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Gabriel N. Gatica, Cristian Inzunza,
Filander A. Sequeira

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New Banach spaces-based fully-mixed finite element methods for pseudostress-assisted diffusion problems *

Gabriel N. Gatica[†] Cristian Inzunza [‡] Filánder A. Sequeira [§]

Abstract

In this paper we propose and analyze Banach spaces-based fully-mixed approaches yielding new finite element methods for numerically solving the coupled partial differential equations describing the pseudostress-assisted diffusion of a solute into an elastic material. Two mixed formulations employing the diffusive flux as an additional variable are introduced for the diffusion equation, and the concentration gradient is considered as an auxiliary unknown of the second one of them. The resulting coupled systems are rewritten as equivalent fixed point operator equations, so that the respective unique solvabilities are proved by applying the classical Banach theorem along with the Babuška-Brezzi theory. The nonlinear dependency on the elastic variables of the diffusion coefficient and its source term, as well as the nonlinear dependency on the concentration of the elastic source term, suggest, for appropriate continuous and discrete analyses, that the unknowns be sought in suitable Lebesgue spaces. The associated Galerkin schemes are addressed similarly, and the Brouwer theorem yields the existence of discrete solutions. A priori error estimates are derived for both approaches, and rates of convergence for specific finite element subspaces satisfying the required discrete inf-sup conditions, are established in 2D. Finally, several numerical examples illustrating the performance of the two methods and confirming the theoretical findings, are reported.

Keywords: linear elasticity, pseudostress-assisted diffusion, fixed point, finite element methods

Mathematics subject classifications (2000): 65N30, 65N12, 76R05, 76D07, 65N15, 35Q79

1 Introduction

In the recent paper [14] we employed a Banach spaces-based variational approach to derive a new mixed-primal finite element method for the nearly incompressible case of the pseudostress-assisted diffusion problem, which models the diffusion of a solute into an elastic material. More precisely, the aforementioned phenomenon refers to diffusion processes in deformable solids occupying originally a domain Ω of R^n, $n \in \{2, 3\}$, and arises in diverse applications, including diffusion of boron and arsenic in silicon [19], voiding of aluminum conductor lines in integrated circuits [21], sorption in polymers [20], damage to electrodes in lithium-ion batteries [2], and anisotropy of cardiac dynamics [6], among others. The usual assumptions in most of them are, on one hand, that the solid satisfies an elastic regime, and

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[†]CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: ggatica@ci2ma.udec.cl.

[‡]CI^{2}MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: crinzunza@udec.cl.

[§]Escuela de Matemática, Universidad Nacional de Costa Rica, Campus Omar Dengo, Heredia, Costa Rica, email: filander.sequeira@una.cr.

on the other hand, that the diffusion obeys a Fickean law enriched with further contributions arising from local effects by exerted stresses. This second hypothesis means that the respective diffusion coefficient is a continuous function depending precisely on the stress, which acts then as a coupling variable. Mathematically, the underlying model is usually described by the following system of partial differential equations (cf. [14, eq. (2.1)]):

$$\begin{aligned} \nabla \boldsymbol{u} \,=\, \widehat{\mathcal{C}}^{-1}(\boldsymbol{\sigma}) \quad \text{in} \quad \Omega\,, \quad -\mathbf{div}(\boldsymbol{\sigma}) \,=\, \boldsymbol{f}(\phi) \quad \text{in} \quad \Omega\,, \quad \boldsymbol{u} \,=\, \boldsymbol{u}_D \quad \text{on} \quad \Gamma\,, \\ \widetilde{\boldsymbol{\sigma}} \,=\, \vartheta(\boldsymbol{\sigma})\nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, g(\boldsymbol{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \end{aligned} \tag{1.1}$$

where

$$\widehat{\mathcal{C}}^{-1}(\boldsymbol{\tau}) \,:=\, \frac{1}{\mu}\,\boldsymbol{\tau}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda + (n+1)\mu\big)}\,\mathrm{tr}(\boldsymbol{\tau})\,\mathbb{I} \qquad \forall\,\boldsymbol{\tau} \in \mathrm{R}^{n\times n}\,. \tag{1.2}$$

Here, $\boldsymbol{\sigma}$ is the non-symmetric pseudostress tensor, $\boldsymbol{u}$ is the displacement field, λ, $\mu > 0$ are the Lamé constants (dilation and shear moduli), which characterize the properties of the material, and $\mathbb{I}$ is the identity tensor of $\mathrm{R}^{n\times n}$. In turn, ϕ represents the local concentration of species, $\widetilde{\boldsymbol{\sigma}}$ is the diffusive flux, and $\vartheta : \mathrm{R}^{n\times n} \to \mathrm{R}^{n\times n}$ is a tensorial diffusivity function. In addition, $\boldsymbol{f} : \mathrm{R} \to \mathrm{R}^n$ is a vector field of body loads (which depends on the species concentration), $g : \mathrm{R}^n \to \mathrm{R}$ denotes an additional source term depending on the solid displacement $\boldsymbol{u}$, and $\boldsymbol{u}_D$ is the Dirichlet datum for $\boldsymbol{u}$, which belongs to a suitable trace space to be identified later on.

The purpose of the present work is to continue contributing in the direction of [14] by introducing and analyzing new fully-mixed finite element methods for the numerical solution of (1.1) - (1.2). In this way, the main novelty with respect to [14] is the utilization of a mixed variational formulation for the diffusion equation. As a consequence, and regarding the mixed approach for the elasticity equation, we certainly make use of the corresponding results from [14] either by stating or referring to them throughout the analysis. In some cases, and just for sake of completeness, the main aspects of the respective proofs are explicitly recalled. Needless to say, we remark that a fully-mixed approach for this model had basically been employed already in [13]. However, to be able to carry out the respective analysis within a Hilbertian framework, it was necessary to incorporate there augmented terms, thus increasing the complexity of the resulting discrete method. According to the above, and motivated by recent works using Banach spaces-based formulations (see, e.g. [3], [14], [15] and [16]), which do not need to resort to augmentation techniques, we proceed similarly to them and propose two mixed variational formulations for the diffusion equation in terms of suitable Lebesgue and Sobolev-type Banach spaces. For the first approach we perform integration by parts on the constitutive equation, while for the second one the diffusion gradient is introduced as an auxiliary unknown.

The paper is organized as follows. The rest of this section collects first some preliminary notations, definitions, and results to be utilized throughout the paper. In Section 2, we derive the two fully-mixed variational formulations of the problem. Suitable integration by parts formulae jointly with the Cauchy-Schwarz and Hölder inequalities are crucial for determining the right Lebesgue and related spaces to which the unknowns and corresponding test functions are required to belong. In Section 3, fixed-point strategies are adopted to analyze the solvability of the continuous formulations. The Babuška-Brezzi theory in Banach spaces is employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution of the respective formulations. Analogue fixed-point approaches to those of Section 3 are utilized in Section 4 to study the well-posedness of the associated Galerkin scheme. In this way, and along with the corresponding versions of the theoretical tools employed in Section 3, a straightforward application of Brouwer's theorem allows us to conclude the existence of discrete solution. A priori error estimates in the form of Cea's estimate are also derived here. Next, in Section 5 we restrict ourselves to the 2D case and introduce specific finite element subspaces satisfying the theoretical hypotheses that were assumed in Section 4. The fact that a required boundedness property for a particular projector involved is still an open problem in 3D, stop us of extending the 2D analysis from Section 5 to that dimension. Finally, several numerical results illustrating the performance of the method and confirming the theoretical rates of convergence provided in Section 5, are reported in Section 6

1.1 Preliminaries

Throughout the paper, Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, which is star shaped with respect to a ball, and whose outward normal at $\Gamma := \partial\Omega$ is denoted by $\boldsymbol{\nu}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$ and Sobolev spaces $\mathrm{W}^{l,t}(\Omega)$ and $\mathrm{W}^{l,t}_0(\Omega)$, with $l \geqslant 0$ and $t \in [1,+\infty)$, whose corresponding norms, either for the scalar and vectorial case, are denoted by $\|\cdot\|_{0,t;\Omega}$ and $\|\cdot\|_{l,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and if $t = 2$ we write $\mathrm{H}^l(\Omega)$ instead of $\mathrm{W}^{l,2}(\Omega)$, with the corresponding norm and seminorm denoted by $\|\cdot\|_{l,\Omega}$ and $|\cdot|_{l,\Omega}$, respectively. In addition, letting t, $t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we denote by $\mathrm{W}^{1/t',t}(\Gamma)$ the trace space of $\mathrm{W}^{1,t}(\Omega)$, and let $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the dual of $\mathrm{W}^{1/t',t}(\Gamma)$ endowed with the norms $\|\cdot\|_{-1/t',t';\Gamma}$ and $\|\cdot\|_{1/t',t;\Gamma}$, respectively. On the other hand, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be the corresponding vectorial and tensorial counterparts. In particular, we set $\mathbf{R} := \mathrm{R}^n$ and $\mathbb{R} := \mathrm{R}^{n\times n}$. Furthermore, $\|\cdot\|$ is employed for the norm of any element or operator whenever there is no confusion about the spaces to which they belong, and $|\cdot|$ stands for the Euclidean norm in $\mathbf{R}$. Also, for any vector field $\boldsymbol{v} = (v_i)_{i=1,n}$ we set the gradient and divergence operators, respectively, as $\nabla\boldsymbol{v} := \left(\dfrac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}$ and $\operatorname{div}(\boldsymbol{v}) := \displaystyle\sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}$. Additionally, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product operators, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} = (\tau_{ji})_{i,j=1,n}, \quad \operatorname{tr}(\boldsymbol{\tau}) = \sum_{i=1}^{n} \boldsymbol{\tau}_{ii}, \quad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,, \quad \text{and} \quad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,.$$

On the other hand, for each $t \in [1,+\infty)$ we introduce the Banach spaces

$$\mathbf{H}(\operatorname{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \quad \operatorname{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}, \quad \text{and} \tag{1.3}$$

$$\mathbb{H}^t(\operatorname{div}_t;\Omega) := \Big\{\boldsymbol{\tau} \in \mathbb{L}^t(\Omega) : \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}, \tag{1.4}$$

which are endowed with the natural norms

$$\|\boldsymbol{\tau}\|_{\operatorname{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,\Omega} + \|\operatorname{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{H}(\operatorname{div}_t;\Omega)\,, \quad \text{and} \tag{1.5}$$

$$\|\boldsymbol{\tau}\|_{t,\mathbf{div}_t;\Omega} := \|\boldsymbol{\tau}\|_{0,t;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^t(\mathbf{div}_t;\Omega)\,. \tag{1.6}$$

Then, we recall that, proceeding as in [11, eq. (1.43), Section 1.3.4] (see also [5, Section 4.1] and [8, Section 3.1]), one can prove that for each $t \in \begin{cases} (1,+\infty) & \text{if } n = 2\,, \\ [6/5,+\infty) & \text{if } n = 3\,, \end{cases}$ there holds

$$\langle \boldsymbol{\tau}\cdot\boldsymbol{\nu}, v\rangle = \int_{\Omega} \Big\{\boldsymbol{\tau}\cdot\nabla v + v\operatorname{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau}, v) \in \mathbf{H}(\operatorname{div}_t;\Omega)\times\mathrm{H}^1(\Omega)\,, \tag{1.7}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.7) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. In addition, throughout this work we suppose that ϑ is of class C^1 and uniformly positive definite, meaning the latter that there exists $\vartheta_0 > 0$ such that

$$\vartheta(\boldsymbol{\tau})\boldsymbol{w}\cdot\boldsymbol{w} \geqslant \vartheta_0\,|\boldsymbol{w}|^2 \quad \forall\,\boldsymbol{w} \in \mathbf{R}\,, \quad \forall\,\boldsymbol{\tau} \in \mathbb{R}\,. \tag{1.8}$$

We also require uniform boundedness and Lipschitz continuity of ϑ, that is that there exist positive constants ϑ_1, ϑ_2, and L_ϑ, such that

$$\vartheta_1 \,\leqslant\, |\vartheta(\boldsymbol{\tau})| \,\leqslant\, \vartheta_2 \quad \text{and} \quad |\vartheta(\boldsymbol{\tau}) - \vartheta(\boldsymbol{\zeta})| \,\leqslant\, L_\vartheta\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{1.9}$$

Moreover, thanks to (1.8), we have that the inverse of ϑ is uniformly positive definite as well, specifically, denoting from now on $\widetilde{\vartheta}(\boldsymbol{\tau}) := \vartheta(\boldsymbol{\tau})^{-1}$, there exists $\widetilde{\vartheta}_0 > 0$ such that

$$\widetilde{\vartheta}(\boldsymbol{\tau})\boldsymbol{w}\cdot\boldsymbol{w} \,\geqslant\, \widetilde{\vartheta}_0\,|\boldsymbol{w}|^2 \quad \forall\,\boldsymbol{w}\in\mathbf{R}\,, \quad \forall\,\boldsymbol{\tau}\in\mathbb{R}\,. \tag{1.10}$$

We also require uniform boundedness and Lipschitz continuity of $\widetilde{\vartheta}$, that is that there exist positive constants $\widetilde{\vartheta}_1$, $\widetilde{\vartheta}_2$, and $L_{\widetilde{\vartheta}}$, such that

$$\widetilde{\vartheta}_1 \,\leqslant\, |\widetilde{\vartheta}(\boldsymbol{\tau})| \,\leqslant\, \widetilde{\vartheta}_2 \quad \text{and} \quad |\widetilde{\vartheta}(\boldsymbol{\tau}) - \widetilde{\vartheta}(\boldsymbol{\zeta})| \,\leqslant\, L_{\widetilde{\vartheta}}\,|\boldsymbol{\tau}-\boldsymbol{\zeta}| \quad \forall\,\boldsymbol{\tau},\boldsymbol{\zeta}\in\mathbb{R}\,. \tag{1.11}$$

Similar hypotheses are assumed on the source functions $\boldsymbol{f}$ and g, which means that there exist positive constants f_1, f_2, L_f, g_1, g_2 and L_g, such that

$$f_1 \,\leqslant\, |\boldsymbol{f}(s)| \,\leqslant\, f_2\,, \quad |\boldsymbol{f}(s)-\boldsymbol{f}(t)| \,\leqslant\, L_f\,|s-t| \quad \forall\,s,t\in\mathrm{R}\,, \tag{1.12}$$

$$g_1 \,\leqslant\, |g(\boldsymbol{w})| \,\leqslant\, g_2\,, \quad \text{and} \quad |g(\boldsymbol{v})-g(\boldsymbol{w})| \,\leqslant\, L_g\,|\boldsymbol{v}-\boldsymbol{w}| \quad \forall\,\boldsymbol{v},\boldsymbol{w}\in\mathbf{R}\,. \tag{1.13}$$

2 The fully-mixed formulations

In this section we introduce two Banach spaces-based fully-mixed formulations of (1.1)-(1.2), which arise from a common formulation for elasticity (see Section 2.1 below) and two different approaches for the diffusion equation (see Sections 2.2 and 2.3 below). The integration by parts formulae provided by (1.7), along with the Cauchy-Schwarz and Hölder inequalities, play key roles in the derivation of the Banach spaces where the respective unknowns will be sought.

2.1 The elasticity equation

As explained in [14, Section 3], given

$$r\in\begin{cases} (2,+\infty) & \text{if } n=2\,,\\ (2,6] & \text{if } n=3\,,\end{cases} \qquad \text{and} \qquad s\in\begin{cases} (1,2) & \text{if } n=2\,,\\ [6/5,2) & \text{if } n=3\,,\end{cases} \tag{2.1}$$

conjugate to each other, and given ϕ in a suitable space to be determined next, the Banach spaces-based mixed formulation for the elasticity equation reads: Find $(\boldsymbol{\sigma},\boldsymbol{u})\in\mathbf{X}_2\times\mathbf{M}_1$ such that

$$\begin{aligned} \mathbf{a}(\boldsymbol{\sigma},\boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau},\boldsymbol{u}) \;&=\; G(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbf{X}_1,\\ \mathbf{b}_2(\boldsymbol{\sigma},\boldsymbol{v}) \;&=\; F_\phi(\boldsymbol{v}) \qquad \forall\,\boldsymbol{v}\in\mathbf{M}_2,\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} \mathbf{X}_2 := \mathbb{H}^r_0(\mathbf{div}_r;\Omega) \;&:=\; \Big\{\boldsymbol{\tau}\in\mathbb{H}^r(\mathbf{div}_r;\Omega):\ \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\}\,, \quad \mathbf{M}_1 \,:=\, \mathbf{L}^r(\Omega)\,,\\ \mathbf{X}_1 := \mathbb{H}^s_0(\mathbf{div}_s;\Omega) \;&:=\; \Big\{\boldsymbol{\tau}\in\mathbb{H}^s(\mathbf{div}_s;\Omega):\ \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\}\,, \quad \mathbf{M}_2 \,:=\, \mathbf{L}^s(\Omega)\,,\end{aligned}$$

and the bilinear forms $\mathbf{a}:\mathbf{X}_2\times\mathbf{X}_1\to\mathrm{R}$ and $\mathbf{b}_i:\mathbf{X}_i\times\mathbf{M}_i\to\mathrm{R}$, $i\in\{1,2\}$, and the functionals $G\in\mathbf{X}_1'$ and $F_\phi\in\mathbf{M}_2'$, are defined, respectively, as

$$\mathbf{a}(\boldsymbol{\zeta},\boldsymbol{\tau}) := \frac{1}{\mu}\int_\Omega \boldsymbol{\zeta}^{\mathrm{d}}:\boldsymbol{\tau}^{\mathrm{d}} \,+\, \frac{1}{n\big(n\lambda+(n+1)\mu\big)}\int_\Omega \mathrm{tr}(\boldsymbol{\zeta})\,\mathrm{tr}(\boldsymbol{\tau}) \qquad \forall\,(\boldsymbol{\zeta},\boldsymbol{\tau})\in\mathbf{X}_2\times\mathbf{X}_1\,,$$

$$\mathbf{b}_i(\boldsymbol{\tau}, \boldsymbol{v}) := \int_\Omega \boldsymbol{v} \cdot \mathbf{div}(\boldsymbol{\tau}) \qquad \forall\, (\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbf{X}_i \times \mathbf{M}_i\,,$$

$$G(\boldsymbol{\tau}) := \langle \boldsymbol{\tau\nu}, \boldsymbol{u}_D \rangle_\Gamma, \qquad \forall\, \boldsymbol{\tau} \in \mathbf{X}_1\,, \tag{2.3}$$

and

$$F_\phi(\boldsymbol{v}) := -\int_\Omega \boldsymbol{f}(\phi) \cdot \boldsymbol{v} \qquad \forall\, \boldsymbol{v} \in \mathbf{M}_2\,. \tag{2.4}$$

Furthermore, we have from [14, eq. (3.39)] that $\mathbf{a}$, $\mathbf{b}_1$, $\mathbf{b}_2$, G and F_ϕ are all bounded with respective constants given by

$$\|\mathbf{a}\| = \frac{2}{\mu}\,, \quad \|\mathbf{b}_1\| = \|\mathbf{b}_2\| = 1\,, \quad \|G\| = C_r\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma}\,, \quad \text{and} \quad \|F_\phi\| = |\Omega|^{1/r} f_2\,,$$

where C_r is a positive constant such that (cf. [14, eq. (3.9)])

$$\|\boldsymbol{\tau}\,\boldsymbol{\nu}\|_{-1/r,r;\Gamma} \leqslant C_r\, \|\boldsymbol{\tau}\|_{r,\mathbf{div}_r;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}^r(\mathbf{div}_r;\Omega)\,.$$

Having recalled the above from [14], we remark that in order to analyze the elasticity equation, we need to be able to control the expression

$$\int_\Omega (\boldsymbol{f}(\psi) - \boldsymbol{f}(\varphi)) \cdot \boldsymbol{v}\,, \tag{2.5}$$

where $\boldsymbol{v} \in \mathbf{M}_2$, and ψ and φ are generic functions belonging to the same space in which we will seek the unknown ϕ. In this regard, employing the Lipschitz-continuity property of $\boldsymbol{f}$ (cf. (1.12)), a straightforward application of the Hölder inequality yields

$$\left| \int_\Omega (\boldsymbol{f}(\psi) - \boldsymbol{f}(\varphi)) \cdot \boldsymbol{v} \right| \leqslant L_f \|\psi - \varphi\|_{0,r;\Omega} \|\boldsymbol{v}\|_{0,s;\Omega}\,, \tag{2.6}$$

from which we deduce that we must look for the unknown ϕ in $\mathrm{L}^r(\Omega)$.

2.2 A first approach for the diffusion equation

In what follows we derive a first mixed variational formulation for the diffusion equation

$$\widetilde{\vartheta}(\boldsymbol{\sigma})\, \widetilde{\boldsymbol{\sigma}} = \nabla\phi \quad \text{in} \quad \Omega\,, \quad -\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\boldsymbol{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,, \tag{2.7}$$

where $\widetilde{\vartheta}(\boldsymbol{\sigma}) = \vartheta(\boldsymbol{\sigma})^{-1}$. To this end, we begin by considering $\phi \in \mathrm{H}^1(\Omega)$, which, thanks to the continuous embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$, does not contradict what was discussed at the end of the previous section. Then, applying (1.7) with s specified in (2.1) and $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s;\Omega)$ (cf. (1.3)), and using the Dirichlet condition satisfied by ϕ, we get

$$\int_\Omega \widetilde{\boldsymbol{\tau}} \cdot \nabla\phi = -\int_\Omega \phi\, \mathrm{div}(\widetilde{\boldsymbol{\tau}})\,,$$

whence the corresponding testing of the first equation of (2.7) becomes

$$\int_\Omega \widetilde{\vartheta}(\boldsymbol{\sigma})\, \widetilde{\boldsymbol{\sigma}} \cdot \widetilde{\boldsymbol{\tau}} + \int_\Omega \phi\, \mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s;\Omega)\,. \tag{2.8}$$

It is clear, thanks to (1.9) and Cauchy-Schwarz's inequality, that the first term of (2.8) makes sense for $\widetilde{\boldsymbol{\sigma}} \in \mathbf{L}^2(\Omega)$. In addition, formally testing the second equation of the second row of (1.1) against a function ψ, yields

$$\int_\Omega \psi\, \mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = -\int_\Omega g(\boldsymbol{u})\psi\,, \tag{2.9}$$

whose right hand side has a similar structure to (2.4). Hence, analogously to (2.5) and (2.6), and since $\boldsymbol{u} \in \mathbf{L}^r(\Omega)$ and $r > s$, Hölder's inequality allows us to conclude that it suffices to take ψ in $\mathrm{L}^r(\Omega)$. In fact, thanks to the Lipschitz continuity property of g (cf. (1.13)), we get

$$\left|\int_\Omega (g(\boldsymbol{u}) - g(\boldsymbol{v}))\psi\right| \leqslant L_g \,\|\boldsymbol{u}-\boldsymbol{v}\|_{0,r;\Omega}\,\|\psi\|_{0,s;\Omega} \leqslant |\Omega|^{\frac{r-s}{rs}}\, L_g\,\|\boldsymbol{u}-\boldsymbol{v}\|_{0,r;\Omega}\,\|\psi\|_{0,r;\Omega}\,, \tag{2.10}$$

from which we deduce that the left hand side of (2.9) is finite if $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{L}^s(\Omega)$, and hence we will look for $\widetilde{\boldsymbol{\sigma}}$ in $\mathbf{H}(\mathrm{div}_s;\Omega)$ (cf. (1.3)). According to the foregoing discussion, we now set the following Banach spaces

$$\mathbf{Q} := \mathbf{H}(\mathrm{div}_s;\Omega) \quad \text{and} \quad \mathrm{M} := \mathrm{L}^r(\Omega)\,, \tag{2.11}$$

so that, given $(\boldsymbol{\sigma},\boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$, the mixed formulation for (2.7) reduces to: Find $(\widetilde{\boldsymbol{\sigma}},\phi) \in \mathbf{Q}\times\mathrm{M}$ such that

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}},\phi) &= 0 && \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\psi) &= \widetilde{G}_{\boldsymbol{u}}(\psi) && \forall\,\psi\in\mathrm{M}\,, \end{aligned} \tag{2.12}$$

where, the bilinear forms $\widetilde{a}_{\boldsymbol{\sigma}} : \mathbf{Q}\times\mathbf{Q} \to \mathrm{R}$, $\widetilde{b} : \mathbf{Q}\times\mathrm{M} \to \mathrm{R}$, and the functional $G_{\boldsymbol{u}} \in \mathrm{M}$, are defined, respectively, as

$$\widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) := \int_\Omega \widetilde{\vartheta}(\boldsymbol{\sigma})\,\widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\tau}} \qquad \forall\,\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \tag{2.13}$$

$$\widetilde{b}(\widetilde{\boldsymbol{\tau}},\psi) = \int_\Omega \psi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\psi) \in \mathbf{Q}\times\mathrm{M}\,, \tag{2.14}$$

and

$$\widetilde{G}_{\boldsymbol{u}}(\psi) := -\int_\Omega g(\boldsymbol{u})\,\psi \qquad \forall\,\psi\in\mathrm{M}\,. \tag{2.15}$$

Next, a direct application of Hölder's inequality, and the bounds given by (1.11) and (1.13), allow to conclude that the bilinear forms $\widetilde{a}$ and $\widetilde{b}$, and the functional $\widetilde{G}_{\boldsymbol{u}}$, are all bounded with the corresponding norms given by

$$\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} := \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_s;\Omega} \quad \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q} \quad \text{and} \quad \|\psi\|_{\mathrm{M}} := \|\psi\|_{0,r;\Omega} \quad \forall\,\psi\in\mathrm{M}\,.$$

In fact, there exist positive constants, given by

$$\|\widetilde{a}_{\boldsymbol{\sigma}}\| = \widetilde{\vartheta}_2\,, \quad \|\widetilde{b}\| = 1\,, \quad \text{and} \quad \|\widetilde{G}_{\boldsymbol{u}}\| = g_2\,|\Omega|^{1/s}\,, \tag{2.16}$$

such that

$$\begin{aligned} |\widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}})| &\leqslant \|\widetilde{a}_{\boldsymbol{\sigma}}\|\,\|\widetilde{\boldsymbol{\zeta}}\|_{\mathbf{Q}}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} && \forall\,\widetilde{\boldsymbol{\zeta}},\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,,\\ |\widetilde{b}(\widetilde{\boldsymbol{\tau}},\psi)| &\leqslant \|\widetilde{b}\|\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}}\,\|\psi\|_{\mathrm{M}} && \forall\,(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M}\,, \end{aligned}$$

and

$$|\widetilde{G}_{\boldsymbol{u}}(\psi)| \leqslant \|\widetilde{G}_{\boldsymbol{u}}\|\,\|\psi\|_{\mathrm{M}} \qquad \forall\,\psi\in\mathrm{M}\,.$$

2.3 A second approach for the diffusion equation

As an alternative to the previous formulation for the diffusion equation, and in order to obtain a more accurate approximation for the diffusion gradient, as well as to avoid inverting ϑ, we introduce the unknown $\boldsymbol{t} := \nabla\phi$ in Ω. Thus, the second row of (1.1) becomes

$$\begin{aligned} &\boldsymbol{t} = \nabla\phi \quad \text{in} \quad \Omega\,, \qquad \widetilde{\boldsymbol{\sigma}} = \vartheta(\boldsymbol{\sigma})\,\boldsymbol{t} \quad \text{in} \quad \Omega\,,\\ &\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) = g(\boldsymbol{u}) \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \phi = 0 \quad \text{on} \quad \Gamma\,. \end{aligned} \tag{2.17}$$

Then, bearing in mind that ϕ must be sought in $\mathrm{L}^r(\Omega)$, and thanks to the continuous embedding of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$, we initially look for ϕ in $\mathrm{H}^1(\Omega)$. In this way, testing the first equation of (2.17) against $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s;\Omega)$, applying (1.7), with s specified in (2.1), and employing the Dirichlet boundary condition for ϕ, we obtain

$$\int_\Omega \boldsymbol{t}\cdot\widetilde{\boldsymbol{\tau}} \,+\, \int_\Omega \phi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \;=\; 0 \qquad \forall\,\widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s;\Omega)\,,$$

whence the first term makes sense for $\boldsymbol{t} \in \mathbf{L}^2(\Omega)$. In turn, testing the second equation of (2.17) against $\boldsymbol{s} \in \mathbf{L}^2(\Omega)$, we formally get

$$\int_\Omega \vartheta(\boldsymbol{\sigma})\,\boldsymbol{t}\cdot\boldsymbol{s} \,-\, \int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\boldsymbol{s} \;=\; 0 \qquad \forall\,\boldsymbol{s} \in \mathbf{L}^2(\Omega)\,, \tag{2.18}$$

from which we notice, thanks to Cauchy-Schwarz's inequality and (1.9), that the first term of (2.18) is finite, whereas its second term makes sense is $\widetilde{\boldsymbol{\sigma}}$ is sought in $\mathbf{L}^2(\Omega)$. Now, testing the third equation of (2.17) against a function φ, we have

$$\int_\Omega \varphi\,\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \;=\; \int_\Omega g(\boldsymbol{u})\,\varphi\,, \tag{2.19}$$

and, similarly to (2.10), we deduce from the right side of (2.19) that φ can be considered in $\mathrm{L}^r(\Omega)$. Hence, in order for the left hand side of (2.19) to be well defined we need that $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{L}^s(\Omega)$, which yields to look for $\widetilde{\boldsymbol{\sigma}}$ in $\mathbf{H}(\mathrm{div}_s;\Omega)$. Consequently, recalling from (2.11) the definition of M, we introduce the following notation

$$\vec{\phi} \,:=\, (\phi,\boldsymbol{t}), \quad \vec{\varphi} \,:=\, (\varphi,\boldsymbol{s}) \,\in\, \mathbf{H} \,:=\, \mathrm{M}\times\mathbf{L}^2(\Omega)\,.$$

Thus, given $(\boldsymbol{\sigma},\boldsymbol{u}) \in \mathbf{X}_2\times\mathbf{M}_1$, we arrive at the following mixed formulation for (2.17): Find $(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi}) \,+\, b(\vec{\varphi},\widetilde{\boldsymbol{\sigma}}) \;&=\; G_{\boldsymbol{u}}(\vec{\varphi}) \quad \forall\,\vec{\varphi}\in\mathbf{H}\,,\\ b(\vec{\phi},\widetilde{\boldsymbol{\tau}}) \;&=\; 0 \qquad\quad \forall\,\widetilde{\boldsymbol{\tau}}\in\mathbf{Q}\,, \end{aligned} \tag{2.20}$$

where the bilinear forms $a_{\boldsymbol{\sigma}} : \mathbf{H}\times\mathbf{H} \to \mathrm{R}$ and $b : \mathbf{H}\times\mathbf{Q} \to \mathrm{R}$ are defined as

$$a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi}) \;=\; \int_\Omega \vartheta(\boldsymbol{\sigma})\,\boldsymbol{t}\cdot\boldsymbol{s} \quad \forall\,\vec{\phi},\,\vec{\varphi}\in\mathbf{H}\,, \quad \text{and} \tag{2.21}$$

$$b(\vec{\varphi},\widetilde{\boldsymbol{\tau}}) \;=\; -\int_\Omega \widetilde{\boldsymbol{\tau}}\cdot\boldsymbol{s} \,-\, \int_\Omega \varphi\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \quad \forall\,(\vec{\varphi},\widetilde{\boldsymbol{\tau}})\in\mathbf{H}\times\mathbf{Q}\,, \tag{2.22}$$

whereas the linear functional $G_{\boldsymbol{u}} : \mathbf{H} \to \mathrm{R}$ is given by

$$G_{\boldsymbol{u}}(\vec{\varphi}) \;=\; -\int_\Omega g(\boldsymbol{u})\,\varphi \quad \forall\,\vec{\varphi}\in\mathbf{H}\,. \tag{2.23}$$

Next, it is easily seen that $a_{\boldsymbol{\sigma}}$, b and $G_{\boldsymbol{u}}$ are bounded. In fact, endowing $\mathbf{H}$ with the product norm

$$\|\vec{\varphi}\|_{\mathbf{H}} \,:=\, \|\varphi\|_{0,r;\Omega} \,+\, \|\boldsymbol{s}\|_{0,\Omega} \quad \forall\,\vec{\varphi} := (\varphi,\boldsymbol{s})\in\mathbf{H}\,,$$

and applying (1.9), (1.13), and the Cauchy-Schwarz and Hölder inequalities, we find that there exist positive constants, denoted and given by

$$\|a_{\boldsymbol{\sigma}}\| \;=\; \vartheta_2\,, \quad \|b\| \;=\; 1\,, \quad \text{and} \quad \|G_{\boldsymbol{u}}\| \;=\; g_2\,|\Omega|^{1/s}\,, \tag{2.24}$$

such that

$$\begin{aligned} |a_{\boldsymbol{\sigma}}(\vec{\phi},\vec{\varphi})| \;&\leqslant\; \|a_{\boldsymbol{\sigma}}\|\,\|\vec{\phi}\|_{\mathbf{H}}\,\|\vec{\varphi}\|_{\mathbf{H}} \qquad \forall\,\vec{\phi},\,\vec{\varphi}\in\mathbf{H}\,,\\ |b(\vec{\varphi},\widetilde{\boldsymbol{\tau}})| \;&\leqslant\; \|b\|\,\|\vec{\varphi}\|_{\mathbf{H}}\,\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \qquad \forall\,(\vec{\varphi},\widetilde{\boldsymbol{\tau}})\in\mathbf{H}\times\mathbf{Q}\,, \end{aligned}$$

and

$$|G_{\boldsymbol{u}}(\vec{\varphi})| \;\leqslant\; \|G_{\boldsymbol{u}}|\,\|\vec{\varphi}\|_{\mathbf{H}} \qquad \forall\,\vec{\varphi}\in\mathbf{H}\,.$$

2.4 The coupled fully-mixed formulations

According to the analyses in Sections 2.1 and 2.2, our first fully-mixed formulation for (1.1)-(1.2) reduces to gathering (2.2) and (2.12), that is: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ and $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q} \times \mathrm{M}$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \boldsymbol{u}) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbf{X}_1, \\
\mathbf{b}_2(\boldsymbol{\sigma}, \boldsymbol{v}) &= F_\phi(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in \mathbf{M}_2, \\
\widetilde{a}_{\boldsymbol{\sigma}}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \phi) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}, \\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \psi) &= \widetilde{G}_{\boldsymbol{u}}(\psi) && \forall\, \psi \in \mathrm{M}.
\end{aligned}
\tag{2.25}
$$

In turn, as a consequence of the discussions in Sections 2.1 and 2.3, the second fully-mixed formulation for (1.1)-(1.2) is given by (2.2) jointly with (2.20), that is: Find $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ and $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \boldsymbol{u}) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbf{X}_1, \\
\mathbf{b}_2(\boldsymbol{\sigma}, \boldsymbol{v}) &= F_\phi(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in \mathbf{M}_2, \\
a_{\boldsymbol{\sigma}}(\vec{\phi}, \vec{\varphi}) + b(\vec{\varphi}, \widetilde{\boldsymbol{\sigma}}) &= G_{\boldsymbol{u}}(\vec{\varphi}) && \forall\, \vec{\varphi} \in \mathbf{H}, \\
b(\vec{\phi}, \widetilde{\boldsymbol{\tau}}) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}.
\end{aligned}
\tag{2.26}
$$

3 The continuous solvability analysis

In this section we proceed similarly as in [8] and [15] (see also [5], [17], and some of the references therein), and adopt a fixed-point strategy to analyze the solvability of (2.25) and (2.26). To this end, we use the Babuska-Brezzi theory in Banach spaces (cf. [4, Theorem 2.1, Corollary 2.1, Section 2.1] for the general case, and [10, Theorem 2.34] for a particular one) to prove the well-posedness of the uncoupled problems (2.2), (2.12), and (2.20).

3.1 Well-posedness of the elasticity equation

We begin by letting $\mathbf{S} : \mathrm{M} \to \mathbf{X}_2 \times \mathbf{M}_1$ be the operator defined by

$$
\mathbf{S}(\varphi) = (\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)) := (\boldsymbol{\sigma}, \boldsymbol{u}) \qquad \forall\, \varphi \in \mathrm{M}, \tag{3.1}
$$

where $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ is the unique solution (to be confirmed below) of the mixed formulation for the elasticity equation (cf. (2.2)) with φ instead of ϕ, that is

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}, \boldsymbol{\tau}) + \mathbf{b}_1(\boldsymbol{\tau}, \widetilde{\boldsymbol{u}}) &= G(\boldsymbol{\tau}) && \forall\, \boldsymbol{\tau} \in \mathbf{X}_1, \\
\mathbf{b}_2(\boldsymbol{\sigma}, \boldsymbol{v}) &= F_\varphi(\boldsymbol{v}) && \forall\, \boldsymbol{v} \in \mathbf{M}_2.
\end{aligned}
\tag{3.2}
$$

Then, assuming that the Lamé parameter λ is sufficiently large, namely $\lambda > M$, where M is specified in [14, Lemmas 3.4], we can establish that the operator $\mathbf{S}$ (cf. (3.1)) is well defined. Indeed, letting $\boldsymbol{\alpha}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$ be the constants yielding the continuous inf-sup conditions for $\mathbf{a}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ (cf. [14, Lemmas 3.4 and 3.5]), a simple application of [4, Theorem 2.1, Corollary 2.1, Section 2.1] leads to the following result (cf. [14, Lemma 3.6]).

Lemma 3.1. *For each $\varphi \in \mathrm{M}$ there exists a unique $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$ solution of (3.2), and hence one can define $\mathbf{S}(\varphi) = \big(\mathbf{S}_1(\varphi), \mathbf{S}_2(\varphi)\big) := (\boldsymbol{\sigma}, \boldsymbol{u}) \in \mathbf{X}_2 \times \mathbf{M}_1$. Moreover, there hold*

$$
\begin{aligned}
\|\mathbf{S}_1(\varphi)\|_{\mathbf{X}_2} &= \|\boldsymbol{\sigma}\|_{\mathbf{X}_2} \leqslant \frac{C_r}{\boldsymbol{\alpha}} \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} + \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2, \quad \textit{and} \\
\|\mathbf{S}_2(\varphi)\|_{\mathbf{M}_1} &= \|\boldsymbol{u}\|_{\mathbf{M}_1} \leqslant \frac{C_r}{\boldsymbol{\beta}_1} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} + \frac{2|\Omega|^{1/r}}{\mu \boldsymbol{\beta}_1 \boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2.
\end{aligned}
\tag{3.3}
$$

3.2 Well-posedness of the first approach for the diffusion equation

We now let $\widetilde{\mathrm{S}} : \mathbf{X}_2 \times \mathbf{M}_1 \to \mathbf{Q} \times \mathrm{M}$ be the operator defined by

$$\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) \,=\, (\widetilde{\mathrm{S}}_1(\boldsymbol{\zeta}, \boldsymbol{w}), \widetilde{\mathrm{S}}_2(\boldsymbol{\zeta}, \boldsymbol{w})) \,:=\, (\widetilde{\boldsymbol{\sigma}}, \phi) \qquad \forall\, (\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1, \tag{3.4}$$

where $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q} \times \mathrm{M}$ is the unique solution (to be confirmed below) of (2.12) with $(\boldsymbol{\zeta}, \boldsymbol{w})$ instead of $(\boldsymbol{\sigma}, \boldsymbol{u})$, that is

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \phi) &\;=\; 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \psi) &\;=\; G_{\boldsymbol{w}}(\psi) && \forall\, \psi \in \mathrm{M}\,. \end{aligned} \tag{3.5}$$

Next, we let $\widetilde{\mathcal{K}}$ be the kernel of the bilinear form $\widetilde{b}$ (cf. (2.14)), which reduces to

$$\widetilde{\mathcal{K}} \,:=\, \Big\{ \widetilde{\boldsymbol{\tau}} \in \mathbf{H}(\mathrm{div}_s; \Omega) : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}) = 0 \Big\}\,.$$

Then, bearing in mind the uniform positiveness of $\widetilde{\vartheta}$ (cf. (1.10)), the definition of $\widetilde{a}_{\boldsymbol{\zeta}}$ (cf. (2.13)), and the norm of $\mathbf{H}(\mathrm{div}_s; \Omega)$ (cf. (1.5)), we readily deduce that

$$\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\tau}}, \widetilde{\boldsymbol{\tau}}) \,\geqslant\, \widetilde{\vartheta}_0 \|\widetilde{\boldsymbol{\tau}}\|^2_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \widetilde{\mathcal{K}}, \quad \forall\, \boldsymbol{\zeta} \in \mathbf{X}_2\,, \tag{3.6}$$

which yields the continuous inf-sup condition for $\widetilde{a}_{\boldsymbol{\zeta}}$ (cf. [10, eq. (2.28), Theorem 2.34]) with constant $\widetilde{\alpha} = \widetilde{\vartheta}_0$. In addition, we know from [15, Lemma 2.9] that there exists a positive constant $\widetilde{\beta}$ such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}} \in \mathbf{Q} \\ \widetilde{\boldsymbol{\tau}} \neq 0}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \psi)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}}} \,\geqslant\, \widetilde{\beta} \|\psi\|_{\mathrm{M}} \qquad \forall\, \psi \in \mathrm{M}\,, \tag{3.7}$$

which establishes the continuous inf-sup condition for $\widetilde{b}$.

Hence, we are in position to state that the operator $\widetilde{\mathrm{S}}$ is well-defined.

Lemma 3.2. *For each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there exists a unique $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q} \times \mathrm{M}$ solution of (3.5), and hence one can define $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) := (\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q} \times \mathrm{M}$. Moreover, there hold*

$$\|\widetilde{\mathrm{S}}_1(\boldsymbol{\zeta}, \boldsymbol{w})\|_{\mathbf{Q}} \,=\, \|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,\leqslant\, \frac{1}{\widetilde{\beta}} \Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big) |\Omega|^{1/s} g_2\,, \quad \textit{and} \tag{3.8}$$

$$\|\widetilde{\mathrm{S}}_2(\boldsymbol{\zeta}, \boldsymbol{w})\|_{\mathrm{M}} \,=\, \|\phi\|_{\mathrm{M}} \,\leqslant\, \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}^2} \Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big) |\Omega|^{1/s} g_2\,. \tag{3.9}$$

Proof. Knowing from (3.6) and (3.7) that, given $(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$, $\widetilde{a}_{\boldsymbol{\zeta}}$ and $\widetilde{b}$ satisfies the hypotheses of [10, Theorem 2.34], and noting that $\mathbf{Q} := \mathbf{H}(\mathrm{div}_s; \Omega)$ and $\mathrm{M} := \mathrm{L}^r(\Omega)$ are reflexive Banach spaces, the proof reduces to a straightforward application of the aforementioned theorem. In this way, the a priori estimates (3.8) and (3.9) follow from [10, eq. (2.30), Theorem 2.34] and (2.16). $\square$

3.3 Well-posedness of the second approach for the diffusion equation

Similarly to the analysis of previous sections, we let $\mathrm{S} : \mathbf{X}_2 \times \mathbf{M}_1 \to \mathbf{H}$ be the operator given by

$$\mathrm{S}(\boldsymbol{\zeta}, \boldsymbol{w}) \,=\, (\mathrm{S}_1(\boldsymbol{\zeta}, \boldsymbol{w}), \mathrm{S}_2(\boldsymbol{\zeta}, \boldsymbol{w})) \,:=\, \vec{\phi} \quad \forall\, (\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1\,, \tag{3.10}$$

where $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) := ((\phi, \boldsymbol{t}), \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of problem (2.20) with $(\boldsymbol{\zeta}, \boldsymbol{w})$ instead of $(\boldsymbol{\sigma}, \boldsymbol{u})$, that is

$$\begin{aligned} a_{\boldsymbol{\zeta}}(\vec{\phi}, \vec{\varphi}) \,+\, b(\vec{\varphi}, \widetilde{\boldsymbol{\sigma}}) &\;=\; G_{\boldsymbol{w}}(\vec{\varphi}) && \forall\, \vec{\varphi} \in \mathbf{H}\,, \\ b(\vec{\phi}, \widetilde{\boldsymbol{\tau}}) &\;=\; 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,. \end{aligned} \tag{3.11}$$

Here we apply [10, Theorem 2.34] to prove that problem (3.11) is well-posed (equivalently, that S is well-defined). In this regard, it is important to stress that the structure of (3.11) is similar to the one of [8, eq. (3.23)], and hence, several results and techniques from there will be employed in what follows. Indeed, let V the kernel of the operator induced by b (cf. (2.22), which reduces to

$$V \,:=\, \Big\{ \vec{\varphi} = (\varphi, \boldsymbol{s}) \in \mathbf{H} := \mathrm{M} \times \mathbf{L}^2(\Omega) : \quad \nabla\varphi \,=\, \boldsymbol{s} \Big\} . \tag{3.12}$$

Now, we let c_P be the positive constant yielding the Friedrichs-Poincaré inequality, which states that $|\varphi|^2_{1,\Omega} \geqslant c_P \|\varphi\|^2_{1,\Omega}$ for all $\varphi \in \mathrm{H}^1_0(\Omega)$, and denote by i_r the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$. In addition, we consider an arbitrary $\boldsymbol{\zeta} \in \mathbf{X}_2$. Then, bearing in mind (1.8) and proceeding analogously to the proof of [8, eq. (3.41), Lemma 3.2], we find that

$$a_{\boldsymbol{\zeta}}(\vec{\varphi}, \vec{\varphi}) \,\geqslant\, \alpha \|\vec{\varphi}\|^2_{\mathbf{H}} \quad \forall\, \vec{\varphi} \in V\,, \tag{3.13}$$

with

$$\alpha \,:=\, \frac{\vartheta_0}{2} \min\{1, \frac{c_P}{\|i_r\|}\}\,,$$

which proves the V-ellipticity of $a_{\boldsymbol{\zeta}}$. In turn, a slight modification of the proof of [8, Lemma 3.3] allows us to prove the existence of a positive constant β such that

$$\sup_{\substack{\vec{\varphi} \in \mathbf{H} \\ \vec{\varphi} \neq \mathbf{0}}} \frac{b(\vec{\varphi}, \widetilde{\boldsymbol{\tau}})}{\|\vec{\varphi}\|_{\mathbf{H}}} \,\geqslant\, \beta \|\widetilde{\boldsymbol{\tau}}\|_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \tag{3.14}$$

whence the bilinear form b satisfies the continuous inf-sup condition required by [10, Theorem 2.34].

We are now in position to confirm that the operator S is well-defined.

Lemma 3.3. *For each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there exists a unique $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ solution of (3.11), and hence one can define $\mathrm{S}(\boldsymbol{\zeta}, \boldsymbol{w}) := \vec{\phi} \in \mathbf{H}$. Moreover, there holds*

$$\|\mathrm{S}(\boldsymbol{\zeta}, \boldsymbol{w})\|_{\mathbf{H}} \,=\, \|\vec{\phi}\|_{\mathbf{H}} \,=\, \|\phi\|_{0,r;\Omega} + \|\boldsymbol{t}\|_{0,\Omega} \,\leqslant\, \frac{|\Omega|^{1/s}}{\alpha}\, g_2\,. \tag{3.15}$$

Proof. Thanks to (2.24), (3.13) and (3.14), a straightforward application of [10, Theorem 2.34] yields the existence of a unique solution $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) \in \mathbf{H} \times \mathbf{Q}$ to (2.20). Moreover, the corresponding a priori estimate given by the first inequality of [10, eq. (2.30)], along with the expression for $\|G_{\boldsymbol{w}}\|$ provided by (2.24), lead to (3.15). □

Regarding the a priori estimate for the component $\widetilde{\boldsymbol{\sigma}}$ of the unique solution of (2.20), which will be used later on, we recall that the second inequality in [10, eq. (2.30)] and (2.24) implies

$$\|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,\leqslant\, \frac{|\Omega|^{1/s}}{\beta} \left(1 \,+\, \frac{\vartheta_2}{\alpha}\right) g_2\,. \tag{3.16}$$

3.4 Solvability of the first fully-mixed formulation

We begin by defining the compose operator $\Xi : \mathrm{M} \to \mathrm{M}$ as

$$\Xi(\psi) \,:=\, \widetilde{\mathrm{S}}_2\big(\mathbf{S}(\psi)\big) \qquad \forall\, \psi \in\, \mathrm{M}\,. \tag{3.17}$$

Then, knowing that the operators $\widetilde{\mathrm{S}}$ and $\mathbf{S}$, and hence Ξ as well, are well-defined, we notice that solving (2.25) is equivalent to seeking a fixed point of Ξ, that is: Find $\psi \in \mathrm{M}$ such that

$$\Xi(\psi) \,=\, \psi\,. \tag{3.18}$$

Next, in order to address the solvability of (3.18) (equivalently of (2.25)), we verify the hypotheses of the Banach fixed-point theorem. For this purpose, let us first introduce the ball

$$\widetilde{W} := \left\{ \phi \in \mathrm{M} : \quad \|\phi\|_{\mathrm{M}} \leqslant \widetilde{\delta} \right\}, \tag{3.19}$$

with

$$\widetilde{\delta} := \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}^2} \left(1 + \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}} \right) |\Omega|^{1/s} g_2 \,.$$

It follows from the definition of Ξ (cf. (3.17)) and the a priori estimate for $\widetilde{\mathrm{S}}_2$ (cf. (3.9)) that

$$\Xi(\widetilde{W}) \subseteq \widetilde{W} \,. \tag{3.20}$$

Now, in order to establish the continuity of Ξ, we previously establish those of $\mathbf{S}$ and $\widetilde{\mathrm{S}}$. Indeed, resorting to a slight modification of [14, Lemma 3.9], we deduce the existence of a positive constant $C_{\mathbf{S}}$, depending only on μ, $\boldsymbol{\alpha}$, $\boldsymbol{\beta}_1$, and $\boldsymbol{\beta}_2$, such that

$$\|\mathbf{S}(\phi) - \mathbf{S}(\varphi)\|_{\mathbf{X}_2 \times \mathbf{M}_1} \leqslant C_{\mathbf{S}} \, L_f \, \|\phi - \varphi\|_{\mathrm{M}} \qquad \forall \, \phi, \, \varphi \in \mathrm{M} \,, \tag{3.21}$$

which proves the Lipschitz-continuity of $\mathbf{S}$. Furthermore, for the same property of $\widetilde{\mathrm{S}}$, the approach from several previous works (see, e.g. [1], [9], [12], [13], and [15]) is adopted here, so that a regularity assumption on the solution of the problem defining this operator is introduced. More precisely, from now on we suppose that there exists $\varepsilon \geqslant \frac{n}{r}$ and a positive constant $\widetilde{C}_\varepsilon$, such that

($\mathbf{RA}_1$) for each $(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there holds $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) = (\widetilde{\boldsymbol{\sigma}}, \phi) \in (\mathbf{Q} \cap \mathbf{H}^\varepsilon(\Omega)) \times \mathrm{W}^{\varepsilon,r}(\Omega)$, and

$$\|\widetilde{\boldsymbol{\sigma}}\|_{\varepsilon,\Omega} + \|\phi\|_{\varepsilon,r;\Omega} \leqslant \widetilde{C}_\varepsilon \, g_2 \,. \tag{3.22}$$

The aforementioned lower bound of ε is explained within the proof of Lemma 3.4 below, which provides the Lipschitz-continuity of $\widetilde{\mathrm{S}}$. In this regard, we recall now that for each $\varepsilon < \frac{n}{2}$ there holds $\mathbf{H}^\varepsilon(\Omega) \subset \mathbf{L}^{\varepsilon^*}(\Omega)$, with continuous injection

$$i_\varepsilon : \mathbf{H}^\varepsilon(\Omega) \longrightarrow \mathbf{L}^{\varepsilon^*}(\Omega) \,, \quad \text{where} \quad \varepsilon^* = \frac{2n}{n - 2\varepsilon} \,. \tag{3.23}$$

Note that the indicated lower and upper bounds for the additional regularity ε, which turn out to require that $\varepsilon \in [\frac{n}{r}, \frac{n}{2})$, are compatible if and only if $r > 2$, which is coherent with the range stipulated in (2.1). Thus, we have the following result.

Lemma 3.4. *There exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, $|\Omega|$, r, ε, $\|i_\varepsilon\|$ (cf. (3.23)), and $\widetilde{C}_\varepsilon$ (cf. (3.22)), such that*

$$\|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v})\|_{\mathbf{Q} \times \mathrm{M}} \leqslant C_{\widetilde{\mathrm{S}}} \left\{ L_{\widetilde{\vartheta}} \, g_2 + L_g \right\} \|(\boldsymbol{\zeta}, \boldsymbol{w}) - (\boldsymbol{\tau}, \boldsymbol{v})\|_{\mathbf{X}_2 \times \mathbf{M}_1} \tag{3.24}$$

for all $(\boldsymbol{\zeta}, \boldsymbol{w})$, $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbf{X}_2 \times \mathbf{M}_1$.

Proof. We begin by noticing that the a priori estimates (3.8) and (3.9) of problem (3.5), with a given $(\boldsymbol{\zeta}, \boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$, are equivalent to stating that

$$\|(\widetilde{\boldsymbol{\zeta}}, \varphi)\|_{\mathbf{Q} \times \mathrm{M}} \leqslant C \sup_{\substack{(\widetilde{\boldsymbol{\tau}}, \psi) \in \mathbf{Q} \times \mathrm{M} \\ (\widetilde{\boldsymbol{\tau}}, \psi) \neq 0}} \frac{\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \varphi) + \widetilde{b}(\widetilde{\boldsymbol{\zeta}}, \psi)}{\|(\widetilde{\boldsymbol{\tau}}, \psi)\|_{\mathbf{Q} \times \mathrm{M}}} \quad \forall \, (\widetilde{\boldsymbol{\zeta}}, \varphi) \in \mathbf{Q} \times \mathrm{M} \,, \tag{3.25}$$

with a positive constant C that depends only on $\widetilde{\vartheta}_2$, $\widetilde{\alpha}$, and $\widetilde{\beta}$, and hence independent of $(\boldsymbol{\zeta}, \boldsymbol{w})$. Next, given $(\boldsymbol{\zeta}, \boldsymbol{w})$, $(\boldsymbol{\tau}, \boldsymbol{v}) \in \mathbf{X}_2 \times \mathbf{M}_1$, we let

$$\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) := (\widetilde{\boldsymbol{\sigma}}, \phi) \quad \text{and} \quad \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v}) := (\widetilde{\boldsymbol{\zeta}}, \varphi) \,,$$

which, according to (3.4) and (3.5), means, respectively, that

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \phi) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \psi) &= G_{\boldsymbol{w}}(\psi) && \forall\, \psi \in \mathrm{M}\,, \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} \widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \varphi) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}} \in \mathbf{Q}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\zeta}}, \psi) &= G_{\boldsymbol{v}}(\psi) && \forall\, \psi \in \mathrm{M}\,. \end{aligned} \tag{3.27}$$

Then, applying (3.25) to $\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v}) = (\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\zeta}}, \phi - \varphi)$, and using (3.26) and (3.27), we get

$$\begin{aligned} \|\widetilde{\mathrm{S}}(\boldsymbol{\zeta}, \boldsymbol{w}) - \widetilde{\mathrm{S}}(\boldsymbol{\tau}, \boldsymbol{v})\|_{\mathbf{Q}\times\mathrm{M}} &\leqslant C \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M} \\ (\widetilde{\boldsymbol{\tau}},\psi)\neq 0}} \frac{\widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \phi - \varphi) + \widetilde{b}(\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\zeta}}, \psi)}{\|(\widetilde{\boldsymbol{\tau}}, \psi)\|_{\mathbf{Q}\times\mathrm{M}}} \\ &\leqslant C \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\psi)\in\mathbf{Q}\times\mathrm{M} \\ (\widetilde{\boldsymbol{\tau}},\psi)\neq 0}} \frac{\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) - \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) + (G_{\boldsymbol{w}} - G_{\boldsymbol{v}})(\psi)}{\|(\widetilde{\boldsymbol{\tau}}, \psi)\|_{\mathbf{Q}\times\mathrm{M}}}\,. \end{aligned} \tag{3.28}$$

Thus, bearing in mind the definitions of $\widetilde{a}_{\boldsymbol{\tau}}$ and $\widetilde{a}_{\boldsymbol{\zeta}}$, and using the Lipschitz-continuity of $\widetilde{\vartheta}$ (cf. (1.11)) along with the Cauchy-Schwarz and Hölder inequalities, we find that

$$|\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) - \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}})| \leqslant L_{\widetilde{\vartheta}}\, \|(\boldsymbol{\tau} - \boldsymbol{\zeta})\, \widetilde{\boldsymbol{\zeta}}\|_{0,\Omega}\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \leqslant L_{\widetilde{\vartheta}}\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,2q;\Omega}\, \|\widetilde{\boldsymbol{\zeta}}\|_{0,2p,\Omega}\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\,, \tag{3.29}$$

where p, $q \in (1, +\infty)$ are conjugate to each other. Now, choosing p such that $2p = \varepsilon^*$ (cf. (3.23)), we get $2q = \frac{n}{\varepsilon}$, which, according to the range stipulated for ε, yields $2q \leqslant r$, and thus the norm of the embedding of $\mathbf{L}^r(\Omega)$ into $\mathbf{L}^{2q}(\Omega) = \mathbf{L}^{\frac{n}{\varepsilon}}(\Omega)$ is given by $C_{r,\varepsilon} := |\Omega|^{\frac{r\varepsilon - n}{rn}}$. In this way, using additionally the continuity of i_ε (cf. (3.23)) along with the regularity estimate (3.22), the inequality (3.29) becomes

$$\begin{aligned} |\widetilde{a}_{\boldsymbol{\tau}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) - \widetilde{a}_{\boldsymbol{\zeta}}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}})| &\leqslant L_{\widetilde{\vartheta}}\, C_{r,\varepsilon}\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{0,r;\Omega}\, \|i_\varepsilon\|\, \|\widetilde{\boldsymbol{\zeta}}\|_{\varepsilon,\Omega}\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \\ &\leqslant L_{\widetilde{\vartheta}}\, C_{r,\varepsilon}\, \|i_\varepsilon\|\, \widetilde{C}_\varepsilon\, g_2\, \|\boldsymbol{\tau} - \boldsymbol{\zeta}\|_{\mathbf{X}_2}\, \|(\widetilde{\boldsymbol{\tau}}, \psi)\|_{\mathbf{Q}\times\mathrm{M}}\,. \end{aligned} \tag{3.30}$$

In turn, the Lipschitz-continuity of g (cf. (1.13)), the fact that $s < r$ (cf. (2.1)), and Hölder's inequality, yield

$$\begin{aligned} |(G_{\boldsymbol{w}} - G_{\boldsymbol{v}})(\psi)| &\leqslant L_g\, \|\boldsymbol{w} - \boldsymbol{v}\|_{0,r;\Omega}\, \|\psi\|_{0,s;\Omega} \leqslant L_g\, |\Omega|^{\frac{r-s}{rs}}\, \|\boldsymbol{w} - \boldsymbol{v}\|_{0,r;\Omega}\, \|\psi\|_{0,r;\Omega} \\ &\leqslant L_g\, |\Omega|^{\frac{r-s}{rs}}\, \|\boldsymbol{w} - \boldsymbol{v}\|_{\mathbf{M}_2}\, \|(\widetilde{\boldsymbol{\tau}}, \psi)\|_{\mathbf{Q}\times\mathrm{M}}\,. \end{aligned} \tag{3.31}$$

Finally, replacing (3.30) and (3.31) back into (3.28), we arrive at (3.24), which ends the proof. □

We are able to prove now the Lipschitz-continuity of Ξ in the closed ball $\widetilde{W}$ of $\mathrm{M} := \mathrm{L}^r(\Omega)$.

Lemma 3.5. *There exists a positive constant C_Ξ, depending only on $C_{\mathbf{S}}$ and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\|\Xi(\phi) - \Xi(\varphi)\|_{\mathrm{M}} \leqslant C_\Xi\, L_f \left\{ L_g + L_{\widetilde{\vartheta}}\, g_2 \right\} \|\phi - \varphi\|_{\mathrm{M}} \qquad \forall\, \phi,\, \varphi \in \mathrm{M}\,. \tag{3.32}$$

Proof. It readily follows from the definition of Ξ (cf. (3.17)), and the estimates (3.21) and (3.24), which yields $C_\Xi := C_{\mathbf{S}}\, C_{\widetilde{\mathrm{S}}}$. □

Consequently, the main result of this subsection is stated as follows.

Theorem 3.6. *Assume the regularity assumption* ($\mathbf{RA_1}$) *(cf. (3.22)), and that the data L_f, L_g, L_θ, and g_2 are sufficiently small so that*

$$C_\Xi\, L_f \left\{ L_g + L_{\widetilde{\vartheta}}\, g_2 \right\} \,<\, 1\,. \tag{3.33}$$

Then, Ξ has a unique fixed point ϕ in $\widetilde{W}$. Equivalently, the coupled problem (2.25) has a unique solution $\big((\boldsymbol{\sigma},\boldsymbol{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$, with $\phi \in \widetilde{W}$ (cf. (3.19)). Moreover, there hold

$$\begin{aligned}
\|\boldsymbol{\sigma}\|_{\mathbf{X}_2} \,&\leqslant\, \frac{C_r}{\boldsymbol{\alpha}}\,\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,,\\
\|\boldsymbol{u}\|_{\mathbf{M}_1} \,&\leqslant\, \frac{C_r}{\boldsymbol{\beta}_1}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right)\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_1\boldsymbol{\beta}_2}\left(1+\frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,, \quad \text{and}\\
\|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,&\leqslant\, \frac{1}{\widetilde{\beta}_2}\Big(1\,+\,\frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}}\Big)|\Omega|^{1/s}g_2\,.
\end{aligned} \tag{3.34}$$

Proof. Thanks to (3.20), Lemma 3.5, and the assumption (3.33), the existence of a unique $\phi \in \widetilde{W}$ solution to (3.18) (equivalently, the existence of a unique $\big((\boldsymbol{\sigma},\boldsymbol{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$ solution to (2.25)), follows from a straightforward application of the Banach fixed point Theorem. In addition, noting that $(\boldsymbol{\sigma},\boldsymbol{u}) = \mathbf{S}(\phi)$ and $(\widetilde{\boldsymbol{\sigma}},\phi) = \widetilde{\mathrm{S}}(\boldsymbol{\sigma},\boldsymbol{u})$, the a priori estimates (3.3) and (3.8) yield (3.34), which ends the proof. $\square$

3.5 Solvability of the second fully-mixed formulation

Similarly to Section 3.4, for the solvability analysis of (2.26) we define the operator $\Lambda : \mathrm{M} \to \mathrm{M}$ as

$$\Lambda(\psi) \,:=\, \mathrm{S}_1\big(\mathbf{S}(\psi)\big) \qquad \forall\, \psi \in \mathrm{M}\,. \tag{3.35}$$

Then, noticing that S and $\mathbf{S}$, and hence Λ as well, are well-defined, we realize that solving (2.26) is equivalent to finding a fixed point of Λ, that is: Find $\psi \in \mathrm{M}$ such that

$$\Lambda(\psi) \,=\, \psi\,. \tag{3.36}$$

In what follows we show that Λ verifies the hypotheses of the respective Banach Theorem. We begin by defining the ball

$$W \,:=\, \Big\{\phi \in \mathrm{M} : \quad \|\phi\|_{\mathrm{M}} \,\leqslant\, \delta\Big\}\,, \tag{3.37}$$

with

$$\delta \,:=\, \frac{|\Omega|^{1/s}}{\alpha}\, g_2\,,$$

so that from the definition of Λ (cf. (3.35)) and the a priori estimate for S_1 (cf. (3.15)), we get

$$\Lambda(W) \,\subseteq\, W\,. \tag{3.38}$$

Next, in order to prove that Λ is Lipschitz-continuous, and similarly to ($\mathbf{RA}_1$)), we need to introduce a regularity hypothesis on the solution of the problem defining the operator S. More precisely, we assume that there exists $\varepsilon \,\geqslant\, \frac{n}{r}$ and a positive constant C_ε such that

($\mathbf{RA}_2$) for each $(\boldsymbol{\zeta},\boldsymbol{w}) \in \mathbf{X}_2 \times \mathbf{M}_1$ there hold $S(\boldsymbol{\zeta},\boldsymbol{w}) \,:=\, (\phi,\boldsymbol{t}) \in \mathrm{W}^{\varepsilon,r}(\Omega) \times \mathbf{H}^{\varepsilon}(\Omega)$, and

$$\|\phi\|_{\varepsilon,r;\Omega} \,+\, \|\boldsymbol{t}\|_{\varepsilon,\Omega} \,\leqslant\, C_\varepsilon\, g_2\,. \tag{3.39}$$

The Lipschitz-continuity of S is addressed next, whose corresponding proof requires again the lower bound of ε, and the embedding specified in (3.23). Therefore, following the same arguments from the previous section, we conclude that the feasible range for r and s are given by (2.1). The announced result is established as follows.

Lemma 3.7. *There exists a positive constant C_S, depending on α, $|\Omega|$, r, s, ε, $\|i_\varepsilon\|$ (cf. (3.23)), and C_ε (cf. (3.39)) such that*

$$\begin{aligned} &\|\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) \,-\, \mathrm{S}(\boldsymbol{\tau},\boldsymbol{v})\|_{\mathbf{H}} \\ &\qquad \leqslant\, C_{\mathrm{S}}\left\{L_g + L_\vartheta\, g_2\right\} \|(\boldsymbol{\zeta},\boldsymbol{w}) - (\boldsymbol{\tau},\boldsymbol{v})\|_{\mathbf{X}_2\times\mathbf{M}_1} \quad \forall\,(\boldsymbol{\zeta},\boldsymbol{w}),\,(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbf{X}_2\times\mathbf{M}_1\,. \end{aligned} \tag{3.40}$$

Proof. Given $(\boldsymbol{\zeta},\boldsymbol{w})$, $(\boldsymbol{\tau},\boldsymbol{v}) \in \mathbf{X}_2\times\mathbf{M}_1$, we let

$$\mathrm{S}(\boldsymbol{\zeta},\boldsymbol{w}) \,:=\, \vec{\phi} \quad \text{and} \quad \mathrm{S}(\boldsymbol{\tau},\boldsymbol{v}) \,:=\, \vec{\psi}\,,$$

where $(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) := ((\phi,\boldsymbol{t}),\widetilde{\boldsymbol{\sigma}}) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\psi},\widetilde{\boldsymbol{\zeta}}) := ((\psi,\boldsymbol{r}),\widetilde{\boldsymbol{\zeta}}) \in \mathbf{H}\times\mathbf{Q}$ are the respective solutions of (3.11). It follows from the corresponding second equations of (3.11) that $\vec{\phi}-\vec{\psi} \in V$ (cf. (3.12)), and thus the V-ellipticity of $a_{\boldsymbol{\zeta}}$ (cf. (2.21)) gives

$$\alpha\,\|\vec{\phi}-\vec{\psi}\|^2_{\mathbf{H}} \,\leqslant\, a_{\boldsymbol{\zeta}}(\vec{\phi}-\vec{\psi},\vec{\phi}-\vec{\psi})\,. \tag{3.41}$$

In turn, applying the corresponding first equations of (3.11) to $\vec{\varphi} = \vec{\phi}-\vec{\psi}$, we obtain

$$a_{\boldsymbol{\zeta}}(\vec{\phi},\vec{\phi}-\vec{\psi}) \,=\, G_{\boldsymbol{w}}(\vec{\phi}-\vec{\psi})\,, \quad \text{and} \tag{3.42}$$

$$a_{\boldsymbol{\tau}}(\vec{\psi},\vec{\phi}-\vec{\psi}) \,=\, G_{\boldsymbol{v}}(\vec{\phi}-\vec{\psi})\,, \tag{3.43}$$

so that employing (3.42), and then subtracting and adding $a_{\boldsymbol{\tau}}(\vec{\psi},\vec{\phi}-\vec{\psi})$ (cf. (3.43)), (3.41) becomes

$$\alpha\,\|\vec{\phi}-\vec{\psi}\|^2_{\mathbf{H}} \,\leqslant\, (G_{\boldsymbol{w}}-G_{\boldsymbol{v}})(\vec{\phi}-\vec{\psi}) \,+\, (a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi})\,. \tag{3.44}$$

Next, proceeding as for (3.31), we easily get

$$(G_{\boldsymbol{w}}-G_{\boldsymbol{v}})(\vec{\phi}-\vec{\psi}) \,\leqslant\, L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\boldsymbol{w}-\boldsymbol{v}\|_{\mathbf{M}_1}\,\|\phi-\psi\|_{\mathrm{M}}\,. \tag{3.45}$$

On the other hand, recalling that r and s are conjugate to each other with $s<r$ (cf. (2.1)), and employing the Lipschitz continuity of ϑ (cf. (1.9)) along with Hölder's inequality, we find that

$$(a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi}) \,\leqslant L_\vartheta\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,2q;\Omega}\,\|\boldsymbol{r}\|_{0,2p;\Omega}\,\|\boldsymbol{t}-\boldsymbol{r}\|_{0,\Omega}\,, \tag{3.46}$$

where p, $q \in (1,+\infty)$ are conjugate to each other as well. Then, similarly to the proof of Lemma 3.4, we choose p such that $2p = \varepsilon^*$ (cf. (3.23)), so that $2q = \frac{n}{\varepsilon} \leqslant r$, and hence

$$(a_{\boldsymbol{\tau}}-a_{\boldsymbol{\zeta}})(\vec{\psi},\vec{\phi}-\vec{\psi}) \,\leqslant\, L_\vartheta\,C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\,g_2\,\|\boldsymbol{\tau}-\boldsymbol{\zeta}\|_{0,r;\Omega}\,\|\boldsymbol{t}-\boldsymbol{r}\|_{0,\Omega}\,. \tag{3.47}$$

Thus, replacing the estimates (3.45) and (3.47) back into (3.44), we arrive at (3.40) with the constant $C_S \,:=\, \max\left\{|\Omega|^{\frac{r-s}{rs}}, C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon\right\}$. □

We are now in position to conclude the Lipschitz-continuity of Λ.

Lemma 3.8. *There exists a positive constant C_Λ, depending only on $C_{\mathbf{S}}$ and C_S, such that*

$$\|\Lambda(\phi) \,-\, \Lambda(\varphi)\|_{\mathrm{M}} \,\leqslant\, C_\Lambda\,L_f\left\{L_g + L_\vartheta\,g_2\right\}\|\phi-\varphi\|_{\mathrm{M}} \qquad \forall\,\phi,\,\varphi \in \mathrm{M}\,. \tag{3.48}$$

Proof. It is a direct consequence of the definition of Λ (cf. (3.35)) and the continuity properties given by (3.21) and Lemma 3.7. □

Finally, the well-posedness of (2.26) is established as follows.

Theorem 3.9. *Assume the regularity assumption* $(\mathbf{RA_2})$ *(cf. (3.39)), and that the data* L_f, L_g, L_ϑ, *and* g_2 *are sufficiently small so that*

$$C_T\, L_f \left\{ L_g + L_\vartheta\, g_2 \right\} \,<\, 1\,. \tag{3.49}$$

Then, Λ *has a unique fixed point* $\phi \in W$. *Equivalently, the coupled problem (2.26) has a unique solution* $\big((\boldsymbol{\sigma}, \boldsymbol{u}), (\vec{\phi}, \widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{H} \times \mathbf{Q}\big)$, *with* $\vec{\phi} := (\phi, \boldsymbol{t}) \in \mathbf{H}$ *and* $\phi \in W$ *(cf. (3.37)). Moreover, there hold*

$$\begin{aligned}
\|\boldsymbol{\sigma}\|_{\mathbf{X}_2} \,&\leqslant\, \frac{C_r}{\boldsymbol{\alpha}}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,,\\
\|\boldsymbol{u}\|_{\mathbf{M}_1} \,&\leqslant\, \frac{C_r}{\boldsymbol{\beta}_1} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \;+\, \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_1\boldsymbol{\beta}_2} \left(1 + \frac{2}{\boldsymbol{\alpha}\mu}\right) f_2\,,\\
\|\vec{\phi}\|_{\mathbf{H}} \,&=\, \|\phi\|_{\mathrm{M}} + \|\boldsymbol{t}\|_{0,\Omega} \,\leqslant\, \frac{|\Omega|^{1/s}}{\alpha}\, g_2\,, \quad \textit{and}\\
\|\widetilde{\boldsymbol{\sigma}}\|_{\mathbf{Q}} \,&\leqslant\, \frac{|\Omega|^{1/s}}{\beta} \left(1 \,+\, \frac{\vartheta_2}{\alpha}\right) g_2\,.
\end{aligned} \tag{3.50}$$

Proof. Bearing in mind (3.38), Lemma 3.8, and the hypothesis (3.49), a direct application of the Banach fixed point Theorem implies the existence of a unique $\phi \in W$ solution to (3.36) (equivalently, the existence of a unique solution $\big((\boldsymbol{\sigma}, \boldsymbol{u}), (\vec{\phi}, \widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{H} \times \mathbf{Q}\big)$ to (2.26)). In addition, recalling that $(\boldsymbol{\sigma}, \boldsymbol{u}) = \mathbf{S}(\phi)$ and $(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) = \mathrm{S}(\boldsymbol{\sigma}, \boldsymbol{u})$, the a priori estimates (3.3), (3.15), and (3.16) yield (3.50) and conclude the proof. □

4 The Galerkin schemes

In this section we introduce and analyze the Galerkin schemes of the fully-mixed formulations (2.25) and (2.26). In particular, for the solvability analyses of the discrete versions of the decoupled problems studied in Sections 3.1, 3.2, and 3.3, we employ the corresponding analogues of [4, Theorem 2.1, Corollary 2.1, Section 2.1] and [10, Theorem 2.34], which are given by [4, Corollary 2.2, eqs. (2.24), (2.25)] and [10, Proposition 2.42], respectively.

4.1 Preliminaries

We begin by letting $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{X}_{1,h}$, and $\mathbf{M}_{2,h}$ be the finite element subspaces of $\mathbf{X}_2$, $\mathbf{M}_1$, $\mathbf{X}_1$, and $\mathbf{M}_2$, respectively, that are described in [14, Section 5.2, eq. (5.9)]. In addition, let $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\boldsymbol{t}}$ be arbitrary finite element subspaces of $\mathbf{Q}$, M, and $\mathbf{L}^2(\Omega)$, respectively. Hereafter, h stands for both the sub-index of each subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, that is, $h := \max\left\{h_K : \; K \in \mathcal{T}_h\right\}$. Then, the Galerkin scheme associated with (2.25) reads: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ and $(\widetilde{\boldsymbol{\sigma}}, \phi) \in \mathbf{Q}_h \times \mathrm{M}_h$ such that

$$\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) \;&=\; G(\boldsymbol{\tau}_h) &&\forall\, \boldsymbol{\tau}_h \in \mathbf{X}_{1,h},\\
\mathbf{b}_2(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) \;&=\; F_{\phi_h}(\boldsymbol{v}_h) &&\forall\, \boldsymbol{v}_h \in \mathbf{M}_{2,h},\\
\widetilde{a}_{\boldsymbol{\sigma}_h}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) \;&=\; 0 &&\forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,,\\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \psi_h) \;&=\; \widetilde{G}_{\boldsymbol{u}_h}(\psi_h) &&\forall\, \psi_h \in \mathrm{M}_h\,.
\end{aligned} \tag{4.1}$$

In turn, defining the product space $\mathbf{H}_h := \mathrm{M}_h \times \mathbf{H}_h^{\boldsymbol{t}}$ and setting the notation

$$\vec{\phi}_h \,:=\, (\phi_h, \boldsymbol{t}_h)\,, \quad \vec{\varphi}_h \,:=\, (\varphi_h, \boldsymbol{s}_h) \in \mathbf{H}_h\,,$$

the Galerkin scheme associated with (2.26) reduces to: Find $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ and $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ such that

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{X}_{1,h}, \\
\mathbf{b}_2(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) &= F_{\phi_h}(\boldsymbol{v}_h) && \forall\, \boldsymbol{v}_h \in \mathbf{M}_{2,h}, \\
a_{\boldsymbol{\sigma}_h}(\vec{\phi}_h, \vec{\varphi}_h) + b(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) &= G_{\boldsymbol{u}_h}(\vec{\varphi}_h) && \forall\, \vec{\varphi}_h \in \mathbf{H}_h\,, \\
b(\vec{\phi}_h, \widetilde{\boldsymbol{\tau}}_h) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,.
\end{aligned}
\tag{4.2}
$$

The aforementioned subspaces $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{X}_{1,h}$, and $\mathbf{M}_{2,h}$, along with specific examples of $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\boldsymbol{t}}$ satisfying the hypotheses to be assumed below in Sections 4.3 and 4.4, are described later on in Section 5.1.

4.2 Discrete well-posedness of the elasticity equation

We let $\mathbf{S}_h : \mathrm{M}_h \to \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ be the discrete version of the operator $\mathbf{S}$ (cf. (3.1)), that is

$$
\mathbf{S}_h(\varphi_h) \;=\; (\mathbf{S}_{1,h}(\varphi_h), \mathbf{S}_{2,h}(\varphi_h)) \;:=\; (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \quad \forall\, \varphi_h \in \mathrm{M}_h\,,
\tag{4.3}
$$

where $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ is the unique solution (to be confirmed below) of the first two rows of (4.1) (or (4.2)) with φ_h instead of ϕ_h, namely

$$
\begin{aligned}
\mathbf{a}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + \mathbf{b}_1(\boldsymbol{\tau}_h, \boldsymbol{u}_h) &= G(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{X}_{1,h}\,, \\
\mathbf{b}_2(\boldsymbol{\sigma}_h, \boldsymbol{v}_h) &= F_{\varphi_h}(\boldsymbol{v}_h) && \forall\, \boldsymbol{v}_h \in \mathbf{M}_{2,h}\,.
\end{aligned}
\tag{4.4}
$$

Then, under the same assumption on the Lamé parameter λ stipulated in Section 3.1, and letting $\boldsymbol{\alpha}_{\mathtt{d}}$, $\boldsymbol{\beta}_{1,\mathtt{d}}$, and $\boldsymbol{\beta}_{2,\mathtt{d}}$ be the constants yielding the discrete inf-sup conditions for $\mathbf{a}$, $\mathbf{b}_1$, and $\mathbf{b}_2$ (cf. [14, Lemmas 5.3 and 5.4]), a direct application of [4, Corollary 2.2, eqs. (2.24), (2.25)] yields the following result (cf. [14, Lemma 4.1]).

Lemma 4.1. *For each $\varphi_h \in \mathrm{M}_h$ there exists a unique $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ solution to (4.4), and hence one can define $\mathbf{S}_h(\varphi_h) = \big(\mathbf{S}_{1,h}(\varphi_h), \mathbf{S}_{2,h}(\varphi_h)\big) := (\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$. Moreover, there hold*

$$
\begin{aligned}
\|\mathbf{S}_{1,h}(\varphi_h)\|_{\mathbf{X}_2} = \|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} &\leqslant \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} + \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}} \left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\, \mu}\right) f_2\,, \quad \textit{and} \\
\|\mathbf{S}_{2,h}(\varphi_h)\|_{\mathbf{M}_1} = \|\boldsymbol{u}_h\|_{\mathbf{M}_1} &\leqslant \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}} \left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\, \mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \;+\; \frac{2|\Omega|^{1/r}}{\mu \boldsymbol{\beta}_{1,\mathrm{d}}\, \boldsymbol{\beta}_{2,\mathrm{d}}} \left(1 + \frac{2}{\alpha_{\mathrm{d}}\, \mu}\right) f_2\,.
\end{aligned}
\tag{4.5}
$$

We stress here that the lack of a required boundedness property for a projector involved in the proof of the previous lemma, restricts the present discrete analysis to the 2D case. We refer to [14, Section 5] for further details.

4.3 Discrete well-posedness of the first approach for the diffusion equation

We now let $\widetilde{\mathrm{S}}_h : \mathbf{X}_{2,h} \times \mathbf{M}_{1,h} \to \mathbf{Q}_h \times \mathrm{M}_h$ be the discrete version of $\widetilde{\mathrm{S}}$ (cf. (3.4)), that is

$$
\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \;:=\; (\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \quad \forall\, (\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\,,
\tag{4.6}
$$

where $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ is the unique solution (to be confirmed below) of the third and fourth rows of (4.1) with $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$ instead of $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$, namely

$$
\begin{aligned}
\widetilde{a}_{\boldsymbol{\zeta}_h}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) &= 0 && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,, \\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \psi_h) &= \widetilde{G}_{\boldsymbol{w}_h}(\psi_h) && \forall\, \psi_h \in \mathrm{M}_h\,.
\end{aligned}
\tag{4.7}
$$

In order to establish the well-posedness of (4.7), we first consider the discrete kernel of $\widetilde{b}$, that is

$$\widetilde{\mathcal{K}}_h \;:=\; \left\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h : \quad \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \phi_h) = 0 \quad \forall\, \phi_h \in \mathrm{M}_h \right\}, \tag{4.8}$$

and suppose that

($\mathbf{H.1}$) $\mathrm{div}(\mathbf{Q}_h) \,\subseteq\, \mathrm{M}_h$.

Then, bearing mind the definition of $\widetilde{b}$ (cf. (2.14)), and employing ($\mathbf{H.1}$), we readily deduce from (4.8) that

$$\widetilde{\mathcal{K}}_h \;:=\; \left\{ \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h) \;=\; 0 \right\},$$

which yields the discrete analogue of (3.6), and hence the $\widetilde{\mathcal{K}}_h$-ellipticity of $\widetilde{a}_{\boldsymbol{\zeta}_h}$ with constant $\widetilde{\alpha}_{\mathtt{d}} = \widetilde{\vartheta}_0$.

Next, we also assume that

($\mathbf{H.2}$) there exists a positive constant $\widetilde{\beta}_{\mathrm{d}}$, independent of h, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h \\ \widetilde{\boldsymbol{\tau}}_h \neq \mathbf{0}}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \psi_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathbf{Q}}} \;\geqslant\; \widetilde{\beta}_{\mathrm{d}}\, \|\psi_h\|_{\mathrm{M}} \qquad \forall\, \psi_h \in \mathrm{M}_h\,.$$

Thus, straightforward applications of [10, Theorem 2.42] and the abstract estimates from [10, eq. (2.30)] imply the discrete analogue of Lemma 3.2, which is stated as follows.

Lemma 4.2. *For each $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$, there exists a unique $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ solution of (4.7), and hence one can define $\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) = \big(\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h), \widetilde{\mathrm{S}}_{2,h}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\big) := (\widetilde{\boldsymbol{\sigma}}_h, \phi_h) \in \mathbf{Q}_h \times \mathrm{M}_h$. Moreover, there hold*

$$\begin{aligned} \|\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{\mathbf{Q}} \;&=\; \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \;\leqslant\; \frac{1}{\widetilde{\beta}_{\mathrm{d}}}\Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big)|\Omega|^{1/s} g_2\,, \quad \textit{and} \\ \|\widetilde{\mathrm{S}}_{2,h}(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{\mathrm{M}} \;&=\; \|\phi_h\|_{\mathrm{M}} \;\leqslant\; \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}_{\mathrm{d}}^2}\Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big)|\Omega|^{1/s} g_2\,. \end{aligned} \tag{4.9}$$

4.4 Discrete well-posedness of the second approach for the diffusion equation

Here we introduce the discrete operator $\mathrm{S}_h : \mathbf{X}_{2,h} \times \mathbf{M}_{1,h} \to \mathbf{H}_h$ given by

$$\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \;:=\; \vec{\phi}_h \quad \forall\, (\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\,, \tag{4.10}$$

where $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) := ((\phi_h, \boldsymbol{t}_h), \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed below) of the third and fourth rows of (4.2) with $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$ instead of $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h$, that is

$$\begin{aligned} a_{\boldsymbol{\zeta}_h}(\vec{\phi}_h, \vec{\varphi}_h) \,+\, b(\vec{\varphi}_h, \widetilde{\boldsymbol{\sigma}}_h) \;&=\; G_{\boldsymbol{w}_h}(\vec{\varphi}_h) \qquad \forall\, \vec{\varphi}_h \in \mathbf{H}_h\,, \\ b(\vec{\phi}_h, \widetilde{\boldsymbol{\tau}}_h) \;&=\; 0 \qquad\qquad\quad\; \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,. \end{aligned} \tag{4.11}$$

In order to prove that (4.11) is well-posed, we need to incorporate a couple of suitable hypotheses on the discrete spaces. Indeed, we first assume that

($\mathbf{H.3}$) there exists a positive constant $\beta_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\varphi}_h \in \mathbf{H}_h \\ \vec{\varphi}_h \neq \mathbf{0}}} \frac{b(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h)}{\|\vec{\varphi}_h\|_{\mathbf{H}}} \;\geqslant\; \beta_{\mathtt{d}}\, \|\widetilde{\boldsymbol{\tau}}_h\|_{\mathbf{Q}} \qquad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h\,.$$

Next, we let V_h be the discrete kernel of the bilinear form b, that is

$$V_h \;:=\; \Big\{ \vec{\varphi}_h \in \mathbf{H}_h : \quad b(\vec{\varphi}_h, \widetilde{\boldsymbol{\tau}}_h) \;=\; 0 \quad \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathbf{Q}_h \Big\} \,,$$

and suppose that

(**H.4**) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\boldsymbol{s}_h\|_{0,\Omega} \;\geqslant\; C_{\mathtt{d}}\, \|\varphi_h\|_{0,r,\Omega} \qquad \forall\, \vec{\varphi}_h := (\varphi_h, \boldsymbol{s}_h) \in V_h \,.$$

In this way, bearing in mind the definition of $a_{\boldsymbol{\zeta}_h}$ (cf. (2.21)), and employing the positive definiteness property of ϑ (cf. (1.8)) and (**H.4**), we deduce for each $\boldsymbol{\zeta}_h \in \mathbf{X}_{2,h}$ that

$$a_{\boldsymbol{\zeta}_h}(\vec{\varphi}_h, \vec{\varphi}_h) \;\geqslant\; \vartheta_0\, \|\boldsymbol{s}_h\|^2_{0,\Omega} \;\geqslant\; \frac{\vartheta_0}{2}\, C^2_{\mathtt{d}}\, \|\varphi_h\|^2_{0,r;\Omega} \;+\; \frac{\vartheta_0}{2}\, \|\boldsymbol{s}_h\|^2_{0,r;\Omega} \quad \forall\, \vec{\varphi}_h := (\varphi_h, \boldsymbol{s}_h) \in V_h \,, \tag{4.12}$$

from which it readily follows the V_h-ellipticity of $a_{\boldsymbol{\zeta}_h}$ with constant $\alpha_{\mathtt{d}} := \frac{\vartheta_0}{2} \min\{C^2_{\mathtt{d}}, 1\}$.

Consequently, applying [10, Proposition 2.42], and making use of the a priori estimate provided by [10, eq. (2.30)], we are lead to the discrete analogue of Lemma 3.3.

Lemma 4.3. *For each $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$ there exists a unique $(\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$ solution of (4.11), and hence one can define $\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) := \vec{\phi}_h \in \mathbf{H}_h$. Moreover, there holds*

$$\|\mathrm{S}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)\|_{\mathbf{H}} \;=\; \|\vec{\phi}_h\|_{\mathbf{H}} \;=\; \|\phi_h\|_{0,r;\Omega} + \|\boldsymbol{t}_h\|_{0,\Omega} \;\leqslant\; \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}}\, g_2 \,. \tag{4.13}$$

We end this section by remarking that the discrete version of (3.16) becomes

$$\|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \;=\; \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_s;\Omega} \;\leqslant\; \frac{|\Omega|^{1/s}}{\beta_{\mathrm{d}}} \left(1 \;+\; \frac{\vartheta_2}{\alpha_{\mathrm{d}}}\right) g_2 \,. \tag{4.14}$$

4.5 Discrete solvability of the first fully-mixed formulation

In this section we adopt the discrete analogue of the fixed point strategy introduced in Section 3.4 to analyze the solvability of (4.1). According to it, we define the operator $\Xi_h : \mathrm{M}_h \to \mathrm{M}_h$ as

$$\Xi_h(\varphi_h) \;:=\; \widetilde{\mathrm{S}}_{2,h}\big(\mathbf{S}_h(\varphi_h)\big) \qquad \forall\, \varphi_h \in \mathrm{M}_h \,, \tag{4.15}$$

and observe, being $\widetilde{\mathrm{S}}_h$ and $\mathbf{S}_h$, and hence Ξ_h as well, well-defined, that solving (4.1) is equivalent to seeking a fixed point of Ξ_h, that is: Find $\phi_h \in \mathrm{M}_h$ such that

$$\Xi_h(\phi_h) \;=\; \phi_h \,. \tag{4.16}$$

Thus, in what follows we show that Ξ_h verifies the hypotheses of the Brouwer theorem. In fact, introducing the ball

$$\widetilde{W}_h \;:=\; \Big\{ \phi_h \in \mathrm{M}_{1,h} : \quad \|\phi_h\|_{0,r;\Omega} \;\leqslant\; \widetilde{\delta}_{\mathrm{d}} \Big\} \,, \tag{4.17}$$

with

$$\widetilde{\delta}_{\mathrm{d}} \;:=\; \frac{\widetilde{\vartheta}_2}{\widetilde{\beta}^2_{\mathrm{d}}} \Big(1 \;+\; \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big) |\Omega|^{1/s} g_2 \,,$$

we realize, according to the definition of Ξ_h (cf. (4.15)) and the second a priori estimate in (4.9), that

$$\Xi_h\big(\widetilde{W}_h\big) \;\subseteq\; \widetilde{W}_h \,. \tag{4.18}$$

Next, in order to derive the continuity of Ξ_h, we first recall from [14, eq. (4.11)] that there exists a positive constant $C_{\mathbf{S},\mathrm{d}}$, independent of h, such that

$$\|\mathbf{S}_h(\phi_h) - \mathbf{S}_h(\varphi_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \,\leqslant\, C_{\mathbf{S},\mathrm{d}}\, L_f\, \|\phi_h - \varphi_h\|_{0,r;\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{M}_h\,. \tag{4.19}$$

On the other hand, for the continuity of $\widetilde{\mathrm{S}}_h$ the reasoning of the proof of Lemma 3.4 is slightly modified. Indeed, knowing that the regularity assumption $(\mathbf{RA}_1)$ is certainly not applicable in the present discrete context, we proceed to utilize a $\mathrm{L}^{2q} - \mathrm{L}^{2p} - \mathrm{L}^2$ argument to derive the discrete version of (3.24), where $p,\, q \in (1, +\infty)$, conjugate to each other, are chosen such that $2q = r$. The above is a feasible choice since, as stipulated in (2.1), there holds $r > 2$, which yields $r^* := 2p = \frac{2r}{r-2}$. In this way, given $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$ $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$, and denoting $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$ and $(\widetilde{\boldsymbol{\zeta}}_h, \varphi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbf{Q}_h \times \mathrm{M}_h$, the discrete analogue of (3.29) becomes

$$\begin{aligned} |\widetilde{a}_{\boldsymbol{\tau}_h}(\widetilde{\boldsymbol{\zeta}}_h, \widetilde{\boldsymbol{\tau}}_h) - \widetilde{a}_{\boldsymbol{\zeta}_h}(\widetilde{\boldsymbol{\zeta}}_h, \widetilde{\boldsymbol{\tau}}_h)| \,&\leqslant\, L_{\widetilde{\vartheta}}\, \|(\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h)\widetilde{\boldsymbol{\zeta}}_h\|_{0,\Omega}\, \|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega} \\ &\leqslant\, L_{\widetilde{\vartheta}}\, \|\boldsymbol{\tau}_h - \boldsymbol{\zeta}_h\|_{0,2q;\Omega}\, \|\widetilde{\boldsymbol{\zeta}}_h\|_{0,2p,\Omega}\, \|\widetilde{\boldsymbol{\tau}}_h\|_{0,\Omega}\,. \end{aligned} \tag{4.20}$$

The foregoing inequality, along with the discrete versions of (3.28) and (3.31), whose details we omit here, imply the existence of a positive constant $C_{\widetilde{\mathrm{S}},\mathrm{d}}$, depending only on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, and $|\Omega|$, and hence independent of h, such that

$$\begin{aligned} &\|\widetilde{\mathrm{S}}_h(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - \widetilde{\mathrm{S}}_h(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ &\qquad \leqslant\, C_{\widetilde{\mathrm{S}},\mathrm{d}} \left\{ L_g + L_{\widetilde{\vartheta}}\, \|\widetilde{\mathrm{S}}_{1,h}(\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{0,r^*;\Omega} \right\} \|(\boldsymbol{\zeta}_h, \boldsymbol{w}_h) - (\boldsymbol{\tau}_h, \boldsymbol{v}_h)\|_{\mathbf{Q}\times\mathrm{M}} \end{aligned} \tag{4.21}$$

for all $(\boldsymbol{\zeta}_h, \boldsymbol{w}_h)$, $(\boldsymbol{\tau}_h, \boldsymbol{v}_h) \in \mathbf{X}_{2,h} \times \mathbf{M}_{1,h}$. In this way, recalling the definition of Ξ_h (cf. (4.15)), and employing the estimates (4.19) and (4.21), we conclude that

$$\begin{aligned} &\|\Xi_h(\phi_h) - \Xi_h(\varphi_h)\|_{0,r;\Omega} \\ &\quad \leqslant\, C_{\Xi,\mathrm{d}}\, L_f \left\{ L_g + L_{\widetilde{\vartheta}}\, \|\widetilde{\mathrm{S}}_{1,h}(\mathbf{S}_h(\varphi_h))\|_{0,r^*;\Omega} \right\} \|\phi_h - \varphi_h\|_{0,r;\Omega} \qquad \forall\, \phi_h,\, \varphi_h \in \mathrm{M}_h\,, \end{aligned} \tag{4.22}$$

with the positive constant $C_{\Xi,\mathrm{d}} \,:=\, C_{\mathbf{S},\mathrm{d}}\, C_{\widetilde{\mathrm{S}},\mathrm{d}}$. While the estimate (4.22) implies that Ξ_h is continuous, we emphasize that the lack of control of the term $\|\widetilde{\mathrm{S}}_{1,h}(\mathbf{S}_h(\varphi_h))\|_{0,r^*;\Omega}$ stop us of concluding Lipschitz-continuity and hence nor contractivity of this operator.

We are now in position to establish the following main result.

Theorem 4.4. *The operator Ξ_h has at least one fixed point $\phi_h \in \widetilde{W}_h$. Equivalently, the Galerkin scheme (4.1) has at least one solution $\big((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{Q}_h \times \mathrm{M}_h\big)$, with $\phi_h \in \widetilde{W}_h$ (cf. (4.17)). Moreover, there hold*

$$\begin{aligned} \|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} \,&\leqslant\, \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\, \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}} \left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\, \mu}\right) f_2\,, \\ \|\boldsymbol{u}_h\|_{\mathbf{M}_1} \,&\leqslant\, \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}} \left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\, \mu}\right) \|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \,+\, \frac{2|\Omega|^{1/r}}{\mu \boldsymbol{\beta}_{1,\mathrm{d}}\, \boldsymbol{\beta}_{2,\mathrm{d}}} \left(1 + \frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\, \mu}\right) f_2\,, \quad \textit{and} \\ \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \,&\leqslant\, \frac{1}{\widetilde{\beta}_{\mathrm{d}}} \Big(1 \,+\, \frac{\widetilde{\vartheta}_2}{\widetilde{\alpha}_{\mathrm{d}}}\Big) |\Omega|^{1/s} g_2\,. \end{aligned} \tag{4.23}$$

Proof. Thanks to (4.18), the continuity of Ξ_h (cf. (4.22)), and the equivalence between (4.1) and (4.16), a straightforward application of Brouwer's theorem (cf. [7, Theorem 9.9-2]) implies the first conclusion of this theorem. Next, noting that $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) = \mathbf{S}_h(\phi_h)$ and $(\widetilde{\boldsymbol{\sigma}}_h, \phi_h) = \widetilde{\mathrm{S}}_h(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$, the a priori estimate (4.23) follows from (4.5) and (4.9). □

4.6 A priori error analysis for the first fully-mixed formulation

In this section we establish the Céa estimate for the global error

$$\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \,+\, \|(\widetilde{\boldsymbol{\sigma}},\phi)-(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}}\,,$$

where $\big((\boldsymbol{\sigma},\boldsymbol{u}),(\widetilde{\boldsymbol{\sigma}},\phi)\big)\in\big(\mathbf{X}_2\times\mathbf{M}_1\big)\times\big(\mathbf{Q}\times\mathrm{M}\big)$ and $\big((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\big)\in\big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{Q}_h\times\mathrm{M}_h\big)$ are the unique solutions of (2.25) and (4.1), respectively, with $\phi\in\widetilde{W}$ (cf. (3.19)) and $\phi_h\in\widetilde{W}_h$ (cf. (4.17)). In what follows, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set

$$\mathrm{dist}(z,Z_h)\,:=\,\inf_{z_h\in Z_h}\|z-z_h\|_Z\qquad\forall\,z\in Z\,.$$

Then, applying the Strang a priori error estimate provided by [4, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the pair of associated continuous and discrete formulations given by the first and second rows of (2.25) and (4.1), respectively, and proceeding as for the derivation of [14, Section 4.4, eq. (4.20)], but without using the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^r(\Omega)$ as done there, we deduce that there exists a positive constant $\widehat{C}_{ST}$, depending only on $\boldsymbol{\alpha}_{\mathrm{d}}$, $\boldsymbol{\beta}_{1,\mathrm{d}}$, $\boldsymbol{\beta}_{2,\mathrm{d}}$, $\|\mathbf{a}\|$, $\|\mathbf{b}_1\|$, and $\|\mathbf{b}_2\|$, and hence independent of h, such that

$$\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}\,\leqslant\,\widehat{C}_{ST}\Big\{\mathrm{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h})+\mathrm{dist}(\boldsymbol{u},\mathbf{M}_{1,h})+L_f\,\|\phi-\phi_h\|_{\mathrm{M}}\Big\}\,. \tag{4.24}$$

Similarly, applying the Strang a priori error estimate from [4, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the pair of associated continuous and discrete formulations given by the third and fourth rows of (2.25) and (4.1), respectively, we find that there exists a positive constant $\widetilde{C}_{ST}$, depending only on $\widetilde{\alpha}_{\mathrm{d}}$, $\widetilde{\beta}_{\mathrm{d}}$, $\|\widetilde{a}_{\boldsymbol{\sigma}}\|$, and $\|\widetilde{b}\|$, and hence independent of h, as well, such that

$$\begin{aligned}&\|(\widetilde{\boldsymbol{\sigma}},\phi)-(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}}\\&\quad\leqslant\,\widetilde{C}_{ST}\Big\{\mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)+\mathrm{dist}(\phi,\mathrm{M}_h)+\|(\widetilde{a}_{\boldsymbol{\sigma}}-\widetilde{a}_{\boldsymbol{\sigma}_h})(\widetilde{\boldsymbol{\sigma}},\cdot)\|_{\mathbf{Q}_h'}+\|\widetilde{G}_{\boldsymbol{u}}-\widetilde{G}_{\boldsymbol{u}_h}\|_{\mathrm{M}_h'}\Big\}\,.\end{aligned} \tag{4.25}$$

Next, proceeding exactly as for the derivations of (3.30) and (3.31), we find that

$$\|(\widetilde{a}_{\boldsymbol{\sigma}}-\widetilde{a}_{\boldsymbol{\sigma}_h})(\widetilde{\boldsymbol{\sigma}},\cdot)\|_{\mathbf{Q}_h'}\,\leqslant\,\widetilde{L}_{\widetilde{\mathbb{S}}}\,L_{\widetilde{\vartheta}}\,g_2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{X}_2}\,, \tag{4.26}$$

where $\widetilde{L}_{\widetilde{\mathbb{S}}}:=C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon$, and

$$\|G_{\boldsymbol{u}}-G_{\boldsymbol{u}_h}\|_{\mathrm{M}_h'}\,\leqslant\,L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{M}_1}\,. \tag{4.27}$$

In this way, replacing (4.26) and (4.27) back into (4.25), we conclude that

$$\begin{aligned}&\|(\widetilde{\boldsymbol{\sigma}},\phi)-(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}}\\&\quad\leqslant\,\widetilde{C}_{ST}\Big\{\mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)+\mathrm{dist}(\phi,\mathrm{M}_h)+\widetilde{L}_{\widetilde{\mathbb{S}}}\,L_{\widetilde{\vartheta}}\,g_2\,\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{\mathbf{X}_2}+L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathbf{M}_1}\Big\}\,.\end{aligned} \tag{4.28}$$

Thus, adding (4.24) and (4.28), we arrive at

$$\begin{aligned}&\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|(\widetilde{\boldsymbol{\sigma}},\phi)-(\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}}\\&\quad\leqslant\,C_{ST}\Big\{\mathrm{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h})+\mathrm{dist}(\boldsymbol{u},\mathbf{M}_{1,h})+\mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)+\mathrm{dist}(\phi,\mathrm{M}_h)\Big\}\\&\quad\quad+\mathcal{C}(\mathtt{data})\Big\{\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}+\|\phi-\phi_h\|_{\mathrm{M}}\Big\}\,,\end{aligned} \tag{4.29}$$

where $C_{ST}\,:=\,\max\big\{\widehat{C}_{ST},\widetilde{C}_{ST}\big\}$, and

$$\mathcal{C}(\mathtt{data})\,:=\,\max\Big\{\widehat{C}_{ST}\,L_f,\,\widetilde{C}_{ST}\,\widetilde{L}_{\widetilde{\mathbb{S}}}\,L_{\widetilde{\vartheta}}\,g_2,\,\widetilde{C}_{ST}\,L_g\,|\Omega|^{\frac{r-s}{rs}}\Big\}\,. \tag{4.30}$$

We are now in a position to state the announced Céa estimate for our first approach.

Theorem 4.5. *Assume that the data satisfy (cf. (4.30))*

$$\mathcal{C}(\mathtt{data}) \,\leqslant\, \frac{1}{2}\,. \tag{4.31}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} & \|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\widetilde{\boldsymbol{\sigma}},\phi) - (\widetilde{\boldsymbol{\sigma}}_h,\phi_h)\|_{\mathbf{Q}\times\mathrm{M}} \\ & \quad\leqslant\, C\,\Big\{\mathrm{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h}) + \mathrm{dist}(\boldsymbol{u},\mathbf{M}_{1,h}) + \mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h) + \mathrm{dist}(\phi,\mathrm{M}_h)\Big\}\,. \end{aligned} \tag{4.32}$$

Proof. It follows directly from (4.29) and (4.31). □

4.7 Discrete solvability of the second fully-mixed formulation

The discrete analogue of the fixed point approach employed in Section 3.5 is adopted here to establish the solvability of (4.2). Thus, we now define the operator $\Lambda_h : \mathrm{M}_h \to \mathrm{M}_h$ as

$$\Lambda_h(\psi_h) \,:=\, \mathrm{S}_{1,h}\big(\mathbf{S}_h(\psi_h)\big) \qquad \forall\,\psi_h \in \mathrm{M}_h\,, \tag{4.33}$$

which is clearly well-defined since S_h and $\mathbf{S}_h$ are, and hence, solving (4.2) is equivalent to finding a fixed point of Λ_h, that is $\phi_h \in \mathrm{M}_h$ such that

$$\Lambda_h(\phi_h) \,=\, \phi_h\,. \tag{4.34}$$

Similarly to the analysis in Section 4.5, in what follows we prove that Λ_h verifies the hypotheses of the Brouwer theorem. Indeed, defining

$$W_h \,:=\, \Big\{\phi_h \in \mathrm{M}_h : \quad \|\phi_h\|_{0,r;\Omega} \,\leqslant\, \delta_{\mathrm{d}}\Big\}\,, \tag{4.35}$$

with

$$\delta_{\mathrm{d}} \,:=\, \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}}\,g_2\,,$$

it is straightforward to see, from the definition of Λ_h (cf. (4.33)) and the a priori estimate for $\mathrm{S}_{1,h}$ (cf. (4.13)), that

$$\Lambda_h(W_h) \,\subseteq\, W_h\,. \tag{4.36}$$

Next, proceeding analogously to the proof of Lemma 3.7, but without using the regularity assumption $(\mathbf{RA}_2)$, which is not valid in the present discrete case, and letting $C_{\mathrm{S,d}} \,:=\, \max\big\{|\Omega|^{\frac{r-s}{rs}},\,1\big\}$, we are able to show that

$$\begin{aligned} & \|\mathrm{S}_h(\boldsymbol{\zeta}_h,\boldsymbol{w}_h) \,-\, \mathrm{S}_h(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|_{\mathbf{H}} \\ & \quad\leqslant\, C_{\mathrm{S,d}}\,\Big\{L_g + L_\vartheta\,\|\mathrm{S}_{2,h}(\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|_{0,r*;\Omega}\Big\}\,\|(\boldsymbol{\zeta}_h,\boldsymbol{w}_h) - (\boldsymbol{\tau}_h,\boldsymbol{v}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \end{aligned} \tag{4.37}$$

for all $(\boldsymbol{\zeta}_h,\boldsymbol{w}_h),\,(\boldsymbol{\tau}_h,\boldsymbol{v}_h) \in \mathbf{X}_{2,h}\times\mathbf{M}_{1,h}$. In this way, bearing in mind the definition of Λ_h (cf. (4.33)), and combining (4.37) with the Lipschitz-continuity of $\mathbf{S}_h$ (cf. (4.19)), we obtain

$$\begin{aligned} & \|\Lambda_h(\phi_h) - \Lambda_h(\varphi_h)\|_{0,r;\Omega} \\ & \quad\leqslant\, L_{\Lambda,\mathrm{d}}\,L_f\,\Big\{L_g + L_\vartheta\,\|\mathrm{S}_{2,h}(\mathbf{S}_h(\varphi_h))\|_{0,r*;\Omega}\Big\}\,\|\phi_h - \varphi_h\|_{0,r;\Omega} \quad \forall\,\phi_h,\,\varphi_h \in \mathrm{M}_h\,, \end{aligned} \tag{4.38}$$

with $L_{\Lambda,\mathrm{d}} \,:=\, C_{\mathbf{S},\mathrm{d}}\,C_{\mathrm{S,d}}$.

The main result of this section is then stated as follows.

Theorem 4.6. *The operator Λ_h has at least one fixed point $\phi_h \in \mathrm{M}_h$. Equivalently, the Galerkin scheme (4.2) has at least one solution $\big((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{H}_h \times \mathbf{Q}_h\big)$, with $\phi_h \in W_h$ (cf. (4.35)). Moreover, there hold*

$$
\begin{aligned}
\|\boldsymbol{\sigma}_h\|_{\mathbf{X}_2} \;&\leqslant\; \frac{C_r}{\boldsymbol{\alpha}_{\mathrm{d}}}\,\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \;+\; \frac{|\Omega|^{1/r}}{\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,,\\
\|\boldsymbol{u}_h\|_{\mathbf{M}_1} \;&\leqslant\; \frac{C_r}{\boldsymbol{\beta}_{1,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right)\|\boldsymbol{u}_D\|_{1/s,r;\Gamma} \;+\; \frac{2|\Omega|^{1/r}}{\mu\boldsymbol{\beta}_{1,\mathrm{d}}\,\boldsymbol{\beta}_{2,\mathrm{d}}}\left(1+\frac{2}{\boldsymbol{\alpha}_{\mathrm{d}}\,\mu}\right) f_2\,,\\
\|\vec{\phi}_h\|_{\mathbf{H}} \;&=\; \|\phi_h\|_{0,r;\Omega} + \|\boldsymbol{t}_h\|_{0,\Omega} \;\leqslant\; \frac{|\Omega|^{1/s}}{\alpha_{\mathrm{d}}}\,g_2\,, \quad \textit{and}\\
\|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathbf{Q}} \;&=\; \|\widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_r;\Omega} \;\leqslant\; \frac{|\Omega|^{1/s}}{\beta_{\mathrm{d}}}\left(1\,+\,\frac{\vartheta_2}{\alpha_{\mathrm{d}}}\right) g_2\,.
\end{aligned}
\tag{4.39}
$$

Proof. Thanks to (4.36), the continuity of Λ_h (cf. (4.38)), and the fact that (4.2) and (4.34) are equivalent, the existence of solution follows from a direct application of the Brouwer theorem (cf. [7, Theorem 9.9-2]). In turn, the a priori estimates (4.5), (4.13), and (4.14) yield (4.39), which finishes the proof. □

4.8 A priori error analysis for the second fully-mixed formulation

In what follows we derive the Céa estimate for the global error

$$
\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} \;+\; \|(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) - (\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\,,
$$

where $\big((\boldsymbol{\sigma},\boldsymbol{u}),(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2\times\mathbf{M}_1\big)\times\big(\mathbf{H}\times\mathbf{Q}\big)$ and $\big((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\big) \in \big(\mathbf{X}_{2,h}\times\mathbf{M}_{1,h}\big)\times\big(\mathbf{H}_h\times\mathbf{Q}_h\big)$ are the unique solutions of (2.26) and (4.2), respectively, with $\phi \in W$ (cf. (3.37)) and $\phi_h \in W_h$ (cf. (4.35)).

Since the first two rows of (2.25) and (4.1) coincide with those of (2.26) and (4.2), we realize that the a priori estimate for $\|(\boldsymbol{\sigma},\boldsymbol{u})-(\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1}$ is exactly the one given by (4.24). In turn, applying the Strang estimate provided by [8, Lemma 6.1] (whose proof is a simple modification of that of [11, Theorem 2.6]) to the pair of associated continuous and discrete formulations given by the last two rows of (2.26) and (4.2), we deduce the existence of a positive constant $\bar{C}_{ST}$, depending only on $\alpha_{\mathtt{d}}$, $\beta_{\mathtt{d}}$, $\|a_{\boldsymbol{\sigma}}\|$, and $\|b\|$, such that

$$
\begin{aligned}
&\|(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) \,-\, (\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\
&\quad\leqslant\; \bar{C}_{ST}\left\{\mathrm{dist}(\vec{\phi},\mathbf{H}_h) \,+\, \mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h) \,+\, \|(a_{\boldsymbol{\sigma}} - a_{\boldsymbol{\sigma}_h})(\vec{\phi},\cdot)\|_{\mathbf{H}_h'} \,+\, \|G_{\boldsymbol{u}} - G_{\boldsymbol{u}_h}\|_{\mathbf{H}_h'}\right\}.
\end{aligned}
\tag{4.40}
$$

Then, proceeding exactly as for the derivations of (3.47) and (3.45), we readily obtain

$$
\|(a_{\boldsymbol{\sigma}} - a_{\boldsymbol{\sigma}_h})(\vec{\phi},\cdot)\|_{\mathbf{H}_h'} \;\leqslant\; L_{\mathrm{S}}\,L_{\vartheta}\,g_2\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r;\Omega}\,,
$$

where $L_{\mathrm{S}} := C_{r,\varepsilon}\,\|i_\varepsilon\|\,C_\varepsilon$, and

$$
\|G_{\boldsymbol{u}} - G_{\boldsymbol{u}_h}\|_{\mathbf{H}_h'} \;\leqslant\; L_g\,|\Omega|^{\frac{r-s}{rs}}\,\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\mathbf{M}_1}\,.
$$

In this way, replacing the foregoing estimates back into (4.40), and adding the resulting inequality to (4.24), we arrive at

$$
\begin{aligned}
&\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) \,-\, (\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}}\\
&\qquad\leqslant\; C_{ST}\left\{\mathrm{dist}(\boldsymbol{\sigma},\mathbf{X}_{2,h}) + \mathrm{dist}(\boldsymbol{u},\mathbf{M}_{1,h}) + \mathrm{dist}(\vec{\phi},\mathbf{H}_h) + \mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathbf{Q}_h)\right\}\\
&\qquad+\,\mathcal{D}(\mathtt{data})\left\{\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|\phi - \phi_h\|_{\mathrm{M}}\right\},
\end{aligned}
\tag{4.41}
$$

where $C_{ST} := \max\left\{\widehat{C}_{ST}, \bar{C}_{ST}\right\}$, and

$$\mathcal{D}(\mathtt{data}) := \max\left\{\widehat{C}_{ST}\, L_f,\, \bar{C}_{ST}\, L_{\mathrm{S}}\, L_\theta\, g_2,\, \bar{C}_{ST}\, L_g\, |\Omega|^{\frac{r-s}{rs}}\right\}. \tag{4.42}$$

Thus, we conclude the Céa estimate for our second approach.

Theorem 4.7. *Assume that the data satisfy (cf. (4.42))*

$$\mathcal{D}(\mathtt{data}) \leqslant \frac{1}{2}. \tag{4.43}$$

Then, there exists a positive constant C, independent of h, such that

$$\begin{aligned} &\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\vec{\phi}, \widetilde{\boldsymbol{\sigma}}) - (\vec{\phi}_h, \widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \\ &\quad\leqslant C\left\{\mathrm{dist}(\boldsymbol{\sigma}, \mathbf{X}_{2,h}) + \mathrm{dist}(\boldsymbol{u}, \mathbf{M}_{1,h}) + \mathrm{dist}(\vec{\phi}, \mathbf{H}_h) + \mathrm{dist}(\widetilde{\boldsymbol{\sigma}}, \mathbf{Q}_h)\right\}. \end{aligned} \tag{4.44}$$

Proof. It is a straightforward consequence of (4.41) and (4.43). □

5 Specific finite element subspaces

We now define specific finite element subspaces satisfying the stability conditions required by the respective discrete analyses developed in Section 4, and provide the rates of convergence of the resulting Galerkin schemes.

5.1 Preliminaries

Bearing in mind the mesh notations introduced at the beginning of Section 4.1, and given an integer $k \geqslant 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials defined on K of degree $\leqslant k$, and denote its vector version by $\mathbf{P}_k(K)$. In addition, we let $\widetilde{\mathrm{P}}_k(K)$ be the space of polynomials defined on K of degree $= k$. Furthermore, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) \oplus \widetilde{\mathrm{P}}_k(K)\boldsymbol{x}$ be the local Raviart-Thomas space defined on K of order k, where $\boldsymbol{x}$ stands for a generic vector in R^2, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart. In turn, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$, and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbf{RT}_k(K)$, and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \left\{\psi_h \in \mathrm{L}^2(\Omega) : \quad \psi_h|_K \in \mathrm{P}_k(K) \quad \forall K \in \mathcal{T}_h\right\},$$

$$\mathbf{P}_k(\mathcal{T}_h) := \left\{\boldsymbol{v}_h \in \mathbf{L}^2(\Omega) : \quad \boldsymbol{v}_h|_K \in \mathbf{P}_k(K) \quad \forall K \in \mathcal{T}_h\right\},$$

$$\mathbf{RT}_k(\mathcal{T}_h) := \left\{\widetilde{\boldsymbol{\tau}}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \widetilde{\boldsymbol{\tau}}_h|_K \in \mathbf{RT}_k(K) \quad \forall K \in \mathcal{T}_h\right\},$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall K \in \mathcal{T}_h\right\}.$$

We stress here that for each $t \in [1, +\infty]$ there hold $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{P}_k(\mathcal{T}_h) \subseteq \mathbf{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, and $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}^t(\mathbf{div}_t;\Omega)$, inclusions that are implicitly utilized in what follows.

As announced in Section 4.1, we first recall from [14, Section 5.2, eq. (5.9)] that the finite element subspaces of $\mathbf{X}_2$, $\mathbf{M}_1$, $\mathbf{X}_1$, and $\mathbf{M}_2$, are given, respectively, by

$$\begin{aligned} \mathbf{X}_{2,h} &:= \mathbb{H}_0^r(\mathbf{div}_r;\Omega) \cap \mathbb{RT}_k(\mathcal{T}_h), \qquad \mathbf{M}_{1,h} := \mathbf{P}_k(\mathcal{T}_h), \\ \mathbf{X}_{1,h} &:= \mathbb{H}_0^s(\mathbf{div}_s;\Omega) \cap \mathbb{RT}_k(\mathcal{T}_h), \quad \text{and} \quad \mathbf{M}_{2,h} := \mathbf{P}_k(\mathcal{T}_h), \end{aligned} \tag{5.1}$$

whereas those of $\mathbf{Q}$, M, and $\mathbf{L}^2(\Omega)$, are defined as

$$\mathbf{Q}_h \,:=\, \mathbf{RT}_k(\mathcal{T}_h)\,, \quad \mathrm{M}_h \,:=\, \mathrm{P}_k(\mathcal{T}_h)\,, \quad \text{and} \quad \mathbf{H}_h^{\boldsymbol{t}} \,:=\, \mathbf{P}_k(\mathcal{T}_h)\,. \tag{5.2}$$

We stress here that $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\boldsymbol{t}}$ verify the assumptions (**H.1**) - (**H.4**). In fact, it is readily seen that $\mathrm{div}(\mathbf{Q}_h) \subseteq \mathrm{M}_h$, which confirms (**H.1**), whereas (**H.2**) is proved in [15, Lemma 4.5]. In turn, the assumptions **(H.3)** and **(H.4)** are shown in [3, Lemma 4.2].

5.2 The rates of convergence

The rates of convergence of the Galerkin schemes (4.1) and (4.2), with the specific finite element subspaces introduced in Section 5.1, are provided next. To this end, we require the approximation properties of $\mathbf{X}_{2,h}$, $\mathbf{M}_{1,h}$, $\mathbf{Q}_h$, M_h, and $\mathbf{H}_h^{\boldsymbol{t}}$, which are collected as follows (cf. [15, Section 4.5]):

($\mathbf{AP}_h^{\boldsymbol{\sigma}}$) there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\tau} \in \mathbb{W}^{l,r}(\Omega)$ with $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{\tau}, \mathbf{X}_{2,h}) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbf{X}_{2,h}} \|\boldsymbol{\tau} - \boldsymbol{\tau}_h\|_{r,\mathrm{div}_r;\Omega} \,\leqslant\, C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,r;\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|_{l,r;\Omega} \right\}.$$

($\mathbf{AP}_h^{\boldsymbol{u}}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{v} \in \mathbf{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{v}, \mathbf{M}_{1,h}) \,:=\, \inf_{\boldsymbol{v}_h \in \mathbf{M}_{1,h}} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{0,r;\Omega} \,\leqslant\, C\, h^l\, \|\boldsymbol{v}\|_{l,r;\Omega}\,.$$

($\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}}$) there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^l(\Omega)$ with $\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{W}^{l,s}(\Omega)$, there holds

$$\mathrm{dist}(\widetilde{\boldsymbol{\tau}}, \mathbf{Q}_h) \,:=\, \inf_{\widetilde{\boldsymbol{\tau}}_h \in X_{2,h}} \|\widetilde{\boldsymbol{\tau}} - \widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{div}_s;\Omega} \,\leqslant\, C\, h^l \left\{ \|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\tau}})\|_{l,s;\Omega} \right\}.$$

($\mathbf{AP}_h^{\phi}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\mathrm{dist}(\psi, \mathrm{M}_h) \,:=\, \inf_{\psi_h \in \mathrm{M}_h} \|\psi - \psi_h\|_{0,r;\Omega} \,\leqslant\, C\, h^l\, \|\psi\|_{l,r;\Omega}\,.$$

($\mathbf{AP}_h^{\boldsymbol{t}}$) there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\boldsymbol{s} \in \mathbf{H}^l(\Omega)$, there holds

$$\mathrm{dist}(\boldsymbol{s}, \mathbf{H}_h^{\boldsymbol{t}}) \,:=\, \inf_{\boldsymbol{s}_h \in \mathbf{H}_h^{\boldsymbol{t}}} \|\boldsymbol{s} - \boldsymbol{s}_h\|_{0,\Omega} \,\leqslant\, C\, h^l\, \|\boldsymbol{s}\|_{l,\Omega}\,.$$

Thus, the following two theorems establish the rates of convergence of (4.1) and (4.2).

Theorem 5.1. *Let* $\big((\boldsymbol{\sigma}, \boldsymbol{u}), (\widetilde{\boldsymbol{\sigma}}, \phi)\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{Q} \times \mathrm{M}\big)$ *be the unique solution of* (2.25)*, with* $\phi \in \widetilde{W}$ *(cf.* (3.19)*), and let* $\big((\boldsymbol{\sigma}_h, \boldsymbol{u}_h), (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{Q}_h \times \mathrm{M}_h\big)$ *be a solution of* (4.1)*, with* $\phi_h \in \widetilde{W}_h$ *(cf.* (4.17)*), whose existences are guaranteed by Theorems* 3.6 *and* 4.4*, respectively. Assume that* (4.31) *(cf. Theorem* 4.5*) holds, and that there exists* $l \in [1, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{W}^{l,r}(\Omega)$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,r}(\Omega)$*,* $\boldsymbol{u} \in \mathbf{W}^{l,r}(\Omega)$*,* $\widetilde{\boldsymbol{\sigma}} \in \mathbf{H}^l(\Omega)$*,* $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{W}^{l,s}(\Omega)$*, and* $\phi \in \mathrm{W}^{l,r}(\Omega)$*. Then, there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned}
&\|(\boldsymbol{\sigma}, \boldsymbol{u}) - (\boldsymbol{\sigma}_h, \boldsymbol{u}_h)\|_{\mathbf{X}_2 \times \mathbf{M}_1} + \|(\widetilde{\boldsymbol{\sigma}}, \phi) - (\widetilde{\boldsymbol{\sigma}}_h, \phi_h)\|_{\mathbf{Q} \times \mathrm{M}} \\
&\qquad \leqslant\, C\, h^l \left\{ \|\boldsymbol{\sigma}\|_{l,r;\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega} \,+\, \|\boldsymbol{u}\|_{l,r;\Omega} \,+\, \|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega} \,+\, \|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,s;\Omega} \,+\, \|\phi\|_{l,r;\Omega} \right\}.
\end{aligned}$$

Proof. It follows from the Céa estimate (4.32) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ - $(\mathbf{AP}_h^{\phi})$. $\square$

Theorem 5.2. *Let* $\big((\boldsymbol{\sigma},\boldsymbol{u}),(\vec{\phi},\widetilde{\boldsymbol{\sigma}})\big) \in \big(\mathbf{X}_2 \times \mathbf{M}_1\big) \times \big(\mathbf{H} \times \mathbf{Q}\big)$ *be the unique solution of* (2.26)*, with* $\phi \in W$ *(cf.* (3.37)*), and let* $\big((\boldsymbol{\sigma}_h,\boldsymbol{u}_h),(\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\big) \in \big(\mathbf{X}_{2,h} \times \mathbf{M}_{1,h}\big) \times \big(\mathbf{H}_h \times \mathbf{Q}_h\big)$ *be a solution of* (4.2)*, with* $\phi_h \in W_h$ *(cf.* (4.35)*), whose existences are guaranteed by Theorems* 3.9 *and* 4.6*, respectively. Assume that* (4.43) *(cf. Theorem* 4.7*) holds, and that there exists* $l \in [1, k+1]$ *such that* $\boldsymbol{\sigma} \in \mathbb{W}^{l,r}(\Omega)$*,* $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,r}(\Omega)$*,* $\boldsymbol{u} \in \mathbf{W}^{l,r}(\Omega)$*,* $\phi \in \mathrm{W}^{l,r}(\Omega)$*,* $\boldsymbol{t} \in \mathbf{H}^l(\Omega)$*,* $\widetilde{\boldsymbol{\sigma}} \in \mathbf{H}^l(\Omega)$*, and* $\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{W}^{l,s}(\Omega)$*. Then there exists a positive constant* C*, independent of* h*, such that*

$$\begin{aligned}&\|(\boldsymbol{\sigma},\boldsymbol{u}) - (\boldsymbol{\sigma}_h,\boldsymbol{u}_h)\|_{\mathbf{X}_2\times\mathbf{M}_1} + \|(\vec{\phi},\widetilde{\boldsymbol{\sigma}}) - (\vec{\phi}_h,\widetilde{\boldsymbol{\sigma}}_h)\|_{\mathbf{H}\times\mathbf{Q}} \\ &\quad\leqslant C\,h^l\Big\{\|\boldsymbol{\sigma}\|_{l,r;\Omega} + \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,r;\Omega} + \|\boldsymbol{u}\|_{l,r;\Omega} + \|\phi\|_{l,r;\Omega} + \|\boldsymbol{t}\|_{l,\Omega} + \|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega} + \|\mathrm{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,s;\Omega}\Big\}.\end{aligned}$$

Proof. It follows from the Céa estimate (4.44) and the approximation properties $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ - $(\mathbf{AP}_h^{\boldsymbol{t}})$. $\square$

6 Numerical results

In this section we present three examples illustrating the performance of the fully-mixed finite schemes (4.1) and (4.2) with the finite element subspaces defined in Section 5.1 for $k \in \{0,1\}$, and confirming the rates of convergence provided by Theorems 5.1 and 5.2 on uniform refinements of the respective domains. The resulting nonlinear algebraic systems are solved employing the Picard iterative process suggested by the respective discrete fixed-point strategy (cf. Sections 4.5 and 4.6), whose computational implementation was done making use of a `FreeFem++` code [18]. We take as initial guess the trivial solution, and, denoting by `DOF` the total number of degrees of freedom (or unknowns) of each approach, the iterations are stopped when the relative error between two consecutive vectors containing the full solutions of the aforementioned systems, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leqslant \mathsf{tol}\,,$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$, and tol is a given tolerance. In this regard, we remark in advance that for each one of the examples to be reported below, 3 iterations are required to achieve $\mathsf{tol} = 1\mathrm{e}-6$.

We now recall that the original Cauchy stress tensor $\boldsymbol{\rho}$ of our model can be computed in terms of $\boldsymbol{\sigma}$ according to the formula derived from [14, eqs. (2.9) and (2.10)] and [14, eq. (3.14)], namely

$$\boldsymbol{\rho} := \boldsymbol{\sigma} + \boldsymbol{\sigma}^{\mathrm{t}} - \left(\frac{\lambda+2\mu}{n\lambda+(n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma}) - \frac{n\lambda+2\mu}{n|\Omega|}\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}\,, \tag{6.1}$$

which naturally suggests to approximate this tensor by (cf. [14, eq. (6.1)])

$$\boldsymbol{\rho}_h := \boldsymbol{\sigma}_h + \boldsymbol{\sigma}_h^{\mathrm{t}} - \left(\frac{\lambda+2\mu}{n\lambda+(n+1)\mu}\,\mathrm{tr}(\boldsymbol{\sigma}_h) - \frac{n\lambda+2\mu}{n|\Omega|}\int_\Gamma \boldsymbol{u}_D\cdot\boldsymbol{\nu}\right)\mathbb{I}\,. \tag{6.2}$$

It follows from (6.1) and (6.2) that there exists a constant $C > 0$, independent of h and λ, such that

$$\|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega} \leqslant C\,\|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{0,r;\Omega}\,,$$

whence the rate of convergence for $\boldsymbol{\rho}_h$ is at least the same of $\boldsymbol{\sigma}_h$.

Some additional notation is introduced next. We begin by defining the individual errors:

$$\begin{gathered}\mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{r,\mathbf{div}_r;\Omega}\,, \quad \mathsf{e}(\boldsymbol{u}) := \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,r;\Omega}\,, \quad \mathsf{e}(\boldsymbol{\rho}) := \|\boldsymbol{\rho} - \boldsymbol{\rho}_h\|_{0,r;\Omega}\,, \\ \mathsf{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_s;\Omega}\,, \quad \mathsf{e}(\phi) := \|\phi - \phi_h\|_{0,r;\Omega}\,, \quad \text{and} \quad \mathsf{e}(\boldsymbol{t}) := \|\boldsymbol{t} - \boldsymbol{t}_h\|_{0,\Omega}\,,\end{gathered}$$

where r and s, taken from (2.1), will be specified in the examples below. In turn, for each $\star \in \{\boldsymbol{\sigma}, \boldsymbol{u}, \boldsymbol{\rho}, \widetilde{\boldsymbol{\sigma}}, \phi, \boldsymbol{t}\}$ we let $\mathsf{r}(\star)$ be its experimental rate of convergence, which is defined as

$$\mathsf{r}(\star) := \log\big(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star)\big)/\log(h/\widehat{h}),$$

where e and $\widehat{\mathsf{e}}$ denote two consecutive errors with mesh sizes h and $\widehat{h}$, respectively.

The examples to be considered in this section are described next. In each case we let E and ν be the Young modulus and Poisson ratio, respectively, of the isotropic linear elastic solid occupying the region Ω, so that the corresponding Lamé parameters are given by

$$\mu := \frac{E}{2(1+\nu)} \quad \text{and} \quad \lambda := \frac{E\nu}{(1+\nu)(1-2\nu)}. \tag{6.3}$$

In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a real Lagrange multiplier, which reduces to adding one row and one column to the matrix system that solves (4.4) for $\boldsymbol{\sigma}_h$ and $\boldsymbol{u}_h$.

Example 1: Convergence in a 2D domain

We begin by corroborating the rates of convergence against a smooth exact solution in the two-dimensional domain $\Omega = (0,1)^2$. To this end, we adequately manufacture the data so that the solution of (1.1)-(1.2) is given by

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{pmatrix} 0.05\,\cos(\pi\,x_1)\,\sin(\pi\,x_2) + \dfrac{x_1^2(1-x_2)^2}{2\lambda} \\[2ex] -0.05\,\sin(\pi\,x_1)\,\cos(\pi\,x_2) + \dfrac{x_1^3(1-x_2)^3}{2\lambda} \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) := (1-x_1)^2 x_1 (1-x_2) x_2^2,$$

for all $\boldsymbol{x} := (x_1, x_2)^{\mathrm{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\mathbf{f}(\phi) := \frac{1}{10}\begin{pmatrix} \cos(\phi) \\ -\sin(\phi) \end{pmatrix}, \quad g(\boldsymbol{u}) := 2 + \frac{1}{1+|\boldsymbol{u}|^2}, \quad \vartheta(\boldsymbol{\sigma}) := \mathbb{I} + \frac{1}{10}\boldsymbol{\sigma}^2.$$

We note here that the second and fifth equation of (1.1), actually include additional explicit source terms that are added to $\mathbf{f}(\phi)$ and $g(\boldsymbol{u})$, respectively. However, yielding only slight modifications of the functionals G, F_ϕ, $\widetilde{G}_{\boldsymbol{u}}$ and $G_{\boldsymbol{u}}$ (cf. (2.3), (2.4), (2.15) and (2.23), respectively), this fact does not compromise the continuous and discrete analyses. Thus, in Tables 6.1 and 6.2 we summarize the convergence of (4.1) and (4.2), respectively, considering the Young's modulus $E = 1$ and the Poisson's ratio $\nu = 0.4999$, which, according to (6.3), yield $\lambda = 1666.44$ and $\mu = 0.3334$. The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorems 5.1 and 5.2 are attained for $k \in \{0,1\}$. Some components and magnitudes of the discrete solutions of the first approach (4.1) are displayed in Figure 6.1.

Example 2: Convergence in a non-convex 2D domain

We consider the L-shaped domain $\Omega = (-1,1)^2 \backslash [0,1]^2$, and suitable perturbations of the definitions of the functionals G, F_ϕ, $\widetilde{G}_{\boldsymbol{u}}$, and $G_{\boldsymbol{u}}$, so that the exact solution of (1.1) - (1.2) reduces to the non-smooth one defined as:

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{pmatrix} |\boldsymbol{x}|^{2/3}\sin(\theta) \\ -|\boldsymbol{x}|^{2/3}\cos(\theta) \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) := \exp(x_1+x_2)\sin(\pi x_1)\sin(\pi x_2),$$

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$
0	0.0471	13680	4.12e-2	−−	1.44e-3	−−	1.83e-2	−−	1.03e-2	−−	5.26e-4	−−
	0.0393	19656	3.43e-2	1.01	1.20e-3	1.03	1.57e-2	0.99	8.65e-3	1.00	4.38e-4	1.00
	0.0337	26712	2.93e-2	1.01	1.02e-3	1.02	1.34e-2	0.99	7.40e-3	1.00	3.76e-4	1.00
	0.0295	34848	2.57e-2	1.01	8.93e-4	1.02	1.18e-2	0.99	6.48e-3	1.00	3.29e-4	1.00
	0.0262	44064	2.29e-2	1.01	7.92e-4	1.01	1.05e-2	1.00	5.76e-3	1.00	2.92e-4	1.00
1	0.0471	43560	5.33e-4	−−	2.83e-5	−−	2.63e-4	−−	2.53e-4	−−	1.53e-5	−−
	0.0393	62640	3.70e-4	2.00	1.96e-5	2.00	1.83e-4	1.99	1.76e-4	2.00	1.06e-5	2.00
	0.0337	85176	2.72e-4	2.00	1.44e-5	2.00	1.35e-4	1.99	1.29e-4	2.00	7.82e-6	2.00
	0.0295	111168	2.08e-4	2.00	1.10e-5	2.00	1.03e-4	1.99	9.89e-5	2.00	5.99e-6	2.00
	0.0262	140616	1.65e-4	2.00	8.71e-6	2.00	8.16e-5	1.99	7.82e-5	2.00	4.73e-6	2.00
0	0.0471	13680	5.14e-2	−−	1.59e-3	−−	2.31e-2	−−	9.96e-3	−−	6.47e-4	−−
	0.0393	19656	4.27e-2	1.01	1.32e-3	1.03	1.93e-2	0.99	8.29e-3	1.00	5.40e-4	1.00
	0.0337	26712	3.66e-2	1.01	1.12e-3	1.02	1.66e-2	0.99	7.11e-3	1.00	4.63-4	1.00
	0.0295	34848	3.20e-2	1.01	9.82e-4	1.02	1.45e-2	0.99	6.22e-3	1.00	4.05e-4	1.00
	0.0262	44064	2.84e-2	1.01	8.72e-4	1.01	1.29e-2	1.00	5.53e-3	1.00	3.60e-4	1.00
1	0.0471	43560	6.15e-4	−−	3.08e-5	−−	3.10e-4	−−	2.45e-4	−−	1.92e-5	−−
	0.0393	62640	4.27e-4	2.00	2.14e-5	2.00	2.15e-4	1.99	1.70e-4	2.00	1.33e-5	2.00
	0.0337	85176	3.14e-4	2.00	1.57e-5	2.00	1.58e-4	2.00	1.25e-4	2.00	9.79e-6	2.00
	0.0295	111168	2.40e-4	2.00	1.20e-5	2.00	1.21e-4	2.00	9.68e-5	2.00	7.50e-6	2.00
	0.0262	140616	1.90e-4	2.00	9.50e-6	2.00	9.59e-5	2.00	7.57e-5	2.00	5.93e-6	2.00

Table 6.1: Example 1: History of convergence for the Galerkin scheme (4.1) with $r = 3$ (upper half), and $r = 4$ (lower half).

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$
0	0.0471	17280	4.12e-1	−−	1.44e-3	−−	1.88e-2	−−	1.04e-2	−−	5.25e-4	−−	3.24-03	−−
	0.0393	24840	3.43e-2	1.01	1.20e-3	1.03	1.57e-2	0.98	8.63e-3	1.00	4.38e-4	1.00	2.70e-3	1.03
	0.0337	33768	2.93e-2	1.01	1.02e-3	1.02	1.34e-2	0.99	7.40e-3	1.00	3.76e-4	1.00	2.31e-3	1.02
	0.0295	44064	2.57e-2	1.01	8.93e-4	1.02	1.18e-2	0.99	6.48e-3	1.00	3.29e-4	1.00	2.02e-3	1.02
	0.0262	55728	2.28e-2	1.01	7.92e-4	1.01	1.05e-2	1.00	5.76e-3	1.00	2.92e-4	1.00	1.18e-3	1.01
1	0.0471	54360	5.33e-4	−−	2.82e-5	−−	2.63e-4	−−	2.53e-4	−−	1.53e-5	−−	7.69e-5	2.00
	0.0393	78192	3.70e-4	2.00	1.96e-5	2.00	1.83e-4	1.99	1.76e-4	2.00	1.06e-5	2.00	5.34e-5	2.00
	0.0337	106344	2.72e-4	2.00	1.44e-5	2.00	1.35e-4	1.99	1.29e-4	2.00	7.82e-6	2.00	3.93e-5	2.00
	0.0295	138816	2.08e-4	2.00	1.10e-5	2.00	1.03e-4	1.99	9.89e-5	2.00	5.99e-6	2.00	3.01e-5	2.00
	0.0262	175608	1.65e-4	2.00	8.71e-6	2.00	8.16e-5	1.99	7.82e-5	2.00	4.73e-6	2.00	2.48e-5	2.00
0	0.0471	17280	5.14e-2	−−	1.59e-3	−−	2.31e-2	−−	9.95e-3	−−	6.47e-4	−−	3.24-03	1.03
	0.0393	24840	4.27e-2	1.01	1.32e-3	1.03	1.93e-2	0.99	8.29e-3	1.00	5.39e-4	1.00	2.70e-3	1.02
	0.0337	33768	3.66e-2	1.01	1.12e-3	1.02	1.66e-2	0.99	7.11e-3	1.00	4.63e-4	1.00	2.31e-3	1.02
	0.0295	44064	3.20e-2	1.01	9.82e-4	1.01	1.45e-2	0.99	6.22e-3	1.00	4.05e-4	1.00	2.02e-3	1.01
	0.0262	55728	2.84e-2	1.01	8.72e-4	1.01	1.29e-2	1.00	5.53e-3	1.00	3.60e-4	1.00	1.18e-3	1.01
1	0.0471	54360	6.15e-3	−−	3.08e-5	−−	3.10e-4	−−	2.45e-4	−−	1.92e-5	−−	7.69e-5	2.00
	0.0393	78192	4.27e-4	2.00	2.14e-5	2.00	2.15e-4	1.99	1.70e-4	2.00	1.33e-5	2.00	5.34e-5	2.00
	0.0337	106344	3.14e-4	2.00	1.57e-5	2.00	1.58e-4	2.00	1.25e-4	2.00	9.79e-6	2.00	3.93e-5	2.00
	0.0295	138816	2.40e-4	2.00	1.20e-5	2.00	1.21e-4	2.00	9.58e-5	2.00	7.50e-6	2.00	3.01e-5	2.00
	0.0262	175608	1.90e-4	2.00	9.50e-6	2.00	9.59e-5	2.00	7.57e-5	2.00	5.93e-6	2.00	2.48e-5	2.00

Table 6.2: Example 1: History of convergence for the Galerkin scheme (4.2) with $r = 3$ (upper half), and $r = 4$ (lower half).

where $\theta = \arctan\left(\frac{x_2}{x_1}\right)$ for all $\boldsymbol{x} = (x_1, x_2)^{\mathtt{t}} \in \Omega$. In turn, the tensorial diffusivity is considered the same from the previous example, whereas the body load and the diffusive source are given, respectively, by

$$\boldsymbol{f}(\phi) := \begin{pmatrix} \dfrac{1}{40}\phi \\ \dfrac{1}{40}\phi(1-\phi) \end{pmatrix} \quad \text{and} \quad g(\boldsymbol{u}) := -|\boldsymbol{u}|\,.$$

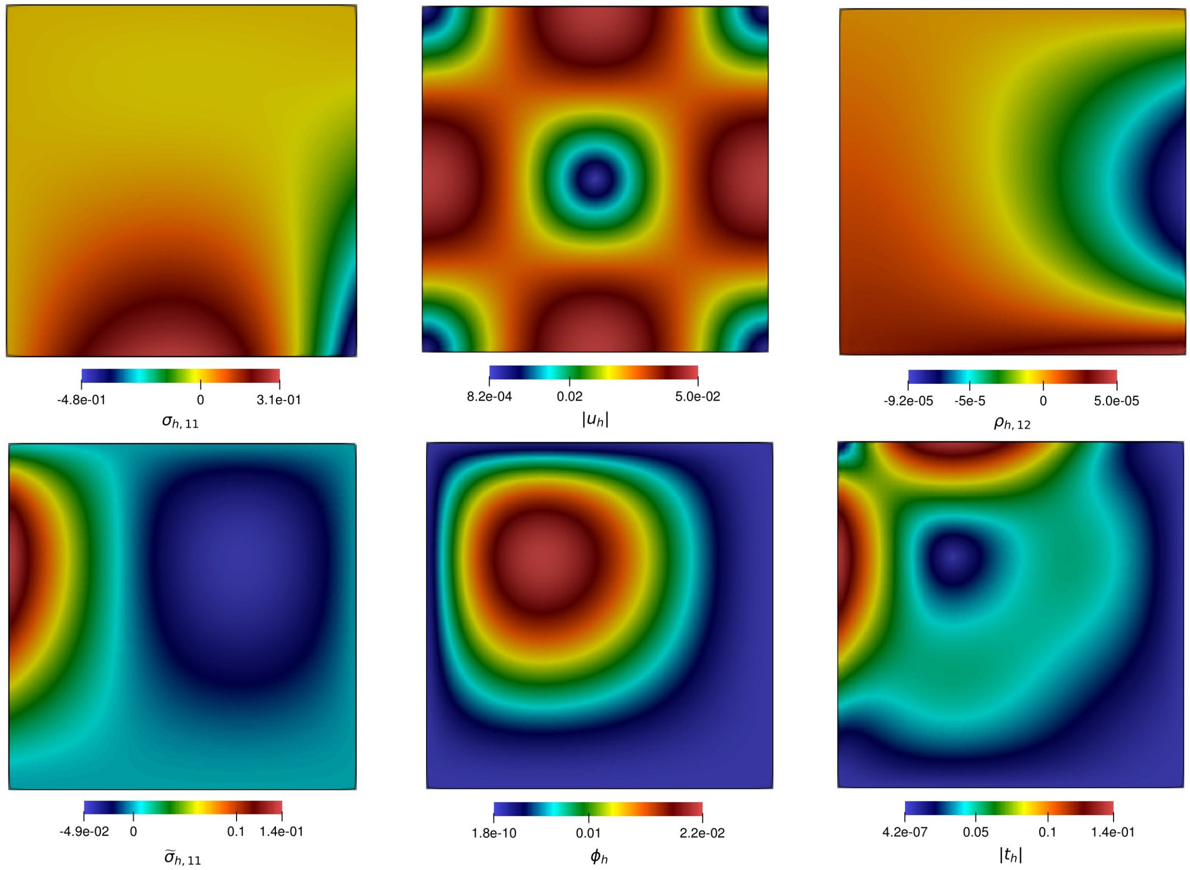

Figure 6.1: Example 1: Some components and magnitudes of the solution of the first approach (4.1) with $k = 1$, $\lambda = 1666.44$, and $\mu = 0.3334$.

In this case, we take $E = 100$ and $\nu = 0.4999$, which yields $\mu = 33.33$ and $\lambda = 166644.44$. Here we can see in Tables 6.3 and 6.4 that it was not possible to reach the convergence order $k + 1$ indicated by Theorems 5.1 and 5.2. In particular, we notice that, for both formulations (cf. (4.1) and (4.2)), negative convergence orders are obtained for $\boldsymbol{\sigma}$, while for $\boldsymbol{u}$, $\boldsymbol{\rho}$, and $\widetilde{\boldsymbol{\sigma}}$, suboptimal ones are attained. Furthermore, as it was observed in [14, Section 6], we remark that the convergence ratios depend not only on k but also on r and its conjugate s, which could be related to the $\mathbf{W}^{l,r}$- regularity of the solution, most likely with a non-integer l depending on r. We refer to [15, Lemma B.1] for a similar situation holding with a regularity result for the Poisson problem with homogeneous Neumann boundary conditions and source term in a Lebesgue space. In order to recover the optimal rates of convergence, one could apply an adaptive strategy based on a posteriori error estimates, subject that we plan to address in a forthcoming work.

Example 3: Convergence in a 3D domain

In this example we confirm the rates of convergence in the three dimensional domain $\Omega = (0,1)^3$ with the indexes $r = 3$ and $s = 3/2$ (cf. (2.1)). As in Example 1, we consider $\mu = 0.3334$ and $\lambda = 1666.44$, and suitably manufacture the data so that the exact solution is given by

$$\boldsymbol{u}(\boldsymbol{x}) := \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix} \quad \text{and} \quad \phi(\boldsymbol{x}) := x_1 x_2^2 x_3 (x_1 - 1)^2 (x_2 - 1)(x_3 - 1)^2 \,,$$

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$
	0.0471	40860	9.56e+2	−−	1.85e-2	−−	7.48e+0	−−	2.62e+1	−−	3.20e-2	−−
	0.0404	55545	1.06e+3	-0.66	1.59e-2	0.99	7.10e+0	0.34	2.31e+1	0.82	2.72e-2	1.05
0	0.0354	72480	1.16e+3	-0.66	1.39e-2	0.99	6.79e+0	0.33	2.07e+1	0.81	2.37e-2	1.04
	0.0314	91665	1.25e+3	-0.66	1.23e-2	0.99	6.53e+0	0.33	1.88e+1	0.80	2.10e-2	1.03
	0.0283	113100	1.34e+3	-0.66	1.11e-2	0.99	6.30e+0	0.33	1.73e+1	0.80	1.88e-2	1.03
	0.0471	130320	4.73e+2	−−	4.92e-4	−−	4.78e+0	−−	9.10e+0	−−	6.78e-4	−−
	0.0404	177240	5.24e+2	-0.66	4.00e-4	1.33	4.54e+0	0.33	8.15e+0	0.71	4.99e-4	1.99
1	0.0354	231360	5.73e+2	-0.66	3.35e-4	1.33	4.34e+0	0.33	7.42e+0	0.71	3.83e-4	1.99
	0.0314	292680	6.19e+2	-0.66	2.87e-4	1.33	4.18e+0	0.33	6.83e+0	0.70	3.03e-4	1.99
	0.0283	361200	6.64e+2	-0.66	2.49e-4	1.33	4.03e+0	0.33	6.34e+0	0.70	2.46e-4	1.98
	0.0471	40860	1.92e+3	−−	1.88e-2	−−	1.28e+1	−−	2.35e+1	−−	3.48e-2	−−
	0.0404	55545	2.18e+3	-0.83	1.61e-2	0.99	1.25e+1	0.17	2.04e+1	0.91	2.96e-2	1.03
0	0.0354	72480	2.44e+3	-0.83	1.41e-2	0.99	1.22e+1	0.17	1.81e+1	0.90	2.58e-2	1.03
	0.0314	91665	2.69e+3	-0.83	1.26e-2	0.99	1.20e+1	0.17	1.63e+1	0.90	2.29e-2	1.02
	0.0283	113100	2.91e+3	-0.83	1.13e-2	0.99	1.18e+1	0.17	1.48e+1	0.90	2.06e-2	1.02
	0.0471	130320	8.76e+2	−−	8.02e-4	−−	9.04e+0	−−	5.88e+0	−−	7.17e-4	−−
	0.0404	177240	9.95e+2	-0.83	6.70e-4	1.17	8.81e+0	0.17	5.12e+0	0.90	5.27e-4	2.00
1	0.0354	231360	1.11e+3	-0.83	5.74e-4	1.17	8.62e+0	0.17	4.55e+0	0.89	4.04e-4	2.00
	0.0314	292680	1.23e+3	-0.83	5.00e-4	1.17	8.45e+0	0.17	4.10e+0	0.89	3.19e-4	1.99
	0.0283	361200	1.34e+3	-0.83	4.42e-4	1.17	8.30e+0	0.17	3.74e+0	0.88	2.59e-4	1.99

Table 6.3: Example 2: History of convergence for the Galerkin scheme (4.1) with $r = 3$ (first half), and $r = 4$ (second half).

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$
	0.0471	51660	9.56e+2	−−	1.85e-2	−−	7.48e+0	−−	2.55e+1	−−	3.19e-2	−−	2.11e-1	−−
	0.0404	70245	1.06e+3	-0.66	1.59e-2	1.00	7.10e+0	0.34	2.25e+1	0.80	2.72e-2	1.04	1.80e-1	1.02
0	0.0354	91680	1.16e+3	-0.66	1.39e-2	1.00	6.79e+0	0.33	2.02e+1	0.80	2.37e-2	1.04	1.58e-1	1.02
	0.0314	115965	1.25e+3	-0.66	1.24e-2	1.00	6.53e+0	0.33	1.85e+1	0.79	2.10e-2	1.03	1.40e-1	1.01
	0.0283	143100	1.34e+3	-0.66	1.11e-2	1.00	6.30e+0	0.33	1.70e+1	0.79	1.88e-2	1.02	1.26e-1	1.01
	0.0471	162720	4.73e+2	−−	4.92e-4	−−	4.78e+0	−−	8.92e+0	−−	6.74e-04	−−	6.98e-3	−−
	0.0404	221340	5.24e+2	-0.66	4.01e-4	1.33	4.54e+0	0.33	8.01e+0	0.70	4.96e-04	1.99	5.24e-3	1.86
1	0.0354	288960	5.73e+2	-0.66	3.35e-4	1.33	4.34e+0	0.33	7.30e+0	0.70	3.80e-04	1.99	4.09e-3	1.86
	0.0314	365580	6.19e+2	-0.66	2.87e-4	1.33	4.18e+0	0.33	6.73e+0	0.69	3.01e-04	1.99	3.28e-3	1.87
	0.0283	451200	6.64e+2	-0.66	2.49e-4	1.33	4.03e+0	0.33	6.25e+0	0.69	2.44e-04	1.99	2.69e-3	1.87
	0.0471	51660	1.92e+3	−−	1.88e-2	−−	1.28e+1	−−	2.28e+1	−−	3.47e-2	−−	2.11e-1	−−
	0.0404	70245	2.18e+3	-0.83	1.61e-2	0.99	1.25e+1	0.17	1.99e+1	0.89	2.96e-2	1.04	1.80e-1	1.02
	0.0354	91680	2.44e+3	-0.83	1.41e-2	0.99	1.22e+1	0.17	1.77e+1	0.89	2.58e-2	1.04	1.58e-1	1.02
0	0.0314	115965	2.69e+3	-0.83	1.26e-2	0.99	1.20e+1	0.17	1.59e+1	0.89	2.29e-2	1.03	1.40e-1	1.01
	0.0283	143100	2.94e+3	-0.83	1.13e-2	0.99	1.18e+1	0.17	1.45e+1	0.89	2.06e-2	1.02	1.26e-1	1.01
	0.0471	162720	8.76e+2	−−	8.02e-4	−−	9.04e+0	−−	5.70e+0	−−	7.14e-04	−−	6.98e-3	−−
	0.0404	221340	9.95e+2	-0.83	6.70e-4	1.17	8.81e+0	0.17	4.98e+0	0.88	5.25e-04	2.00	5.24e-3	1.86
1	0.0354	288960	1.11e+3	-0.83	5.74e-4	1.17	8.62e+0	0.17	4.43e+0	0.88	4.02e-04	2.00	4.09e-3	1.86
	0.0314	365580	1.23e+3	-0.83	5.00e-4	1.17	8.45e+0	0.17	4.00e+0	0.87	3.18e-04	1.99	3.28e-3	1.87
	0.0283	451200	1.34e+3	-0.83	4.42e-4	1.17	8.30e+0	0.17	3.65e+0	0.87	2.58e-04	1.99	2.69e-3	1.87

Table 6.4: Example 2: History of convergence for the Galerkin scheme (4.2) with $r = 3$ (first half), and $r = 4$ (second half).

for all $\boldsymbol{x} := (x_1, x_2, x_3)^{\mathtt{t}} \in \Omega$, whereas the body load, the diffusive source, and the tensorial diffusivity, are given, respectively, by

$$\boldsymbol{f}(\phi) := \frac{1}{10}\begin{pmatrix} \cos(\phi) \\ -\sin(\phi) \\ \cos(\phi) \end{pmatrix}, \quad g(\boldsymbol{u}) := u_1 + u_2 + u_3\,, \quad \vartheta(\boldsymbol{\sigma}) := \frac{1}{2}\left(1 + \frac{1}{(1+|\boldsymbol{\sigma}|^2)^{1/2}}\right)\mathbb{I}\,.$$

The convergence histories for quasi-uniform refinements using $k = 0$ are reported in Tables 6.5 and 6.6. Again, the mixed finite element methods converge optimally, that is with order $\mathcal{O}(h)$ in this case,

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$
	0.4330	4992	3.11e+2	−−	2.69e-1	−−	6.51e+1	−−	7.35e-3	−−	4.00e-4	−−
	0.3464	9600	2.51e+2	0.95	2.18e-1	0.95	5.26e+1	0.96	5.58e-3	1.23	3.30e-4	0.87
	0.2887	16416	2.11e+2	0.97	1.83e-1	0.97	4.40e+1	0.97	4.50e-3	1.18	2.79e-4	0.92
0	0.2474	25872	1.81e+2	0.98	1.57e-1	0.98	3.79e+1	0.98	3.78e-3	1.13	2.41e-4	0.95
	0.2165	38400	1.59e+2	0.98	1.38e-1	0.98	3.32e+1	0.99	3.25e-3	1.13	2.12e-4	0.96
	0.1925	54432	1.41e+2	0.99	1.23e-1	0.99	2.96e+1	0.99	2.87e-3	1.07	1.89e-4	0.97
	0.1732	74400	1.27e+2	0.99	1.11e-1	0.99	2.66e+1	0.99	2.55e-3	1.12	1.70e-4	0.98

Table 6.5: Example 3: History of convergence for the Galerkin scheme (4.1) with $r = 3$ and $s = 3/2$.

k	h	DOF	$\mathtt{e}(\boldsymbol{\sigma})$	$\mathtt{r}(\boldsymbol{\sigma})$	$\mathtt{e}(\boldsymbol{u})$	$\mathtt{r}(\boldsymbol{u})$	$\mathtt{e}(\boldsymbol{\rho})$	$\mathtt{r}(\boldsymbol{\rho})$	$\mathtt{e}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{r}(\widetilde{\boldsymbol{\sigma}})$	$\mathtt{e}(\phi)$	$\mathtt{r}(\phi)$	$\mathtt{e}(\boldsymbol{t})$	$\mathtt{r}(\boldsymbol{t})$
	0.4330	6144	3.11e+2	−−	2.69e-1	−−	6.52e+1	−−	7.35e-3	−−	3.89e-4	−−	2.60e-3	−−
	0.3464	11850	2.51e+2	0.95	2.18e-1	0.95	5.26e+1	0.96	5.58e-3	1.23	3.23e-4	0.83	2.14e-3	0.89
	0.2887	20304	2.11e+2	0.97	1.83e-1	0.97	4.40e+1	0.97	4.50e-3	1.18	2.75e-4	0.89	1.81e-3	0.92
0	0.2474	32046	1.81e+2	0.98	1.57e-1	0.98	3.79e+1	0.98	3.78e-3	1.13	2.38e-4	0.93	1.56e-3	0.95
	0.2165	47616	1.59e+2	0.98	1.38e-1	0.98	3.32e+1	0.99	3.25e-3	1.13	2.10e-4	0.95	1.38e-3	0.96
	0.1925	67554	1.41e+2	0.99	1.23e-1	0.99	2.96e+1	0.99	2.87e-3	1.07	1.87e-4	0.96	1.23e-3	0.97
	0.1732	92400	1.27e+2	0.99	1.11e-1	0.99	2.66e+1	0.99	2.55e-3	1.12	1.69e-4	0.97	1.11e-3	0.98

Table 6.6: Example 3: History of convergence for the Galerkin scheme (4.2) with $r = 3$ and $s = 3/2$.

as it was proved by Theorems 5.1 and 5.2. This fact suggests that perhaps only technical difficulties stop us from extending the analysis to the 3D framework. Finally, some components and magnitudes of the solution of the second approach (4.2) are displayed in Figure 6.2.

Concluding remarks

In this paper we have continued advancing in the direction of [14] by introducing and analyzing two new Banach spaces-based fully-mixed finite element methods for the numerical solution of pseudostress-assisted diffusion problems. As compared with the mixed-primal method from [14], the main advantages of the schemes proposed here, which actually arise from the use of two different mixed approaches for the diffusion equation, are given by the fact that some additional variables of physical interest, such as the diffusive flux and the concentration gradient, are approximated directly. In this way, and differently from what one would do to obtain approximations of those variables starting from the numerical solutions provided by the method from [14], no numerical differentiation, with the consequent loss of accuracy, is employed in the present case. Regarding a comparison between the two fully-mixed finite element methods developed here, we first notice from the respective theoretical results, which are confirmed by the reported numerical results, that, under assumed regularities of the exact solution, they provide the same rates of convergence. However, we also observe from the tables that in order to attain a given accuracy, the second method requires a bit higher number of degrees of freedom, which is explained by the fact that the latter incorporates one more unknown than the first one. A minor aspect, though not that relevant, is that the tensorial diffusivity function does not need to be inverted in the second approach. Therefore, both methods are fully comparable, and deciding which one to employ for practical computations will depend on whether, besides the diffusive flux, the user is interested or not in obtaining also direct approximations of the concentration gradient.

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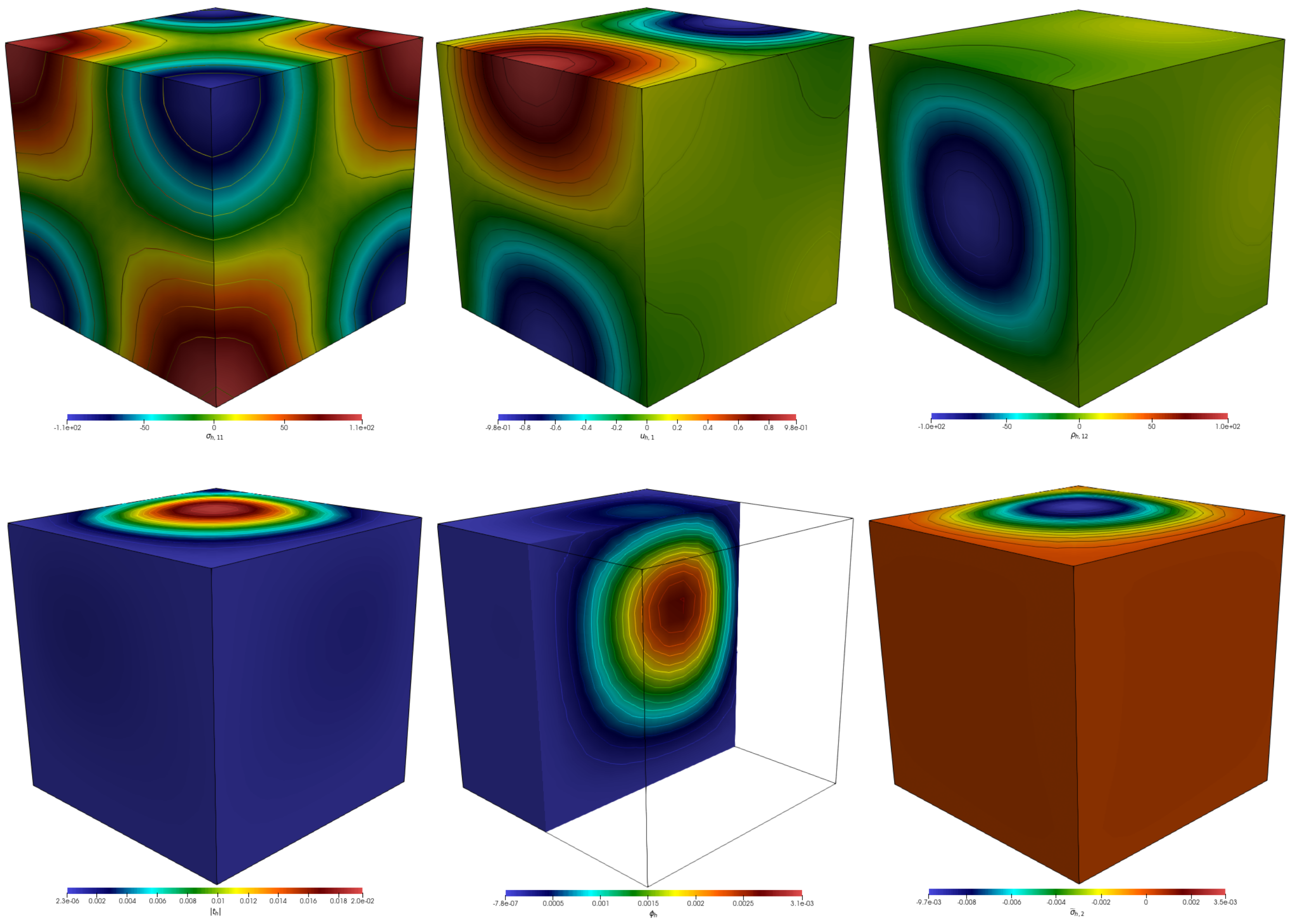

Figure 6.2: Example 3: Some components and magnitudes of the solution of the second approach (4.2) with $k = 0$, $\lambda = 1666.44$, and $\mu = 0.3334$.

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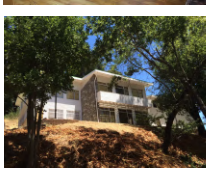

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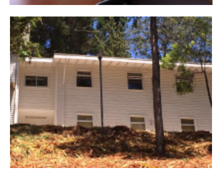

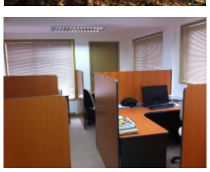

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