UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Mixed-primal and fully-mixed formulations for the convection-diffusion-reaction system based upon Brinkman–Forchheimer equations

Alonso J. Bustos, Sergio Caucao,
Gabriel N. Gatica

PREPRINT 2025-13

SERIE DE PRE-PUBLICACIONES

Mixed-primal and fully-mixed formulations for the convection-diffusion-reaction system based upon Brinkman–Forchheimer equations*

Alonso J. Bustos† Sergio Caucao‡ Gabriel N. Gatica§

Abstract

We introduce and analyze new mixed formulations, within Banach spaces-based frameworks, for numerically solving the model given by the coupling of the Brinkman–Forchheimer equations with a convection-diffusion-reaction phenomenon. Specifically, for the former, we consider a pseudostress-velocity mixed formulation, whereas for the latter we analyze both primal and mixed approaches. In particular, for the mixed one the convection-diffusion-reaction part is reformulated by introducing the pseudodiffusion vector as an additional unknown, thus leading to a fully-mixed formulation of the coupling. On the other hand, in the mixed-primal setting, the Dirichlet boundary condition for the concentration is enforced through a suitable Lagrange multiplier. In contrast, this requirement is avoided with the fully-mixed approach, but an additional theoretical constraint on the data needs to be assumed. We establish the well-posedness of both formulations using a fixed-point strategy and prove the well-posedness of the uncoupled problems by relying on recently established solvability results for perturbed saddle-point problems in Banach spaces, together with the Banach–Nečas–Babuška theorem and the Babuška–Brezzi theory. Additionally, we provide a discrete analysis for both approaches under specific hypotheses on arbitrary finite element spaces. For instance, for each integer $k \geqslant 0$, we consider tensor and vector Raviart–Thomas subspaces of order k for the pseudostress and pseudodiffusion, respectively, along with piecewise polynomial subspaces of degree $\leqslant k$ for the velocity and concentration. This choice yields stable Galerkin schemes for the fully-mixed approach, for which optimal theoretical convergence rates are achieved. Finally, we illustrate the theoretical results through several numerical examples, comparing both approaches and testing the associated data assumptions.

Key words: Brinkman–Forchheimer, convection-diffusion-reaction, pseudostress-velocity formulation, fixed point theory, mixed finite element methods, *a priori* error analysis

Mathematics subject classifications (2020): 65N30, 65N12, 65N15, 35Q79, 80A19, 76R50, 76D07

---

*This research was partially supported by ANID-Chile through Centro de Modelamiento Matemático (FB210005), Anillo of Computational Mathematics for Desalination Processes (ACT210087), and Fondecyt project 1250937; by Grupo de Investigación en Análisis Numérico y Cálculo Científico (GIANuC^2), Universidad Católica de la Santísima Concepción; and by Centro de Investigación en Ingeniería Matemática (CI^{2}MA), Universidad de Concepción.

†CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `albustos2020@udec.cl`.

‡GIANuC^2 and Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, email: `scaucao@ucsc.cl`.

§CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: `ggatica@ci2ma.udec.cl`.

1 Introduction

The transport of chemical species in a saturated porous medium often involves complex interactions between fluid flow, pressure distribution, and reactive processes occurring within the porous structure. These coupled phenomena play a key role in a wide range of applications, including groundwater contamination, reactive filtration, catalytic reactors, biomedical flows, and enhanced oil recovery. Accurate modeling and numerical simulation of such systems are essential for process optimization, environmental protection, and risk assessment. Over the years, various mathematical models have been developed to capture different aspects of these flows, with considerable focus placed on coupling the Stokes (or Brinkman) model with convection-diffusion transport. However, models based on Darcy or Stokes flow may fail to adequately represent the behavior of the fluid in highly porous media or at moderate-to-high Reynolds numbers. To address these limitations, the Brinkman–Forchheimer equations have been introduced as a generalization that incorporates both viscous effects and inertial corrections (see, e.g., [19], [26], [18], and [17]). On the transport side, the evolution of chemical species can be more accurately described by a convection-diffusion-reaction (CDR) equation (see, e.g., [20], [8], [33]), which accounts for advective transport by the fluid, molecular diffusion, and local reaction kinetics. Based on the preceding discussion, the present work focuses on the analysis and numerical simulation of a coupled flow and transport system, where the Brinkman–Forchheimer equations govern the velocity field, which in turn drives the evolution of the concentration governed by a CDR equation.

Regarding the literature, several works address the mathematical and numerical analysis of coupled systems involving Stokes (Brinkman or Darcy–Forchheimer) flow and transport (or CDR) equations. To begin, [1] proposed and analyzed an augmented mixed formulation for the fluid equations combined with a standard primal scheme for the transport equation. This approach was later extended in [2] to strongly coupled flow and transport systems, modeled by the Brinkman problem with variable viscosity expressed through Cauchy pseudo-stresses and the bulk velocity of the mixture, along with a nonlinear advection-diffusion equation representing the transport of the solids volume fraction. Additionally, [7] established the existence of solutions for a related model describing chemically reacting non-Newtonian fluids. More recently, [3] analyzed a flow-transport interaction model in a porous-fluidic domain by employing techniques developed in [1] and [2]. This model considers a highly permeable medium where the flow of an incompressible viscous fluid is governed by the Brinkman equations formulated in terms of vorticity, velocity, and pressure, alongside a porous medium where Darcy's law describes the fluid motion via filtration velocity and pressure. Furthermore, an augmented fully mixed variational formulation for the model initially introduced in [1] was proposed and studied in [30], where a dual-mixed method combined with an augmentation technique was employed for both the Stokes and transport equations. In [33], the authors investigated the coupling of the CDR problem with Darcy–Forchheimer flow, considering a nonlinear external force dependent on concentration. They proved existence and uniqueness of solutions using a Galerkin method and developed a finite element numerical scheme accompanied by optimal a priori error estimates. On the other hand, we highlight [4] and [5], where non-augmented mixed-primal and fully mixed formulations for the coupled problems analyzed in [1] and [30], respectively, were introduced and studied within Banach space frameworks. We conclude by mentioning [9], where the authors analyze the convective Brinkman–Forchheimer equations coupled with a nonlinear transport phenomenon. This approach relies on the incorporation of the fluid velocity gradient, the incomplete nonlinear fluid pseudostress, the concentration gradient, the total (diffusive plus advective) flux of the concentration, as well as the velocity and the concentration themselves, as auxiliary variables, leading to a Banach spaces-based fully-mixed formulation.

The purpose of this work is to develop and analyze mixed formulations within an appropriate Banach space framework for the coupling of the Brinkman–Forchheimer and CDR equations, as well as to study suitable numerical discretizations. Motivated by [23], [22], [15], [18], and [12], we propose and analyze a pseudostress-velocity mixed formulation for the Brinkman–Forchheimer equations. In turn, for the CDR equation, we consider two distinct strategies. First, we formulate the coupled problem using a mixed-primal approach, as in [22], but without employing any augmentation procedure. Next, similarly to [15], we reformulate the CDR equation by introducing the pseudodiffusion vector as an additional unknown, resulting in a fully-mixed formulation of the coupled problem within a complete Banach space framework. In the mixed-primal approach, the Dirichlet boundary condition for the concentration is enforced via a suitable Lagrange multiplier. The fully-mixed formulation, by contrast, avoids this requirement but entails an additional assumption on the data. Following the ideas in [24], [22], and [12], we combine fixed-point arguments, the abstract results from [23], the Banach–Nečas–Babuška theorem, the Babuška–Brezzi theory, small data assumptions, and the Banach fixed-point theorem to establish existence and uniqueness of solutions for both formulations. Additionally, we perform a discrete analysis of both approaches under specific assumptions on general finite element spaces. In particular, for each integer $k \geqslant 0$, we consider tensor and vector Raviart–Thomas subspaces of order k for the pseudostress and pseudodiffusion, respectively, along with piecewise polynomial subspaces of degree $\leqslant k$ for the velocity and concentration. This choice yields stable Galerkin schemes for the fully-mixed approach, for which optimal theoretical convergence rates are achieved. Analogously, optimal convergence rates are also obtained for the mixed-primal approach when using continuous piecewise polynomial subspaces of degree $k+1$ for the concentration.

This work is organized as follows. The remainder of this section introduces the standard notation and functional spaces used throughout the paper. In Section 2, we present the model problem. Section 3 is dedicated to the derivation and analysis of the mixed-primal variational formulation in Banach spaces. We establish the well-posedness of both the continuous problem and the corresponding Galerkin scheme, applying the discrete counterpart of the continuous theory to prove existence, uniqueness, and *a priori* error estimates for general discrete spaces. Convergence rates are then obtained by considering specific finite element subspaces. Section 4 focuses on the fully-mixed variational formulation and its associated Galerkin scheme. In a similar way, we prove their well-posedness and derive convergence rates based on suitable choices of finite element spaces. Finally, Section 5 illustrates the performance of the proposed methods through numerical examples in both 2D and 3D, including test cases with and without manufactured solutions, validating the accuracy and flexibility of the Banach-space-based mixed finite element methods and comparing the numerical approaches.

Preliminary notations

Let $\Omega \subset \mathrm{R}^n, n \in \{2,3\}$, be a bounded domain with polyhedral boundary Γ, and let $\mathbf{n}$ be the outward unit normal vector on Γ. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$ and Sobolev spaces $\mathrm{W}^{s,p}(\Omega)$, with $s \in \mathrm{R}$ and $p > 1$, whose corresponding norms, either for the scalar, vectorial, or tensorial case, are denoted by $\|\cdot\|_{0,p;\Omega}$ and $\|\cdot\|_{s,p;\Omega}$, respectively. In particular, given a non-negative integer m, $\mathrm{W}^{m,2}(\Omega)$ is also denoted by $\mathrm{H}^m(\Omega)$, and the notations of its norm and seminorm are simplified to $||\cdot||_{m,\Omega}$ and $|\cdot|_{m,\Omega}$, respectively. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the space of traces of functions of $\mathrm{H}^1(\Omega)$, and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. On the other hand, given any generic scalar functional space S, we let $\mathbf{S}$ and $\mathbb{S}$ be the corresponding vectorial and tensorial counterparts, whereas $\|\cdot\|$, with no subscripts, will be employed for the norm of any element or operator whenever there is no confusion about the space to which they belong. Also, $|\cdot|$ denotes the Euclidean norm in both R^n and $\mathrm{R}^{n\times n}$, and as usual, $\mathbb{I}$ stands for the identity tensor in $\mathrm{R}^{n\times n}$. In turn, for any vector field $\mathbf{v} = (v_i)_{i=1,n}$, we set the gradient and divergence, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j} \right)_{i,j=1,n} \quad \text{and} \quad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j} \,,$$

whereas for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the deviatoric tensor, and the tensor inner product, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\tau_{ji})_{i,j=1,n} \,, \quad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii} \,, \quad \boldsymbol{\tau}^{\mathrm{d}} := \boldsymbol{\tau} - \frac{1}{n} \operatorname{tr}(\boldsymbol{\tau}) \, \mathbb{I} \,, \quad \text{and} \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij} \, \zeta_{ij} \,.$$

Furthermore, for each $t \in [1, +\infty)$ we introduce the Banach spaces

$$\mathbf{H}(\operatorname{div}_t; \Omega) := \Big\{ \mathbf{v} \in \mathbf{L}^2(\Omega) : \quad \operatorname{div}(\mathbf{v}) \in \mathrm{L}^t(\Omega) \Big\} \,, \quad \text{and} \tag{1.1}$$

$$\mathbb{H}(\mathbf{div}_t; \Omega) := \Big\{ \boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega) \Big\} \,, \tag{1.2}$$

endowed with the natural norms

$$\|\mathbf{v}\|_{\operatorname{div}_t;\Omega} := \|\mathbf{v}\|_{0,\Omega} + \|\operatorname{div}(\mathbf{v})\|_{0,t;\Omega} \quad \forall\, \mathbf{v} \in \mathbf{H}(\operatorname{div}_t; \Omega) \,, \quad \text{and}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t; \Omega) \,.$$

In addition, we consider the canonical injections $i_{p,q} : \mathrm{L}^p(\Omega) \to \mathrm{L}^q(\Omega)$ for all $p, q \in [1, +\infty)$, $p \geqslant q$, and $i_p : \mathrm{H}^1(\Omega) \to \mathrm{L}^p(\Omega)$ for all $p \in [1, +\infty)$ when $n = 2$, and for all $p \in [1, 6]$ when $n = 3$, which are continuous with norms depending on the domain. In particular, we have

$$\|i_{p,q}\| \leqslant |\Omega|^{(p-q)/(pq)} \,. \tag{1.3}$$

In turn, we let $\boldsymbol{i}_{p,q}$ and $\boldsymbol{i}_p$ be the corresponding vector counterparts of $i_{p,q}$ and i_p, respectively. Note that the norm of $\boldsymbol{i}_{p,q}$ also achieves the bound (1.3). Additionally, we recall that, proceeding as in [28, eq. (1.43), Section 1.3.4] (see also [11, Section 4.1] and [21, Section 3.1]), one can prove that for $t \in \begin{cases} (1, +\infty] \text{ in } \mathrm{R}^2 \,, \\ [\frac{6}{5}, +\infty] \text{ in } \mathrm{R}^3 \,, \end{cases}$ there holds

$$\langle \boldsymbol{\xi} \cdot \mathbf{n}, \varphi \rangle = \int_{\Omega} \Big\{ \boldsymbol{\xi} \cdot \nabla \varphi + \varphi \operatorname{div}(\boldsymbol{\xi}) \Big\} \quad \forall\, (\boldsymbol{\xi}, \varphi) \in \mathbf{H}(\operatorname{div}_t; \Omega) \times \mathrm{H}^1(\Omega) \,, \quad \text{and} \tag{1.4}$$

$$\langle \boldsymbol{\tau} \mathbf{n}, \mathbf{v} \rangle = \int_{\Omega} \Big\{ \boldsymbol{\tau} : \nabla \mathbf{v} + \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \Big\} \quad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}(\mathbf{div}_t; \Omega) \times \mathbf{H}^1(\Omega) \,, \tag{1.5}$$

where $\langle \cdot, \cdot \rangle$ in (1.4) and (1.5) denotes the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, and between $\mathbf{H}^{-1/2}(\Gamma)$ and $\mathbf{H}^{1/2}(\Gamma)$, respectively.

2 The model problem

We consider the physical process of fluid flow and reactive transport in a saturated porous medium occupying the region Ω. The fluid flow is governed by the Brinkman–Forchheimer equations (cf. [19], [25], [18]), characterized by the velocity $\mathbf{u}$ and the pressure p. In addition, following the approach in

[33], the scalar field ϕ denotes the concentration of a chemical species transported by the fluid and modeled by a convection-diffusion-reaction equation. As a result, the coupled model of interest is described by the following system of partial differential equations:

$$-\mathbf{div}\big(\nu\,\nabla\mathbf{u}\big) \,+\, \mathtt{D}\,\mathbf{u} \,+\, \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} \,+\, \nabla p = \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \tag{2.1a}$$

$$\mathrm{div}(\mathbf{u}) = f \quad \text{in} \quad \Omega\,, \tag{2.1b}$$

$$-\kappa\,\Delta\,\phi \,+\, \mathbf{u}\cdot\nabla\phi \,+\, \eta\,\phi = g \quad \text{in} \quad \Omega\,, \tag{2.1c}$$

where $\nu > 0$ is the Brinkman coefficient (or effective viscosity), $\mathtt{D} > 0$ is the Darcy coefficient, $\mathtt{F} > 0$ is the Forchheimer coefficient, $\rho \in [3,4]$ is a given number, $\kappa > 0$ is the diffusion coefficient, and $\eta > 0$ is the reaction coefficient. We assume that ν, $\mathtt{D}$, and $\mathtt{F}$ may vary spatially and are bounded in terms of positive constants $\nu_0, \nu_1, \mathtt{D}_0, \mathtt{D}_1, \mathtt{F}_0$, and $\mathtt{F}_1$ satisfying

$$\nu_0 \,\leqslant\, \nu(\mathbf{x}) \,\leqslant\, \nu_1\,, \quad \mathtt{D}_0 \,\leqslant\, \mathtt{D}(\mathbf{x}) \,\leqslant\, \mathtt{D}_1\,, \quad \text{and} \quad \mathtt{F}_0 \,\leqslant\, \mathtt{F}(\mathbf{x}) \,\leqslant\, \mathtt{F}_1\,, \qquad \forall\,\mathbf{x}\in\Omega\,. \tag{2.2}$$

The source terms f and g belong to suitable function spaces to be specified later. In addition, the external force $\mathbf{f}(\phi)$ is defined by

$$\mathbf{f}(\phi) \,:=\, -(\phi - \phi_{\mathtt{r}})\,\mathbf{g}\,, \tag{2.3}$$

where $\mathbf{g}$ represents the gravitational acceleration of potential type, and $\phi_{\mathtt{r}}$ is the reference concentration of the solute.

Equations (2.1) are complemented with Dirichlet boundary conditions for the velocity and concentration fields, namely,

$$\mathbf{u} \,=\, \mathbf{u}_{\mathrm{D}} \quad \text{and} \quad \phi \,=\, \phi_{\mathrm{D}} \quad \text{on} \quad \Gamma\,, \tag{2.4}$$

with given data $\mathbf{u}_{\mathrm{D}} \in \mathbf{H}^{1/2}(\Gamma)$ and $\phi_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$. Due to condition (2.1b) and the Dirichlet boundary condition for $\mathbf{u}$, the datum $\mathbf{u}_{\mathrm{D}}$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_{\mathrm{D}}\cdot\mathbf{n} \,=\, \int_\Omega f\,. \tag{2.5}$$

Additionally, to ensure uniqueness of the pressure p in (2.1a), we seek p in the space

$$\mathrm{L}^2_0(\Omega) \,:=\, \left\{ q\in\mathrm{L}^2(\Omega) : \quad \int_\Omega q = 0 \right\}\,.$$

3 The mixed-primal approach

In this section, we derive a mixed-primal formulation for the model problem (2.1). To this end, we introduce a pseudostress-velocity mixed formulation for the Brinkman–Forchheimer equations (2.1a)–(2.1b), while a primal approach is employed for the convection-diffusion-reaction equation (2.1c). We then establish the well-posedness of the coupled system using a fixed-point strategy. Next, we present a Galerkin scheme, prove its well-posedness, and derive a Céa estimate. Finally, we introduce specific finite element spaces and establish convergence rates.

3.1 The continuous formulation

Following the approach in [18] (see also [12, 14, 15]), we first introduce the pseudostress tensor $\boldsymbol{\sigma}$ as an additional unknown, defined by

$$\boldsymbol{\sigma} \,:=\, \nu\,\nabla\mathbf{u} \,-\, p\,\mathbb{I} \quad \text{in} \quad \Omega\,. \tag{3.1}$$

Thus, by taking the matrix trace and using the fact that $\mathrm{tr}(\nu\nabla\mathbf{u}) = \nu\,\mathrm{div}(\mathbf{u}) = \nu f$ (cf. (2.1b)), along with the application of the deviatoric operator to $\boldsymbol{\sigma}$, we deduce from (3.1) that

$$p = -\frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}) + \frac{\nu}{n}\,f \quad \text{and} \quad \frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} = \nabla\mathbf{u} - \frac{1}{n}\,f\,\mathbb{I}\,. \tag{3.2}$$

We note that (3.2) is equivalent to the combination of (3.1) and (2.1b). Next, by taking the divergence of $\boldsymbol{\sigma}$, substituting it into (2.1a), and eliminating the unknown p, which is subsequently computed using the identity in (3.2), we obtain a system equivalent to (2.1)–(2.4): Find $\mathbf{u}$, $\boldsymbol{\sigma}$, and ϕ in suitable spaces to be indicated below, such that

$$-\mathbf{div}(\boldsymbol{\sigma}) + \mathtt{D}\,\mathbf{u} + \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u} = \mathbf{f}(\phi) \quad \text{in} \quad \Omega\,, \tag{3.3a}$$

$$\frac{1}{\nu}\,\boldsymbol{\sigma}^{\mathrm{d}} - \nabla\mathbf{u} = -\frac{1}{n}\,f\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{3.3b}$$

$$-\kappa\,\Delta\,\phi + \mathbf{u}\cdot\nabla\phi + \eta\,\phi = g \quad \text{in} \quad \Omega\,, \tag{3.3c}$$

$$\mathbf{u} = \mathbf{u}_{\mathrm{D}}, \quad \phi = \phi_{\mathrm{D}} \quad \text{on} \quad \Gamma\,, \tag{3.3d}$$

$$\int_\Omega \Big\{\mathrm{tr}(\boldsymbol{\sigma}) - \nu\,f\Big\} = 0. \tag{3.3e}$$

We remark that the constraint $p \in \mathrm{L}^2_0(\Omega)$ is equivalently enforced by equation (3.3e).

We now proceed with the derivation of the variational formulation for our mixed-primal system (3.3). We begin with the Brinkman-Forchheimer part by testing (3.3a) against a vector field $\mathbf{v}$, formally obtaining

$$\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) - \int_\Omega \mathtt{D}\,\mathbf{u}\cdot\mathbf{v} - \int_\Omega \mathtt{F}\,|\mathbf{u}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v} = -\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\,. \tag{3.4}$$

Regarding the Forchheimer term, given by the third expression in (3.4), we observe that it can be bounded directly by applying Hölder's inequality twice and invoking the boundedness of $\mathtt{F}$ (cf. (2.2)), thereby obtaining

$$\left|\int_\Omega \mathtt{F}\,|\mathbf{z}|^{\rho-2}\,\mathbf{u}\cdot\mathbf{v}\right| \leqslant \mathtt{F}_1\,\left\{\int_\Omega |\mathbf{z}|^{\ell(\rho-2)}\,|\mathbf{u}|^{\ell}\right\}^{1/\ell}\,\|\mathbf{v}\|_{0,j;\Omega} \leqslant \mathtt{F}_1\,\|\mathbf{z}\|^{\rho-2}_{0,\ell(\rho-1);\Omega}\,\|\mathbf{u}\|_{0,\ell(\rho-1);\Omega}\,\|\mathbf{v}\|_{0,j;\Omega}\,,$$

where $j,\ell \in (1,+\infty)$ are Hölder conjugates to each other, meaning that $1/j + 1/\ell = 1$. Here, we introduced the field $\mathbf{z}$, which will be used to handle the nonlinearity of this term. Further details are provided in Section 3.2. Naturally, if $\mathbf{z} = \mathbf{u}$, we recover the original term given in (3.4). For this reason, we assume that both fields belong to the same space. We may continue our analysis by considering arbitrary values of j and ℓ, which leads to the requirement that $\mathbf{z},\mathbf{u} \in \mathbf{L}^{\ell(\rho-1)}(\Omega)$ and $\mathbf{v} \in \mathbf{L}^{j}(\Omega)$. However, in order to derive a formulation that yields a classical Galerkin method with symmetry in the function spaces, we simplify our setting by assuming that $\ell(\rho-1) = j$. In this way, since j and ℓ are conjugates to each other, we discover that $j = \rho \in [3,4]$ and $\ell = \rho/(\rho-1) \in [4/3, 3/2]$. Consequently, we require $\mathbf{z},\mathbf{u},\mathbf{v} \in \mathbf{L}^{\rho}(\Omega)$. Having established these function spaces, we observe that the second term of (3.4) is finite due to the boundedness of $\mathtt{D}$ (cf. (2.2)), the Cauchy–Schwarz inequality and the Sobolev embedding of $\mathbf{L}^{\rho}(\Omega)$ into $\mathbf{L}^2(\Omega)$ (cf. (1.3)) since $\rho > 2$. Moreover, the first term of (3.4) is well-defined provided that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{\ell}(\Omega)$, so we require $\boldsymbol{\sigma} \in \mathbb{H}(\mathbf{div}_{\ell};\Omega)$ (cf. (1.2)). On the other hand, recalling the definition of $\mathbf{f}(\phi)$ (cf. (2.3)), and applying the Cauchy-Schwarz and Hölder inequalities, the later with conjugate indexes $\frac{\rho}{\rho-2}$ and $\frac{\rho}{2}$, we find that the term on the right-hand side of (3.4) is bounded as

$$\begin{aligned}\left|\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\right| &\leqslant \|(\phi-\phi_{\mathbf{r}})\,\mathbf{v}\|_{0,\Omega}\,\|\mathbf{g}\|_{0,\Omega} \leqslant \|\phi-\phi_{\mathbf{r}}\|_{0,2\rho/(\rho-2);\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega}\,\|\mathbf{g}\|_{0,\Omega} \\ &\leqslant \big(\|\phi\|_{0,s;\Omega}+\|\phi_{\mathbf{r}}\|_{0,s;\Omega}\big)\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega}\,,\end{aligned} \tag{3.5}$$

where $s := 2\rho/(\rho-2) \in [4,6]$. Thus, we consider the data $\phi_{\mathbf{r}} \in \mathrm{L}^s(\Omega)$ and $\mathbf{g} \in \mathbf{L}^2(\Omega)$. While it would be sufficient to seek $\phi \in \mathrm{L}^s(\Omega)$, we shall see below that this unknown must instead be sought in $\mathrm{H}^1(\Omega)$. This is consistent with (3.5), since, by invoking the continuous embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^s(\Omega)$, we find that

$$\left|\int_\Omega \mathbf{f}(\phi)\cdot\mathbf{v}\right| \leqslant \big(\|i_s\|\,\|\phi\|_{1,\Omega}+\|\phi_{\mathbf{r}}\|_{0,s;\Omega}\big)\,\|\mathbf{g}\|_{0,\Omega}\,\|\mathbf{v}\|_{0,\rho;\Omega}\,.$$

Now, since $\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega)$, we deduce from (3.3b) that $\nabla\mathbf{u} \in \mathbb{L}^2(\Omega)$, provided that the datum f belongs to $\mathrm{L}^2(\Omega)$ as well. Moreover, since $\mathbf{u}$ lies in $\mathbf{L}^\rho(\Omega)$, which is continuously embedded into $\mathbf{L}^2(\Omega)$ for $\rho \in [3,4]$, it follows that $\mathbf{u} \in \mathbf{H}^1(\Omega)$. Consequently, we multiply equation (3.3b) by a test function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, where t lies in a suitable range that allows us to integrate by parts according to (1.5), thereby obtaining

$$\int_\Omega \frac{1}{\nu}\boldsymbol{\sigma}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}} + \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) = \langle \boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_{\mathrm{D}}\rangle - \frac{1}{n}\int_\Omega f\,\mathrm{tr}(\boldsymbol{\tau})\,. \tag{3.6}$$

Recalling that $\ell \in [4/3, 3/2]$ is the Hölder conjugate of ρ, it suffices to seek $\mathbf{u}$ in $\mathbf{L}^\rho(\Omega)$ and set $t = \ell$, so that every term in (3.6) is well-defined. On the other hand, we now consider the decomposition $\mathbb{H}(\mathbf{div}_\ell;\Omega) = \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \oplus \mathrm{R}\,\mathbb{I}$, with

$$\mathbb{H}_0(\mathbf{div}_\ell;\Omega) := \left\{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_\ell;\Omega) : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\},$$

which implies, in particular, that there exist unique components $\boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_\ell;\,\Omega)$ and $d_{\boldsymbol{\sigma}} \in \mathrm{R}$ such that $\boldsymbol{\sigma} = \boldsymbol{\sigma}_0 + d_{\boldsymbol{\sigma}}\mathbb{I}$. Moreover, employing the uniqueness condition for p (cf. (3.3e)), we deduce that $d_{\boldsymbol{\sigma}}$ can be computed explicitly as

$$d_{\boldsymbol{\sigma}} = \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = \frac{1}{n\,|\Omega|}\int_\Omega \nu\,f\,, \tag{3.7}$$

so that, in order to complete $\boldsymbol{\sigma}$, it would only remain to find $\boldsymbol{\sigma}_0$. In this regard, we notice that (3.4) and (3.6) remain unaltered if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$, and hence from now on we simply redefine $\boldsymbol{\sigma} := \boldsymbol{\sigma}_0$ and seek $\boldsymbol{\sigma} \in \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$. The original $\boldsymbol{\sigma}$ can then be recovered through post-processing by employing the aforementioned decomposition and (3.7). Furthermore, invoking the compatibility condition (cf. (2.5)), we note that equation (3.6) is trivially satisfied for all $\boldsymbol{\tau} \in \mathrm{R}\,\mathbb{I}$. Therefore, we may restrict the test space from $\mathbb{H}(\mathbf{div}_\ell;\Omega)$ to $\mathbb{H}_0(\mathbf{div}_\ell;\Omega)$.

Now we aim to derive a primal formulation for the convection-diffusion-reaction equation (cf. (3.3c)). To this end, following the approach of [22], we seek $\phi \in \mathrm{H}^1(\Omega)$, test equation (3.3c) against $\psi \in \mathrm{H}^1(\Omega)$, integrate by parts, and introduce the additional unknown $\lambda := -\kappa\,\nabla\phi\cdot\mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$, obtaining

$$\kappa\int_\Omega \nabla\phi\cdot\nabla\psi + \int_\Omega (\mathbf{u}\cdot\nabla\phi)\,\psi + \eta\int_\Omega \phi\,\psi + \langle\lambda,\psi\rangle_\Gamma = \int_\Omega g\,\psi \qquad \forall\,\psi \in \mathrm{H}^1(\Omega)\,. \tag{3.8}$$

Regarding the well-definedness of the second term, we proceed similarly as for the derivation of (3.5), so that applying the Cauchy–Schwarz and Hölder inequalities along with the continuous embedding $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, we arrive at

$$\int_\Omega (\mathbf{u}\cdot\nabla\phi)\,\psi \leqslant \|\mathbf{u}\|_{0,\rho;\Omega}\,\|\nabla\phi\|_{0,\Omega}\,\|\psi\|_{0,s;\Omega} \leqslant \|i_s\|\,\|\mathbf{u}\|_{0,\rho;\Omega}\,\|\phi\|_{1,\Omega}\,\|\psi\|_{1,\Omega}\,. \tag{3.9}$$

In turn, assuming that the datum g belongs to $\mathrm{L}^2(\Omega)$, and recalling that $\phi, \psi \in \mathrm{H}^1(\Omega)$, the remaining terms in (3.8) are well defined. Finally, the Dirichlet condition for the concentration, given in (3.3d), is imposed weakly via

$$\langle \xi, \phi \rangle_\Gamma = \langle \xi, \phi_\mathrm{D} \rangle_\Gamma \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,. \tag{3.10}$$

Therefore, denoting from now on $\mathcal{H} := \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ and $\mathcal{Q} := \mathbf{L}^\rho(\Omega)$, and suitably grouping the equations (3.4), (3.6), (3.8) and (3.10), the aforementioned mixed-primal formulation reads: Find $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ and $(\phi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathcal{H}\,,\\
\mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) - \mathbf{c}_\mathbf{u}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) && \forall\, \mathbf{v} \in \mathcal{Q}\,,\\
a_\mathbf{u}(\phi, \psi) + b(\psi, \lambda) &= \mathrm{F}(\psi) && \forall\, \psi \in \mathrm{H}^1(\Omega)\,,\\
b(\phi, \xi) &= \mathrm{G}(\xi) && \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,,
\end{aligned} \tag{3.11}$$

where the bilinear forms $\mathbf{a} : \mathcal{H} \times \mathcal{H} \to \mathrm{R}$, $\mathbf{b} : \mathcal{H} \times \mathcal{Q} \to \mathrm{R}$, and $\mathbf{c}_\mathbf{z} : \mathcal{Q} \times \mathcal{Q} \to \mathrm{R}$, for each $\mathbf{z} \in \mathcal{Q}$, and the linear functionals $\mathbf{F} : \mathcal{H} \to \mathrm{R}$ and $\mathbf{G}_\varphi : \mathcal{Q} \to \mathrm{R}$, for each $\varphi \in \mathcal{H}$, are defined as

$$\mathbf{a}(\boldsymbol{\chi}, \boldsymbol{\tau}) := \int_\Omega \frac{1}{\nu} \boldsymbol{\chi}^\mathrm{d} : \boldsymbol{\tau}^\mathrm{d}\,, \qquad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) := \int_\Omega \mathbf{v} \cdot \mathbf{div}(\boldsymbol{\tau})\,, \tag{3.12}$$

$$\mathbf{c}_\mathbf{z}(\mathbf{w}, \mathbf{v}) := \int_\Omega \mathtt{D}\, \mathbf{w} \cdot \mathbf{v} + \int_\Omega \mathtt{F}\, |\mathbf{z}|^{\rho-2} \mathbf{w} \cdot \mathbf{v}\,, \tag{3.13}$$

$$\mathbf{F}(\boldsymbol{\tau}) := \langle \boldsymbol{\tau}\, \mathbf{n}, \mathbf{u}_\mathrm{D} \rangle - \frac{1}{n} \int_\Omega f\, \mathrm{tr}(\boldsymbol{\tau})\,, \qquad \mathbf{G}_\varphi(\mathbf{v}) := - \int_\Omega \mathbf{f}(\varphi) \cdot \mathbf{v}\,, \tag{3.14}$$

while the bilinear forms $a_\mathbf{z} : \mathrm{H}^1(\Omega) \times \mathrm{H}^1(\Omega) \to \mathrm{R}$ and $b : \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{R}$, and linear functionals $\mathrm{F} : \mathrm{H}^1(\Omega) \to \mathrm{R}$ and $\mathrm{G} : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{R}$, are given by

$$a_\mathbf{z}(\varphi, \psi) := \kappa \int_\Omega \nabla\varphi \cdot \nabla\psi + \int_\Omega (\mathbf{z} \cdot \nabla\varphi)\, \psi + \eta \int_\Omega \varphi\, \psi\,, \tag{3.15}$$

$$b(\psi, \xi) := \langle \xi, \psi \rangle_\Gamma\,, \quad \mathrm{F}(\psi) := \int_\Omega g\, \psi \quad \text{and} \quad \mathrm{G}(\xi) := \langle \xi, \phi_\mathrm{D} \rangle_\Gamma\,. \tag{3.16}$$

Now, for the stability properties of the bilinear forms and functionals associated with (3.11), we apply again the Cauchy–Schwarz and Hölder inequalities, the continuous embeddings (1.3), the continuity of the canonical and normal trace operators, and the data assumptions (2.2), to deduce that

$$|\mathbf{a}(\boldsymbol{\chi}, \boldsymbol{\tau})| \leqslant \frac{1}{\nu_0} \|\boldsymbol{\chi}\|_\mathcal{H} \|\boldsymbol{\tau}\|_\mathcal{H}\,, \qquad |\mathbf{b}(\boldsymbol{\tau}, \mathbf{v})| \leqslant \|\boldsymbol{\tau}\|_\mathcal{H} \|\mathbf{v}\|_\mathcal{Q}\,, \tag{3.17a}$$

$$|\mathbf{c}_\mathbf{z}(\mathbf{w}, \mathbf{v})| \leqslant \Big(\mathtt{D}_1\, |\Omega|^{(\rho-2)/\rho} + \mathtt{F}_1\, \|\mathbf{z}\|_{0,\rho;\Omega}^{\rho-2} \Big) \|\mathbf{w}\|_\mathcal{Q} \|\mathbf{v}\|_\mathcal{Q}\,, \tag{3.17b}$$

$$|\mathbf{F}(\boldsymbol{\tau})| \leqslant \Big(\max\{1, \|\boldsymbol{i}_\rho\|\}\, \|\mathbf{u}_\mathrm{D}\|_{1/2,\Gamma} + \frac{1}{\sqrt{n}} \|f\|_{0,\Omega} \Big) \|\boldsymbol{\tau}\|_\mathcal{H}\,, \tag{3.17c}$$

$$|\mathbf{G}_\varphi(\mathbf{v})| \leqslant \|\mathbf{g}\|_{0,\Omega} \big(\|i_s\|\, \|\varphi\|_{1,\Omega} + \|\phi_\mathbf{r}\|_{0,s;\Omega} \big) \|\mathbf{v}\|_\mathcal{Q}\,, \tag{3.17d}$$

$$|a_\mathbf{z}(\varphi, \psi)| \leqslant \big(\kappa + \|i_s\|\, \|\mathbf{z}\|_{0,\rho;\Omega} + \eta \big) \|\varphi\|_{1,\Omega} \|\psi\|_{1,\Omega}\,, \tag{3.17e}$$

$$|b(\psi, \xi)| \leqslant \|\xi\|_{-1/2,\Gamma} \|\psi\|_{1,\Omega}\,, \qquad |\mathrm{F}(\psi)| \leqslant \|g\|_{0,\Omega} \|\psi\|_{1,\Omega}\,, \tag{3.17f}$$

$$\text{and} \quad |\mathrm{G}(\xi)| \leqslant \|\phi_\mathrm{D}\|_{1/2,\Gamma} \|\xi\|_{-1/2,\Gamma}\,. \tag{3.17g}$$

3.2 Solvability analysis

In order to establish the well-posedness of (3.11), we propose a fixed-point strategy. To this end, we first define the operator $\mathbf{S} : \mathcal{Q} \times \mathrm{H}^1(\Omega) \to \mathcal{Q}$ by $\mathbf{S}(\mathbf{z}, \varphi) := \mathbf{u}$, where, given $(\mathbf{z}, \varphi) \in \mathcal{Q} \times \mathrm{H}^1(\Omega)$, $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ denotes the unique solution, as will be shown below in Lemma 3.1, to the uncoupled Brinkman–Forchheimer component arising from the formulation (3.11) when $\mathbf{c_u}$ and $\mathbf{G}_\phi$ are replaced by $\mathbf{c_z}$ and $\mathbf{G}_\varphi$, respectively, that is

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathcal{H}\,, \\
\mathbf{b}(\boldsymbol{\sigma}, \mathbf{v}) - \mathbf{c_z}(\mathbf{u}, \mathbf{v}) &= \mathbf{G}_\varphi(\mathbf{v}) \qquad \forall\, \mathbf{v} \in \mathcal{Q}\,.
\end{aligned}
\tag{3.18}
$$

Equivalently, $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of

$$
\mathbf{A_z}\big((\boldsymbol{\sigma}, \mathbf{u}), (\boldsymbol{\tau}, \mathbf{v})\big) = \mathbf{R}_\varphi(\boldsymbol{\tau}, \mathbf{v}) \qquad \forall\, (\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{H} \times \mathcal{Q}\,, \tag{3.19}
$$

where

$$
\mathbf{A_z}\big((\boldsymbol{\chi}, \mathbf{w}), (\boldsymbol{\tau}, \mathbf{v})\big) := \mathbf{a}(\boldsymbol{\chi}, \boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau}, \mathbf{w}) + \mathbf{b}(\boldsymbol{\chi}, \mathbf{v}) - \mathbf{c_z}(\mathbf{w}, \mathbf{v})
$$

and

$$
\mathbf{R}_\varphi(\boldsymbol{\tau}, \mathbf{v}) := \mathbf{F}(\boldsymbol{\tau}) + \mathbf{G}_\varphi(\mathbf{v})\,,
$$

for all $(\boldsymbol{\chi}, \mathbf{w}), (\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{H} \times \mathcal{Q}$. Secondly, we define $\widetilde{\mathbf{S}} : \mathcal{Q} \to \mathrm{H}^1(\Omega)$ by $\widetilde{\mathbf{S}}(\mathbf{z}) := \phi$, where, given $\mathbf{z} \in \mathcal{Q}$, $(\phi, \lambda) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ is the unique solution, as will be confirmed below in Lemma 3.2, to the convection-diffusion-reaction part arising from (3.11) when $a_{\mathbf{u}}$ is replaced by $a_{\mathbf{z}}$, that is

$$
\begin{aligned}
a_{\mathbf{z}}(\phi, \psi) + b(\psi, \lambda) &= \mathrm{F}(\psi) \qquad \forall\, \psi \in \mathrm{H}^1(\Omega)\,, \\
b(\phi, \xi) &= \mathrm{G}(\xi) \qquad \forall\, \xi \in \mathrm{H}^{-1/2}(\Gamma)\,.
\end{aligned}
\tag{3.20}
$$

Finally, we introduce the operator $\mathbf{T} : \mathcal{Q} \to \mathcal{Q}$ defined by

$$
\mathbf{T}(\mathbf{z}) := \mathbf{S}(\mathbf{z}, \widetilde{\mathbf{S}}(\mathbf{z})) \qquad \forall\, \mathbf{z} \in \mathcal{Q}\,,
$$

and realize that solving (3.11) is equivalent to finding a fixed point of the operator $\mathbf{T}$, namely, seeking $\mathbf{u} \in \mathcal{Q}$ such that

$$
\mathbf{T}(\mathbf{u}) = \mathbf{u}\,. \tag{3.21}
$$

In what follows, we prove that the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ are well-defined, meaning that the problems (3.18) and (3.20) are well-posed. As a consequence, the operator $\mathbf{T}$ is well-defined as well.

We begin by applying [23, Theorem 3.4] to problem (3.18). Indeed, note that the null space of the operator $\boldsymbol{\tau} \mapsto \mathbf{b}(\boldsymbol{\tau}, \cdot) \in \mathcal{Q}'$ is given by

$$
\mathbb{V} := \Big\{ \boldsymbol{\tau} \in \mathcal{H} : \quad \mathbf{b}(\boldsymbol{\tau}, \mathbf{v}) = 0 \quad \forall\, \mathbf{v} \in \mathcal{Q} \Big\} = \Big\{ \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) = 0 \Big\}\,. \tag{3.22}
$$

In turn, from a slight modification of [28, Lemma 2.3], we have the existence of a positive constant c_1 such that

$$
\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,\ell;\Omega} \geqslant c_1\, \|\boldsymbol{\tau}\|_{0,\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_\ell; \Omega)\,,
$$

which implies $\|\boldsymbol{\chi}^{\mathrm{d}}\|_{0,\Omega} \geqslant c_1\, \|\boldsymbol{\chi}\|_{0,\Omega} = c_1\, \|\boldsymbol{\chi}\|_{\mathbf{div}_\ell;\Omega}$ for all $\boldsymbol{\chi} \in \mathbb{V}$. This inequality and the boundedness of ν (cf. (2.2)) allow us to infer that

$$
\sup_{0 \neq \boldsymbol{\tau} \in \mathbb{V}} \frac{\mathbf{a}(\boldsymbol{\chi}, \boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_\ell;\Omega}} = \sup_{0 \neq \boldsymbol{\tau} \in \mathbb{V}} \frac{\displaystyle\int_\Omega \frac{1}{\nu}\, \boldsymbol{\chi}^{\mathrm{d}} : \boldsymbol{\tau}^{\mathrm{d}}}{\|\boldsymbol{\tau}\|_{\mathbf{div}_\ell;\Omega}} \geqslant \frac{\|\boldsymbol{\chi}^{\mathrm{d}}\|_{0,\Omega}^2}{\nu_1\, \|\boldsymbol{\chi}\|_{\mathbf{div}_\ell;\Omega}} \geqslant \frac{c_1^2}{\nu_1}\, \|\boldsymbol{\chi}\|_{\mathbf{div}_\ell;\Omega}\,,
$$

for all $\boldsymbol{\chi} \in \mathbb{V}\backslash\{0\}$. Since the resulting inequality holds trivially for $\boldsymbol{\chi} = 0$, we have established the inf-sup condition for $\mathbf{a}$ (cf. (3.12)) required by [23, Theorem 3.4], namely,

$$\sup_{0\neq\boldsymbol{\tau}\in\mathbb{V}} \frac{\mathbf{a}(\boldsymbol{\chi},\boldsymbol{\tau})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_\ell;\Omega}} \geqslant \alpha\,\|\boldsymbol{\chi}\|_{\mathbf{div}_\ell;\Omega} \qquad \forall\,\boldsymbol{\chi}\in\mathbb{V}\,, \tag{3.23}$$

where $\alpha := c_1^2/\nu_1$. On the other hand, by extending the argument employed in [31, Lemma 2.9] to the tensorial case (see also [10, Lemma 3.3]), we establish the inf-sup condition for $\mathbf{b}$ (cf. (3.12)) that is needed by [23, Theorem 3.4], namely:

$$\sup_{0\neq\boldsymbol{\tau}\in\mathcal{H}} \frac{\mathbf{b}(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_\ell;\Omega}} \geqslant \beta\,\|\mathbf{v}\|_{0,\rho;\Omega} \qquad \forall\,\mathbf{v}\in\mathcal{Q}\,, \tag{3.24}$$

where β is a positive constant depending only on Ω. Finally, the bilinear forms $\mathbf{a}$ and $\mathbf{c_z}$ (cf. (3.12), (3.13)) are certainly symmetric and satisfy

$$\mathbf{a}(\boldsymbol{\tau},\boldsymbol{\tau}) \geqslant \frac{1}{\nu_1}\,\|\boldsymbol{\tau}^{\mathrm{d}}\|_{0,\Omega}^2 \geqslant 0 \quad \text{and} \quad \mathbf{c_z}(\mathbf{v},\mathbf{v}) \geqslant \mathtt{D}_0\,\|\mathbf{v}\|_{0,\Omega}^2 + \mathtt{F}_0\int_\Omega |\mathbf{z}|^{\rho-2}\,|\mathbf{v}|^2 \geqslant 0\,, \tag{3.25}$$

for all $\boldsymbol{\tau}\in\mathcal{H}$ and $\mathbf{v}\in\mathcal{Q}$, which says that $\mathbf{a}$ and $\mathbf{c_z}$ are positive semi-definite.

Consequently, the well-definedness of the operator $\mathbf{S}$ is stated as follows.

Lemma 3.1 *Given $\delta > 0$ and $(\mathbf{z},\varphi)\in\mathcal{Q}\times\mathrm{H}^1(\Omega)$ such that $\|\mathbf{z}\|_{0,\rho;\Omega}\leqslant\delta$, the problem (3.18) has a unique solution $(\boldsymbol{\sigma},\mathbf{u})\in\mathcal{H}\times\mathcal{Q}$, and, consequently, $\mathbf{S}(\mathbf{z},\varphi)$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending on δ, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, β, and $|\Omega|$, such that*

$$\|\mathbf{S}(\mathbf{z},\varphi)\|_{0,\rho;\Omega} \leqslant \|(\boldsymbol{\sigma},\mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}} \leqslant C_{\mathbf{S}}\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|\varphi\|_{1,\Omega} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\right)\Big\}. \tag{3.26}$$

Proof. Bearing in mind (3.23), (3.24), and the symmetry and positive semi-definiteness of $\mathbf{a}$ and $\mathbf{c_z}$ (see (3.25)), the existence of a unique solution for (3.18) follows from a straightforward application of [23, Theorem 3.4]. Moreover, (3.26) is obtained from the *a priori* bound established in [23, eq. (3.51) in Theorem 3.4] and the stability bounds (3.17a)–(3.17d). In particular, note that the bound for $\|\mathbf{c_z}\|$ provided by (3.17b) depends on $\mathtt{D}_1$, $|\Omega|$, ρ, $\mathtt{F}_1$, and δ. $\square$

Having proved the well-posedness of (3.18), the analysis from [23, Theorem 3.4] also gives us the inf-sup condition for $\mathbf{A_z}$. More precisely, given $\delta > 0$, there exists a constant $\alpha_{\mathbf{A}} > 0$, depending only on δ, ρ, ν_0, $\mathtt{D}_1$, $\mathtt{F}_1$, α, β, and $|\Omega|$, such that for each $\mathbf{z}\in\mathcal{Q}$ satisfying $\|\mathbf{z}\|_{0,\rho;\Omega}\leqslant\delta$, there holds

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathcal{H}\times\mathcal{Q}} \frac{\mathbf{A_z}\left((\boldsymbol{\chi},\mathbf{w}),(\boldsymbol{\tau},\mathbf{v})\right)}{\|(\boldsymbol{\tau},\mathbf{v})\|_{\mathcal{H}\times\mathcal{Q}}} \geqslant \alpha_{\mathbf{A}}\,\|(\boldsymbol{\chi},\mathbf{w})\|_{\mathcal{H}\times\mathcal{Q}} \qquad \forall\,(\boldsymbol{\chi},\mathbf{w})\in\mathcal{H}\times\mathcal{Q}\,. \tag{3.27}$$

In order to show next the well-posedness of (3.20), we need to invoke the classical Poincaré inequality, which establishes the existence of a positive constant c_P such that

$$|\psi|_{1,\Omega}^2 \geqslant c_P\,\|\psi\|_{1,\Omega}^2 \qquad \forall\,\psi\in\mathrm{H}_0^1(\Omega)\,. \tag{3.28}$$

In addition, and in contrast to Lemma 3.1, it is required that $\|\mathbf{z}\|_{0,\rho;\Omega}$ be bounded by a specific constant, which, in turn, depends on c_P.

Lemma 3.2 *Given $\mathbf{z}\in\mathcal{Q}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega}\leqslant\delta_0 := \frac{1}{2}\,\|i_s\|^{-1}\,\kappa\,c_P$, the problem (3.20) has a unique solution $(\phi,\lambda)\in\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$, and, consequently, $\widetilde{\mathbf{S}}(\mathbf{z})$ is well-defined. In addition, there exists a positive constant $C_{\widetilde{\mathbf{S}}}$, depending only on κ, η, c_P, and $|\Omega|$, such that*

$$\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1,\Omega} \leqslant \|(\phi,\lambda)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant C_{\widetilde{\mathbf{S}}}\Big\{\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,. \tag{3.29}$$

Proof. While this proof is a slight adaptation of that of [22, Lemma 3.4], we include the details here for the sake of completeness. In fact, let $\mathcal{B} : \mathrm{H}^1(\Omega) \to \mathrm{H}^{-1/2}(\Gamma)$ be the linear and bounded operator defined by $\langle \mathcal{B}(\psi), \xi\rangle_{-1/2,\Gamma} = b(\psi, \xi)$, for all $(\psi, \xi) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$, where $\langle\cdot,\cdot\rangle_{-1/2,\Gamma}$ stands for the inner product of $\mathrm{H}^{-1/2}(\Gamma)$. It can be readily shown that $\mathcal{B} = \mathcal{R}^*_{-1/2} \circ \gamma_0$, where $\mathcal{R}_{-1/2} : \mathrm{H}^{-1/2}(\Gamma) \to \mathrm{H}^{1/2}(\Gamma)$ denotes the Riesz operator, $\mathcal{R}^*_{-1/2}$ its adjoint, and $\gamma_0 : \mathrm{H}^1(\Omega) \to \mathrm{H}^{1/2}(\Gamma)$ the trace operator. Thus, being the composition of two surjective operators, $\mathcal{B}$ becomes surjective as well. Certainly, this is equivalent (cf. [28, Lemma 2.1]) to the existence of a positive constant $\widetilde{\beta}$, depending only on Ω, such that

$$\sup_{0\neq\psi\in\mathrm{H}^1(\Omega)} \frac{b(\psi,\xi)}{\|\psi\|_{1,\Omega}} \geqslant \widetilde{\beta}\,\|\xi\|_{-1/2,\Gamma} \qquad \forall\,\xi\in\mathrm{H}^{-1/2}(\Gamma)\,.$$

We now address the ellipticity of $a_{\mathbf{z}}$ in the null space of $\mathcal{B}$, which is easily seen to be given by $\mathrm{H}^1_0(\Omega)$. Indeed, employing the Cauchy–Schwarz and Hölder inequalities, along with (3.28) and the continuous embedding $i_s : \mathrm{H}^1(\Omega) \to \mathrm{L}^s(\Omega)$, we find that

$$a_{\mathbf{z}}(\psi,\psi) \,\geqslant\, \big(\kappa\, c_P \,-\, \|i_s\|\,\|\mathbf{z}\|_{0,\rho;\Omega}\big)\,\|\psi\|^2_{1,\Omega} \,\geqslant\, \frac{1}{2}\,\kappa\, c_P\,\|\psi\|^2_{1,\Omega} \qquad \forall\,\psi\in\mathrm{H}^1_0(\Omega)\,, \tag{3.30}$$

where we have used the assumption $\|\mathbf{z}\|_{0,\rho;\Omega} \leqslant \delta_0 = \frac{1}{2}\,\|i_s\|^{-1}\,\kappa\, c_P$. Consequently, by applying the Babuška–Brezzi theory in Hilbert spaces (see, for instance, [28, Theorem 2.3]), we deduce the well-posedness of (3.20), as well as the corresponding *a priori* estimate (3.29), using the stability bounds (3.17e)–(3.17g). Note, in particular, that the bound for $\|a_{\mathbf{z}}\|$ provided by (3.17e), and hence neither the a priori estimate (3.29), does not depend on $\|i_s\|$. ▫

Having established the well-definedness of the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, our next goal is to prove the well-posedness of (3.11), equivalently that (3.21) admits a unique solution, for which we aim below to apply the Banach fixed-point theorem. In fact, given δ and δ_0 as in Lemmas 3.1 and 3.2, we now consider $r \in (0, r_0]$, where

$$r_0 \,:=\, \min\big\{\delta,\delta_0\big\}\,, \tag{3.31}$$

and introduce the closed and convex subset of $\mathcal{Q}$ given by

$$\mathbf{W}(r) \,:=\, \Big\{\mathbf{z}\in\mathcal{Q}: \quad \|\mathbf{z}\|_{0,\rho;\,\Omega}\leqslant r\Big\}\,.$$

The following lemma proves that $\mathbf{T}$ maps $\mathbf{W}(r)$ into itself.

Lemma 3.3 *Let $r \in (0, r_0]$, with r_0 as in (3.31), and assume that the data satisfy*

$$C_{\mathbf{T}}\,\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\,\big(\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\,\Omega}\big)\Big\} \,\leqslant\, r\,, \tag{3.32}$$

where $C_{\mathbf{T}} := C_{\mathbf{S}}\,\max\big\{C_{\widetilde{\mathbf{S}}},1\big\}$, (cf. Lemmas 3.1 and 3.2). Then, $\mathbf{T}(\mathbf{W}(r)) \subset \mathbf{W}(r)$ and the restricted operator $\mathbf{T}|_{\mathbf{W}(r)} : \mathbf{W}(r) \to \mathbf{W}(r)$ is well-defined.

Proof. Given $\mathbf{z} \in \mathbf{W}(r)$, it is clear from Lemmas 3.1 and 3.2 that $\mathbf{T}(\mathbf{z})$ is well-defined. Moreover, employing the estimate (3.26) in combination with (3.29), we deduce that

$$\begin{aligned}\|\mathbf{T}(\mathbf{z})\|_{0,\rho;\Omega} = \|\mathbf{S}(\mathbf{z},\widetilde{\mathbf{S}}(\mathbf{z}))\|_{0,\rho;\Omega} &\leqslant C_{\mathbf{S}}\,\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\,\big(\|\widetilde{\mathbf{S}}(\mathbf{z})\|_{1,\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\,\Omega}\big)\Big\}\\ &\leqslant C_{\mathbf{T}}\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\,\big(\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\,\Omega}\big)\Big\}\,,\end{aligned}$$

which, thanks to assumption (3.32), implies that $\mathbf{T}(\mathbf{z}) \in \mathbf{W}(r)$, thus completing the proof. ▫

Hereafter, we simplify the notation by denoting $\mathbf{T}$ as the restricted operator $\mathbf{T}\big|_{\mathbf{W}(r)} : \mathbf{W}(r) \to \mathbf{W}(r)$. The two following results establish the Lipschitz continuity of the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$, respectively.

Lemma 3.4 *Let $r \in (0, r_0]$, with r_0 as in (3.31). Then, there exists a positive constant $L_{\mathbf{S}}$, depending on r, ρ, $\mathtt{F}_1$, $|\Omega|$, $C_{\mathbf{S}}$, $\|i_s\|$, and $\alpha_{\mathbf{A}}$, such that*

$$\|\mathbf{S}(\mathbf{z}_1,\varphi_1) - \mathbf{S}(\mathbf{z}_2,\varphi_2)\|_{0,\rho;\Omega} \leqslant L_{\mathbf{S}} \Big\{ C\big(\mathbf{u}_{\mathrm{D}}, f, \mathbf{g}, \phi_{\mathbf{r}}, \varphi_2\big) \, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega} + \|\mathbf{g}\|_{0,\Omega} \, \|\varphi_1 - \varphi_2\|_{1,\Omega} \Big\}, \qquad (3.33)$$

for all $(\mathbf{z}_1,\varphi_1), (\mathbf{z}_2,\varphi_2) \in \mathbf{W}(r) \times \mathrm{H}^1(\Omega)$, where

$$C\big(\mathbf{u}_{\mathrm{D}}, f, \mathbf{g}, \phi_{\mathbf{r}}, \varphi\big) := \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|\varphi\|_{1,\Omega} + \|\phi_{\mathbf{r}}\|_{1,\Omega}\right) \quad \forall\, \varphi \in \mathrm{H}^1(\Omega)\,.$$

Proof. Let $(\mathbf{z}_1,\varphi_1), (\mathbf{z}_2,\varphi_2) \in \mathbf{W}(r) \times \mathrm{H}^1(\Omega)$ such that $\mathbf{S}(\mathbf{z}_1,\varphi_1) = \mathbf{u}_1$ and $\mathbf{S}(\mathbf{z}_2,\varphi_2) = \mathbf{u}_2$, where, for each $i \in \{1,2\}$, $(\boldsymbol{\sigma}_i, \mathbf{u}_i) \in \mathcal{H} \times \mathcal{Q}$ is the unique solution of (3.18), or equivalently, the unique solution of (3.19), with the given $(\mathbf{z}_i, \varphi_i)$ instead of $\mathbf{z}$ and φ there. Thus, making use of (3.27) with $\mathbf{z} = \mathbf{z}_1$ and $(\boldsymbol{\chi}, \mathbf{w}) = (\boldsymbol{\sigma}_1, \mathbf{u}_1) - (\boldsymbol{\sigma}_2, \mathbf{u}_2)$, we obtain

$$\alpha_{\mathbf{A}} \, \|(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2)\|_{\mathcal{H}\times\mathcal{Q}} \leqslant \sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathcal{H}\times\mathcal{Q}} \frac{\mathbf{A}_{\mathbf{z}_1}\left((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v})\right)}{\|(\boldsymbol{\tau}, \mathbf{v})\|_{\mathcal{H}\times\mathcal{Q}}}\,. \qquad (3.34)$$

In turn, setting problem (3.19) with $(\mathbf{z}, \varphi)$ equal to both $(\mathbf{z}_1, \varphi_1)$ and $(\mathbf{z}_2, \varphi_2)$, and then subtracting the resulting equations, we obtain

$$\mathbf{A}_{\mathbf{z}_1}\big((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v})\big) = \mathbf{c}_{\mathbf{z}_1}(\mathbf{u}_2, \mathbf{v}) - \mathbf{c}_{\mathbf{z}_2}(\mathbf{u}_2, \mathbf{v}) + \mathbf{G}_{\varphi_1}(\mathbf{v}) - \mathbf{G}_{\varphi_2}(\mathbf{v})\,, \qquad (3.35)$$

for all $(\boldsymbol{\tau}, \mathbf{v}) \in \mathcal{H} \times \mathcal{Q}$. Next, from a slight modification of [12, Lemma 4.4], one deduces that there exists a positive constant $L_{\mathbf{c}}$, depending only on ρ, $\mathtt{F}_1$ and $|\Omega|$, such that

$$\big|\mathbf{c}_{\mathbf{z}_1}(\mathbf{u}_2, \mathbf{v}) - \mathbf{c}_{\mathbf{z}_2}(\mathbf{u}_2, \mathbf{v})\big| \leqslant L_{\mathbf{c}} \Big\{ \|\mathbf{z}_1\|_{0,\rho;\Omega} + \|\mathbf{z}_2\|_{0,\rho;\Omega} \Big\}^{\rho-3} \|\mathbf{u}_2\|_{0,\rho;\Omega} \, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega} \, \|\mathbf{v}\|_{0,\rho;\Omega}\,, \qquad (3.36)$$

for all $\mathbf{v} \in \mathcal{Q}$, whereas the definitions of $\mathbf{G}_\varphi$ and $\mathbf{f}$ (cf. (3.14), (2.3)), together with Hölder's inequality, allow us to deduce that

$$\big|\mathbf{G}_{\varphi_1}(\mathbf{v}) - \mathbf{G}_{\varphi_2}(\mathbf{v})\big| = \int_\Omega \big(\mathbf{f}(\varphi_1) - \mathbf{f}(\varphi_2)\big) \cdot \mathbf{v} \leqslant \|i_s\| \, \|\mathbf{g}\|_{0,\Omega} \, \|\varphi_1 - \varphi_2\|_{1,\Omega} \, \|\mathbf{v}\|_{0,\rho;\Omega}\,. \qquad (3.37)$$

Finally, substituting (3.35) back into (3.34), using the bounds given in (3.36) and (3.37), along with the fact that $\mathbf{z}_i \in \mathbf{W}(r)$ for each $i \in \{1,2\}$, and applying the *a priori* estimate (3.26) to $\mathbf{S}(\mathbf{z}_2, \varphi_2) = \mathbf{u}_2$, we derive (3.33) with $L_{\mathbf{S}} = \max\big\{L_{\mathbf{c}}\, C_{\mathbf{S}}\, (2r)^{\rho-3}, \|i_s\|\big\}/\alpha_{\mathbf{A}}$, thus completing the proof. □

Lemma 3.5 *Let $r \in (0, r_0]$, with r_0 as in (3.31). Then, there exists a positive constant $L_{\widetilde{\mathbf{S}}}$, depending only on κ, c_P, $C_{\widetilde{\mathbf{S}}}$, and $\|i_s\|$, such that*

$$\|\widetilde{\mathbf{S}}(\mathbf{z}_1) - \widetilde{\mathbf{S}}(\mathbf{z}_2)\|_{1,\Omega} \leqslant L_{\widetilde{\mathbf{S}}} \Big\{ \|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\} \, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega} \quad \forall\, \mathbf{z}_1,\, \mathbf{z}_2 \in \mathbf{W}(r)\,. \qquad (3.38)$$

Proof. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}(r)$ such that $\widetilde{\mathbf{S}}(\mathbf{z}_1) = \phi_1$ and $\widetilde{\mathbf{S}}(\mathbf{z}_2) = \phi_2$, where, for each $i \in \{1,2\}$, $(\phi_i, \lambda_i) \in \mathrm{H}^1(\Omega) \times \mathrm{H}^{-1/2}(\Gamma)$ is the unique solution of the problem (3.20), with the given $\mathbf{z}_i$ instead of $\mathbf{z}$ there. Then, subtracting both problems, we easily deduce that $b(\phi_1 - \phi_2, \xi) = 0$ for all $\xi \in \mathrm{H}^{-1/2}(\Gamma)$, and

$$a_{\mathbf{z}_1}(\phi_1, \psi) - a_{\mathbf{z}_2}(\phi_2, \psi) + b(\psi, \lambda_1 - \lambda_2) = 0 \quad \forall\, \psi \in \mathrm{H}^1(\Omega)\,.$$

Next, taking $\psi = \phi_1 - \phi_2$, we obtain the identity $a_{\mathbf{z}_1}(\phi_1, \phi_1 - \phi_2) = a_{\mathbf{z}_2}(\phi_2, \phi_1 - \phi_2)$, which, combined with the coercivity of the bilinear form $a_{\mathbf{z}_2}$ (cf. (3.30)), yields

$$\frac{1}{2}\, \kappa\, c_P \, \|\phi_1 - \phi_2\|_{1,\Omega}^2 \leqslant a_{\mathbf{z}_2}(\phi_1 - \phi_2, \phi_1 - \phi_2) = a_{\mathbf{z}_2}(\phi_1, \phi_1 - \phi_2) - a_{\mathbf{z}_1}(\phi_1, \phi_1 - \phi_2)\,,$$

and using the definition of the bilinear form $a_{\mathbf{z}}$ (cf. (3.15)) and (3.9), we deduce that

$$\begin{aligned}\frac{1}{2}\,\kappa\, c_P\,\|\phi_1-\phi_2\|_{1,\Omega}^2 &\leqslant \int_\Omega \big\{(\mathbf{z}_2-\mathbf{z}_1)\cdot\nabla\phi_1\big\}(\phi_1-\phi_2)\\ &\leqslant \|i_s\|\,\|\phi_1\|_{1,\Omega}\,\|\mathbf{z}_1-\mathbf{z}_2\|_{0,\rho;\,\Omega}\,\|\phi_1-\phi_2\|_{1,\Omega}\,.\end{aligned}\tag{3.39}$$

Finally, applying the *a priori* estimate (3.29) (cf. Lemma 3.2) to $\widetilde{\mathbf{S}}(\mathbf{z}_1)=\phi_1$ in (3.39), we arrive at (3.38), with $L_{\widetilde{\mathbf{S}}}=2\,\|i_s\|\,C_{\widetilde{\mathbf{S}}}/(\kappa\,c_P)$, and conclude the proof. □

The following lemma establishes that the operator $\mathbf{T}$ is indeed Lipschitz continuous.

Lemma 3.6 *Let $r\in(0,r_0]$, with r_0 as in (3.31). Then, there exists a positive constant $L_{\mathbf{T}}$, depending on $L_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$, and $L_{\widetilde{\mathbf{S}}}$, such that*

$$\begin{aligned}&\|\mathbf{T}(\mathbf{z}_1)-\mathbf{T}(\mathbf{z}_2)\|_{0,\rho;\,\Omega}\\ &\quad\leqslant L_{\mathbf{T}}\,\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|f\|_{0,\Omega}+\|\mathbf{g}\|_{0,\Omega}\big(\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}+\|\phi_{\mathbf{r}}\|_{0,s;\,\Omega}\big)\Big\}\,\|\mathbf{z}_1-\mathbf{z}_2\|_{0,\rho;\,\Omega}\,,\end{aligned}\tag{3.40}$$

for all $\mathbf{z}_1,\mathbf{z}_2\in\mathbf{W}(r)$.

Proof. First, given $\mathbf{z}_1,\mathbf{z}_2\in\mathbf{W}(r)$, we observe from (3.33) that

$$\begin{aligned}\|\mathbf{T}(\mathbf{z}_1)-\mathbf{T}(\mathbf{z}_2)\|_{0,\rho;\,\Omega} &= \|\mathbf{S}(\mathbf{z}_1,\widetilde{\mathbf{S}}(\mathbf{z}_1))-\mathbf{S}(\mathbf{z}_2,\widetilde{\mathbf{S}}(\mathbf{z}_2))\|_{0,\rho;\,\Omega}\\ &\leqslant L_{\mathbf{S}}\,\Big\{C\big(\mathbf{u}_{\mathrm{D}},f,\mathbf{g},\phi_{\mathbf{r}},\widetilde{\mathbf{S}}(\mathbf{z}_2)\big)\,\|\mathbf{z}_1-\mathbf{z}_2\|_{0,\rho;\Omega}+\|\mathbf{g}\|_{0,\Omega}\,\|\widetilde{\mathbf{S}}(\mathbf{z}_2)-\widetilde{\mathbf{S}}(\mathbf{z}_1)\|_{1,\Omega}\Big\}.\end{aligned}\tag{3.41}$$

In turn, using the estimate (3.29), we certainly have

$$C\big(\mathbf{u}_{\mathrm{D}},f,\mathbf{g},\phi_{\mathbf{r}},\widetilde{\mathbf{S}}(\mathbf{z}_2)\big)\leqslant \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|f\|_{0,\Omega}+\|\mathbf{g}\|_{0,\Omega}\,\Big\{C_{\widetilde{\mathbf{S}}}\,(\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma})+\|\phi_{\mathbf{r}}\|_{0,s;\,\Omega}\Big\}\,,$$

and using (3.38), the last term in (3.41) can be bounded as

$$\|\widetilde{\mathbf{S}}(\mathbf{z}_2)-\widetilde{\mathbf{S}}(\mathbf{z}_1)\|_{1,\Omega}\;\leqslant\; L_{\widetilde{\mathbf{S}}}\,\Big\{\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,\|\mathbf{z}_2-\mathbf{z}_1\|_{0,\rho;\,\Omega}\,.\tag{3.42}$$

Thus, replacing back (3.42) into (3.41), and performing simple algebraic manipulations, we get (3.40), with $L_{\mathbf{T}}:=L_{\mathbf{S}}\,\max\{1,C_{\widetilde{\mathbf{S}}}+L_{\widetilde{\mathbf{S}}}\}$. □

Theorem 3.7 *Let $r\in(0,r_0]$, with r_0 as in (3.31), and assume that the data satisfy (3.32) and*

$$L_{\mathbf{T}}\,\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|f\|_{0,\Omega}+\|\mathbf{g}\|_{0,\Omega}\,\big(\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}+\|\phi_{\mathbf{r}}\|_{0,s;\,\Omega}\big)\Big\}\;<\;1\,.\tag{3.43}$$

Then, there exists a unique $\mathbf{u}\in\mathbf{W}(r)$ such that $\mathbf{T}(\mathbf{u})=\mathbf{u}$, or equivalently, the problem (3.11) has a unique solution $(\boldsymbol{\sigma},\mathbf{u},\phi,\lambda)\in\mathcal{H}\times\mathcal{Q}\times\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$, with $\mathbf{u}\in\mathbf{W}(r)$. Moreover, there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$, such that

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}} &\leqslant \mathcal{C}_1\,\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|f\|_{0,\Omega}+\|\mathbf{g}\|_{0,\Omega}\,\Big(\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}+\|\phi_{\mathbf{r}}\|_{0,s;\,\Omega}\Big)\Big\}\,,\\ \textit{and}\qquad \|(\phi,\lambda)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} &\leqslant \mathcal{C}_2\,\Big\{\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}\,.\end{aligned}\tag{3.44}$$

Proof. By (3.32) and Lemma 3.3, we know that $\mathbf{T}:\mathbf{W}(r)\to\mathbf{W}(r)$ is well-defined. Since $\mathbf{W}(r)$ is a closed and convex subset of $\mathbf{L}^\rho(\Omega)$, it is a complete metric space. Moreover, since $\mathbf{T}$ is Lipschitz continuous with constant $L_{\mathbf{T}}$ (cf. Lemma 3.6), under the assumption (3.43) we conclude that $\mathbf{T}$ is a contraction. Therefore, by the Banach fixed-point theorem, the operator $\mathbf{T}$ admits a unique fixed point, or equivalently, the problem (3.11) has a unique solution. In addition, the *a priori* estimates (3.44) follows from Lemmas 3.1 and 3.2. We omit further details. □

3.3 The Galerkin scheme

In this section, we analyze a Galerkin scheme associated with the mixed-primal formulation (3.11). To do so, we first consider a regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, and set $h := \max\{h_K : K \in \mathcal{T}_h\}$. Additionally, let $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, H_h^{ϕ}, and H_h^{λ} be generic finite-dimensional subspaces of $\mathbb{H}(\mathbf{div}_\ell;\Omega)$, $\mathbf{L}^\rho(\Omega)$, $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$, respectively. Specific choices of these subspaces, satisfying suitable hypotheses to be introduced later in the discussion, will be described below. Finally, to obtain a conforming approximation setting, we also define the space $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$.

Under this notation, we introduce the Galerkin scheme associated with (3.11): Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\phi_h, \lambda_h) \in \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$ such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,\\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}_{\mathbf{u}_h}(\mathbf{u}_h, \mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,,\\
a_{\mathbf{u}_h}(\phi_h, \psi_h) + b(\psi_h, \lambda_h) &= \mathrm{F}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,,\\
b(\phi_h, \xi_h) &= \mathrm{G}(\xi_h) && \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,.
\end{aligned} \tag{3.45}$$

In order to address the solvability of (3.45), we adopt the discrete analogue of the fixed-point strategy employed in the continuous case (cf. Section 3.2). We first define the operator $\mathbf{S}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi} \to \mathbf{H}_h^{\mathbf{u}}$ by $\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \varphi_h) := \mathbf{u}_h$, where $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution, which will be confirmed below (cf. Lemma 3.8), of the uncoupled problem arising from the first two rows of (3.45), after replacing $(\mathbf{u}_h, \phi_h)$ by the given $(\mathbf{z}_h, \varphi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi}$, that is

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h, \mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,\\
\mathbf{b}(\boldsymbol{\sigma}_h, \mathbf{v}_h) - \mathbf{c}_{\mathbf{z}_h}(\mathbf{u}_h, \mathbf{v}_h) &= \mathbf{G}_{\varphi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,.
\end{aligned} \tag{3.46}$$

Equivalently, $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ is the unique solution of

$$\mathbf{A}_{\mathbf{z}_h}\big((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\big) = \mathbf{R}_{\varphi_h}(\boldsymbol{\tau}_h, \mathbf{v}_h) \qquad \forall\, (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\,, \tag{3.47}$$

where $\mathbf{A}_{\mathbf{z}_h} : (\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}) \times (\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}) \to \mathrm{R}$ and $\mathbf{R}_{\varphi_h} : (\mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}) \to \mathrm{R}$ are defined according to (3.19), when restricted to the finite-dimensional subspaces. Additionally, we define $\widetilde{\mathbf{S}}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \to \mathrm{H}_h^{\phi}$ by $\widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h) := \phi_h$, where $(\phi_h, \lambda_h) \in \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$ is the unique solution, which will be confirmed below (cf. Lemma 3.9), of the problem arising from the third and fourth rows of (3.45), after replacing $\mathbf{u}_h$ by the given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, that is

$$\begin{aligned}
a_{\mathbf{z}_h}(\phi_h, \psi_h) + b(\psi_h, \lambda_h) &= \mathrm{F}(\psi_h) && \forall\, \psi_h \in \mathrm{H}_h^{\phi}\,,\\
b(\phi_h, \xi_h) &= \mathrm{G}(\xi_h) && \forall\, \xi_h \in \mathrm{H}_h^{\lambda}\,.
\end{aligned} \tag{3.48}$$

Finally, we introduce the operator $\mathbf{T}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ defined by

$$\mathbf{T}_{\mathtt{d}}(\mathbf{z}_h) := \mathbf{S}_{\mathtt{d}}\big(\mathbf{z}_h, \widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)\big) \qquad \forall\, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}\,,$$

and realize that solving (3.45) is equivalent to finding a fixed point of the operator $\mathbf{T}_{\mathtt{d}}$, that is, seeking $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$\mathbf{T}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h\,. \tag{3.49}$$

As in the continuous fixed-point strategy, it remains to prove that $\mathbf{S}_{\mathtt{d}}$ and $\widetilde{\mathbf{S}}_{\mathtt{d}}$ are well-defined, that is, that the problems (3.46) and (3.48) are well-posed, thus implying that $\mathbf{T}_{\mathtt{d}}$ is well-defined as well.

In order to achieve the well-definedness of the operators $\mathbf{S}_{\mathtt{d}}$ and $\widetilde{\mathbf{S}}_{\mathtt{d}}$, we analyze the uncoupled problems (3.46) and (3.48). For this purpose, we introduce in what follows several assumptions on the finite element subspaces, all of which are assumed to be valid throughout the rest of this section. We start with $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$:

(H.0) $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiples of the identity tensor $\mathbb{I}$.

(H.1) $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}}) \subset \mathbf{H}_h^{\mathbf{u}}$.

The hypothesis **(H.0)**, together with the decomposition $\mathbb{H}(\mathbf{div}_\ell;\Omega) = \mathbb{H}_0(\mathbf{div}_\ell;\Omega) \oplus \mathrm{R}\,\mathbb{I}$, allows us to rewrite $\mathbb{H}_h^{\boldsymbol{\sigma}}$ as

$$\mathbb{H}_h^{\boldsymbol{\sigma}} = \left\{ \boldsymbol{\tau}_h - \left(\frac{1}{n\,|\Omega|} \int_\Omega \mathrm{tr}(\boldsymbol{\tau}_h) \right) \mathbb{I} : \quad \boldsymbol{\tau}_h \in \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \right\}.$$

Next, to obtain the discrete analogue of Lemma 3.1, it remains to verify the assumptions of [23, Theorem 3.5], namely, that $\mathbf{a}$ and $\mathbf{c}_{\mathbf{z}_h}$ are symmetric and positive semi-definite, and that the discrete versions of (3.23) and (3.24) hold. The first statement clearly follows from (3.25), whereas for the second one we first notice, thanks to **(H.1)**, that the discrete kernel of the operator induced by $\mathbb{H}_h^{\boldsymbol{\sigma}} \ni \boldsymbol{\tau}_h \mapsto \mathbf{b}(\boldsymbol{\tau}_h,\cdot) \in (\mathbf{H}_h^{\mathbf{u}})'$ is given by

$$\mathbb{V}_h := \left\{ \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}} : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \right\},$$

which is certainly contained in the continuous kernel $\mathbb{V}$ (see (3.22)). Consequently, proceeding similarly as for the continuous inf-sup condition for $\mathbf{a}$ (cf. (3.23)), we readily deduce the existence of a positive constant $\alpha_{\mathtt{d}}$, which actually coincides with $\alpha = c_1^2/\nu_1$, and hence independent of h, such that

$$\sup_{0 \neq \boldsymbol{\tau}_h \in \mathbb{V}_h} \frac{\mathbf{a}(\boldsymbol{\chi}_h, \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_\ell;\Omega}} \geqslant \alpha_{\mathtt{d}}\, \|\boldsymbol{\chi}_h\|_{\mathbf{div}_\ell;\Omega} \qquad \forall\, \boldsymbol{\chi}_h \in \mathbb{V}_h\,.$$

Now we introduce the third assumption concerning the finite element subspaces, namely the discrete inf-sup condition for $\mathbf{b}$:

(H.2) there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{0 \neq \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}} \frac{\mathbf{b}(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_\ell;\Omega}} \geqslant \beta_{\mathtt{d}}\, \|\mathbf{v}_h\|_{0,\rho;\Omega} \qquad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,. \tag{3.50}$$

We are now in a position to present the discrete version of Lemma 3.1.

Lemma 3.8 *Given $\delta_{\mathtt{d}} > 0$ and $(\mathbf{z}_h, \varphi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \delta_{\mathtt{d}}$, the problem (3.46) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$, and consequently, $\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \varphi_h)$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S}_{\mathtt{d}}}$, depending only on $\delta_{\mathtt{d}}$, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, $\beta_{\mathtt{d}}$, and $|\Omega|$, such that*

$$\begin{aligned} \|\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \psi_h)\|_{0,\rho;\Omega} &\leqslant \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \\ &\leqslant C_{\mathbf{S}_{\mathtt{d}}} \left\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|\varphi_h\|_{1,\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \right) \right\}. \end{aligned} \tag{3.51}$$

Proof. It suffices to see, according to the previous analysis, that the hypotheses of [23, Theorem 3.5] are satisfied. Consequently, and analogously to Lemma 3.1, we obtain the existence and uniqueness of the solution, along with the *a priori* estimate (3.51). ▫

In order to establish the well-definedness of $\widetilde{\mathbf{S}}_{\mathtt{d}}$, we shall prove that the problem (3.48) is well-posed. To achieve this, we need to assume the following hypothesis concerning the finite element subspaces H_h^ϕ and H_h^λ:

(H.3) there exists a positive constant $\widetilde{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{0\neq\psi_h\in\mathrm{H}_h^\phi} \frac{b(\psi_h,\xi_h)}{\|\psi_h\|_{1,\Omega}} \geqslant \widetilde{\beta}_{\mathtt{d}}\,\|\xi_h\|_{-1/2,\Gamma} \qquad \forall\,\xi_h\in\mathrm{H}_h^\lambda\,. \tag{3.52}$$

Lemma 3.9 *Given δ_0 as in Lemma 3.2, and $\mathbf{z}_h\in\mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega}\leqslant\delta_0$, the problem (3.48) has a unique solution $(\phi_h,\lambda_h)\in\mathrm{H}_h^\phi\times\mathrm{H}_h^\lambda$, and, consequently, $\widetilde{\mathbf{S}}_{\mathtt{d}}(\phi_h,\lambda_h)$ is well-defined. In addition, there exists a positive constant $C_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$, depending only on κ, η, c_P, $\widetilde{\beta}_{\mathtt{d}}$, and $|\Omega|$, such that*

$$\|\widetilde{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} \leqslant C_{\widetilde{\mathbf{S}}_{\mathtt{d}}}\left\{\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\}.$$

Proof. Since $\|\mathbf{z}_h\|_{0,\rho;\Omega}\leqslant\delta_0=\frac{1}{2}\,\|i_s\|^{-1}\,\kappa\,c_P$, the bilinear form $a_{\mathbf{z}_h}$ is $\mathrm{H}^1(\Omega)$-elliptic (cf. (3.30)), and hence, in particular, its restriction to $\mathrm{H}_h^\phi\times\mathrm{H}_h^\phi$ becomes coercive. In turn, thanks to assumption **(H.3)**, the discrete inf-sup condition for b holds (cf. (3.52)). Therefore, by Babuška–Brezzi theory (see, for instance, [28, Theorem 2.3]), we conclude the well-posedness of (3.48), along with the corresponding *a priori* estimate. □

As in the continuous case (cf. Section 3.2), we now study the solvability of the fixed-point equation (3.49). Indeed, in order to ensure the well-definedness of $\mathbf{T}_{\mathtt{d}}$, we first set $r\in(0,r_0^{\mathtt{d}}]$, where

$$r_0^{\mathtt{d}} := \min\{\delta_{\mathtt{d}},\delta_0\}\,,\quad \text{with}\ \ \delta_{\mathtt{d}}>0 \ \ \text{and}\quad \delta_0 := \frac{1}{2}\,\|i_s\|^{-1}\,\kappa\,c_P\,, \tag{3.53}$$

satisfying Lemmas 3.8 and 3.9. We remark that δ_0 is the same radius considered in Lemma 3.2. We then introduce the discrete ball

$$\mathbf{W}_h(r) := \left\{\mathbf{z}_h\in\mathbf{H}_h^{\mathbf{u}}: \quad \|\mathbf{z}_h\|_{0,\rho;\Omega}\leqslant r\right\}. \tag{3.54}$$

The following result, analogous to Lemma 3.3, establishes that $\mathbf{T}_{\mathtt{d}}$ is well-defined when restricted to $\mathbf{W}_h(r)$, and maps it into itself.

Lemma 3.10 *Let $r\in(0,r_0^{\mathtt{d}}]$, with $r_0^{\mathtt{d}}$ as in (3.53), and assume that the data fulfill*

$$C_{\mathbf{T}_{\mathtt{d}}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma}+\|f\|_{0,\Omega}+\|\mathbf{g}\|_{0,\Omega}\left(\|g\|_{0,\Omega}+\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}+\|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\right\}\leqslant r\,, \tag{3.55}$$

where $C_{\mathbf{T}_{\mathtt{d}}} := C_{\mathbf{S}_{\mathtt{d}}}\max\left\{C_{\widetilde{\mathbf{S}}_{\mathtt{d}}},1\right\}$ (cf. Lemmas 3.8 and 3.9). Then, $\mathbf{T}_{\mathtt{d}}(\mathbf{W}_h(r))\subset\mathbf{W}_h(r)$ and the restricted operator $\mathbf{T}_{\mathtt{d}}|_{\mathbf{W}_h(r)}:\mathbf{W}_h(r)\to\mathbf{W}_h(r)$ is well-defined.

Proof. The argument is analogous to that of Lemma 3.3, and the details are omitted. □

With the previous result established, we can now state the discrete analogues of Lemmas 3.4 and 3.5 with corresponding constants denoted by $L_{\mathbf{S}_{\mathtt{d}}}$ and $L_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$. However, since the discrete versions are direct counterparts of their continuous analogues, we omit the proofs and focus instead on presenting the Lipschitz continuity result for the discrete global fixed-point operator $\mathbf{T}_{\mathtt{d}}$. We then conclude with the main result of this section: the solvability of the Galerkin scheme (3.45). In other words, the discrete versions of Lemma 3.6 and Theorem 3.7 read as follows.

Lemma 3.11 *Let $r \in (0, r_0^{\mathtt{d}}]$, with $r_0^{\mathtt{d}}$ as in (3.53), and assume that the data satisfy (3.55). Then, there exists a positive constant $L_{\mathbf{T}_{\mathtt{d}}}$, depending on $L_{\mathbf{S}_{\mathtt{d}}}$, $C_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$, and $L_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$, such that*

$$\begin{aligned}&\|\mathbf{T}_{\mathtt{d}}(\mathbf{z}_{1,h}) - \mathbf{T}_{\mathtt{d}}(\mathbf{z}_{2,h})\|_{0,\rho;\Omega} \\ &\quad \leqslant L_{\mathbf{T}_{\mathtt{d}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \right) \Big\} \|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{0,\rho;\Omega}\end{aligned}$$

for all $\mathbf{z}_{1,h}, \mathbf{z}_{2,h} \in \mathbf{W}_h(r)$.

Theorem 3.12 *Let $r \in (0, r_0^{\mathtt{d}}]$, with $r_0^{\mathtt{d}}$ as in (3.53), and assume that the data satisfy (3.55) and*

$$L_{\mathbf{T}_{\mathtt{d}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \big) \Big\} < 1 \,.$$

Then, there exists a unique $\mathbf{u}_h \in \mathbf{W}_h(r)$ (cf. (3.54)) such that $\mathbf{T}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h$, or equivalently, the problem (3.45) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \phi_h, \lambda_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathrm{H}_h^{\phi} \times \mathrm{H}_h^{\lambda}$, with $\mathbf{u}_h \in \mathbf{W}_h(r)$. Moreover, there exist positive constants $\mathcal{C}_{1,\mathtt{d}}$ and $\mathcal{C}_{2,\mathtt{d}}$, depending only on $C_{\mathbf{S}_{\mathtt{d}}}$ and $C_{\widetilde{\mathbf{S}}_{\mathtt{d}}}$, such that

$$\begin{aligned}\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} &\leqslant \mathcal{C}_{1,\mathtt{d}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \right) \Big\} \,, \\ \textit{and} \qquad \|(\phi_h, \lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)} &\leqslant \mathcal{C}_{2,\mathtt{d}} \Big\{ \|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\} \,.\end{aligned}$$

3.4 *A priori* error analysis

In this section, we derive an *a priori* error estimate for the Galerkin scheme (3.45). To this end, we set $r \in (0, \min\{r_0, r_0^{\mathtt{d}}\}]$, with r_0, $r_0^{\mathtt{d}}$ satisfying (3.31), (3.53), and let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\vartheta}, \phi) \in \mathcal{H}\times\mathcal{Q}\times\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)$, with $\mathbf{u} \in \mathbf{W}(r)$, and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\vartheta}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \mathrm{H}_h^{\phi}$, with $\mathbf{u}_h \in \mathbf{W}_h(r)$, be the unique solutions of the continuous problem (3.11) and the Galerkin Scheme (3.45), respectively. In what follows, given a subspace V_h of a generic Banach space $(V, \|\cdot\|_V)$, we set the distance of $v \in V$ to V_h as

$$\mathrm{dist}\,(v, V_h) \,:=\, \inf_{v_h \in V_h} \|v - v_h\|_V \,. \tag{3.56}$$

We begin our analysis by estimating the error for the Brinkman–Forchheimer unknowns, namely $\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}$. To achieve this, we recall the equivalent form of the uncoupled Brinkman–Forchheimer formulation, namely (3.19) with $(\mathbf{z}, \varphi) = (\mathbf{u}, \phi)$, and its discrete counterpart (3.47) with $(\mathbf{z}_h, \varphi_h) = (\mathbf{u}_h, \phi_h)$. Bearing in mind the well-posedness of these problems, and employing the same arguments used to infer (3.27), we derive the existence of a constant $\alpha_{\mathbf{A},\mathtt{d}} > 0$, depending only on $\delta_{\mathtt{d}}$, ρ, ν_0, $\mathtt{D}_1$, $\mathtt{F}_1$, $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, and $|\Omega|$, such that for each $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ satisfying $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \delta_{\mathtt{d}}$, there holds the inf-sup condition for $\mathbf{A}_{\mathbf{z}_h}$, that is,

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau}_h, \mathbf{v}_h)\in\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}} \frac{\mathbf{A}_{\mathbf{z}_h}\left((\boldsymbol{\chi}_h, \mathbf{w}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h)\right)}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|_{\mathcal{H}\times\mathcal{Q}}} \,\geqslant\, \alpha_{\mathbf{A},\mathtt{d}} \,\|(\boldsymbol{\chi}_h, \mathbf{w}_h)\|_{\mathcal{H}\times\mathcal{Q}} \quad \forall\, (\boldsymbol{\chi}_h, \mathbf{w}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \,.$$

In particular, since $\|\mathbf{u}_h\|_{0,\rho;\Omega} \leqslant r \leqslant \delta_{\mathtt{d}}$, the above holds for $\mathbf{A}_{\mathbf{u}_h}$, and hence, employing the Strang-type estimate provided by [15, Lemma 5.1], we arrive at

$$\begin{aligned}\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \,\leqslant\, \mathcal{C}_{\mathtt{ST,BF}} \Big\{ &\mathrm{dist}\left((\boldsymbol{\sigma}, \mathbf{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\right) + \|\mathbf{R}_{\phi} - \mathbf{R}_{\phi_h}\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}})'} \\ + \quad &\|\mathbf{A}_{\mathbf{u}}\big((\boldsymbol{\sigma}, \mathbf{u}), (\cdot,\cdot)\big) - \mathbf{A}_{\mathbf{u}_h}\big((\boldsymbol{\sigma}, \mathbf{u}), (\cdot,\cdot)\big)\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}})'} \Big\} \,,\end{aligned} \tag{3.57}$$

where $\mathcal{C}_{\mathtt{ST,BF}}$ is a positive constant depending only on $\alpha_{\mathbf{A},\mathtt{d}}$, ν_0, $\mathtt{D}_1$, $\mathtt{F}_1$, r, ρ, and $|\Omega|$, and hence independent of h. In order to estimate the consistency terms of the right hand side of (3.57), we first use (3.37) to discover

$$\|\mathbf{R}_\phi - \mathbf{R}_{\phi_h}\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}})'} = \|\mathbf{G}_\phi - \mathbf{G}_{\phi_h}\|_{(\mathbf{H}_h^{\mathbf{u}})'} \leqslant \|i_s\|\,\|\mathbf{g}\|_{0,\Omega}\,\|\phi-\phi_h\|_{1,\Omega}\,. \tag{3.58}$$

In turn, using the definition of $\mathbf{A}_{\mathbf{u}}$ and $\mathbf{A}_{\mathbf{u}_h}$, together with the estimate (3.36), we obtain that

$$\begin{aligned}\|\mathbf{A}_{\mathbf{u}}\big((\boldsymbol{\sigma},\mathbf{u}),(\cdot,\cdot)\big) - \mathbf{A}_{\mathbf{u}_h}\big((\boldsymbol{\sigma},\mathbf{u}),(\cdot,\cdot)\big)\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}})'} &= \|\mathbf{c}_{\mathbf{u}_h}(\mathbf{u},\cdot) - \mathbf{c}_{\mathbf{u}}(\mathbf{u},\cdot)\|_{(\mathbf{H}_h^{\mathbf{u}})'}\\ &\leqslant L_{\mathbf{c}}\left(\|\mathbf{u}_h\|_{0,\rho;\Omega} + \|\mathbf{u}\|_{0,\rho;\Omega}\right)^{\rho-3}\|\mathbf{u}\|_{0,\rho;\Omega}\,\|\mathbf{u}_h-\mathbf{u}\|_{0,\rho;\Omega}\,,\end{aligned}$$

which, since $\mathbf{u}\in\mathbf{W}(r)$ and $\mathbf{u}_h\in\mathbf{W}_h(r)$, implies that

$$\|\mathbf{A}_{\mathbf{u}}\big((\boldsymbol{\sigma},\mathbf{u}),(\cdot,\cdot)\big) - \mathbf{A}_{\mathbf{u}_h}\big((\boldsymbol{\sigma},\mathbf{u}),(\cdot,\cdot)\big)\|_{(\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}})'} \leqslant L_{\mathbf{c}}\,(2r)^{\rho-3}\,\|\mathbf{u}\|_{0,\rho;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;\Omega}\,. \tag{3.59}$$

Then, replacing back (3.58) and (3.59) into (3.57), it follows that

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}\\ &\leqslant \widetilde{\mathcal{C}}_{\mathtt{ST,BF}}\left\{\mathrm{dist}\left((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\right) + \|\phi-\phi_h\|_{1,\Omega} + \|\mathbf{u}\|_{0,\rho;\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;\Omega}\right\},\end{aligned} \tag{3.60}$$

where $\widetilde{\mathcal{C}}_{\mathtt{ST,BF}}$ is a positive constant depending only on $\mathcal{C}_{\mathtt{ST,BF}}$, $\|i_s\|$, $\|\mathbf{g}\|_{0,\Omega}$, $L_{\mathbf{c}}$, r, and ρ.

Now, our goal is to obtain an estimate for $\|(\phi,\lambda)-(\phi_h,\lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)}$. For this, we now employ the Strang-type estimate from [29, Theorem 2.2], so that after using the coercivity constant of $a_{\mathbf{u}_h}$ (cf. (3.30)), the stability properties (3.17e)–(3.17g), and the fact that $\mathbf{u}\in\mathbf{W}(r)$ and $\mathbf{u}_h\in\mathbf{W}_h(r)$, we deduce that

$$\|(\phi,\lambda)-(\phi_h,\lambda_h)\| \leqslant C_1\,\mathrm{dist}\,(\phi,\mathrm{H}_h^{\phi}) + C_2\,\mathrm{dist}\,(\lambda,\mathrm{H}_h^{\lambda}) + C_3\,\|\,(a_{\mathbf{u}}-a_{\mathbf{u}_h})\,(\phi,\cdot)\|_{(\mathrm{H}_h^{\phi})'}\,, \tag{3.61}$$

where, C_1, C_2, and C_3 are positive constants depending only on η, κ, c_P, $\|i_s\|$, and $\widetilde{\beta}_{\mathtt{d}}$. Regarding the third term on the right-hand side of (3.61), we proceed as in (3.39), that is, we use the definition of the bilinear form $a_{\mathbf{z}}$ (cf. (3.15)), Hölder's inequality, and the continuity of the embedding operator i_s. As a consequence, estimate (3.61) becomes

$$\|(\phi,\lambda)-(\phi_h,\lambda_h)\| \leqslant \widetilde{\mathcal{C}}_{\mathtt{ST,CDR}}\left\{\mathrm{dist}\left((\phi,\lambda),\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\lambda}\right) + \|\phi\|_{1,\Omega}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;\Omega}\right\}, \tag{3.62}$$

where $\widetilde{\mathcal{C}}_{\mathtt{ST,CDR}} := \max\left\{C_1, C_2, C_3\,\|i_s\|\right\}$. Thus, multiplying (3.60) by $\frac{1}{2\,\widetilde{\mathcal{C}}_{\mathtt{ST,BF}}}$, summing up with (3.62), bounding $\|\mathbf{u}\|_{0,\rho;\Omega}$ and $\|\phi\|_{1,\Omega}$ according to (3.44), and performing some algebraic arrangements, we find that

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\phi,\lambda)-(\phi_h,\lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)}\\ &\quad\leqslant\ \widehat{\mathcal{C}}\left\{\mathrm{dist}\left((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\right) + \mathrm{dist}\left((\phi,\lambda),\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\lambda}\right)\right\}\\ &\quad+\ \widetilde{\mathcal{C}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right\}\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;\Omega}\,,\end{aligned} \tag{3.63}$$

where $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are positive constants depending only on $\widetilde{\mathcal{C}}_{\mathtt{ST,BF}}$, $\widetilde{\mathcal{C}}_{\mathtt{ST,CDR}}$, $\mathcal{C}_1$, $\mathcal{C}_2$ (cf. (3.44)), and $\|\mathbf{g}\|_{0,\Omega}$.

We conclude this section with its main result, which is the Céa estimate associated with the Galerkin scheme (3.45).

Theorem 3.13 *In addition to the hypotheses of Theorems 3.7 and 3.12, assume that*

$$\widetilde{\mathcal{C}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|g\|_{0,\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\right\} \leqslant \frac{1}{2}\,. \tag{3.64}$$

Then, there holds

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \,+\, \|(\phi,\lambda) - (\phi_h,\lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)}\\ &\leqslant\, 2\,\widehat{\mathcal{C}}\left\{\mathrm{dist}\left((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\right) + \mathrm{dist}\left((\phi,\lambda_h),\mathrm{H}_h^{\phi}\times\mathrm{H}_h^{\lambda}\right)\right\}.\end{aligned} \tag{3.65}$$

Proof. It follows straightforwardly from (3.63) by bounding $\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;,\Omega}$ by $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}$, and then using assumption (3.64). We omit further details. ▫

3.5 Specific finite element subspaces and rates of convergence

Given an integer $l \geqslant 0$ and a subset S of R^n, we denote by $\mathrm{P}_l(S)$ the space of polynomials of total degree at most l defined on S, and $\mathbf{P}_l(S)$ its vectorial counterpart. In turn, for each integer $k \geqslant 0$ and $K \in \mathcal{T}_h$, we define the local Raviart–Thomas spaces of order k as $\mathbf{RT}_k(K) := \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\,\mathbf{x}$, where $\mathbf{x} := (x_1,\dots,x_n)^{\mathbf{t}}$ is a generic vector of R^n and $\widetilde{\mathrm{P}}_k(K)$ is the space of polynomials of total degree equal to k defined on K. Furthermore, define $\mathbb{RT}_k(K)$ as the tensor space in which each row lies in $\mathbf{RT}_k(K)$.

Under this notation, we define the following finite element subspaces for the Brinkman–Forchheimer unknowns:

$$\begin{aligned}\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} &:= \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}_\ell;\Omega) : \quad \boldsymbol{\tau}_h\big|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\right\},\\ \mathbf{H}_h^{\mathbf{u}} &:= \left\{\mathbf{v}_h \in \mathbf{L}^\rho(\Omega) : \quad \mathbf{v}_h\big|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\right\}.\end{aligned} \tag{3.66}$$

Certainly, it is easy to see that $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ contains the multiples of the identity and that $\mathbf{div}(\mathbb{H}_h^{\boldsymbol{\sigma}}) \subset \mathbf{H}_h^{\mathbf{u}}$, so that hypotheses **(H.0)** and **(H.1)** hold. Furthermore, the inf-sup condition associated with **(H.2)** was established for $\rho = 4$ in [21, Lemma 5.5], and its proof can be easily extended to the present range of ρ (see also [10, Lemma 4.4] or [11, Lemma 3.3] for the vector version of it).

For the convection-diffusion-reaction part, we approximate ϕ with the classical Lagrange finite element space of order $k+1$,

$$\mathrm{H}_h^{\phi} \,:=\, \left\{\psi_h \in C(\overline{\Omega}) : \quad \psi_h|_K \in \mathrm{P}_{k+1}(K) \quad \forall\, K \in \mathcal{T}_h\right\}. \tag{3.67}$$

On the other hand, in order to approximate λ, we introduce an independent triangulation of Γ (made of straight segments in R^2, or triangles in R^3), namely $\{\Gamma_1,\Gamma_2,\dots,\Gamma_m\}$, set $\widetilde{h} := \max\limits_{j\in\{1,\dots,m\}} |\Gamma_j|$, and define

$$\mathrm{H}_{\widetilde{h}}^{\lambda} \,:=\, \left\{\xi_{\widetilde{h}} \in \mathrm{L}^2(\Gamma) : \quad \xi_{\widetilde{h}}|_{\Gamma_j} \in \mathrm{P}_k(\Gamma_j) \quad \forall\, j \in \{1,\dots,m\}\right\} \tag{3.68}$$

as the approximation subspace of λ. Then, under certain conditions on the mesh sizes, H_h^{ϕ} and $\mathrm{H}_{\widetilde{h}}^{\lambda}$ constitute a stable pair of finite element subspaces for the convection-diffusion-reaction part of our Galerkin scheme (3.45). More precisely, one can prove (cf. [22, Lemma 4.10] or [28, Lemma 4.7]) that there exists a positive constant C_0 such that whenever $h \leqslant C_0\,\widetilde{h}$, the discrete inf-sup condition (3.52), corresponding to hypothesis **(H.3)**, is satisfied. Therefore, when using the aforementioned subspaces, it is necessary to assume this mesh-size restriction in order to ensure the theoretical results established earlier.

Now we aim to obtain the rates of convergence of our Galerkin scheme (3.45) with the specific finite element subspaces defined previously. To this end, approximation properties of the finite element subspaces $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$, H_h^{ϕ} and $\mathrm{H}_{\widetilde{h}}^{\lambda}$ are presented below, which follow from interpolation estimates of Sobolev spaces and the approximation properties of the orthogonal projectors and the interpolation operators involved in their definitions (see, for instance, [6], [11], [21], [27], [28]).

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,\ell}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\tau}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \leqslant C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,\ell;\Omega} \right\},$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,\rho}(\Omega)$ there holds

$$\mathrm{dist}\,(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}) \leqslant C\, h^l\, \|\mathbf{v}\|_{l,\rho;\Omega}\,,$$

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in (0, k+1]$, and for each $\psi \in \mathrm{H}^{l+1}(\Omega)$, there holds

$$\mathrm{dist}\,(\psi, \mathrm{H}_h^{\phi}) \leqslant C\, h^l\, \|\psi\|_{l+1,\Omega}\,,$$

$(\mathbf{AP}_{\widetilde{h}}^{\lambda})$ there exists $C > 0$, independent of $\widetilde{h}$, such that for each $l \in (0, k+1]$, and for each $\xi \in \mathrm{H}^{-1/2+l}(\Gamma)$, there holds

$$\mathrm{dist}\,(\xi, \mathrm{H}_{\widetilde{h}}^{\lambda}) \leqslant C\, \widetilde{h}^l\, \|\xi\|_{-1/2+l,\Gamma}\,.$$

These approximation properties, together with the Céa estimate (3.65), yield the following result, which summarizes the convergence rates of our Galerkin scheme (3.45).

Theorem 3.14 *In addition to the hypotheses of Theorems 3.7, 3.12, and 3.13, assume that there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,\ell}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,\rho}(\Omega)$, $\phi \in \mathrm{H}^{l+1}(\Omega)$ and $\lambda \in \mathrm{H}^{-1/2+l}(\Gamma)$. Then, there exists a positive constant C, independent of h and $\widetilde{h}$, such that for all $h \leqslant C_0\,\widetilde{h}$ there holds*

$$\begin{aligned}&\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \;+\; \|(\phi,\lambda) - (\phi_h,\lambda_h)\|_{\mathrm{H}^1(\Omega)\times\mathrm{H}^{-1/2}(\Gamma)}\\ &\quad\leqslant\; C\, h^l \left\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\ell;\Omega} + \|\mathbf{u}\|_{l,\rho;\Omega} + \|\phi\|_{l+1,\Omega} \right\} \;+\; C\,\widetilde{h}^l\, \|\lambda\|_{-1/2+l,\Gamma}\,.\end{aligned}$$

4 The fully-mixed approach

In this section, and as an alternative to the approach presented in Section 3, we introduce and analyze a fully-mixed method for the system (2.1). Following the same structure as in that section, we begin by introducing the corresponding variational formulation and establishing its well-posedness through a fixed-point strategy. Then, we prove the stability of the associated Galerkin scheme and derive a Céa-type estimate for the discrete approximations. Finally, we provide an example of finite element subspaces yielding a stable associated Galerkin scheme.

4.1 The continuous formulation

In what follows we basically adopt the strategy of [15], but without employing the augmentation procedure used therein. Instead, we employ a fully Banach space framework, as in [14], [12], and

[9]. Specifically, in addition to the pseudostress tensor $\boldsymbol{\sigma}$ defined in (3.1), we now introduce the pseudodiffusion vector $\boldsymbol{\vartheta}$ by

$$\boldsymbol{\vartheta} \,:=\, \kappa\,\nabla\phi - \phi\,\mathbf{u} \quad \text{in} \quad \Omega\,. \tag{4.1}$$

We emphasize that this new unknown does not modify the formulation of the Brinkman–Forchheimer part (cf. (3.4) and (3.6)). Thus, we focus on deriving the mixed formulation for the convection-diffusion-reaction component of the coupled problem. In this way, we take the divergence of $\boldsymbol{\vartheta}$ in (4.1), apply (2.1b) to the resulting equation, and then use (2.1c), obtaining

$$\operatorname{div}(\boldsymbol{\vartheta}) = \kappa\,\Delta\phi - \mathbf{u}\cdot\nabla\phi - \operatorname{div}(\mathbf{u})\,\phi = (\eta - f)\,\phi - g \quad \text{in} \quad \Omega\,.$$

The resulting equation, along with (4.1) and the Dirichlet condition for the concentration, yields an equivalent system for the convection-diffusion-reaction equations, given by

$$\begin{aligned} \kappa^{-1}\,\boldsymbol{\vartheta} \,-\, \nabla\phi \,+\, \kappa^{-1}\,\phi\,\mathbf{u} \;&=\; \mathbf{0} \qquad \text{in} \quad \Omega\,,\\ \operatorname{div}\big(\boldsymbol{\vartheta}\big) \,-\, (\eta \,-\, f)\,\phi \;&=\; -\,g \qquad \text{in} \quad \Omega\,,\\ \phi \;&=\; \phi_{\mathrm{D}} \qquad \text{on} \quad \Gamma\,. \end{aligned} \tag{4.2}$$

Seeking ϕ originally in $\mathrm{H}^1(\Omega)$, we multiply the first equation of (4.2) by a function $\boldsymbol{\psi} \in \mathbf{H}(\operatorname{div}_t;\,\Omega)$ (cf. (1.1)), where t lies in the range specified right before (1.4), and integrate by parts, to arrive at

$$\int_\Omega \kappa^{-1}\,\boldsymbol{\vartheta}\cdot\boldsymbol{\psi} + \int_\Omega \phi\operatorname{div}(\boldsymbol{\psi}) + \int_\Omega \kappa^{-1}\phi\,\mathbf{u}\cdot\boldsymbol{\psi} = \langle \boldsymbol{\psi}\cdot\mathbf{n}, \phi_{\mathrm{D}}\rangle \qquad \forall\,\boldsymbol{\psi} \in \mathbf{H}(\operatorname{div}_t;\,\Omega)\,, \tag{4.3}$$

where we use, additionally, the Dirichlet condition from (4.2), with the datum $\phi_{\mathrm{D}} \in \mathrm{H}^{1/2}(\Gamma)$. On the other hand, testing the second equation of (4.2) against a scalar field ξ, we formally obtain

$$\int_\Omega \xi\operatorname{div}(\boldsymbol{\vartheta}) - \int_\Omega (\eta - f)\,\phi\,\xi = -\int_\Omega g\,\xi\,. \tag{4.4}$$

Certainly, for the equations (4.3) and (4.4) to be well-defined, it is not necessary that ϕ belong to $\mathrm{H}^1(\Omega)$, since the gradient was eliminated by introducing the pseudodiffusion vector. In this context, returning to the Brinkman–Forchheimer equations, and more precisely to the right-hand side of (3.4), the bound given in (3.5) reveals that it suffices for ϕ to lie in $\mathrm{L}^s(\Omega)$, where we recall that $s = 2\rho/(\rho-2) \in [4,6]$. Bearing this in mind, the second term of (4.3) is well-defined if t is chosen to be the Hölder conjugate of s, that is, $t := 2\rho/(\rho+2) \in [6/5, 4/3]$. Note that this choice is indeed consistent with the range of values specified in (1.4). If, additionally, we seek $\boldsymbol{\vartheta}$ in $\mathbf{L}^2(\Omega)$, then every term of (4.3) is well-defined.

Next, taking ξ in the same space to which ϕ belongs, that is $\xi \in \mathrm{L}^s(\Omega)$, we realize from the first term of (4.4) that, besides requiring $\boldsymbol{\vartheta}$ in $\mathbf{L}^2(\Omega)$, we also need that $\operatorname{div}(\boldsymbol{\vartheta}) \in \mathrm{L}^t(\Omega)$, whence $\boldsymbol{\vartheta}$ must be sought in $\mathbf{H}(\operatorname{div}_t;\Omega)$. Additionally, the right-hand side of the same equation motivates the assumption that the datum g belongs to $\mathrm{L}^t(\Omega)$. Regarding the second term, we apply the triangle inequality and use the injections $i_{s,2}$ and $i_{s,4}$ (cf. (1.3)) to observe that

$$\Big|\int_\Omega (\eta - f)\,\phi\,\xi\Big| \leqslant \Big(\eta\,|\Omega|^{2/\rho} + \|f\|_{0,\Omega}\,|\Omega|^{(4-\rho)/(2\rho)}\Big)\;\|\phi\|_{0,s;\Omega}\,\|\xi\|_{0,s;\Omega}\,.$$

We remark that the space setting depends completely on $\rho \in [3,4]$. More precisely, we introduce three new parameters, which have been discussed earlier, that depend on ρ, namely,

$$\ell := \frac{\rho}{\rho-1} \in \left[\frac{4}{3},\frac{3}{2}\right], \qquad s := \frac{2\rho}{\rho-2} \in [4,6] \quad \text{and} \quad t := \frac{2\rho}{\rho+2} \in \left[\frac{6}{5},\frac{4}{3}\right], \tag{4.5}$$

which help to define the spaces $\mathcal{H} := \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, $\mathcal{Q} := \mathbf{L}^\rho(\Omega)$, $\mathrm{X} := \mathbf{H}(\mathrm{div}_t;\Omega)$ and $\mathrm{Y} := \mathrm{L}^s(\Omega)$. This, along with (3.4), (3.6), (4.3), and (4.4), leads to the fully-mixed formulation of the problem (2.1): Find $(\boldsymbol{\sigma},\mathbf{u}) \in \mathcal{H}\times\mathcal{Q}$ and $(\boldsymbol{\vartheta},\phi) \in \mathrm{X}\times\mathrm{Y}$ such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}(\boldsymbol{\tau},\mathbf{u}) &= \mathbf{F}(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathcal{H}\,,\\
\mathbf{b}(\boldsymbol{\sigma},\mathbf{v}) - \mathbf{c}_{\mathbf{u}}(\mathbf{u},\mathbf{v}) &= \mathbf{G}_\phi(\mathbf{v}) && \forall\,\mathbf{v}\in\mathcal{Q}\,,\\
\widehat{a}(\boldsymbol{\vartheta},\boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi},\phi) + d_{\mathbf{u}}(\boldsymbol{\psi},\phi) &= \widehat{\mathrm{F}}(\boldsymbol{\psi}) && \forall\,\boldsymbol{\psi}\in\mathrm{X}\,,\\
\widehat{b}(\boldsymbol{\vartheta},\xi) - \widehat{c}_f(\phi,\xi) &= \widehat{\mathrm{G}}(\xi) && \forall\,\xi\in\mathrm{Y}\,,
\end{aligned} \tag{4.6}$$

where the bilinear forms $\mathbf{a}:\mathcal{H}\times\mathcal{H}\to\mathrm{R}$, $\mathbf{b}:\mathcal{H}\times\mathcal{Q}\to\mathrm{R}$ and $\mathbf{c}_{\mathbf{z}}:\mathcal{Q}\times\mathcal{Q}\to\mathrm{R}$, for each $\mathbf{z}\in\mathcal{Q}$, and the linear functionals $\mathbf{F}:\mathcal{H}\to\mathrm{R}$ and $\mathbf{G}_\varphi:\mathcal{Q}\to\mathrm{R}$, for each $\varphi\in\mathrm{Y}$, are already defined in (3.12)–(3.14). In turn, the bilinear forms $\widehat{a}:\mathrm{X}\times\mathrm{X}\to\mathrm{R}$, $\widehat{b}:\mathrm{X}\times\mathrm{Y}\to\mathrm{R}$, $d_{\mathbf{z}}:\mathrm{X}\times\mathrm{Y}\to\mathrm{R}$, for each $\mathbf{z}\in\mathcal{Q}$, $\widehat{c}_f:\mathrm{Y}\times\mathrm{Y}\to\mathrm{R}$, and the linear functionals $\widehat{\mathrm{F}}:\mathrm{X}\to\mathrm{R}$ and $\widehat{\mathrm{G}}:\mathrm{Y}\to\mathrm{R}$, are defined as

$$\widehat{a}(\boldsymbol{\zeta},\boldsymbol{\psi}) := \int_\Omega \kappa^{-1}\,\boldsymbol{\zeta}\cdot\boldsymbol{\psi}\,,\quad \widehat{b}(\boldsymbol{\psi},\xi) := \int_\Omega \xi\,\mathrm{div}(\boldsymbol{\psi})\,,\quad d_{\mathbf{z}}(\boldsymbol{\psi},\xi) := \int_\Omega \kappa^{-1}\,\xi\,\mathbf{z}\cdot\boldsymbol{\psi}\,,$$

$$\widehat{c}_f(\zeta,\xi) := \int_\Omega (\eta - f)\,\zeta\,\xi\,,\quad \widehat{\mathrm{F}}(\boldsymbol{\psi}) := \langle\boldsymbol{\psi}\cdot\mathbf{n},\phi_{\mathrm{D}}\rangle\,,\quad \widehat{\mathrm{G}}(\xi) := -\int_\Omega g\,\xi\,.$$

We finally remark that the only change for the Brinkman–Forchheimer mixed formulation is that ϕ now lies in $\mathrm{L}^s(\Omega)$ instead of $\mathrm{H}^1(\Omega)$. This implies that the stability estimate for $\mathbf{G}_\varphi$ must be slightly modified. More precisely, in addition to the stability properties for $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}_{\mathbf{z}}$, and $\mathbf{F}$ (cf. (3.17a), (3.17b), (3.17c)), we also have

$$|\mathbf{G}_\varphi(\mathbf{v})| \leqslant \|\mathbf{g}\|_{0,\Omega}\Big(\|\varphi\|_{0,s;\Omega} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\Big)\,\|\mathbf{v}\|_{\mathcal{Q}}, \tag{4.7a}$$

$$|\widehat{a}(\boldsymbol{\zeta},\boldsymbol{\psi})| \leqslant \kappa^{-1}\,\|\boldsymbol{\zeta}\|_{\mathrm{X}}\,\|\boldsymbol{\psi}\|_{\mathrm{X}}, \qquad |\widehat{b}(\boldsymbol{\psi},\xi)| \leqslant \|\boldsymbol{\psi}\|_{\mathrm{X}}\,\|\xi\|_{\mathrm{Y}}, \tag{4.7b}$$

$$|d_{\mathbf{z}}(\boldsymbol{\psi},\xi)| \leqslant \kappa^{-1}\,\|\mathbf{z}\|_{0,\rho;\Omega}\,\|\boldsymbol{\psi}\|_{\mathrm{X}}\,\|\xi\|_{\mathrm{Y}}, \tag{4.7c}$$

$$|\widehat{c}_f(\zeta,\xi)| \leqslant \Big(\eta\,|\Omega|^{2/\rho} + \|f\|_{0,\Omega}\,|\Omega|^{(4-\rho)/(2\rho)}\Big)\,\|\zeta\|_{\mathrm{Y}}\,\|\xi\|_{\mathrm{Y}}, \tag{4.7d}$$

$$|\widehat{\mathrm{F}}(\boldsymbol{\psi})| \leqslant \max\{1,\|i_s\|\}\,\|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\,\|\boldsymbol{\psi}\|_{\mathrm{X}}, \qquad \text{and} \qquad |\widehat{\mathrm{G}}(\xi)| \leqslant \|g\|_{0,t;\Omega}\,\|\xi\|_{\mathrm{Y}}. \tag{4.7e}$$

4.2 Solvability analysis

To prove the well-posedness of (4.6), we proceed analogously as in Section 3.2. We keep the same notation for the operator $\mathbf{S}$ defined according to (3.18), but understanding that now the space $\mathrm{H}^1(\Omega)$ becomes Y. Additionally, we define the operator $\widehat{\mathbf{S}}:\mathcal{Q}\to\mathrm{Y}$ by $\widehat{\mathbf{S}}(\mathbf{z}) := \phi$, where $(\boldsymbol{\vartheta},\phi)\in\mathrm{X}\times\mathrm{Y}$ is the unique solution, to be confirmed below, of the uncoupled mixed formulation arising from (4.6) when $d_{\mathbf{u}}$ is replaced by $d_{\mathbf{z}}$, that is

$$\begin{aligned}
\widehat{a}(\boldsymbol{\vartheta},\boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi},\phi) + d_{\mathbf{z}}(\boldsymbol{\psi},\phi) &= \widehat{\mathrm{F}}(\boldsymbol{\psi}) && \forall\,\boldsymbol{\psi}\in\mathrm{X}\,,\\
\widehat{b}(\boldsymbol{\vartheta},\xi) - \widehat{c}_f(\phi,\xi) &= \widehat{\mathrm{G}}(\xi) && \forall\,\xi\in\mathrm{Y}\,.
\end{aligned} \tag{4.8}$$

Equivalently, $(\boldsymbol{\vartheta},\phi)\in\mathrm{X}\times\mathrm{Y}$ is the unique solution of

$$\widehat{\mathbf{A}}\big((\boldsymbol{\vartheta},\phi),(\boldsymbol{\psi},\xi)\big) + d_{\mathbf{z}}(\boldsymbol{\psi},\phi) = \widehat{\mathrm{F}}(\boldsymbol{\psi}) + \widehat{\mathrm{G}}(\xi) \qquad \forall\,(\boldsymbol{\psi},\xi)\in\mathrm{X}\times\mathrm{Y}, \tag{4.9}$$

where $\widehat{\mathbf{A}} : (\mathrm{X} \times \mathrm{Y}) \times (\mathrm{X} \times \mathrm{Y}) \to \mathrm{R}$ is the bilinear form defined by

$$\widehat{\mathbf{A}}\big((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi)\big) := \widehat{a}(\boldsymbol{\varrho}, \boldsymbol{\psi}) + \widehat{b}(\boldsymbol{\psi}, \zeta) + \widehat{b}(\boldsymbol{\varrho}, \xi) - \widehat{c}_f(\zeta, \xi), \tag{4.10}$$

for all $\big((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi)\big) \in \mathrm{X} \times \mathrm{Y}$. Finally, we define the operator $\widehat{\mathbf{T}} : \mathcal{Q} \to \mathcal{Q}$ as

$$\widehat{\mathbf{T}}(\mathbf{z}) := \mathbf{S}\big(\mathbf{z}, \widehat{\mathbf{S}}(\mathbf{z})\big) \qquad \forall\, \mathbf{z} \in \mathcal{Q}.$$

We remark here that if $\mathbf{S}$ and $\widehat{\mathbf{S}}$ are well-defined, then $\widehat{\mathbf{T}}$ is well-defined as well. In addition, it is clear that solving (4.6) is equivalent to finding a fixed point of the operator $\widehat{\mathbf{T}}$, that is, seeking $\mathbf{u} \in \mathcal{Q}$ such that

$$\widehat{\mathbf{T}}(\mathbf{u}) = \mathbf{u}. \tag{4.11}$$

In the analysis presented in Section 3, we have already established that the operator $\mathbf{S}$ is well-defined. It is worth noting that the constants appearing in the associated estimates may differ in the current setting, due to the modified stability properties of the operator $\mathbf{G}_\varphi$ (cf. (4.7a)). Nevertheless, the only change arises in the norm of the embedding i_s, which is specific to the mixed-primal approach. The following result shows the slight modification of Lemma 3.1.

Lemma 4.1 *Given $\delta > 0$ and $(\mathbf{z}, \varphi) \in \mathcal{Q} \times \mathrm{Y}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \leqslant \delta$, the problem (3.18) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}) \in \mathcal{H} \times \mathcal{Q}$ and hence, $\mathbf{S}(\mathbf{z}, \varphi)$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S}}$, depending only on δ, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, β (cf. (3.24)) and $|\Omega|$, such that*

$$\|\mathbf{S}(\mathbf{z}, \varphi)\|_{0,\rho;\Omega} \leqslant \|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}} \leqslant C_{\mathbf{S}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|\varphi\|_{0,s;\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \big) \Big\}. \tag{4.12}$$

In addition, we also have the global inf-sup condition, given by (3.27), which we shall use later.

Our next goal is to prove that the uncoupled problem (4.8) is well-posed, and, consequently, the operator $\widehat{\mathbf{S}}$ is well-defined. To achieve this, we will actually prove the well-posedness of the problem (4.9) by applying the well-known Banach–Nečas–Babuška theorem in combination with [23, Theorem 3.4]. We start with the following lemma, which establishes an inf-sup condition for $\widehat{\mathbf{A}}$. We remark in advance that applying [23, Theorem 3.4] will require an additional assumption on the data, which is not needed in the mixed-primal approach studied in Section 3, namely

$$f(\mathbf{x}) \leqslant \eta \qquad \forall\, \mathbf{x} \in \Omega. \tag{4.13}$$

In particular, note that (4.13) is trivially satisfied for incompressible fluids, that is when $f = 0$. The aforementioned result reads as follows.

Lemma 4.2 *Assume that the data satisfy (4.13). Then, there exists a positive constant $\alpha_{\widehat{\mathbf{A}}}$, depending on κ, η, ρ, and $|\Omega|$, such that*

$$\sup_{\mathbf{0} \neq (\boldsymbol{\psi},\xi) \in \mathrm{X}\times\mathrm{Y}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi))}{\|(\boldsymbol{\psi}, \xi)\|_{\mathrm{X}\times\mathrm{Y}}} \geqslant \alpha_{\widehat{\mathbf{A}}} \, \|(\boldsymbol{\varrho}, \zeta)\|_{\mathrm{X}\times\mathrm{Y}} \qquad \forall\, (\boldsymbol{\varrho}, \zeta) \in \mathrm{X} \times \mathrm{Y}. \tag{4.14}$$

Proof. According to the structure of $\widehat{\mathbf{A}}$ (cf. (4.10)), which was studied in [23], and bearing in mind that the global inf-sup condition [23, eq. (3.33)] follows from the verification of the hypotheses of [23, Theorem 3.1] (or of its particular case given by [23, Theorem 3.4]), we realize that in order to prove

([4.14](#)), it suffices to check that $\widehat{a}$, $\widehat{b}$, and $\widehat{c}_f$ satisfy the hypotheses of [23, Theorem 3.4]. Indeed, we start by noting that $\widehat{a}$ and $\widehat{c}_f$ are symmetric. In addition, there clearly holds

$$\widehat{a}(\boldsymbol{\psi}, \boldsymbol{\psi}) \,=\, \kappa^{-1}\, \|\boldsymbol{\psi}\|_{0,\Omega}^2 \,\geqslant\, 0 \qquad \forall\, \boldsymbol{\psi} \in \mathrm{X}\,,$$

whereas employing (4.13) we deduce that

$$\widehat{c}_f(\xi, \xi) \,=\, \int_\Omega (\eta - f)\, |\xi|^2 \,\geqslant\, 0 \qquad \forall\, \xi \in \mathrm{Y}\,,$$

so that $\widehat{a}$ and $\widehat{c}_f$ are both positive semi-definite. On the other hand, the kernel $\widehat{V}$ of the operator $\mathrm{X} \ni \boldsymbol{\psi} \mapsto \widehat{b}(\boldsymbol{\psi}, \cdot) \in \mathrm{Y}'$ is characterized by

$$\widehat{V} \,:=\, \Big\{ \boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\Omega) : \quad \mathrm{div}(\boldsymbol{\psi}) = 0 \quad \text{in} \quad \Omega \Big\}\,. \tag{4.15}$$

It readily follows from the definition of $\widehat{a}$ and the above characterization of $\widehat{V}$ that

$$\sup_{0 \neq \boldsymbol{\psi} \in \widehat{V}} \frac{\widehat{a}(\widehat{\boldsymbol{\vartheta}}, \boldsymbol{\psi})}{\|\boldsymbol{\psi}\|_{\mathrm{div}_t;\Omega}} \,\geqslant\, \kappa^{-1}\, \|\widehat{\boldsymbol{\vartheta}}\|_{\mathrm{div}_t;\Omega} \qquad \forall\, \widehat{\boldsymbol{\vartheta}} \in \widehat{V}\,, \tag{4.16}$$

which constitutes the required inf-sup condition for $\widehat{a}$. In turn, we know from [31, Lemma 2.9] that there exists a positive constant $C_{\widehat{b}}$, depending only on $|\Omega|$, such that

$$\sup_{0 \neq \boldsymbol{\psi} \in \mathbf{H}(\mathrm{div}_t;\Omega)} \frac{\widehat{b}(\boldsymbol{\psi}, \widehat{\phi})}{\|\boldsymbol{\psi}\|_{\mathrm{div}_t;\Omega}} \,\geqslant\, C_{\widehat{b}}\, \|\widehat{\phi}\|_{0,s;\Omega} \qquad \forall\, \widehat{\phi} \in \mathrm{L}^s(\Omega)\,,$$

thus establishing the continuous inf-sup condition for $\widehat{b}$. In this way, a straightforward application of [23, Theorem 3.4] yields (4.14) with a positive constant $\alpha_{\widehat{\mathbf{A}}}$ depending only on κ, $C_{\widehat{b}}$, and the stability properties associated with $\widehat{a}$, $\widehat{b}$, and $\widehat{c}_f$ (cf. (4.7b) and (4.7d)), and hence only on κ, $C_{\widehat{b}}$, η, $|\Omega|$, ρ, and $\|f\|_{0,\Omega}$, where the latter can be bounded by $\|f\|_{0,\Omega} \leqslant \eta\, |\Omega|^{1/2}$ thanks to (4.13). This ends the proof. □

Now, as a consequence of (4.14) and the stability property for $d_{\mathbf{z}}$ (cf. (4.7c)), we easily deduce that for each $\mathbf{z} \in \mathcal{Q}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \,\leqslant\, \widehat{\delta}_0 \,:=\, \kappa\, \alpha_{\widehat{\mathbf{A}}}/2$, there holds

$$\sup_{\mathbf{0} \neq (\boldsymbol{\psi}, \xi) \in \mathrm{X}\times\mathrm{Y}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi)) + d_{\mathbf{z}}(\boldsymbol{\psi}, \zeta)}{\|(\boldsymbol{\psi}, \xi)\|_{\mathrm{X}\times\mathrm{Y}}} \,\geqslant\, \frac{\alpha_{\widehat{\mathbf{A}}}}{2}\, \|(\boldsymbol{\varrho}, \zeta)\|_{\mathrm{X}\times\mathrm{Y}} \qquad \forall\, (\boldsymbol{\varrho}, \zeta) \in \mathrm{X}\times\mathrm{Y}\,. \tag{4.17}$$

Similarly, thanks to the symmetry of $\widehat{\mathbf{A}}$ and (4.7c), for each $\mathbf{z} \in \mathcal{Q}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \,\leqslant\, \widehat{\delta}_0$, there also holds

$$\sup_{\mathbf{0} \neq (\boldsymbol{\varrho}, \zeta) \in \mathrm{X}\times\mathrm{Y}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}, \zeta), (\boldsymbol{\psi}, \xi)) + d_{\mathbf{z}}(\boldsymbol{\psi}, \zeta)}{\|(\boldsymbol{\varrho}, \zeta)\|_{\mathrm{X}\times\mathrm{Y}}} \,\geqslant\, \frac{\alpha_{\widehat{\mathbf{A}}}}{2}\, \|(\boldsymbol{\psi}, \xi)\|_{\mathrm{X}\times\mathrm{Y}} \qquad \forall\, (\boldsymbol{\psi}, \xi) \in \mathrm{X}\times\mathrm{Y}\,. \tag{4.18}$$

Hence, we are now in a position to show the well-posedness of (4.8).

Lemma 4.3 *Let $\mathbf{z} \in \mathcal{Q}$ such that $\|\mathbf{z}\|_{0,\rho;\Omega} \,\leqslant\, \widehat{\delta}_0 \,:=\, \kappa\, \alpha_{\widehat{\mathbf{A}}}/2$, and assume that the data satisfy (4.13). Then, the problem (4.8) has a unique solution $(\boldsymbol{\vartheta}, \phi) \in \mathrm{X}\times\mathrm{Y}$ and, hence, $\widehat{\mathbf{S}}(\mathbf{z})$ is well-defined. Moreover, there exists a positive constant $C_{\widehat{\mathbf{S}}}$, depending only on κ, η, ρ and $|\Omega|$, such that*

$$\|\widehat{\mathbf{S}}(\mathbf{z})\|_{0,s;\Omega} \leqslant \|(\boldsymbol{\vartheta}, \phi)\|_{\mathrm{X}\times\mathrm{Y}} \,\leqslant\, C_{\widehat{\mathbf{S}}} \Big\{ \|g\|_{0,t;\Omega} \,+\, \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\}\,. \tag{4.19}$$

Proof. From the previous discussion we know that the assumption on $\mathbf{z}$ implies (4.17) and (4.18), and hence the result follows from a straightforward application of the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.6]) to the problem (4.9), which is equivalent to (4.8). Moreover, the bound (4.19) is derived from the *a priori* bound provided by the aforementioned theorem. More precisely, using the indicated upper bound for $\|\mathbf{z}\|_{0,\rho;\Omega}$ along with the stability properties (4.7e), we deduce that

$$\|\widehat{\mathbf{S}}(\mathbf{z})\|_{0,s;\Omega} \leqslant \|(\boldsymbol{\vartheta},\phi)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant \frac{2}{\alpha_{\widehat{\mathbf{A}}}}\,\|(\widehat{\mathrm{F}},\widehat{\mathrm{G}})\|_{\mathrm{X}'\times\mathrm{Y}'} \leqslant C_{\widehat{\mathbf{S}}}\left\{\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\right\},$$

with $C_{\widehat{\mathbf{S}}} := 2\,\alpha_{\widehat{\mathbf{A}}}^{-1}\max\{1,\|i_s\|\}$. □

Next, we aim to prove that the equation $\widehat{\mathbf{T}}(\mathbf{u}) = \mathbf{u}$ has a unique solution under certain conditions on the data. To this end, we take $r \in (0,\widehat{r}_0]$, where

$$\widehat{r}_0 := \min\{\delta,\widehat{\delta}_0\}\,, \quad \text{with } \delta > 0 \quad \text{and} \quad \widehat{\delta}_0 := \kappa\,\alpha_{\widehat{\mathbf{A}}}/2\,. \tag{4.20}$$

We remark that δ is the same radius considered in Lemma 3.1. Then, we define

$$\mathbf{W}(r) := \left\{\mathbf{z}\in\mathcal{Q}: \quad \|\mathbf{z}\|_{0,\rho;\,\Omega} \leqslant r\right\},$$

and prove below that, under sufficiently small data, $\widehat{\mathbf{T}}$ maps $\mathbf{W}(r)$ into itself.

Lemma 4.4 *Let $r \in (0,\widehat{r}_0]$, with $\widehat{r}_0$ as in (4.20), and assume that the data satisfy (4.13) and*

$$C_{\widehat{\mathbf{T}}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\right)\right\} \leqslant r\,, \tag{4.21}$$

where $C_{\widehat{\mathbf{T}}} := C_{\mathbf{S}}\max\left\{C_{\widehat{\mathbf{S}}},1\right\}$ (cf. Lemmas 4.1 and 4.3). Then, $\widehat{\mathbf{T}}(\mathbf{W}(r)) \subset \mathbf{W}(r)$ and the restricted operator $\widehat{\mathbf{T}}|_{\mathbf{W}(r)} : \mathbf{W}(r) \to \mathbf{W}(r)$ is well-defined.

Proof. Having established the well-definedness of $\mathbf{S}$ and $\widehat{\mathbf{S}}|_{\mathbf{W}(r)}$ (cf. Lemmas 4.1 and 4.3), the operator $\widehat{\mathbf{T}}|_{\mathbf{W}(r)}$ is well-defined. In turn, given $\mathbf{z}\in\mathbf{W}(r)$, the estimate (4.12) in combination with (4.19), yields

$$\begin{aligned}\|\widehat{\mathbf{T}}(\mathbf{z})\|_{0,\rho;\Omega} &= \|\mathbf{S}(\mathbf{z},\widehat{\mathbf{S}}(\mathbf{z}))\|_{0,\rho;\Omega} \leqslant C_{\mathbf{S}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|\widehat{\mathbf{S}}(\mathbf{z})\|_{0,s;\Omega} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\right)\right\}\\ &\leqslant C_{\widehat{\mathbf{T}}}\left\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\right)\right\},\end{aligned}$$

which, thanks to (4.21), allows us to conclude that $\widehat{\mathbf{T}}(\mathbf{z}) \in \mathbf{W}(r)$, thus ending the proof. □

Hereafter, the restricted operator $\widehat{\mathbf{T}}|_{\mathbf{W}(r)} : \mathbf{W}(r) \to \mathbf{W}(r)$ is simply denoted by $\widehat{\mathbf{T}}$. We now aim to prove that $\widehat{\mathbf{T}}$ is a contraction, which will enable us to apply the well-known Banach fixed-point theorem. We start by proving two preliminary results, which will be instrumental in showing that $\widehat{\mathbf{T}}$ is Lipschitz continuous.

Lemma 4.5 *Let $r \in (0,\widehat{r}_0]$, with $\widehat{r}_0$ as in (4.20), and assume that the data satisfy (4.13). Then, there exists a positive constant $L_{\mathbf{S}}$, depending only on r, ρ, $\mathtt{F}_1$, $|\Omega|$, $C_{\mathbf{S}}$, and $\alpha_{\mathbf{A}}$, such that*

$$\|\mathbf{S}(\mathbf{z}_1,\varphi_1) - \mathbf{S}(\mathbf{z}_2,\varphi_2)\|_{0,\rho;\,\Omega} \leqslant L_{\mathbf{S}}\left\{C\big(\mathbf{u}_{\mathrm{D}},f,\mathbf{g},\phi_{\mathbf{r}},\varphi_2\big)\,\|\mathbf{z}_1-\mathbf{z}_2\|_{0,\rho;\Omega} + \|\mathbf{g}\|_{0,\Omega}\,\|\varphi_1-\varphi_2\|_{0,s;\,\Omega}\right\}, \tag{4.22}$$

for all $(\mathbf{z}_1,\varphi_1),(\mathbf{z}_2,\varphi_2) \in \mathbf{W}(r)\times\mathrm{Y}$, where

$$C\left(\mathbf{u}_{\mathrm{D}},f,\mathbf{g},\phi_{\mathbf{r}},\varphi\right) := \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|\varphi\|_{0,s;\,\Omega} + \|\phi_{\mathbf{r}}\|_{0,s;\,\Omega}\right) \quad \forall\,\varphi\in\mathrm{Y}\,.$$

Proof. The argument is analogous to that of Lemma 3.4, with $L_{\mathbf{S}} = \max\big\{L_{\mathbf{c}}\, C_{\mathbf{S}}\,(2r)^{\rho-3}, 1\big\}/\alpha_{\mathbf{A}}$. $\square$

Lemma 4.6 *Let $r \in (0, \widehat{r}_0]$, with $\widehat{r}_0$ as in (4.20), and assume that the data satisfy (4.13). Then, there exists a positive constant $L_{\widehat{\mathbf{S}}}$, depending only on κ, $\alpha_{\widehat{\mathbf{A}}}$, and $C_{\widehat{\mathbf{S}}}$, such that*

$$\|\widehat{\mathbf{S}}(\mathbf{z}_1) - \widehat{\mathbf{S}}(\mathbf{z}_2)\|_{0,s;\Omega} \,\leqslant\, L_{\widehat{\mathbf{S}}} \Big\{ \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\} \, \|\mathbf{z}_2 - \mathbf{z}_1\|_{0,\rho;\Omega} \,, \tag{4.23}$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}(r)$.

Proof. Let $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}(r)$ such that $\widehat{\mathbf{S}}(\mathbf{z}_1) = \phi_1$ and $\widehat{\mathbf{S}}(\mathbf{z}_2) = \phi_2$, where, for each $i \in \{1,2\}$, $(\boldsymbol{\vartheta}_i, \phi_i) \in \mathrm{X} \times \mathrm{Y}$ is the unique solution of the problem (4.9). Thus, it is easy to see that

$$\widehat{\mathbf{A}}\big((\boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2, \phi_1 - \phi_2), (\boldsymbol{\psi}, \xi)\big) + d_{\mathbf{z}_1}(\boldsymbol{\psi}, \phi_1) - d_{\mathbf{z}_2}(\boldsymbol{\psi}, \phi_2) = 0 \qquad \forall (\boldsymbol{\psi}, \xi) \in \mathrm{X} \times \mathrm{Y}\,,$$

which, along with (4.14) applied to $\boldsymbol{\varrho} = \boldsymbol{\vartheta}_1 - \boldsymbol{\vartheta}_2 \in \mathrm{X}$ and $\zeta = \phi_1 - \phi_2 \in \mathrm{Y}$, allows us to write

$$\|\phi_1 - \phi_2\|_{0,s;\Omega} \,\leqslant\, \frac{1}{\alpha_{\widehat{\mathbf{A}}}} \sup_{0 \neq \boldsymbol{\psi} \in \mathrm{X}} \frac{d_{\mathbf{z}_1}(\boldsymbol{\psi}, \phi_2) - d_{\mathbf{z}_2}(\boldsymbol{\psi}, \phi_2)}{\|\boldsymbol{\psi}\|_{\mathrm{div}_t;\Omega}}\,. \tag{4.24}$$

In turn, given $\boldsymbol{\psi} \in \mathrm{X} = \mathbf{H}(\mathrm{div}_t;\Omega)$, a straightforward application of the Cauchy–Schwarz and Hölder inequalities, along with the estimate (4.19), yields

$$\begin{aligned} \big|d_{\mathbf{z}_1}(\boldsymbol{\psi}, \phi_2) - d_{\mathbf{z}_2}(\boldsymbol{\psi}, \phi_2)\big| \,&\leqslant\, \kappa^{-1}\, \|\phi_2\|_{0,s;\Omega}\, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega}\, \|\boldsymbol{\psi}\|_{\mathrm{div}_t;\Omega} \\ &\leqslant\, \kappa^{-1}\, C_{\widehat{\mathbf{S}}} \Big\{ \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\}\, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega}\, \|\boldsymbol{\psi}\|_{\mathrm{div}_t;\Omega}\,, \end{aligned} \tag{4.25}$$

so that, replacing back (4.25) into (4.24), we obtain (4.23) with $L_{\widehat{\mathbf{S}}} := \kappa^{-1}\, \alpha_{\widehat{\mathbf{A}}}^{-1}\, C_{\widehat{\mathbf{S}}}$. $\square$

Next, as a consequence of Lemmas 4.5 and 4.6, we are able to prove the Lipschitz continuity of $\widehat{\mathbf{T}}$.

Lemma 4.7 *Let $r \in (0, \widehat{r}_0]$, with $\widehat{r}_0$ as in (4.20), and assume that the data satisfy (4.13). Then, there exists a positive constant $L_{\widehat{\mathbf{T}}}$, depending only on $L_{\mathbf{S}}$, $C_{\widehat{\mathbf{S}}}$, and $L_{\widehat{\mathbf{S}}}$, such that*

$$\begin{aligned} &\|\widehat{\mathbf{T}}(\mathbf{z}_1) - \widehat{\mathbf{T}}(\mathbf{z}_2)\|_{0,\rho;\Omega} \\ &\quad \leqslant\, L_{\widehat{\mathbf{T}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega} \big) \Big\}\, \|\mathbf{z}_1 - \mathbf{z}_2\|_{0,\rho;\Omega}\,, \end{aligned} \tag{4.26}$$

for all $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}(r)$.

Proof. Letting $\mathbf{z}_1, \mathbf{z}_2 \in \mathbf{W}(r)$ and employing (4.22), (4.19), and (4.23), the proof follows the same steps as in Lemma 3.6, with $L_{\widehat{\mathbf{T}}} := L_{\mathbf{S}} \max\big\{1, C_{\widehat{\mathbf{S}}} + L_{\widehat{\mathbf{S}}}\big\}$. Further details are omitted. $\square$

We conclude this section with the main result for the continuous problem, namely, the solvability of the fixed-point equation (4.11). The proof follows analogous arguments to those used in the proof of Theorem 3.7, and is therefore omitted.

Theorem 4.8 *Let $r \in (0, \widehat{r}_0]$, with $\widehat{r}_0$ as in (4.20), and assume that the data satisfy (4.13), (4.21), and*

$$L_{\widehat{\mathbf{T}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega} \big) \Big\} \,<\, 1\,.$$

Then, there exists a unique $\mathbf{u} \in \mathbf{W}(r)$ such that $\widehat{\mathbf{T}}(\mathbf{u}) = \mathbf{u}$, or, equivalently, the problem (4.6) has a unique solution $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\vartheta}, \phi) \in \mathcal{H} \times \mathcal{Q} \times \mathrm{X} \times \mathrm{Y}$, with $\mathbf{u} \in \mathbf{W}(r)$. Moreover, there exist positive constants $\mathcal{C}_1$ and $\mathcal{C}_2$, depending only on $C_{\mathbf{S}}$ and $C_{\widehat{\mathbf{S}}}$, such that

$$\begin{aligned} \|(\boldsymbol{\sigma}, \mathbf{u})\|_{\mathcal{H}\times\mathcal{Q}} \,&\leqslant\, \mathcal{C}_1 \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega} \big) \Big\}\,, \\ \textit{and} \qquad \|(\boldsymbol{\vartheta}, \phi)\|_{\mathrm{X}\times\mathrm{Y}} \,&\leqslant\, \mathcal{C}_2 \Big\{ \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\}\,. \end{aligned} \tag{4.27}$$

4.3 The Galerkin scheme

In this section, we introduce and analyze a Galerkin scheme for the fully-mixed variational formulation (4.6). To this end, we focus mainly on the discrete scheme arising from the convection-diffusion-reaction equations since the one corresponding to the Brinkman–Forchheimer part is exactly as derived in Section 3.3, except the space where the given discrete concentration is taken now. The above means that in what follows we consider the same generic finite-dimensional subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \subset \mathbb{H}(\mathbf{div}_\ell;\Omega)$ and $\mathbf{H}_h^{\mathbf{u}} \subset \mathbf{L}^\rho(\Omega)$ from before, and set $\mathbb{H}_h^{\boldsymbol{\sigma}} := \widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, whereas for the convection-diffusion-reaction component, we introduce finite-dimensional subspaces $\mathbf{H}_h^{\boldsymbol{\vartheta}} \subset \mathbf{H}(\mathrm{div}_t;\Omega)$ and $\widehat{\mathrm{H}}_h^{\phi} \subset \mathrm{L}^s(\Omega)$. In this way, the Galerkin scheme associated with (4.6) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}_h) + \mathbf{b}(\boldsymbol{\tau}_h,\mathbf{u}_h) &= \mathbf{F}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_h^{\boldsymbol{\sigma}}\,,\\
\mathbf{b}(\boldsymbol{\sigma}_h,\mathbf{v}_h) - \mathbf{c}_{\mathbf{u}_h}(\mathbf{u}_h,\mathbf{v}_h) &= \mathbf{G}_{\phi_h}(\mathbf{v}_h) && \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}\,,\\
\widehat{a}(\boldsymbol{\vartheta}_h,\boldsymbol{\psi}_h) + \widehat{b}(\boldsymbol{\psi}_h,\phi_h) + d_{\mathbf{u}_h}(\boldsymbol{\psi}_h,\phi_h) &= \widehat{\mathrm{F}}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}}\,,\\
\widehat{b}(\boldsymbol{\vartheta}_h,\xi_h) - \widehat{c}_f(\phi_h,\xi_h) &= \widehat{\mathrm{G}}(\xi_h) && \forall\, \xi_h \in \widehat{\mathrm{H}}_h^{\phi}\,.
\end{aligned}
\tag{4.28}
$$

Then, following the discrete analogue of the approach from Section 4.2, we use here a fixed-point strategy to address the well-posedness of (4.28). More precisely, we let $\mathbf{S}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \times \widehat{\mathrm{H}}_h^{\phi} \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined as in the mixed-primal approach (cf. (3.46)), where we remark, as previously announced, that the only change is the use of the space $\widehat{\mathrm{H}}_h^{\phi}$ instead of H_h^{ϕ}. In turn, we define the operator $\widehat{\mathbf{S}}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \to \widehat{\mathrm{H}}_h^{\phi}$ by $\widehat{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h) := \phi_h$, where $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}$ is the unique solution, to be confirmed later, of the problem arising from the third and fourth equations of (4.28), after replacing $\mathbf{u}_h$ by the given $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$, that is

$$
\begin{aligned}
\widehat{a}(\boldsymbol{\vartheta}_h,\boldsymbol{\psi}_h) + \widehat{b}(\boldsymbol{\psi}_h,\phi_h) + d_{\mathbf{z}_h}(\boldsymbol{\psi}_h,\phi_h) &= \widehat{\mathrm{F}}(\boldsymbol{\psi}_h) && \forall\, \boldsymbol{\psi}_h \in \mathbf{H}_h^{\boldsymbol{\vartheta}}\,,\\
\widehat{b}(\boldsymbol{\vartheta}_h,\xi_h) - \widehat{c}_f(\phi_h,\xi_h) &= \widehat{\mathrm{G}}(\xi_h) && \forall\, \xi_h \in \widehat{\mathrm{H}}_h^{\phi}\,.
\end{aligned}
\tag{4.29}
$$

Finally, we define the operator $\widehat{\mathbf{T}}_{\mathtt{d}} : \mathbf{H}_h^{\mathbf{u}} \to \mathbf{H}_h^{\mathbf{u}}$ as

$$
\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{z}_h) := \mathbf{S}_{\mathtt{d}}(\mathbf{z}_h, \widehat{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)) \qquad \forall\, \mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}\,,
$$

and realize that solving (4.28) is equivalent to finding a fixed point of the operator $\widehat{\mathbf{T}}_{\mathtt{d}}$, that is, seeking $\mathbf{u}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that

$$
\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{u}_h) \,=\, \mathbf{u}_h\,.
$$

Next, we aim to establish the well-definedness of the discrete operators $\mathbf{S}_{\mathtt{d}}$ and $\widehat{\mathbf{S}}_{\mathtt{d}}$. We begin with $\mathbf{S}_{\mathtt{d}}$ by assuming throughout the rest of this section the same hypotheses **(H.0)**, **(H.1)**, and **(H.2)** on the subspaces $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and $\mathbf{H}_h^{\mathbf{u}}$, that were introduced in Section 3.3. In this way, and employing the same arguments from the proof of Lemma 3.8, we are able to state the following result.

Lemma 4.9 *Given $\delta_{\mathtt{d}} > 0$ and $(\mathbf{z}_h, \varphi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \widehat{\mathrm{H}}_h^{\phi}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \delta_{\mathtt{d}}$, the problem (3.46) has a unique solution $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}$ and, consequently, $\mathbf{S}_{\mathtt{d}}(\mathbf{z}, \varphi_h)$ is well-defined. Moreover, there exists a positive constant $C_{\mathbf{S}_{\mathtt{d}}}$, depending only on $\delta_{\mathtt{d}}$, ρ, ν_0, ν_1, $\mathtt{D}_1$, $\mathtt{F}_1$, $\beta_{\mathtt{d}}$ (cf. (3.50)), and $|\Omega|$, such that*

$$
\begin{aligned}
\|\mathbf{S}_{\mathtt{d}}(\mathbf{z}_h,\psi_h)\|_{0,\rho;\Omega} &\leqslant \|(\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}\\
&\leqslant C_{\mathbf{S}_{\mathtt{d}}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \left(\|\varphi_h\|_{0,s;\Omega} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega} \right) \Big\}\,.
\end{aligned}
$$

In order to deal with the well-definedness of $\widehat{\mathbf{S}}_{\mathbf{d}}$, equivalently the well-posedness of (4.29), we now introduce two hypotheses concerning the finite element subspaces $\mathbf{H}_h^{\boldsymbol{\vartheta}}$ and $\widehat{\mathrm{H}}_h^{\phi}$, namely:

$\widehat{\mathbf{(H.3)}}$ $\mathrm{div}(\mathbf{H}_h^{\boldsymbol{\vartheta}}) \subset \widehat{\mathrm{H}}_h^{\phi}$,

$\widehat{\mathbf{(H.4)}}$ there exists a positive constant $C_{\widehat{b},\mathbf{d}}$, independent of h, such that

$$\sup_{0\neq\boldsymbol{\psi}_h\in\mathbf{H}_h^{\boldsymbol{\vartheta}}} \frac{\widehat{b}(\boldsymbol{\psi}_h,\zeta_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{div}_t;\Omega}} \geqslant C_{\widehat{b},\mathbf{d}}\,\|\zeta_h\|_{0,s;\Omega} \qquad \forall\,\zeta_h\in\widehat{\mathrm{H}}_h^{\phi}\,, \tag{4.30}$$

which are also assumed to hold throughout the rest of this section.

The following result constitutes the discrete analogue of Lemma 4.2.

Lemma 4.10 *Assume that the data satisfy* (4.13)*. Then, there exists a positive constant* $\alpha_{\widehat{\mathbf{A}},\mathbf{d}}$*, depending only on* κ*,* η*,* ρ*,* $C_{\widehat{b},\mathbf{d}}$*, and* $|\Omega|$*, such that*

$$\sup_{\mathbf{0}\neq(\boldsymbol{\psi}_h,\xi_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}_h,\zeta_h),(\boldsymbol{\psi}_h,\xi_h))}{\|(\boldsymbol{\psi}_h,\xi_h)\|_{\mathrm{X}\times\mathrm{Y}}} \geqslant \alpha_{\widehat{\mathbf{A}},\mathbf{d}}\,\|(\boldsymbol{\varrho}_h,\zeta_h)\|_{\mathrm{X}\times\mathrm{Y}} \quad \forall\,(\boldsymbol{\varrho}_h,\zeta_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}\,. \tag{4.31}$$

Proof. We proceed as in the proof of Lemma 4.2 by applying now [23, Theorem 3.5], which is the discrete version of [23, Theorem 3.4]. Indeed, we first recall, as established at the beginning of that proof, that $\widehat{a}$ and $\widehat{c}_f$ are symmetric and positive semi-definite. In addition, employing $\widehat{\mathbf{(H.3)}}$, we readily find that the kernel $\widehat{V}_h$ of the discrete operator $\mathbf{H}_h^{\boldsymbol{\vartheta}}\ni\boldsymbol{\psi}_h\mapsto\widehat{b}(\boldsymbol{\psi}_h,\cdot)\in\big(\widehat{\mathrm{H}}_h^{\phi}\big)'$ reduces to

$$\widehat{V}_h \;=\; \Big\{\boldsymbol{\psi}_h\in\mathbf{H}_h^{\boldsymbol{\vartheta}}: \quad \mathrm{div}(\boldsymbol{\psi}_h)=0 \quad \text{in} \quad \Omega\Big\}\,,$$

which is certainly contained in the continuous kernel $\widehat{V}$ (cf. (4.15)). It follows that the discrete version of (4.16) holds with the same constant, that is

$$\sup_{0\neq\boldsymbol{\psi}_h\in\widehat{V}_h} \frac{\widehat{a}(\widehat{\boldsymbol{\varrho}}_h,\boldsymbol{\psi}_h)}{\|\boldsymbol{\psi}_h\|_{\mathrm{div}_t;\Omega}} \geqslant \kappa^{-1}\,\|\widehat{\boldsymbol{\varrho}}_h\|_{\mathrm{div}_t;\Omega} \qquad \forall\,\widehat{\boldsymbol{\varrho}}_h\in\widehat{V}_h\,.$$

In turn, the hypothesis $\widehat{\mathbf{(H.4)}}$ (cf. (4.30)) establishes the discrete inf-sup condition for $\widehat{b}$. Therefore, a straightforward application of [23, Theorem 3.5] yields (4.31) with a constant $\alpha_{\widehat{\mathbf{A}},\mathbf{d}}$ as announced. ▫

As a consequence of (4.31), the stability property (4.7) again, and the symmetry of $\widehat{\mathbf{A}}$, we easily deduce the discrete analogues of (4.17) and (4.18). More precisely, for each $\mathbf{z}_h\in\mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega}\leqslant\widehat{\delta}_0^{\mathrm{d}}:=\kappa\,\alpha_{\widehat{\mathbf{A}},\mathbf{d}}/2$, there hold

$$\sup_{\mathbf{0}\neq(\boldsymbol{\psi}_h,\xi_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}_h,\zeta_h),(\boldsymbol{\psi}_h,\xi_h))+d_{\mathbf{z}_h}(\boldsymbol{\psi}_h,\zeta_h)}{\|(\boldsymbol{\psi}_h,\xi_h)\|_{\mathrm{X}\times\mathrm{Y}}} \geqslant \frac{\alpha_{\widehat{\mathbf{A}},\mathbf{d}}}{2}\,\|(\boldsymbol{\varrho}_h,\zeta_h)\|_{\mathrm{X}\times\mathrm{Y}}\,, \tag{4.32}$$

for all $(\boldsymbol{\varrho}_h,\zeta_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}$, and

$$\sup_{\mathbf{0}\neq(\boldsymbol{\varrho}_h,\zeta_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}} \frac{\widehat{\mathbf{A}}((\boldsymbol{\varrho}_h,\zeta_h),(\boldsymbol{\psi}_h,\xi_h))+d_{\mathbf{z}_h}(\boldsymbol{\psi}_h,\zeta_h)}{\|(\boldsymbol{\varrho}_h,\zeta_h)\|_{\mathrm{X}\times\mathrm{Y}}} \geqslant \frac{\alpha_{\widehat{\mathbf{A}},\mathbf{d}}}{2}\,\|(\boldsymbol{\psi}_h,\xi_h)\|_{\mathrm{X}\times\mathrm{Y}}\,, \tag{4.33}$$

for all $(\boldsymbol{\psi}_h,\xi_h)\in\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}$.

We are now in a position to establish the discrete analogue of Lemma 4.3.

Lemma 4.11 *Let $\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}}$ such that $\|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant \widehat{\delta}_0^{\mathtt{d}} := \kappa\,\alpha_{\widehat{\mathbf{A}},\mathtt{d}}/2$, and assume that the data satisfy (4.13). Then, the problem (4.29) has a unique solution $(\boldsymbol{\vartheta}_h, \phi_h) \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}$ and, hence, $\widehat{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)$ is well-defined. Moreover, there exists a positive constant $C_{\widehat{\mathbf{S}}_{\mathtt{d}}}$, depending only on κ, η, ρ, $C_{\widehat{b},\mathtt{d}}$, and $|\Omega|$, such that*

$$\|\widehat{\mathbf{S}}_{\mathtt{d}}(\mathbf{z}_h)\|_{0,s;\Omega} \leqslant \|(\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant C_{\widehat{\mathbf{S}}_{\mathtt{d}}} \Big\{\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma}\Big\}.$$

Proof. It follows similarly to the proof of Lemma 4.3, by applying now the discrete version of the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.22]), that is, by taking into account either (4.32) or (4.33). We recall that in the finite-dimensional case these discrete inf-sup conditions are equivalent, and hence just one of them suffices to conclude. Further details are omitted. ▫

Having established the well-definedness of the operators $\mathbf{S}_{\mathtt{d}}$ and $\widehat{\mathbf{S}}_{\mathtt{d}}$, we now proceed to show that the equation $\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h$ admits a unique solution under certain conditions on the data. To this end, we take $r \in (0, \widehat{r}_0^{\mathtt{d}}]$, where

$$\widehat{r}_0^{\mathtt{d}} := \min\{\delta_{\mathtt{d}}, \widehat{\delta}_0^{\mathtt{d}}\}, \quad \text{with } \delta_{\mathtt{d}} > 0 \quad \text{and} \quad \widehat{\delta}_0^{\mathtt{d}} := \kappa\,\alpha_{\widehat{\mathbf{A}},\mathtt{d}}/2, \tag{4.34}$$

and define

$$\mathbf{W}_h(r) := \Big\{\mathbf{z}_h \in \mathbf{H}_h^{\mathbf{u}} : \quad \|\mathbf{z}_h\|_{0,\rho;\Omega} \leqslant r\Big\}.$$

The following result shows that, under sufficiently small data, $\widehat{\mathbf{T}}_{\mathtt{d}}$ maps $\mathbf{W}_h(r)$ into itself.

Lemma 4.12 *Let $r \in (0, \widehat{r}_0^{\mathtt{d}}]$, with $\widehat{r}_0^{\mathtt{d}}$ as in (4.34), and assume that the data satisfy (4.13) and*

$$C_{\widehat{\mathbf{T}}_{\mathtt{d}}} \Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,t;\Omega}\left(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\Big\} \leqslant r, \tag{4.35}$$

where $C_{\widehat{\mathbf{T}}_{\mathtt{d}}} := C_{\mathbf{S}_{\mathtt{d}}} \max\big\{C_{\widehat{\mathbf{S}}_{\mathtt{d}}}, 1\big\}$ (cf. Lemmas 4.9 and 4.11). Then, $\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{W}_h(r)) \subseteq \mathbf{W}_h(r)$ and the restricted operator $\widehat{\mathbf{T}}_{\mathtt{d}}|_{\mathbf{W}_h(r)} : \mathbf{W}_h(r) \to \mathbf{W}_h(r)$ is well-defined.

Proof. It is analogous to the proof of Lemma 4.4. ▫

We can establish the discrete analogues of Lemmas 4.5 and 4.6, with constants denoted $L_{\mathbf{S}_{\mathtt{d}}}$ and $L_{\widehat{\mathbf{S}}_{\mathtt{d}}}$, respectively, and subsequently derive the remaining results. However, since their proofs closely follow those of their continuous counterparts, we omit them and restrict ourselves to stating next the discrete analogue of Lemma 4.7 together with the well-posedness of the Galerkin scheme, without providing detailed proofs either.

Theorem 4.13 *Let $r \in (0, \widehat{r}_0^{\mathtt{d}}]$, with $\widehat{r}_0^{\mathtt{d}}$ as in (4.34) and assume that the data satisfy (4.13) and (4.35). Then, there exists a positive constant $L_{\widehat{\mathbf{T}}_{\mathtt{d}}}$, depending only on $L_{\mathbf{S}_{\mathtt{d}}}$, $C_{\widehat{\mathbf{S}}_{\mathtt{d}}}$, and $L_{\widehat{\mathbf{S}}_{\mathtt{d}}}$, such that*

$$\begin{aligned}
&\|\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{z}_{1,h}) - \widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{z}_{2,h})\|_{0,\rho;\Omega} \\
&\qquad \leqslant L_{\widehat{\mathbf{T}}_{\mathtt{d}}} \Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\Big\} \|\mathbf{z}_{1,h} - \mathbf{z}_{2,h}\|_{0,\rho;\Omega}
\end{aligned}$$

for all $\mathbf{z}_{1,h}, \mathbf{z}_{2,h} \in \mathbf{W}(r)$. Moreover, if

$$L_{\widehat{\mathbf{T}}_{\mathtt{d}}} \Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega}\left(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathtt{r}}\|_{0,s;\Omega}\right)\Big\} < 1,$$

there exists a unique $\mathbf{u}_h \in \mathbf{W}_h(r)$ *such that* $\widehat{\mathbf{T}}_{\mathtt{d}}(\mathbf{u}_h) = \mathbf{u}_h$*, or, equivalently, the problem* (4.28) *has a unique solution* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\vartheta}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \mathrm{H}_h^{\phi}$*, with* $\mathbf{u}_h \in \mathbf{W}_h(r)$*. Moreover, there exist positive constants* $\mathcal{C}_{1,\mathrm{d}}$ *and* $\mathcal{C}_{2,\mathrm{d}}$*, depending only on* $C_{\mathbf{S}_{\mathrm{d}}}$ *and* $C_{\widehat{\mathbf{S}}_{\mathrm{d}}}$*, such that*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \leqslant \mathcal{C}_{1,\mathrm{d}} \Big\{ \|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|\mathbf{g}\|_{0,\Omega} \big(\|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega} \big) \Big\},$$

$$\textit{and} \qquad \|(\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant \mathcal{C}_{2,\mathrm{d}} \Big\{ \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} \Big\}.$$

4.4 *A priori* error analysis

We now aim to derive an *a priori* error estimate for the Galerkin scheme (4.28). For this purpose, we set $r \in \big(0, \min\{\widehat{r}_0, \widehat{r}_0^{\mathtt{d}}\}\big]$, with $\widehat{r}_0$, $\widehat{r}_0^{\mathtt{d}}$ satisfying (4.20), (4.34), and let $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\vartheta}, \phi) \in \mathcal{H} \times \mathcal{Q} \times \mathrm{X} \times \mathrm{Y}$, with $\mathbf{u} \in \mathbf{W}(r)$, and $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\vartheta}_h, \phi_h) \in \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}$, with $\mathbf{u}_h \in \mathbf{W}_h(r)$, be the unique solutions of the continuous problem (4.6) and the Galerkin scheme (4.28), respectively. We begin with the estimate for the Brinkman–Forchheimer part of error, that is, $\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}$, for which, proceeding analogously to the analysis in Section 3.4, and using again the notation from (3.56), we derive the existence of a positive constant $\widehat{\mathcal{C}}_{\mathtt{ST,BF}}$, depending only on $\mathcal{C}_{\mathtt{ST,BF}}$ (cf. (3.57)), $\|i_s\|$, $\|\mathbf{g}\|_{0,\Omega}$, $L_{\mathbf{c}}$, r, and ρ, such that the new version of (3.60) reads

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \mathbf{u}) - (\boldsymbol{\sigma}_h, \mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} \\ &\quad \leqslant \widehat{\mathcal{C}}_{\mathtt{ST,BF}} \Big\{ \mathrm{dist}\,\big((\boldsymbol{\sigma}, \mathbf{u}), \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}}\big) + \|\phi - \phi_h\|_{0,s;\Omega} + \|\mathbf{u}\|_{0,\rho;\Omega}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\rho;\Omega} \Big\}. \end{aligned} \tag{4.36}$$

Furthermore, in order to derive the error estimate for the convection-diffusion-reaction part, that is, $\|(\boldsymbol{\vartheta}, \phi) - (\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}}$, we first rewrite the last two rows of (4.6) and (4.28), respectively, as

$$\begin{aligned} \mathcal{A}_{\mathbf{u}}\big((\boldsymbol{\vartheta}, \phi), (\boldsymbol{\psi}, \xi)\big) &= \mathcal{F}(\boldsymbol{\psi}, \xi) \qquad \forall\, (\boldsymbol{\psi}, \xi) \in \mathrm{X} \times \mathrm{Y}, \\ \mathcal{A}_{\mathbf{u}_h}\big((\boldsymbol{\vartheta}_h, \phi_h), (\boldsymbol{\psi}_h, \xi_h)\big) &= \mathcal{F}(\boldsymbol{\psi}_h, \xi_h) \quad \forall\, (\boldsymbol{\psi}_h, \xi_h) \in \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}, \end{aligned}$$

where the bilinear form $\mathcal{A}_{\mathbf{z}} : (\mathrm{X}\times\mathrm{Y}) \times (\mathrm{X}\times\mathrm{Y}) \to \mathrm{R}$, for each $\mathbf{z} \in \mathcal{Q} := \mathbf{L}^\rho(\Omega)$, and the linear functional $\mathcal{F} : \mathrm{X} \times \mathrm{Y} \to \mathrm{R}$, are defined, respectively, by

$$\mathcal{A}_{\mathbf{z}}\big((\boldsymbol{\vartheta}, \phi), (\boldsymbol{\psi}, \xi)\big) := \widehat{\mathbf{A}}\big((\boldsymbol{\vartheta}, \phi), (\boldsymbol{\psi}, \xi)\big) + d_{\mathbf{z}}(\boldsymbol{\psi}, \phi) \quad \text{and} \quad \mathcal{F}(\boldsymbol{\psi}, \xi) := \widehat{\mathrm{F}}(\boldsymbol{\psi}) + \widehat{\mathrm{G}}(\xi),$$

for all $(\boldsymbol{\vartheta}, \phi)$, $(\boldsymbol{\psi}, \xi) \in \mathrm{X} \times \mathrm{Y}$. Then, knowing from (4.32) or (4.33), that $\mathcal{A}_{\mathbf{u}_h}$ satisfies the hypotheses of the discrete version of the Banach–Nečas–Babuška theorem (cf. [27, Theorem 2.22]) with constant $\alpha_{\widehat{\mathbf{A}},\mathbf{d}}/2$, we can apply the Strang-type estimate given by [15, Lemma 5.1] to conclude the existence of a positive constant $\mathcal{C}_{\mathtt{ST,CDR}}$, depending only on $\alpha_{\widehat{\mathbf{A}},\mathbf{d}}$, κ, η, $|\Omega|$, r, and ρ, such that

$$\|(\boldsymbol{\vartheta}, \phi) - (\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant \mathcal{C}_{\mathtt{ST,CDR}} \Big\{ \mathrm{dist}\,\big((\boldsymbol{\vartheta}, \phi), \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}\big) + \|d_{\mathbf{u}}(\cdot, \phi) - d_{\mathbf{u}_h}(\cdot, \phi)\|_{(\mathbf{H}_h^{\boldsymbol{\vartheta}})'} \Big\}. \tag{4.37}$$

Next, employing the estimate provided by the first row of (4.25), we get

$$\|d_{\mathbf{u}}(\cdot, \phi) - d_{\mathbf{u}_h}(\cdot, \phi)\|_{(\mathbf{H}_h^{\boldsymbol{\vartheta}})'} \leqslant \kappa^{-1}\, \|\phi\|_{0,s;\Omega}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\rho\,\Omega},$$

which, replaced back into (4.37), leads to

$$\|(\boldsymbol{\vartheta}, \phi) - (\boldsymbol{\vartheta}_h, \phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \leqslant \widehat{\mathcal{C}}_{\mathtt{ST,CDR}} \Big\{ \mathrm{dist}\,\big((\boldsymbol{\vartheta}, \phi), \mathbf{H}_h^{\boldsymbol{\vartheta}} \times \widehat{\mathrm{H}}_h^{\phi}\big) + \|\phi\|_{0,s;\Omega}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,\rho\,\Omega} \Big\}, \tag{4.38}$$

where $\widehat{\mathcal{C}}_{\mathtt{ST,CDR}} := \mathcal{C}_{\mathtt{ST,CDR}} \max\{\kappa^{-1}, 1\}$. Thus, multiplying now (4.36) by $\frac{1}{2\,\widehat{\mathcal{C}}_{\mathtt{ST,BF}}}$, adding the resulting inequality to (4.38), and bounding $\|\mathbf{u}\|_{0,\rho;\Omega}$ and $\|\phi\|_{0,s;\Omega}$ according to (4.27), we arrive at

$$\begin{aligned} &\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\boldsymbol{\vartheta},\phi) - (\boldsymbol{\vartheta}_h,\phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \\ &\quad\leqslant \widehat{\mathcal{C}}\Big\{\mathrm{dist}\,\big((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\big) + \mathrm{dist}\,\big((\boldsymbol{\vartheta},\phi),\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}\big)\Big\} \\ &\quad+ \widetilde{\mathcal{C}}\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\Big\}\,\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;\Omega}\,, \end{aligned} \tag{4.39}$$

where $\widehat{\mathcal{C}}$ and $\widetilde{\mathcal{C}}$ are positive constants depending only on $\widehat{\mathcal{C}}_{\mathtt{ST,BF}}$, $\widehat{\mathcal{C}}_{\mathtt{ST,CDR}}$, $\mathcal{C}_1$, $\mathcal{C}_2$ (cf. (4.27)), and $\|\mathbf{g}\|_{0,\Omega}$.

We conclude this section with the Céa estimate associated with the Galerkin scheme (4.28).

Theorem 4.14 *In addition to the hypotheses of Theorems 4.8 and 4.13, assume that*

$$\widetilde{\mathcal{C}}\Big\{\|\mathbf{u}_{\mathrm{D}}\|_{1/2,\Gamma} + \|f\|_{0,\Omega} + \|g\|_{0,t;\Omega} + \|\phi_{\mathrm{D}}\|_{1/2,\Gamma} + \|\phi_{\mathbf{r}}\|_{0,s;\Omega}\Big\} \leqslant \frac{1}{2}\,. \tag{4.40}$$

Then, there holds

$$\begin{aligned} &\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} + \|(\boldsymbol{\vartheta},\phi) - (\boldsymbol{\vartheta}_h,\phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \\ &\leqslant 2\,\widehat{\mathcal{C}}\Big\{\mathrm{dist}\,\big((\boldsymbol{\sigma},\mathbf{u}),\mathbb{H}_h^{\boldsymbol{\sigma}}\times\mathbf{H}_h^{\mathbf{u}}\big) + \mathrm{dist}\,\big((\boldsymbol{\vartheta},\phi),\mathbf{H}_h^{\boldsymbol{\vartheta}}\times\widehat{\mathrm{H}}_h^{\phi}\big)\Big\}\,. \end{aligned} \tag{4.41}$$

Proof. It suffices to bound $\|\mathbf{u}-\mathbf{u}_h\|_{0,\rho;,\Omega}$ in (4.39) by $\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}}$, and then use assumption (4.40). We omit further details. □

4.5 Specific finite element subspaces and rates of convergence

In this section, we present an example of finite element subspaces that satisfy the hypotheses introduced in Section 4.3. Using the same notations as in Section 3.5, we consider the tensor Raviart–Thomas space of order k for $\widetilde{\mathbb{H}}_h^{\boldsymbol{\sigma}}$ and the discontinuous polynomial space of order k for $\mathbf{H}_h^{\mathbf{u}}$ to approximate $\boldsymbol{\sigma}$ and $\mathbf{u}$, respectively (cf. (3.66)). As we noticed in Section 3.5, these spaces satisfy hypotheses **(H.0)**, **(H.1)**, and **(H.2)**. Additionally, we introduce the subspaces

$$\begin{aligned} \mathbf{H}_h^{\boldsymbol{\vartheta}} &:= \Big\{\boldsymbol{\psi}_h \in \mathbf{H}(\mathrm{div}_t;\Omega): \quad \boldsymbol{\psi}_h\big|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \\ \widehat{\mathrm{H}}_h^{\phi} &:= \Big\{\xi_h \in \mathrm{L}^s(\Omega): \quad \xi_h\big|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h\Big\}\,, \end{aligned} \tag{4.42}$$

which we employ to approximate the solutions $\boldsymbol{\vartheta}$ and ϕ of the problem (4.6). These spaces are stable for the Galerkin scheme (4.28), as they satisfy the hypotheses $\widehat{\textbf{(H.3)}}$ and $\widehat{\textbf{(H.4)}}$. In fact, it is straightforward to deduce that $\mathrm{div}(\mathbf{H}_h^{\boldsymbol{\vartheta}}) \subset \widehat{\mathrm{H}}_h^{\phi}$, thus proving $\widehat{\textbf{(H.3)}}$, whereas $\widehat{\textbf{(H.4)}}$ has already been established by [11, Lemma 3.3] for $\rho \in (3,4]$, which can be easily extended to the case $\rho = 3$.

We now collect the approximation properties associated with $\mathbf{H}_h^{\boldsymbol{\vartheta}}$ and $\widehat{\mathrm{H}}_h^{\phi}$ (see, e.g., [6], [11], [21], [27], and [28]). Those regarding the remaining spaces are provided in Section 3.5.

$(\mathbf{AP}_h^{\boldsymbol{\vartheta}})$ there exists $C > 0$, independent of h, such that, for each $l \in (0, k+1]$, and for each $\boldsymbol{\psi} \in \mathbf{H}^l(\Omega)$ with $\mathrm{div}(\boldsymbol{\psi}) \in \mathrm{W}^{l,t}(\Omega)$, there holds

$$\mathrm{dist}\,(\boldsymbol{\psi},\mathbf{H}_h^{\boldsymbol{\vartheta}}) \leqslant C\,h^l\Big\{\|\boldsymbol{\psi}\|_{l,\Omega} + \|\mathrm{div}(\boldsymbol{\psi})\|_{l,t;\Omega}\Big\}\,,$$

$(\mathbf{AP}_h^{\phi})$ there exists $C > 0$, independent of h, such that for each $l \in [0, k+1]$, and for each $\xi \in \mathrm{W}^{l,s}(\Omega)$, there holds

$$\operatorname{dist}(\xi, \widehat{\mathrm{H}}_h^{\phi}) \leqslant C\, h^l\, \|\xi\|_{l,s;\Omega}\,.$$

We end this section with the rates of convergence of our Galerkin scheme (4.28), which follow from the Céa estimate (4.41) and the approximation properties of the subspaces involved.

Theorem 4.15 *In addition to the hypotheses of Theorems 4.8, 4.13, and 4.14, assume that there exists $l \in (0, k+1]$ such that $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_\ell;\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,\ell}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{l,\rho}(\Omega)$, $\boldsymbol{\vartheta} \in \mathbf{H}^l(\Omega)$, $\operatorname{div}(\boldsymbol{\vartheta}) \in \mathrm{W}^{l,t}(\Omega)$, and $\phi \in \mathrm{W}^{l,s}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}\|(\boldsymbol{\sigma},\mathbf{u}) - (\boldsymbol{\sigma}_h,\mathbf{u}_h)\|_{\mathcal{H}\times\mathcal{Q}} &+ \|(\boldsymbol{\vartheta},\phi) - (\boldsymbol{\vartheta}_h,\phi_h)\|_{\mathrm{X}\times\mathrm{Y}} \\ &\leqslant C\, h^l \Big\{ \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,\ell;\Omega} + \|\mathbf{u}\|_{l,\rho;\Omega} + \|\boldsymbol{\vartheta}\|_{l,\Omega} + \|\operatorname{div}(\boldsymbol{\vartheta})\|_{l,t;\,\Omega} + \|\phi\|_{l,s;\Omega} \Big\}\,.\end{aligned}$$

5 Numerical tests

In this section we present three examples illustrating the performance of the mixed finite element methods (3.45) and (4.28) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (3.66), (3.67), (3.68) and (4.42) (cf. Sections 3.5 and 4.5). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k \in \{0,1\}$ in the mixed-primal and fully-mixed schemes as simply $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1} - \mathrm{P}_k$ and $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$, respectively. The numerical methods are implemented using `FreeFEM` [32]. A Newton–Raphson algorithm with a fixed tolerance of $\mathsf{tol} = 1\mathrm{E}-06$ is used to solve both nonlinear problems (3.45) and (4.28). As usual, the iterative process is terminated when the relative error between two successive iterates of the full coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, becomes sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|_{\mathtt{DoF}}}{\|\mathbf{coeff}^{m+1}\|_{\mathtt{DoF}}} \leqslant \mathsf{tol}\,,$$

where $\|\cdot\|_{\mathtt{DoF}}$ denotes the standard Euclidean norm in $\mathrm{R}^{\mathtt{DoF}}$, and `DoF` represents the total number of degrees of freedom associated with the finite element subspaces involved.

We now introduce some additional notation. The individual errors are denoted by

$$\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_\ell;\Omega}\,, \quad \mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,\rho;\Omega}\,, \quad \mathsf{e}(\phi) := \|\phi - \phi_h\|_{1,\Omega}\,,$$

$$\mathsf{e}(\lambda) := \|\lambda - \lambda_{\widetilde{h}}\|_{0,\Gamma}\,, \quad \mathsf{e}(\boldsymbol{\vartheta}) := \|\boldsymbol{\vartheta} - \boldsymbol{\vartheta}_h\|_{\operatorname{div}_t;\Omega}\,, \quad \text{and} \quad \widehat{\mathsf{e}}(\phi) := \|\phi - \phi_h\|_{0,s;\Omega}\,,$$

where ℓ, ρ, t and s are described in (4.5), and will be specified in the examples below. We emphasize that other physically relevant variables, such as the pressure, velocity gradient, vorticity, and shear stress tensor, can be computed using suitable postprocessing formulae, such as those in (3.2) for the pressure and the velocity gradient. However, to avoid overcharging this section, in the examples below we only present plots of the pressure obtained from this formula (cf. first equation in (3.2)):

$$p_h = -\frac{1}{n}\operatorname{tr}(\boldsymbol{\sigma}_h) + \frac{\nu}{n}\, f\,.$$

As usual, for each $\diamond \in \{\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\vartheta}, \phi\}$ and for λ, we denote by $\mathsf{r}(\diamond)$ and $\mathsf{r}(\lambda)$ the corresponding experimental rates of convergence, defined by

$$\mathsf{r}(\diamond) := \frac{\log(\mathsf{e}(\diamond)/\mathsf{e}'(\diamond))}{\log(h/h')} \quad \text{and} \quad \mathsf{r}(\lambda) := \frac{\log(\mathsf{e}(\lambda)/\mathsf{e}'(\lambda))}{\log(\widetilde{h}/\widetilde{h}')}\,,$$

where e, e' denote errors computed on two consecutive meshes of sizes h, h' ($\widetilde{h}$ and $\widetilde{h}'$ for λ), respectively.

The examples considered in this section are described below. In all of them, for the sake of simplicity, we set $\kappa = 1$, $\eta = 1$, and $\phi_{\mathtt{r}} = 0$, and choose the Brinkman, Darcy, and Forchheimer coefficients as follows:

$$\nu(\mathbf{x}) = \exp\left(-\prod_{i=1}^{n} x_i\right), \quad \mathtt{D}(\mathbf{x}) = \exp\left(-\sum_{i=1}^{n} x_i\right), \quad \text{and} \quad \mathtt{F}(\mathbf{x}) = \exp\left(\sum_{i=1}^{n} x_i\right),$$

respectively, which satisfy (2.2). In addition, the mean value of $\operatorname{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a Lagrange multiplier strategy, which means adding one row and one column to the matrix system that solves (3.46) (cf. (4.6)) for $\boldsymbol{\sigma}_h$ and $\mathbf{u}_h$.

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test, we verify the rates of convergence in a two-dimensional domain. The domain is the square $\Omega = (0,1)^2$. We choose $\rho = 3$, from which the remaining parameters follow as $\ell = 3/2$, $s = 6$, and $t = 6/5$ (cf. (4.5)). We then consider the potential gravitational acceleration $\mathbf{g} = (0, -1)^{\mathrm{t}}$ and adjust the data $\mathbf{f}(\phi)$ (cf. (2.3)), f, and g in (2.1) so that the exact solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \cos(\pi x_1)\, \sin(\pi x_2) \\ \sin(\pi x_1)\, \exp(x_2) \end{pmatrix}, \quad p(\mathbf{x}) = \cos(\pi x_1)\, \sin(\pi x_2)\,,$$

$$\text{and} \quad \phi(\mathbf{x}) = 0.1 + 0.3\, \exp(x_1 x_2)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary condition. Tables 5.1 and 5.2 report the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations, for both the mixed-primal and fully-mixed schemes. The results confirm that the optimal convergence rates $\mathcal{O}(h^{k+1})$, predicted by Theorems 3.14 and 4.15, are attained for both approaches with $k \in \{0, 1\}$. The Newton method exhibits mesh-independent performance, converging in four iterations in all cases. We remark that the data assumption (4.13), required for the fully-mixed approach, is not satisfied in this example, since $f(\mathbf{x}) = \operatorname{div}(\mathbf{u}) = \sin(\pi x_1)\big(\exp(x_2) - \pi \sin(\pi x_2)\big) \in [-1.5346, e]$ and $\eta = 1$. Nevertheless, optimal convergence rates are still achieved for the $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme, as previously noted, suggesting that assumption (4.13) is merely a theoretical requirement. We note that the errors for the pseudostress and velocity are nearly identical, as the same mixed formulation is employed in both methods. Regarding the concentration, optimal convergence is observed from the initial mesh levels in the mixed-primal approach, whereas in the fully-mixed case, it is attained from the second mesh refinement. This behavior is justified by the use of higher polynomial degree approximations in the former. In Figure 5.1, we present the computed magnitude of the velocity, the pressure and concentration fields obtained using the mixed-primal scheme, together with the pseudodiffusion vector computed with the fully-mixed scheme, all corresponding to approximations with $k = 1$. Both mixed methods are applied on a mesh with $41,146$ triangles, resulting in $743,263$ and $989,088$ degrees of freedom for the mixed-primal and fully-mixed schemes, respectively.

We stress that, on the one hand, the mixed-primal approach is less expensive in terms of degrees of freedom and provides better approximations for the concentration, as it employs higher-degree polynomial approximations. However, it requires a different mesh for the Lagrange multiplier λ, which complicates the extension to three-dimensional problems from a computational standpoint, even though no theoretical difficulties arise (cf. Section 3.5). In contrast, the fully-mixed scheme uses a single mesh for all variables, which not only simplifies the computational implementation but also facilitates its adaptation to three-dimensional settings. This advantage will be exploited in the next example. Moreover, the fully-mixed approach yields a direct approximation of a physically meaningful variable, namely the pseudodiffusion vector, although at the cost of a larger number of degrees of freedom.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example, we consider the cubic domain $\Omega = (0,1)^3$ and choose the parameter $\rho = 7/2$, whence $\ell = 7/5$, $s = 14/3$, and $t = 14/11$ (cf. (4.5)). The solution is given by

$$\mathbf{u}(\mathbf{x}) = \begin{pmatrix} \sin(\pi x_1)\,\sin(\pi x_2)\,\sin(\pi x_3) \\ -\cos(\pi x_1)\,\cos(\pi x_2)\,\cos(\pi x_3) \\ \cos(\pi x_1)\,\cos(\pi x_2)\,\sin(\pi x_3) \end{pmatrix}, \quad p(\mathbf{x}) = \cos(\pi x_1)\,\exp(x_2 + x_3)\,,$$

$$\text{and} \quad \phi(\mathbf{x}) = 0.1 + 0.3\,\exp(x_1 x_2 x_3)\,.$$

Similarly to the first example, we consider the potential-type gravitational acceleration $\mathbf{g} = (0,0,-1)^{\mathrm{t}}$, while the data $\mathbf{f}(\phi)$, f, and g are computed from (2.1) using the solution above. As mentioned earlier, for computational simplicity, in this three-dimensional example we focus on the fully-mixed scheme (4.28). The convergence history for a set of quasi-uniform mesh refinements with $k = 0$ is shown in Table 5.3. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as established by Theorem 4.15. Additionally, some components of the numerical solution are displayed in Figure 5.2, obtained using the fully-mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation with mesh size $h = 0.0866$ and $48,000$ tetrahedral elements, representing $585,600$ degrees of freedom.

Example 3: Fluid flow through a rectangular domain with circular obstacles

In the final example, inspired by [9, Example 3], we focus on flow through a rectangular porous medium with circular obstacles and a non-manufactured solution of the unsteady version of problem (2.1) (cf. (3.3a)–(3.3b) and (4.2)). To that end, we consider the domain $\Omega = (0,2)\times(0,0.25)\backslash\Omega_c$, where $\Omega_c := \bigcup_{i=1}^3 \Omega_c^{\mathrm{up},i} \cup \bigcup_{j=1}^2 \Omega_c^{\mathrm{down},j}$,

$$\Omega_c^{\mathrm{up},i} = \Big\{(x_1,x_2): \quad (x_1 - 0.8\,i + 0.6)^2 + (x_2 - 0.15)^2 < 0.05^2\Big\}, \quad i = \{1,2,3\}\,,$$

and

$$\Omega_c^{\mathrm{down},j} = \Big\{(x_1,x_2): \quad (x_1 - 0.8\,j + 0.2)^2 + (x_2 - 0.1)^2 < 0.05^2\Big\}, \quad j = \{1,2\}\,,$$

with boundary $\Gamma = \partial\Omega$, where the input and output parts are defined as $\Gamma_{\mathrm{in}} = \{0\}\times(0,0.25)$ and $\Gamma_{\mathrm{out}} = \{2\}\times(0,0.25)$, respectively. We consider the parameter $\rho = 4$, and set the data as $\mathbf{g} = (0,-9.81)^{\mathrm{t}}$, $f = 0$, and $g = 0$. The initial conditions for both the velocity and concentration are taken to be zero. Denoting $u_{\mathrm{in}} := -10\,x_2(x_2 - 0.25)\big(1 + 0.5\sin(2\pi\,t/T)\big)$ and $\phi_{\mathrm{in}} := 5 + 0.5\sin(2\pi\,t)$, the boundary conditions are given by

$$\mathbf{u} = (u_{\mathrm{in}},0)^{\mathrm{t}} \quad \text{on} \quad \Gamma_{\mathrm{in}}\,, \quad \mathbf{u} = \mathbf{0} \quad \text{on} \quad \Gamma\backslash\big(\Gamma_{\mathrm{in}}\cup\Gamma_{\mathrm{out}}\big)\,, \quad \boldsymbol{\sigma}\mathbf{n} = \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{out}}\,,$$

$$\phi = \phi_{\mathrm{in}} \quad \text{on} \quad \Gamma_{\mathrm{in}}\,, \quad \text{and} \quad \boldsymbol{\vartheta}\cdot\mathbf{n} = 0 \quad \text{on} \quad \Gamma\backslash\Gamma_{\mathrm{in}}\,,$$

which drive the flow across the rectangular domain Ω through an oscillatory parabolic velocity profile from left to right, with an oscillatory concentration prescribed at the left boundary. We employ a suitable backward Euler time discretization, with time step $\Delta t = 0.02$ and final time $T = 2$. We observe that at each time step we are solving a slight adaptation of the stationary fully-mixed discrete problem (4.28). We remark that the analysis presented in the previous sections can be extended, with minor modifications, to the case of mixed boundary conditions considered in this example (see, e.g., [13, Section 2.4] and [16] for details). The well-posedness analysis for the unsteady version of (2.1) can be addressed by following similar arguments to the ones developed in [18]. This is a topic of current research.

In Figures 5.3, 5.4, and 5.5, we show the computed velocity magnitudes, and the pressure and concentration fields, respectively. These results were obtained using the fully-mixed $\mathbb{RT}_0 - \mathbf{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ scheme on a mesh with $h = 0.0126$ and $18,916$ triangular elements (corresponding to $143,094$ degrees of freedom). As expected, the velocity flows from left to right, exhibiting an oscillatory behavior as time increases. In addition, due to the gravitational force $\mathbf{g}$ and the impermeability of the top, bottom, and circular boundaries, a sinusoidal flow pattern develops within the domain. This behavior is consistent with the pressure distribution, which decreases from left to right. Similarly, the concentration is higher near the left boundary and decreases towards the right, also following an oscillatory pattern as time progresses.

Mixed-primal $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1} - \mathrm{P}_k$ scheme with $k = 0$												
DoF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	$\widetilde{h}$	e(λ)	r(λ)
914	0.196	4	1.6E+00	–	1.5E-01	–	3.3E-02	–	0.250	1.1E-01	–
2010	0.127	4	1.0E+00	0.966	1.0E-01	0.930	2.2E-02	0.896	0.167	7.0E-02	1.052
5434	0.078	4	6.2E-01	1.063	6.0E-02	1.082	1.3E-02	1.076	0.100	4.1E-02	1.050
17551	0.044	4	3.4E-01	1.064	3.3E-02	1.083	7.0E-03	1.098	0.056	2.2E-02	1.044
60936	0.024	4	1.8E-01	1.054	1.8E-02	1.044	3.7E-03	1.060	0.029	1.2E-02	1.023
227621	0.014	4	9.4E-02	1.108	9.1E-03	1.107	1.9E-03	1.096	0.015	5.9E-03	1.009

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 0$											
DoF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	$\widehat{\mathrm{e}}(\phi)$	$\widehat{\mathrm{r}}(\phi)$
1188	0.196	4	1.6E+00	–	1.5E-01	–	1.7E-01	–	1.8E-02	–
2652	0.127	4	1.0E+00	0.966	1.0E-01	0.930	1.2E-01	0.914	1.4E-02	0.575
7260	0.078	4	6.2E-01	1.063	6.0E-02	1.082	6.7E-02	1.086	8.6E-03	1.003
23661	0.044	4	3.4E-01	1.064	3.3E-02	1.083	3.6E-02	1.104	4.6E-03	1.121
82578	0.024	4	1.8E-01	1.054	1.8E-02	1.044	2.0E-02	1.031	2.6E-03	0.948
309387	0.014	4	9.4E-02	1.108	9.1E-03	1.107	1.0E-02	1.118	1.3E-03	1.132

Table 5.1: [Example 1, $k = 0$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the mixed approximations.

References

[1] M. Alvarez, G. N. Gatica, and R. Ruiz-Baier, *An augmented mixed-primal finite element method for a coupled flow-transport problem.* ESAIM Math. Model. Numer. Anal. 49 (2015), no. 5, 1399–1427.

[2] M. Alvarez, G. N. Gatica, and R. Ruiz-Baier, *A mixed-primal finite element approximation of a sedimentation-consolidation system.* M3AS: Math. Models Methods Appl. Sci. 26 (2016), no. 5, 867–900.

Mixed-primal $\mathbb{RT}_k - \mathbf{P}_k - \mathrm{P}_{k+1} - \mathrm{P}_k$ scheme with $k = 1$											
DoF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e(ϕ)	r(ϕ)	$\widetilde{h}$	e(λ)	r(λ)
2891	0.196	4	9.3E-02	–	7.2E-03	–	1.1E-03	–	0.250	4.8E-03	–
6427	0.127	4	4.0E-02	1.927	3.2E-03	1.883	4.8E-04	1.986	0.167	2.1E-03	2.013
17531	0.078	4	1.5E-02	2.060	1.2E-03	1.960	1.6E-04	2.210	0.100	7.6E-04	2.025
56983	0.044	4	4.4E-03	2.116	3.7E-04	2.106	4.5E-05	2.275	0.056	2.3E-04	2.006
198563	0.024	4	1.3E-03	2.111	1.0E-04	2.135	1.3E-05	2.158	0.029	6.5E-05	2.003
743263	0.014	4	3.4E-04	2.220	2.8E-05	2.238	3.3E-06	2.270	0.015	1.7E-05	2.002

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 1$										
DoF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	$\widehat{\mathsf{e}}(\phi)$	$\widehat{\mathsf{r}}(\phi)$
3744	0.196	4	9.3E-02	–	7.2E-03	–	1.1E-02	–	5.3E-04	–
8400	0.127	4	4.0E-02	1.927	3.2E-03	1.883	4.8E-03	1.776	2.8E-04	1.446
23088	0.078	4	1.5E-02	2.060	1.2E-03	1.960	1.7E-03	2.082	1.1E-04	1.856
75456	0.044	4	4.4E-03	2.116	3.7E-04	2.106	5.3E-04	2.123	3.3E-05	2.227
263760	0.024	4	1.3E-03	2.111	1.0E-04	2.135	1.5E-04	2.097	1.1E-05	1.908
989088	0.014	4	3.4E-04	2.220	2.8E-05	2.238	4.1E-05	2.229	2.7E-06	2.274

Table 5.2: [Example 1, $k = 1$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the mixed approximations.

Fully-mixed $\mathbb{RT}_k - \mathbf{P}_k - \mathbf{RT}_k - \mathrm{P}_k$ scheme with $k = 0$										
DoF	h	it	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\boldsymbol{\vartheta}$)	r($\boldsymbol{\vartheta}$)	e(ϕ)	r(ϕ)
672	0.866	4	1.0E+01	–	3.8E-01	–	5.1E-01	–	6.6E-02	–
4992	0.433	4	5.6E+00	0.887	2.0E-01	0.916	2.7E-01	0.906	3.6E-02	0.880
38400	0.217	4	2.8E+00	0.999	1.0E-01	0.980	1.4E-01	0.974	1.8E-02	0.973
301056	0.108	4	1.4E+00	1.027	5.0E-02	1.001	6.9E-02	0.995	9.2E-03	0.994
585600	0.087	4	1.1E+00	1.022	4.0E-02	1.002	5.6E-02	0.998	7.4E-03	0.998

Table 5.3: [Example 2, $k = 0$] Number of degrees of freedom, meshsizes, Newton iteration count, errors, and rates of convergence for the fully-mixed approximation.

[3] M. Alvarez, G.N. Gatica, and R. Ruiz-Baier, *A mixed-primal finite element method for the coupling of Brinkman-Darcy flow and nonlinear transport.* IMA J. Numer. Anal. 41 (2021), no. 1, 381–411.

[4] G.A. Benavides, S. Caucao, G.N. Gatica, and A.A. Hopper, *A Banach spaces-based analysis of a new mixed-primal finite element method for a coupled flow-transport problem.* Comput. Methods Appl. Mech. Engrg. 371 (2020), 113285, 29 pp.

[5] G.A. Benavides, S. Caucao, G.N. Gatica, and A.A. Hopper, *A new non-augmented and momentum-conserving fully-mixed finite element method for a coupled flow-transport problem.* Calcolo 59 (2022), no. 1, Paper No. 6, 44 pp.

[6] F. Brezzi and M. Fortin, *Mixed and Hybrid Finite Element Methods.* Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

[7] M. Bulíček and P. Pustějovská, *Existence analysis for a model describing flow of an incompressible chemically reacting non-Newtonian fluid.* SIAM J. Math. Anal. 46 (2014), no. 5, 3223–3240.

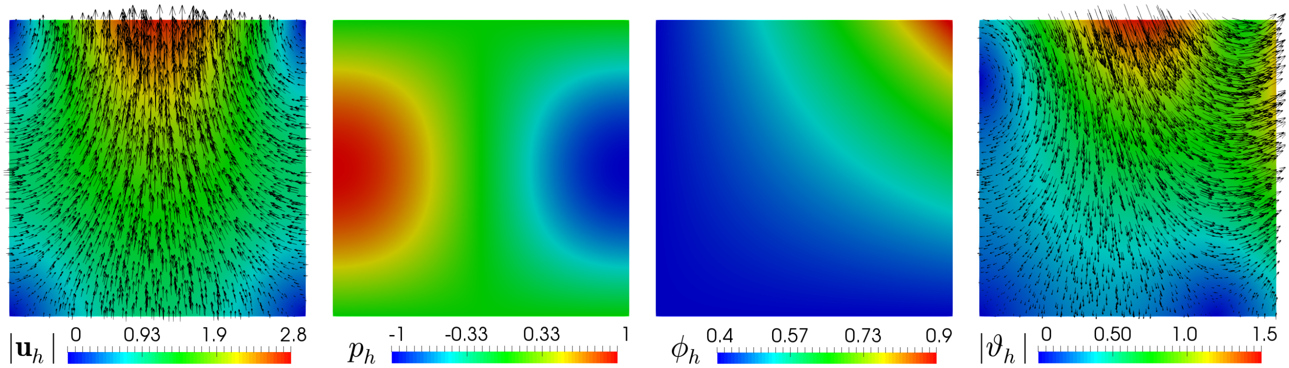


Figure 5.1: [Example 1] Computed velocity magnitude, pressure and concentration fields, and the magnitude of the pseudodiffusion vector.

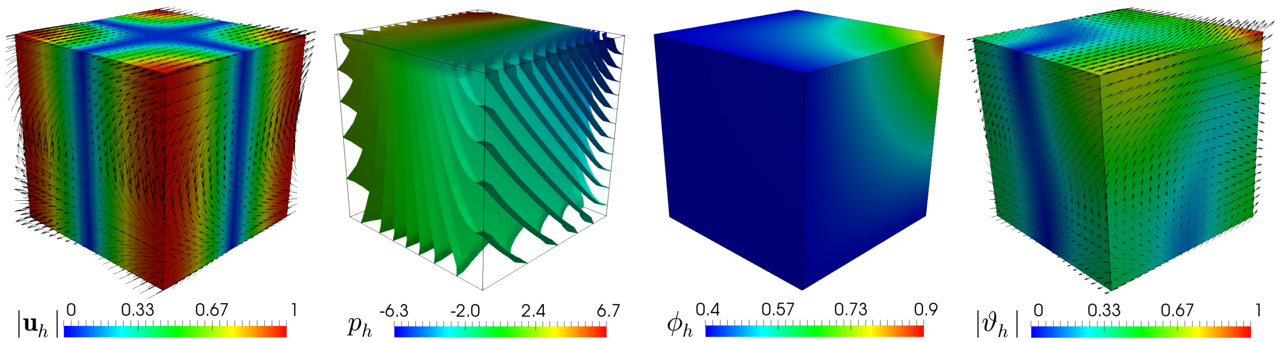


Figure 5.2: [Example 2] Computed velocity magnitude, pressure and concentration fields, and the magnitude of the pseudodiffusion vector.

[8] E. Burman and A. Ern, *Stabilized Galerkin approximation of convection-diffusion-reaction equations: discrete maximum principle and convergence.* Mathematics of Computation 74 (2005), no. 252, 1637–1652.

[9] A.J. Bustos, S. Caucao, G.N. Gatica, and B.N. Venegas, *New fully mixed finite element methods for the coupled convective Brinkman–Forchheimer and nonlinear transport equations.* J. Sci. Comput. 104 (2025), no. 2, Paper No. 64.

[10] J. Camaño, C. García and R. Oyarzúa, *Analysis of a momentum conservative mixed-FEM for the stationary Navier-Stokes problem.* Numer. Methods Partial Differential Equations 37 (2021), no. 5, 2895–2923.

[11] J. Camaño, C. Muñoz, and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electron. Trans. Numer. Anal. 48 (2018), 114–130.

[12] S. Carrasco, S. Caucao, and G.N. Gatica, *New mixed finite element methods for the coupled convective Brinkman–Forchheimer and double-diffusion equations.* J. Sci. Comput. 97 (2023), no. 3, Paper No. 61.

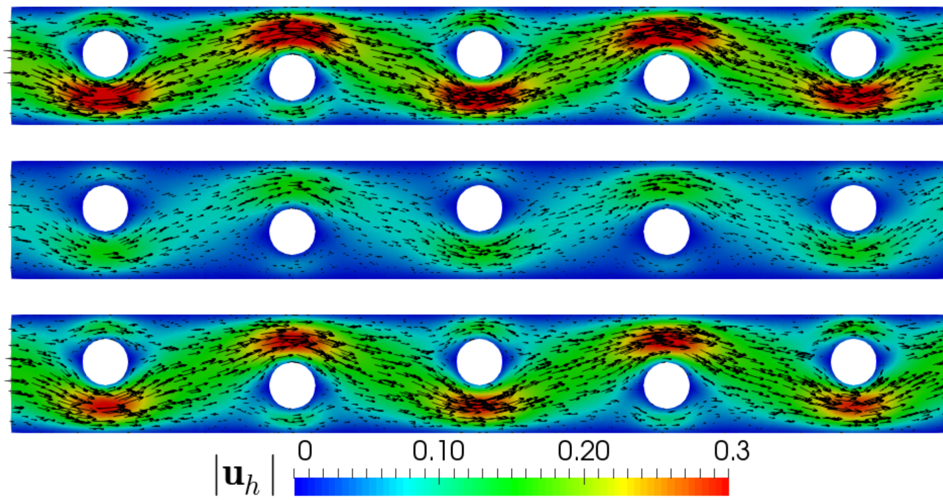


Figure 5.3: [Example 3] Computed velocity magnitude at times $t \in \{0.1, 1.5, 2\}$ (from top to bottom).

[13] S. Caucao, E. Colmenares, G.N. Gatica, and C. Inzunza, *A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier–Stokes problem*. Comput. Math. Appl. 145 (2023), 65–89.

[14] S. Caucao, G.N. Gatica, and J.P. Ortega, *A fully-mixed formulation in Banach spaces for the coupling of the steady Brinkman–Forchheimer and double-diffusion equations*. ESAIM Math. Model. Numer. Anal. 55 (2021), no. 6, 2725–2758.

[15] S. Caucao, G.N. Gatica, R. Oyarzúa, and N. Sánchez, *A fully-mixed formulation for the steady double-diffusive convection system based upon Brinkman–Forchheimer equations*. J. Sci. Comput. 85 (2020), no. 2, Paper No. 44, 37 pp.

[16] S. Caucao, R. Oyarzúa, and S. Villa-Fuentes, *A new mixed-FEM for steady-state natural convection models allowing conservation of momentum and thermal energy*. Calcolo 57 (2020), no. 4, Paper No. 36, 39 pp.

[17] S. Caucao, R. Oyarzúa, S. Villa-Fuentes, and I. Yotov, *A three-field Banach spaces-based mixed formulation for the unsteady Brinkman–Forchheimer equations*. Computer Methods in Applied Mechanics and Engineering 394 (2022), Art. Num. 114895.

[18] S. Caucao and I. Yotov, *A Banach space mixed formulation for the unsteady Brinkman–Forchheimer equations*. IMA J. Numer. Anal. 41 (2021), no. 4, 2708–2743.

[19] A.O. Celebi, V.K. Kalantarov, and D. Ugurlu, *On continuous dependence on coefficients of the Brinkman–Forchheimer equations*. Appl. Math. Lett. 19 (2006), no. 8, 801–807.

[20] R. Codina, *Comparison of some finite element methods for solving the diffusion-convection-reaction equation*. Comput. Methods Appl. Mech. Engrg. 156 (1998), no. (1-4), 185–210.

[21] E. Colmenares, G.N. Gatica, and S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem*. ESAIM Math. Model. Numer. Anal. 54 (2020), no. 5, 1525–1568.

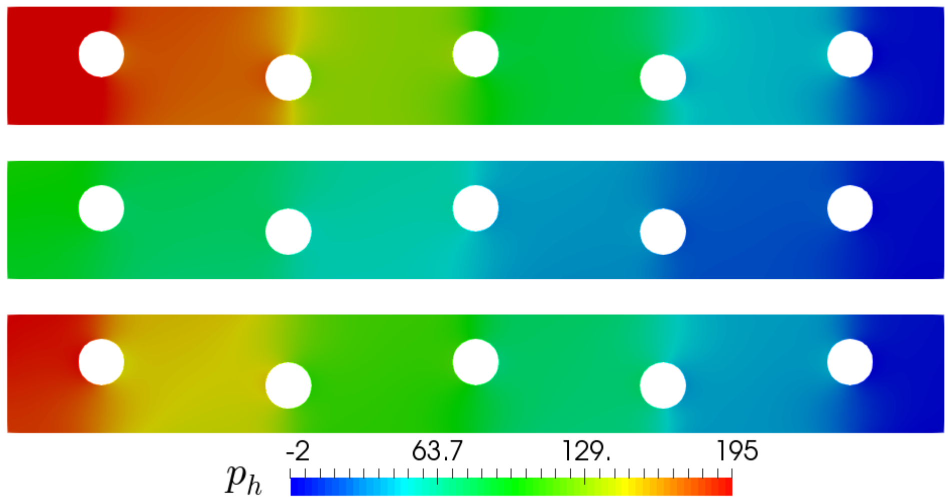


Figure 5.4: [Example 3] Computed pressure field at times $t \in \{0.1, 1.5, 2\}$ (from top to bottom).

[22] E. Colmenares, G.N. Gatica, and R. Oyarzúa, *Analysis of an augmented mixed-primal formulation for the stationary Boussinesq problem.* Numer. Methods Partial Differ. Equ. 32 (2016), no. 2, 445–478.

[23] C.I. Correa and G.N. Gatica, *On the continuous and discrete well-posedness of perturbed saddle-point formulations in Banach spaces.* Comput. Math. Appl. 117 (2022), 14–23.

[24] C.I. Correa, G.N. Gatica, and R. Ruiz-Baier, *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces.* ESAIM Math. Model. Numer. Anal. 57 (2023), no. 3, pp. 1511-1551.

[25] J.K. Djoko and P.A. Razafimandimby, *Analysis of the Brinkman–Forchheimer equations with slip boundary conditions.* Appl. Anal. 93 (2014), no.7, 1477–1494.

[26] W. Ehlers, *Darcy, Forchheimer, Brinkman and Richards: classical hydromechanical equations and their significance in the light of the TPM.* Archive of Applied Mechanics (2022), no. 92, 619–639.

[27] A. Ern and J.-L. Guermond, *Theory and Practice of Finite Elements.* Applied Mathematical Sciences, 159. Springer-Verlag, New York, 2004.

[28] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications.* SpringerBriefs in Mathematics. Springer, Cham, 2014.

[29] G.N. Gatica, *A note on the generalized Babuška–Brezzi theory: Revisiting the proof of the associated Strang error estimates.* Appl. Math. Lett. 157 (2024), 109197.

[30] G.N. Gatica and C. Inzunza, *An augmented fully-mixed finite element method for a coupled flow-transport problem.* Calcolo 57 (2020), no. 1, Art. 8.

[31] G.N. Gatica, S. Meddahi, and R. Ruiz-Baier, *An L^p-spaces-based formulation yielding a new fully mixed finite element method for the coupled Darcy and heat equations.* IMA J. Numer. Anal. 42 (2022), no. 4, 3154–3206.

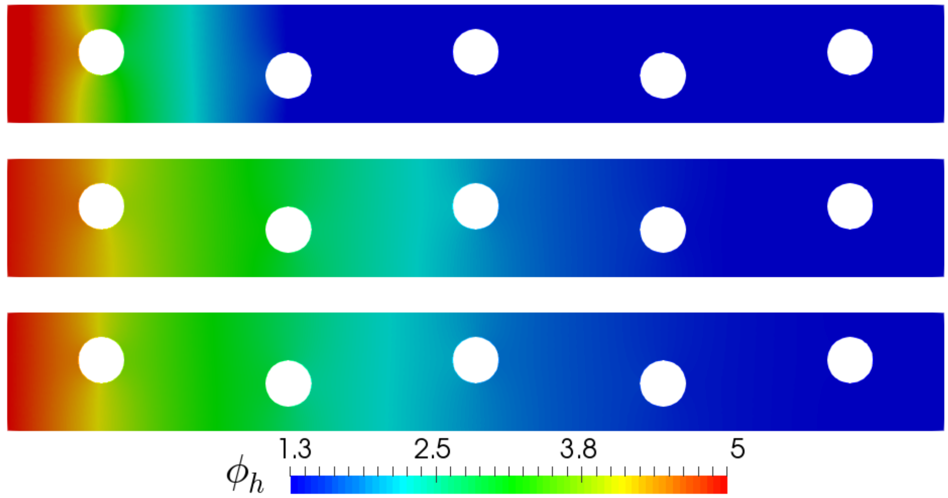


Figure 5.5: [Example 3] Computed concentration field at times $t \in \{0.1, 1.5, 2\}$ (from top to bottom).

[32] F. Hecht, *New development in FreeFem++*. J. Numer. Math. 20 (2012), 251–265.

[33] T. Sayah, G. Semaan, and F. Triki, *Finite element methods for the Darcy–Forchheimer problem coupled with the convection-diffusion-reaction problem.* ESAIM Math. Model. Numer. Anal. 55 (2021), no.6, 2643–2678.

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2025

2025-02 Julio Aracena, Arturo Zapata-Cortés: *Hamiltonian dynamics of boolean networks*

2025-03 Raimund Bürger, Andrés Guerra, Carlos A. Vega: *An entropy stable and well-balanced scheme for an augmented blood flow model with variable geometrical and mechanical properties*

2025-04 Alonso Bustos, Sergio Caucao, Gabriel N. Gatica, Benjamín Venegas: *New fully mixed finite element methods for the coupled convective Brinkman-Forchheimer and nonlinear transport equations*

2025-05 Fahim Aslam, Zayd Hajjej, Jianghao Hao, Mauricio Sepúlveda: *Global Existence and Asymptotic Profile of an Infinite Memory Logarithmic Wave Equation with Fractional Derivative and Strong Damping*

2025-06 Jessika Camaño, Ricardo Oyarzúa, Katherine Rojo: *A momentum and mass conservative pseudostress-based mixed finite element method for the Stokes problem*

2025-07 Aníbal Coronel, Fernando Huancas, Mauricio Sepúlveda: *Theoretical and numerical approaches of reaction term identification in a SIS reaction diffusion system*

2025-08 Isaac Bermudez, Gabriel N. Gatica, Juan P. Silva: *A new Banach spaces-based mixed finite element method for the coupled Navier–Stokes and Darcy equations*

2025-09 Sergio Caucao, Gabriel N. Gatica, Saulo Medrado, Yuri D. Sobral: *A posteriori error analysis of mixed finite element methods for a regularized mu(I)-rheology model of granular materials*

2025-10 Veronica Anaya, Gerardo Chowell, Felipe Jara, Mauricio Sepúlveda: *Optimal Control of Inter-Population Disease Spread via Reaction–Diffusion Models*

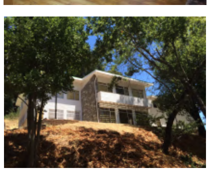
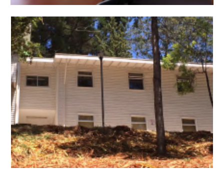
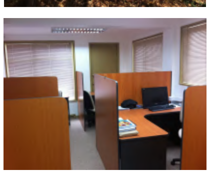
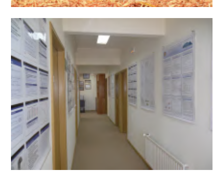
2025-11 Ramiro Acevedo, Rommel Bustinza, Christian Gómez: *A Transient Eddy Current Problem via Potential Formulation with Current Excitation*

2025-12 Abraham J. Arenas, Juan Barajas-Calonge, Gilberto González-Parra, Luis M. Villada: *A second-order nonstandard finite difference scheme for eco-epidemiological predator-prey models*

2025-13 Alonso Bustos, Sergio Caucao, Gabriel N. Gatica: *Mixed-primal and fully-mixed formulations for the convection-diffusion-reaction system based upon Brinkman–Forchheimer equations*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



Centro de Investigación en
Ingeniería Matemática (CI²MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl