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A conforming finite element method for a nonisothermal fluid-membrane interaction *

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Abstract

We propose and analyze a conforming finite element method for a two-dimensional nonisothermal fluid-membrane interaction problem. The problem consists of a Navier-Stokes/heat system, commonly known as the Boussinesq system, in the free-fluid region, and a Darcy-heat coupled system in the membrane. These systems are coupled through buoyancy terms and a set of transmission conditions on the fluid-membrane interface, including mass conservation, balance of normal forces, the Beavers-Joseph-Saffman law, and continuity of heat flux and fluid temperature. We consider the standard velocity-pressure-temperature variational formulation for the Boussinesq system, along with a dual-mixed scheme coupled with a primal formulation for the Darcy and Heat equations in the membrane region. The latter yields the introduction of the trace of the porous medium pressure as a suitable Lagrange multiplier. For the associated Galerkin scheme, we employ Bernardi-Raugel and Raviart-Thomas elements for velocities, piecewise constant elements for pressures, continuous piecewise linear functions for temperatures, and continuous piecewise linear functions for the Lagrange multiplier on a partition of the interface. We prove well-posedness for both the continuous and discrete schemes and derive corresponding error estimates. Finally, we present numerical examples to confirm the predicted convergence rates and demonstrate the performance of the method.

Key words: nonisothermal fluid-membrane, Navier–Stokes equation, Darcy equation, heat equation, mixed finite element method.

Mathematics subject classifications (2020): 65N15, 65N30, 35K05, 76D05, 76S05, 74K15, 76R05, 76B03.

1 Introduction

Membrane-based water filtration devices have been widely used in recent years to produce clean water for human consumption. Among the different types of water treatment processes that use a

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membrane to eliminate impurities, micro-filtration (MF), ultra-filtration (UF), nano-filtration (NF), reverse osmosis (RO), and membrane distillation (MD) are commonly employed.

Numerous experimental studies have been conducted to enhance the efficiency of these processes. However, experimentation can be costly, involving expensive equipment and staff training. In this regard, computational fluid dynamics (CFD) provides a cost-effective means for conducting numerical simulations that can guide important decisions for process optimization (see, for example, [1, 24, 29, 35, 40, 41, 45, 48, 49] and references therein). In particular, a detailed review of different mathematical models for simulating water treatment processes can be found in [40].

In this work, our focus is on proposing and analyzing a conforming numerical scheme for a non-isothermal fluid-membrane model that arises in membrane desalination processes [41, 48]. To this end, we adopt the mesoscopic model given by the coupled Navier-Stokes/Darcy model (see [40, Section 3.3.1]) and consider suitable coupling conditions prescribed on the common free flow-porous interface, including conservation of mass, balance of the normal stresses, and the Beavers-Joseph-Saffman condition. In the literature, there exists an extensive list of numerical methods for approximating the solution of the Navier-Stokes/Darcy system, including conforming and nonconforming schemes, such as those introduced in [2, 12, 13, 15, 20, 28, 30, 32].

In particular, in [32], the authors introduce and analyze a discontinuous Galerkin (DG) discretization for the nonlinear coupled problem, employing the nonsymmetric interior penalty Galerkin (NIPG), symmetric interior penalty Galerkin (SIPG), and incomplete interior penalty Galerkin (IIPG) bilinear forms for the discretization of the Laplacian in both media, and the upwind Lesaint-Raviart discretization of the convective term in the free fluid domain. On the other hand, in [2], the authors propose an iterative subdomain method that uses the velocity-pressure formulation for the Navier-Stokes equation and the primal formulation for the Darcy equation. Finally, in [20], the authors extend the work in [26] to the Navier-Stokes/Darcy model and introduce a conforming numerical scheme to approximate the solution of the problem. The variational formulation is based on the standard velocity-pressure formulation for the Navier-Stokes equation and the dual-mixed formulation for the Darcy equation, resulting in the velocity and pressure of the fluid in both media as the main unknowns of the coupled system. Since one of the interface conditions becomes essential, they proceed similarly to [26] and incorporate the trace of the porous medium pressure as an additional unknown.

Now, when it comes to numerical methods for coupling fluid flow with the heat equation, it is noteworthy that the literature offers an extensive list of contributions for the Navier-Stokes/heat coupled problem, commonly referred to as the Boussinesq problem [6, 8, 14, 16, 17, 22, 23, 37, 38, 39] and the references therein. However, the number of contributions for the Darcy-heat coupled system is relatively limited. In fact, the first contribution on the analysis of a finite element method for Darcy's problem coupled with the heat equation is presented in [4] (see also [5]). There, the authors introduce two finite element discretizations for the coupled system with temperature-dependent viscosity. One of the difficulties in the analysis of [4] is the fact that the velocity lives in H(div), which forces the trial and test spaces for the temperature to be different, preventing the utilization of classical results for elliptic problems to obtain the well-posedness of the continuous formulation. More recently, in [27] it is introduced a new fully-mixed finite element method for the model studied in [4]. There, the authors employ a Banach spaces-based analysis to prove well-posedness of the continuous problem and its corresponding finite element discretization.

Here, we focus on analyzing a finite element discretization for the Navier-Stokes/Darcy/Heat coupled system in two dimension, which, to the best of our knowledge, represents the first contribution in this direction. Specifically, we study the steady-state case of the model previously studied in [36], where a coupled system is given by the Boussinesq system in the free-fluid domain and the Darcy-heat coupled system in the membrane region. These equations are supplemented with appropriate interface conditions, including continuity of heat-flux, temperature, mass conservation, balance of normal stresses, and the Beavers-Joseph-Saffman condition.

We use a velocity-pressure-temperature variational formulation in both domains, which yields the introduction of a suitable Lagrange multiplier representing the trace of the porous media pressure on the interface. Then, we combine the theory developed in [20] and [4], and similarly to [17] (see also [6]), make use of a suitable lifting of the temperature data to prove the existence of a solution by means of a fixed-point strategy and under a smallness assumption on data. In addition, under a restrictive assumption on the temperature solution, we prove uniqueness.

In terms of the discretization of the formulation, we utilize Bernardi-Raugel and Raviart-Thomas elements for the velocities in the free-fluid and porous media domains, respectively. For the pressure and temperature, we employ piece-wise constant and Lagrange elements, respectively, in both domains. Additionally, for the Lagrange multiplier, we use continuous and piece-wise linear functions. The analysis of the discrete scheme employs a similar approach to the continuous case, and under a smallness assumption on data and on the temperature in the membrane, we obtain the convergence of the Galerkin scheme and its corresponding rate.

The remainder of the paper is organized as follows. In Section 2, we present the model problem, derive the corresponding variational formulation, and analyze the existence and uniqueness of the solution. Next, in Section 3, we define a conforming numerical scheme and analyze its well-posedness. In Section 4, we perform the error analysis and derive the corresponding order of convergence of the scheme. Finally, in Section 5, we provide some numerical results illustrating the performance of the method and confirming the theoretical rate of convergence.

We end this section by introducing definitions and fixing some notations. Let $\mathcal{O} \subseteq \mathbb{R}^d$, $d \in \{2,3\}$, denote a domain with Lipschitz boundary Γ. For $s \geq 0$ and $p \in [1,+\infty]$, we denote by $\mathrm{W}^{s,p}(\mathcal{O})$ the usual Sobolev space endowed with the norm $\|\cdot\|_{s,p,\mathcal{O}}$. If $s = 0$, $\mathrm{W}^{0,p}(\mathcal{O})$ corresponds to the usual Lebesgue space $\mathrm{L}^p(\mathcal{O})$, which is endowed with the norm $\|\cdot\|_{0,p,\mathcal{O}}$. If $p = 2$, we write $\mathrm{H}^s(\mathcal{O})$ in place of $\mathrm{W}^{s,2}(\mathcal{O})$, and denote the corresponding Lebesgue and Sobolev norms by $\|\cdot\|_{0,\mathcal{O}}$ and $\|\cdot\|_{s,\mathcal{O}}$, respectively, and the seminorm by $|\cdot|_{s,\mathcal{O}}$. In addition, $\mathrm{H}^1_0(\mathcal{O})$ will denote the space of functions in $\mathrm{H}^1(\mathcal{O})$ with null trace on Γ, and $\mathrm{L}^2_0(\mathcal{O})$ will be the space of $\mathrm{L}^2(\mathcal{O})$ functions with zero mean value over $\mathcal{O}$, that is

$$\mathrm{L}^2_0(\mathcal{O}) := \left\{ v \in \mathrm{L}^2(\mathcal{O}) : \int_{\mathcal{O}} v = 0 \right\}.$$

Given $p, q \in (1,+\infty)$ satisfying $1/p + 1/q = 1$, in what follows, we will denote by $\mathrm{W}^{1/q,p}(\Gamma)$ the trace space of $\mathrm{W}^{1,p}(\mathcal{O})$ and by $\mathrm{W}^{-1/q,q}(\Gamma)$ the dual space of $\mathrm{W}^{1/q,p}(\Gamma)$ endowed with the norms $\|\cdot\|_{1/q,p;\Gamma}$ and $\|\cdot\|_{-1/q,q;\Gamma}$, defined respectively by

$$\|\phi\|_{1/q,p,\Gamma} := \inf\left\{ \|\psi\|_{1,p,\mathcal{O}} : \psi \in \mathrm{W}^{1,p}(\mathcal{O}), \psi|_{\Gamma} = \phi \right\} \qquad \forall \phi \in \mathrm{W}^{1/q,p}(\Gamma),$$

and

$$\|\psi\|_{-1/q,p,\Gamma} = \sup_{\xi \in \mathrm{W}^{1/q,p}(\Gamma)\setminus\{\mathbf{0}\}} \frac{\langle \psi, \xi \rangle_{\Gamma}}{\|\xi\|_{1/q,p,\Gamma}} \qquad \forall \psi \in \mathrm{W}^{-1/q,q}(\Gamma).$$

where $\langle \cdot, \cdot \rangle_{\Gamma}$ denotes the duality parity between $\mathrm{W}^{-1/q,q}(\Gamma)$ and $\mathrm{W}^{1/q,p}(\Gamma)$, which coincides with the inner product on $\mathrm{L}^2(\Gamma)$ when restricted to $\mathrm{L}^2(\Gamma)$. When $p = 2$, we will write $\mathrm{H}^{1/2}(\Gamma) := \mathrm{W}^{1/2,2}(\Gamma)$, $\|\cdot\|_{1/2,2,\Gamma} = \|\cdot\|_{1/2,\Gamma}$, $\mathrm{H}^{-1/2}(\Gamma) := \mathrm{W}^{-1/2,2}(\Gamma)$ and $\|\cdot\|_{-1/2,2,\Gamma} = \|\cdot\|_{-1/2,\Gamma}$.

Additionally, we recall that $\mathbf{H}(\mathrm{div};\mathcal{O}) := \left\{ \mathbf{w} \in \mathbf{L}^2(\mathcal{O}) : \mathrm{div}\,\mathbf{w} \in \mathrm{L}^2(\mathcal{O}) \right\}$, endowed with the norm $\|\mathbf{w}\|_{\mathrm{div};\mathcal{O}} := \left(\|\mathbf{w}\|^2_{0,\mathcal{O}} + \|\mathrm{div}\,\mathbf{w}\|^2_{0,\mathcal{O}} \right)^{1/2}$ is a standard Hilbert space in the realm of mixed problems (see, e.g., [11]).

For simplicity, in what follows for any scalar fields v and w, vector fields $\mathbf{v} = (v_i)_{i=1,d}$ and $\mathbf{w} = (w_i)_{i=1,d}$, and tensor fields $\mathbf{A} = (a_{ij})_{i,j=1,d}$ and $\mathbf{B} = (b_{ij})_{i,j=1,d}$, we will denote

$$(v, w)_{\mathcal{O}} := \int_{\mathcal{O}} vw, \quad (\mathbf{v}, \mathbf{w})_{\mathcal{O}} := \int_{\mathcal{O}} \mathbf{v} \cdot \mathbf{w}, \quad \text{and} \quad (\mathbf{A}, \mathbf{B})_{\mathcal{O}} := \int_{\mathcal{O}} \mathbf{A} : \mathbf{B},$$

where $\mathbf{A} : \mathbf{B} := \sum_{i,j=1}^{d} a_{ij} b_{ij}$.

By $\mathbf{M}$ and $\mathbb{M}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space M, In turn, when no confusion arises, $|\cdot|$ will denote the Euclidean norm in $\mathbb{R}^d$ or $\mathbb{R}^{d\times d}$. Furthermore, given a non-negative integer k and a subset S of $\mathbb{R}^d$, $\mathrm{P}_k(S)$ stands for the space of polynomials defined on S of degree $\leq k$.

In the sequel we will employ $\mathbf{0}$ as a generic null vector, and use C and c, with or without subscripts, bars, tildes or hats, to denote generic positive constants independent of the discretization parameters, which may take different values at different places.

Now we recall some useful estimates that will be employed in the forthcoming analysis. We begin with the well-known Hölder inequality:

$$|(f, g)_{\mathcal{O}}| \leq \|f\|_{0,p,\mathcal{O}} \|g\|_{0,q,\mathcal{O}}, \qquad \forall\, f \in \mathrm{L}^p(\mathcal{O}),\, \forall\, g \in \mathrm{L}^q(\mathcal{O}), \quad \text{with} \quad 1/p + 1/q = 1. \tag{1.1}$$

We further recall that the following Sobolev inequality holds: there exists a constant $C_{\mathrm{Sob},\mathcal{O}} > 0$, depending only on $|\mathcal{O}|$ and q, such that

$$\|w\|_{0,q,\mathcal{O}} \leq C_{\mathrm{Sob},\mathcal{O}} \|w\|_{1,\mathcal{O}} \qquad \forall\, w \in \mathrm{H}^1(\mathcal{O}), \tag{1.2}$$

for $1 \leq q < \infty$ when $d = 2$ and $1 \leq q \leq 6$ when $d = 3$. This means that $\mathrm{H}^1(\mathcal{O})$ is continuously embedded into $\mathrm{L}^q(\mathcal{O})$ for the aforementioned ranges of q. In addition, we have the following continuous embeddings (see [42, Theorem 1.3.4] and [9, Proposition 1.4.2.]):

$$\mathrm{W}^{q,r}(\mathcal{O}) \hookrightarrow \mathrm{C}^0(\overline{\mathcal{O}}) \quad \text{and} \quad \mathrm{W}^{1,s}(\mathcal{O}) \hookrightarrow \mathrm{W}^{1,t}(\mathcal{O}), \tag{1.3}$$

for all $q > \frac{d}{r}$, and for all $1 \leq t \leq s \leq \infty$, respectively.

Finally, given a real number $l > 0$, we recall from [4] the truncation function τ_l and its primitive π_l, defined by

$$\tau_l(t) := \begin{cases} t & \text{if } |t| \leq l, \\ l\,\mathrm{sgn}\,(t) & \text{if } |t| > l, \end{cases} \quad \forall\, t \in \mathbb{R} \qquad \text{and} \qquad \pi_l(t) := \int_0^t \tau_l(s)\, ds, \tag{1.4}$$

respectively, where $\mathrm{sgn}\,(t) = 1$ if $t \geq 0$, or $\mathrm{sgn}\,(t) = -1$ if $t < 0$. It is clear that τ_l belongs to $\mathrm{W}^{1,\infty}(\mathbb{R})$. Moreover, for any $\psi \in \mathrm{H}^1_0(\mathcal{O})$, we have $\tau_l(\psi) \in \mathrm{H}^1_0(\mathcal{O}) \cap L^\infty(\mathcal{O})$ and

$$\nabla \tau_l(\psi) := \begin{cases} \nabla\psi & \text{if } |\psi| \leq l, \\ 0 & \text{if } |\psi| > l, \end{cases} \qquad \text{a.e. in } \mathcal{O}. \tag{1.5}$$

On the other hand, we observe that π_l is a Lipschitz continuous function, piecewise $\mathrm{C}^1(\mathbb{R})$, and satisfies $\pi_l(0) = 0$. In addition, for all $\psi \in \mathrm{H}^1_0(\mathcal{O})$, we have that $\pi_l(\psi) \in \mathrm{H}^1_0(\mathcal{O})$ and

$$\nabla \pi_l(\psi) = \tau_l(\psi) \nabla\psi. \tag{1.6}$$

2 Continuous problem

In this section, we present the model problem and derive the corresponding variational formulation. Next, we discuss the stability properties of the different forms involved and analyze the well-posedness of the resulting formulation.

2.1 The model problem

In order to describe the geometry of the problem, we let $\Omega_{\rm f}$ and $\Omega_{\rm m}$ be two bounded and simply connected polygonal domains in $\mathbb{R}^2$, such that $\partial\Omega_{\rm f} \cap \partial\Omega_{\rm m} = \Sigma \neq \emptyset$, and $\Omega_{\rm f} \cap \Omega_{\rm m} = \emptyset$. Then, we let $\Gamma_{\rm f} := \partial\Omega_{\rm f} \setminus \overline{\Sigma}$, $\Gamma_{\rm m} := \partial\Omega_{\rm m} \setminus \overline{\Sigma}$, and denote by $\mathbf{n}$ the unit normal vector on the boundaries, which is chosen pointing outward from $\Omega := \Omega_{\rm f} \cup \Omega_{\rm m} \cup \Sigma$ and $\Omega_{\rm f}$ (and hence inward to $\Omega_{\rm m}$ when seen on Σ). On Σ we also consider a unit tangent vector $\mathbf{t}$ (see Fig. 2.1).

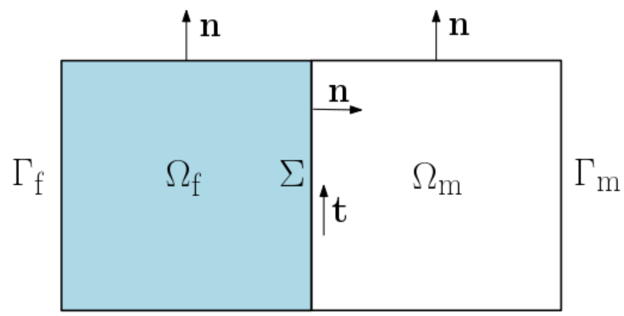

Figure 2.1: Sketch of the geometry of the domains.

The problem we are interested in consists of the movement of an incompressible viscous fluid subject to a heat source occupying $\Omega_{\rm f}$ which flows towards and from a porous membrane $\Omega_{\rm m}$ through Σ, where $\Omega_{\rm m}$ is saturated with the same fluid (see [36, 41]). The mathematical model is defined by two separate groups of equations and a set of coupling terms. In the free fluid domain $\Omega_{\rm f}$, the motion of the fluid can be described by the following Navier–Stokes/Heat system:

$$\boldsymbol{\sigma}_{\rm f} = 2\mu\,\mathbf{e}(\mathbf{u}_{\rm f}) - p_{\rm f}\,\mathbf{I} \quad \text{in} \quad \Omega_{\rm f}, \tag{2.1a}$$

$$-\mathbf{div}\,\boldsymbol{\sigma}_{\rm f} + (\mathbf{u}_{\rm f}\cdot\nabla)\,\mathbf{u}_{\rm f} - \mathbf{g}_{\rm f}\,\theta_{\rm f} = \mathbf{0} \quad \text{in} \quad \Omega_{\rm f}, \tag{2.1b}$$

$$\operatorname{div}\mathbf{u}_{\rm f} = 0 \quad \text{in} \quad \Omega_{\rm f}, \tag{2.1c}$$

$$-\kappa_{\rm f}\,\Delta\,\theta_{\rm f} + \mathbf{u}_{\rm f}\cdot\nabla\,\theta_{\rm f} = 0 \quad \text{in} \quad \Omega_{\rm f}, \tag{2.1d}$$

where $\mu > 0$ is the dynamic viscosity of the fluid, $\mathbf{u}_{\rm f}$ is the fluid velocity, $p_{\rm f}$ is the fluid pressure, $\boldsymbol{\sigma}_{\rm f}$ is the Cauchy stress tensor, $\mathbf{I}$ is the 2×2 identity matrix, $\theta_{\rm f}$ is the fluid temperature, $\kappa_{\rm f} > 0$ is the fluid thermal conductivity, $\mathbf{g}_{\rm f} \in \mathbf{L}^2(\Omega_{\rm f})$ is the external force per unit mass, $\mathbf{div}$ is the usual divergence operator div acting row-wise on each tensor, and $\mathbf{e}(\mathbf{u}_{\rm f})$ is the strain rate tensor given by $\mathbf{e}(\mathbf{u}_{\rm f}) := \frac{1}{2}\left(\nabla\mathbf{u}_{\rm f} + (\nabla\mathbf{u}_{\rm f})^t\right)$, where the superscript t denotes transposition.

In the porous membrane $\Omega_{\rm m}$ the behavior of the fluid can be described by the following Darcy-Heat system,

$$\mathbf{K}^{-1}\,\mathbf{u}_{\rm m} + \nabla\,p_{\rm m} - \mathbf{g}_{\rm m}\,\theta_{\rm m} = \mathbf{0} \quad \text{in} \quad \Omega_{\rm m}, \tag{2.2a}$$

$$\operatorname{div}\mathbf{u}_{\rm m} = 0 \quad \text{in} \quad \Omega_{\rm m}, \tag{2.2b}$$

$$-\kappa_{\rm m}\,\Delta\,\theta_{\rm m} + \mathbf{u}_{\rm m}\cdot\nabla\,\theta_{\rm m} = 0 \quad \text{in} \quad \Omega_{\rm m}, \tag{2.2c}$$

where $\mathbf{u}_{\mathrm{m}}$ represents the fluid velocity, p_{m} the fluid pressure, θ_{m} the fluid temperature, $\mathbf{g}_{\mathrm{m}} \in \mathbf{L}^3(\Omega_{\mathrm{m}})$ a given external force, $\kappa_{\mathrm{m}} > 0$ the thermal conductivity, and $\mathbf{K} \in [\mathrm{L}^\infty(\Omega_{\mathrm{m}})]^{2\times 2}$ is a symmetric and uniformly positive definite tensor in Ω_{m} representing the intrinsic permeability $\boldsymbol{\kappa}$ of the membrane divided by the dynamic viscosity μ of the fluid. Throughout the paper we assume that there exists $C_{\mathbf{K}} > 0$ such that

$$\boldsymbol{\xi}^t \mathbf{K}(x)\, \boldsymbol{\xi} \;\geq\; C_{\mathbf{K}}\, |\boldsymbol{\xi}|^2, \tag{2.3}$$

for almost all $x \in \Omega_{\mathrm{m}}$, and for all $\boldsymbol{\xi} \in \mathbb{R}^2$.

The transmission conditions that couple the systems (2.1) and (2.2) on the interface Σ are given by

$$\theta_{\mathrm{f}} = \theta_{\mathrm{m}} \quad \text{on} \quad \Sigma, \tag{2.4a}$$

$$\kappa_{\mathrm{f}}\, \nabla \theta_{\mathrm{f}} \cdot \mathbf{n} = \kappa_{\mathrm{m}}\, \nabla \theta_{\mathrm{m}} \cdot \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4b}$$

$$\mathbf{u}_{\mathrm{f}} \cdot \mathbf{n} = \mathbf{u}_{\mathrm{m}} \cdot \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4c}$$

$$\boldsymbol{\sigma}_{\mathrm{f}}\, \mathbf{n} + \frac{\alpha_d\, \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}} (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t})\, \mathbf{t} = -p_{\mathrm{m}}\, \mathbf{n} \quad \text{on} \quad \Sigma, \tag{2.4d}$$

where α_d is a dimensionless constant which depends only on the geometrical characteristics of the membrane (see [3]).

The first and second conditions represent the continuity of the temperature and the heat flux, respectively, while (2.4c) is a consequence of the incompressibility of the fluid and the conservation of mass across Σ (see [36]). In turn, the fourth condition (2.4d) can be decomposed, at least formally, into its normal and tangential components as follows:

$$(\boldsymbol{\sigma}_{\mathrm{f}}\, \mathbf{n}) \cdot \mathbf{n} \;=\; -p_{\mathrm{m}} \quad \text{and} \quad (\boldsymbol{\sigma}_{\mathrm{f}}\, \mathbf{n}) \cdot \mathbf{t} \;=\; -\frac{\alpha_d\, \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}} (\mathbf{u}_{\mathrm{f}} \cdot \mathbf{t}) \quad \text{on} \quad \Sigma. \tag{2.5}$$

The first equation in (2.5) corresponds to the balance of normal forces, whereas the second one is known as the Beavers–Joseph–Saffman law, which establishes that the slip velocity along Σ is proportional to the shear stress along Σ (assuming also, based on experimental evidence, that $\mathbf{u}_{\mathrm{m}} \cdot \mathbf{t}$ is negligible). We refer to [3, 34, 44] for further details on this interface condition. Finally, the Navier–Stokes/Darcy/Heat system (2.1), (2.2) and (2.4) is complemented with suitable boundary conditions:

$$\begin{aligned} \mathbf{u}_{\mathrm{f}} &= \mathbf{0} \quad \text{on} \quad \Gamma_{\mathrm{f}}, \qquad & \mathbf{u}_{\mathrm{m}} \cdot \mathbf{n} &= 0 \quad \text{on} \quad \Gamma_{\mathrm{m}}, \\ \theta_{\mathrm{f}} &= \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{f}}} \quad \text{on} \quad \Gamma_{\mathrm{f}}, \qquad & \theta_{\mathrm{m}} &= \theta_{\mathrm{D}}|_{\Gamma_{\mathrm{m}}} \quad \text{on} \quad \Gamma_{\mathrm{m}}, \end{aligned} \tag{2.6}$$

where $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ is given function defined on $\Gamma := \Gamma_{\mathrm{f}} \cup \Gamma_{\mathrm{m}}$.

2.2 The variational formulation

In this section we proceed similarly to [26, Section 2] and derive a weak formulation for the coupled problem given by (2.1), (2.2), (2.4) and (2.6). To this end, let us first introduce further notations and definitions. In the sequel we will employ the following subspaces of $\mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}})$ and $\mathrm{H}^1(\Omega_\star)$, respectively

$$\begin{aligned} \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}) &:= \Big\{ \mathbf{v} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}}) : \quad \mathbf{v} \cdot \mathbf{n} = 0 \quad \text{on} \quad \Gamma_{\mathrm{m}} \Big\}, \\ \mathrm{H}^1_{\Gamma_\star}(\Omega_\star) &:= \Big\{ \mathrm{v} \in \mathrm{H}^1(\Omega_\star) : \quad \mathrm{v} = 0 \quad \text{on} \quad \Gamma_\star \Big\}, \end{aligned}$$

with $\star \in \{\mathrm{f}, \mathrm{m}\}$. Notice that the latter implies the definition of the following subspace of $\mathbf{H}^1(\Omega_\mathrm{f})$

$$\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) := [\mathrm{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})]^2.$$

To derive our weak formulation, first we multiply (2.1b) by a test function $\mathbf{v}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, integrate by parts and employ (2.1a) and (2.4d), to obtain

$$\begin{aligned} 2\mu\,(\mathbf{e}(\mathbf{u}_\mathrm{f}), \mathbf{e}(\mathbf{v}_\mathrm{f}))_{\Omega_\mathrm{f}} + \left\langle \frac{\alpha_d\,\mu}{\sqrt{\mathbf{t}\cdot\boldsymbol{\kappa}\cdot\mathbf{t}}}(\mathbf{u}_\mathrm{f}\cdot\mathbf{t}), \mathbf{v}_\mathrm{f}\cdot\mathbf{t} \right\rangle_\Sigma + ((\mathbf{u}_\mathrm{f}\cdot\nabla)\,\mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} - (p_\mathrm{f}, \operatorname{div}\mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} \\ + \langle \lambda, \mathbf{v}_\mathrm{f}\cdot\mathbf{n}\rangle_\Sigma - (\theta_\mathrm{f}\,\mathbf{g}_\mathrm{f}, \mathbf{v}_\mathrm{f})_{\Omega_\mathrm{f}} = 0, \end{aligned} \tag{2.7a}$$

for all $\mathbf{v}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, where $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ is a further unknown representing the trace of the porous medium pressure on Σ, that is $\lambda = p_\mathrm{m}|_\Sigma$.

Next, we multiply (2.2a) by $\mathbf{v}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\operatorname{div};\Omega_\mathrm{m})$, and integrate by parts to obtain

$$(\mathbf{K}_\mathrm{m}^{-1}\mathbf{u}_\mathrm{m}, \mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} - \langle \mathbf{v}_\mathrm{m}\cdot\mathbf{n}, \lambda\rangle_\Sigma - (p_\mathrm{m}, \operatorname{div}\mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} - (\theta_\mathrm{m}\,\mathbf{g}_\mathrm{m}, \mathbf{v}_\mathrm{m})_{\Omega_\mathrm{m}} = 0 \quad \forall\, \mathbf{v}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\operatorname{div};\Omega_\mathrm{m}). \tag{2.7b}$$

Now, to incorporate (2.1d) and (2.2c) to the variational system, we first define the spaces

$$\Psi_\infty := \{\psi \in \mathrm{H}^1(\Omega) \,:\, \psi|_{\Omega_\mathrm{m}} \in \mathrm{L}^\infty(\Omega_\mathrm{m})\} \quad \text{and} \quad \Psi_{\infty,0} := \Psi_\infty \cap \mathrm{H}^1_0(\Omega),$$

and notice that if $\psi \in \Psi_{\infty,0}$, then $\psi|_{\Omega_\mathrm{f}} \in \mathrm{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ and $\psi|_{\Omega_\mathrm{m}} \in \mathrm{H}^1_{\Gamma_\mathrm{m}}(\Omega_\mathrm{m}) \cap \mathrm{L}^\infty(\Omega_\mathrm{m})$. Then, we let $\psi \in \Psi_{\infty,0}$, and multiply (2.1d) and (2.2c) by $\psi|_{\Omega_\mathrm{f}}$ and $\psi|_{\Omega_\mathrm{m}}$, respectively, to obtain

$$\kappa_\mathrm{f}\,(\nabla\theta_\mathrm{f}, \nabla\psi)_{\Omega_\mathrm{f}} - \kappa_\mathrm{f}\,\langle \nabla\theta_\mathrm{f}\cdot\mathbf{n}, \psi\rangle_\Sigma + (\mathbf{u}_\mathrm{f}\cdot\nabla\theta_\mathrm{f}, \psi)_{\Omega_\mathrm{f}} = 0,$$

and

$$\kappa_\mathrm{m}\,(\nabla\theta_\mathrm{m}, \nabla\psi)_{\Omega_\mathrm{m}} + \kappa_\mathrm{m}\,\langle \nabla\theta_\mathrm{m}\cdot\mathbf{n}, \psi\rangle_\Sigma + (\mathbf{u}_\mathrm{m}\cdot\nabla\theta_\mathrm{m}, \psi)_{\Omega_\mathrm{m}} = 0,$$

and summing up both equations, and using (2.4b), we finally get

$$\kappa_\mathrm{f}\,(\nabla\theta_\mathrm{f}, \nabla\psi)_{\Omega_\mathrm{f}} + \kappa_\mathrm{m}\,(\nabla\theta_\mathrm{m}, \nabla\psi)_{\Omega_\mathrm{m}} + (\mathbf{u}_\mathrm{f}\cdot\nabla\theta_\mathrm{f}, \psi)_{\Omega_\mathrm{f}} + (\mathbf{u}_\mathrm{m}\cdot\nabla\theta_\mathrm{m}, \psi)_{\Omega_\mathrm{m}} = 0 \quad \forall\, \psi \in \Psi_\infty. \tag{2.7c}$$

Finally, we incorporate the equations (2.1c), (2.2b), and (2.4c), weakly as follows

$$(q_\mathrm{f}, \operatorname{div}\mathbf{u}_\mathrm{f})_{\Omega_\mathrm{f}} = 0, \qquad (q_\mathrm{m}, \operatorname{div}\mathbf{u}_\mathrm{m})_{\Omega_\mathrm{m}} = 0 \quad \text{and} \quad \langle \mathbf{u}_\mathrm{f}\cdot\mathbf{n} - \mathbf{u}_\mathrm{m}\cdot\mathbf{n}, \xi\rangle_\Sigma = 0, \tag{2.7d}$$

for all $q_\mathrm{f} \in \mathrm{L}^2(\Omega_\mathrm{f})$, $q_\mathrm{m} \in \mathrm{L}^2(\Omega_\mathrm{m})$, and $\xi \in \mathrm{H}^{1/2}(\Sigma)$, respectively.

As a consequence of the above, we define $p := p_\mathrm{f}\chi_\mathrm{f} + p_\mathrm{m}\chi_\mathrm{m}$, $\theta := \theta_\mathrm{f}\chi_\mathrm{f} + \theta_\mathrm{m}\chi_\mathrm{m}$, with $\chi_\star$ being the characteristic function:

$$\chi_\star := \begin{cases} 1 & \text{in} \quad \Omega_\star, \\ 0 & \text{in} \quad \Omega \setminus \overline{\Omega}_\star, \end{cases}$$

for $\star \in \{\mathrm{f}, \mathrm{m}\}$, to obtain the variational problem: Find $\mathbf{u}_\mathrm{f} \in \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, $\mathbf{u}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\operatorname{div};\Omega_\mathrm{m})$, $p \in \mathrm{L}^2(\Omega)$, $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_\Gamma = \theta_\mathrm{D}$, such that (2.7a)–(2.7d) hold.

We observe that $\theta \in \mathrm{H}^1(\Omega)$ if and only if (2.4a) holds, so the interface condition (2.4a) is imposed on the temperature space. In turn, we notice that since $\mathbf{u}_\mathrm{m} \in \mathbf{H}_{\Gamma_\mathrm{m}}(\operatorname{div};\Omega_\mathrm{m})$ and $\nabla\theta|_{\Omega_\mathrm{m}} = \nabla\theta_\mathrm{m} \in \mathbf{L}^2(\Omega_\mathrm{m})$, then $\mathbf{u}_\mathrm{m}\cdot\nabla\theta_\mathrm{m} \in \mathrm{L}^1(\Omega)$, which justify the introduction of the space $\Psi_{\infty,0}$ for the test function ψ in (2.7c).

Now, let us observe that if $(\mathbf{u}_{\rm f}, \mathbf{u}_{\rm m}, p, \lambda, \theta)$ is a solution of the resulting variational problem, then for all $c \in \mathbb{R}$, $(\mathbf{u}_{\rm f}, \mathbf{u}_{\rm m}, p + c, \lambda + c, \theta)$ is also a solution. Consequently, we avoid the non-uniqueness of (2.7a)–(2.7d) by requiring from now on that $p \in \mathrm{L}^2_0(\Omega)$.

In this way, we let:

$$\mathbf{u} := (\mathbf{u}_{\rm f}, \mathbf{u}_{\rm m}) \in \mathbf{H} := \mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) \times \mathbf{H}_{\Gamma_{\rm m}}(\mathrm{div}; \Omega_{\rm m}),$$

$$(p, \lambda) \in \mathrm{Q} := \mathrm{L}^2_0(\Omega) \times \mathrm{H}^{1/2}(\Sigma),$$

where $\mathbf{H}$ and Q are endowed with the norms

$$\|\mathbf{v}\|_{\mathbf{H}} := \|\mathbf{v}_{\rm f}\|_{1,\Omega_{\rm f}} + \|\mathbf{v}_{\rm m}\|_{\mathrm{div};\Omega_{\rm m}} \qquad \forall\, \mathbf{v} \in \mathbf{H},$$

$$\|(q, \xi)\|_{\mathrm{Q}} := \|q\|_{0,\Omega} + \|\xi\|_{1/2,\Sigma} \qquad \forall\, (q, \xi) \in \mathrm{Q},$$

and arrive at the following variational problem: Find $(\mathbf{u}, (p, \lambda)) \in \mathbf{H} \times \mathrm{Q}$ and $\theta \in \mathrm{H}^1(\Omega)$, with $\theta|_\Gamma = \theta_{\rm D}$, such that

$$\begin{aligned} A_{\rm F}(\mathbf{u}, \mathbf{v}) + O_{\rm F}(\mathbf{u}_{\rm f}; \mathbf{u}_{\rm f}, \mathbf{v}_{\rm f}) + B(\mathbf{v}, (p, \lambda)) - D(\theta, \mathbf{v}) &= 0 \qquad \forall\, \mathbf{v} = (\mathbf{v}_{\rm f}, \mathbf{v}_{\rm m}) \in \mathbf{H}, \\ B(\mathbf{u}, (q, \xi)) &= 0 \qquad \forall\, (q, \xi) \in \mathrm{Q}, \\ A_{\rm T}(\theta, \psi) + O_{\rm T}(\mathbf{u}; \theta, \psi) &= 0 \qquad \forall\, \psi \in \Psi_{\infty,0}, \end{aligned} \tag{2.8}$$

where the forms $A_{\rm F} : \mathbf{H} \times \mathbf{H} \to \mathbb{R}$, $O_{\rm F} : \mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) \times \mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) \times \mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) \to \mathbb{R}$, $B : \mathbf{H} \times \mathrm{Q} \to \mathbb{R}$, $D : \mathrm{H}^1(\Omega) \times \mathbf{H} \to \mathbb{R}$, $A_{\rm T} : \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, and $O_{\rm T} : \mathbf{H} \times \mathrm{H}^1(\Omega) \times \Psi_{\infty,0} \to \mathbb{R}$, are defined respectively, as

$$\begin{aligned} A_{\rm F}(\mathbf{u}, \mathbf{v}) &:= a_{\rm F,f}(\mathbf{u}_{\rm f}, \mathbf{v}_{\rm f}) + a_{\rm F,m}(\mathbf{u}_{\rm m}, \mathbf{v}_{\rm m}), \\ O_{\rm F}(\mathbf{w}_{\rm f}; \mathbf{u}_{\rm f}, \mathbf{v}_{\rm f}) &:= ((\mathbf{w}_{\rm f} \cdot \nabla)\mathbf{u}_{\rm f}, \mathbf{v}_{\rm f})_{\Omega_{\rm f}}, \\ B(\mathbf{v}, (q, \xi)) &:= -(q, \mathrm{div}\, \mathbf{v}_{\rm f})_{\Omega_{\rm f}} - (q, \mathrm{div}\, \mathbf{v}_{\rm m})_{\Omega_{\rm m}} + \langle \mathbf{v}_{\rm f} \cdot \mathbf{n} - \mathbf{v}_{\rm m} \cdot \mathbf{n}, \xi \rangle_\Sigma, \\ D(\theta, \mathbf{v}) &:= (\theta\, \mathbf{g}_{\rm f}, \mathbf{v}_{\rm f})_{\Omega_{\rm f}} + (\theta\, \mathbf{g}_{\rm m}, \mathbf{v}_{\rm m})_{\Omega_{\rm m}}, \\ A_{\rm T}(\theta, \psi) &:= \kappa_{\rm f}(\nabla\theta, \nabla\psi)_{\Omega_{\rm f}} + \kappa_{\rm m}(\nabla\theta, \nabla\psi)_{\Omega_{\rm m}}, \\ O_{\rm T}(\mathbf{w}; \theta, \psi) &:= (\mathbf{w}_{\rm f} \cdot \nabla\theta, \psi)_{\Omega_{\rm f}} + (\mathbf{w}_{\rm m} \cdot \nabla\theta, \psi)_{\Omega_{\rm m}}, \end{aligned} \tag{2.9}$$

with

$$\begin{aligned} a_{\rm F,f}(\mathbf{u}_{\rm f}, \mathbf{v}_{\rm f}) &:= 2\mu\, (\mathbf{e}(\mathbf{u}_{\rm f}), \mathbf{e}(\mathbf{v}_{\rm f}))_{\Omega_{\rm f}} + \left\langle \frac{\alpha_d\, \mu}{\sqrt{\mathbf{t} \cdot \boldsymbol{\kappa} \cdot \mathbf{t}}}(\mathbf{u}_{\rm f} \cdot \mathbf{t}), \mathbf{v}_{\rm f} \cdot \mathbf{t} \right\rangle_\Sigma, \\ a_{\rm F,m}(\mathbf{u}_{\rm m}, \mathbf{v}_{\rm m}) &:= (\mathbf{K}^{-1}\, \mathbf{u}_{\rm m}, \mathbf{v}_{\rm m})_{\Omega_{\rm m}}. \end{aligned}$$

2.3 Existence and stability of solution

Now we address the existence and stability of solution of problem (2.8). We start the analysis by deriving the stability properties of the forms involved.

2.3.1 Stability properties

We begin by observing that, after simple computations, the bilinear forms $A_{\rm F}$, B and $A_{\rm T}$ are bounded, that is,

$$|A_{\rm F}(\mathbf{u}, \mathbf{v})| \le C_{A_{\rm F}}\, \|\mathbf{u}\|_{\mathbf{H}}\, \|\mathbf{v}\|_{\mathbf{H}} \qquad \forall\, \mathbf{u}, \mathbf{v} \in \mathbf{H}, \tag{2.10}$$

$$|B(\mathbf{v}, (q, \xi))| \le C_B \|\mathbf{v}\|_{\mathbf{H}}\, \|(q, \xi)\|_{\mathrm{Q}} \qquad \forall\, \mathbf{v} \in \mathbf{H}, \forall\, (q, \xi) \in \mathrm{Q},$$

$$|A_{\rm T}(\theta, \psi)| \le C_{A_{\rm T}}\, \|\theta\|_{1,\Omega}\, \|\psi\|_{1,\Omega} \qquad \forall\, \theta \in \mathrm{H}^1(\Omega), \forall\, \psi \in \Psi_\infty. \tag{2.11}$$

In turn, employing (1.1) and (1.2), it is easy to see that

$$|D(\theta,\mathbf{v})| \,\le\, C_D\,(\|\mathbf{g}_{\rm f}\|_{0,\Omega_{\rm f}} + \|\mathbf{g}_{\rm m}\|_{0,3,\Omega_{\rm m}})\,\|\theta\|_{1,\Omega}\,\|\mathbf{v}\|_{\mathbf{H}} \qquad \forall\,\theta\in\Psi_\infty,\,\forall\,\mathbf{v}\in\mathbf{H}. \tag{2.12}$$

Similarly, for $O_{\rm F}$ we utilize (1.1) and (1.2), to obtain

$$|O_{\rm F}(\mathbf{w}_{\rm f};\mathbf{u}_{\rm f},\mathbf{v}_{\rm f})| \,\le\, C_{O_{\rm F}}\,\|\mathbf{w}_{\rm f}\|_{1,\Omega_{\rm f}}\,\|\mathbf{u}_{\rm f}\|_{1,\Omega_{\rm f}}\,\|\mathbf{v}_{\rm f}\|_{1,\Omega_{\rm f}} \quad \forall\,\mathbf{w}_{\rm f},\mathbf{u}_{\rm f},\mathbf{v}_{\rm f}\in\mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}).$$

Now, we let $\mathbf{V}$ be the kernel of the bilinear form B, that is

$$\mathbf{V} := \{\mathbf{v}\in\mathbf{H} : B(\mathbf{v},(q,\xi)) = 0 \quad \forall\,(q,\xi)\in\mathrm{Q}\}. \tag{2.13}$$

From the definition of B we observe that $\mathbf{v} = (\mathbf{v}_{\rm f},\mathbf{v}_{\rm m})\in\mathbf{V}$ if and only if

$$(q,\operatorname{div}\mathbf{v}_{\rm f})_{\Omega_{\rm f}} + (q,\operatorname{div}\mathbf{v}_{\rm m})_{\Omega_{\rm m}} = 0 \quad \forall\,q\in\mathrm{L}^2_0(\Omega) \quad\text{and}\quad \langle\mathbf{v}_{\rm f}\cdot\mathbf{n} - \mathbf{v}_{\rm m}\cdot\mathbf{n},\xi\rangle_\Sigma = 0 \quad \forall\,\xi\in\mathrm{H}^{1/2}(\Sigma).$$

Then, noting that $\mathrm{L}^2(\Omega) = \mathrm{L}^2_0(\Omega)\oplus\mathbb{R}$, and taking $\xi\in\mathbb{R}$ in the latter equation, we deduce that

$$(q,\operatorname{div}\mathbf{v}_{\rm f})_{\Omega_{\rm f}} + (q,\operatorname{div}\mathbf{v}_{\rm m})_{\Omega_{\rm m}} = 0 \qquad \forall\,q\in\mathrm{L}^2(\Omega),$$

which implies

$$\operatorname{div}\mathbf{v}_{\rm f} = 0 \quad\text{in}\quad \Omega_{\rm f} \quad\text{and}\quad \operatorname{div}\mathbf{v}_{\rm m} = 0 \quad\text{in}\quad \Omega_{\rm m}.$$

According to the above, we can rewrite $\mathbf{V}$ as

$$\mathbf{V} := \{\mathbf{v} = (\mathbf{v}_{\rm f},\mathbf{v}_{\rm m})\in\mathbf{V}_{\rm f}\times\mathbf{V}_{\rm m} : \mathbf{v}_{\rm f}\cdot\mathbf{n} - \mathbf{v}_{\rm m}\cdot\mathbf{n} = 0 \quad\text{on}\quad \Sigma\},$$

with

$$\begin{aligned}\mathbf{V}_{\rm f} &:= \{\mathbf{v}_{\rm f}\in\mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f}) : \operatorname{div}\mathbf{v}_{\rm f} = 0 \quad\text{in}\quad \Omega_{\rm f}\},\\ \mathbf{V}_{\rm m} &:= \{\mathbf{v}_{\rm m}\in\mathbf{H}_{\Gamma_{\rm m}}(\operatorname{div};\Omega_{\rm m}) : \operatorname{div}\mathbf{v}_{\rm m} = 0 \quad\text{in}\quad \Omega_{\rm m}\}.\end{aligned}$$

Next, we employ the well-known Korn's inequality (see, e.g., [21]) for the bilinear form $a_{\rm F,f}$, and the fact that $\mathbf{K}^{-1}$ is symmetric and positive definite (cf. (2.3)) for $a_{\rm F,m}$, to deduce that

$$a_{\rm F,f}(\mathbf{v}_{\rm f},\mathbf{v}_{\rm f}) \,\ge\, 2\mu\alpha_{\rm f}\,\|\mathbf{v}_{\rm f}\|^2_{1,\Omega_{\rm f}} \quad\text{and}\quad a_{\rm F,m}(\mathbf{v}_{\rm m},\mathbf{v}_{\rm m}) \,\ge\, C_{\mathbf{K}}\,\|\mathbf{v}_{\rm m}\|^2_{\operatorname{div};\Omega_{\rm m}}, \tag{2.14}$$

for all $\mathbf{v}_{\rm f}\in\mathbf{H}^1_{\Gamma_{\rm f}}(\Omega_{\rm f})$, and for all $\mathbf{v}_{\rm m}\in\mathbf{V}_{\rm m}$, with $\alpha_{\rm f} > 0$. Using these estimates we deduce that the form $A_{\rm F}(\cdot,\cdot) + O_{\rm F}(\mathbf{w}_{\rm f};\cdot,\cdot) : \mathbf{H}\times\mathbf{H}\to\mathbb{R}$, is elliptic on $\mathbf{V}$ for suitable $\mathbf{w}_{\rm f}\in\mathbf{V}_{\rm f}$. More precisely, we have the following lemma. For its proof, we refer the reader to [20, Lemma 2].

Lemma 2.1 *Let* $\mathbf{w}_{\rm f}\in\mathbf{V}_{\rm f}$ *be such that*

$$\|\mathbf{w}_{\rm f}\cdot\mathbf{n}\|_{0,\Sigma} \,\le\, \frac{2\mu\alpha_{\rm f}}{C^2_{\rm tr}C^2_{\mathrm{Sob},\Sigma}}, \tag{2.15}$$

where $C_{\rm tr} > 0$ *is the constant of the well-known trace inequality (see [25, Theorem 1.4]). There holds*

$$A_{\rm F}(\mathbf{v},\mathbf{v}) \,+\, O_{\rm F}(\mathbf{w}_{\rm f};\mathbf{v}_{\rm f},\mathbf{v}_{\rm f}) \,\ge\, \alpha_{\rm F}\,\|\mathbf{v}\|^2_{\mathbf{H}} \qquad \forall\,\mathbf{v}\in\mathbf{V}, \tag{2.16}$$

with $\alpha_{\rm F} := \frac{1}{2}\,\min\{\mu\alpha_{\rm f}, C_{\mathbf{K}}\}$.

We also recall from [20, Lemma 1] that the bilinear form B satisfies the following inf-sup condition

$$\sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(q,\xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \,\geq\, \beta\,\|(q,\xi)\|_{\mathrm{Q}} \qquad \forall\,(q,\xi)\in\mathrm{Q}, \tag{2.17}$$

with $\beta > 0$.

We continue by introducing the following lemma that summarizes some properties of the form O_{T}.

Lemma 2.2 *The following identity holds true:*

$$O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \,=\, -\,O_{\mathrm{T}}(\mathbf{w};\psi,\theta) \qquad \forall\,\mathbf{w}\in\mathbf{V},\,\forall\,\theta,\psi\in\Psi_{\infty,0}. \tag{2.18}$$

In addition, there exist positive constants $C_{O_{\mathrm{T}}}, \widetilde{C}_{O_{\mathrm{T}}}$*, such that*

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \,\leq\, C_{O_{\mathrm{T}}}\|\mathbf{w}\|_{\mathbf{H}}\|\psi\|_{1,\Omega}\,(\|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\,\mathbf{w}\in\mathbf{H},\,\forall\,\theta,\psi\in\Psi_{\infty,0}, \tag{2.19a}$$

$$|O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \,\leq\, \widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{w}\|_{\mathbf{H}}\,\|\psi\|_{1,\Omega}\,\,(\|\theta\|_{1,\Omega_{\mathrm{f}}} \,+\, \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\,\mathbf{w}\in\mathbf{H},\,\forall\,\theta,\psi\in\Psi_{\infty,0}. \tag{2.19b}$$

Proof. Let $\mathbf{w}\in\mathbf{V}$ and $\theta,\psi\in\Psi_{\infty,0}$ be given. Noticing that $\psi|_{\Omega_{\mathrm{m}}}\in\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})\cap\mathrm{L}^\infty(\Omega_{\mathrm{m}})$, it readily follows that $\nabla(\theta|_{\Omega_{\mathrm{m}}}\psi|_{\Omega_{\mathrm{m}}}) = \theta|_{\Omega_{\mathrm{m}}}\nabla(\psi|_{\Omega_{\mathrm{m}}}) + \psi|_{\Omega_{\mathrm{m}}}\nabla(\theta|_{\Omega_{\mathrm{m}}})\in\mathbf{L}^2(\Omega_{\mathrm{m}})$. Then, integrating by parts the two terms defining the form O_{T} (cf. (2.9)) and using the fact that $\operatorname{div}\mathbf{w}_\star = 0$ in $\Omega_\star$, for $\star\in\{\mathrm{f},\mathrm{m}\}$, and $\mathbf{w}_{\mathrm{f}}\cdot\mathbf{n} - \mathbf{w}_{\mathrm{m}}\cdot\mathbf{n} = 0$ on Σ, we easily obtain (2.18).

Now, for (2.19a) we employ (2.18) and (1.1), to obtain

$$\begin{aligned} |O_{\mathrm{T}}(\mathbf{w};\theta,\psi)| \;&=\; |O_{\mathrm{T}}(\mathbf{w};\psi,\theta)| \\ &\leq |(\mathbf{w}_{\mathrm{f}}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{f}}}| \,+\, |(\mathbf{w}_{\mathrm{m}}\cdot\nabla\psi,\theta)_{\Omega_{\mathrm{m}}}| \\ &\leq \|\mathbf{w}_{\mathrm{f}}\|_{0,6,\Omega_{\mathrm{f}}}\,|\psi|_{1,\Omega_{\mathrm{f}}}\,\|\theta\|_{0,3,\Omega_{\mathrm{f}}} \,+\, \|\mathbf{w}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}\,|\psi|_{1,\Omega_{\mathrm{m}}}\,\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}. \end{aligned}$$

Then, applying (1.2) to $\|\mathbf{w}_{\mathrm{f}}\|_{0,6,\Omega_{\mathrm{f}}}$ we easily deduce (2.19a). Finally, from (2.19a) and (1.2) applied to $\|\theta\|_{0,3,\Omega_{\mathrm{f}}}$ we easily obtain (2.19b), which concludes the proof. □

On the other hand, using the Poincaré inequality, we obtain

$$A_{\mathrm{T}}(\psi,\psi) \,\geq\, \alpha_{\mathrm{T}}\,\|\psi\|^2_{1,\Omega} \qquad \forall\,\psi\in\mathrm{H}^1_0(\Omega), \tag{2.20}$$

with constant $\alpha_{\mathrm{T}} := C_{\mathrm{P}}\min\{\kappa_{\mathrm{f}}\,,\kappa_{\mathrm{m}}\}$, where $C_{\mathrm{P}} > 0$ is the corresponding Poincaré's constant. Combining this estimate and (2.18) one easily deduce that for a given $\mathbf{w}\in\mathbf{V}$, $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ is elliptic on $\Psi_{\infty,0}$, that is

$$A_{\mathrm{T}}(\psi,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\psi,\psi) \,\geq\, \alpha_{\mathrm{T}}\,\|\psi\|^2_{1,\Omega} \qquad \forall\,\psi\in\Psi_{\infty,0}. \tag{2.21}$$

However, the latter is not valid for any $\psi\in\mathrm{H}^1_0(\Omega)$ since $\psi|_{\Omega_{\mathrm{m}}}$ would not belong to $\mathrm{L}^\infty(\Omega_{\mathrm{m}})$ and consequently $O_{\mathrm{T}}(\mathbf{w};\psi,\psi)$ would not be well-defined. Moreover, since the space for the temperature unknown θ is different to the one for the test functions ψ in the third equation of (2.8), namely $\mathrm{H}^1(\Omega)$ and $\Psi_{\infty,0}$, respectively, estimate (2.21) is not sufficient to study the well-posedness of our problem. According to the above, now we employ the truncation function defined in (1.4) to prove that for any $\mathbf{w}\in\mathbf{V}$, $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot) : \mathrm{H}^1(\Omega)\times\Psi_\infty\to\mathbb{R}$ induces an invertible operator. More precisely, we have the following lemma.

Lemma 2.3 *Let $\mathbf{w} \in \mathbf{V}$. The following inf-sup conditions hold*

$$\sup_{\psi \in \Psi_{\infty,0} \setminus \{0\}} \frac{A_{\mathrm{T}}(\theta,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\psi)}{\|\psi\|_{1,\Omega}} \,\geq\, \alpha_{\mathrm{T}}\,\|\theta\|_{1,\Omega} \qquad \forall\,\theta \in \mathrm{H}_0^1(\Omega), \tag{2.22a}$$

and

$$\sup_{\theta \in \mathrm{H}_0^1(\Omega)} A_{\mathrm{T}}(\theta,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \,>\, 0 \qquad \forall\,\psi \in \Psi_{\infty,0} \setminus \{0\}. \tag{2.22b}$$

Proof. Let $\mathbf{w} \in \mathbf{V}$. First, we recall that for any $l > 0$ and $\phi \in \mathrm{H}_0^1(\Omega)$, $\tau_l(\phi)$ belongs to $\mathrm{H}_0^1(\Omega) \cap \mathrm{L}^\infty(\Omega)$ and $\nabla \pi_l(\phi) = \nabla\phi\, \tau_l(\phi)$, which implies that

$$O_{\mathrm{T}}(\mathbf{w};\phi,\tau_l(\phi)) = (\mathbf{w}_{\mathrm{f}}, \nabla\phi\,\tau_l(\phi))_{\Omega_{\mathrm{f}}} + (\mathbf{w}_{\mathrm{m}}, \nabla\phi\,\tau_l(\phi))_{\Omega_{\mathrm{m}}} = (\mathbf{w}_{\mathrm{f}}, \nabla\pi_l(\phi))_{\Omega_{\mathrm{f}}} + (\mathbf{w}_{\mathrm{m}}, \nabla\pi_l(\phi))_{\Omega_{\mathrm{m}}},$$

and then, integrating by parts and using the fact that $\mathrm{div}\,\mathbf{w}_\star = 0$ in $\Omega_\star$, for $\star \in \{\mathrm{f},\mathrm{m}\}$, and $\mathbf{w}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{w}_{\mathrm{m}} \cdot \mathbf{n} = 0$ on Σ, we obtain

$$O_{\mathrm{T}}(\mathbf{w};\phi,\tau_l(\phi)) \,=\, 0 \qquad \forall\,\phi \in \mathrm{H}_0^1(\Omega). \tag{2.23}$$

In this way, from (2.20), (2.23) and the fact that $A_{\mathrm{T}}(\phi,\tau_l(\phi)) = A_{\mathrm{T}}(\tau_l(\phi),\tau_l(\phi))$, for all $\phi \in \mathrm{H}_0^1(\Omega)$, it readily follows that for any $l > 0$ and $\theta \in \mathrm{H}_0^1(\Omega)$, there holds

$$\sup_{\psi \in \Psi_{\infty,0} \setminus \{0\}} \frac{A_{\mathrm{T}}(\theta,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\psi)}{\|\psi\|_{1,\Omega}} \,\geq\, \frac{A_{\mathrm{T}}(\theta,\tau_l(\theta)) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\tau_l(\theta))}{\|\tau_l(\theta)\|_{1,\Omega}} \,\geq\, \alpha_{\mathrm{T}}\,\|\tau_l(\theta)\|_{1,\Omega}.$$

The latter and the strong convergence of $\tau_l(\theta)$ to θ in $\mathrm{H}^1(\Omega)$, imply (2.22a) (see [4, Lemma 2.5]).

Finally, for (2.22b) we recall that $\Psi_{\infty,0} \subseteq \mathrm{H}_0^1(\Omega)$ and employing the coercivity of $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ given in (2.21), to obtain that for any $\psi \in \Psi_{\infty,0} \setminus \{0\}$, there holds

$$\sup_{\theta \in \mathrm{H}_0^1(\Omega)} A_{\mathrm{T}}(\theta,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \,\geq\, A_{\mathrm{T}}(\psi,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\psi,\psi) \,\geq\, \alpha_{\mathrm{T}}\,\|\psi\|_{1,\Omega}^2 \,>\, 0,$$

which concludes the proof. □

2.3.2 An equivalent reduced problem

To simplify the analysis of existence and stability of solutions of (2.8), we now introduce a reduced equivalent version of the problem. To do this, we let $\mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega)$ be the usual lifting operator (see for instance [21, Corollary B.53]), satisfying

$$\gamma_0(\mathrm{E}(\zeta)) = \zeta \quad \text{and} \quad \|\mathrm{E}(\zeta)\|_{1,4,\Omega} \leq c\|\zeta\|_{3/4,4,\Gamma} \quad \forall\,\zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{2.24}$$

where $\gamma_0 : \mathrm{W}^{1,4}(\Omega) \to \mathrm{W}^{3/4,4}(\Gamma)$ is the trace operator. In turn, we let $\delta > 0$ and, similarly to [6, Lemma 2.8], define the function $\beta_\delta : \mathbb{R}^2 \to \mathbb{R}$ given by

$$\beta_\delta(\mathbf{x}) := \begin{cases} 1 & \text{if } 0 \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq \delta, \\ 2 - \delta^{-1}\mathrm{dist}(\mathbf{x},\Gamma) & \text{if } \delta \leq \mathrm{dist}(\mathbf{x},\Gamma) \leq 2\delta, \\ 0 & \text{if } \mathrm{dist}(\mathbf{x},\Gamma) \geq 2\delta, \end{cases} \tag{2.25}$$

where $\mathrm{dist}(\mathbf{x},\Gamma)$ denotes the distance from the point $\mathbf{x}$ to the boundary Γ. Observe that β_δ is continuous and satisfies

$$\beta_\delta \in \mathrm{W}^{1,\infty}(\Omega), \qquad 0 \le \beta_\delta \le 1 \quad \text{in} \quad \Omega_\delta, \quad \beta_\delta \equiv 0 \quad \text{in} \quad \Omega \setminus \Omega_\delta, \quad \text{and} \quad \|\nabla \beta_\delta\|_{0,4,\Omega_\delta} \le \delta^{-1} |\Omega_\delta|^{1/4}, \tag{2.26}$$

where $\Omega_\delta := \{\mathbf{x} \in \Omega : \mathrm{dist}(\mathbf{x},\Gamma) < 2\delta\}$. In this way, in order to handle the non-homogeneous Dirichlet boundary condition for the temperature, we introduce the extension operator

$$\mathrm{E}_\delta := \beta_\delta \mathrm{E} : \mathrm{W}^{3/4,4}(\Gamma) \to \mathrm{W}^{1,4}(\Omega). \tag{2.27}$$

In the following lemma we summarize some properties of this operator.

Lemma 2.4 *For all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, $\mathrm{E}_\delta(\zeta)$ lies in $\mathrm{L}^\infty(\Omega)$ and satisfies the estimate*

$$\|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega} \le \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}. \tag{2.28}$$

In addition, there exist $C_{\mathrm{lift},1}$, $C_{\mathrm{lift},2} > 0$, such that

$$\|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \le C_{\mathrm{lift},1}\, \delta^{1/12}\, \|\zeta\|_{3/4,4,\Gamma} \quad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma), \tag{2.29a}$$

$$\|\mathrm{E}_\delta(\zeta)\|_{1,\Omega} \le C_{\mathrm{lift},2}\, \delta^{-3/2}(1+\delta^2)^{1/2}\, \|\zeta\|_{3/4,4,\Gamma} \quad \forall \zeta \in \mathrm{W}^{3/4,4}(\Gamma). \tag{2.29b}$$

Proof. We begin the proof by observing that the first Sobolev's embedding in (1.3) with $q = 1$ and $r = 4$ guarantees the fact that $\mathrm{E}_\delta(\zeta) \in \mathrm{C}^0(\overline{\Omega}) \subseteq \mathrm{L}^\infty(\Omega)$, for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$. In addition, using the second and third properties of β_δ (cf. (2.25)) given in (2.26), there holds

$$\|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega} = \|\mathrm{E}_\delta(\zeta)\|_{0,\infty,\Omega_\delta} \le \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta},$$

which implies (2.28).

Now, to derive (2.29a) and (2.29b) we proceed similarly to [17, Lemma 3.2]. In fact, we first apply the Hölder inequality (1.1) with $p = 4$ and $q = 4/3$ and the estimate in (2.24), to obtain

$$\|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega}^3 \le \|\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta}^3 \le |\Omega_\delta|^{1/4} \|\mathrm{E}(\zeta)\|_{0,4,\Omega}^3 \le (2\delta)^{1/4}\, \|\mathrm{E}(\zeta)\|_{1,4,\Omega}^3 \le c\, \delta^{1/4} \|\zeta\|_{3/4,4,\Gamma}^3,$$

which implies (2.29a). Similarly, but now applying Hölder's inequality with $p = q = 2$ and the properties of β_δ in (2.26), it is follows that

$$\begin{aligned} \|\nabla(\mathrm{E}_\delta(\zeta))\|_{0,\Omega} &\le \|\nabla(\beta_\delta)\mathrm{E}(\zeta)\|_{0,\Omega_\delta} + \|\beta_\delta \nabla(\mathrm{E}(\zeta))\|_{0,\Omega_\delta} \\ &\le c\, \delta^{-1}\, |\Omega_\delta|^{1/4}\, \|\mathrm{E}(\zeta)\|_{0,4,\Omega} + |\Omega_\delta|^{1/4}\, \|\nabla(\mathrm{E}(\zeta))\|_{0,4,\Omega} \\ &\le c\, \delta^{-3/4}(1+\delta^2)^{1/2}\, \|\zeta\|_{3/4,4,\Gamma}. \end{aligned}$$

which gives (2.29b).

□

Given a fixed $\delta > 0$ now we define the following lifting for the Dirichlet datum $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$:

$$\theta_1 := \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega), \tag{2.30}$$

and decompose the unknown $\theta \in \mathrm{H}^1(\Omega)$ as $\theta = \theta_0 + \theta_1$, with $\theta_0 \in \mathrm{H}_0^1(\Omega)$. In turn, we recall from the second equation of (2.8) that the unknown $\mathbf{u} = (\mathbf{u}_{\mathrm{f}}, \mathbf{u}_{\mathrm{m}}) \in \mathbf{H}$ satisfies $B(\mathbf{u},(q,\xi)) = 0 \quad \forall (q,\xi) \in \mathrm{Q}$, which implies that $\mathbf{u} \in \mathbf{V}$ (cf. (2.13)). According to the above, now we introduce the reduced version of problem (2.8) on the kernel $\mathbf{V}$, which consists in finding $(\mathbf{u},\theta_0) \in \mathbf{V} \times \mathrm{H}_0^1(\Omega)$, such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) - D(\theta_0,\mathbf{v}) &= D(\theta_1,\mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}, \\ A_{\mathrm{T}}(\theta_0,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\psi) &= -\, A_{\mathrm{T}}(\theta_1,\psi) && \forall\, \psi \in \Psi_{\infty,0}. \end{aligned} \tag{2.31}$$

It is not difficult to see that problems (2.31) and (2.8) are equivalent. This result is established next.

Lemma 2.5 *If* $(\mathbf{u},(p,\lambda),\theta) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ *is a solution of* (2.8)*, then* $\mathbf{u} \in \mathbf{V}$ *and* $(\mathbf{u},\theta_0) = (\mathbf{u},\theta - \theta_1)$*, with* θ_1 *defined in* (2.30) *is a solution to* (2.31)*. Conversely, if* $(\mathbf{u},\theta_0) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ *is a solution to* (2.31)*, then there exists* $p \in \mathrm{L}^2_0(\Omega)$ *and* $\lambda \in \mathrm{H}^{1/2}(\Sigma)$ *such that* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1)$ *is a solution to* (2.8)*.*

Proof. The proof follows from the definition of the lifting θ_1 (cf. (2.30)) and the inf-sup condition (2.17). We omit further details and refer the reader to [39, Lemma 2.1] for a similar result. □

According to the previous lemma, to prove existence of solution of problem (2.8), it suffices to prove existence of solution of problem (2.31). In addition, by deriving the stability of solution of problem (2.31) one can easily obtain the corresponding stability for (2.8). We begin with the latter.

2.3.3 Stability of solution

The following theorem addresses the stability of solution of the reduced problem (2.31).

Theorem 2.6 *Let* $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$*, with* $\delta > 0$ *satisfying*

$$\frac{C_D C_{O_\mathrm{T}}}{\alpha_\mathrm{F}\alpha_\mathrm{T}}(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})C_{\mathrm{lift},1}\,\delta^{1/12}\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma} \le \frac{1}{4}, \tag{2.32}$$

and let $(\mathbf{u},\theta_0) = ((\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{m}),\theta_0)$ *be a solution of* (2.31)*. If we assume that*

$$\|\mathbf{u}_\mathrm{f}\cdot\mathbf{n}\|_{0,\Sigma} \le \frac{2\mu\alpha_\mathrm{f}}{C^2_\mathrm{tr}C^2_{\mathrm{Sob},\Sigma}}, \tag{2.33}$$

and that the lifting θ_1 *satisfies the estimate*

$$\frac{C_D\, C_{O_\mathrm{T}}}{\alpha_\mathrm{F}\,\alpha_\mathrm{T}}\,(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\,\|\theta_1\|_{0,\infty,\Omega} \le \frac{1}{4}, \tag{2.34}$$

then, there holds

$$\|\mathbf{u}\|_\mathbf{H} \le C_\mathbf{u}\,(\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}})\,\|\theta_1\|_{1,\Omega}, \tag{2.35}$$

and

$$\|\theta_0\|_{1,\Omega} \le C_\theta\,\|\theta_1\|_{1,\Omega}, \tag{2.36}$$

with $C_\mathbf{u}$ *and* C_θ*, being positive constants depending on the stability constants given in Section 2.3.1 (see* (2.40) *and* (2.44) *below for explicit expressions of* $C_\mathbf{u}$ *and* C_θ*, respectively).*

Proof. Let $(\mathbf{u},\theta_0) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ be a solution of (2.31) and to simplify the notation, let

$$\gamma_\mathbf{g} := \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}. \tag{2.37}$$

Noticing that the first equation of (2.31) can be written as

$$A_\mathrm{F}(\mathbf{u},\mathbf{v}) + O_\mathrm{F}(\mathbf{u}_\mathrm{f};\mathbf{v}_\mathrm{f},\mathbf{u}_\mathrm{f}) = D(\theta_0,\mathbf{v}) + D(\theta_1,\mathbf{v}), \quad \forall\,\mathbf{v} \in \mathbf{V},$$

we take $\mathbf{v} = \mathbf{u}$, and owing to (2.33) we can make use of the ellipticity of $A_\mathrm{F}(\cdot,\cdot) + O_\mathrm{F}(\mathbf{u}_\mathrm{f};\cdot,\cdot)$ given in (2.16), and the continuity of D, to obtain

$$\alpha_\mathrm{F}\,\|\mathbf{u}\|_\mathbf{H} \le C_D\,\gamma_\mathbf{g}\,\|\theta_0\|_{1,\Omega} + C_D\gamma_\mathbf{g}\,\|\theta_1\|_{1,\Omega}. \tag{2.38}$$

In turn, from the second equation of (2.31), the inf-sup condition (2.22a), and the continuity of A_{T} and O_{T} (cf. (2.11) and (2.19a), respectively), we obtain

$$\begin{aligned}\alpha_{\mathrm{T}}\,\|\theta_0\|_{1,\Omega} &\le \sup_{\psi\in\Psi_{\infty,0}\backslash\{0\}}\frac{|A_{\mathrm{T}}(\theta_0,\psi)\,+\,O_{\mathrm{T}}(\mathbf{u};\theta_0,\psi)|}{\|\psi\|_{1,\Omega}} = \sup_{\psi\in\Psi_{\infty,0}\backslash\{0\}}\frac{|-\,A_{\mathrm{T}}(\theta_1,\psi)\,-\,O_{\mathrm{T}}(\mathbf{u};\theta_1,\psi)|}{\|\psi\|_{1,\Omega}}\\ &\le C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega}\,+\,C_{O_{\mathrm{T}}}\,\|\mathbf{u}\|_{\mathbf{H}}\,\left(\|\theta_1\|_{0,3,\Omega_{\mathrm{f}}}\,+\,\|\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}\right).\end{aligned} \tag{2.39}$$

In this way, from (2.38), (2.39), there holds

$$\begin{aligned}\|\mathbf{u}\|_{\mathbf{H}} &\le \frac{C_D(C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}})}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega} + \frac{C_D C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\gamma_{\mathbf{g}}\|\mathbf{u}\|_{\mathbf{H}}\left(\|\theta_1\|_{0,3,\Omega_{\mathrm{f}}}+\|\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}\right)\\ &\le \frac{C_D(C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}})}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega} + \frac{C_D C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\gamma_{\mathbf{g}}\|\mathbf{u}\|_{\mathbf{H}}\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}+\|\theta_1\|_{0,\infty,\Omega}\right),\end{aligned}$$

where in the last inequality we employed (2.29a). In this way, from (2.34) and (2.32) we deduce (2.35), with

$$C_{\mathbf{u}} := 2\,\alpha_{\mathrm{F}}^{-1}\,\alpha_{\mathrm{T}}^{-1}\,C_D\,\left(C_{A_{\mathrm{T}}}\,+\,\alpha_{\mathrm{T}}\right). \tag{2.40}$$

In turn, by combining (2.39) and (2.35), we get

$$\|\theta_0\|_{1,\Omega}\,\le\,\alpha_{\mathrm{T}}^{-1}\,\left[C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega}\,+\,C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}+\|\theta_1\|_{0,\infty,\Omega}\right)\right]. \tag{2.41}$$

In addition, using the definition of $C_{\mathbf{u}}$ in (2.40), from (2.32) and (2.34), we obtain, respectively

$$\alpha_{\mathrm{T}}^{-1}\,C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,C_{\mathrm{lift},1}\,\delta^{1/12}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}\,\le\,\left(\frac{C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}}}{2\alpha_{\mathrm{T}}}\right)\,\|\theta_1\|_{1,\Omega}, \tag{2.42}$$

and

$$\alpha_{\mathrm{T}}^{-1}\,C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,\|\theta_1\|_{0,\infty,\Omega}\,\le\,\left(\frac{C_{A_{\mathrm{T}}}+\alpha_{\mathrm{T}}}{2\alpha_{\mathrm{T}}}\right)\,\|\theta_1\|_{1,\Omega}. \tag{2.43}$$

Therefore, by combining (2.41), (2.42), and (2.43), we achieve (2.36) with

$$C_\theta := \alpha_{\mathrm{T}}^{-1}\,(2\,C_{A_{\mathrm{T}}}\,+\,\alpha_{\mathrm{T}}). \tag{2.44}$$

□

Remark 2.1 *Observe that, according to* (2.28), *the condition* (2.34) *can be replaced by*

$$\frac{C_D\,C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right)\|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta}\,\le\,\frac{1}{4},$$

which, in other words, means that the L^∞*-norm of the extension of the datum* θ_{D} *must be small enough on* $\Omega_\delta := \{\mathbf{x}\in\Omega : \mathrm{dist}(\mathbf{x},\Gamma) < 2\delta\}$, *where* Ω_δ *is a small portion of the domain* Ω *near the boundary* Γ. *In particular, if* δ *is small enough so that* $\|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta}\approx\|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}$, *one could simply assume that*

$$\frac{C_D\,C_{O_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}\right)\|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}\,\le\,\frac{1}{4}.$$

On the other hand, assumption (2.33) *suggests that the magnitude of the inflow on the interface must be bounded, which is a reasonable assumption for this kind of phenomena. Otherwise, the porous medium would act as a wall which would prevent the fluid to penetrate.*

We end this section by deriving the corresponding estimate for the pressure p and the Lagrange multiplier λ.

Corollary 2.7 *Let* $(\mathbf{u},\theta_0) = ((\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{m}),\theta_0) \in \mathbf{V}\times \mathrm{H}^1_0(\Omega)$ *be a solution of* (2.31) *and let* $(p,\lambda)\in \mathrm{Q}$ *be such that* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *is a solution to* (2.8)*. If we assume that the hypotheses of Theorem 2.6 hold, then there exist* C_1, $C_2>0$, *such that*

$$\|(p,\lambda)\|_\mathrm{Q} \leq (C_1 + C_2\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega})\gamma_\mathbf{g}\|\theta_1\|_{1,\Omega},$$

with $\gamma_\mathbf{g} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$.

Proof. The result is a direct consequence of the inf-sup condition (2.17) and the first equation of (2.8). In fact, it is easy to see that

$$\begin{aligned}
\beta\,\|(p,\lambda)\|_\mathrm{Q} &\leq \sup_{\mathbf{v}\in\mathbf{H}\backslash\{\mathbf{0}\}} \frac{B(\mathbf{v},(p,\lambda))}{\|\mathbf{v}\|_\mathbf{H}}\\
&= \sup_{\mathbf{v}\in\mathbf{H}\backslash\{\mathbf{0}\}} \frac{-A_\mathrm{F}(\mathbf{u},\mathbf{v}) \,-\, O_\mathrm{F}(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) \,+\, D(\theta,\mathbf{v})}{\|\mathbf{v}\|_\mathbf{H}}\\
&\leq \big(C_{A_\mathrm{F}}\,\|\mathbf{u}\|_\mathbf{H} \,+\, C_{O_\mathrm{F}}\,\|\mathbf{u}_\mathrm{f}\|^2_{1,\Omega_\mathrm{f}} \,+\, C_D\gamma_\mathbf{g}\,\|\theta\|_{1,\Omega}\big)\\
&\leq (C_{A_\mathrm{F}} \,+\, C_{O_\mathrm{F}}\,C_\mathbf{u}\,\gamma_\mathbf{g}\,\|\theta_1\|_{1,\Omega})C_\mathbf{u}\gamma_\mathbf{g}\,\|\theta_1\|_{1,\Omega} \,+\, C_D\,\gamma_\mathbf{g}(C_\theta+1)\|\theta_1\|_{1,\Omega},
\end{aligned}$$

which implies the result. □

2.3.4 Existence of solution

In what follows we proceed similarly to [4, Section 2.3] to prove existence of solution of (2.31) by means of a Galerkin's method and a fixed-point strategy. More precisely, since the trial and test spaces are different, we introduce a Galerkin scheme for (2.31) to obtain a finite-dimensional square system of nonlinear equations. Then, we apply the Brouwer Fixed Point Theorem to prove existence of solution of the resulting finite-dimensional problem and pass to the limit to obtain the desired solution.

We begin by recalling from [10, Propositions 9.1 and 3.25] that $\mathrm{W}^{1,4}_0(\Omega)$ is separable and has a countable basis. In turn, since $\mathbf{V}$ is a closed subspace of $\mathbf{H} = \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})\times\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$ and $\mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ and $\mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$ are separable, $\mathbf{V}$ is also separable and has a countable basis. Then, we let $\{\varphi_i\}_{i\in\mathbb{N}}$ and $\{\mathbf{z}_i\}_{i\in\mathbb{N}} := \{(\mathbf{z}_{i,\mathrm{f}},\mathbf{z}_{i,\mathrm{m}})\}_{i\in\mathbb{N}}$ be the countable bases of $\mathrm{W}^{1,4}_0(\Omega)$ and $\mathbf{V}$, respectively and for a fixed $n\in\mathbb{N}$, we let $\Psi_n = \langle\{\varphi_1,\ldots,\varphi_n\}\rangle$ and $\mathbf{V}_n = \langle\{\mathbf{z}_1,\ldots,\mathbf{z}_n\}\rangle$. We define the following finite-dimensional nonlinear problem: Find $(\mathbf{u}_n,\theta_{n,0}) := ((\mathbf{u}_{n,\mathrm{f}},\mathbf{u}_{n,\mathrm{m}}),\theta_{n,0}) \in \mathbf{V}_n\times\Psi_n$, such that

$$\begin{aligned}
A_\mathrm{F}(\mathbf{u}_n,\mathbf{v}) \,+\, O_\mathrm{F}(\mathbf{u}_{n,\mathrm{f}};\mathbf{u}_{n,\mathrm{f}},\mathbf{v}_\mathrm{f}) \,-\, D(\theta_{n,0},\mathbf{v}) &= D(\theta_1,\mathbf{v}),\\
A_\mathrm{T}(\theta_{n,0},\psi) \,+\, O_\mathrm{T}(\mathbf{u}_n;\theta_{n,0}+\theta_1,\psi) &= -\,A_\mathrm{T}(\theta_1,\psi),
\end{aligned}\tag{2.45}$$

for all $(\mathbf{v},\psi) := ((\mathbf{v}_\mathrm{f},\mathbf{v}_\mathrm{m}),\psi)\in\mathbf{V}_n\times\Psi_n$, with θ_1 defined as in (2.30).

Notice that (2.45) is a discrete version of (2.31) since $\mathrm{W}^{1,4}_0(\Omega)\subseteq\mathrm{L}^\infty(\Omega)$ and $\mathrm{W}^{1,4}_0(\Omega)\subseteq\mathrm{H}^1_0(\Omega)$ (owing to (1.3)).

In what follows, we prove that problem (2.45) has at least one solution by means of the classical Brouwer's fixed point theorem in the following form (see [10]):

Theorem 2.8 *Let* Y *be a compact and convex subset of a finite dimensional Banach space* X*, and let* $f:\mathrm{Y}\to\mathrm{Y}$ *be a continuous mapping. Then,* f *has at least one fixed point.*

To apply Theorem 2.8 to the context of problem (2.45), we first define the compact and convex set

$$\mathbf{X}_n := \{(\mathbf{w},\phi) \in \mathbf{V}_n \times \Psi_n : \|\mathbf{w}\|_{\mathbf{H}} \,\leq\, C_{\mathbf{u}}\,(\|\mathbf{g}_{\rm f}\|_{0,\Omega_{\rm f}} + \|\mathbf{g}_{\rm m}\|_{0,3,\Omega_{\rm m}})\|\theta_1\|_{1,\Omega}, \quad \|\phi\|_{1,\Omega} \,\leq\, C_\theta\,\|\theta_1\|_{1,\Omega}\}\,,$$

with $C_{\mathbf{u}}$ and C_θ defined in (2.40) and (2.44), respectively. In turn, we let $\mathcal{J}_n : \mathbf{X}_n \to \mathbf{V}_n \times \Psi_n$ be the operator defined by

$$\mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u}_n, \theta_{n,0}) \qquad \forall\,(\mathbf{w},\phi) = ((\mathbf{w}_{\rm f},\mathbf{w}_{\rm m}),\phi) \in \mathbf{X}_n, \tag{2.46}$$

where $(\mathbf{u}_n, \theta_{n,0})$ is the unique solution (to be verified below) of the linearized version of problem (2.45): Find $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{V}_n \times \Psi_n$, such that

$$\begin{aligned} A_{\rm F}(\mathbf{u}_n,\mathbf{v}) \,+\, O_{\rm F}(\mathbf{w}_{\rm f};\mathbf{u}_{n,{\rm f}},\mathbf{v}_{\rm f}) &= D(\phi,\mathbf{v}) \,+\, D(\theta_1,\mathbf{v}) && \forall\,\mathbf{v} \in \mathbf{V}_n, \\ A_{\rm T}(\theta_{n,0},\psi) \,+\, O_{\rm T}(\mathbf{w};\theta_{n,0},\psi) &= -\,A_{\rm T}(\theta_1,\psi) \,-\, O_{\rm T}(\mathbf{w};\theta_1,\psi) && \forall\,\psi \in \Psi_n. \end{aligned} \tag{2.47}$$

Then, it is clear that $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{V}_n \times \Psi_n$ is a solution of problem (2.45), if and only if, $\mathcal{J}_n(\mathbf{u}_n, \theta_{n,0}) = (\mathbf{u}_n, \theta_{n,0})$.

According to the above, to prove existence of solution to (2.45) in what follows we equivalently prove that $\mathcal{J}_n$ satisfies the hypotheses of Theorem 2.8. Before doing that, we must verify that $\mathcal{J}_n$ is well-defined. This is addressed in the following Lemma.

Lemma 2.9 *Let $\delta > 0$ satisfying (2.32) and let $\theta_1 = {\rm E}_\delta(\theta_{\rm D}) \in {\rm W}^{1,4}(\Omega)$ be such that (2.34) holds. If we assume further that θ_D satisfies the following estimate*

$$C_{\mathbf{u}}\,(\|\mathbf{g}_{\rm f}\|_{0,\Omega_{\rm f}} + \|\mathbf{g}_{\rm m}\|_{0,3,\Omega_{\rm m}})\,C_{{\rm lift},2}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_{\rm D}\|_{3/4,4,\Gamma} \,\leq\, \frac{2\mu\alpha_{\rm f}}{C_{\rm tr}^3\,C_{{\rm Sob},\Sigma}^2} \tag{2.48}$$

then, $\mathcal{J}_n(\mathbf{X}_n) \subseteq \mathbf{X}_n$ and for each $(\mathbf{w},\phi) \in \mathbf{X}_n$, there exists a unique $(\mathbf{u},\theta) \in \mathbf{X}_n$, such that $\mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u},\theta)$.

Proof. Given $(\mathbf{w},\phi) = ((\mathbf{w}_{\rm f},\mathbf{w}_{\rm m}),\phi) \in \mathbf{X}_n$, we first observe that (2.47) is an uncoupled system of linear equations. Thus, to prove that operator $\mathcal{J}_n$ is well-defined it suffices to prove the well-posedness of the two equations in (2.47) separately.

For the subsequent analysis, we let $\gamma_{\mathbf{g}}$ as in (2.37), that is $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\rm f}\|_{0,\Omega_{\rm f}} + \|\mathbf{g}_{\rm m}\|_{0,3,\Omega_{\rm m}}$.

First, we use the well-known trace inequality with constant $C_{\rm tr} > 0$ (see [25, Theorem 1.4]), and estimates (2.29b) and (2.48), to obtain

$$\begin{aligned} \|\mathbf{w}_{\rm f}\cdot\mathbf{n}\|_{0,\Sigma} &\leq\, C_{\rm tr}\|\mathbf{w}_{\rm f}\|_{1,\Omega_{\rm f}} \,\leq\, C_{\rm tr}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega} \\ &\leq\, C_{\rm tr}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,C_{{\rm lift},2}\,\delta^{-3/4}(1+\delta^2)^{1/2}\,\|\theta_{\rm D}\|_{3/4,4,\Gamma} \\ &\leq\, \tfrac{2\mu\alpha_{\rm f}}{C_{\rm tr}^2\,C_{{\rm Sob},\Sigma}^2}, \end{aligned} \tag{2.49}$$

which implies that $\mathbf{w}_{\rm f}$ satisfies (2.15). Then, thanks to Lemma 2.1 we have that $A_{\rm F}(\cdot,\cdot) + O_{\rm F}(\mathbf{w}_{\rm f};\cdot,\cdot)$ is elliptic on $\mathbf{V}_n$, which together with the Lax–Milgram Lemma implies that there exists a unique $\mathbf{u} \in \mathbf{V}_n$ solution to the first equation of (2.47). Similarly, since $\Psi_n \subseteq \Psi_{\infty,0}$ and $\mathbf{V}_n \subseteq \mathbf{V}$, from (2.21) we have that $A_{\rm T}(\cdot,\cdot) + O_{\rm T}(\mathbf{w};\cdot,\cdot)$ is Ψ_n-elliptic. Then, owing to the Lax–Milgram Lemma we obtain that there exists a unique $\theta \in \Psi_n$, solution to the second equation of (2.47). According to the above, we have proved that there exists a unique $(\mathbf{u},\theta) \in \mathbf{V}_n \times \Psi_n$, such that $\mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u},\theta)$.

To conclude that $\mathcal{J}_n(\mathbf{X}_n) \subseteq \mathbf{X}_n$, it remains to prove that the aforementioned solution $(\mathbf{u}, \theta)$ belongs to $\mathbf{X}_n$. To that end, we first notice that, since $\mathbf{u}$ satisfies the first equation in (2.47), from the ellipticity of $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\cdot,\cdot)$, the continuity of D, the fact that $\phi \in \mathbf{X}_n$, and the definitions of C_θ and $C_{\mathbf{u}}$ (cf. (2.44) and (2.40), respectively), we have

$$\|\mathbf{u}\|_{\mathbf{H}} \,\leq\, \alpha_{\mathrm{F}}^{-1}\,\left(C_D\,\|\phi\|_{1,\Omega}\,\gamma_{\mathbf{g}} \,+\, C_D\,\|\theta_1\|_{1,\Omega}\,\gamma_{\mathbf{g}}\right) \,\leq\, \alpha_{\mathrm{F}}^{-1}\,C_D\,(C_\theta + 1)\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega} \leq C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}.$$

Similarly, since θ satisfies the second equation of (2.47), from the estimate above, the ellipticity of $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w};\cdot,\cdot)$ (cf. (2.21)), the continuity of A_{T} (cf. (2.11)) and O_{T} (cf. (2.19a)), and estimate (2.29a), it follows that

$$\begin{aligned}\|\theta\|_{1,\Omega} &\leq \alpha_{\mathrm{T}}^{-1}\,\left[C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega} \,+\, C_{O_{\mathrm{T}}}\,\|\mathbf{u}\|_{\mathbf{H}}\,\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + \|\theta_1\|_{0,\infty,\Omega}\right)\right] \\ &\leq \alpha_{\mathrm{T}}^{-1}\,\left[C_{A_{\mathrm{T}}}\|\theta_1\|_{1,\Omega} \,+\, C_{O_{\mathrm{T}}}\,C_{\mathbf{u}}\,\gamma_{\mathbf{g}}\,\|\theta_1\|_{1,\Omega}\,\left(C_{\mathrm{lift},1}\delta^{1/12}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + \|\theta_1\|_{0,\infty,\Omega}\right)\right].\end{aligned}$$

In this way, noticing that the estimate above coincides with estimate (2.41), analogously to the proof of Theorem 2.6, we easily obtain $\|\theta\|_{1,\Omega} \leq C_\theta\|\theta_1\|_{1,\Omega}$. According to the above, $(\mathbf{u},\theta) \in \mathbf{X}_n$ which implies that $\mathcal{J}_n(\mathbf{X}_n) \subseteq \mathbf{X}_n$, which concludes the proof. □

Now we establish the continuity of operator $\mathcal{J}_n$.

Lemma 2.10 *If we assume that the hypotheses of Lemma 2.9 hold, then $\mathcal{J}_n$ is a continuous operator.*

Proof. Let $(\mathbf{w},\phi) = ((\mathbf{w}_{\mathrm{f}},\mathbf{w}_{\mathrm{m}}),\phi) \in \mathbf{X}_n$ and $\{(\mathbf{w}_j,\phi_j)\}_{j\in\mathbb{N}} = \{((\mathbf{w}_{\mathrm{f},j},\mathbf{w}_{\mathrm{m},j}),\phi_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_n$, be such that $\|\mathbf{w}_j - \mathbf{w}\|_{\mathbf{H}} \stackrel{j\to\infty}{\longrightarrow} 0$ and $\|\phi_j - \phi\|_{1,\Omega} \stackrel{j\to\infty}{\longrightarrow} 0$, and let $\{(\mathbf{u}_j,\theta_j)\}_{j\in\mathbb{N}} = \{((\mathbf{u}_{\mathrm{f},j},\mathbf{u}_{\mathrm{m},j}),\theta_j)\}_{j\in\mathbb{N}} \subseteq \mathbf{X}_n$ and $(\mathbf{u},\theta) = ((\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{m}}),\theta) \in \mathbf{X}_n$ given, respectively, by

$$\mathcal{J}_n(\mathbf{w}_j,\phi_j) = (\mathbf{u}_j,\theta_j) \quad \forall\, j \in \mathbb{N} \quad \text{and} \quad \mathcal{J}_n(\mathbf{w},\phi) = (\mathbf{u},\theta).$$

To prove the continuity of $\mathcal{J}_n$ it suffices to prove that $\|\mathbf{u}_j - \mathbf{u}\|_{\mathbf{H}} \stackrel{j\to\infty}{\longrightarrow} 0$ and $\|\theta_j - \theta\|_{1,\Omega} \stackrel{j\to\infty}{\longrightarrow} 0$. To that end, given $j \in \mathbb{N}$, from (2.47) and the definition of $\mathcal{J}_n$ (cf. (2.46)), we first observe that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}-\mathbf{u}_j,\mathbf{v}) \,+\, O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,-\, O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f},j};\mathbf{u}_{\mathrm{f},j},\mathbf{v}_{\mathrm{f}}) &= D(\phi-\phi_j,\mathbf{v}) & \forall\,\mathbf{v} = (\mathbf{v}_{\mathrm{f}},\mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_n, \\ A_{\mathrm{T}}(\theta-\theta_j,\psi) \,+\, O_{\mathrm{T}}(\mathbf{w};\theta,\psi) \,-\, O_{\mathrm{T}}(\mathbf{w}_j;\theta_j,\psi) &= -\,O_{\mathrm{T}}(\mathbf{w}-\mathbf{w}_j;\theta_1,\psi) & \forall\,\psi \in \Psi_n. \end{aligned} \tag{2.50}$$

In turn, noticing that $\mathbf{w}_{\mathrm{f}}$ satisfies (2.49), we have that $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}};\cdot,\cdot)$ is elliptic (cf. (2.16)) on $\mathbf{V}_n$. Then, from (2.50) with $\mathbf{v} = \mathbf{u} - \mathbf{u}_j$, adding and subtracting $O_{\mathrm{F}}(\mathbf{w}_{\mathrm{f},j};\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{f}} - \mathbf{u}_{\mathrm{f},j})$ and employing the continuity of O_{F} and D, and the fact that $\mathbf{w}_{\mathrm{f},j} \stackrel{j\to\infty}{\longrightarrow} \mathbf{w}_{\mathrm{f}}$ and $\phi_j \stackrel{j\to\infty}{\longrightarrow} \phi$, we arrive at

$$\alpha_{\mathrm{F}}\,\|\mathbf{u}-\mathbf{u}_j\|_{\mathbf{H}} \,\leq\, C_{O_{\mathrm{F}}}\|\mathbf{w}_{\mathrm{f}} - \mathbf{w}_{\mathrm{f},j}\|_{1,\Omega_{\mathrm{f}}}\,\|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \,+\, C_D\,\|\phi - \phi_j\|_{1,\Omega}\,\gamma_{\mathbf{g}} \stackrel{j\to\infty}{\longrightarrow} 0, \tag{2.51}$$

where $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$.

Similarly, in the second equation of (2.50), we let $\psi = \theta - \theta_j$, add and subtract $O_{\mathrm{T}}(\mathbf{w}_j;\theta,\theta-\theta_j)$, and employ the coercivity of $A_{\mathrm{T}}(\cdot,\cdot) + O_{\mathrm{T}}(\mathbf{w}_j;\cdot,\cdot)$ given in (2.21), the continuity of O_{T} (cf. (2.19b)), and the fact $\mathbf{w}_j \stackrel{j\to\infty}{\longrightarrow} \mathbf{w}$, to obtain

$$\alpha_{\mathrm{T}}\,\|\theta-\theta_j\|_{1,\Omega} \,\leq\, \widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{w}-\mathbf{w}_j\|_{\mathbf{H}}\,(\|\theta+\theta_1\|_{1,\Omega_{\mathrm{f}}} + \|\theta+\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}) \stackrel{j\to\infty}{\longrightarrow} 0. \tag{2.52}$$

In this way, according to the definition of $\mathcal{J}_n$ (cf. (2.46)), from (2.51) and (2.52) we obtain that $\mathcal{J}_n(\mathbf{w}_j,\phi_j) \stackrel{j\to\infty}{\longrightarrow} \mathcal{J}_n(\mathbf{w},\phi)$, which implies the continuity of $\mathcal{J}_n$. □

Now we are in position of establishing the solvability result for the finite-dimensional nonlinear problem (2.45).

Theorem 2.11 *Let* $\delta > 0$ *satisfying* (2.32) *and let* $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ *be such that* (2.34) *holds. Assume further that* (2.48) *holds. There exists at least one* $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{X}_n$ *solution to problem* (2.45).

Proof. The proof follows from Lemmas 2.9, 2.10 and Theorem 2.8. □

Now we address the solvability of the reduced problem (2.31). This result is established in the following Theorem.

Theorem 2.12 *Let* $\delta > 0$ *satisfying* (2.32) *and let* $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ *be such that* (2.34) *holds. Assume further that* (2.48) *holds. Then, there exists at least one solution* $(\mathbf{u}, \theta_0) \in \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ *to problem* (2.31).

Proof. In what follow we proceed similarly to the proof of [4, Theorem 2.3]. To that end, for each $n \in \mathbb{N}$, we let $(\mathbf{u}_n, \theta_{n,0}) := ((\mathbf{u}_{\mathrm{f},n}, \mathbf{u}_{\mathrm{m},n}), \theta_{n,0}) \in \mathbf{X}_n$ be a solution of problem (2.45) and let $\{(\mathbf{u}_n, \theta_{n,0})\}_{n\in\mathbb{N}} \subseteq \mathbf{V} \times \mathrm{H}^1_0(\Omega)$ be the resulting sequence. In turn, for a fixed $1 \le i \le n$, we let $(\mathbf{z}_i, \varphi_i) := ((\mathbf{z}_{i,\mathrm{f}}, \mathbf{z}_{i,\mathrm{m}}), \varphi_i) \in \mathbf{V}_n \times \Psi_n$ be the i-th basis function of $\mathbf{V}_n \times \Psi_n$.

First we notice that, since $(\mathbf{u}_n, \theta_{n,0}) \in \mathbf{X}_n$, then for all $n \in \mathbb{N}$,

$$\|\mathbf{u}_n\|_\mathbf{H} \,\le\, C_\mathbf{u}\, (\|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}) \|\theta_1\|_{1,\Omega} \quad \text{and} \quad \|\theta_{n,0}\|_{1,\Omega} \,\le\, C_\theta\, \|\theta_1\|_{1,\Omega},$$

thus $\{(\mathbf{u}_n, \theta_{n,0})\}_{n\in\mathbb{N}}$ is a uniformly bounded sequence of $\mathbf{V} \times \mathrm{H}^1_0(\Omega)$, which together with the fact that $\mathbf{V}$ is a closed subspace of $\mathbf{H}$, implies that there exists a subsequence, namely $\{(\widehat{\mathbf{u}}_n, \widehat{\theta}_{n,0})\}_{n\in\mathbb{N}} \subseteq \{(\mathbf{u}_n, \theta_{n,0})\}_{n\in\mathbb{N}}$, that weakly converges to some function $(\mathbf{u}, \theta_0) = ((\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}), \theta_0)$ in $\mathbf{V} \times \mathrm{H}^1_0(\Omega)$, that is,

$$\widehat{\mathbf{u}}_n = (\widehat{\mathbf{u}}_{\mathrm{f},n}, \widehat{\mathbf{u}}_{\mathrm{m},n}) \overset{n\to\infty}{\rightharpoonup} \mathbf{u} = (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V} \subseteq \mathbf{H} = \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f}) \times \mathrm{H}_{\Gamma_\mathrm{m}}(\mathrm{div}; \Omega_\mathrm{m}) \quad \text{and} \quad \widehat{\theta}_{n,0} \overset{n\to\infty}{\rightharpoonup} \theta_0 \in \mathrm{H}^1_0(\Omega). \tag{2.53}$$

In the sequel we prove that $(\mathbf{u}, \theta_0)$ is a solution to (2.31). In fact, from the second weak convergence in (2.53) we have that for each $i \in \mathbb{N}$, there holds

$$|A_\mathrm{T}(\widehat{\theta}_{n,0} - \theta_0, \varphi_i)| \le \max\{\kappa_\mathrm{f}, \kappa_\mathrm{m}\} |(\nabla(\widehat{\theta}_{n,0} - \theta_0), \nabla\varphi_i)_\Omega| \overset{n\to\infty}{\longrightarrow} 0,$$

thus

$$\lim_{n\to\infty} A_\mathrm{T}(\widehat{\theta}_{n,0}, \varphi_i) \,=\, A_\mathrm{T}(\theta_0, \varphi_i). \tag{2.54}$$

Now, recalling that $\varphi_i \in \Psi_{\infty,0}$ for all $i \in \mathbb{N}$, by applying the Green formula (2.18), we deduce that

$$O_\mathrm{T}(\widehat{\mathbf{u}}_n; \widehat{\theta}_{n,0} + \theta_1, \varphi_i) \,=\, -\,O_\mathrm{T}(\widehat{\mathbf{u}}_n; \varphi_i, \widehat{\theta}_{n,0} + \theta_1) \,=\, -\,(\widehat{\mathbf{u}}_{\mathrm{f},n}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}} - (\widehat{\mathbf{u}}_{\mathrm{m},n}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}}. \tag{2.55}$$

In turn, since $\{\widehat{\mathbf{u}}_{\mathrm{f},n}\}_{n\in\mathbb{N}}$ converges weakly to $\mathbf{u}_\mathrm{f}$ in $\mathbf{H}^1(\Omega_\mathrm{f})$ and since $\mathbf{H}^1(\Omega_\mathrm{f})$ is compactly embedded in $\mathbf{L}^4(\Omega_\mathrm{f})$, it follows that $\{\widehat{\mathbf{u}}_{\mathrm{f},n}\}_{n\in\mathbb{N}}$ converges strongly to $\mathbf{u}_\mathrm{f}$ in $\mathbf{L}^4(\Omega_\mathrm{f})$, and analogously we have that $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ converges strongly to θ_0 in $\mathrm{L}^4(\Omega)$. These strong convergences and the fact that $\theta_1 \in \mathrm{W}^{1,4}(\Omega)$, imply that

$$\lim_{n\to\infty} (\widehat{\mathbf{u}}_{\mathrm{f},n}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}} = (\mathbf{u}_\mathrm{f}\,(\theta_0 + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{f}}. \tag{2.56}$$

On the other hand, the strong convergence of $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ to θ_0 in $\mathrm{L}^4(\Omega)$ and the fact that $\nabla\varphi_i \in \mathbf{L}^4(\Omega)$ imply that $\{\widehat{\theta}_{n,0}\nabla\varphi_i\}_{n\in\mathbb{N}}$ convergences strongly to $\theta_0\nabla\varphi_i$ in $\mathbf{L}^2(\Omega)$. Then, similarly to (2.56), this strong convergence and the weak convergence of $\{\widehat{\mathbf{u}}_{\mathrm{m},n}\}_{n\in\mathbb{N}}$ to $\mathbf{u}_\mathrm{m}$ in $\mathbf{L}^2(\Omega_\mathrm{m})$, imply that

$$\lim_{n\to\infty} (\widehat{\mathbf{u}}_{\mathrm{m},n}(\widehat{\theta}_{n,0} + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}} = (\mathbf{u}_\mathrm{m}\,(\theta_0 + \theta_1), \nabla\varphi_i)_{\Omega_\mathrm{m}}. \tag{2.57}$$

In this way, from the second equation of (2.45), and from (2.54), (2.55), (2.56) and (2.57), it follows that

$$\begin{aligned} -A_{\mathrm{T}}(\theta_1,\varphi_i) &= \lim_{n\to\infty}\Big[A_{\mathrm{T}}(\widehat{\theta}_{n,0},\varphi_i) + O_{\mathrm{T}}(\widehat{\mathbf{u}}_n;\widehat{\theta}_{n,0}+\theta_1,\varphi_i)\Big] \\ &= \lim_{n\to\infty}\Big[A_{\mathrm{T}}(\widehat{\theta}_{n,0},\varphi_i) - O_{\mathrm{T}}(\widehat{\mathbf{u}}_n;\varphi_i,\widehat{\theta}_{n,0}+\theta_1)\Big] \\ &= A_{\mathrm{T}}(\theta_0,\varphi_i) - O_{\mathrm{T}}(\mathbf{u};\varphi_i,\theta_0+\theta_1) = A_{\mathrm{T}}(\theta_0,\varphi_i) + O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\varphi_i). \end{aligned} \tag{2.58}$$

Analogously to the above, from the first equation of (2.45) and using the fact that $\{\widehat{\mathbf{u}}_{\mathrm{f},n}\}_{n\in\mathbb{N}}$ and $\{\widehat{\theta}_{n,0}\}_{n\in\mathbb{N}}$ converge strongly to $\mathbf{u}$ in $\mathbf{L}^4(\Omega_{\mathrm{f}})$ and θ_0 in $\mathrm{L}^4(\Omega)$, respectively, we deduce that

$$\begin{aligned} D(\theta_1,\mathbf{z}_i) &= \lim_{n\to\infty}\Big[A_{\mathrm{F}}(\widehat{\mathbf{u}}_n,\mathbf{z}_i) + O_{\mathrm{F}}(\widehat{\mathbf{u}}_{n,\mathrm{f}};\widehat{\mathbf{u}}_{n,\mathrm{f}},\mathbf{z}_{i,\mathrm{f}}) - D(\widehat{\theta}_{n,0},\mathbf{z}_i)\Big] \\ &= A_{\mathrm{F}}(\mathbf{u},\mathbf{z}_i) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{z}_{i,\mathrm{f}}) - D(\theta_0,\mathbf{z}_i). \end{aligned} \tag{2.59}$$

In this way, from (2.58), (2.59), and the fact that the basis $\{(\mathbf{z}_i,\varphi_i)\}_{i\in\mathbb{N}}$ is dense in $\mathbf{V}\times\mathrm{W}^{1,4}_0(\Omega)$, we obtain

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) - D(\theta_0,\mathbf{v}) &= D(\theta_1,\mathbf{v}) \qquad \forall\,\mathbf{v}\in\mathbf{V}, \\ A_{\mathrm{T}}(\theta_0,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\psi) &= -A_{\mathrm{T}}(\theta_1,\psi) \qquad \forall\,\psi\in\mathrm{W}^{1,4}_0(\Omega). \end{aligned} \tag{2.60}$$

Since each term in the first and second equation of (2.60) defines a continuous linear functional on $\mathbf{V}$ and $\mathrm{W}^{1,4}_0(\Omega)$, respectively, particularly from the second equation of (2.60), we deduce (2.1d) and (2.2c) in the sense of distributions, that is

$$-\kappa_{\mathrm{f}}\,\Delta\,\theta_{\mathrm{f}} + \mathbf{u}_{\mathrm{f}}\cdot\nabla\,\theta_{\mathrm{f}} = 0 \quad\text{and}\quad -\kappa_{\mathrm{m}}\,\Delta\,\theta_{\mathrm{m}} + \mathbf{u}_{\mathrm{m}}\cdot\nabla\,\theta_{\mathrm{m}} = 0.$$

Hence, recalling that $\mathbf{u}_{\mathrm{f}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{f}}}\in\mathrm{L}^2(\Omega_{\mathrm{f}})$ and $\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{m}}}\in\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})'$, we observe that the second equation of (2.60) implies the identity

$$A_{\mathrm{T}}(\theta_0,\psi) + (\mathbf{u}_{\mathrm{f}}\cdot\nabla\theta,\psi)_{\Omega_{\mathrm{f}}} + \langle\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta,\psi\rangle_{\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})',\mathrm{H}^1_{\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})} + O_{\mathrm{T}}(\mathbf{u};\theta_1,\psi) = -A_{\mathrm{T}}(\theta_1,\psi) \quad \forall\,\psi\in\mathrm{H}^1_0(\Omega).$$

Therefore, since $\mathbf{u}_{\mathrm{m}}\cdot\nabla\theta_0|_{\Omega_{\mathrm{m}}}$ also belongs to $\mathrm{L}^1(\Omega_{\mathrm{m}})$, from the latter we can recover the equation

$$A_{\mathrm{T}}(\theta_0,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta_0+\theta_1,\psi) = -A_{\mathrm{T}}(\theta_1,\psi) \quad \forall\,\psi\in\Psi_{\infty,0},$$

thus, $(\mathbf{u},\theta_0)$ satisfies (2.31), which concludes the proof. □

2.4 Uniqueness of solution

The uniqueness result for problem (2.8) is established in the following theorem.

Theorem 2.13 *Assume that the hypotheses of Theorems 2.6 and 2.12 hold and let* $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1)\in\mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be a solution to* (2.8)*, with* $\theta_1=\mathrm{E}_\delta(\theta_{\mathrm{D}})\in\mathrm{W}^{1,4}(\Omega)$ *and* $\theta_0\in\mathrm{H}^1_0(\Omega)$*. Assume further that* $\theta_0|_{\Omega_{\mathrm{m}}}\in\mathrm{L}^\infty(\Omega_{\mathrm{m}})$ *and that*

$$(C_1\,\gamma_{\mathbf{g}}+C_2)\,C_{\mathrm{lift},2}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} + C_3\|\theta_0\|_{0,\infty,\Omega_{\mathrm{m}}} + C_3\|\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}} + C_4\gamma_{\mathbf{g}} < 1, \tag{2.61}$$

with $\gamma_{\mathbf{g}} = \|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$ *and* C_1*,* C_2*,* C_3 *and* C_4 *the positive constants given in* (2.66)*. Then the solution of problem* (2.8) *is unique.*

Proof. Let $(\mathbf{u},(p,\lambda),\theta) = (\mathbf{u},(p,\lambda),\theta_0+\theta_1) \in \mathbf{H}\times \mathrm{Q}\times \mathrm{H}^1(\Omega)$ and $(\overline{\mathbf{u}},(\overline{p},\overline{\lambda}),\overline{\theta}) = (\overline{\mathbf{u}},(\overline{p},\overline{\lambda}),\overline{\theta}_0+\theta_1) \in \mathbf{H}\times \mathrm{Q}\times \mathrm{H}^1(\Omega)$ be two solutions of problem (2.8). It follows that $(\mathbf{u},\theta_0),(\overline{\mathbf{u}},\overline{\theta}_0) \in \mathbf{V}\times \mathrm{H}^1_0(\Omega)$ are solutions of (2.31), which implies

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}-\overline{\mathbf{u}},\mathbf{v}) + O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) - O_{\mathrm{F}}(\overline{\mathbf{u}}_{\mathrm{f}};\overline{\mathbf{u}}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) - D(\theta_0-\overline{\theta}_0,\mathbf{v}) &= 0 \quad \forall\, \mathbf{v}\in\mathbf{V},\\ A_{\mathrm{T}}(\theta_0-\overline{\theta}_0,\psi) + O_{\mathrm{T}}(\mathbf{u};\theta_0,\psi) - O_{\mathrm{T}}(\overline{\mathbf{u}};\overline{\theta}_0,\psi) + O_{\mathrm{T}}(\mathbf{u}-\overline{\mathbf{u}};\theta_1,\psi) &= 0 \quad \forall\, \psi\in\Psi_{\infty,0}. \end{aligned} \tag{2.62}$$

From the first equation of (2.62) we observe that by adding and subtracting $O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\overline{\mathbf{u}}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}})$, taking $\mathbf{v}=\mathbf{u}-\overline{\mathbf{u}}$, employing the coercivity of $A_{\mathrm{F}}(\cdot,\cdot)+O_{\mathrm{F}}(\mathbf{u};\cdot,\cdot)$ (cf. (2.16)) , and the continuity of O_{F} and D, we have

$$\alpha_{\mathrm{F}}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}} \le C_{O_{\mathrm{F}}}\|\mathbf{u}_{\mathrm{f}}-\overline{\mathbf{u}}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\,\|\overline{\mathbf{u}}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + C_D\,\gamma_{\mathbf{g}}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega}. \tag{2.63}$$

In turn, in the second equation of (2.62) we add and subtract $O_{\mathrm{T}}(\mathbf{u};\overline{\theta}_0,\psi)$, recall the fact that $\theta_0|_{\Omega_{\mathrm{m}}},\overline{\theta}_0|_{\Omega_{\mathrm{m}}} \in \mathrm{L}^\infty(\Omega_{\mathrm{m}})$ to define $\psi=\theta_0-\overline{\theta}_0\in\Psi_{\infty,0}$, and then employ the coercivity of $A_{\mathrm{T}}(\cdot,\cdot)+O_{\mathrm{T}}(\mathbf{u};\cdot,\cdot)$ (cf. (2.21)) and the continuity of O_{T} (cf. (2.19b)), to get

$$\alpha_{\mathrm{T}}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega} \le \widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}}\left(\|\overline{\theta}_0+\theta_1\|_{1,\Omega_{\mathrm{f}}} + \|\overline{\theta}_0+\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}\right). \tag{2.64}$$

Then, summing up (2.63) and (2.64), we arrive at

$$\begin{aligned} \|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}} + \|\theta_0-\overline{\theta}_0\|_{1,\Omega} \le{}& \alpha_{\mathrm{F}}^{-1}\left[C_{O_{\mathrm{F}}}\|\mathbf{u}_{\mathrm{f}}-\overline{\mathbf{u}}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\,\|\overline{\mathbf{u}}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + C_D\,\gamma_{\mathbf{g}}\,\|\theta_0-\overline{\theta}_0\|_{1,\Omega}\right]\\ &+\alpha_{\mathrm{T}}^{-1}\left[\widetilde{C}_{O_{\mathrm{T}}}\,\|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}}\left(\|\overline{\theta}_0+\theta_1\|_{1,\Omega_{\mathrm{f}}} + \|\overline{\theta}_0+\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}\right)\right], \end{aligned}$$

and then, using the fact that $\|\overline{\mathbf{u}}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \le \|\overline{\mathbf{u}}\|_{\mathbf{H}} \le C_{\mathbf{u}}\gamma_{\mathbf{g}}\|\theta_1\|_{1,\Omega}$ (cf. (2.35)) and $\|\overline{\theta}_0+\theta_1\|_{1,\Omega_{\mathrm{f}}} \le \|\overline{\theta}_0+\theta_1\|_{1,\Omega} \le (C_\theta+1)\|\theta_1\|_{1,\Omega}$ (cf. (2.36)), we obtain

$$\begin{aligned} \|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}} + \|\theta_0-\overline{\theta}_0\|_{1,\Omega} \le{}& \left((C_1\gamma_{\mathbf{g}}+C_2)\|\theta_1\|_{1,\Omega} + C_3\|\overline{\theta}_0\|_{0,\infty,\Omega_{\mathrm{m}}} + C_3\|\theta_1\|_{0,\infty,\Omega_{\mathrm{m}}}\right)\|\mathbf{u}-\overline{\mathbf{u}}\|_{\mathbf{H}}\\ &+C_4\gamma_{\mathbf{g}}\|\theta_0-\overline{\theta}_0\|_{1,\Omega}, \end{aligned} \tag{2.65}$$

where

$$C_1 := \alpha_{\mathrm{F}}^{-1}\,C_{O_{\mathrm{F}}}\,C_{\mathbf{u}}, \quad C_2 := \alpha_{\mathrm{T}}^{-1}\,\widetilde{C}_{O_{\mathrm{T}}}\,(C_\theta+1), \quad C_3 := \alpha_{\mathrm{T}}^{-1}\,\widetilde{C}_{O_{\mathrm{T}}}, \quad \text{and} \quad C_4 := \alpha_{\mathrm{F}}^{-1}\,C_D. \tag{2.66}$$

In this way, recalling that estimate (2.29b), implies

$$\|\theta_1\|_{1,\Omega} \le C_{\mathrm{lift},2}\,\delta^{-3/2}(1+\delta^2)^{1/2}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma},$$

from (2.65) and (2.61) we readily obtain that $\mathbf{u}=\overline{\mathbf{u}}$ and $\theta_0=\overline{\theta}_0$. Now, for the pressure and the Lagrange multiplier, from the inf-sup condition (2.17) we have that

$$\begin{aligned} \beta\,\|(p-\overline{p},\lambda-\overline{\lambda})\|_{\mathrm{Q}} &\le \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(p-\overline{p},\lambda-\overline{\lambda}))}{\|\mathbf{v}\|_{\mathbf{H}}}\\ &= \sup_{\mathbf{v}\in\mathbf{H}\setminus\{\mathbf{0}\}} \frac{-A_{\mathrm{F}}(\mathbf{u}-\overline{\mathbf{u}},\mathbf{v}) - O_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + O_{\mathrm{F}}(\overline{\mathbf{u}}_{\mathrm{f}};\overline{\mathbf{u}}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) + D(\theta_0-\overline{\theta}_0,\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}}}, \end{aligned}$$

which after simple computations implies that $\|(p-\overline{p},\lambda-\overline{\lambda})\|_{\mathrm{Q}} \le 0$, thus $p=\overline{p}$ and $\lambda=\overline{\lambda}$, which concludes the proof. □

3 Galerkin scheme

In this section we introduce and analyze a finite element scheme to approximate the solution of problem (2.8). We start by introducing the Galerkin scheme and reviewing the discrete stability properties of the forms involved. As we shall see next in the forthcoming sections, the analysis of the associated discrete scheme is analogous to the analysis of the finite-dimensional problem (2.47), employed to study the continuous problem (2.31).

3.1 Discrete problem

Let $\mathcal{T}_h^{\mathrm{f}}$ and $\mathcal{T}_h^{\mathrm{m}}$ be the respective triangulations of the domains Ω_{f} and Ω_{m} formed by shape-regular triangles of diameter h_T and denote by h_{f} and h_{m} their corresponding mesh sizes. Assume that they match on Σ so that $\mathcal{T}_h := \mathcal{T}_h^{\mathrm{f}} \cup \mathcal{T}_h^{\mathrm{m}}$ is a triangulation of $\Omega := \Omega_{\mathrm{f}} \cup \Sigma \cup \Omega_{\mathrm{m}}$. Hereafter $h := \max\{h_{\mathrm{f}}, h_{\mathrm{m}}\}$.

Given an integer $l \geq 0$, for each $T \in \mathcal{T}_h$, we let $\mathrm{P}_l(T)$ be the space of polynomials functions on T of degree equal or less than l. Moreover, for each $T \in \mathcal{T}_h^{\mathrm{f}}$, we denote by $\mathbf{BR}(T)$ the local Bernardi–Raugel space (see [7, 31]),

$$\mathbf{BR}(T) := [\mathrm{P}_1(T)]^2 \oplus \{\eta_2\, \eta_3\, \mathbf{n}_1, \eta_1\, \eta_3\, \mathbf{n}_2, \eta_1\, \eta_2\, \mathbf{n}_3\},$$

where $\{\eta_1, \eta_2, \eta_3\}$ are the baricentric coordinates of T, and $\{\mathbf{n}_1, \mathbf{n}_2, \mathbf{n}_3\}$ are the unit outward normals to the opposite sides of the corresponding vertices of T. In turn, for each $T \in \mathcal{T}_h^{\mathrm{m}}$ we consider the local Raviart–Thomas space of the lowest order (see [43])

$$\mathbf{RT}_0(T) := \mathrm{span}\{(1,0), (0,1), (x_1, x_2)\}.$$

Hence, we define the following finite element subspaces:

$$\begin{aligned}
&\mathbf{H}_h(\Omega_{\mathrm{f}}) := \{\mathbf{v}_{\mathrm{f}} \in \mathbf{H}^1(\Omega_{\mathrm{f}}) : \mathbf{v}_{\mathrm{f}}|_T \in \mathbf{BR}(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}}\}, && \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) := \mathbf{H}_h(\Omega_{\mathrm{f}}) \cap \mathbf{H}^1_{\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}),\\
&\mathbf{H}_h(\Omega_{\mathrm{m}}) := \{\mathbf{v}_{\mathrm{m}} \in \mathbf{H}(\mathrm{div};\Omega_{\mathrm{m}}) : \mathbf{v}_{\mathrm{m}}|_T \in \mathbf{RT}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{m}}\}, && \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) := \mathbf{H}_h(\Omega_{\mathrm{m}}) \cap \mathbf{H}_{\Gamma_{\mathrm{m}}}(\mathrm{div};\Omega_{\mathrm{m}}),\\
&\mathrm{L}_h(\Omega) := \{q \in \mathrm{L}^2(\Omega) : q|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h\}, && \mathrm{L}_{h,0}(\Omega) := \mathrm{L}_h(\Omega) \cap \mathrm{L}^2_0(\Omega),\\
&\Psi_h := \{\psi \in \mathrm{H}^1(\Omega) : \psi|_T \in \mathrm{P}_1(T) \quad \forall\, T \in \mathcal{T}_h\}, && \Psi_{h,0} := \Psi_h \cap \mathrm{H}^1_0(\Omega).
\end{aligned}$$

It remains to introduce the finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$. To do that we denote by Σ_h the partition of Σ inherited from $\mathcal{T}_h^{\mathrm{f}}$ (or $\mathcal{T}_h^{\mathrm{m}}$) and assume, without loss of generality, that the number of edges of Σ_h is even. Then, we let Σ_{2h} be the partition of Σ arising by joining pairs of adjacent edges of Σ_h. If the number of edges of Σ_h is odd, we simply reduce it to the even case by adding one node to the discretization of the interface and locally modify the triangulation to keep the mesh conformity and regularity. According to the above, we define the following finite element subspace for $\mathrm{H}^{1/2}(\Sigma)$

$$\Lambda_h(\Sigma) := \{\xi_h \in \mathrm{C}^0(\Sigma) : \xi_h|_e \in \mathrm{P}_1(e) \quad \forall\, e \in \Sigma_{2h}\}.$$

Now, we let $\mathrm{I}_h : \mathrm{C}^0(\overline{\Omega}) \to \Psi_h$ be the well-known Lagrange interpolation operator and recall that, under the assumption $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ and for a given $\delta > 0$, $\mathrm{E}_\delta(\theta_{\mathrm{D}})$ belongs to $\mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$ (cf. (1.3)). For a fixed $\delta > 0$ (to be specified below), we define the following approximation to θ_{D}:

$$\theta^\delta_{\mathrm{D},h} = \mathrm{I}_h(\mathrm{E}_\delta(\theta_{\mathrm{D}}))|_\Gamma \in \{\psi_{\mathrm{D},h} \in \mathrm{C}^0(\Gamma) : \psi_{\mathrm{D},h}|_e \in \mathrm{P}_1(e) \text{ for all } e \in \mathcal{E}_\Gamma\}, \tag{3.1}$$

where $\mathcal{E}_\Gamma$ stands for the set of edges on Γ.

Let us observe that since Ω is a polygonal domain, Ω_δ is also a polygon that can be discretized by shaped-regular triangles. According to this, for the forthcoming analysis we let $\mathcal{T}_h^\delta$ be a triangulation of Ω_δ and assume that $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$.

In this way, defining the global spaces

$$\mathbf{H}_h := \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) \times \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) \quad \text{and} \quad \mathrm{Q}_h := \mathrm{L}_{h,0}(\Omega) \times \Lambda_h(\Sigma),$$

the Galerkin scheme associated to (2.8) reads: Find $\mathbf{u}_h := (\mathbf{u}_{h,\mathrm{f}}, \mathbf{u}_{h,\mathrm{m}}) \in \mathbf{H}_h$, $(p_h, \lambda_h) \in \mathrm{Q}_h$ and $\theta_h \in \Psi_h$, such that $\theta_h|_\Gamma = \theta^\delta_{\mathrm{D},h}$, and

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}_h) + O^h_{\mathrm{F}}(\mathbf{u}_{h,\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{h,\mathrm{f}}) - B(\mathbf{v}_h, (p_h, \lambda_h)) - D(\theta_h, \mathbf{v}_h) &= 0 \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h, \\ B(\mathbf{u}_h, (q_h, \xi_h)) &= 0 \quad \forall\, (q_h, \xi_h) \in \mathrm{Q}_h, \\ A_{\mathrm{T}}(\theta_h, \psi_h) + O^h_{\mathrm{T}}(\mathbf{u}_h; \theta_h, \psi_h) &= 0 \quad \forall\, \psi_h \in \Psi_{h,0}, \end{aligned} \tag{3.2}$$

where A_{F}, B, D, and A_{T} are the form defined in Section 2.2, while O^h_{F} and O^h_{T} are the skew-symmetric convection forms (see [47]), defined by

$$O^h_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) := ((\mathbf{w}_{\mathrm{f}} \cdot \nabla)\mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} + \frac{1}{2}(\operatorname{div} \mathbf{w}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}}\, \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}},$$

for all $\mathbf{w}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, and

$$O^h_{\mathrm{T}}(\mathbf{w}; \theta, \psi) := (\mathbf{w}_{\mathrm{f}} \cdot \nabla\theta, \psi)_{\Omega_{\mathrm{f}}} + (\mathbf{w}_{\mathrm{m}} \cdot \nabla\theta, \psi)_{\Omega_{\mathrm{m}}} + \frac{1}{2}(\operatorname{div} \mathbf{w}_{\mathrm{f}}, \theta\, \psi)_{\Omega_{\mathrm{f}}},$$

for all $\mathbf{w} = (\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}) \in \mathbf{H}_h$ and for all $\theta, \psi \in \Psi_h$. The motivation for this choice is given later on in Remark 3.1.

3.2 Existence of solution of the discrete scheme

In what follows we prove that the discrete problem (3.2) has at least one solution under suitable assumptions on the data. We begin the discussion by establishing the stability properties of the forms involved restricted to the corresponding discrete spaces.

3.2.1 Discrete stability properties

We begin by observing that the forms A_{F}, B, D, and A_{T} are continuous with the same constants described in Section 2.3.1 (see (2.10)-(2.11)). In turn, by using estimate (1.2) with $p = 4$, it is easy to see that

$$|O^h_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}})| \le \widehat{C}_{O_{\mathrm{F}}} \|\mathbf{w}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \qquad \forall\, \mathbf{w}_{\mathrm{f}}, \mathbf{u}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}), \tag{3.3}$$

with $\widehat{C}_{O_{\mathrm{F}}} := C^2_{\mathrm{Sob},\Omega_{\mathrm{f}}}\left(1 + \frac{\sqrt{2}}{2}\right)$. Furthermore, we observe that integrating by parts, there holds

$$O^h_{\mathrm{F}}(\mathbf{w}_{\mathrm{f}}; \mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) = \frac{1}{2}\left\langle \mathbf{w}_{\mathrm{f}} \cdot \mathbf{n}, |\mathbf{v}_{\mathrm{f}}|^2 \right\rangle_\Sigma \qquad \forall\, \mathbf{w}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}).$$

Now, let $\mathbf{V}_h$ be the discrete kernel of B, that is

$$\mathbf{V}_h := \{\mathbf{v} \in \mathbf{H}_h : B(\mathbf{v}, (q, \xi)) = 0 \qquad \forall\, (q, \xi) \in \mathrm{Q}_h\}. \tag{3.4}$$

Similarly to the continuous case, $\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_h$ if and only if

$$(q, \operatorname{div} \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} + (q, \operatorname{div} \mathbf{v}_{\mathrm{m}})_{\Omega_{\mathrm{m}}} = 0 \quad \forall\, q \in \mathrm{L}_{h,0}(\Omega) \quad \text{and} \quad \langle \mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_\Sigma = 0 \quad \forall\, \xi \in \Lambda_h(\Sigma),$$

which imply that

$$(\operatorname{div} \mathbf{v}_{\mathrm{f}}, q)_{\Omega_{\mathrm{f}}} \;=\; 0 \quad \forall\, q \in \mathrm{L}_h(\Omega_{\mathrm{f}}) \quad \text{and} \quad \operatorname{div} \mathbf{v}_{\mathrm{m}} \;=\; 0 \quad \text{in} \quad \Omega_{\mathrm{m}},$$

where $\mathrm{L}_h(\Omega_{\mathrm{f}})$ is the set of functions of $\mathrm{L}_h(\Omega)$ restricted to Ω_{f}, more precisely,

$$\mathrm{L}_h(\Omega_{\mathrm{f}}) := \{q \in \mathrm{L}^2(\Omega_{\mathrm{f}}) : q|_T \in \mathrm{P}_0(T) \quad \forall\, T \in \mathcal{T}_h^{\mathrm{f}}\}.$$

According to the above, we observe that the discrete kernel (3.4) can be written as

$$\mathbf{V}_h := \{\mathbf{v} = (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_{h,\mathrm{f}} \times \mathbf{V}_{h,\mathrm{m}} : \langle \mathbf{v}_{\mathrm{f}} \cdot \mathbf{n} - \mathbf{v}_{\mathrm{m}} \cdot \mathbf{n}, \xi \rangle_\Sigma = 0 \quad \forall\, \xi \in \Lambda_h(\Sigma)\}, \tag{3.5}$$

where

$$\begin{aligned} \mathbf{V}_{h,\mathrm{f}} &:= \quad \{\mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}}) : (q, \operatorname{div} \mathbf{v}_{\mathrm{f}})_{\Omega_{\mathrm{f}}} = 0 \quad \forall\, q \in \mathrm{L}_h(\Omega_{\mathrm{f}})\}, \\ \mathbf{V}_{h,\mathrm{m}} &:= \quad \{\mathbf{v}_{\mathrm{m}} \in \mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}}) : \operatorname{div} \mathbf{v}_{\mathrm{m}} = 0 \quad \text{in} \quad \Omega_{\mathrm{m}}\}. \end{aligned}$$

Remark 3.1 *We observe here that if $\mathbf{v} := (\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{m}}) \in \mathbf{V}_h$, then $\mathbf{v}_{\mathrm{f}}$ is not necessarily divergence-free, which motivates the introduction of the convective forms O_{F}^h and O_{T}^h.*

On the other hand, our election of finite element spaces and the definition of the bilinear form B allow us to prove that there exist $h_0 > 0$ and $\widehat{\beta} > 0$, independent of h, such that for any $h_{\mathrm{m}} < h_0$, there holds:

$$\sup_{\mathbf{v} \in \mathbf{H}_h \setminus \{\mathbf{0}\}} \frac{B(\mathbf{v}, (q, \xi))}{\|\mathbf{v}\|_{\mathbf{H}}} \;\geq\; \widehat{\beta}\, \|(q, \xi)\|_{\mathrm{Q}} \quad \forall\, (q, \xi) \in \mathrm{Q}_h. \tag{3.6}$$

Its proof can be found in [20, Lemma 9], which is based on the proof of [26, Lemma 4.3].

Let us observe now that the forms $a_{\mathrm{F,f}}$, $a_{\mathrm{F,m}}$, and A_{T}, are elliptic with the same constants of the continuous case (see (2.14) and (2.20)), that is,

$$a_{\mathrm{F,f}}(\mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) \;\geq\; 2\mu\alpha_{\mathrm{f}}\, \|\mathbf{v}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}^2, \quad a_{\mathrm{F,m}}(\mathbf{v}_{\mathrm{m}}, \mathbf{v}_{\mathrm{m}}) \;\geq\; C_{\mathbf{K}}\, \|\mathbf{v}_{\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}}^2, \quad \text{and} \quad A_{\mathrm{T}}(\psi, \psi) \;\geq\; \alpha_{\mathrm{T}}\, \|\psi\|_{1,\Omega}^2, \tag{3.7}$$

for all $\mathbf{v}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, for all $\mathbf{v}_{\mathrm{m}} \in \mathbf{V}_{h,\mathrm{m}}$ and for all $\psi \in \Psi_{h,0}$, respectively. In particular, using the ellipticity of $a_{\mathrm{F,f}}$ and $a_{\mathrm{F,m}}$ one can deduce that the form $A_{\mathrm{F}}(\cdot,\cdot) + O_{\mathrm{F}}^h(\mathbf{w}_{\mathrm{f}};\cdot,\cdot)$, is elliptic on $\mathbf{V}_h$ for suitable $\mathbf{w}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$. More precisely, we have the following discrete version of Lemma 2.1. For its proof we refer the reader to [20, Lemma 10].

Lemma 3.1 *Let $\mathbf{w}_{\mathrm{f}} \in \mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})$, be such that*

$$\|\mathbf{w}_{\mathrm{f}} \cdot \mathbf{n}\|_{0,\Sigma} \;\leq\; \frac{2\mu\alpha_{\mathrm{f}}}{C_{\mathrm{tr}}^2\, C_{\mathrm{Sob},\Sigma}^2}.$$

There holds

$$A_{\mathrm{F}}(\mathbf{v}, \mathbf{v}) \;+\; O_{\mathrm{F}}^h(\mathbf{w}_{\mathrm{f}}; \mathbf{v}_{\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) \;\geq\; \alpha_{\mathrm{F}}\, \|\mathbf{v}\|_{\mathbf{H}}^2 \quad \forall\, \mathbf{v} \in \mathbf{V}_h.$$

We conclude this section by establishing some useful properties of O_{T}^h, similar to the ones provided in Lemma 2.2.

Lemma 3.2 *If $\mathbf{w} \in \mathbf{V}_h$ is given, then the following identities hold*

$$O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi) \,=\, -\,O_{\mathrm{T}}^h(\mathbf{w};\psi,\theta) \qquad \forall\,\theta,\psi \in \Psi_{h,0}, \tag{3.8a}$$

$$|O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi)| \,\leq\, \widehat{C}_{O_{\mathrm{T}}^1}\|\mathbf{w}\|_{\mathbf{H}}\|\psi\|_{1,\Omega}\,(\|\theta\|_{0,3,\Omega_{\mathrm{f}}} + \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\,\theta,\psi \in \Psi_{h,0}, \tag{3.8b}$$

$$|O_{\mathrm{T}}^h(\mathbf{w};\theta,\psi)| \,\leq\, \widehat{C}_{O_{\mathrm{T}}^2}\,\|\mathbf{w}\|_{\mathbf{H}}\,\|\psi\|_{1,\Omega}\,(\|\theta\|_{1,\Omega_{\mathrm{f}}} \,+\, \|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}) \qquad \forall\,\theta,\psi \in \Psi_{h,0}, \tag{3.8c}$$

where $\widehat{C}_{O_{\mathrm{T}}^1} := C_{\mathrm{Sob},\Omega_{\mathrm{f}}}\left(1+\frac{\sqrt{2}}{2}\right)$ *and* $\widehat{C}_{O_{\mathrm{T}}^2} := C_{\mathrm{Sob},\Omega_{\mathrm{f}}}^2\left(1+\frac{\sqrt{2}}{2}\right)$.

Proof. Given $\mathbf{w} \in \mathbf{V}_h$, we recall that $\operatorname{div}\mathbf{w}_{\mathrm{m}} = 0$ in Ω_{m}. Then, integrating by parts one can easily obtain (3.8a). In turn, using (3.8a) and proceeding analogously to the proof of estimate (2.19a), it is easy to deduce (3.8b). Finally, by combining (3.8b) and (1.2), we obtain (3.8c). □

Observe that, similarly to the continuous case, by combining (3.8a) and the third estimate in (3.7), for a given $\mathbf{w} \in \mathbf{V}_h$, it is possible to obtain

$$A_{\mathrm{T}}(\psi,\psi) \,+\, O_{\mathrm{T}}^h(\mathbf{w};\psi,\psi) \,\geq\, \alpha_{\mathrm{T}}\,\|\psi\|_{1,\Omega}^2 \qquad \forall\,\psi \in \Psi_{h,0}. \tag{3.9}$$

3.2.2 The discrete lifting

For the sake of the subsequent analysis, and analogously to the continuous case, given $\delta > 0$ now we introduce the discrete extension operator $\mathrm{E}_{\delta,h} : \mathrm{W}^{3/4,4}(\Gamma) \to \Psi_h$ given by $\mathrm{E}_{\delta,h} := \mathrm{I}_h\mathrm{E}_\delta$, where E_δ is the extension operator defined in (2.27) and I_h is the Lagrange interpolation operator. Then, it is clear from (3.1) that there holds

$$\theta_{\mathrm{D},h}^{\delta} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})|_{\Gamma}. \tag{3.10}$$

In what follows we derive some useful estimates for the operator $\mathrm{E}_{\delta,h}$ that will allow us to prove existence and stability of solution of problem (3.2). To that end we first recall that the Lagrange operator I_h satisfies the following approximation property (see, eg. [21, Theorem 1.103]): Given $p > 2$, $l \in \{0,1\}$, $0 \leq m \leq l+1$ and $T \in \mathcal{T}_h$, there holds

$$|\mathrm{I}_h(\psi) - \psi|_{m,p,T} \leq c h_T^{l+1-m}|\psi|_{l+1,p,T} \quad \forall\,\psi \in \mathrm{W}^{l+1,p}(T). \tag{3.11}$$

In addition, we recall the following inverse inequality (see [21, Lemma 1.138]):

$$\|\psi_h\|_{l,p,T} \leq C h_T^{m-l+2(\frac{1}{p}-\frac{1}{q})}\|\psi_h\|_{m,q,T} \qquad \forall\,\psi_h \in \Psi_h, \tag{3.12}$$

for any $T \in \mathcal{T}_h$, $l \geq 0$, $0 \leq m \leq l$ and $p,q \geq 1$.

Finally, for all $\psi \in \mathrm{W}^{1,4}(\Omega) \subseteq \mathrm{C}^0(\overline{\Omega})$, it is easy to see that

$$\|\mathrm{I}_h(\psi)\|_{0,\infty,T} \leq 3\|\psi\|_{0,\infty,T}, \quad \forall\,T \in \mathcal{T}_h. \tag{3.13}$$

Lemma 3.3 *The following estimates hold:*

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} \,\leq\, \widehat{C}_{\mathrm{lift},1}\,\delta^{1/12}(h\delta^{-1}+h+1)\,\|\zeta\|_{3/4,4,\Gamma}, \tag{3.14a}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{1,\Omega} \,\leq\, \widehat{C}_{\mathrm{lift},2}\,\delta^{1/4}(2+\delta^{-1})\,\|\zeta\|_{3/4,4,\Gamma}, \tag{3.14b}$$

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} \,\leq\, 3\|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta}, \tag{3.14c}$$

for all $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, where $\widehat{C}_{\mathrm{lift},1}$, $\widehat{C}_{\mathrm{lift},2} > 0$ are constants indepedent of h and δ.

Proof. Given $\zeta \in \mathrm{W}^{3/4,4}(\Gamma)$, we first use estimate (3.11) and recall that $\mathrm{E}_\delta = \beta_\delta \mathrm{E}$ (cf. (2.27)), to obtain

$$\begin{aligned}\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,3,\Omega} &= \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{0,3,\Omega} \le \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta)) - \mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \\ &\le ch|\mathrm{E}_\delta(\zeta)|_{1,3,\Omega} + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega} \\ &\le ch(\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} + \|\beta_\delta \nabla \mathrm{E}(\zeta)\|_{0,3,\Omega_\delta}) + \|\mathrm{E}_\delta(\zeta)\|_{0,3,\Omega}.\end{aligned} \tag{3.15}$$

Now, for the first term in the above inequality Hölder's inequality we have

$$\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|^3_{0,3,\Omega_\delta} = \int_{\Omega_\delta} |\nabla\beta_\delta|^3 |\mathrm{E}(\zeta)|^3 \le \||\nabla\beta_\delta|^3\|_{0,4,\Omega_\delta} \||\mathrm{E}(\zeta)|^3\|_{0,4/3,\Omega} = \|\nabla\beta_\delta\|^3_{0,12,\Omega_\delta}\|\mathrm{E}(\zeta)\|^3_{0,4,\Omega}.$$

Then, we recall that β_δ (cf. (2.25)) satisfies $|\nabla\beta_\delta| = \delta^{-1}$ a.e in $\{\mathbf{x} \in \Omega_\delta : \delta \le \mathrm{dist}(\mathbf{x}, \Gamma) \le 2\delta\}$ and $\nabla\beta_\delta$ vanishes elsewhere, to obtain

$$\|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} \le \delta^{-1}|\Omega_\delta|^{1/12}\|\mathrm{E}(\zeta)\|_{0,4,\Omega}.$$

Similarly, using again Hölder's inequality and the fact that $\beta_\delta \le 1$, we have

$$\|\beta_\delta \nabla \mathrm{E}(\zeta)\|^3_{0,3,\Omega_\delta} \le \||\beta_\delta|^3\|_{0,4,\Omega_\delta}\||\nabla\mathrm{E}(\zeta)|^3\|_{0,4/3,\Omega} \le |\Omega_\delta|^{1/4}\|\nabla\mathrm{E}(\zeta)\|^3_{0,4,\Omega},$$

which implies

$$\|\beta_\delta\nabla\mathrm{E}(\zeta)\|_{0,3,\Omega_\delta} \le |\Omega_\delta|^{1/12}|\mathrm{E}(\zeta)|_{1,4,\Omega}. \tag{3.16}$$

In this way, combining (3.15)–(3.16), applying (2.29a) and employing (2.24) and the fact that $|\Omega_\delta| \approx \delta$, we readily obtain (3.14a).

To derive (3.14b) we first recall that $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$, make use the fact that $\mathrm{E}_\delta(\zeta) = 0$ in $\Omega\backslash\Omega_\delta$ and employ estimate (3.12), to obtain

$$\begin{aligned}|\mathrm{E}_{\delta,h}(\zeta)|^2_{1,\Omega} = |\mathrm{I}_h(\mathrm{E}_\delta(\zeta))|^2_{1,\Omega_\delta} &= \sum_{T\in\mathcal{T}_h^\delta} |\mathrm{I}_h(\mathrm{E}_\delta(\zeta))|^2_{1,T} \le \sum_{T\in\mathcal{T}_h^\delta} h_T\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|^2_{1,4,T} \\ &\le \left(\sum_{T\in\mathcal{T}_h^\delta} h_T^2\right)^{1/2} \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|^2_{1,4,\Omega_\delta},\end{aligned}$$

which combined with the fact that $h_T^2 \approx |T|$ and estimate (3.11) with $p = 4$, implies

$$\begin{aligned}|\mathrm{E}_{\delta,h}(\zeta)|_{1,\Omega} \le\;& C|\Omega_\delta|^{1/4}\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{1,4,\Omega_\delta} \le C\delta^{1/4}(\|\mathrm{I}_h(\mathrm{E}_\delta(\zeta)) - \mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} + \|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta}) \\ \le\;& C_1\delta^{1/4}|\mathrm{E}_\delta(\zeta)|_{1,4,\Omega_\delta} + C_2\delta^{1/4}\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} \le \delta^{1/4}\hat{C}\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta}.\end{aligned} \tag{3.17}$$

Then, using again that $\beta_\delta \le 1$ in Ω_δ, $|\nabla\beta_\delta| = \delta^{-1}$ a.e in $\{\mathbf{x} \in \Omega_\delta : \delta \le \mathrm{dist}(\mathbf{x},\Gamma) \le 2\delta\}$ and $\nabla\beta_\delta$ vanishes elsewhere, there holds

$$\begin{aligned}\|\mathrm{E}_\delta(\zeta)\|_{1,4,\Omega_\delta} = \|\beta_\delta\mathrm{E}(\zeta)\|_{1,4,\Omega_\delta} &\le \|\beta_\delta\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} + \|\beta_\delta\nabla\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} + \|(\nabla\beta_\delta)\mathrm{E}(\zeta)\|_{0,4,\Omega_\delta} \\ &\le (2 + \delta^{-1})\|\mathrm{E}(\zeta)\|_{1,4,\Omega},\end{aligned}$$

which together with (3.17) and (2.24), imply (3.14b).

Finally, for (3.14c) we first notice that, since $\mathcal{T}_h^\delta \subseteq \mathcal{T}_h$ and $\mathrm{E}_\delta(\zeta) = 0$ in $\Omega\backslash\Omega_\delta$, then

$$\|\mathrm{E}_{\delta,h}(\zeta)\|_{0,\infty,\Omega} = \|\mathrm{I}_h(\mathrm{E}_\delta(\zeta))\|_{0,\infty,\Omega_\delta} = \|\mathrm{I}_h(\beta_\delta\mathrm{E}(\zeta))\|_{0,\infty,\Omega_\delta}.$$

Then, employing the second property of β_δ in (2.26) and using the fact that $\beta_\delta\mathrm{E}(\zeta)$ is continuous, from the identity above we readily obtain (3.14c).

□

3.2.3 Main result

Similarly to the continuous case, let us fix $\delta > 0$ (to be specified below in Theorem 3.5) and decompose the discrete temperature θ_h as $\theta_h = \theta_{h,0} + \theta_{h,1}$, with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D},h}) \in \Psi_h$ and $\theta_{h,0} = \theta_h - \theta_{h,1} \in \Psi_{h,0}$ and analogously to the analysis of the continuous problem we introduce the reduced version of problem (3.2): Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{u}_{h,\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) - D(\theta_{h,0}, \mathbf{v}) &= D(\theta_{h,1}, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}_h, \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) + O_{\mathrm{T}}^h(\mathbf{u}_h; \theta_{h,0} + \theta_{h,1}, \psi) &= -\, A_{\mathrm{T}}(\theta_{h,1}, \psi) && \forall\, \psi \in \Psi_{h,0}, \end{aligned} \tag{3.18}$$

where $\mathbf{V}_h$ is the discrete kernel of B defined in (3.5).

Using the discrete inf-sup condition (3.6) and analogously to the continuous case we readily obtain that both problems (3.2) and (3.18) are equivalent.

Lemma 3.4 *Assume that $h_{\mathrm{m}} < h_0$, with h_0 being the positive constant that allows us to derive the inf-sup condition (3.6). If $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ is a solution of (3.2), then $\mathbf{u}_h \in \mathbf{V}_h$ and $(\mathbf{u}_h, \theta_{h,0}) = (\mathbf{u}_h, \theta_h - \theta_{h,1})$ is a solution to (3.18). Conversely, if $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ is a solution of (3.18), then there exists $(p_h, \lambda_h) \in \mathrm{Q}_h$, such that $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = (\mathbf{u}_h, (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1})$ is a solution to (3.2).*

Notice that since (3.18) is a finite-dimensional problem, we can use the same strategy employed in Section 2.3.4 to analyze problem (2.45). To that end, let us now define the compact and convex set

$$\mathbf{X}_h := \{(\mathbf{w}, \phi) \in \mathbf{V}_h \times \Psi_{h,0} : \|\mathbf{w}\|_{\mathbf{H}} \,\le\, C_{\mathbf{u}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{1,\Omega}, \quad \|\phi\|_{1,\Omega} \,\le\, C_\theta\,\|\theta_{h,1}\|_{1,\Omega}\}\,, \tag{3.19}$$

with $C_{\mathbf{u}}$ and C_θ defined in (2.40) and (2.44), respectively, and the operator $\mathcal{J}_h : \mathbf{X}_h \to \mathbf{V}_h \times \Psi_{h,0}$ given by

$$\mathcal{J}_h(\mathbf{w}, \phi) = (\mathbf{u}_h, \theta_{h,0}) \qquad \forall\,(\mathbf{w}, \phi) \in \mathbf{X}_h,$$

where $(\mathbf{u}_h, \theta_{h,0})$ is the solution of the linearized version of problem (3.18): Find $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{V}_h \times \Psi_{h,0}$ such that

$$\begin{aligned} A_{\mathrm{F}}(\mathbf{u}_h, \mathbf{v}) + O_{\mathrm{F}}^h(\mathbf{w}_{\mathrm{f}}; \mathbf{u}_{h,\mathrm{f}}, \mathbf{v}_{\mathrm{f}}) &= D(\phi, \mathbf{v}) + D(\theta_{h,1}, \mathbf{v}) && \forall\, \mathbf{v} \in \mathbf{V}_h, \\ A_{\mathrm{T}}(\theta_{h,0}, \psi) + O_{\mathrm{T}}^h(\mathbf{w}; \theta_{h,0}, \psi) &= -\, A_{\mathrm{T}}(\theta_{h,1}, \psi) - O_{\mathrm{T}}^h(\mathbf{w}; \theta_{h,1}, \psi) && \forall\, \psi \in \Psi_{h,0}. \end{aligned}$$

Now we state the main result of this section.

Theorem 3.5 *Let h_0 be the positive constant that allows to derive the inf-sup condition (3.6), and assume that $h \le \min\{h_0, \delta\}$, with $\delta > 0$ be such that*

$$\frac{C_D \widehat{C}_{O_{\mathrm{T}}^1}}{\alpha_{\mathrm{F}}\alpha_{\mathrm{T}}}(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\widehat{C}_{\mathrm{lift},1}\,\delta^{1/12}(2+\delta)\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \le \frac{1}{4}. \tag{3.20}$$

Let $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ be such that

$$\frac{C_D\,\widehat{C}_{O_{\mathrm{T}}^1}}{\alpha_{\mathrm{F}}\,\alpha_{\mathrm{T}}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\|\theta_{h,1}\|_{0,\infty,\Omega} \,\le\, \frac{1}{4}, \tag{3.21}$$

and assume further that $\theta_{\mathrm{D}} \in \mathrm{W}^{3/4,4}(\Gamma)$ satisfies

$$C_{\mathbf{u}}\,(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\,\widehat{C}_{\mathrm{lift},2}\,\delta^{1/4}(2+\delta^{-1})\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \,\le\, \frac{2\mu\alpha_{\mathrm{f}}}{C_{\mathrm{tr}}^3\,C_{\mathrm{Sob},\Sigma}^2}. \tag{3.22}$$

Then, there exists at least one $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ *solution to* (3.2)*. Moreover, there exists* $C > 0$*, independent of* h*, such that*

$$\|\mathbf{u}_h\|_{\mathbf{H}} + \|(p_h, \lambda_h)\|_{\mathrm{Q}} + \|\theta_h\|_{1,\Omega} \le C \left(\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}} + \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \right). \tag{3.23}$$

Proof. First, let us observe that assumption $h \le \min\{h_0, \delta\}$ implies that estimate (3.14a) with $\zeta = \theta_{\mathrm{D}}$ becomes

$$\|\theta_{h,1}\|_{0,3,\Omega} = \|\mathrm{E}_{\delta,h}(\theta_{\mathrm{D}})\|_{0,3,\Omega} \le \widehat{C}_{\mathrm{lift},1}\, \delta^{1/12}(2+\delta)\, \|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}.$$

Then, for a given $(\mathbf{w}, \phi) = ((\mathbf{w}_{\mathrm{f}}, \mathbf{w}_{\mathrm{m}}), \phi) \in \mathbf{X}_h$, analogously to the proof of Lemma 2.9 we make use of assumptions (3.20), (3.21), (3.22) and the aforementioned inequality to deduce that there exists a unique $(\mathbf{u}, \theta) \in \mathbf{X}_h$, such that $\mathscr{J}_h(\mathbf{w}, \phi) = (\mathbf{u}, \theta)$, thus $\mathscr{J}_h$ is well-defined and satisfies $\mathscr{J}_h(\mathbf{X}_h) \subseteq \mathbf{X}_h$. In addition, using the same arguments employed in the proof of Lemma 2.10, we can deduce that $\mathscr{J}_h$ is continuous. According to the above, and analogously to the proof of Theorem 2.11, we employ Theorem 2.8 to obtain that there exists $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ such that (3.18) holds, which together with Lemma 3.4 implies that there exists $(p_h, \lambda_h) \in \mathrm{Q}_h$, such that $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = (\mathbf{u}_h, (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1})$ is a solution to (3.2).

Finally, using the fact that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ and proceeding analogously to the proof of Corollary 2.7 we easily deduce that $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h)$ satisfies (3.23), which concludes the proof. □

Remark 3.2 *Observe that, owing to* (3.14c) *and similarly to Remark 2.1, assumption* (3.21) *becomes*

$$\frac{3C_D\, \widehat{C}_{O^1_{\mathrm{T}}}}{\alpha_{\mathrm{F}}\, \alpha_{\mathrm{T}}} (\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}) \|\mathrm{E}(\zeta)\|_{0,\infty,\Omega_\delta} \le \frac{1}{4},$$

and again one could take an small enough δ *in such a way* $\|\mathrm{E}(\theta_{\mathrm{D}})\|_{0,\infty,\Omega_\delta} \approx \|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma}$*, and simply assume that*

$$\frac{3C_D\, \widehat{C}_{O^1_T}}{\alpha_{\mathrm{F}}\, \alpha_{\mathrm{T}}} (\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}} + \|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}})\, \|\theta_{\mathrm{D}}\|_{0,\infty,\Gamma} \le \frac{1}{4}.$$

4 Error analysis

In this section we address the error analysis and provide the theoretical rate of convergence for the Galerkin scheme (3.2). We begin with some notations and preliminary results.

Let us assume that the hypotheses of Theorem 2.13 hold and let $(\mathbf{u}, (p, \lambda), \theta) = (\mathbf{u}, (p, \lambda), \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ be a solution to (2.8), with $\theta_1 = \mathrm{E}_\delta(\theta_{\mathrm{D}}) \in \mathrm{W}^{1,4}(\Omega)$ and $\theta_0 \in \mathrm{H}^1_0(\Omega)$. In addition, we assume that the hypotheses of Theorem 3.5 and let $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = (\mathbf{u}_h, (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be a solution to (3.2), with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_{\mathrm{D}}) \in \Psi_h$ and $\theta_{h,0} \in \Psi_{h,0}$. Then to simplify the subsequent analysis, we write

$$\mathbf{e}_{\mathbf{u}_{\mathrm{f}}} = \mathbf{u}_{\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}, \qquad \mathbf{e}_{\mathbf{u}_{\mathrm{m}}} = \mathbf{u}_{\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}, \qquad \mathrm{e}_p = p - p_h, \qquad \mathrm{e}_\lambda = \lambda - \lambda_h, \qquad \mathrm{e}_\theta = \theta - \theta_h. \tag{4.1}$$

We decompose these errors as

$$\mathbf{e}_{\mathbf{u}_{\mathrm{f}}} = \boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{f}}} + \boldsymbol{\chi}_{\mathbf{u}_{\mathrm{f}}}, \quad \mathbf{e}_{\mathbf{u}_{\mathrm{m}}} = \boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{m}}} + \boldsymbol{\chi}_{\mathbf{u}_{\mathrm{m}}}, \quad \mathrm{e}_p = \varrho_p + \chi_p, \quad \mathrm{e}_\lambda = \varrho_\lambda + \chi_\lambda, \quad \mathrm{e}_\theta = \varrho_\theta + \chi_\theta, \tag{4.2}$$

with

$$\begin{aligned} &\boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{f}}} = \mathbf{u}_{\mathrm{f}} - \widehat{\mathbf{v}}_{h,\mathrm{f}}, \quad \boldsymbol{\varrho}_{\mathbf{u}_{\mathrm{m}}} = \mathbf{u}_{\mathrm{m}} - \widehat{\mathbf{v}}_{h,\mathrm{m}}, \quad \varrho_p = p - \widehat{q}_h, \quad \varrho_\lambda = \lambda - \widehat{\xi}_h, \quad \varrho_\theta = \theta - \widehat{\psi}_h, \\ &\boldsymbol{\chi}_{\mathbf{u}_{\mathrm{f}}} = \widehat{\mathbf{v}}_{h,\mathrm{f}} - \mathbf{u}_{h,\mathrm{f}}, \quad \boldsymbol{\chi}_{\mathbf{u}_{\mathrm{m}}} = \widehat{\mathbf{v}}_{h,\mathrm{m}} - \mathbf{u}_{h,\mathrm{m}}, \quad \chi_p = \widehat{q}_h - p_h, \quad \chi_\lambda = \widehat{\xi}_h - \lambda_h, \quad \chi_\theta = \widehat{\psi}_h - \theta_h, \end{aligned} \tag{4.3}$$

where $\widehat{\mathbf{v}}_h = (\widehat{\mathbf{v}}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{h,\mathrm{m}}) \in \mathbf{V}_h$, $(\widehat{q}_h, \widehat{\xi}_h) \in \mathrm{Q}_h$, and $\widehat{\psi}_h \in \Psi_h^\Gamma$. Here, Ψ_h^Γ is the set of functions in Ψ_h that coincide with $\theta^\delta_{\mathrm{D},h}$ (cf. (3.10)) on Γ, that is

$$\Psi_h^\Gamma := \{\psi_h \in \Psi_h \,:\, \psi_h|_\Gamma = \theta^\delta_{\mathrm{D},h}\}.$$

Finally, we let

$$\mathbf{e}_{\mathbf{u}} = (\mathbf{e}_{\mathbf{u}_\mathrm{f}}, \mathbf{e}_{\mathbf{u}_\mathrm{m}}), \quad \boldsymbol{\varrho}_{\mathbf{u}} = (\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}}, \boldsymbol{\varrho}_{\mathbf{u}_\mathrm{m}}) \quad \text{and} \quad \boldsymbol{\chi}_{\mathbf{u}} = (\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}}, \boldsymbol{\chi}_{\mathbf{u}_\mathrm{m}}).$$

Let us recall that, since the inf-sup condition (3.6) holds, it is possible to prove that there exists $c > 0$, independent of h, such that (see for instance [25, Theorem 2.6])

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}} \,\leq\, c \inf_{\mathbf{v}_h \in \mathbf{H}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{H}}. \tag{4.4}$$

Now we provide some useful properties of the convective terms that will allow us to derive the desired error estimates.

Lemma 4.1 *Let $\mathbf{u} := (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V}$ and $\theta \in \mathrm{H}^1(\Omega)$. The following identities hold*

$$O^h_\mathrm{F}(\mathbf{u}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \,=\, O_\mathrm{F}(\mathbf{u}_\mathrm{f}; \mathbf{u}_\mathrm{f}, \mathbf{v}_\mathrm{f}) \quad \text{and} \quad O^h_\mathrm{T}(\mathbf{u}; \theta, \psi) \,=\, O_\mathrm{T}(\mathbf{u}; \theta, \psi), \tag{4.5}$$

for all $\mathbf{v}_\mathrm{f} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$ and for all $\psi \in \Psi_h$, respectively. In addition, for any $\mathbf{u}_h \in \mathbf{H}_h$ and $\theta \in \mathrm{H}^1(\Omega)$, such that $\theta|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$, there holds

$$|O^h_\mathrm{T}(\mathbf{u}_h; \theta, \psi)| \,\leq\, \widehat{C}_{O^3_\mathrm{T}} \|\mathbf{u}_h\|_{\mathbf{H}} \left(\|\theta\|_{1,\Omega_\mathrm{f}} \,+\, \|\theta\|_{1,4,\Omega_\mathrm{m}}\right) \|\psi\|_{1,\Omega} \quad \forall\, \psi \in \Psi_{h,0}, \tag{4.6}$$

where $\widehat{C}_{O^3_\mathrm{T}}$ is a positive constant independent of h.

Proof. Given $\mathbf{u} := (\mathbf{u}_\mathrm{f}, \mathbf{u}_\mathrm{m}) \in \mathbf{V}$ and $\theta \in \mathrm{H}^1(\Omega)$, it is clear that $\operatorname{div} \mathbf{u}_\star = 0$ in $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$, which readily implies (4.5). On the other hand, given $\mathbf{u}_h \in \mathbf{H}_h$ and employing the Hölder inequality (1.1), we have that for all $\psi \in \Psi_{h,0}$, there holds

$$\begin{aligned} |O^h_\mathrm{T}(\mathbf{u}_h; \theta, \psi)| &\leq |(\mathbf{u}_{h,\mathrm{f}} \cdot \nabla\theta, \psi)_{\Omega_\mathrm{f}}| \,+\, |(\mathbf{u}_{h,\mathrm{m}} \cdot \nabla\theta, \psi)_{\Omega_\mathrm{m}}| + \frac{1}{2}|(\operatorname{div}\mathbf{u}_h, \theta\psi)_{\Omega_\mathrm{f}}| \\ &\leq \|\mathbf{u}_{h,\mathrm{f}}\|_{0,4,\Omega_\mathrm{f}} \, |\theta|_{1,\Omega_\mathrm{f}} \, \|\psi\|_{0,4,\Omega_\mathrm{f}} \,+\, \|\mathbf{u}_\mathrm{m}\|_{\mathrm{div};\Omega_\mathrm{m}} \, |\theta|_{1,4,\Omega_\mathrm{m}} \, \|\psi\|_{0,4,\Omega_\mathrm{m}} \\ &\quad + \frac{1}{2}\|\operatorname{div} \mathbf{u}_{h,\mathrm{f}}\|_{0,\Omega_\mathrm{f}} \|\theta\|_{0,4,\Omega_\mathrm{f}} \|\psi\|_{0,4,\Omega_\mathrm{f}}. \end{aligned}$$

Then, applying the Sobolev inequality (1.2) to $\|\mathbf{u}_\mathrm{f}\|_{0,4,\Omega_\mathrm{f}}$, $\|\theta\|_{0,4,\Omega_\mathrm{f}}$, $\|\psi\|_{0,4,\Omega_\mathrm{f}}$ and $\|\psi\|_{0,4,\Omega_\mathrm{m}}$, we easily deduce (4.6). $\square$

The following preliminary estimate is an intermediate step to obtain the desired convergence result.

Lemma 4.2 *Let us assume that the hypotheses of Theorem 2.13 hold and let $(\mathbf{u}, (p, \lambda), \theta) = (\mathbf{u}, (p, \lambda), \theta_0 + \theta_1) \in \mathbf{H} \times \mathrm{Q} \times \mathrm{H}^1(\Omega)$ be a solution to (2.8), with $\theta_1 = \mathrm{E}_\delta(\theta_\mathrm{D}) \in \mathrm{W}^{1,4}(\Omega)$ and $\theta_0 \in \mathrm{H}^1_0(\Omega)$. In addition, we assume that the hypotheses of Theorem 3.5 and let $(\mathbf{u}_h, (p_h, \lambda_h), \theta_h) = (\mathbf{u}_h, (p_h, \lambda_h), \theta_{h,0} + \theta_{h,1}) \in \mathbf{H}_h \times \mathrm{Q}_h \times \Psi_h$ be a solution to (3.2), with $\theta_{h,1} = \mathrm{E}_{\delta,h}(\theta_\mathrm{D}) \in \Psi_h$ and $\theta_{h,0} \in \Psi_{h,0}$. Finally, let $\gamma_{\mathbf{g}} = \|\mathbf{g}_\mathrm{f}\|_{0,\Omega_\mathrm{f}} + \|\mathbf{g}_\mathrm{m}\|_{0,3,\Omega_\mathrm{m}}$ and assume that $\theta|_{\Omega_\mathrm{m}} \in \mathrm{W}^{1,4}(\Omega_\mathrm{m})$ holds. There hold*

$$\alpha_\mathrm{F} \, \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \,\leq\, \widehat{C}_{O_\mathrm{F}} \, \|\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \, \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \,+\, C_D \, \gamma_{\mathbf{g}} \, \|\chi_\theta\|_{1,\Omega} \,+\, L_1, \tag{4.7}$$

$$\alpha_\mathrm{T} \, \|\chi_\theta\|_{1,\Omega} \,\leq\, \widehat{C}_{O^2_\mathrm{T}} \, \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \, \left(\|\theta\|_{1,\Omega_\mathrm{f}} + \|\theta\|_{0,\infty,\Omega_\mathrm{m}}\right) \,+\, L_2, \tag{4.8}$$

and

$$\widehat{\beta}\,\|(\chi_p,\chi_\lambda)\|_{\mathrm{Q}} \,\leq\, C_{A_\mathrm{F}}\|\mathbf{e_u}\|_{\mathbf{H}} \,+\, \widehat{C}_{O_\mathrm{F}}\|\mathbf{e}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}}\|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}} \,+\, \widehat{C}_{O_\mathrm{F}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}}\|\mathbf{e}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \\ +\,C_D\gamma_{\mathbf{g}}\|\mathrm{e}_\theta\|_{1,\Omega} \,+\, \|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}, \tag{4.9}$$

where L_1 *and* L_2 *are defined later in* (4.11) *and* (4.12) *which depend on the solutions* $\mathbf{u}$*,* $\mathbf{u}_h$ *and* θ*.*

Proof. Using the definition of the errors given in (4.1), employing (4.5) and subtracting (2.8) and (3.2), we readily obtain

$$\begin{aligned} A_\mathrm{F}(\mathbf{e_u},\mathbf{v}) \,+\, \big[O_\mathrm{F}^h(\mathbf{u}_\mathrm{f};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) \,-\, O_\mathrm{F}^h(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_\mathrm{f})\big] \,-\, D(\mathrm{e}_\theta,\mathbf{v}) + B(\mathbf{v},(\varrho_p,\varrho_\lambda)) &= 0,\\ B(\mathbf{e_u},(q,\xi)) &= 0,\\ A_\mathrm{T}(\mathrm{e}_\theta,\psi) \,+\, \big[O_\mathrm{T}^h(\mathbf{u};\theta,\psi) \,-\, O_\mathrm{T}^h(\mathbf{u}_h;\theta_h,\psi)\big] &= 0, \end{aligned} \tag{4.10}$$

for all $\mathbf{v}\in\mathbf{V}_h$, $(q,\xi)\in\mathrm{Q}_h$ and $\psi\in\Psi_{h,0}$. Then, adding and subtracting suitable terms and employing the decompositions (4.2) and (4.3), the first equation in (4.10) can be rewritten as

$$A_\mathrm{F}(\boldsymbol{\chi}_\mathbf{u},\mathbf{v}) \,+\, O_\mathrm{F}^h(\mathbf{u}_{h,\mathrm{f}};\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}},\mathbf{v}_\mathrm{f}) \,=\, -\,O_\mathrm{F}^h(\boldsymbol{\chi}_{\mathbf{u}_\mathrm{f}};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) \,+\, D(\chi_\theta,\mathbf{v}) \,-\, A_\mathrm{F}(\boldsymbol{\varrho}_\mathbf{u},\mathbf{v}) \,-\, O_\mathrm{F}^h(\mathbf{u}_{h,\mathrm{f}};\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}},\mathbf{v}_\mathrm{f}) \\ -\,O_\mathrm{F}^h(\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}};\mathbf{u}_\mathrm{f},\mathbf{v}_\mathrm{f}) \,+\, D(\varrho_\theta,\mathbf{v}) - B(\mathbf{v},(\varrho_p,\varrho_\lambda)),$$

for all $\mathbf{v}\in\mathbf{V}_h$. In particular, for $\mathbf{v}=\boldsymbol{\chi}_\mathbf{u}$, employing the ellipticity of the bilinear form $A_\mathrm{F}(\cdot,\cdot)+O_\mathrm{F}^h(\mathbf{u}_{h,\mathrm{f}};\cdot,\cdot)$, and the continuity of A_F, O_F^h, and D (cf. (2.10), (3.3) and (2.12), respectively), we obtain (4.7) where

$$L_1 \,:=\, C_{A_\mathrm{F}}\,\|\boldsymbol{\varrho}_\mathbf{u}\|_{\mathbf{H}} \,+\, \widehat{C}_{O_\mathrm{F}}\,\left(\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \,+\, \|\mathbf{u}_\mathrm{f}\|_{1,\Omega_\mathrm{f}}\right)\,\|\boldsymbol{\varrho}_{\mathbf{u}_\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \\ +\,C_D\,\gamma_{\mathbf{g}}\,\|\varrho_\theta\|_{1,\Omega} \,+\, C_B\|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}. \tag{4.11}$$

On the other hand, from the third equation of (4.10), after a simple computations it can be obtained the identity

$$A_\mathrm{T}(\chi_\theta,\psi) \,+\, O_\mathrm{T}^h(\mathbf{u}_h;\chi_\theta,\psi) \,=\, -\,A_\mathrm{T}(\varrho_\theta,\psi) \,-\, O_\mathrm{T}^h(\boldsymbol{\varrho}_\mathbf{u};\theta,\psi) \\ -\,O_\mathrm{T}^h(\boldsymbol{\chi}_\mathbf{u};\theta,\psi) \,-\, O_\mathrm{T}^h(\mathbf{u}_h;\varrho_\theta,\psi),$$

for all $\psi\in\Psi_{h,0}$. Then, noticing that $\chi_\theta\in\Psi_{h,0}$, we take $\psi=\chi_\theta$ in the latter identity, employ the ellipticity of the bilinear form $A_\mathrm{T}(\cdot,\cdot)+O_\mathrm{T}^h(\mathbf{u}_h;\cdot,\cdot)$ (cf. (3.9)), the continuity of A_T (cf. (2.11)), and estimates (4.6) and (3.8c), to obtain (4.8) with

$$L_2 \,:=\, C_{A_\mathrm{T}}\,\|\varrho_\theta\|_{1,\Omega} + \widehat{C}_{O_\mathrm{T}^2}\|\boldsymbol{\varrho}_\mathbf{u}\|_{\mathbf{H}}\,(\|\theta\|_{1,\Omega_\mathrm{f}} + \|\theta\|_{0,\infty,\Omega_\mathrm{m}}) + \widehat{C}_{O_\mathrm{T}^3}\|\mathbf{u}_h\|_{\mathbf{H}}\,(\|\varrho_\theta\|_{1,\Omega_\mathrm{f}} \,+\, \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}})\,. \tag{4.12}$$

Now, to estimate χ_p and χ_λ we observe that from the discrete inf-sup condition (3.6), there holds

$$\begin{aligned} \widehat{\beta}\,\|(\chi_p,\chi_\lambda)\|_{\mathrm{Q}} \;&\leq \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\chi_p,\chi_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}}\\ &\leq \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \,+\, \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},-(\varrho_p,\varrho_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}}\\ &\leq \sup_{\mathbf{v}\in\mathbf{H}_h\setminus\{\mathbf{0}\}} \frac{B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda))}{\|\mathbf{v}\|_{\mathbf{H}}} \,+\, \|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}. \end{aligned} \tag{4.13}$$

In turn, from the first equations of (4.10), adding and subtracting suitable terms, we obtain

$$\begin{aligned} B(\mathbf{v},(\mathrm{e}_p,\mathrm{e}_\lambda)) &= A_{\mathrm{F}}(\mathbf{e}_{\mathbf{u}},\mathbf{v}) \,+\, O^h_{\mathrm{F}}(\mathbf{u}_{\mathrm{f}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,-\, O^h_{\mathrm{F}}(\mathbf{u}_{h,\mathrm{f}};\mathbf{u}_{h,\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,-\, D(\mathrm{e}_\theta,\mathbf{v}) \\ &= A_{\mathrm{F}}(\mathbf{e}_{\mathbf{u}},\mathbf{v}) \,+\, O^h_{\mathrm{F}}(\mathbf{e}_{\mathbf{u}_{\mathrm{f}}};\mathbf{u}_{\mathrm{f}},\mathbf{v}_{\mathrm{f}}) \,+\, O^h_{\mathrm{F}}(\mathbf{u}_{h,\mathrm{f}};\mathbf{e}_{\mathbf{u}_{\mathrm{f}}},\mathbf{v}_{\mathrm{f}}) \,-\, D(\mathrm{e}_\theta,\mathbf{v}), \end{aligned}$$

for all $\mathbf{v}\in\mathbf{H}_h$. Then, utilizing the last identity, from (4.13) and the continuity of the forms involved, we obtain

$$\begin{aligned} \widehat{\beta}\,\|(\chi_p,\chi_\lambda)\|_{\mathrm{Q}} \,\le\, C_{A_{\mathrm{F}}}\|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}} \,+\, \widehat{C}_{O_{\mathrm{F}}}\|\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} \,+\, \widehat{C}_{O_{\mathrm{F}}}\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}}\|\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}\|_{1,\Omega_{\mathrm{f}}} \\ +\, C_D\gamma_{\mathbf{g}}\|\mathrm{e}_\theta\|_{1,\Omega} \,+\, \|(\varrho_p,\varrho_\lambda)\|_{\mathrm{Q}}. \end{aligned}$$

□

Theorem 4.3 *Let us assume that the hypotheses of Lemma 4.2 hold and let* $(\mathbf{u},(p,\lambda),\theta)=((\mathbf{u}_{\mathrm{f}},\mathbf{u}_{\mathrm{m}}),(p,\lambda),\theta_0+\theta_1)\in\mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be the unique solution of* (2.8) *and* $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h)=((\mathbf{u}_{h,\mathrm{f}},\mathbf{u}_{h,\mathrm{m}}),(p_h,\lambda_h),\theta_{h,0}+\theta_{h,1})\in\mathbf{H}_h\times\mathrm{Q}_h\times\Psi_h$ *be a solution to* (3.2). *Finally, let* $\gamma_{\mathbf{g}}=\|\mathbf{g}_{\mathrm{f}}\|_{0,\Omega_{\mathrm{f}}}+\|\mathbf{g}_{\mathrm{m}}\|_{0,3,\Omega_{\mathrm{m}}}$ *and assume further that*

$$C_1\gamma_{\mathbf{g}}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \,+\, C_2\gamma_{\mathbf{g}}\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}} \le \frac{1}{2}, \tag{4.14}$$

with $C_1\,,C_2>0$, *independent of* h, δ, $\gamma_{\mathbf{g}}$ *and* θ_{D}. *There exists* $C>0$, *independent of the aforementioned datum, such that*

$$\begin{aligned} \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{H}}+\|(\mathrm{e}_p,\mathrm{e}_\lambda)\|_{\mathrm{Q}}+\|\mathrm{e}_\theta\|_{1,\Omega} \le C\Bigg\{ \inf_{\mathbf{v}_{h,\mathrm{f}}\in\mathbf{H}_{h,\Gamma_{\mathrm{f}}}(\Omega_{\mathrm{f}})}\|\mathbf{u}_{\mathrm{f}}-\mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} + \inf_{\mathbf{v}_{h,\mathrm{m}}\in\mathbf{H}_{h,\Gamma_{\mathrm{m}}}(\Omega_{\mathrm{m}})}\|\mathbf{u}_{\mathrm{m}}-\mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_{\mathrm{m}}} \\ + \inf_{q_h\in\mathrm{L}_{h,0}(\Omega)}\|p-q_h\|_{0,\Omega} + \inf_{\xi_h\in\Lambda_h(\Sigma)}\|\lambda-\xi_h\|_{1/2,\Sigma} + \inf_{\psi_h\in\Psi_h^\Gamma}\big(\|\theta-\psi_h\|_{1,\Omega_{\mathrm{f}}}+\|\theta-\psi_h\|_{1,4,\Omega_{\mathrm{m}}}\big)\Bigg\}. \end{aligned} \tag{4.15}$$

Proof. Let us first recall that from (2.35), (2.36) and (2.29b) with $\zeta=\theta_{\mathrm{D}}$, the following estimates hold

$$\begin{aligned} \|\mathbf{u}_{\mathrm{f}}\|_{1,\Omega_{\mathrm{f}}} &\le C_{\mathbf{u}}\gamma_{\mathbf{g}}C_{\mathrm{lift},2}\,\delta^{-3/2}(1+\delta^2)^{1/2}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}, \\ \|\theta\|_{1,\Omega_{\mathrm{f}}} &\le (C_\theta+1)C_{\mathrm{lift},2}\,\delta^{-3/2}(1+\delta^2)^{1/2}\,\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma}. \end{aligned} \tag{4.16}$$

Then, combining (4.7), (4.8), and (4.16), we obtain

$$\begin{aligned} \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \le \Big(C_1\gamma_{\mathbf{g}}\delta^{-3/2}(1+\delta^2)^{1/2}\|\theta_{\mathrm{D}}\|_{3/4,4,\Gamma} \,+\, C_2\gamma_{\mathbf{g}}\|\theta\|_{0,\infty,\Omega_{\mathrm{m}}}\Big)\,\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \\ +\,\alpha_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}C_D\gamma_{\mathbf{g}}L_2 \,+\, \alpha_{\mathrm{F}}^{-1}L_1, \end{aligned}$$

with

$$C_1 := \alpha_{\mathrm{F}}^{-1}C_{\mathrm{lift},2}\Big(\widehat{C}_{O_{\mathrm{F}}}C_{\mathbf{u}}+\alpha_{\mathrm{T}}^{-1}C_D\widehat{C}_{O_{\mathrm{T}}^2}(C_\theta+1)\Big) \quad\text{and}\quad C_2 := \alpha_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}C_D\widehat{C}_{O_{\mathrm{T}}^2},$$

which together with (4.14), implies that

$$\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{H}} \le\, 2\alpha_{\mathrm{F}}^{-1}\alpha_{\mathrm{T}}^{-1}C_D\gamma_{\mathbf{g}}L_2(\boldsymbol{\varrho}_u,\varrho_\theta) \,+\, 2\alpha_{\mathrm{F}}^{-1}L_1. \tag{4.17}$$

Now, to estimate L_1 and L_2 we recall that $(\mathbf{u}_h, \theta_{h,0}) \in \mathbf{X}_h$ (cf. (3.19)) which yields $\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \leq \|\mathbf{u}_h\|_\mathbf{H} \leq C_\mathbf{u}\,\gamma_\mathbf{g}\|\theta_{h,1}\|_{1,\Omega}$, which combined with (3.14b), implies

$$\|\mathbf{u}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \leq \|\mathbf{u}_h\|_\mathbf{H} \leq C_\mathbf{u}\gamma_\mathbf{g}\widehat{C}_{\mathrm{lift},2}\,\delta^{1/4}(2+\delta^{-1})\,\|\theta_\mathrm{D}\|_{3/4,4,\Gamma}. \tag{4.18}$$

Then, from the latter, the definition of L_1 and L_2, we get

$$L_1 \leq c_1\left(\|\boldsymbol{\varrho}_\mathbf{u}\|_\mathbf{H} + \|\varrho_\theta\|_{1,\Omega} + \|(\varrho_p,\varrho_\lambda)\|_\mathrm{Q}\right), \tag{4.19}$$

$$L_2 \leq c_2\left(\|\varrho_\theta\|_{1,\Omega} + \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}} + \|\boldsymbol{\varrho}_\mathbf{u}\|_\mathbf{H}\right), \tag{4.20}$$

with c_1, c_2 being positive constants independent of h. In this way, using (4.17), (4.19), (4.20) and the fact that $\mathrm{W}^{1,4}(\Omega_\mathrm{m})$ is continuously embedded in $\mathrm{H}^1(\Omega_\mathrm{m})$, which implies $\|\varrho_\theta\|_{1,\Omega} \leq c(\|\varrho_\theta\|_{1,\Omega_\mathrm{f}} + \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}})$, we readily obtain

$$\|\boldsymbol{\chi}_\mathbf{u}\|_\mathbf{H} \leq c_3\left(\|\boldsymbol{\varrho}_\mathbf{u}\|_\mathbf{H} + \|\varrho_\theta\|_{1,\Omega_\mathrm{f}} + \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}}\right) + c_4\|(\varrho_p,\varrho_\lambda)\|_\mathrm{Q}, \tag{4.21}$$

with c_3, $c_4 > 0$, independent of h.

Now, to estimate $\|\chi_\theta\|_{1,\Omega}$ we simply substitute (4.21) in (4.8) and employ (4.14), the second estimate in (4.16) and (4.20), to obtain

$$\|\chi_\theta\|_{1,\Omega} \leq c_5\left(\|\boldsymbol{\varrho}_\mathbf{u}\|_\mathbf{H} + \|\varrho_\theta\|_{1,\Omega_\mathrm{f}} + \|\varrho_\theta\|_{1,4,\Omega_\mathrm{m}} + \|(\varrho_p,\varrho_\lambda)\|_\mathrm{Q}\right), \tag{4.22}$$

with $c_5 > 0$, independent of h.

According to the above, from (4.2), (4.21), (4.22), the triangle inequality, the fact that $\widehat{\mathbf{v}}_h = (\widehat{\mathbf{v}}_{h,\mathrm{f}}, \widehat{\mathbf{v}}_{h,\mathrm{m}}) \in \mathbf{V}_h$, $(\widehat{q}_h, \widehat{\xi}_h) \in \mathrm{Q}_h$, and $\widehat{\psi}_h \in \Psi_h^\Gamma$ are arbitrary and (4.4), we obtain

$$\begin{aligned}\|\mathbf{e}_\mathbf{u}\|_\mathbf{H} + \|\mathrm{e}_\theta\|_{1,\Omega} \leq c_6\Bigg\{ &\inf_{\mathbf{v}_{h,\mathrm{f}}\in\mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})}\|\mathbf{u}_\mathrm{f}-\mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} + \inf_{\mathbf{v}_{h,\mathrm{m}}\in\mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m})}\|\mathbf{u}_\mathrm{m}-\mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}} \\ &+ \inf_{q_h\in\mathrm{L}_{h,0}(\Omega)}\|p-q_h\|_{0,\Omega} + \inf_{\xi_h\in\Lambda_h(\Sigma)}\|\lambda-\xi_h\|_{1/2,\Sigma} + \inf_{\psi_h\in\Psi_h^\Gamma}\left(\|\theta-\psi_h\|_{1,\Omega_\mathrm{f}} + \|\theta-\psi_h\|_{1,4,\Omega_\mathrm{m}}\right)\Bigg\},\end{aligned} \tag{4.23}$$

with $c_6 > 0$, independent of h.

Finally, and similarly to the above, from (4.9), the first inequality in (4.16), (4.18), (4.23) and the triangle inequality, we deduce that there exists $c_7 > 0$, independent of h, such that

$$\begin{aligned}\|(\mathrm{e}_p,\mathrm{e}_\lambda)\|_\mathrm{Q} \leq c_7\Bigg\{ &\inf_{\mathbf{v}_{h,\mathrm{f}}\in\mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})}\|\mathbf{u}_\mathrm{f}-\mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} + \inf_{\mathbf{v}_{h,\mathrm{m}}\in\mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m})}\|\mathbf{u}_\mathrm{m}-\mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}} \\ &+ \inf_{q_h\in\mathrm{L}_{h,0}(\Omega)}\|p-q_h\|_{0,\Omega} + \inf_{\xi_h\in\Lambda_h(\Sigma)}\|\lambda-\xi_h\|_{1/2,\Sigma} + \inf_{\psi_h\in\Psi_h^\Gamma}\left(\|\theta-\psi_h\|_{1,\Omega_\mathrm{f}} + \|\theta-\psi_h\|_{1,4,\Omega_\mathrm{m}}\right)\Bigg\},\end{aligned}$$

which together with (4.23) implies (4.15) and concludes the proof. □

We conclude this section by deriving the theoretical rate of convergence for the Galerkin scheme (3.2). To that end we recall that the discrete spaces satisfy the following approximation properties (see [7, 11, 21, 25]):

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{f}})$ For each $\mathbf{v}_\mathrm{f} \in \mathbf{H}^2(\Omega_\mathrm{f}) \cap \mathbf{H}^1_{\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, there exists $\mathbf{v}_{h,\mathrm{f}} \in \mathbf{H}_{h,\Gamma_\mathrm{f}}(\Omega_\mathrm{f})$, such that

$$\|\mathbf{v}_\mathrm{f} - \mathbf{v}_{h,\mathrm{f}}\|_{1,\Omega_\mathrm{f}} \leq Ch\,\|\mathbf{v}_\mathrm{f}\|_{2,\Omega_\mathrm{f}}.$$

$(\mathbf{AP}_h^{\mathbf{u}_\mathrm{m}})$ For each $\mathbf{v}_\mathrm{m} \in \mathbf{H}^1(\Omega_\mathrm{m}) \cap \mathbf{H}_{\Gamma_\mathrm{m}}(\mathrm{div};\Omega_\mathrm{m})$, with $\mathrm{div}\,\mathbf{v}_\mathrm{m} \in \mathrm{H}^1(\Omega_\mathrm{m})$, there exists $\mathbf{v}_{h,\mathrm{m}} \in \mathbf{H}_{h,\Gamma_\mathrm{m}}(\Omega_\mathrm{m})$, such that

$$\|\mathbf{v}_\mathrm{m} - \mathbf{v}_{h,\mathrm{m}}\|_{\mathrm{div};\Omega_\mathrm{m}} \,\leq\, Ch\,\{\|\mathbf{v}_\mathrm{m}\|_{1,\Omega_\mathrm{m}} \,+\, \|\mathrm{div}\,\mathbf{v}_\mathrm{m}\|_{1,\Omega_\mathrm{m}}\}\,.$$

$(\mathbf{AP}_h^p)$ For each $q \in \mathrm{H}^1(\Omega) \cap \mathrm{L}_0^2(\Omega)$, there exists $q_h \in \mathrm{L}_{h,0}(\Omega)$ such that

$$\|q - q_h\|_{0,\Omega} \,\leq\, Ch\,\|q\|_{1,\Omega}.$$

$(\mathbf{AP}_h^\lambda)$ For each $\xi \in \mathrm{W}^{3/2,2}(\Sigma)$, there exists $\xi_h \in \Lambda_h(\Sigma)$ such that

$$\|\xi - \xi_h\|_{1/2,\Sigma} \,\leq\, Ch\,\|\xi\|_{3/2,2,\Sigma}.$$

$(\mathbf{AP}_h^\theta)$ For each $\psi \in \mathrm{H}^2(\Omega)$, with $\psi|_\Gamma = \theta_\mathrm{D}$ and $\psi|_{\Omega_\mathrm{m}} \in \mathrm{W}^{2,4}(\Omega_\mathrm{m})$, there exists $\psi_h \in \Psi_h^\Gamma$, such that

$$\|\psi - \psi_h\|_{1,\Omega_\mathrm{f}} + \|\psi - \psi_h\|_{1,4,\Omega_\mathrm{m}} \,\leq\, Ch\{\|\psi\|_{2,\Omega_\mathrm{f}} \,+\, \|\psi\|_{2,4,\Omega_\mathrm{m}}\}\,.$$

Using this approximation properties we obtain the following result.

Theorem 4.4 *Assume the same hypotheses in Theorem 4.3 hold, let* $(\mathbf{u},(p,\lambda),\theta) = ((\mathbf{u}_\mathrm{f},\mathbf{u}_\mathrm{m}),(p,\lambda), \theta_0+\theta_1) \in \mathbf{H}\times\mathrm{Q}\times\mathrm{H}^1(\Omega)$ *be the unique solution of* (2.8) *and* $(\mathbf{u}_h,(p_h,\lambda_h),\theta_h) = ((\mathbf{u}_{h,\mathrm{f}},\mathbf{u}_{h,\mathrm{m}}),(p_h,\lambda_h), \theta_{h,0}+\theta_{h,1}) \in \mathbf{H}_h\times\mathrm{Q}_h\times\Psi_h$ *be a solution to* (3.2)*. Assume further that* $\mathbf{u}_\mathrm{f} \in \mathbf{H}^2(\Omega_\mathrm{f})$*,* $\mathbf{u}_\mathrm{m} \in \mathbf{H}^1(\Omega_\mathrm{m})$*,* $\mathrm{div}\,\mathbf{u}_\mathrm{m} \in \mathrm{H}^1(\Omega_\mathrm{m})$*,* $p \in \mathrm{H}^1(\Omega)$*,* $\lambda \in \mathrm{H}^{3/2}(\Sigma)$*,* $\theta|_{\Omega_\mathrm{f}} \in \mathrm{H}^2(\Omega_\mathrm{f})$*, and* $\theta|_{\Omega_\mathrm{m}} \in \mathrm{W}^{2,4}(\Omega_\mathrm{m})$*. Then, there exists* $C_{rate} > 0$*, independent of* h *and the continuous and discrete solutions, such that*

$$\begin{aligned}\|\mathbf{e}_\mathbf{u}\|_\mathbf{H} \,+\, \|(\mathrm{e}_p,\mathrm{e}_\lambda)\|_\mathrm{Q} \,+\, \|\mathrm{e}_\theta\|_{1,\Omega} \,\leq\, C_{rate}h\,\{&\|\mathbf{u}_\mathrm{f}\|_{2,\Omega_\mathrm{f}} \,+\, \|\mathbf{u}_\mathrm{m}\|_{1,\Omega_\mathrm{m}} \,+\, \|\mathrm{div}\,\mathbf{u}_\mathrm{m}\|_{1,\Omega_\mathrm{m}} \,+\, \|p\|_{1,\Omega}\\ &+\|\lambda\|_{3/2,2,\Sigma} \,+\, \|\theta\|_{2,\Omega_\mathrm{f}} \,+\, \|\theta\|_{2,4,\Omega_\mathrm{m}}\}\,.\end{aligned}$$

Proof. The result is a direct consequence of Theorem 4.3 and the approximation properties $(\mathbf{AP}_h^{\mathbf{u}_\mathrm{f}})$, $(\mathbf{AP}_h^{\mathbf{u}_\mathrm{m}})$, $(\mathbf{AP}_h^p)$, $(\mathbf{AP}_h^\lambda)$ and $(\mathbf{AP}_h^\theta)$. □

5 Numerical results

In this section we present some numerical results illustrating the performance of our finite element scheme (3.2) on a set of quasi-uniform triangulations of the corresponding domains and considering the finite element spaces introduced in Section 3. Our implementation is based on a *FreeFem++* code [33], in conjunction with the direct linear solver *UMFPACK* [19]. In order to solve the nonlinear problem, we propose the Newton-type strategy: Starting with the initial guess $\mathbf{u}^0 = (\mathbf{u}_\mathrm{f}^0,\mathbf{u}_\mathrm{m}^0) \in \mathbf{H}_h$ and $\theta^0 \in \Psi_h$, for $n \geq 1$, find $\mathbf{u}^n \in \mathbf{H}_h$, $(p^n,\lambda^n) \in \mathrm{Q}_h$, and $\theta^n \in \Psi_h$, such that $\theta_h|_\Gamma = \theta_{\mathrm{D},h}^\delta$ and

$$\begin{aligned} A_\mathrm{F}(\mathbf{u}^n,\mathbf{v}) \,+\, O_\mathrm{F}^h(\mathbf{u}_\mathrm{f}^{n-1};\mathbf{u}_\mathrm{f}^n,\mathbf{v}_\mathrm{f}) \,+\, O_\mathrm{F}^h(\mathbf{u}_\mathrm{f}^n;\mathbf{u}_\mathrm{f}^{n-1},\mathbf{v}_\mathrm{f})&\\ -\,B(\mathbf{v},(p^n,\lambda^n)) \,-\, D(\theta^n,\mathbf{v}) &\,=\, O_\mathrm{F}^h(\mathbf{u}_\mathrm{f}^{n-1};\mathbf{u}_\mathrm{f}^{n-1},\mathbf{v}_\mathrm{f}),\\ B(\mathbf{u}^n,(q,\xi)) &\,=\, 0,\\ A_\mathrm{T}(\theta^n,\psi) \,+\, O_\mathrm{T}^h(\mathbf{u}^{n-1};\theta^n,\psi) \,+\, O_\mathrm{T}^h(\mathbf{u}^n;\theta^{n-1},\psi) &\,=\, O_\mathrm{T}^h(\mathbf{u}^{n-1};\theta^{n-1},\psi),\end{aligned}$$

for all $\mathbf{v} = (\mathbf{v}_\mathrm{f},\mathbf{v}_\mathrm{m}) \in \mathbf{H}_h$, $(q,\xi) \in \mathrm{Q}_h$, and $\psi \in \Psi_h$.

The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is

$$\frac{\|\mathbf{coeff}^{n+1} - \mathbf{coeff}^{n}\|_{l^2}}{\|\mathbf{coeff}^{n+1}\|_{l^2}} \leq tol,$$

where $\|\cdot\|_{l^2}$ stands for the usual euclidean norm in $\mathbb{R}^{\mathrm{dof}}$, with dof denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h(\Omega_\mathrm{f})$, $\mathbf{H}_h(\Omega_\mathrm{m})$, $\mathrm{L}_h(\Omega)$, $\Lambda_h(\Sigma)$, and Ψ_h, and tol is a fixed tolerance. For each example shown below we simply take $\mathbf{u}^0 = \mathbf{0}$ and $\theta^0 = 0$ as initial guess and $tol = 1e-6$.

Now, we introduce some additional notations. We denote by $h_\Sigma := \max\{h_e : e \in \Sigma_{2h}\}$. As in Section 4, the individual errors for each variable are denoted by $\mathbf{e}_{\mathbf{u}_\mathrm{f}}$, $\mathbf{e}_{\mathbf{u}_\mathrm{m}}$, e_λ, e_p and e_θ and let $\mathrm{e}_{p_\star} = \mathrm{e}_p|_{\Omega_\star}$, $\mathrm{e}_{\theta_\star} = \mathrm{e}_\theta|_{\Omega_\star}$, for $\star \in \{\mathrm{f}, \mathrm{m}\}$. In addition, we define the experimental rates of convergence $\mathbf{r}_{\mathbf{u}_\mathrm{f}}$, $\mathbf{r}_{\mathbf{u}_\mathrm{m}}$, $\mathrm{r}_{p_\mathrm{f}}$, $\mathrm{r}_{p_\mathrm{m}}$, r_λ, $\mathrm{r}_{\theta_\mathrm{f}}$ and $\mathrm{r}_{\theta_\mathrm{m}}$, as

$$\mathbf{r}_{\mathbf{u}_\mathrm{f}} := \frac{\log(\mathbf{e}_{\mathbf{u}_\mathrm{f}}/\mathbf{e}'_{\mathbf{u}_\mathrm{f}})}{\log(h_\mathrm{f}/h'_\mathrm{f})}, \quad \mathbf{r}_{\mathbf{u}_\mathrm{m}} := \frac{\log(\mathbf{e}_{\mathbf{u}_\mathrm{m}}/\mathbf{e}'_{\mathbf{u}_\mathrm{m}})}{\log(h_\mathrm{m}/h'_\mathrm{m})}, \quad \mathrm{r}_{p_\mathrm{f}} := \frac{\log(\mathrm{e}_{p_\mathrm{f}}/\mathrm{e}'_{p_\mathrm{f}})}{\log(h_\mathrm{f}/h'_\mathrm{f})}, \quad \mathrm{r}_{p_\mathrm{m}} := \frac{\log(\mathrm{e}_{p_\mathrm{m}}/\mathrm{e}'_{p_\mathrm{m}})}{\log(h_\mathrm{m}/h'_\mathrm{m})},$$

$$\mathrm{r}_\lambda := \frac{\log(\mathrm{e}_\lambda/\mathrm{e}'_\lambda)}{\log(h_\Sigma/h'_\Sigma)}, \quad \mathrm{r}_{\theta_\mathrm{f}} := \frac{\log(\mathrm{e}_{\theta_\mathrm{f}}/\mathrm{e}'_{\theta_\mathrm{f}})}{\log(h_\mathrm{f}/h'_\mathrm{f})}, \quad \mathrm{r}_{\theta_\mathrm{m}} := \frac{\log(\mathrm{e}_{\theta_\mathrm{m}}/\mathrm{e}'_{\theta_\mathrm{m}})}{\log(h_\mathrm{m}/h'_\mathrm{m})},$$

where $h_\star$ and $h'_\star$ ($\star \in \{\mathrm{f}, \mathrm{m}, \Sigma\}$) denote two consecutive mesh sizes with their respective errors $\mathbf{e}$, $\mathbf{e}'$ (or e, e$'$).

Example 1: Manufactured Exact Solution

In our first example we illustrate the accuracy of our method considering a manufactured exact solution defined on $\Omega = \Omega_\mathrm{f} \cup \Sigma \cup \Omega_\mathrm{m}$, with $\Omega_\mathrm{f} := (-1/2, 1/2) \times (0, 1/2)$ and $\Omega_\mathrm{m} := (-1/2, 1/2) \times (-1/2, 0)$. We consider the following parameters $\mu = 1$, $\mathbf{g}_\mathrm{f} = (0, -1)^t$, $\mathbf{g}_\mathrm{m} = (0, -1)^t$, $\alpha_d = 1$, $\kappa_\mathrm{f} = 1$, $\kappa_\mathrm{m} = 1$, $\mathbf{K} = \mathbf{I}$, and $\boldsymbol{\kappa} = \mathbf{I}$ and the terms on the right-hand side are adjusted so that the exact solution is given by the functions

$$\begin{aligned}
\mathbf{u}_\mathrm{f}(x, y) &:= \begin{pmatrix} 16y\cos(\pi x)^2(y^2 - 1/4) \\ 8\pi\cos(\pi x)\sin(\pi x)(y^2 - 1/4)^2 \end{pmatrix} \quad \text{in} \quad \Omega_\mathrm{f}, \\
\mathbf{u}_\mathrm{m}(x, y) &:= \begin{pmatrix} -2y\cos(\pi x)^2 \\ -2\pi\cos(\pi x)\sin(\pi x)(y^2 - 1/4) \end{pmatrix} \quad \text{in} \quad \Omega_\mathrm{m}, \\
p_\star(x, y) &:= \exp(y)\sin(x) \quad \text{in} \quad \Omega_\star, \\
\theta_\star(x, y) &:= \exp(-xy) \quad \text{in} \quad \Omega_\star,
\end{aligned}$$

with $\star \in \{\mathrm{f}, \mathrm{m}\}$. We notice that $\mathbf{u}_\mathrm{f}|_\Sigma = \mathbf{u}_\mathrm{m}|_\Sigma$, $\theta_\mathrm{f}|_\Sigma = \theta_\mathrm{m}|_\Sigma$, and $\kappa_\mathrm{f}\,\nabla\theta_\mathrm{f}|_\Sigma = \kappa_\mathrm{m}\,\nabla\theta_\mathrm{m}|_\Sigma$. We notice also that these functions do not satisfy the interface conditions (2.5), thus the difference must be incorporated as a functional at the right-hand side of the resulting system.

In Table 5.1 we summarize the history of convergence for a sequence of quasi-uniform triangulations. We observe there that the rate of convergence $O(h)$ predicted by Theorem 4.4 is attained in all the cases.

Example 2: Nondimensional problem

In our second example we are interested in studying the phenomenon on a square cavity with differentially heated walls. To that end, we let $\Omega = \Omega_\mathrm{f} \cup \Sigma \cup \Omega_\mathrm{m}$, with $\Omega_\mathrm{f} := (0, 1) \times (0, 3/4)$ and

dof	h_{f}	$\mathbf{e}_{\mathbf{u}_{\mathrm{f}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{f}}}$	$\mathrm{e}_{p_{\mathrm{f}}}$	$\mathrm{r}_{p_{\mathrm{f}}}$	$\mathrm{e}_{\theta_{\mathrm{f}}}$	$\mathrm{r}_{\theta_{\mathrm{f}}}$
216	0.3207	0.5592	–	0.2104	–	0.0813	–
834	0.1804	0.3492	0.8189	0.1133	1.0767	0.0390	1.2762
3026	0.1013	0.1844	1.1057	0.0565	1.2039	0.0199	1.1657
11738	0.0503	0.0855	1.0989	0.0292	0.9398	0.0099	0.9945
45622	0.0247	0.0424	0.9875	0.0145	0.9926	0.0050	0.9674
180930	0.0123	0.0208	1.0226	0.0070	1.0338	0.0025	0.9818
725890	0.0065	0.0103	1.1014	0.0035	1.1007	0.0012	1.1210

dof	h_{m}	$\mathbf{e}_{\mathbf{u}_{\mathrm{m}}}$	$\mathbf{r}_{\mathbf{u}_{\mathrm{m}}}$	$\mathrm{e}_{p_{\mathrm{m}}}$	$\mathrm{r}_{p_{\mathrm{m}}}$	$\mathrm{e}_{\theta_{\mathrm{m}}}$	$\mathrm{r}_{\theta_{\mathrm{m}}}$
216	0.3663	0.1752	–	0.0330	–	0.0953	–
834	0.1804	0.0748	1.2018	0.0137	1.2349	0.0377	1.3073
3026	0.0951	0.0398	0.9856	0.0072	1.0013	0.0200	0.9917
11738	0.0488	0.0198	1.0416	0.0035	1.0802	0.0098	1.0692
45622	0.0247	0.0099	1.0091	0.0017	0.9984	0.0050	0.9764
180930	0.0143	0.0050	1.2604	0.0008	1.2650	0.0025	1.2636
725890	0.0064	0.0024	0.8754	0.0004	0.8686	0.0012	0.8831

dof	h_{Σ}	e_{λ}	r_{λ}	iteration
216	0.2500	0.0470	–	5
834	0.1250	0.0308	0.6107	5
3026	0.0625	0.0114	1.4330	5
11738	0.0312	0.0051	1.1391	5
45622	0.0156	0.0026	0.9634	5
180930	0.0078	0.0013	1.0307	5
725890	0.0039	0.0006	1.0442	5

Table 5.1: Example 1: Degree of Freedom, mesh sizes, errors, rates of convergence and number of iterations for the coupled problem.

$\Omega_{\mathrm{m}} := (0,1) \times (3/4,1)$, and similarly to [36, Section 2.4] we consider the problem with dimensionless

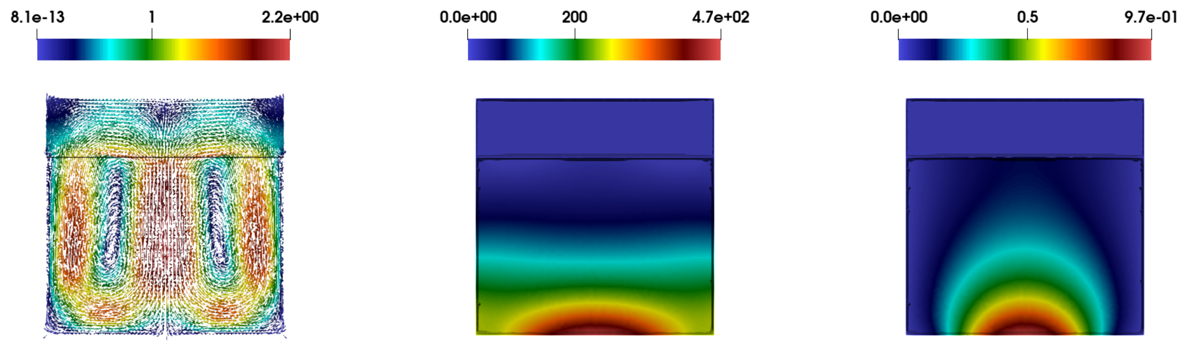

Figure 5.1: Example 2: Velocity vector field (left), pressure (center) and temperature (right).

numbers

$$
\begin{aligned}
\boldsymbol{\sigma}_{\rm f} = 2\,\mathbf{e}(\mathbf{u}_{\rm f}) \,-\, p_{\rm f}\,\mathbf{I} & \quad \text{in} \quad \Omega_{\rm f}, \\
-\mathbf{div}\,\boldsymbol{\sigma}_{\rm f} \,+\, (\mathbf{u}_{\rm f}\cdot\nabla)\,\mathbf{u}_{\rm f} \,-\, \mathrm{Ra}_{\rm f}\,\mathbf{g}_{\rm f}\,\theta_{\rm f} = \mathbf{0} & \quad \text{in} \quad \Omega_{\rm f}, \\
\mathrm{div}\,\mathbf{u}_{\rm f} = 0 & \quad \text{in} \quad \Omega_{\rm f}, \\
-\Delta\,\theta_{\rm f} \,+\, \mathrm{Pr}_{\rm f}\,\mathbf{u}_{\rm f}\cdot\nabla\,\theta_{\rm f} = 0 & \quad \text{in} \quad \Omega_{\rm f}, \\
\mathbf{u}_{\rm m} \,+\, \mathrm{Da}\,\nabla p_{\rm m} \,-\, \mathrm{Ra}_{\rm m}\,\mathbf{g}_{\rm m}\,\theta_{\rm m} \,=\, \mathbf{0} & \quad \text{in} \quad \Omega_{\rm m}, \\
\mathrm{div}\,\mathbf{u}_{\rm m} \,=\, 0 & \quad \text{in} \quad \Omega_{\rm m}, \\
-\Delta\,\theta_{\rm m} \,+\, \mathrm{Pr}_{\rm m}\,\mathbf{u}_{\rm m}\cdot\nabla\theta_{\rm m} \,=\, 0 & \quad \text{in} \quad \Omega_{\rm m}, \\
\nabla\,\theta_{\rm f}\cdot\mathbf{n} = \nabla\,\theta_{\rm m}\cdot\mathbf{n} & \quad \text{on} \quad \Sigma, \\
\boldsymbol{\sigma}_{\rm f}\,\mathbf{n} \,+\, \frac{\alpha_d}{\sqrt{\mathrm{Da}}}(\mathbf{u}_{\rm f}\cdot\mathbf{t})\,\mathbf{t} = \,-p_{\rm m}\,\mathbf{n} & \quad \text{on} \quad \Sigma, \\
\theta = \theta_{\rm D} & \quad \text{on} \quad \Gamma,
\end{aligned}
$$

where $\mathrm{Pr}_\star$ and $\mathrm{Ra}_\star$ represent the Prandtl and Rayleigh numbers in the domain $\Omega_\star$ for $\star \in \{\mathrm{f}, \mathrm{m}\}$ and Da represents the Darcy number. Here, we fix the Prandtl, Rayleigh and Darcy numbers as $\mathrm{Pr}_{\rm f} = 0.5$, $\mathrm{Pr}_{\rm m} = 0.5$, $\mathrm{Ra}_{\rm f} = 2000$, $\mathrm{Ra}_{\rm m} = 2000$, and $\mathrm{Da} = 1$, and consider $\alpha_d = 1$. For the boundary condition, we choose $\theta_{\rm D}(x, y) = 0.5(1 - \cos(2\pi x))(1 - y)$ on Γ and observe that $\theta_{\rm D} = 0$ on the left, bottom and right walls whereas on the top wall $\theta_{\rm D}$ has a sinusoidal profile with a peak of temperature $\theta_{\rm D} = 1$ at $x = 0.5$. In Figure 5.1 we display the approximate solutions obtained with $\mathrm{dof} = 45726$. In Figure 5.1 we show the velocity vector field (left), the pressure (center) and temperature (right). There, it is possible to see the expected physical behavior from [18], that is, convection currents form inside the cavity in a symmetric configuration. However, in our case, the interface plays a role in the phenomenon, and as reported in [46], we observe that the velocity field has an expected velocity decrease when it crosses the interface from the free fluid region to the porous medium.

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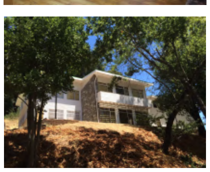

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