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STOKES PROBLEM WITH SLIP BOUNDARY CONDITIONS USING STABILIZED FINITE ELEMENTS COMBINED WITH NITSCHE

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Abstract. We discuss how slip conditions for the Stokes equation can be handled using Nitsche method, for a stabilized finite element discretization. Emphasis is made on the interplay between stabilization and Nitsche terms. Well-posedness of the discrete problem and optimal convergence rates are established, and illustrated with various numerical experiments.

1. Introduction

Slip boundary conditions arise naturally for Stokes or Navier-Stokes equations, for instance when modelling biological surfaces [4], in slide coating [11] or in the context of turbulence modeling [23]. These are essential boundary conditions, and can be in fact considered as generalized Dirichlet conditions. They are not straightforward to implement into standard finite element libraries, with standard techniques such as a discrete lifting or a partitioning of the global matrix. As a result, many works were devoted to study alternative approaches.

This work presents a simple approach based on Nitsche's technique combined with a stabilized equal-order finite element method. To simplify the presentation, we focus on the Stokes equation on a polygonal boundary and without any specific law that involve the tangential components of the velocity, such as a Navier law. We consider both symmetric and non-symmetric variants of Nitsche, since they have different advantages, particularly to enforce accurately the boundary condition, see, *e.g.* [10, 17] and references therein. We take advantage of the stabilization terms to carry out the analysis. Notably we are able to prove the stability with a constant independent of the fluid viscosity. The overall method is consistent, introduces no extra unknown and can be implemented easily. To assess the properties of the method, we propose an implementation in the FEniCS environment [1] and present several numerical experiments.

Let us put our work in a general perspective. The first methods to enforce slip conditions were based on Lagrange multipliers: see, e.g., [22, 27, 28]. In [2] the condition was enforced pointwise at nodal values of the velocity. Many studies have been devoted to the study of penalty methods, to enforce approximately the slip condition with a regularization term. These methods are not consistent, but remain popular and very easy to implement. Moreover, penalty can be interpreted as a penetration condition with a given resistance [18]. A first work has been focused on the Navier-Stokes equation [9], and followed by [12, 13], with emphasis on the case of a curved boundary, where a Babuska-type paradox may appear. Other recent works have been devoted to the usage of penalty terms combined with Lagrange finite elements [20, 30, 31] or Crouzeix-Raviart finite elements [21, 32]. To our knowledge, Nitsche's method has been first considered in [15], as a simple, consistent and primal technique to take into account the slip condition. Notably, it has been noticed that the skew-symmetric variant of Nitsche remains operational even when the Nitsche parameter vanishes (*penalty-free* variants), a result which opened the path to further research on this topic [6, 7, 8]. Later on, in [26], different variants of Nitsche have been proposed and linked, as usual, with stabilized mixed methods (following [25]). Emphasis has been once again made on the curved boundary and a possibly related Babuska-type paradox. More recently, a specific treatment of the Navier boundary condition has been studied in [29], building on the specific Nitsche-type method proposed by Juntunen & Stenberg [19] to discretize robustly Robin-type boundary conditions (see also [33]), and a symmetric Nitsche method with specific, accurate, discretization of the curved boundary, has been designed and studied in [16]. In conclusion, we observe that almost all the aforementioned works have considered inf-sup stable pairs to discretize the Stokes equation, except [20] where the penalty method combined with a $\mathbb{P}_1/\mathbb{P}_1$ finite element pair with pressure stabilization is taken into account.

This paper is structured as follows. Section 2 describes the model equations in strong form. The weak formulation and the corresponding functional setting is object of Section 3. Section 4 presents the discretization with finite elements, stabilization and Nitsche. Section 5 details the stability and convergence analysis. Numerical experiments are provided in Section 6.

2. Model problem

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded domain with Lipschitz continuous boundary $\partial\Omega$. We use standard notation for Lebesgue spaces $L^q(\Omega)$, with norm $\|\cdot\|_{0,q,\Omega}$, for $q > 2$, and $\|\cdot\|_{0,\Omega}$ for $q = 2$ and inner product $(\cdot,\cdot)_\Omega$, and Sobolev spaces $H^m(\Omega)$, with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$. The boundary $\partial\Omega$ is partitionned into a subset Γ_D, where a Dirichlet boundary condition is imposed, with $\text{meas}(\Gamma_D) > 0$, and a subset Γ_S, where the slip condition is enforced. Moreover, we denote with $\boldsymbol{n}$ the outer normal vector to Γ_S, and with $\boldsymbol{t}_i$, $1 \le i \le d-1$ are orthonormal vectors spanning the plane tangent to Γ_S.

We consider the Stokes equations seeking for a velocity field $\boldsymbol{u} : \Omega \to \mathbb{R}^d$ and a pressure field $p : \Omega \to \mathbb{R}$ solutions to

$$\begin{cases} -\nabla\cdot\sigma(\boldsymbol{u},p) &= \boldsymbol{f} \quad \text{in } \Omega, \\ \nabla\cdot\boldsymbol{u} &= 0 \quad \text{in } \Omega, \\ \boldsymbol{u} &= \boldsymbol{0} \quad \text{on } \Gamma_D, \\ \boldsymbol{u}\cdot\boldsymbol{n} &= 0 \quad \text{on } \Gamma_S \\ \sigma(\boldsymbol{u},p)\boldsymbol{n}\cdot\boldsymbol{t}_i &= s_i, \quad 1 \le i \le d-1 \quad \text{on } \Gamma_S. \end{cases} \tag{2.1}$$

In (2.1), the stress tensor is expressed as

$$\sigma(\boldsymbol{u},p) := 2\nu\boldsymbol{\varepsilon}(\boldsymbol{u}) - p\mathrm{I},$$

the parameter $\nu > 0$ denotes the fluid viscosity, $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \frac{1}{2}(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^T)$ stands for the symmetric part of the deformation tensor, $\boldsymbol{f} \in L^2(\Omega)^d$ is a given source term and $s_i \in L^2(\Gamma_S)$.

3. Continuous variational formulation

We define the Hilbert spaces

$$\begin{aligned} \boldsymbol{H} &:= \{\boldsymbol{v} \in H^1(\Omega)^d \ : \ \boldsymbol{v} = \boldsymbol{0} \quad \text{on } \Gamma_D,\ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{on } \Gamma_S \ \}, \\ Q &:= L^2_0(\Omega) \end{aligned}$$

and consider the product space $\boldsymbol{H} \times Q$ equipped with the norm

$$\|(\boldsymbol{v},q)\|^2 := \nu\|\boldsymbol{\varepsilon}(\boldsymbol{v})\|^2_{0,\Omega} + \|q\|^2_{0,\Omega}, \qquad \forall(\boldsymbol{v},q) \in \boldsymbol{H}\times Q.$$

We then introduce the bilinear forms

$$a : \boldsymbol{H}\times\boldsymbol{H} \to \mathbb{R},\ a(\boldsymbol{u},\boldsymbol{v}) := 2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}),\boldsymbol{\varepsilon}(\boldsymbol{v}))_\Omega \qquad \forall\boldsymbol{u},\,\boldsymbol{v} \in \boldsymbol{H},$$

and

$$b : \boldsymbol{H}\times Q \to \mathbb{R},\ b(\boldsymbol{v},q) := -(\nabla\cdot\boldsymbol{v},q)_\Omega \qquad \forall(\boldsymbol{v},q) \in \boldsymbol{H}\times Q,$$

and consider the following variational formulation associated with problem (2.1):

Problem 1. *Find $(\boldsymbol{u},p) \in \boldsymbol{H}\times Q$ such that*

$$B((\boldsymbol{u},p),(\boldsymbol{v},q)) = F(\boldsymbol{v},q), \tag{3.1}$$

for all $(\boldsymbol{v},q) \in \boldsymbol{H}\times Q$, where

$$B((\boldsymbol{u},p),(\boldsymbol{v},q)) := a(\boldsymbol{u},\boldsymbol{v}) + b(\boldsymbol{v},p) - b(\boldsymbol{u},q) \tag{3.2}$$

and

$$F(\boldsymbol{v},q) := (\boldsymbol{f},\boldsymbol{v})_\Omega + \sum_{i=1}^{d-1}\int_{\Gamma_S} s_i\,\boldsymbol{v}\cdot\boldsymbol{t}_i\,\mathrm{d}s.$$

Theorem 3.1. *The variational problem (3.1) has a unique solution $(\boldsymbol{u},p) \in \boldsymbol{H}\times Q$, and there exists a positive constant C such that*

$$\|(\boldsymbol{u},p)\| \le C\left\{\|\boldsymbol{f}\|_{0,\Omega} + \sum_{i=1}^{d-1}\|s_i\|_{0,\Gamma_S}\right\}.$$

Proof. The proof is a direct consequence of the Babuska–Brezzi theory. See also [5]. □

4. Discrete stabilized scheme

In what follows, we denote by $\{\mathcal{T}_h\}_{h>0}$ a regular family of triangulations of $\bar{\Omega}$ composed by simplexes. For a given triangulation $\mathcal{T}_h$, we will denote by $\mathcal{E}_h$ the set of all faces (edges) of $\mathcal{T}_h$, with the partitioning

$$\mathcal{E}_h := \mathcal{E}_\Omega \cup \mathcal{E}_D \cup \mathcal{E}_S,$$

where $\mathcal{E}_\Omega$ stands for the faces (edges) lying in the interior of Ω, $\mathcal{E}_S$ stands for the faces (edges) lying on the boundary Γ_S, and $\mathcal{E}_D$ stands for the edges (faces) lying on the boundary Γ_D. Moreover, we will denote with K a generic element of a triangulation $\mathcal{T}_h$, with h_K the diameter of K and define $h := \max_{K\in\mathcal{T}_h} h_K$.

As next, for a given $l \geq 1$, we introduce the following finite element spaces:

$$\begin{aligned}\boldsymbol{H}_h &:= \left\{\boldsymbol{v} \in C(\overline{\Omega})^d \ : \ \boldsymbol{v}|_K \in \mathbb{P}_l(K)^d, \quad \forall K \in \mathcal{T}_h\right\},\\ Q_h &:= \left\{q \in C(\overline{\Omega}) \ : \ q|_K \in \mathbb{P}_l(K), \quad \forall K \in \mathcal{T}_h\right\} \cap Q,\end{aligned}$$

where $\mathbb{P}_l$ stands for the space of polynomials of total degree less or equal than l.

Remark. Note that Q_h is a subspace of Q, but $\boldsymbol{H}_h$ is not a subspace of $\boldsymbol{H}$. In that sense, imposing weakly the (slip or Dirichlet) boundary conditions using Nitsche, can be considered a non-conforming finite element method.

In the sequel we will need the following well known results:

Lemma 4.1. Let $\boldsymbol{v}_h \in \boldsymbol{H}_h$ then for each $K \in \mathcal{T}_h$; $l, m \in \mathbb{N}$, with $0 \leq m \leq l$, there exists a positive constant C_{in}, independent of K, such that

$$|\boldsymbol{v}_h|_{l,K} \leq C_{\text{in}}\, h_K^{m-l} |\boldsymbol{v}_h|_{m,K}.$$

Proof. See [14, Lemma 12.1]. □

Lemma 4.2. Let $\boldsymbol{v}_h \in \boldsymbol{H}_h$ then for each $K \in \mathcal{T}_h$, $E \subset \partial K$, there exists a positive constant C_{tr}, independent of K, such that

$$\|\boldsymbol{v}_h\|_{0,E} \leq C_{\text{tr}}\, h_K^{-\frac{1}{2}} \|\boldsymbol{v}_h\|_{0,K}.$$

Proof. See [14, Lemma 12.8]. □

Lemma 4.3. Let $v \in H^1(K)$, with $K \in \mathcal{T}_h$. Then for any face (edge) $E \subset \partial K$, there exists a positive constant C, independent of K, such that

$$\|v\|_{0,E} \leq C \left\{ h_K^{-1/2} \|v\|_{0,K} + h_K^{1/2} \|\nabla v\|_{0,K} \right\}$$

Proof. Use [14, Lemma 12.15] and Young's inequality. □

To define our stabilized scheme, we start by defining the following notation

$$\langle \gamma_0\, \phi, \psi \rangle_{1/2,h,\Gamma_i} := \sum_{E\in\mathcal{E}_i} \frac{\gamma_0}{h_E} (\phi, \psi)_E,$$

where $i = D$ or S, and $\gamma_0 > 0$.

Let $\theta = -1, 0, 1$. For a given stabilization parameter $\beta > 0$, our stabilized discrete scheme is given by:

Problem 2. *Find* $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ *such that*

$$B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) = F_S(\boldsymbol{v}_h, q_h) \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h \times Q_h, \tag{4.1}$$

where

$$\begin{aligned} B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) := {}& B((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) - 2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}, \boldsymbol{v}_h)_{\Gamma_D} - 2\theta\nu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}, \boldsymbol{u}_h)_{\Gamma_D} \\ & + \nu\langle \gamma_0\, \boldsymbol{u}_h, \boldsymbol{v}_h \rangle_{1/2,h,\Gamma_D} + \theta(q_h, \boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_D} + (p_h, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_D} \\ & - 2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_S} - 2\theta\nu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}\cdot\boldsymbol{n}, \boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} \\ & + \nu\langle \gamma_0\, \boldsymbol{u}_h\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n} \rangle_{1/2,h,\Gamma_S} + \theta(q_h, \boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} + (p_h, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_S} \\ & + \frac{\beta}{\nu} \sum_{K\in\mathcal{T}_h} h_K^2\, (-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \nabla p_h, \nabla q_h)_K \end{aligned} \tag{4.2}$$

and

$$
\begin{aligned}
F_S(\boldsymbol{v}_h, q_h) := & \, F(\boldsymbol{v}_h, q_h) - 2\nu\theta(\boldsymbol{h}, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n})_{\Gamma_D} + \theta(\boldsymbol{h}\cdot\boldsymbol{n}, q_h)_{\Gamma_D} + \nu\langle\gamma_0\,\boldsymbol{h}, \boldsymbol{v}_h\rangle_{1/2,h,\Gamma_D} \\
& - 2\nu\theta(g, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}\cdot\boldsymbol{n})_{\Gamma_S} + \theta(g, q_h)_{\Gamma_S} + \nu\langle\gamma_0\, g, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{1/2,h,\Gamma_S} \\
& + \frac{\beta}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\,(\boldsymbol{f}, \nabla q_h)_K\,.
\end{aligned}
\tag{4.3}
$$

We state below a consistency result:

Lemma 4.4 (Consistency)**.** Let $(\boldsymbol{u}, p) \in \boldsymbol{H}\times Q$ and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ be the solutions to Problem (3.1) and Problem (4.1), respectively. Assume that $(\boldsymbol{u}, p) \in (H^2(\Omega)^d \cap \boldsymbol{H}) \times (H^1(\Omega)\cap Q)$, then

$$
B_S((\boldsymbol{u}-\boldsymbol{u}_h, p-p_h), (\boldsymbol{v}_h, q_h)) = 0 \qquad \forall (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h\times Q_h.
$$

Proof. The proof is a direct consequence of the definition of B_S, the problem (P), and the regularity assumption. □

Over $\boldsymbol{H}_h \times Q_h$ we consider the discrete norm

$$
|\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_h := \left(\nu\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{E\in\mathcal{E}_D}\frac{\nu}{h_E}\|\boldsymbol{v}_h\|_{0,E}^2 + \sum_{E\in\mathcal{E}_S}\frac{\nu}{h_E}\|\boldsymbol{v}_h\cdot\boldsymbol{n}\|_{0,E}^2 + \sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}\,\|\nabla q_h\|_{0,K}^2\right)^{1/2}.
$$

We state below the continuity of the discrete bilinear form.

Theorem 4.5. *For $\theta = -1, 0, 1$, there exists a positive constant C_a, independent of h, ν, and θ, such that*

$$
B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{v}_h, q_h)) \le C_a\, |\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_h\, |\!|\!|(\boldsymbol{v}_h, q_h)|\!|\!|_h \qquad \forall\, (\boldsymbol{u}_h, p_h),\, (\boldsymbol{v}_h, q_h) \in \boldsymbol{H}_h\times Q_h.
$$

Proof. This is as a direct consequence of lemmas 4.1, 4.2, Cauchy-Schwarz and Hölder inequalities. Moreover, it can be seen that

$$
C_a \sim 2 + C_{\rm tr} + 2\gamma_0 + 2C_{\rm in}C_{\rm tr} + C_{\rm in} + 2\beta
$$

i.e., C_a can bounded by a constant that depends only on the trace and inverse inequality constants, and on the parameters γ_0 and β. □

The well-posedness of our discretization is established as follows.

Theorem 4.6 (Well-posedness)**.** *For $\theta = -1, 0, 1$, for γ_0 large enough and for β small enough, there exists a positive constant $C_S = C_S(\theta, \beta, \gamma_0)$, independent on h and ν, such that,*

$$
B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) \ge C_S |\!|\!|(\boldsymbol{u}_h, p_h)|\!|\!|_h^2 \qquad \forall\ (\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h\times Q_h\,.
$$

The bounds on the parameters β and γ_0 depend only on the trace and inverse inequality constants. Moreover, in the case $\theta = \pm 1$, well-posedness can be proven for any $\gamma_0 > 0$.

Proof. $\underline{\theta = -1.}$

Take $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$. Using Lemma 4.1, Hölder and Young inequalities, we get that

$$
\begin{aligned}
2\beta h_K^2\,(\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \nabla p_h)_K &\le 2\beta h_K^2\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K}\,\|\nabla p_h\|_{0,K} \\
&= 2\beta\,\nu^{\frac12}\,\|\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K}\,h_K^2\nu^{-\frac12}\,\|\nabla p_h\|_{0,K} \\
&\le 2\beta\left(\frac{\delta_1}{2}C_{\rm in}^2\,\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K}^2 + \frac{1}{2\delta_1}\frac{h_K^2}{\nu}\,\|\nabla p_h\|_{0,K}^2\right) \\
&\le \beta\, C_{\rm in}^2\,\delta_1\,\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,K}^2 + \frac{\beta}{\delta_1}\,\frac{h_K^2}{\nu}\,\|\nabla p_h\|_{0,K}^2,
\end{aligned}
\tag{4.4}
$$

with δ_1 a positive parameter to be chosen in a convenient way. Now, using (4.4) and the definition of B_S, with $\theta = -1$, we obtain

$$
\begin{aligned}
&B_S((\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)) = 2\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \nu\langle\gamma_0\,\boldsymbol{u}_h,\boldsymbol{u}_h\rangle_{1/2,h,\Gamma_D} + \nu\langle\gamma_0\,\boldsymbol{u}_h\cdot\boldsymbol{n},\boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{1/2,h,\Gamma_S}\\
&+\frac{\beta}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\,(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h)+\nabla p_h,\nabla p_h)_K\\
&= 2\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \nu\sum_{E\in\mathcal{E}_D}\frac{\gamma_0}{h_E}\|\boldsymbol{u}_h\|^2_{0,E} + \nu\sum_{E\in\mathcal{E}_S}\frac{\gamma_0}{h_E}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\\
&\quad+\frac{\beta}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\,\|\nabla p_h\|^2_{0,K} - 2\beta\sum_{K\in\mathcal{T}_h} h_K^2\,(\nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h),\nabla p_h)_K\\
&\ge (2-\beta\,C_{\rm in}^2\,\delta_1)\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \nu\sum_{E\in\mathcal{E}_D}\frac{\gamma_0}{h_E}\|\boldsymbol{u}_h\|^2_{0,E} + \nu\sum_{E\in\mathcal{E}_S}\frac{\gamma_0}{h_E}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\\
&+\beta\left(1-\frac{1}{\delta_1}\right)\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}\,\|\nabla p_h\|^2_{0,K}.
\end{aligned}
$$

Now, choosing $\delta_1 = 2$ and the stabilization parameter β such that $\beta < C_{\rm in}^{-2}$ we obtain that there exists a positive constant

$$
C_S = \min\left\{2(1-\beta C_{\rm in}^2),\gamma_0,\frac{\beta}{2}\right\}\,, \tag{4.5}
$$

independent of h and ν, such that

$$
B_S((\boldsymbol{u}_h,p_h),(\boldsymbol{u}_h,p_h)) \ge C_S |\!|\!|(\boldsymbol{u}_h,p_h)|\!|\!|^2_h,
$$

which proves that the problem is well posed.

$\underline{\theta = 0.}$ Using Lemma 4.2, Hölder and Young inequalities, we have that

$$
\begin{aligned}
2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}\cdot\boldsymbol{n},\boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} &\le 2\nu\sum_{E\in\mathcal{E}_S}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}\|_{0,E}\,\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|_{0,E}\\
&= 2\nu\sum_{E\in\mathcal{E}_S} h_E^{1/2}\,\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}\|_{0,E}\,h_E^{-1/2}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|_{0,E}\\
&\le 2\nu\sum_{E\in\mathcal{E}_S}\left(\frac{h_E\delta_2}{2}\,\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,E} + \frac{h_E^{-1}}{2\delta_2}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\right)\\
&\le \nu\delta_2 C_{\rm tr}^2\sum_{K\in\mathcal{T}_h}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,K} + \frac{\nu}{\gamma_0\delta_2}\sum_{E\in\mathcal{E}_S}\frac{\gamma_0}{h_E}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\\
&\le \delta_2 C_{\rm tr}^2\,\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \frac{\nu}{\delta_2}\langle\,\boldsymbol{u}_h\cdot\boldsymbol{n},\boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{1/2,h,\Gamma_S}.
\end{aligned} \tag{4.6}
$$

Using again the same arguments it is easy to prove, for any $\delta_3 > 0$, that

$$
2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n},\boldsymbol{u}_h)_{\Gamma_D} \le \delta_3 C_{\rm tr}^2\,\nu\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|^2_{0,\Omega} + \frac{\nu}{\delta_3}\langle\,\boldsymbol{u}_h,\boldsymbol{u}_h\rangle_{1/2,h,\Gamma_D}. \tag{4.7}
$$

On the other hand, using lemmas 4.1 and 4.2, and Hölder's inequality, we have that

$$
\begin{aligned}
(p_h,\boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} &\le \sum_{E\in\mathcal{E}_S}\|p_h\|_{0,E}\,\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|_{0,E}\\
&= \sum_{E\in\mathcal{E}_S}\nu^{-\frac12}h_E^{1/2}\|p_h\|_{0,E}\,\nu^{\frac12}h_E^{-1/2}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|_{0,E}\\
&\le \sum_{E\in\mathcal{E}_S}\frac{h_E\delta_4}{2\nu}\|p_h\|^2_{0,E} + \sum_{E\in\mathcal{E}_S}\nu\frac{h_E^{-1}}{2\delta_4}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\\
&\le \sum_{K\in\mathcal{T}_h}\frac{C_{\rm tr}^2\delta_4}{2\nu}\|p_h\|^2_{0,K} + \frac{1}{2\delta_4}\sum_{E\in\mathcal{E}_S}\frac{\nu}{h_E}\|\boldsymbol{u}_h\cdot\boldsymbol{n}\|^2_{0,E}\\
&\le \frac{C_{\rm in}^2\,C_{\rm tr}^2\delta_4}{2}\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{\nu}\|\nabla p_h\|^2_{0,K} + \frac{\nu}{2\delta_4}\langle\,\boldsymbol{u}_h\cdot\boldsymbol{n},\boldsymbol{u}_h\cdot\boldsymbol{n}\rangle_{1/2,h,\Gamma_S}.
\end{aligned} \tag{4.8}
$$

In the same way, we can prove that

$$(p_h, \boldsymbol{u}_h \cdot \boldsymbol{n})_{\Gamma_D} \leq \frac{C_{\text{in}}^2\, C_{\text{tr}}^2 \delta_5}{2} \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\nu} \|\nabla p_h\|_{0,K}^2 + \frac{\nu}{2\delta_5} \langle \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_{1/2,h,\Gamma_D}. \tag{4.9}$$

Thus, using (4.4)–(4.9), and Young's inequality we get

$$\begin{aligned}
&B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) = 2\nu \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega}^2 - 2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}, \boldsymbol{u}_h)_{\Gamma_D} + \nu\langle \gamma_0\, \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_{1/2,h,\Gamma_D} + (p_h, \boldsymbol{u}_h \cdot \boldsymbol{n})_{\Gamma_D} \\
&- 2\nu(\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\boldsymbol{n}\cdot\boldsymbol{n}, \boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} + \nu\langle \gamma_0\, \boldsymbol{u}_h\cdot\boldsymbol{n}, \boldsymbol{u}_h\cdot\boldsymbol{n} \rangle_{1/2,h,\Gamma_S} + (p_h, \boldsymbol{u}_h\cdot\boldsymbol{n})_{\Gamma_S} \\
&+ \frac{\beta}{\nu} \sum_{K\in\mathcal{T}_h} h_K^2 \left(-2\nu\, \nabla\cdot\boldsymbol{\varepsilon}(\boldsymbol{u}_h) + \nabla p_h, \nabla p_h\right)_K \\
&\geq (2 - \delta_2 C_{\text{tr}}^2 - \delta_3 C_{\text{tr}}^2 - \beta C_{\text{in}}^2 \delta_1)\nu \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega}^2 + \left(\gamma_0 - \frac{1}{\delta_3} - \frac{1}{2\delta_5}\right)\nu\langle\, \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_{1/2,h,\Gamma_D} \\
&+ \left(\gamma_0 - \frac{1}{\delta_2} - \frac{1}{2\delta_4}\right)\nu\langle\, \boldsymbol{u}_h\cdot\boldsymbol{n}, \boldsymbol{u}_h\cdot\boldsymbol{n} \rangle_{1/2,h,\Gamma_S} + \left(\beta - \frac{\beta}{\delta_1} - \frac{C_{\text{in}}^2 C_{\text{tr}}^2 \delta_4}{2} - \frac{C_{\text{in}}^2 C_{\text{tr}}^2 \delta_5}{2}\right) \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\nu} \|\nabla p_h\|_{0,K}^2.
\end{aligned}$$

The positive parameters δ_1, δ_2, δ_3, δ_4, and δ_5 should be now chosen properly. We take $\delta_1 = 2$, $\delta_2 = \delta_3$, $\delta_4 = \delta_5$, obtaining

$$\begin{aligned}
B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) \geq\ & 2\left(1 - \delta_2 C_{\text{tr}}^2 - \beta C_{\text{in}}^2\right)\nu \|\boldsymbol{\varepsilon}(\boldsymbol{u})\|_{0,\Omega}^2 + \left(\gamma_0 - \frac{1}{\delta_2} - \frac{1}{2\delta_4}\right)\nu\langle\, \boldsymbol{u}_h, \boldsymbol{u}_h \rangle_{1/2,h,\Gamma_D} \\
&+ \left(\gamma_0 - \frac{1}{\delta_2} - \frac{1}{2\delta_4}\right)\nu\langle\, \boldsymbol{u}_h\cdot\boldsymbol{n}, \boldsymbol{u}_h\cdot\boldsymbol{n} \rangle_{1/2,h,\Gamma_S} \\
&+ \left(\frac{\beta}{2} - \delta_4 C_{\text{in}}^2 C_{\text{tr}}^2\right) \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{\nu} \|\nabla p_h\|_{0,K}^2.
\end{aligned}$$

Choosing $\delta_2 = \frac{\beta}{C_{\text{tr}}^2}$, $\delta_4 = \frac{\beta}{4C_{\text{tr}}^2 C_{\text{in}}^2}$ one gets

$$\gamma_0 - \frac{1}{\delta_2} - \frac{1}{2\delta_4} = \gamma_0 - C_{\text{tr}}^2 \frac{1 + 2C_{\text{in}}^2}{\beta},$$

$$\frac{\beta}{2} - \delta_4 C_{\text{in}}^2 C_{\text{tr}}^2 = \frac{\beta}{4},$$

$$1 - \delta_2 C_{\text{tr}}^2 - \beta C_{\text{in}}^2 = 1 - \beta(C_{\text{tr}}^2 + C_{\text{in}}^2).$$

Hence, taking $\beta < (C_{\text{tr}}^2 + C_{\text{in}}^2)^{-1}$ and $\gamma_0 > C_{\text{tr}}^2 \frac{1+2C_{\text{in}}^2}{\beta}$ yields

$$B_S((\boldsymbol{u}_h, p_h), (\boldsymbol{u}_h, p_h)) \geq C_S |\!|\!| (\boldsymbol{u}_h, p_h) |\!|\!|_h^2,$$

with

$$C_S := \min\left\{2\left(1 - \beta(C_{\text{tr}}^2 + C_{\text{in}}^2)\right), \frac{\beta}{4}, \gamma_0 - C_{\text{tr}}^2 \frac{1 + 2C_{\text{in}}^2}{\beta}\right\}. \tag{4.10}$$

The case $\theta = 1$ is proved using the same arguments as for $\theta = 0$. □

5. Error analysis

This section is devoted to a priori error analysis based on the arguments of [3] and [7]. To this purpose, let $\boldsymbol{I}_h : \boldsymbol{H} \to \boldsymbol{H}_h$ and $J_h : Q \to Q_h$ be the vectorial and scalar version of the Scott–Zhang interpolant, respectively. Then the following results concerning the approximation properties of these operators hold.

Lemma 5.1. For each $K \in \mathcal{T}_h$ and $k \geq 1$, there exist two positive constants C and $\tilde{C}$, independent of h_K, such that

$$\|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}\|_{0,K} + h_K|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}|_{1,K} + h_K^2|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}|_{2,K} \leq C\, h_K^{k+1} |\boldsymbol{u}|_{k+1,\omega_K} \qquad \forall \boldsymbol{u} \in H^{k+1}(\omega_K)^d,$$

$$\|q - J_h q\|_{0,K} + h_K|q - J_h q|_{1,K} \leq \tilde{C}\, h_K^k |q|_{k,\omega_K} \qquad \forall q \in H^k(\omega_K),$$

where

$$\omega_K := \bigcup_{K\cap K' \neq \emptyset} K'.$$

Proof. See [24]. □

Remark. For a given element $q \in Q$, the Scott–Zhang interpolant $J_h q$ does not belong, in general, to $Q = L_0^2(\Omega)$. However, we can consider its modified version given by

$$J_h q - \frac{1}{|\Omega|}\int_\Omega J_h q \,\mathrm{d}x \in Q,$$

that, with a little abuse of notation, we will also denote as $J_h q$, and that retains all the approximation properties of the original interpolator.

Let us introduce the following notation

$$e^{\boldsymbol{u}_h} := \boldsymbol{I}_h\boldsymbol{u} - \boldsymbol{u}_h, \qquad e^{p_h} := J_h p - p_h$$
$$\eta^{\boldsymbol{u}_h} := \boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}, \qquad \eta^{p_h} := p - J_h p.$$

Note that

$$\boldsymbol{u} - \boldsymbol{u}_h = \eta^{\boldsymbol{u}_h} + e^{\boldsymbol{u}_h} \qquad \text{and} \qquad p - p_h = \eta^{p_h} + e^{p_h}.$$

The following theorem states the convergence of the method.

Theorem 5.2. *Let $(\boldsymbol{u}, p) \in \boldsymbol{H} \times Q$ and $(\boldsymbol{u}_h, p_h) \in \boldsymbol{H}_h \times Q_h$ be the solutions of problems* (3.1) *and* (4.1)*, respectively. Assume that $(\boldsymbol{u}, p) \in (H^{k+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^k(\Omega) \cap Q)$, with $k \geq 1$, then*

$$|||(\boldsymbol{u} - \boldsymbol{u}_h, p - p_h)|||_h \preceq h^k\{|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}\}.$$

Proof. For any $K \in \mathcal{T}_h$ and $E \subset \partial K$, we have, as a direct consequence of lemmas 4.3 and 5.1, that

$$\|\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h})\|_{0,K} \leq C h_K^k |\boldsymbol{u}|_{k+1,K}, \tag{5.1}$$
$$h_E^{1/2}\|\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h})\boldsymbol{n}\|_{0,E} \leq C h_K^k |\boldsymbol{u}|_{k+1,K}, \tag{5.2}$$
$$h_E^{-1/2}\|\eta^{\boldsymbol{u}_h}\cdot\boldsymbol{n}\|_{0,E} \leq C h_K^k |\boldsymbol{u}|_{k+1,K}, \tag{5.3}$$
$$\|\nabla\cdot\eta^{\boldsymbol{u}_h}\|_{0,K} \leq C h_K^k |\boldsymbol{u}|_{k+1,K}, \tag{5.4}$$
$$h_K\|\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h})\|_{0,K} \leq C h_K^k |\boldsymbol{u}|_{k+1,K}, \tag{5.5}$$
$$h_K\|\nabla\eta^{p_h}\|_{0,K} \leq C h_K^k |p|_{k,K}. \tag{5.6}$$

Now, using Cauchy-Schwarz's inequality and (5.1)–(5.6), we obtain

$$\begin{aligned} B_S((\eta^{\boldsymbol{u}_h}, \eta^{p_h}), (\boldsymbol{v}_h, q_h)) := {} & 2\nu(\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h}), \boldsymbol{\varepsilon}(\boldsymbol{v}_h))_\Omega - (\nabla\cdot\boldsymbol{v}_h, \eta^{p_h})_\Omega + (\nabla\cdot\eta^{\boldsymbol{u}_h}, q_h)_\Omega \\ & - 2\nu(\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h})\boldsymbol{n}, \boldsymbol{v}_h)_{\Gamma_D} - 2\theta\nu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}, \eta^{\boldsymbol{u}_h})_{\Gamma_D} \\ & + \nu\langle\gamma_0\,\eta^{\boldsymbol{u}_h}, \boldsymbol{v}_h\rangle_{1/2,h,\Gamma_D} + \theta(q_h, \eta^{\boldsymbol{u}_h}\cdot\boldsymbol{n})_{\Gamma_D} + (\eta^{p_h}, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_D} \\ & - 2\nu(\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h})\boldsymbol{n}\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_S} - 2\theta\nu(\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\boldsymbol{n}\cdot\boldsymbol{n}, \eta^{\boldsymbol{u}_h}\cdot\boldsymbol{n})_{\Gamma_S} \\ & + \nu\langle\gamma_0\,\eta^{\boldsymbol{u}_h}\cdot\boldsymbol{n}, \boldsymbol{v}_h\cdot\boldsymbol{n}\rangle_{1/2,h,\Gamma_S} + \theta(q_h, \eta^{\boldsymbol{u}_h}\cdot\boldsymbol{n})_{\Gamma_S} + (\eta^{p_h}, \boldsymbol{v}_h\cdot\boldsymbol{n})_{\Gamma_S} \\ & + \frac{\beta}{\nu}\sum_{K\in\mathcal{T}_h} h_K^2\,(-2\nu\,\nabla\cdot\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}_h}) + \nabla\eta^{p_h}, \nabla q_h)_K \\ \leq {} & C\,h^k\{|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}\}\,|||(\boldsymbol{v}_h, q_h)|||_h. \end{aligned} \tag{5.7}$$

On the other hand, using again (5.1)– (5.6), and the definition of $|||\cdot|||_h$, we have that

$$|||(\eta^{\boldsymbol{u}_h}, \eta^{p_h})|||_h \preceq h^k\{|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}\}. \tag{5.8}$$

Thus, from Theorem 4.6, lemmas 4.4 and (5.7), we get

$$\begin{aligned} |||(e^{\boldsymbol{u}_h}, e^{p_h})|||_h^2 &\leq C B_S((e^{\boldsymbol{u}_h}, e^{p_h}), (e^{\boldsymbol{u}_h}, e^{p_h})) = -B_S((\eta^{\boldsymbol{u}_h}, \eta^{p_h}), (e^{\boldsymbol{u}_h}, e^{p_h})) \\ &\leq C h^k\{|\boldsymbol{u}|_{k+1,\Omega} + |p|_{k,\Omega}\}\,|||(e^{\boldsymbol{u}_h}, e^{p_h})|||_h. \end{aligned} \tag{5.9}$$

The result follows using triangle inequality and the bounds (5.8) and (5.9). □

6. Numerical experiments

6.1. **Example 1: 2D Cavity.** In this example, taken from [26], we take $\Omega := (-1, 1)^2$ and the slip boundary condition is imposed on $y = -1$, while a Dirichlet boundary condition is enforced on the rest of the boundary. The exact solution of this problem is given by $\boldsymbol{u} := (2y(1-x^2), -2x(1-y^2))$ and $p := 0$. For our computations we use $\mathbb{P}_1$ for all the variables, while the viscosity is set to $\nu = 1$. The corresponding computed velocity field is depicted Figure 1. Table 1 presents the approximation errors for pressure and velocity as well as the computed convergence rate, which are in good agreement with the theory, with a slight superconvergence for the pressure. Table 2 presents the error in L^2 norm on the slip condition on Γ_S, that, for this situation, does not differ significantly between the symmetric and skew-symmetric variants. However, it shows that the larger the Nitsche parameter γ_0 is, the smaller is the error on the slip condition, for both variants.

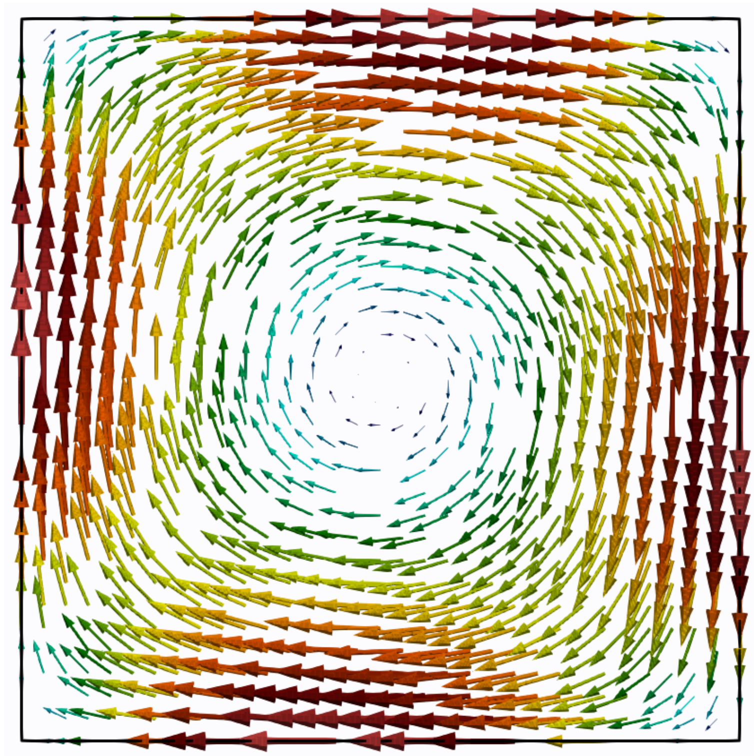

FIGURE 1. The computed velocity field for Example 1.

h	$\|p - p_h\|_{0,\Omega}$	order	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$	order	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$	order
0.353553	0.256600	—	0.055039	—	1.058715	—
0.176777	0.110749	1.21	0.017263	1.67	0.538051	0.97
0.088388	0.040998	1.43	0.004827	1.83	0.270114	0.99
0.044194	0.014566	1.49	0.001276	1.91	0.135161	0.99
0.022097	0.005134	1.50	0.000328	1.96	0.067574	1.00

TABLE 1. Approximation errors and convergence orders for each variable.

	$\theta = -1$			$\theta = 1$		
h	$\gamma_0 = 10^{-3}$	$\gamma_0 = 1$	$\gamma_0 = 10^3$	$\gamma_0 = 10^{-3}$	$\gamma_0 = 1$	$\gamma_0 = 10^3$
0.353553	0.233603	0.187756	0.001221	0.182408	0.158295	0.001222
0.176777	0.043670	0.035254	0.000250	0.039551	0.032317	0.000250
0.088388	0.008092	0.006591	0.000050	0.007483	0.006229	0.000050
0.044194	0.001524	0.001257	0.000010	0.001419	0.001235	0.000010
0.022097	0.000297	0.000250	0.000002	0.000280	0.000256	0.000002

TABLE 2. Computations of $\|\boldsymbol{u}_h \cdot \boldsymbol{n}\|_{0,\Gamma_S}$ for different values of θ and γ_0.

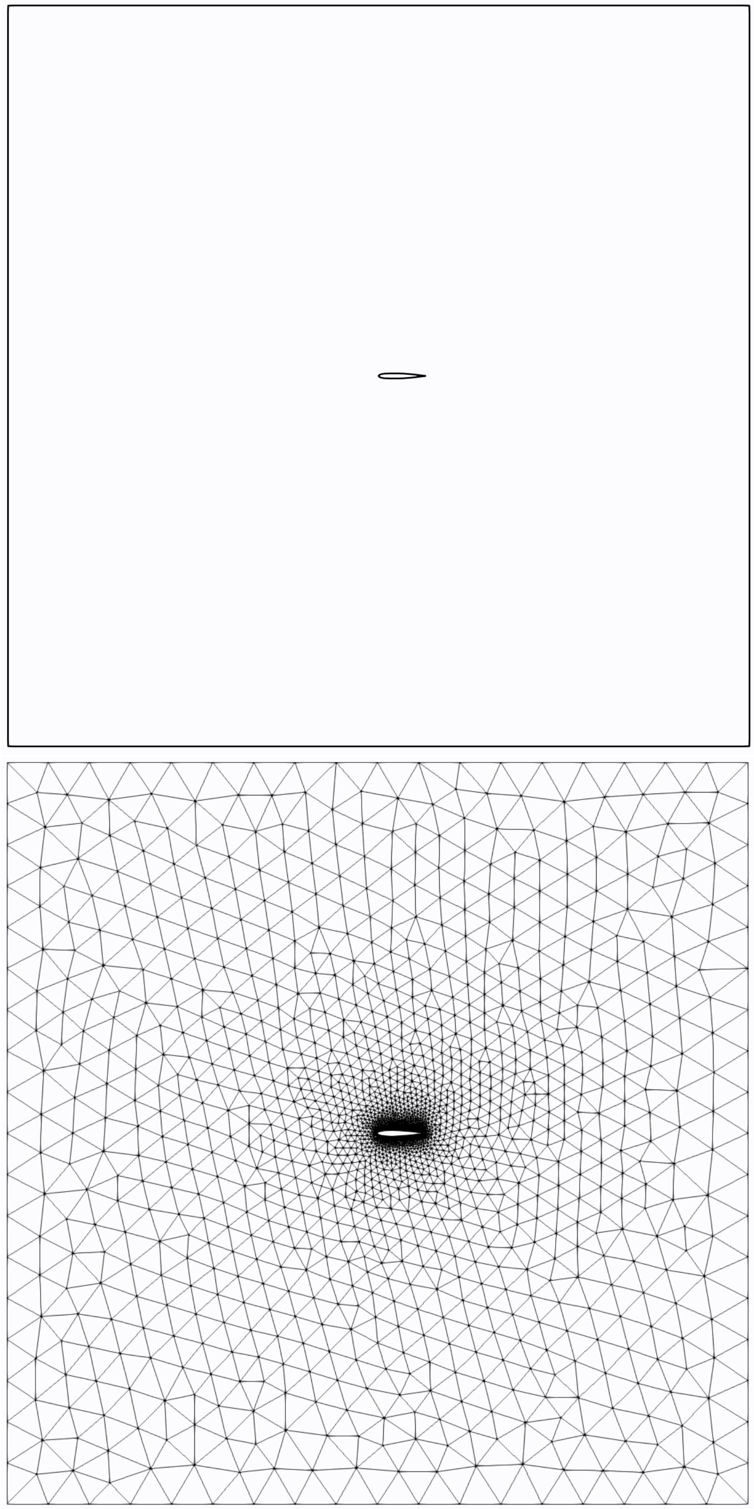

FIGURE 2. Computational domain and corresponding mesh of the Naca problem.

6.2. **Example 2: 2D Naca 0012.** In this case we use the standard Naca 0012 configuration depicted in Figure 2.

The boundary conditions are $\boldsymbol{u} = (51.4814, 0)$ on all the box boundaries and $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ and $\sigma(\boldsymbol{u}, p)\boldsymbol{n} \cdot \boldsymbol{t} = 0$ on the surface of the Naca domain. We use a mesh with 8,808 elements with polynomials of order 1. The predicted pressure and velocity are depicted Figure 3 and Figure 4, which show the method behaves as expected on this classical example, notably for the enforcement on the slip condition on the wing.

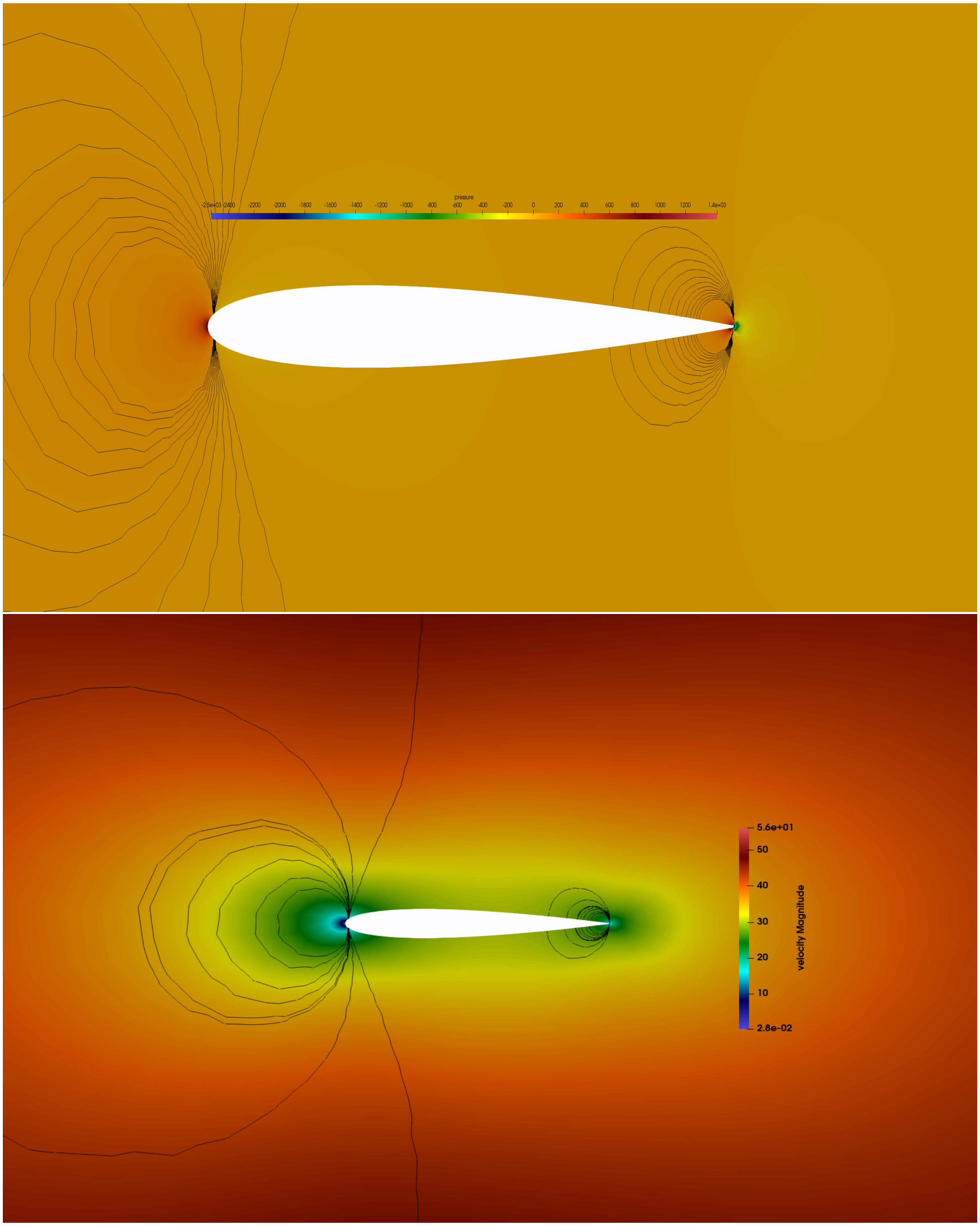

FIGURE 3. Isovalues of the pressure (top) and velocity magnitude (bottom).

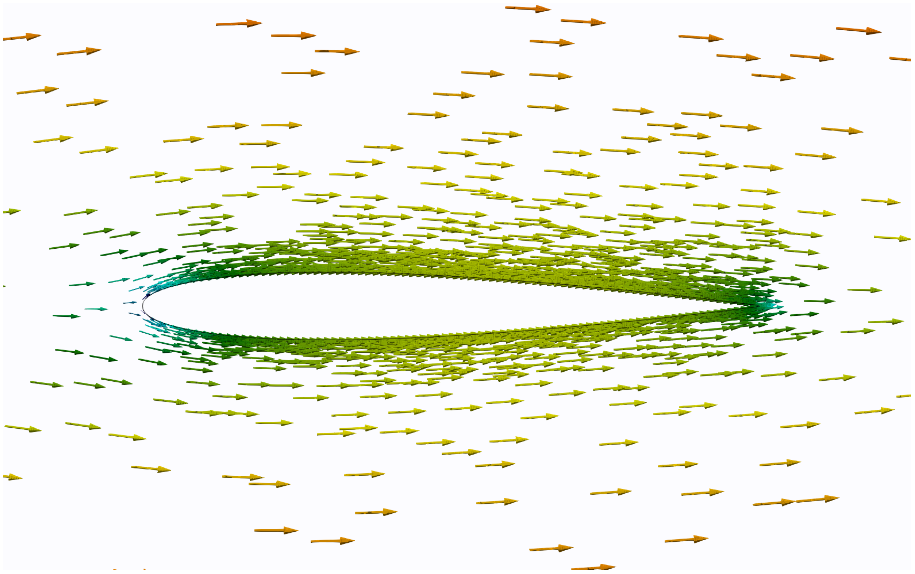

FIGURE 4. Zoom, close to the Naca wing, of the velocity field.

6.3. **Example 3: 3D Cylinder.** The last example is based on a standard three-dimensional CFD benchmark: the cylinder problem. The geometrical settings of the domain are given in Figure 5.

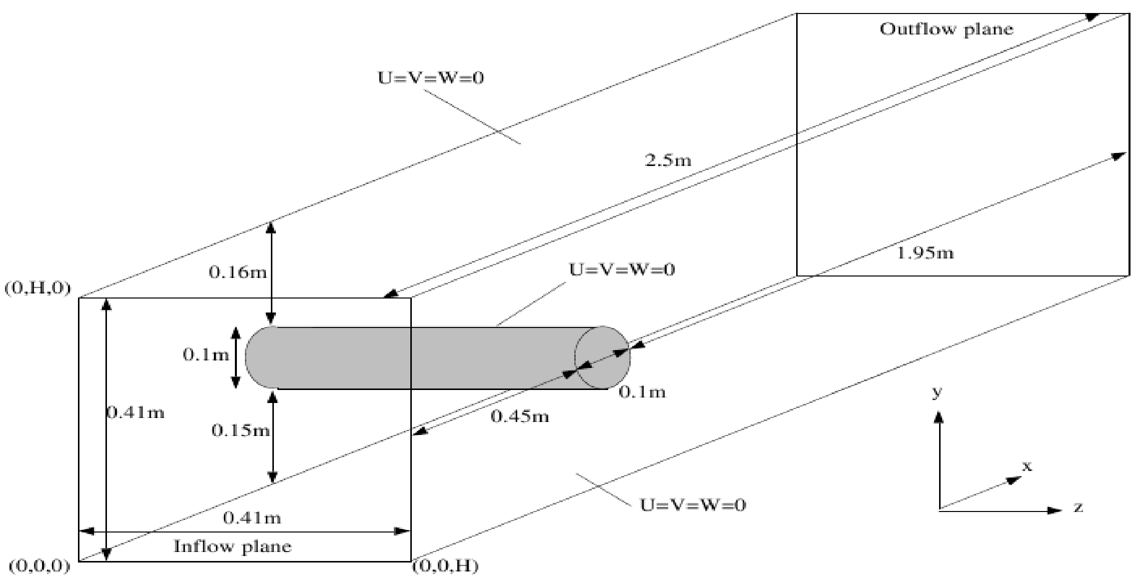

FIGURE 5. Cylinder problem. Domain and boundary conditions.

In this case $H = 0.41\,m$, no-slip boundary conditions are imposed on all the lateral walls of the box, while do-nothing boundary conditions are imposed at the outflow plane. On the surface of the cylinder we impose the conditions $\boldsymbol{u}\cdot\boldsymbol{n} = 0$ and $\displaystyle\sum_{i=1}^{2}\sigma(\boldsymbol{u},p)\boldsymbol{n}\cdot\boldsymbol{t}_i = 0$. Finally, the inflow condition is given by

$$\boldsymbol{u}_D := \left(\frac{16\,U_m\,yz\,(H-y)(H-z)}{H^4}, 0, 0\right)^T,$$

with $U_m := 0.45\,m/s$. We use the viscosity $\nu := 10^{-3}\,m^2/s$. The mesh is depicted Figure 6. The numerical solutions for the pressure and velocity fields are shown in Figure 7, Figure 8, Figure 9 and Figure 10. The results illustrate the good behavior of the method also on this more complex example.

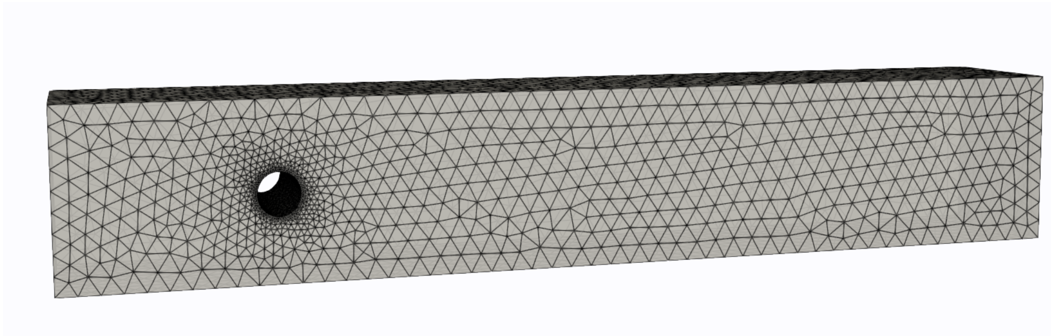

FIGURE 6. Cylinder problem. Surface view of the computational mesh.

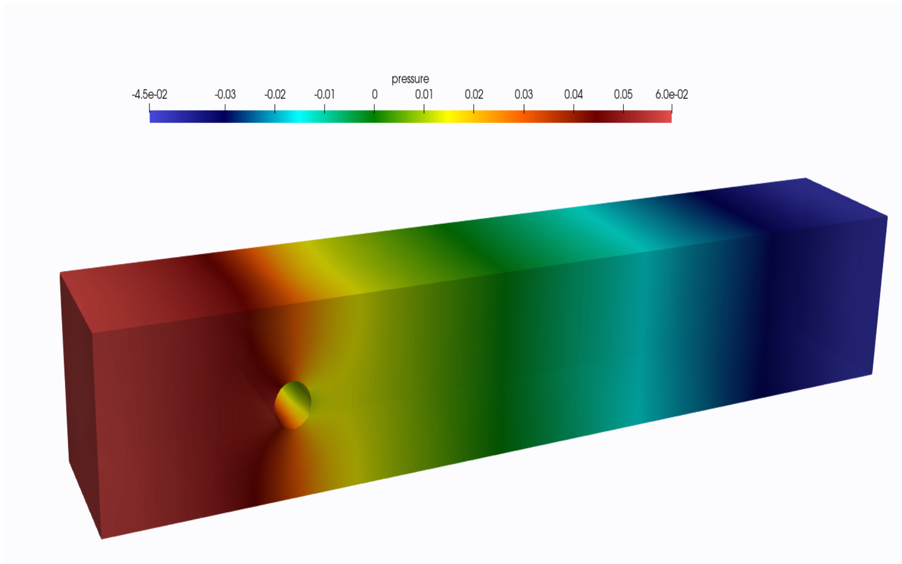

FIGURE 7. Cylinder problem. Isovalues of the pressure.

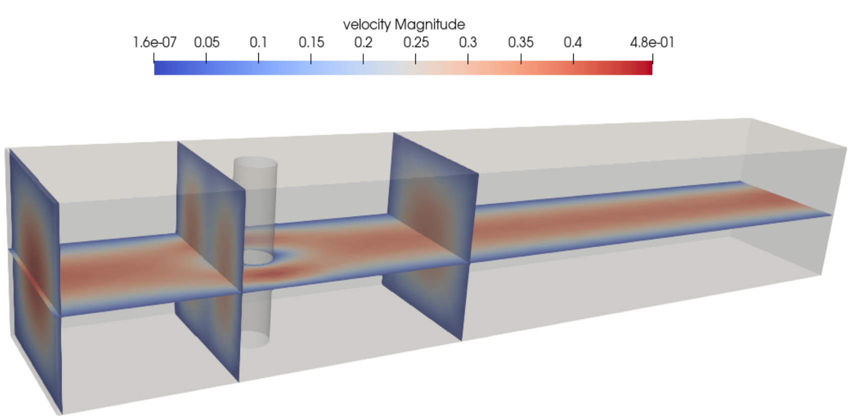

FIGURE 8. Cylinder problem. Velocity magnitude (top view).

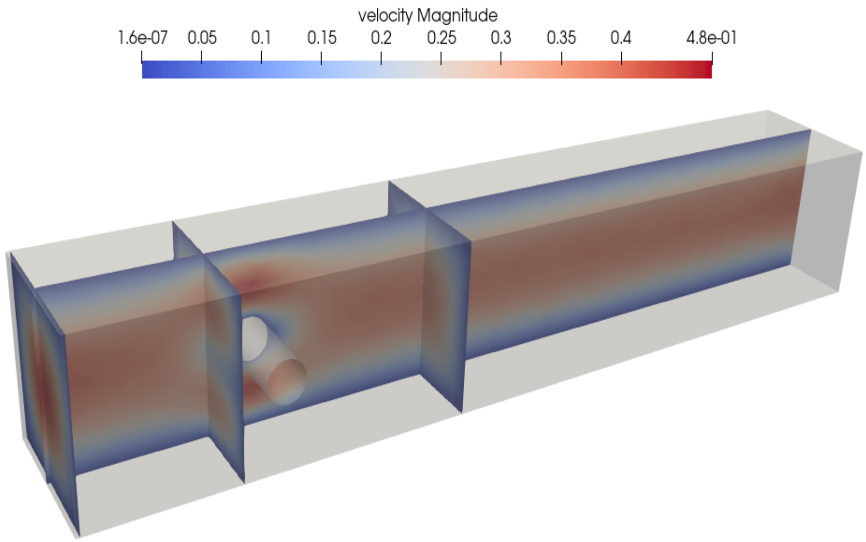

FIGURE 9. Cylinder problem. Velocity magnitude (side view).

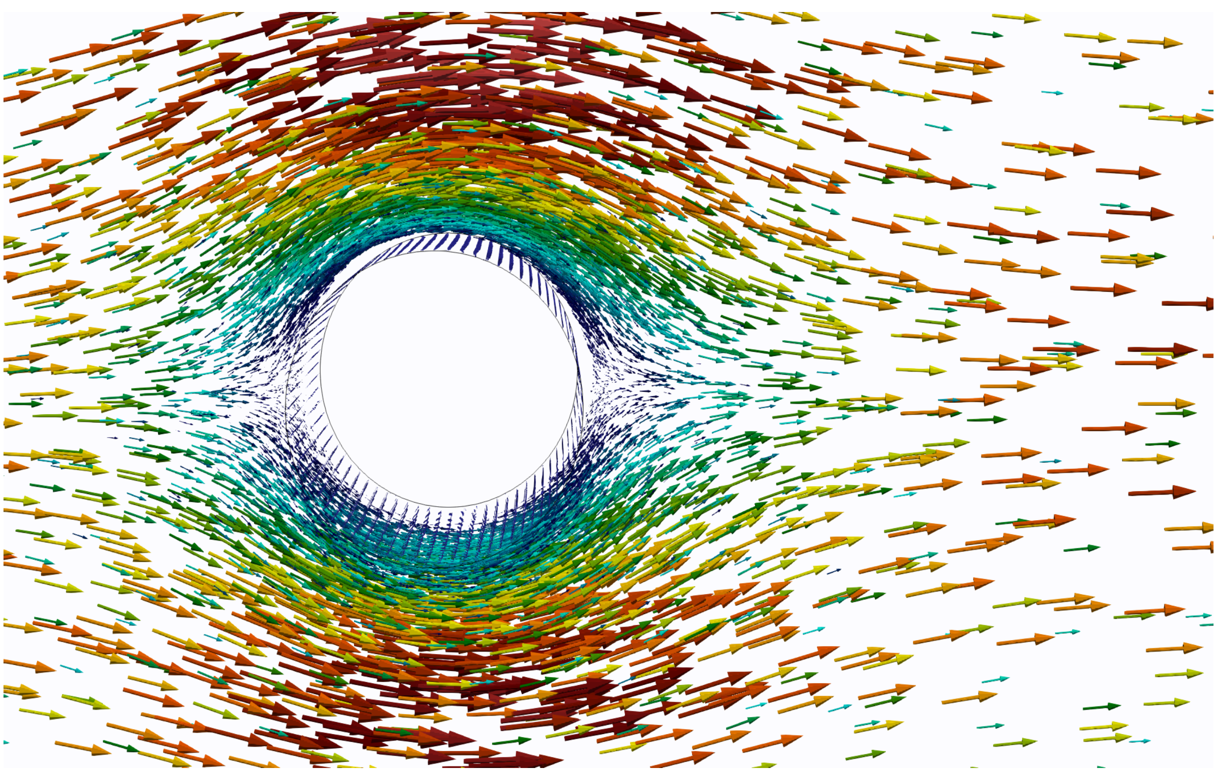

FIGURE 10. Cylinder problem. Zoom, close to the cylinder, of the velocity field.

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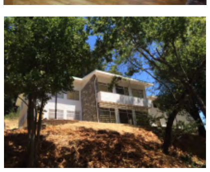

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