\boldsymbol{\sigma}_p, \mathbf{u}, p), (\mathbf{u}_s, \boldsymbol{\gamma}))$ and $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) = ((\boldsymbol{\sigma}_{ph}, \mathbf{u}_h, p_h), (\mathbf{u}_{sh}, \boldsymbol{\gamma}_h))$ denote the solutions of (2.16) and (4.3), respectively. We define the error components as

$$\mathbf{e}_{\underline{\boldsymbol{\sigma}}} := (\mathbf{e}_{\boldsymbol{\sigma}_p}, \mathbf{e}_{\mathbf{u}}, \mathbf{e}_p) = (\boldsymbol{\sigma}_p - \boldsymbol{\sigma}_{ph}, \mathbf{u} - \mathbf{u}_h, p - p_h) \quad \text{and} \quad \mathbf{e}_{\underline{\mathbf{u}}} := (\mathbf{e}_{\mathbf{u}_s}, \mathbf{e}_{\boldsymbol{\gamma}}) = (\mathbf{u}_s - \mathbf{u}_{sh}, \boldsymbol{\gamma} - \boldsymbol{\gamma}_h).$$

In turn, following a standard approach in error analysis, we decompose the errors into the following components:

$$\mathbf{e}_{\boldsymbol{\sigma}_p} = \mathbf{e}_{\boldsymbol{\sigma}_p}^I + \mathbf{e}_{\boldsymbol{\sigma}_p}^h := (\boldsymbol{\sigma}_p - \Pi_h^{\mathrm{BDM}}(\boldsymbol{\sigma}_p)) + (\Pi_h^{\mathrm{BDM}}(\boldsymbol{\sigma}_p) - \boldsymbol{\sigma}_{ph}),$$

$$\mathbf{e}_{\mathbf{u}} = \mathbf{e}_{\mathbf{u}}^I + \mathbf{e}_{\mathbf{u}}^h := (\mathbf{u} - \boldsymbol{\Pi}_{c,h}^{1,k}(\mathbf{u})) + (\boldsymbol{\Pi}_{c,h}^{1,k}(\mathbf{u}) - \mathbf{u}_h),$$

$$\mathbf{e}_p = \mathbf{e}_p^I + \mathbf{e}_p^h := (p - \Pi_{c,h}^{0,k}(p)) + (\Pi_{c,h}^{0,k}(p) - p_h),$$

$$\mathbf{e}_{\mathbf{u}_s} = \mathbf{e}_{\mathbf{u}_s}^I + \mathbf{e}_{\mathbf{u}_s}^h := (\mathbf{u}_s - \mathbf{P}_{d,h}^{0,k}(\mathbf{u}_s)) + (\mathbf{P}_{d,h}^{0,k}(\mathbf{u}_s) - \mathbf{u}_{sh}),$$

$$\text{and} \quad \mathbf{e}_{\boldsymbol{\gamma}} = \mathbf{e}_{\boldsymbol{\gamma}}^I + \mathbf{e}_{\boldsymbol{\gamma}}^h := (\boldsymbol{\gamma} - \mathbb{P}_{d,h}^{0,k}(\boldsymbol{\gamma})) + (\mathbb{P}_{d,h}^{0,k}(\boldsymbol{\gamma}) - \boldsymbol{\gamma}_h).$$

Thus, in order to derive the error equations, we subtract the discrete variational formulation (4.3) from the continuous one (2.16). This yields the following error system:

$$\begin{aligned} \frac{\partial}{\partial t}[\mathcal{E}_1(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}),\underline{\boldsymbol{\tau}}_h] + [\mathcal{A}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}),\underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\mathbf{e}_{\underline{\mathbf{u}}}),\underline{\boldsymbol{\tau}}_h] &= 0 \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{H}_h\,, \\ \frac{\partial}{\partial t}[\mathcal{E}_2(\mathbf{e}_{\underline{\mathbf{u}}}),\underline{\mathbf{v}}_h] - [\mathcal{B}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}),\underline{\mathbf{v}}_h] &= 0 \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h\,. \end{aligned} \tag{4.22}$$

Prior to establishing the main result of this section, we first derive a bound for the initial errors.

Lemma 4.2 *Under the assumptions of Theorems 3.6 and 4.1, there exists a positive constant $\mathcal{C}_{\mathsf{e},0}$, independent of s_0, such that*

$$\begin{aligned} &\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\gamma}}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha \mathbf{e}^h_p \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t \mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} \\ &\quad + \rho_p\|\partial_t \mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} \le \mathcal{C}_{\mathsf{e},0}\bigg\{ \|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha \mathbf{e}^I_p \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \|\partial_t \mathbf{e}^I_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t \mathbf{e}^I_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \left(1 + \frac{1}{s_0}\right)\Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \|\mathbf{e}^I_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^I_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)} \Big) \bigg\}\,. \end{aligned} \tag{4.23}$$

Proof. We begin recalling from Theorems 3.6 and 4.1 that

$$((\boldsymbol{\sigma}_p(0), \mathbf{u}(0), p(0)), \mathbf{u}_s(0)) = ((\boldsymbol{\sigma}_{p,0}, \mathbf{u}_0, p_0), \mathbf{u}_{s,0})$$

and

$$((\boldsymbol{\sigma}_{ph}(0), \mathbf{u}_h(0), p_h(0)), \mathbf{u}_{sh}(0))) = ((\boldsymbol{\sigma}^0_{ph}, \mathbf{u}^0_h, p^0_h), \mathbf{u}^0_{sh})\,.$$

In addition, given the initial data $\boldsymbol{\gamma}_0$ and $\boldsymbol{\gamma}^0_h$ established in Lemma 3.5 and (4.9), respectively. We define its corresponding error

$$\mathbf{e}_{\boldsymbol{\gamma}_0} = \boldsymbol{\gamma}_0 - \boldsymbol{\gamma}^0_h = \mathbf{e}^I_{\boldsymbol{\gamma}_0} + \mathbf{e}^h_{\boldsymbol{\gamma}_0} := (\boldsymbol{\gamma}_0 - \mathbb{P}^{0,k}_{d,h}(\boldsymbol{\gamma}_0)) + (\mathbb{P}^{0,k}_{d,h}(\boldsymbol{\gamma}_0) - \boldsymbol{\gamma}^0_h)\,.$$

Then, by subtracting the continuous and discrete initial condition problems (3.16) and (4.6), we get the following error system for the initial conditions

$$\begin{aligned} (\mathbf{e}_{\boldsymbol{\sigma}_p}(0), \boldsymbol{\tau}_h)_\Omega + (\mathbf{e}_p(0), q_h)_\Omega + [\mathcal{A}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}(0)), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\mathbf{e}_{\mathbf{u}_s}(0), \mathbf{e}_{\boldsymbol{\gamma}_0}), \underline{\boldsymbol{\tau}}_h] &= 0\,, \\ -[\mathcal{B}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}(0)), \underline{\mathbf{v}}_h] &= 0\,, \end{aligned} \tag{4.24}$$

for all $(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$. Thus, testing (4.24) with $\underline{\boldsymbol{\tau}}_h = (\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0), \mathbf{e}^h_{\mathbf{u}}(0), \mathbf{e}^h_p(0))$ and $\underline{\mathbf{v}}_h = (\mathbf{e}^h_{\mathbf{u}_s}(0), \mathbf{e}^h_{\boldsymbol{\gamma}_0})$, and taking into account the projection properties (4.19)–(4.21), we apply the Cauchy–Schwarz and Young inequalities with suitable weights to obtain

$$\begin{aligned} &\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} \le C_1\Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad + \|\mathbf{e}^I_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^I_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}_0}\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{L}^2(\Omega)} \|\mathbf{e}^h_{\boldsymbol{\gamma}_0}\|_{\mathbb{L}^2(\Omega)} \Big)\,, \end{aligned} \tag{4.25}$$

where C_1 is a positive constant depending on ν and $\mathtt{D}$.

Now, in order to control the last term of (4.25), we invoke the discrete inf-sup condition (4.4) and use the first equation in (4.24) to deduce that

$$\begin{aligned}\Big(\|\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\gamma_0}\|^2_{\mathbb{L}^2(\Omega)}\Big)^{1/2} &\leq C_2\Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\mathbf{u}}(0)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^I_p(0)\|_{\mathrm{L}^2(\Omega)} \\ &+\|\mathbf{e}^I_{\mathbf{u}_s}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\gamma_0}\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(0)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^h_p(0)\|_{\mathrm{L}^2(\Omega)}\Big),\end{aligned} \tag{4.26}$$

where $C_2 > 0$ is a constant depending principally on ν, $\mathtt{D}$ and the discrete inf-sup constant $\beta_{1,\mathtt{d}}$. In consequence, combining (4.25) and (4.26), and employing the Young inequality, we obtain

$$\begin{aligned}&\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\gamma_0}\|^2_{\mathbb{L}^2(\Omega)} \\ &\quad\leq C_3\Big(\|\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\mathbf{u}}(0)\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^I_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\gamma_0}\|^2_{\mathbb{L}^2(\Omega)}\Big),\end{aligned} \tag{4.27}$$

where $C_3 > 0$ is a constant depending principally on ν, $\mathtt{D}$ and $\beta_{1,\mathtt{d}}$.

In turn, to the end to bound $\mathbf{e}^h_{\boldsymbol{\gamma}}(0)$, we employ again the discrete inf-sup condition (4.4), but now, the first equation in (4.22) at $t = 0$, obtaining

$$\begin{aligned}\Big(\|\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\gamma}}(0)\|^2_{\mathbb{L}^2(\Omega)}\Big)^{1/2} &\leq C_4\Big(\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}_p\mathbb{I})(0)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}_{\mathbf{u}}(0)\|_{\mathbf{L}^2(\Omega)} \\ &+ s_0\|\partial_t \mathbf{e}_p(0)\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}(0)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}_p(0)\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\mathbf{u}_s}(0)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}(0)\|_{\mathbb{L}^2(\Omega)}\Big),\end{aligned} \tag{4.28}$$

where $C_4 > 0$ is a constant depending principally on ν, $\mathtt{D}$ and $\beta_{1,\mathtt{d}}$.

Now, in order to bound the terms involving time derivatives in (4.28), we first evaluate the error equation (4.22) at $t = 0$ and test it with

$$(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h) = ((\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0), \partial_t \mathbf{e}^h_{\mathbf{u}}(0), \partial_t \mathbf{e}^h_p(0)), (\partial_t \mathbf{e}^h_{\mathbf{u}_s}(0), \mathbf{0})).$$

We then differentiate in time the second row of (4.22), evaluate the resulting identity at $t = 0$, and test it with $\underline{\mathbf{v}}_h = (\mathbf{0}, \mathbf{e}^h_{\boldsymbol{\gamma}}(0))$. As a result, we obtain

$$\begin{aligned}&\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^h_p\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + s_0\|\partial_t \mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{e}_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad= -(\partial_t A(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^I_p\mathbb{I})(0), \partial_t(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^h_p\mathbb{I})(0))_\Omega - (\partial_t \mathbf{e}^I_{\mathbf{u}}(0), \partial_t \mathbf{e}^h_{\mathbf{u}}(0))_\Omega - s_0(\partial_t \mathbf{e}^I_p(0), \partial_t \mathbf{e}^h_p(0))_\Omega \\ &\quad- \rho_p(\partial_t \mathbf{e}^I_{\mathbf{u}_s}(0), \partial_t \mathbf{e}^h_{\mathbf{u}_s}(0))_\Omega - (\mathbf{e}^I_{\boldsymbol{\gamma}}(0), \partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0))_\Omega + (\partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p}(0), \mathbf{e}^h_{\boldsymbol{\gamma}}(0))_\Omega \\ &\quad- [\mathcal{A}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}(0)), \partial_t \mathbf{e}^h_{\underline{\boldsymbol{\sigma}}}(0)] - (\mathbf{e}_{\mathbf{u}_s}(0), \mathbf{div}(\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p})(0))_\Omega + (\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})(0), \partial_t \mathbf{e}^h_{\mathbf{u}_s}(0))_\Omega.\end{aligned} \tag{4.29}$$

In turn, in order to bound the last three terms in (4.29), we test (4.24) with $(\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0), \partial_t \mathbf{e}^h_{\mathbf{u}}(0), \partial_t \mathbf{e}^h_p(0))$ and $(\partial_t \mathbf{e}^h_{\mathbf{u}_s}(0), \mathbf{0})$, respectively, and obtain that

$$\begin{aligned}&[\mathcal{A}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}(0)), \partial_t \mathbf{e}^h_{\underline{\boldsymbol{\sigma}}}(0)] + (\mathbf{e}_{\mathbf{u}_s}(0), \mathbf{div}(\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p})(0))_\Omega - (\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})(0), \partial_t \mathbf{e}^h_{\mathbf{u}_s}(0))_\Omega \\ &\quad= -(\mathbf{e}_{\gamma_0}, \partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0))_\Omega - (\mathbf{e}_{\boldsymbol{\sigma}_p}(0), \partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0))_\Omega - (\mathbf{e}_p(0), \partial_t \mathbf{e}^h_p(0))_\Omega.\end{aligned} \tag{4.30}$$

Moreover, analogous to (3.25), there exists a positive constant C_4, depending only on μ, λ, d and α, such that

$$\|\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} \leq C_4\left(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)}\right). \tag{4.31}$$

Thus, combining (4.27)–(4.31), together with the Cauchy–Schwarz and Young inequalities, we conclude (4.23). □

We are now in a position to state the main result of this section.

Theorem 4.3 *Let* $(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}) : [0,T] \to \mathbb{H} \times \mathbb{Q}$, *with* $\underline{\boldsymbol{\sigma}} \in \mathrm{W}^{1,\infty}(0,T;\mathbb{H}_2')$ *and* $\mathbf{u}_s \in \mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))$ *and* $(\underline{\boldsymbol{\sigma}}_h, \underline{\mathbf{u}}_h) : [0,T] \to \mathbb{H}_h \times \mathbb{Q}_h$ *with* $\underline{\boldsymbol{\sigma}}_h \in \mathrm{W}^{1,\infty}(0,T;\mathbb{H}_h')$ *and* $\mathbf{u}_s \in \mathrm{W}^{1,\infty}(0,T;\mathbf{Q}_h^{\mathbf{u}_s})$ *be the unique solutions of the continuous and semidiscrete problems* (2.16) *and* (4.3), *respectively. Assume further that there exists* $s, l \in [1, k+1]$, *such that* $\boldsymbol{\sigma}_p \in \mathbb{H}^l(\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}_p) \in \mathbf{H}^l(\Omega)$, $\mathbf{u} \in \mathbf{H}^{s+2}(\Omega)$, $p \in \mathrm{H}^{s+1}(\Omega)$, $\mathbf{u}_s \in \mathbf{H}^l(\Omega)$, *and* $\boldsymbol{\eta} \in \mathbb{H}^l(\Omega)$. *Then, there exists a positive constant* $\mathcal{C}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$, *independent of* h, *such that the following estimate holds:*

$$
\begin{aligned}
&\|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}_p \mathbb{I})\|_{\mathrm{W}^{1,\infty}(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\mathrm{L}^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\mathbf{e}_{\mathbf{u}}\|_{\mathrm{H}^1(0,T;\mathbf{H}^1(\Omega))} + \sqrt{s_0}\|\mathbf{e}_p\|_{\mathrm{W}^{1,\infty}(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{e}_p\|_{\mathrm{L}^2(0,T;\mathrm{L}^2(\Omega))} \\
&\quad + \|\mathbf{e}_{\mathbf{u}_s}\|_{\mathrm{W}^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\mathrm{L}^\infty(0,T;\mathbb{L}^2(\Omega))} \le \mathcal{C}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})\, h^{\min\{l,s+1\}} \,.
\end{aligned} \tag{4.32}
$$

Proof. We begin testing the error equation (4.22) with the tuple $((\mathbf{e}^h_{\boldsymbol{\sigma}_p}, \mathbf{e}^h_{\mathbf{u}}, \mathbf{e}^h_p), (\mathbf{e}^h_{\mathbf{u}_s}, \mathbf{e}^h_{\boldsymbol{\gamma}}))$, obtaining

$$
\begin{aligned}
&\frac{1}{2}\partial_t \left(\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\mathbf{e}^h_p\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{e}^h_{\mathbf{u}_s}\|^2_{\mathbf{L}^2(\Omega)}\right) \\
&\quad + \nu\|\nabla \mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} + \mathtt{D}\|\mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} = (\partial_t A(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^I_p \mathbb{I}), \mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})_\Omega \\
&\quad - (\partial_t \mathbf{e}^I_{\mathbf{u}}, \mathbf{e}^h_{\mathbf{u}})_\Omega - \nu(\nabla \mathbf{e}^I_{\mathbf{u}}, \nabla \mathbf{e}^h_{\mathbf{u}})_\Omega - \mathtt{D}(\mathbf{e}^I_{\mathbf{u}}, \mathbf{e}^h_{\mathbf{u}})_\Omega + (\mathbf{e}^I_p, \mathrm{div}(\mathbf{e}^h_{\mathbf{u}}))_\Omega - (\mathrm{div}(\mathbf{e}^I_{\mathbf{u}}), \mathbf{e}^h_p)_\Omega \\
&\quad - (\mathbf{e}^I_{\boldsymbol{\gamma}}, \mathbf{e}^h_{\boldsymbol{\sigma}_p})_\Omega + (\mathbf{e}^h_{\boldsymbol{\gamma}}, \mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega \,.
\end{aligned} \tag{4.33}
$$

Here, the right-hand side of (4.33) has been simplified by using the projection properties (4.19), (4.20), and (4.21), together with the fact that $\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}) \in \mathbf{Q}^{\mathbf{u}}_h$. In particular, the following terms vanish

$$
s_0(\partial_t \mathbf{e}^I_p, \mathbf{e}^h_p)_\Omega \,, \quad (\partial_t \mathbf{e}^I_{\mathbf{u}_s}, \mathbf{e}^h_{\mathbf{u}_s})_\Omega \,, \quad (\mathbf{e}^I_{\mathbf{u}_s}, \mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}))_\Omega \,, \quad (\mathbf{e}^h_{\mathbf{u}_s}, \mathbf{div}(\mathbf{e}^I_{\boldsymbol{\sigma}_p}))_\Omega \,. \tag{4.34}
$$

Then, integrating (4.33) over the interval $(0,t)$ with $t \in (0,T]$, and applying the Cauchy–Schwarz and Young inequalities (cf. (1.2)–(1.3)), we arrive at the estimate

$$
\begin{aligned}
&\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\mathbf{e}^h_p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{e}^h_{\mathbf{u}_s}(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}\, ds \\
&\le C_1\bigg\{ \int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^I_p \mathbb{I})\|_{\mathbb{L}^2(\Omega)}\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\, \|\mathbf{e}^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)}\Big) ds \\
&+ \int_0^t \Big(\|\partial_t \mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}^I_p\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega)}\Big) ds \\
&+ \|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)}\bigg\} \\
&+ \delta \int_0^t \Big(\|\mathbf{e}^h_p\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\gamma}}\|^2_{\mathbb{L}^2(\Omega)}\Big) ds \,,
\end{aligned} \tag{4.35}
$$

where $C_1 > 0$ is a constant depending on δ, ν, $\mathtt{D}$, and, independent of s_0.

Next, similar to (3.25), by a simple algebraic manipulation combined with (2.4) and the triangle inequality, we infer that there exists a constant $C_2 > 0$, depending only on μ, d, λ, and α, such that

$$
\|\mathbf{e}^h_{\boldsymbol{\sigma}_p}(t)\|_{\mathbb{L}^2(\Omega)} \le C_2 \left\{\|A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^h_p(t)\|_{\mathrm{L}^2(\Omega)}\right\} . \tag{4.36}
$$

Then, substituting (4.36) back into (4.35), we obtain

$$\|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^h(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\mathbf{e}_p^h(t)\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}_s}^h(t)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\mathbf{e}_{\mathbf{u}}^h\|^2_{\mathbf{H}^1(\Omega)}\, ds$$
$$\leq C_3\bigg\{ \int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^I + \alpha\, \mathbf{e}_p^I\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^I\|_{\mathbb{L}^2(\Omega)}\Big) \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)}\, ds$$
$$+ \int_0^t \Big(\|\mathbf{e}_{\boldsymbol{\sigma}_p}^I\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}_{\mathbf{u}}^I\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^I\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}_p^I\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^I\|^2_{\mathbb{L}^2(\Omega)}\Big)\, ds \tag{4.37}$$
$$+ \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^h(0)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\mathbf{e}_p^h(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\mathbf{e}_{\mathbf{u}_s}^h(0)\|^2_{\mathbf{L}^2(\Omega)}\bigg\}$$
$$+ \delta \int_0^t \Big(\|\mathbf{e}_p^h\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^h\|^2_{\mathbb{L}^2(\Omega)}\Big)\, ds\,,$$

where $C_3 > 0$ is a constant depending on δ, ν, $\mathtt{D}$, μ, λ, α and independent of s_0. Hence, in order to bound the right-hand side of (4.37), we must estimate the norms of $\mathbf{e}_p^h$ and $\mathbf{e}_{\boldsymbol{\gamma}}^h$. To this end, we invoke the discrete inf-sup conditions (4.4) and (4.5), respectively. In fact, using (4.4) together with the first row of (4.22) tested with $(\boldsymbol{\tau}_h, \mathbf{0}, 0)$, property (2.4), and the Cauchy–Schwarz inequality, we derive that there exists a constant $C_4 > 0$, depending only on $\beta_{1,\mathtt{d}}$ and μ, such that

$$\Big(\|\mathbf{e}_{\mathbf{u}_s}^h(t)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^h(t)\|^2_{\mathbb{L}^2(\Omega)}\Big)^{1/2}$$
$$\leq C_4\left\{\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^I + \alpha \mathbf{e}_p^I \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^I(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha \mathbf{e}_p^h \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)}\right\}. \tag{4.38}$$

In turn, from (4.5), the first row of (4.22) tested with $(\mathbf{0}, \mathbf{v}_h, 0)$, and the Cauchy–Schwarz inequality, we yield that there exists a positive constant C_5, depending on $\beta_{2,\mathtt{d}}$, ν, $\mathtt{D}$, such that

$$\|\mathbf{e}_p^h(t)\|_{\mathrm{L}^2(\Omega)}$$
$$\leq C_5\Big\{\|\partial_t \mathbf{e}_{\mathbf{u}}^I(t)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^I(t)\|_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}_p^I(t)\|_{\mathrm{L}^2(\Omega)} + \|\partial_t \mathbf{e}_{\mathbf{u}}^h(t)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^h(t)\|_{\mathbf{H}^1(\Omega)}\Big\}. \tag{4.39}$$

Thus, similarly to (3.30), taking squares in (4.38) and (4.39), integrating over the interval $(0, t)$ with $t \in (0, T]$, combining them with the estimates (4.37), properly choosing δ, and employing Lemma 3.8 in the context of the non-negative functions $B = \frac{C_3}{2}\big(\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^I + \alpha\, \mathbf{e}_p^I\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^I\|_{\mathbb{L}^2(\Omega)}\big)$ and $\chi = \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)}$, with R and A representing the remaining terms, we deduce

$$\|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^h(t)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\|\mathbf{e}_p^h(t)\|_{\mathrm{L}^2(\Omega)} + \sqrt{\rho_p}\|\mathbf{e}_{\mathbf{u}_s}^h(t)\|_{\mathbf{L}^2(\Omega)}$$
$$+ \bigg(\int_0^t \Big(\|\mathbf{e}_{\mathbf{u}}^h\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}_p^h\|^2_{\mathrm{L}^2(\Omega)}\Big)\, ds\bigg)^{1/2} \leq C_6\bigg\{\int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^I + \alpha\, \mathbf{e}_p^I\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\gamma}}^I\|_{\mathbb{L}^2(\Omega)}\Big)\, ds$$
$$+\bigg(\int_0^t (\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^I + \alpha\, \mathbf{e}_p^I\, \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\boldsymbol{\sigma}_p}^I\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}_{\mathbf{u}}^I\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^I\|^2_{\mathbf{H}^1(\Omega)} + \|\mathbf{e}_p^I\|^2_{\mathrm{L}^2(\Omega)}$$
$$+\|\mathbf{e}_{\boldsymbol{\gamma}}^I\|^2_{\mathbb{L}^2(\Omega)})\, ds\bigg)^{1/2} + \|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})(0)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}_{\mathbf{u}}^h(0)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\|\mathbf{e}_p^h(0)\|_{\mathrm{L}^2(\Omega)}$$
$$+\sqrt{\rho_p}\|\mathbf{e}_{\mathbf{u}_s}^h(0)\|_{\mathbf{L}^2(\Omega)} + \bigg(\int_0^t (\|\partial_t A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p}^h + \alpha\, \mathbf{e}_p^h\, \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}_{\mathbf{u}}^h\|^2_{\mathbf{L}^2(\Omega)})\, ds\bigg)^{1/2}\bigg\}, \tag{4.40}$$

where $C_6 > 0$ is a constant depending only on ν, $\mathtt{D}$, μ, $\beta_{1,\mathtt{d}}$ and $\beta_{2,\mathtt{d}}$. In addition, testing the first row of (4.22) with $\underline{\boldsymbol{\tau}}_h = (\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}), \mathbf{0})$, and bearing in mind (4.21), we derive the following estimate

$$\|\mathbf{div}(\mathbf{e}^h_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega)} \,\leq\, \|\mathbf{div}(\mathbf{e}^I_{\boldsymbol{\sigma}_p})\|_{\mathbf{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{e}^h_{\mathbf{u}_s}\|_{\mathbf{L}^2(\Omega)} \,. \tag{4.41}$$

Bounds on time derivatives on the right-hand side of (4.40) and (4.41).

Now, in order to bound the last terms of (4.40) and (4.41), we proceed as in (3.33). We differentiate (4.22) with respect to time and test it with $\underline{\boldsymbol{\tau}}_h = (\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}, \partial_t \mathbf{e}^h_{\mathbf{u}}, \partial_t \mathbf{e}^h_p)$ and $\underline{\mathbf{v}}_h = (\partial_t \mathbf{e}^h_{\mathbf{u}_s}, \partial_t \mathbf{e}^h_{\boldsymbol{\gamma}})$, which yields

$$\begin{aligned}
&\frac{1}{2}\partial_t\left(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^h_p\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t \mathbf{e}^h_p\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{e}^h_{\mathbf{u}_s}\|^2_{\mathbf{L}^2(\Omega)}\right)\\
&\quad + \nu\|\nabla\partial_t \mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \mathtt{D}\|\partial_t \mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} = (\partial_{tt} A(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I}), \partial_t(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I}))_\Omega\\
&\quad - (\partial_{tt}\mathbf{e}^I_{\mathbf{u}}, \partial_t \mathbf{e}^h_{\mathbf{u}})_\Omega - \nu(\nabla\partial_t \mathbf{e}^I_{\mathbf{u}}, \nabla\partial_t \mathbf{e}^h_{\mathbf{u}})_\Omega - \mathtt{D}(\partial_t \mathbf{e}^I_{\mathbf{u}}, \partial_t \mathbf{e}^h_{\mathbf{u}})_\Omega + (\partial_t \mathbf{e}^I_p, \mathrm{div}(\partial_t \mathbf{e}^h_{\mathbf{u}}))_\Omega\\
&\quad - (\mathrm{div}(\partial_t \mathbf{e}^I_{\mathbf{u}}), \partial_t \mathbf{e}^h_p)_\Omega - (\partial_t \mathbf{e}^I_{\boldsymbol{\gamma}}, \partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p})_\Omega + (\partial_t \mathbf{e}^h_{\boldsymbol{\gamma}}, \partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega\,,
\end{aligned} \tag{4.42}$$

where, analogously to (4.34), the following terms vanish

$$s_0(\partial_{tt}\mathbf{e}^I_p, \partial_t \mathbf{e}^h_p)_\Omega\,,\quad (\partial_{tt}\mathbf{e}^I_{\mathbf{u}_s}, \partial_t \mathbf{e}^h_{\mathbf{u}_s})_\Omega\,,\quad (\partial_t \mathbf{e}^I_{\mathbf{u}_s}, \mathbf{div}(\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}))_\Omega\,,\quad (\partial_t \mathbf{e}^h_{\mathbf{u}_s}, \mathbf{div}(\partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p}))_\Omega\,.$$

Then, integrating (4.42) over $(0,t)$ for $t\in(0,T]$, and employing the Cauchy–Schwarz and Young inequalities, we further obtain the estimate

$$\begin{aligned}
&\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + s_0\|\partial_t \mathbf{e}^h_p(t)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{e}^h_{\mathbf{u}_s}(t)\|^2_{\mathbf{L}^2(\Omega)}\\
&\quad + \int_0^t \|\partial_t \mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} ds \leq C_7\left\{\int_0^t (\|\partial_{tt} A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)}\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} ds\right.\\
&\quad + \int_0^t\left(\|\partial_{tt}\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} + \|\nabla\partial_t \mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbf{L}^2(\Omega)} + \|\partial_t \mathbf{e}^I_p\|^2_{\mathrm{L}^2(\Omega)}\right) ds\\
&\quad + \int_0^t \|\nabla\partial_t \mathbf{e}^I_{\mathbf{u}}\|_{\mathbb{L}^2(\Omega)}\|\partial_t \mathbf{e}^h_p\|_{\mathrm{L}^2(\Omega)} ds + \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0) + \alpha\,\mathbf{e}^h_p(0)\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)}\\
&\quad \left. + s_0\|\partial_t \mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\partial_t \mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \int_0^t \|\partial_t \mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\|\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)} ds + \int_0^t (\partial_t \mathbf{e}^h_{\boldsymbol{\gamma}}, \partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega ds\right\},
\end{aligned} \tag{4.43}$$

where C_7 is a positive constant which depends on ν, $\mathtt{D}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, and μ. So that, to control the last two terms in (4.42), we first notice that similarly to (4.36) the triangle inequality and the bound (2.4), allow us to conclude

$$\begin{aligned}
&\int_0^t \|\partial_t \mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\|\partial_t \mathbf{e}^h_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)} ds\\
&\qquad \leq C_8\int_0^t \|\partial_t \mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\left(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^h_p\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_p\|_{\mathrm{L}^2(\Omega)}\right) ds\,,
\end{aligned} \tag{4.44}$$

where C_8 is a positive constant depending on α, μ and λ. In addition, applying (4.38) and the Cauchy–Schwarz inequality, we obtain for the last term in (4.43) the following estimate

$$\int_0^t (\partial_t \mathbf{e}^h_{\boldsymbol{\gamma}}, \partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega\, ds = (\mathbf{e}^h_{\boldsymbol{\gamma}}, \partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega\Big|_0^t - \int_0^t (\mathbf{e}^h_{\boldsymbol{\gamma}}, \partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p})_\Omega\, ds$$

$$\leq C_4\Big(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^h_p\mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^I_p\mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)}\Big)\|\partial_t\mathbf{e}^I_{\boldsymbol{\sigma}_p}(t)\|_{\mathbb{L}^2(\Omega)}$$

$$+ C_4\int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^h_p\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\mathbf{e}^I_p\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\Big)\|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)}ds$$

$$+ \|\mathbf{e}^h_{\boldsymbol{\gamma}}(0)\|_{\mathbb{L}^2(\Omega)}\|\partial_t\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{L}^2(\Omega)}\,. \tag{4.45}$$

Then, substituting (4.44) and (4.45) into (4.43) and applying Young's inequality, we find that

$$\Big(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t\mathbf{e}^h_p(t)\|_{\mathrm{L}^2(\Omega)}\Big)^2 + \|\partial_t\mathbf{e}^h_{\mathbf{u}}(t)\|^2_{\mathbf{L}^2(\Omega)} + \rho_p\|\partial_t\mathbf{e}^h_{\mathbf{u}_s}(t)\|^2_{\mathbf{L}^2(\Omega)}$$

$$+ \int_0^t \|\partial_t\mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}ds \leq C_9\Big\{\int_0^t \Big(\|\partial_{tt}A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}$$

$$+ \frac{1}{\sqrt{s_0}}\left(\|\nabla\mathbf{e}^I_{\mathbf{u}}\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\right)\Big)\Big(\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t\mathbf{e}^h_p\|_{\mathrm{L}^2(\Omega)}\Big)ds \tag{4.46}$$

$$+ \int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}$$

$$+ \|\partial_t\mathbf{e}^I_p\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}\|^2_{\mathbb{L}^2(\Omega)}\Big)\,ds + \|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\sigma}_p}(t)\|^2_{\mathbb{L}^2(\Omega)}$$

$$+ \|\mathbf{e}^I_{\boldsymbol{\gamma}}(t)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^h_{\mathbf{u}}(0)\|^2_{\mathbf{L}^2(\Omega)}$$

$$+ s_0\|\partial_t\mathbf{e}^h_p(0)\|^2_{\mathrm{L}^2(\Omega)} + \rho_p\|\partial_t\mathbf{e}^h_{\mathbf{u}_s}(0)\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\gamma}}(0)\|^2_{\mathbb{L}^2(\Omega)}\Big\}\,,$$

where C_9, is a positive constant depending on ν, $\mathtt{D}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, α, μ and λ. Hence, given a positive constant C_{10}, depending on ν, $\mathtt{D}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, α, μ and λ, employing Lemma 5.1 with

$$B = \frac{C_{10}}{2}\Bigg(\|\partial_{tt}A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\sqrt{s_0}}\big(\|\nabla\mathbf{e}^I_{\mathbf{u}}\|_{\mathbf{L}^2(\Omega)}$$
$$+\|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\big)\Bigg)\,,\quad\text{and}\quad \chi = \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \sqrt{s_0}\|\partial_t\mathbf{e}^h_p\|_{\mathrm{L}^2(\Omega)}\,,$$

and, R and A the corresponding remaining terms of (4.46), we conclude

$$\|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^h_p\,\mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^h_{\mathbf{u}}(t)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\,\|\partial_t\mathbf{e}^h_p(t)\|_{\mathrm{L}^2(\Omega)} + \sqrt{\rho_p}\,\|\partial_t\mathbf{e}^h_{\mathbf{u}_s}(t)\|_{\mathbf{L}^2(\Omega)}$$

$$+ \left(\int_0^t \|\partial_t\mathbf{e}^h_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)}\,ds\right)^{1/2} \;\leq\; C_{11}\Big\{\int_0^t \Big(\|\partial_{tt}A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|_{\mathbb{L}^2(\Omega)}$$

$$+ \|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)} + \frac{1}{\sqrt{s_0}}\Big(\|\nabla\mathbf{e}^I_{\mathbf{u}}\|_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}\Big)\Big)ds + \Big(\int_0^t \Big(\|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}^I_p\,\mathbb{I})\|^2_{\mathbb{L}^2(\Omega)}$$

$$+ \|\partial_{tt}\mathbf{e}^I_{\boldsymbol{\sigma}_p}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_{tt}\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbb{L}^2(\Omega)} + \|\partial_t\mathbf{e}^I_{\mathbf{u}}\|^2_{\mathbf{H}^1(\Omega)} + \|\partial_t\mathbf{e}^I_p\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}\|^2_{\mathbb{L}^2(\Omega)}\Big)\,ds\Big)^{1/2} \tag{4.47}$$

$$+ \|\partial_t A^{1/2}(\mathbf{e}^I_{\boldsymbol{\sigma}_p} + \alpha\, \mathbf{e}^I_p \mathbb{I})(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p}(t)\|_{\mathbb{L}^2(\Omega)} + \|\mathbf{e}^I_{\boldsymbol{\gamma}}(t)\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^I_{\boldsymbol{\sigma}_p}(0)\|_{\mathbb{L}^2(\Omega)}$$
$$+ \|\partial_t A^{1/2}(\mathbf{e}^h_{\boldsymbol{\sigma}_p}(0) + \alpha\, \mathbf{e}^h_p(0)\, \mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|\partial_t \mathbf{e}^h_{\mathbf{u}}(0)\|_{\mathbf{L}^2(\Omega)} + \sqrt{s_0}\, \|\partial_t \mathbf{e}^h_p(0)\|_{\mathrm{L}^2(\Omega)}$$
$$\left. + \sqrt{\rho_p}\, \|\partial_t \mathbf{e}^h_{\mathbf{u}_s}(0)\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{e}^h_{\boldsymbol{\gamma}}(0)\|_{\mathbb{L}^2(\Omega)} \right\},$$

where C_{11} is a positive constant depending of ν, $\mathtt{D}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$, α, μ and λ. Hence, by combining the estimates (4.38), (4.40), (4.41), and (4.47), together with the bound established in Lemma 4.2, some algebraic manipulations, and the approximation properties introduced at the beginning of Section 4.2, we conclude the proof of (4.32). □

Remark 4.2 *The dependence on* $\dfrac{1}{\sqrt{s_0}}$ *of the constant* $\mathcal{C}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}})$ *in the error estimate* (4.32) *arises from the terms in* (4.23)*, as well as from the quantities* $\|\nabla \partial_t \mathbf{e}^I_{\mathbf{u}}\|_{\mathbb{L}^2(\Omega))}$ *and* $\|\partial_t \mathbf{e}^I_{\boldsymbol{\gamma}}\|_{\mathbb{L}^2(\Omega)}$ *appearing in* (4.47)*.*

5 Fully discrete approximation

In this section, we analyze a fully discrete approximation of (2.16). We first establish the well-posedness of the scheme and then derive the corresponding error estimates. Finally, we provide error bounds for the post-processed variables (cf. (2.10)).

5.1 Well-posedness and error analysis of the fully discrete approximation

We now focus on the fully discrete scheme associated with (2.16) (cf. (4.3)), obtained by applying the backward Euler method for the temporal discretization. Let $\Delta t > 0$ denote the time step, set $T = N\Delta t$, and define $t_n = n\Delta t$ for $n = 0, \ldots, N$. For a sequence $u^n := u(t_n)$, we introduce the first and second backward differences by

$$d_t u^n := \frac{u^n - u^{n-1}}{\Delta t} \quad (n \geq 1), \qquad d_{tt} u^n := \frac{d_t u^n - d_t u^{n-1}}{\Delta t} \quad (n \geq 2).$$

For the sake of presentation, we assume homogeneous initial data

$$\boldsymbol{\sigma}^0_{ph} = \mathbf{0}, \qquad \mathbf{u}^0_h = \mathbf{0}, \qquad p^0_h = 0, \qquad \mathbf{u}^0_{sh} = \mathbf{0}, \qquad \boldsymbol{\gamma}^0_h = \mathbf{0}. \tag{5.1}$$

We will highlight in the proof of Theorem 5.2 the step where this assumption is required in order to simplify the computations. Notice that these initial data are consistent with those used in the numerical experiments reported in Section 6. Accordingly, the fully discrete scheme reads as follows: for $n = 1, \ldots, N$, given $\mathbf{g}^n \in \mathbf{L}^2(\Omega)$, $g^n \in \mathrm{L}^2_0(\Omega)$, and $\mathbf{f}^n_p \in \mathbf{L}^2(\Omega)$, find $(\underline{\boldsymbol{\sigma}}^n_h, \underline{\mathbf{u}}^n_h) \in \mathbb{H}_h \times \mathbb{Q}_h$ such that

$$\begin{aligned} d_t[\mathcal{E}_1(\underline{\boldsymbol{\sigma}}^n_h), \underline{\boldsymbol{\tau}}_h] + [\mathcal{A}(\underline{\boldsymbol{\sigma}}^n_h), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\underline{\mathbf{u}}^n_h), \underline{\boldsymbol{\tau}}_h] &= [\mathbf{F}^n, \underline{\boldsymbol{\tau}}_h] \qquad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{H}_h, \\ d_t[\mathcal{E}_2(\underline{\mathbf{u}}^n_h), \underline{\mathbf{v}}_h] - [\mathcal{B}(\underline{\boldsymbol{\sigma}}^n_h), \underline{\mathbf{v}}_h] &= [\mathbf{G}^n, \underline{\mathbf{v}}_h] \qquad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h, \end{aligned} \tag{5.2}$$

with

$$[\mathbf{F}^n, \underline{\boldsymbol{\tau}}_h] := (\mathbf{g}^n, \mathbf{v}_h)_\Omega + (g^n, q_h)_\Omega, \qquad [\mathbf{G}^n, \underline{\mathbf{v}}_h] := (\mathbf{f}^n_p, \mathbf{v}_{sh})_\Omega.$$

In what follows, for a separable Banach space $(V, \|\cdot\|_V)$ and for $p \in \{1, 2\}$, we use the following discrete-in-time norms:

$$\|u\|^p_{\ell^p(0,T;V)} := \Delta t \sum_{n=1}^{N} \|u^n\|^p_V, \quad \|u\|^2_{h^1(0,T;V)} := \|u\|^2_{\ell^2(0,T;V)} + \|d_t u\|^2_{\ell^2(0,T;V)},$$

$$\|u\|_{\ell^\infty(0,T;V)} := \max_{0\le n\le N} \|u^n\|_V, \quad \|d_t u\|_{\ell^\infty(0,T;V)} := \max_{1\le n\le N} \|d_t u^n\|_V\,,$$

and

$$\|u\|_{w^{1,\infty}(0,T;V)} := \|u\|_{\ell^\infty(0,T;V)} + \|d_t u\|_{\ell^\infty(0,T;V)}.$$

We also recall the standard discrete identities: for $n \ge 1$,

$$(d_t u_h^n,\, u_h^n)_\Omega = \frac{1}{2}\, d_t\|u_h^n\|^2_{\mathrm{L}^2(\Omega)} + \frac{\Delta t}{2}\, \|d_t u_h^n\|^2_{\mathrm{L}^2(\Omega)}, \tag{5.3}$$

which implies, for $n \ge 2$,

$$(d_{tt} u_h^n,\, d_t u_h^n)_\Omega = \frac{1}{2}\, d_t\|d_t u_h^n\|^2_{\mathrm{L}^2(\Omega)} + \frac{\Delta t}{2}\, \|d_{tt} u_h^n\|^2_{\mathrm{L}^2(\Omega)}. \tag{5.4}$$

Next, we state and prove a discrete counterpart of Lemma 3.8.

Lemma 5.1 *Let $\Delta t > 0$ and suppose that for each $n \ge 1$,*

$$\chi_n^2 + R_n \;\le\; A_n \;+\; 2\,\Delta t \sum_{k=1}^{n-1} B_k\, \chi_k\,, \tag{5.5}$$

where $\{\chi_n\}_{n\ge1}$, $\{R_n\}_{n\ge1}$, $\{A_n\}_{n\ge0}$, $\{B_n\}_{n\ge1}$ are nonnegative sequences. Then, for all $n \ge 1$, there holds

$$\sqrt{\chi_n^2 + R_n} \;\le\; \max_{0\le m\le N} \sqrt{A_m} \;+\; \Delta t \sum_{k=1}^{n-1} B_k\,. \tag{5.6}$$

Proof. Define

$$M := \max_{0\le m\le N} \sqrt{A_m}\,, \qquad v_n := \sqrt{\,M^2 + 2\,\Delta t \sum_{k=1}^{n-1} B_k\, \chi_k\,}\,, \quad n \ge 1\,.$$

Note that for $n = 1$ the sum is empty, hence $v_1 = M$. From (5.5) and the definition of M we have, for all $n \ge 1$,

$$\chi_n^2 + R_n \;\le\; M^2 + 2\,\Delta t \sum_{k=1}^{n-1} B_k\, \chi_k \;=\; v_n^2\,,$$

whence

$$\chi_n \le \sqrt{\chi_n^2 + R_n} \le v_n \tag{5.7}$$

For $n \ge 2$, using that $\chi_{n-1} \le v_{n-1}$ and after simple algebraic computations, we deduce that

$$\begin{aligned} v_n^2 &= M^2 + 2\,\Delta t \sum_{k=1}^{n-1} B_k\, \chi_k = v_{n-1}^2 + 2\,\Delta t\, B_{n-1}\, \chi_{n-1} \\ &\le v_{n-1}^2 + 2\,\Delta t\, B_{n-1}\, v_{n-1} \;\le\; \big(v_{n-1} + \Delta t\, B_{n-1}\big)^2\,, \end{aligned}$$

Thus, taking square roots yields the recursion

$$v_n \;\le\; v_{n-1} + \Delta t\, B_{n-1}\,, \qquad n \ge 2\,.$$

Iterating from $n = 2$ to n and using $v_1 = M$ gives

$$v_n \;\leq\; M + \Delta t \sum_{k=1}^{n-1} B_k \,,$$

which combined with (5.7) proves (5.6), and concludes the proof. $\square$

We now present the principal results for the fully discrete scheme (5.2), concerning existence, uniqueness, and stability.

Theorem 5.2 *For $(\underline{\boldsymbol{\sigma}}_h^0, \underline{\mathbf{u}}_h^0)$ satisfying (5.1), $\mathbf{g}^n \in \mathbf{L}^2(\Omega)$, $g^n \in \mathrm{L}^2_0(\Omega)$, and $\mathbf{f}_p^n \in \mathbf{L}^2(\Omega)$, there exists a unique solution $(\underline{\boldsymbol{\sigma}}_h^n, \underline{\mathbf{u}}_h^n) := ((\boldsymbol{\sigma}_{ph}^n, \mathbf{u}_h^n, p_h^n), (\mathbf{u}_{sh}^n, \boldsymbol{\gamma}_h^n)) \in \mathbb{H}_h \times \mathbb{Q}_h$ to (5.2), with $n = 1, ..., N$. Moreover, assuming that g is time-independent there exists a constant $\mathcal{C}_{\mathtt{st,f}} > 0$, such that*

$$\begin{aligned}
&\|A^{1/2}(\boldsymbol{\sigma}_{ph} + \alpha p_h \mathbb{I})\|_{w^{1,\infty}(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{div}(\boldsymbol{\sigma}_{ph})\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{u}_h\|_{w^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\mathbf{u}_h\|_{h^1(0,T;\mathbf{H}^1(\Omega))} + \sqrt{s_0}\,\|p_h\|_{w^{1,\infty}(0,T;\mathrm{L}^2(\Omega))} + \|p_h\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{u}_{sh}\|_{w^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} \\
&\quad + \|\boldsymbol{\gamma}_h\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))} \;\leq\; \mathcal{C}_{\mathtt{st,f}} \Bigg\{ \|\mathbf{g}\|_{h^1(0,T;\mathbf{L}^2(\Omega))} + \left(1 + \frac{1}{\sqrt{s_0}}\right) \|g\|_{\mathrm{L}^2(\Omega)} \\
&\quad + \|\mathbf{f}_p\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{f}_p\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \|d_t \mathbf{f}_p\|_{\ell^1(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{g}^1\|_{\mathbf{L}^2(\Omega)} \Bigg\}.
\end{aligned} \tag{5.8}$$

Proof. Existence and uniqueness of the fully discrete problem (5.2) at each time step t_n, $n = 1, \ldots, N$, may be established by induction. In particular, assuming that a solution is known at t_{n-1}, existence and uniqueness of the solution at t_n follow from arguments analogous to those used in the proof of Lemma 3.4.

For the derivation of (5.8), we take $(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h) = (\underline{\boldsymbol{\sigma}}_h^n, \underline{\mathbf{u}}_h^n)$ in (5.2), use the discrete identity (5.3), and apply the Cauchy–Schwarz and Young inequalities (cf. (1.3)) to obtain

$$\begin{aligned}
&\frac{1}{2} d_t \left(\|A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|p_h^n\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} \right) \\
&\quad + \frac{1}{2} \Delta t \left(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_t p_h^n\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_t \mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} \right) \\
&\quad + \min\left\{ \nu, \frac{\mathtt{D}}{2} \right\} \|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \leq \frac{1}{2} \Big(\frac{1}{\mathtt{D}} \|\mathbf{g}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}_p^n\|^2_{\mathbf{L}^2(\Omega)} \Big) + \|g\|_{\mathrm{L}^2(\Omega)} \|p_h^n\|_{\mathrm{L}^2(\Omega)} + \frac{1}{2} \|\mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} .
\end{aligned} \tag{5.9}$$

In addition, invoking the discrete inf-sup conditions (4.4) and (4.5), and after simple computations, we obtain the discrete counterparts of (3.28) and (3.29):

$$\Big(\|\mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\boldsymbol{\gamma}_h^n\|^2_{\mathbb{L}^2(\Omega)} \Big)^{1/2} \;\leq\; \frac{1}{\sqrt{2\,\mu}\,\beta_{1,\mathtt{d}}} \, \big\| d_t A^{1/2} (\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n\, \mathbb{I}) \big\|_{\mathbb{L}^2(\Omega)} \,, \tag{5.10}$$

and

$$\|p_h^n\|_{\mathrm{L}^2(\Omega)} \;\leq\; \frac{1}{\beta_{2,\mathtt{d}}} \Big(\|d_t \mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)} + \nu\, \|\nabla \mathbf{u}_h^n\|_{\mathbb{L}^2(\Omega)} + \mathtt{D}\, \|\mathbf{u}_h^n\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{g}^n\|_{\mathbf{L}^2(\Omega)} \Big). \tag{5.11}$$

Hence, combining (5.10) and (5.11) with (5.9), employing the Cauchy–Schwarz and Young inequalities, and bearing in mind that the initial conditions are assumed to be zero (cf. (5.1)), summing over $n = 1, \ldots, m$, with $m = 1, \ldots, N$, and multiplying by Δt, we deduce

$$\|A^{1/2}(\boldsymbol{\sigma}_{ph}^m + \alpha\, p_h^m \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|\mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|p_h^m\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|\mathbf{u}_{sh}^m\|^2_{\mathbf{L}^2(\Omega)}$$

$$+ (\Delta t)^2 \sum_{n=1}^{m} \Big(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_t p_h^n\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_t \mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} \Big)$$

$$+ \Delta t \sum_{n=1}^{m} \Big(\|\mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} + \|p_h^n\|^2_{\mathrm{L}^2(\Omega)} \Big) \le C_1 \Big\{ \Delta t \sum_{n=1}^{m} \Big(\|\mathbf{g}^n\|^2_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}_p^n\|^2_{\mathbf{L}^2(\Omega)} \Big) + T \|g\|^2_{\mathrm{L}^2(\Omega)}$$

$$+ \Delta t \sum_{n=1}^{m} \Big(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} \Big) \Big\}, \tag{5.12}$$

where $C_1 > 0$ depends only on $\mathtt{D}$, ν, $\beta_{1,\mathtt{d}}$, and $\beta_{2,\mathtt{d}}$. Furthermore, testing the second row of (5.2) with test function $\underline{\mathbf{v}}_h = (\mathbf{div}(\boldsymbol{\sigma}_{ph}^n), \mathbf{0})$, and applying the Cauchy–Schwarz inequality, we derive

$$\|\mathbf{div}(\boldsymbol{\sigma}_{ph}^n)\|_{\mathbf{L}^2(\Omega)} \;\le\; \|d_t \mathbf{u}_{sh}^n\|_{\mathbf{L}^2(\Omega)} + \|\mathbf{f}_p^n\|_{\mathbf{L}^2(\Omega)}\,. \tag{5.13}$$

Next, to bound the derivative terms appearing in (5.12) and (5.13), we proceed as in (5.9). We subtract (5.2) at time levels t_n and t_{n-1}, test the resulting relations with $\underline{\boldsymbol{\tau}}_h = (d_t \boldsymbol{\sigma}_{ph}^n, d_t \mathbf{u}_h^n, d_t p_h^n)$ and $\underline{\mathbf{v}}_h = (d_t \mathbf{u}_{sh}^n, d_t \boldsymbol{\gamma}_h^n)$, use the discrete identity (5.4), and apply the Cauchy–Schwarz and Young inequalities to obtain

$$\frac{1}{2}\, d_t \Big(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_t p_h^n\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_t \mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} \Big)$$

$$+ \frac{1}{2}\, \Delta t \Big(\|d_{tt} A^{1/2}(\boldsymbol{\sigma}_{ph}^n + \alpha\, p_h^n \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_{tt} \mathbf{u}_h^n\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_{tt} p_h^n\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_{tt} \mathbf{u}_{sh}^n\|^2_{\mathbf{L}^2(\Omega)} \Big)$$

$$+ \min\Big\{ \nu, \frac{\mathtt{D}}{2} \Big\}\, \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \;\le\; \frac{1}{2\mathtt{D}}\, \|d_t \mathbf{g}^n\|^2_{\mathbf{L}^2(\Omega)} + \|d_t \mathbf{f}_p^n\|_{\mathbf{L}^2(\Omega)}\, \|d_t \mathbf{u}_{sh}^n\|_{\mathbf{L}^2(\Omega)}\,, \tag{5.14}$$

for $n = 2, \dots, N$, owing to the presence of the second discrete derivative d_{tt}. We observe that the term involving g vanishes, since it is time-independent. Dropping the nonnegative second line on the left-hand side of (5.14), summing for $n = 2, \dots, m$, with $m = 2, \dots, N$, and multiplying by Δt, we obtain

$$\frac{1}{2} \Big(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^m + \alpha\, p_h^m \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^m\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_t p_h^m\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_t \mathbf{u}_{sh}^m\|^2_{\mathbf{L}^2(\Omega)} \Big)$$

$$+ \min\Big\{ \nu, \frac{\mathtt{D}}{2} \Big\}\, \Delta t \sum_{n=2}^{m} \|d_t \mathbf{u}_h^n\|^2_{\mathbf{H}^1(\Omega)} \le\; \Delta t \sum_{n=2}^{m} \|d_t \mathbf{g}^n\|^2_{\mathbf{L}^2(\Omega)} + \Delta t \sum_{n=2}^{m} \|d_t \mathbf{f}_p^n\|_{\mathbf{L}^2(\Omega)}\, \|d_t \mathbf{u}_{sh}^n\|_{\mathbf{L}^2(\Omega)}$$

$$+ \frac{1}{2} \Big(\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^1 + \alpha\, p_h^1 \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \|d_t \mathbf{u}_h^1\|^2_{\mathbf{L}^2(\Omega)} + s_0 \|d_t p_h^1\|^2_{\mathrm{L}^2(\Omega)} + \rho_p \|d_t \mathbf{u}_{sh}^1\|^2_{\mathbf{L}^2(\Omega)} \Big)\,. \tag{5.15}$$

For the case $m = 1$, taking $n = 1$ in (5.2) and testing with $\underline{\boldsymbol{\tau}}_h = (d_t \boldsymbol{\sigma}_{ph}^1, d_t \mathbf{u}_h^1, d_t p_h^1)$ and $\underline{\mathbf{v}}_h = (d_t \mathbf{u}_{sh}^1, d_t \boldsymbol{\gamma}_h^1)$, applying the Cauchy–Schwarz and Young inequalities yields

$$\begin{aligned} &\|d_t A^{1/2}(\boldsymbol{\sigma}_{ph}^1 + \alpha\, p_h^1 \mathbb{I})\|^2_{\mathbb{L}^2(\Omega)} + \frac{1}{2} \|d_t \mathbf{u}_h^1\|^2_{\mathbf{L}^2(\Omega)} + \frac{s_0}{2} \|d_t p_h^1\|^2_{\mathrm{L}^2(\Omega)} + \frac{1}{2} \|d_t \mathbf{u}_{sh}^1\|^2_{\mathbf{L}^2(\Omega)} \\ &\quad + \frac{\min\{\nu, \mathtt{D}\}}{2} \Delta t\, \|d_t \mathbf{u}_h^1\|^2_{\mathbf{H}^1(\Omega)} \le \frac{1}{2} \Big(\|\mathbf{g}^1\|^2_{\mathbf{L}^2(\Omega)} + \frac{1}{s_0}\, \|g\|^2_{\mathrm{L}^2(\Omega)} + \|\mathbf{f}_p^1\|^2_{\mathbf{L}^2(\Omega)} \Big)\,, \end{aligned} \tag{5.16}$$

where we have simplified the left- and right-hand sides of (5.16) by using that the initial conditions are homogeneous (cf. (5.1)), which makes several terms vanish. More precisely, on the left-hand side we have neglected the term

$$\frac{\min\{\nu, \mathtt{D}\}}{2\,\Delta\, t}\, d_t \|\mathbf{u}_h^1\|^2_{\mathbf{H}^1(\Omega)} \;=\; \frac{\min\{\nu, \mathtt{D}\}}{2\,\Delta\, t} \Big(\|\mathbf{u}_h^1\|^2_{\mathbf{H}^1(\Omega)} - \|\mathbf{u}_h^0\|^2_{\mathbf{H}^1(\Omega)} \Big) \;=\; \frac{\min\{\nu, \mathtt{D}\}}{2\,\Delta\, t}\, \|\mathbf{u}_h^1\|^2_{\mathbf{H}^1(\Omega)} \;\ge\; 0\,,$$

whereas, on the right-hand side, the following terms vanish:

$$\begin{aligned}&(\mathbf{u}_{sh}^1, \mathbf{div}(d_t\boldsymbol{\sigma}_{ph}^1))_\Omega + (\boldsymbol{\gamma}_h^1, d_t\boldsymbol{\sigma}_{ph}^1)_\Omega - (d_t\mathbf{u}_{sh}^1, \mathbf{div}(\boldsymbol{\sigma}_{ph}^1))_\Omega - (d_t\boldsymbol{\gamma}_h^1, \boldsymbol{\sigma}_{ph}^1)_\Omega \\ &\quad = \frac{1}{\Delta t}\big((\mathbf{u}_{sh}^0, \mathbf{div}(\boldsymbol{\sigma}_{ph}^1))_\Omega - (\mathbf{u}_{sh}^1, \mathbf{div}(\boldsymbol{\sigma}_{ph}^0))_\Omega + (\boldsymbol{\gamma}_h^0, \boldsymbol{\sigma}_{ph}^1)_\Omega - (\boldsymbol{\gamma}_h^1, \boldsymbol{\sigma}_{ph}^0)_\Omega\big) = 0\end{aligned} \tag{5.17}$$

and

$$(p_h^1, \operatorname{div}(d_t\mathbf{u}_h^1))_\Omega - (d_tp_h^1, \operatorname{div}(\mathbf{u}_h^1))_\Omega = \frac{1}{\Delta t}\big((p_h^0, \operatorname{div}(\mathbf{u}_h^1))_\Omega - (p_h^1, \operatorname{div}(\mathbf{u}_h^0))_\Omega\big) = 0\,. \tag{5.18}$$

Notice that, although each term in (5.17) and (5.18) can be bounded, if the initial data are not taken to be zero, these bounds involve a factor $(\Delta t)^{-1}$, which then appears on the right-hand side of (5.16) multiplying the corresponding initial data.

Thus, combining (5.15) and (5.16), and applying Lemma 5.1 with $B_k = \frac{1}{\sqrt{\rho_p}}\|d_t\mathbf{f}_p^k\|_{\mathbf{L}^2(\Omega)}$ and $\chi_k = \sqrt{\rho_p}\|d_t\mathbf{u}_{sh}^k\|_{\mathbb{L}^2(\Omega)}$, we obtain for $m = 1, ..., N$,

$$\begin{aligned}&\|d_tA^{1/2}(\boldsymbol{\sigma}_{ph}^m + \alpha\, p_h^m\mathbb{I})\|_{\mathbb{L}^2(\Omega)} + \|d_t\mathbf{u}_h^m\|_{\mathbf{L}^2(\Omega)} + \Delta t\sum_{n=1}^m \|d_t\mathbf{u}_h^n\|_{\mathbf{H}^1(\Omega)}^2 + \sqrt{s_0}\|d_tp_h^m\|_{\mathrm{L}^2(\Omega)} \\ &\quad + \sqrt{\rho_p}\|d_t\mathbf{u}_{sh}^m\|_{\mathbf{L}^2(\Omega)} \leq C_2\Big\{\|\mathbf{g}^1\|_{\mathbf{L}^2(\Omega)} + \|d_t\mathbf{g}\|_{\ell^2(0,T;\mathbf{L}^2(\Omega))} + \frac{1}{\sqrt{s_0}}\|g\|_{\mathrm{L}^2(\Omega)} + \|\mathbf{f}_p^1\|_{\mathbf{L}^2(\Omega)} \\ &\quad + \|d_t\mathbf{f}_p\|_{\ell^1(0,T;\mathbf{L}^2(\Omega))}\Big\}\,,\end{aligned} \tag{5.19}$$

where C_2 is a positive constant depending on ρ_p, ν and $\mathtt{D}$. Finally, by combining (5.12) and (5.13) with (5.19), and using the Cauchy–Schwarz and Young inequalities together with some algebraic manipulations, we arrive at (5.8). □

We now establish convergence rates for the fully discrete scheme (5.2). Subtracting (5.2) from the continuous problem (2.16) at each time level t_n, $n = 1, \ldots, N$, we obtain the error system

$$\begin{aligned} d_t[\mathcal{E}_1(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}^n), \underline{\boldsymbol{\tau}}_h] + [\mathcal{A}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}^n), \underline{\boldsymbol{\tau}}_h] + [\mathcal{B}'(\mathbf{e}_{\underline{\mathbf{u}}}^n), \underline{\boldsymbol{\tau}}_h] &= [r_n^1(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}_h] \quad \forall\, \underline{\boldsymbol{\tau}}_h \in \mathbb{H}_h, \\ d_t[\mathcal{E}_2(\mathbf{e}_{\underline{\mathbf{u}}}^n), \underline{\mathbf{v}}_h] - [\mathcal{B}(\mathbf{e}_{\underline{\boldsymbol{\sigma}}}^n), \underline{\mathbf{v}}_h] &= [r_n^2(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] \quad \forall\, \underline{\mathbf{v}}_h \in \mathbb{Q}_h,\end{aligned} \tag{5.20}$$

where $\mathbf{e}_{\underline{\boldsymbol{\sigma}}}^n := (\boldsymbol{\sigma}_p^n - \boldsymbol{\sigma}_{ph}^n,\, \mathbf{u}^n - \mathbf{u}_h^n,\, p^n - p_h^n)$ and $\mathbf{e}_{\underline{\mathbf{u}}}^n := (\mathbf{u}_s^n - \mathbf{u}_{sh}^n,\, \boldsymbol{\gamma}^n - \boldsymbol{\gamma}_h^n)$. The residual functionals, collecting the time-discretization defects, are given by

$$\begin{aligned}[r_n^1(\underline{\boldsymbol{\sigma}}), \underline{\boldsymbol{\tau}}_h] &:= (d_tA(\boldsymbol{\sigma}_p^n + \alpha p^n\mathbb{I}) - \partial_tA(\boldsymbol{\sigma}_p(t_n) + \alpha p(t_n)\mathbb{I}), \boldsymbol{\tau}_h + \alpha\, q\mathbb{I})_\Omega \\ &\quad + (d_t\mathbf{u}^n - \partial_t\mathbf{u}(t_n), \mathbf{v}_h)_\Omega + s_0(d_tp^n - \partial_tp(t_n), q_h)_\Omega\end{aligned}$$

and

$$[r_n^2(\underline{\mathbf{u}}), \underline{\mathbf{v}}_h] := \rho_p(d_t\mathbf{u}_s^n - \partial_t\mathbf{u}_s(t_n), \mathbf{v}_{sh})_\Omega$$

for all $(\underline{\boldsymbol{\tau}}_h, \underline{\mathbf{v}}_h) \in \mathbb{H}_h \times \mathbb{Q}_h$.

In addition, by invoking [14, Lemma 4] and using (2.4) to bound the first term of $r_n^1(\underline{\boldsymbol{\sigma}})$, we obtain, for sufficiently smooth $\underline{\boldsymbol{\sigma}}$ and $\mathbf{u}_s$, the estimates

$$\Delta t\sum_{n=1}^N \|r_n^1(\underline{\boldsymbol{\sigma}})\|_{\mathbb{H}_2'}^2 \;\leq\; \mathcal{C}(\partial_{tt}\underline{\boldsymbol{\sigma}})\,(\Delta t)^2\,, \quad \Delta t\sum_{n=1}^N \|r_n^2(\mathbf{u}_s)\|_{\mathbf{L}^2(\Omega)}^2 \;\leq\; \mathcal{C}(\partial_{tt}\mathbf{u}_s)\,(\Delta t)^2, \tag{5.21}$$

and

$$\Delta t\sum_{n=1}^{N}\|d_t r_n^1(\underline{\boldsymbol{\sigma}})\|^2_{\mathbb{H}_2'} \,\leq\, \mathcal{C}(\partial_{ttt}\underline{\boldsymbol{\sigma}})\,(\Delta t)^2\,,\quad \Delta t\sum_{n=1}^{N}\|d_t r_n^2(\mathbf{u}_s)\|^2_{\mathbf{L}^2(\Omega)} \,\leq\, \mathcal{C}(\partial_{ttt}\mathbf{u}_s)\,(\Delta t)^2\,, \tag{5.22}$$

where

$$\mathcal{C}(\partial_{tt}\underline{\boldsymbol{\sigma}}) := C\,\|\partial_{tt}\underline{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_2')}, \qquad \mathcal{C}(\partial_{tt}\mathbf{u}_s) := C\,\|\partial_{tt}\mathbf{u}_s\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))},$$
$$\mathcal{C}(\partial_{ttt}\underline{\boldsymbol{\sigma}}) := C\,\|\partial_{ttt}\underline{\boldsymbol{\sigma}}\|_{\mathrm{L}^2(0,T;\mathbb{H}_2')}, \qquad \mathcal{C}(\partial_{ttt}\mathbf{u}_s) := C\,\|\partial_{ttt}\mathbf{u}_s\|_{\mathrm{L}^2(0,T;\mathbf{L}^2(\Omega))}\,.$$

Thus, the proof of the theoretical convergence rates for the fully discrete scheme (5.2) follows the structure of the proof of Theorem 4.3, now with the error system (5.20) and the discrete-in-time arguments as in the proof of Theorem 5.2, the discrete inequality from Lemma 5.1, and the temporal estimates (5.21). We note that the bounds in (5.22) are instrumental in controlling the fully discrete counterparts of (4.42)–(4.43), where the terms $d_t r_n^1$ and $d_t r_n^2$ appear. See also [20, Theorem 5.3] for a related approach.

Theorem 5.3 *Let the assumptions of Theorem 4.3 hold. Then, for the solution of the fully discrete problem (5.2), there exists a constant $\mathcal{C}_{\mathtt{f}}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}}) > 0$, depending only on the regularity of the exact solution but independent of h and Δt, such that*

$$\begin{aligned}&\|A^{1/2}(\mathbf{e}_{\boldsymbol{\sigma}_p} + \alpha\,\mathbf{e}_p\,\mathbb{I})\|_{w^{1,\infty}(0,T;\mathbb{L}^2(\Omega))} + \|\mathbf{div}(\mathbf{e}_{\boldsymbol{\sigma}_p})\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}}\|_{w^{1,\infty}(0,T;\mathbf{L}^2(\Omega))}\\ &\quad + \sqrt{s_0}\,\|\mathbf{e}_p\|_{w^{1,\infty}(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{e}_p\|_{\ell^2(0,T;\mathrm{L}^2(\Omega))} + \|\mathbf{e}_{\mathbf{u}_s}\|_{w^{1,\infty}(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}\\ &\quad \leq \mathcal{C}_{\mathtt{f}}(\underline{\boldsymbol{\sigma}},\underline{\mathbf{u}})\,\big(h^{\min\{l,s+1\}} + \Delta t\big).\end{aligned} \tag{5.23}$$

5.2 Recovering post-processed variables

We next describe how the displacement and rotation, which are not treated as primary unknowns, may be reconstructed. In particular, following (2.10), at times t_m, $m = 1,\dots,N$, they are obtained, respectively, via the post-processing formulas

$$\boldsymbol{\eta}_h^m \;=\; \boldsymbol{\eta}_h^0 + \frac{\Delta t}{2}\sum_{n=1}^{m}\big(\mathbf{u}_{sh}^{n-1} + \mathbf{u}_{sh}^n\big) \quad\text{and}\quad \boldsymbol{\rho}_h^m \;=\; \boldsymbol{\rho}_h^0 + \frac{\Delta t}{2}\sum_{n=1}^{m}\big(\boldsymbol{\gamma}_h^{n-1} + \boldsymbol{\gamma}_h^n\big), \tag{5.24}$$

where the integral terms are approximated by the composite trapezoidal rule. We also establish the corresponding theoretical convergence rates. To this end, we assume that $\boldsymbol{\eta}_h \in \mathbf{Q}_h^{\mathbf{u}_s}$ and $\boldsymbol{\rho}_h \in \mathbf{Q}_h^{\boldsymbol{\gamma}}$, and, following the same convention as in (5.1), we consider $\boldsymbol{\eta}_0 = \boldsymbol{\eta}_h^0 = \mathbf{0}$ and $\boldsymbol{\rho}_0 = \boldsymbol{\rho}_h^0 = \mathbf{0}$. Then, recalling (2.10) and (5.24), and after adding and subtracting suitable terms, we obtain, for any time t_m,

$$\begin{aligned}\mathbf{e}_{\boldsymbol{\eta}}^m &= \int_0^{t_m}\mathbf{u}_s(s)\,ds \,-\, \frac{\Delta t}{2}\sum_{n=1}^{m}\big(\mathbf{u}_{sh}^{n-1} + \mathbf{u}_{sh}^n\big)\\ &= \sum_{n=1}^{m}\Big(\int_{t_{n-1}}^{t_n}\mathbf{u}_s(s)\,ds - \frac{\Delta t}{2}\big(\mathbf{u}_s^{n-1} + \mathbf{u}_s^n\big)\Big) + \frac{\Delta t}{2}\sum_{n=1}^{m}\Big((\mathbf{u}_s^{n-1} - \mathbf{u}_{sh}^{n-1}) + (\mathbf{u}_s^n - \mathbf{u}_{sh}^n)\Big).\end{aligned} \tag{5.25}$$

Define the trapezoidal-rule quadrature error, for $\boldsymbol{\zeta} \in \{\mathbf{u}_s, \boldsymbol{\gamma}\}$, as

$$Q_n(\boldsymbol{\zeta}) := \int_{t_{n-1}}^{t_n}\boldsymbol{\zeta}(s)\,ds \,-\, \frac{\Delta t}{2}\big(\boldsymbol{\zeta}^{\,n-1} + \boldsymbol{\zeta}^{\,n}\big)\,,$$

and recall the nodal errors $\mathbf{e}_{\mathbf{u}_s}^n := \mathbf{u}_s^n - \mathbf{u}_{sh}^n$ and $\mathbf{e}_{\boldsymbol{\gamma}}^n := \boldsymbol{\gamma}^n - \boldsymbol{\gamma}_h^n$. Applying the $\mathbf{L}^2(\Omega)$-norm to both sides of (5.25) and using the Minkowski and triangle inequalities, we obtain, for $m = 1, \ldots, N$, that

$$\|\mathbf{e}_{\boldsymbol{\eta}}^m\|_{\mathbf{L}^2(\Omega)} \;\leq\; \Delta t \sum_{n=0}^{m} \|\mathbf{e}_{\mathbf{u}_s}^n\|_{\mathbf{L}^2(\Omega)} + \sum_{n=1}^{m} \|Q_n(\mathbf{u}_s)\|_{\mathbf{L}^2(\Omega)}\,, \tag{5.26}$$

and, arguing analogously,

$$\|\mathbf{e}_{\boldsymbol{\rho}}^m\|_{\mathbb{L}^2(\Omega)} \;\leq\; \Delta t \sum_{n=0}^{m} \|\mathbf{e}_{\boldsymbol{\gamma}}^n\|_{\mathbb{L}^2(\Omega)} + \sum_{n=1}^{m} \|Q_n(\boldsymbol{\gamma})\|_{\mathbb{L}^2(\Omega)}\,. \tag{5.27}$$

Moreover, assuming that $\boldsymbol{\eta}$ and $\boldsymbol{\rho}$ are sufficiently regular, the standard trapezoidal-rule estimate (cf. [36, Section 9.2.2]) yields, for each n and for $\boldsymbol{\zeta} \in \{\boldsymbol{\eta}, \boldsymbol{\rho}\}$, the local bound

$$\|Q_n(\boldsymbol{\zeta})\|_{\mathbf{L}^2(\Omega)} \;\leq\; \frac{(\Delta t)^3}{12}\,\|\partial_{tt}\boldsymbol{\zeta}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}.$$

Consequently, since $m\Delta t \leq T$,

$$\sum_{n=1}^{m} \|Q_n(\boldsymbol{\zeta})\|_{\mathbf{L}^2(\Omega)} \;\leq\; \mathcal{C}(\partial_{tt}\boldsymbol{\zeta})\,(\Delta t)^2, \tag{5.28}$$

where $\mathcal{C}(\partial_{tt}\boldsymbol{\zeta}) := \frac{T}{12}\,\|\partial_{tt}\boldsymbol{\zeta}\|_{L^\infty(0,T;\mathbf{L}^2(\Omega))}$.

Next, invoking $(\mathbf{AP}_h^{\mathbf{u}_s})$, $(\mathbf{AP}_h^{\boldsymbol{\gamma}})$, and the trapezoidal-rule estimate (5.28), together with (5.23), and (5.26)–(5.27), we obtain the following result.

Lemma 5.4 *Let the assumptions of Theorem 5.2 hold, and let $\boldsymbol{\eta}_h$ and $\boldsymbol{\rho}_h$ be defined by (5.24). Then there exists a constant $\mathcal{C}_{\mathrm{p,f}} > 0$, depending only on the regularity of $\underline{\boldsymbol{\sigma}}$, $\underline{\mathbf{u}}$, $\boldsymbol{\eta}$, and $\boldsymbol{\rho}$ and independent of h and Δt, such that*

$$\|\mathbf{e}_{\boldsymbol{\eta}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))} + \|\mathbf{e}_{\boldsymbol{\rho}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))} \;\leq\; \mathcal{C}_{\mathrm{p,f}}(\underline{\boldsymbol{\sigma}}, \underline{\mathbf{u}}, \boldsymbol{\eta}, \boldsymbol{\rho})\left(h^{\min\{l,\,s+1\}} + \Delta t + (\Delta t)^2\right).$$

Remark 5.1 *Following a procedure similar to that in Lemma 3.5 and (4.6)–(4.9), one may also construct appropriate initial conditions $\boldsymbol{\rho}_0$, $\boldsymbol{\eta}_h^0$, and $\boldsymbol{\rho}_h^0$ from $\boldsymbol{\eta}_0$. For simplicity of the presentation, and consistently with the numerical experiments, however, these quantities were assumed to be zero.*

6 Numerical results

In this section we present numerical experiments illustrating the performance of the fully discrete method (5.2). The implementation is carried out in the FEniCS library [1], using triangular meshes in two dimensions and tetrahedral meshes in three dimensions. For the spatial discretization we employ the Arnold–Falk–Winther spaces (4.1) in combination with the Taylor–Hood pair (4.2). The condition $(p, 1)_\Omega = 0$ is enforced via a scalar Lagrange multiplier, which adds one row and one column to the linear system associated with (5.2). The examples considered in this section are as follows.

Examples 1 and 2 confirm the spatial convergence rates in two- and three-dimensional domains, respectively. In both cases, the parameters are set to $\alpha = 1$, $\rho_p = 1$, $\lambda = 1$, $\mu = 1$, $\nu = 1$, $\mathtt{D} = 1$, and $s_0 = 1$. For the lowest order, $k = 0$, we set $T = 10^{-2}$ and $\Delta t = 10^{-3}$. For $k = 1$, we further reduce the final time and time step to $T = 10^{-4}$ and $\Delta t = 10^{-5}$, which are small enough to ensure that temporal errors do not affect the observed spatial rates.

Example 3 considers a cantilever-type configuration to investigate locking with $s_0 = 0$, assessing whether the scheme remains locking-free. Example 4 is the classical Mandel–Cryer benchmark, used to validate the transient poroelastic response (including the characteristic pressure overshoot) and to further assess the method's accuracy and robustness across parameter regimes. In both examples, due to the high computational cost of these configurations, we employ Arnold–Falk–Winther and Taylor–Hood elements of order $k = 0$.

Example 1: Two-dimensional smooth exact solution

We examine the spatial-convergence behavior using an analytical solution on the unit square $\Omega = (0,1)^2$. The source terms $\mathbf{g}$, g, and $\mathbf{f}_p$ are chosen so that the exact solution in (2.9) is:

$$p = \exp(t)\left(\sin(\pi x)\cos\left(\frac{\pi y}{2}\right) - \frac{4}{\pi^2}\right), \qquad \mathbf{u} = \exp(t)\begin{pmatrix}\sin(\pi x)\,\sin(2\pi y)\\ -\sin(2\pi x)\,\sin(\pi y)\end{pmatrix},$$
$$\text{and} \quad \boldsymbol{\eta} = \exp(t)\begin{pmatrix}\sin(\pi x)\,\cos(\pi y)\\ \cos(\pi x)\,\sin(\pi y)\end{pmatrix},$$

with $\boldsymbol{\sigma}_p$, $\mathbf{u}_s$, and $\boldsymbol{\gamma}$ defined by (2.2) and (2.8), respectively. The model is complemented with appropriate Dirichlet boundary conditions and consistent initial data (cf. (5.1)). Tables 6.1 and 6.2 report the convergence history for a sequence of quasi-uniform mesh refinements. We also computed the convergence rates for the original unknowns $\boldsymbol{\eta}_h$ and $\boldsymbol{\rho}_h$. According to (5.24), for each $m \in \{1, \ldots, N\}$, we observe that these variables can be computed recursively as

$$\boldsymbol{\eta}_h^m = \boldsymbol{\eta}_h^{m-1} + \frac{\Delta t}{2}\,(\mathbf{u}_{sh}^{m-1} + \mathbf{u}_{sh}^m)\,, \qquad \boldsymbol{\rho}_h^m = \boldsymbol{\rho}_h^{m-1} + \frac{\Delta t}{2}\,(\boldsymbol{\gamma}_{sh}^{m-1} + \boldsymbol{\gamma}_{sh}^m)\,. \tag{6.1}$$

The results confirm the expected optimal spatial convergence rates: $\mathcal{O}(h^{k+1})$ for the poroelastic stress tensor in $\mathbb{H}_h^{\boldsymbol{\sigma}_p}$, the structural velocity in $\mathbf{Q}_h^{\mathbf{u}_s}$, the rate of rotation in $\mathbb{Q}_h^{\gamma}$ (cf. (4.1)), and for the displacement and rotation recovered by post-processing; and $\mathcal{O}(h^{k+2})$ for the fluid velocity in $\mathbf{H}_h^{\mathbf{u}}$ and the pressure in H_h^p (cf. (4.2)) in agreement with Theorems 4.3 and 5.3 and Lemma 5.4. In Figure 6.1, we display the solution obtained with 20,000 triangular elements and 271,804 `DoF`, using order $k = 1$ and final time $T = 10^{-5}$.

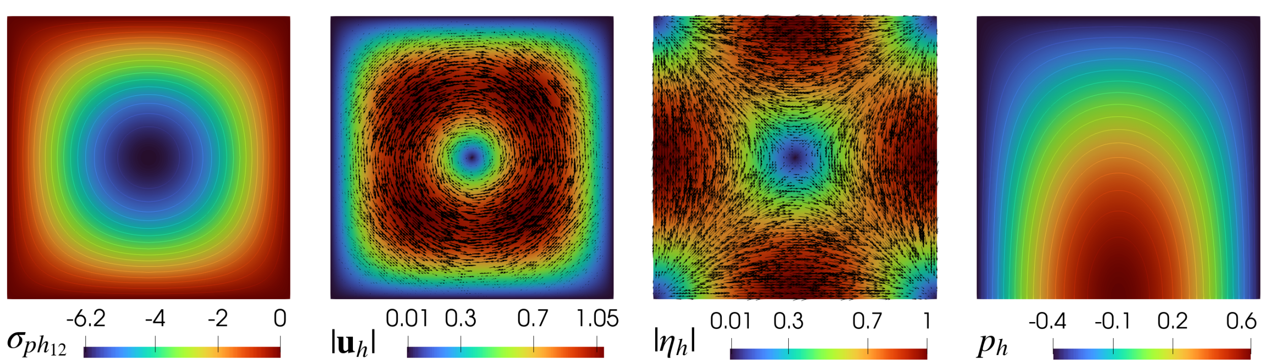

Figure 6.1: [Example 1]: Computed component (1,2) of the poroelasticity stress tensor, magnitude of the velocity and displacement, and pressure field.

#elements	h	DoF	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{H}(\mathbf{div};\Omega))}$ error	rate	$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$ error	rate	$\|\mathbf{e}_p\|_{\ell^\infty(0,T;\mathrm{L}^2(\Omega))}$ error	rate
32	0.3536	508	1.23e+01	0	6.54e-01	0	1.65e-02	0
128	0.1768	1,876	6.13e+00	1.008	1.71e-01	1.933	3.99e-03	2.050
512	0.0884	7,204	3.05e+00	1.009	4.36e-02	1.975	9.87e-04	2.017
2048	0.0442	28,228	1.51e+00	1.015	1.09e-02	1.993	2.46e-04	2.006
7200	0.0236	98,284	7.86e-01	1.036	3.12e-03	1.998	6.98e-05	2.003
20000	0.0141	271,804	4.58e-01	1.056	1.12e-03	1.999	2.51e-05	1.999

$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\eta}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\rho}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$ error	rate
1.92e-01	0	3.36e+01	0	1.83e-01	0	3.46e-02	0
9.62e-02	0.997	5.21e+00	2.689	9.30e-02	0.979	5.68e-03	2.606
4.74e-02	1.019	7.35e-01	2.825	4.67e-02	0.995	9.28e-04	2.613
2.34e-02	1.021	1.04e-01	2.816	2.34e-02	0.999	1.54e-04	2.592
1.25e-02	1.000	1.86e-02	2.748	1.25e-02	1.000	2.49e-05	2.900
7.48e-03	1.000	4.74e-03	2.674	7.48e-03	1.000	6.05e-06	2.768

Table 6.1: [Example 1, $k = 0$] Convergence history for the fully discrete scheme with AFW–Taylor–Hood approximations.

#elements	h	DoF	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{H}(\mathbf{div};\Omega))}$ error	rate	$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$ error	rate	$\|\mathbf{e}_p\|_{\ell^\infty(0,T;\mathrm{L}^2(\Omega))}$ error	rate
32	0.3536	1,236	1.89e+00	–	1.18e-01	–	1.53e-03	0.000
128	0.1768	4,708	4.83e-01	1.967	1.42e-02	3.050	2.26e-04	2.753
512	0.0884	18,372	1.21e-01	1.995	1.69e-03	3.070	3.07e-05	2.882
2048	0.0442	72,580	3.03e-02	2.000	2.08e-04	3.026	3.99e-06	2.944
7200	0.0236	253,684	8.61e-03	2.001	3.15e-05	3.004	6.23e-07	2.955
20000	0.0141	702,804	3.09e-03	2.003	6.80e-06	3.000	1.40e-07	2.918

$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\eta}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$ error	rate	$\|\mathbf{e}_{\boldsymbol{\rho}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$ error	rate
2.76e-02	–	4.00e+01	–	2.76e-02	–	4.00e-03	–
7.01e-03	1.977	8.48e+00	2.237	7.01e-03	1.977	8.50e-04	2.236
1.76e-03	1.994	1.28e+00	2.732	1.76e-03	1.994	1.28e-04	2.729
4.40e-04	1.999	1.71e-01	2.898	4.40e-04	1.999	1.73e-05	2.886
1.25e-04	2.000	2.67e-02	2.956	1.25e-04	2.000	2.80e-06	2.902
4.51e-05	2.000	5.84e-03	2.975	4.51e-05	2.000	6.81e-07	2.766

Table 6.2: [Example 1, $k = 1$] Convergence history for the fully discrete scheme with AFW–Taylor–Hood approximations.

Example 2: Three-dimensional smooth exact solution

In this second example we consider the cube $\Omega = (0,1)^3$ and the exact solution

$$p = \exp(t)\,\cos(\pi x)\exp(y+z)\,, \qquad \mathbf{u} = \exp(t)\begin{pmatrix} \sin(2\pi x)\cos(\pi y)\sin(\pi z) \\ -2\sin(\pi x)\cos(2\pi y)\sin(\pi z) \\ \sin(\pi x)\cos(\pi y)\sin(2\pi z) \end{pmatrix},$$

$$\text{and} \quad \boldsymbol{\eta} = \exp(t)\begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ 2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix},$$

where the source terms $\mathbf{f}_p$, $\mathbf{g}$, and g are determined by substituting this exact solution into (2.9).

The convergence history for a sequence of quasi-uniform tetrahedral meshes with $k = 0$ is reported in Table 6.3. The scheme exhibits optimal convergence: the poroelastic stress, structural velocity, the rotation rate, and the postprocessed variables converge with order $\mathcal{O}(h)$, whereas the fluid velocity and the pressure converge with order $\mathcal{O}(h^2)$. These rates agree with Theorems 4.3 and 5.3 and Lemma 5.4. Figure 6.2 displays several solutions obtained on a mesh with 10,368 tetrahedral elements and 305,681 `DoF`, using order $k = 0$ and final time $T = 10^{-3}$.

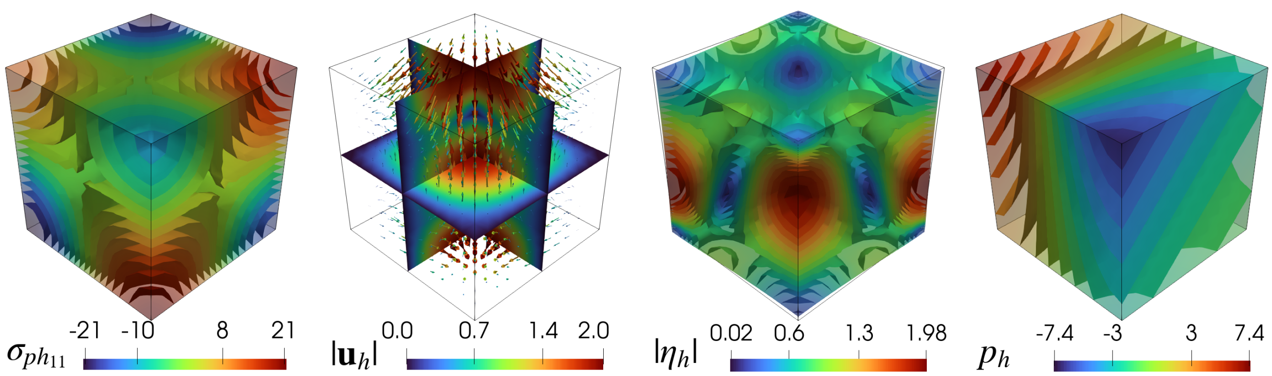

Figure 6.2: [Example 2]: Computed component (1,1) of the poroelasticity stress tensor, magnitude of the velocity and displacement, and pressure field.

Example 3: The cantilever bracket problem

Inspired by [35, Section 3] (see also [25]), we consider a cantilever bracket test to assess the ability of the proposed five-field mixed formulation to suppress the nonphysical pressure oscillations that typically arise in nearly incompressible and low-permeability regimes. The computational domain is the unit square $\Omega = (0,1)^2$. On the fluid side we impose the natural Neumann boundary condition $(\nabla\mathbf{u} - p\,\mathbb{I})\cdot\boldsymbol{\nu} = \mathbf{0}$. The structural displacement is fixed along the right edge ($x = 1$), whereas a uniform downward traction is applied on the top boundary ($y = 1$), namely $\mathbf{t} = (0, -b_y)$ with $b_y = 5.0 \times 10^3$. The left and bottom edges are traction-free. For this experiment we consider $k = 0$, set the time step to $\Delta t = 10^{-4}$, and, as usual, take the body forces $\mathbf{f}_p$, $\mathbf{g}$, and the scalar source g to be zero. We choose material parameters known to exacerbate locking:

$$s_0 = 0, \quad \alpha = 0.93, \quad \nu = 10^{-3}, \quad \mathtt{D} = 1.0\times 10^4, \quad \lambda = 1.4\times 10^8, \quad \mu = 3.6\times 10^5, \quad \rho_p = 2.5\times 10^3.$$

#elements	h	DoF	$\|\mathbf{e}_{\boldsymbol{\sigma}_p}\|_{\ell^\infty(0,T;\mathbb{H}(\mathbf{div};\Omega))}$		$\|\mathbf{e}_{\mathbf{u}}\|_{\ell^\infty(0,T;\mathbf{H}^1(\Omega))}$		$\|\mathbf{e}_p\|_{\ell^\infty(0,T;\mathrm{L}^2(\Omega))}$	
			error	rate	error	rate	error	rate
384	0.4330	12,393	2.23e+01	0	1.09e+00	0	7.59e-02	0
1296	0.2887	39,983	1.50e+01	0.975	5.06e-01	1.900	3.30e-02	2.057
6000	0.1732	178,515	9.01e+00	0.997	1.89e-01	1.922	1.17e-02	2.027
16464	0.1237	482,263	6.42e+00	1.004	9.79e-02	1.961	5.95e-03	2.012
34992	0.0962	1,016,123	4.99e+00	1.009	5.96e-02	1.977	3.59e-03	2.007

$\|\mathbf{e}_{\mathbf{u}_s}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\gamma}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\eta}}\|_{\ell^\infty(0,T;\mathbf{L}^2(\Omega))}$		$\|\mathbf{e}_{\boldsymbol{\rho}}\|_{\ell^\infty(0,T;\mathbb{L}^2(\Omega))}$	
error	rate	error	rate	error	rate	error	rate
2.58e-01	0	7.61e+01	0	2.37e-01	0	2.29e-01	0
1.73e-01	0.986	3.34e+01	2.030	1.60e-01	0.965	1.50e-01	1.048
1.02e-01	1.024	1.15e+01	2.083	9.68e-02	0.986	8.89e-02	1.020
7.19e-02	1.051	5.83e+00	2.030	6.93e-02	0.994	6.34e-02	1.007
5.51e-02	1.060	3.53e+00	1.992	5.39e-02	0.997	4.92e-02	1.004

Table 6.3: [Example 2, $k = 0$] Convergence history for the fully discrete scheme with AFW–Taylor–Hood approximations.

A classical signature of locking is the emergence of spurious, oscillatory (zigzag) pressure patterns in discrete solutions [35, Figure 5]. In our computations (see Figure 6.3, top), no such behavior is observed, the discrete pressure p_h remains smooth along vertical cuts. This indicates that the proposed five-field mixed formulation effectively eliminates locking-induced pressure oscillations. The bottom panel of Figure 6.3 reports the magnitude of the pressure at times $T = 0.0005$, 0.001, 0.003, and 0.005, obtained on a mesh with 1,800 triangular elements and 24,844 DoF, further confirming the stability and accuracy of the method in this challenging regime.

Example 4: The Mandel–Cryer effect

This numerical experiment reproduces the classical Mandel–Cryer effect (see [24, 32, 44]), a standard benchmark in poroelasticity that exhibits the transient pore-pressure overshoot during consolidation of a confined, fluid-saturated medium. The overshoot arises from the coupling between solid compressibility and pore-fluid diffusion: immediately after a sudden load is applied, the pore pressure increases before gradually dissipating as the system relaxes toward equilibrium.

We consider a spherical domain $\Omega = \{(x, y, z) \in \mathbb{R}^3 :\ x^2 + y^2 + z^2 < R^2\}$ of radius $R = 0.4\,\mathrm{m}$, representing a confined saturated sample. The outer boundary Γ_R is subjected to a uniform compressive surface traction $\mathbf{t} = -F\,\boldsymbol{\nu}$ applied instantaneously at $t = 0$, with $F = 10^3$ Pa. We assume $\mathbf{f}_p$, $\mathbf{g}$, and g to be zero, and consider a final time $T = 5\,\mathrm{s}$ with a time step $\Delta t = 10^{-2}\,\mathrm{s}$. The boundary is taken to be permeable, enforcing $p = 0$ on Γ_R. The material parameters used in this test are:

$$s_0 = 9.6 \times 10^{-11}, \qquad \alpha = 1.0, \qquad \nu = 10^{-3}\ \mathrm{Pa}\cdot\mathrm{s}, \qquad \mathtt{D} = 1.0 \times 10^8\ \mathrm{N}\cdot\mathrm{s/m}^4,$$

$$\lambda = 1.136 \times 10^5\ \mathrm{Pa}, \qquad \mu = 4.545 \times 10^5\ \mathrm{Pa}, \qquad \rho_p = 2.5 \times 10^3\ \mathrm{kg/m}^3.$$

Figure 6.4 displays the evolution of the pore pressure at 0.05 s, 0.5 s, and 1 s (top row), and at 2 s, 4 s, and 5 s (bottom row), computed on a mesh with 3,963 tetrahedral elements and 117,159 DoF.

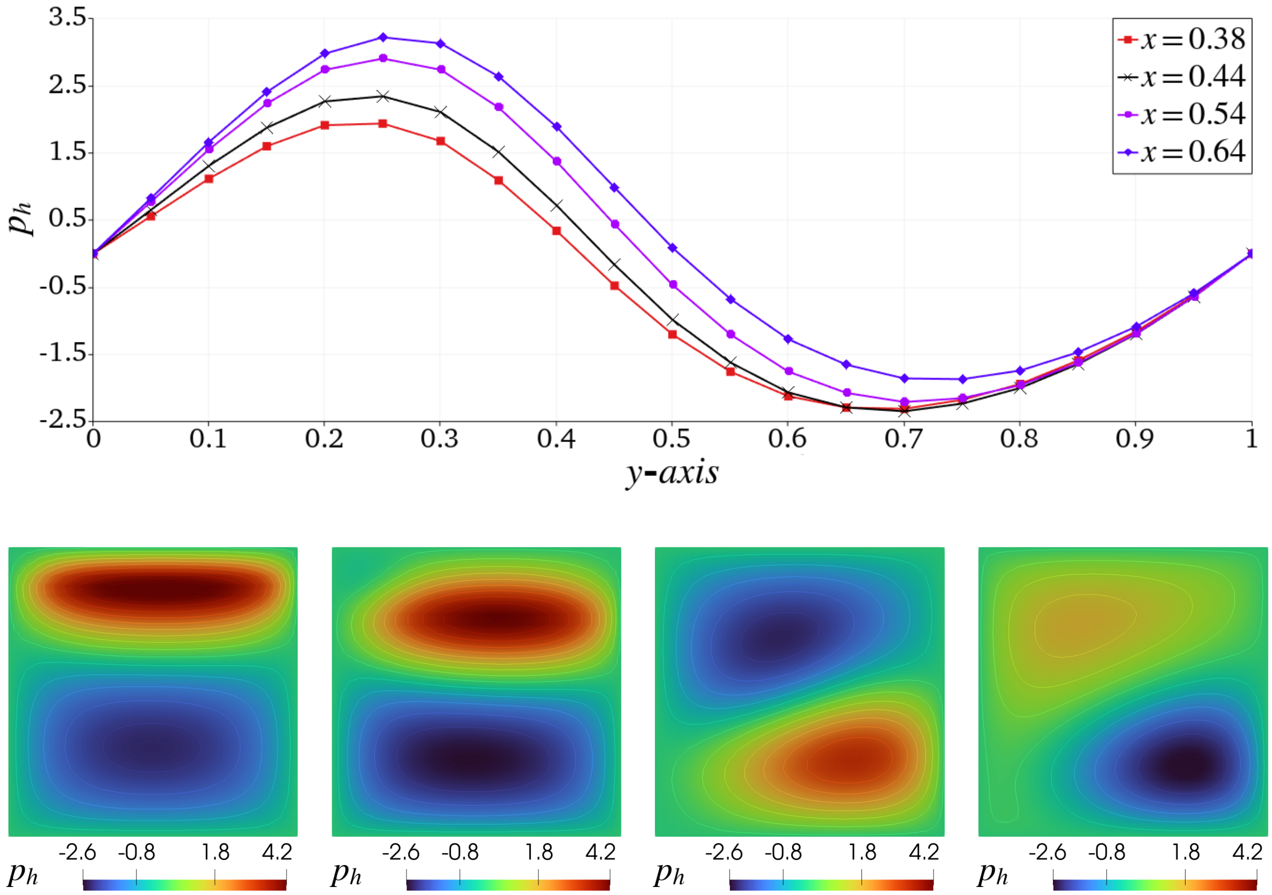

Figure 6.3: [Example 3]: Pressure field p_h along different x-lines at $t = 0.003$ (top), and at $T = 0.0005$, 0.001, 0.003, and 0.005 (bottom).

The pressure at the center of the sample exhibits the characteristic Mandel–Cryer response: an initial rise above the undrained value, followed by a gradual decay toward zero as fluid drains through the boundary. This overshoot emerges naturally in our simulations, without spurious oscillations or numerical instabilities. Moreover, the inclusion of the Brinkman viscous term produces a slightly slower dissipation compared with the classical Biot model, as expected due to the additional shear-related resistance. Altogether, these results show that the proposed formulation reproduces the transient poromechanical behavior and robustly captures the Mandel–Cryer phenomenon.

7 Conclusions

We have presented a five-field mixed formulation for the fully dynamic Biot–Brinkman model together with its conforming mixed finite element discretizations. The system couples second-order poroelasticity with a Brinkman-type momentum balance for the pore fluid, thereby capturing internal viscous diffusion and its interaction with skeleton deformation. By reformulating the model in terms of the poroelastic stress tensor, fluid velocity, pore pressure, structural velocity, and rotation rate, the approach enables the direct and accurate approximation of all primary fields. Moreover, the physical displacement and rotation tensor are recovered through a simple post-processing step. On the an-

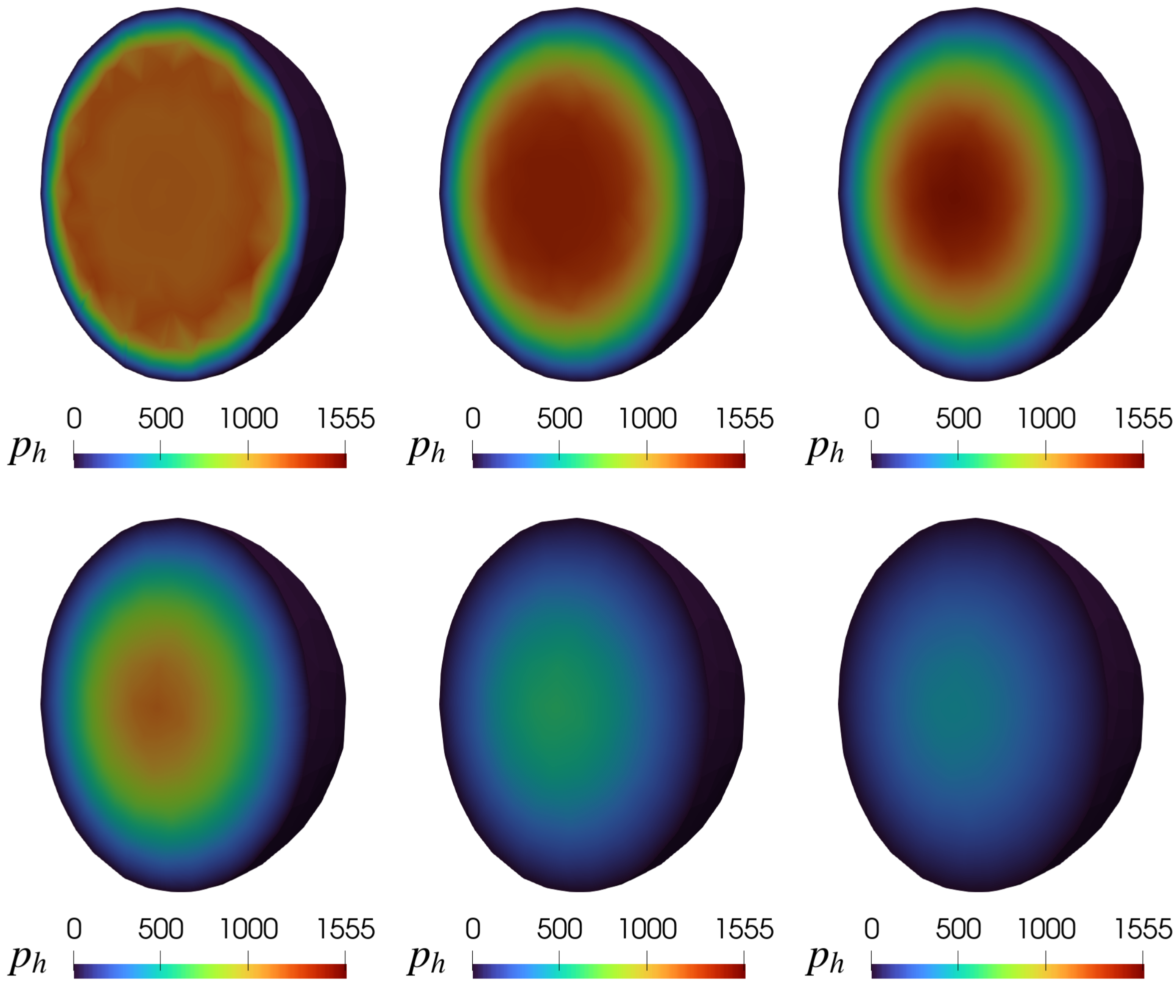

Figure 6.4: [Example 4]: Pressure field p_h at $T = 0.1, 0.5, 1$ (top) and $T = 2, 4, 5$ (bottom), highlighting the transient peak and subsequent relaxation.

alytical side, we established well-posedness of the weak formulation and derived stability estimates. We further analyzed a semidiscrete continuous-in-time scheme and a fully discrete scheme, proving stability bounds and optimal *a priori* error estimates. The numerical experiments corroborate the theoretical convergence rates, illustrate robustness over a wide range of parameters, including small storage coefficient s_0, and indicate the absence of spurious pressure oscillations typically associated with locking. Future directions include addressing more complex geometries through *a posteriori* error analysis and the associated adaptive refinement strategies. Subsequently, we will couple the present formulation with a free-flow model to study fluid-poroelastic structure interaction (FPSI), thereby extending the applicability of the method to a broader class of multiphysics problems.

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