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Residual a posteriori error estimation for frictional contact with Nitsche method

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Abstract

We consider frictional contact problems in small strain elasticity discretized with finite elements and Nitsche method. Both bilateral and unilateral contact problems are taken into account, as well as both Tresca and Coulomb models for the friction. We derive residual a posteriori error estimates for each friction model, following [Chouly et al, IMA J. Numer. Anal. (38) 2018, pp. 921-954]. For the incomplete variant of Nitsche, we prove an upper bound for the dual norm of the residual, for Tresca and Coulomb friction, without any extra regularity and without a saturation assumption. Numerical experiments allow to assess the accuracy of the estimates and their interest for adaptive meshing in different situations.

Keywords: unilateral contact; Tresca friction; Coulomb friction; elasticity; Lagrange finite elements; Nitsche method; residual a posteriori error estimates.

AMS Subject Classification: 65N12, 65N15, 65N30, 74M10, 74M15, 74M20.

1 Introduction

For a wide range of systems in structural mechanics, it is crucial to consider account contact and friction between rigid or elastic bodies. Among numerous applications, let us mention foundations in civil engineering, metal forming processes, crash-tests of cars, design of car tires (see, *e.g.*, [66]). Contact and friction conditions are usually formulated with a set of inequalities and non-linear equations on the boundary of each body, with corresponding unknowns that are displacements, velocities and surface stresses. Basically, contact conditions allow to enforce non-penetration on the whole candidate contact surface, and the actual contact surface is not known in advance. A friction law may be taken into account additionally, and various models exist that correspond to different surface properties, the most popular one being Coulomb's friction (see, *e.g.*, [49] and references therein).

Frictional contact problems can be formulated weakly within the framework of variational inequalities (see, *e.g.*, [26, 31, 49]). Those are the very basis of most existing Finite Element Methods (FEM), see, *e.g.*, [33, 36, 40, 49, 52, 65, 66]. For numerical computations with the FEM, various techniques have been devised to enforce contact and friction conditions at the discrete level, and the foremost are penalty methods (see, *e.g.*, [12, 18, 49, 50, 58, 59]) and mixed methods (see, *e.g.*, [7, 40, 42, 48, 51, 65]).

Nitsche's method has been considered recently to discretize contact and friction conditions. The Nitsche method orginally proposed in [57] aims at treating the boundary or interface conditions in a weak sense, with appropriate consistent terms that involve only the primal variables. It differs in this aspect from standard penalization techniques which are generally non-consistent [49]. Moreover, no additional unknown (Lagrange multiplier) is needed and, therefore, no discrete inf–sup condition must be fulfilled, contrarily to mixed methods (see, *e.g.*, [40, 65]). Most of the applications of Nitsche's method during the last two decades involved linear conditions on the boundary of a domain or at the interface between sub-domains: see, e.g,. [63] for the Dirichlet problem, [6] for domain decomposition with non-matching meshes and [38] for a global review. In some previous works [37, 41] it has been adapted for bilateral (persistent) contact, which still corresponds to linear boundary conditions on the contact zone. We remark furthermore that an algorithm for unilateral contact which makes use of Nitsche's method in its original form is presented and implemented in [37], An extension to large strain bilateral contact has been performed in [67]. In [17, 21] a new Nitsche-based FEM was proposed and analyzed for Signorini's problem, where a linear elastic body is in frictionless contact with a rigid foundation. Conversely to bilateral (persistent) contact, Signorini's problem involves non-linear boundary conditions associated to unilateral contact, with an unknown actual contact region.

For this Nitsche-based FEM, optimal convergence in the $H^1(\Omega)$-norm of order $\mathcal{O}(h^{s-1})$ has been proved, provided the solution has a regularity $H^s(\Omega)$, $3/2 < s \leq 1+k$ ($k = 1, 2$ is the polynomial degree of the Lagrange finite elements). To this purpose there is no need of additional assumption on the contact/friction zone.

Moreover the Nitsche-based FEM encompasses symmetric and nonsymmetric variants depending upon a parameter called θ. The symmetric case of [17] is recovered when $\theta = 1$. When $\theta \neq 1$ positivity of the contact term in the Nitsche variational formulation is generally lost. Nevertheless some other advantages are recovered, mostly from the numerical viewpoint. Namely, one of the variants ($\theta = 0$) involves a reduced quantity of terms, which makes it easier to implement and to extend to contact problems involving non-linear elasticity. In addition, this nonsymmetric variant $\theta = 0$ performs better in the sense that it requires less Newton iterations to converge, for a wider range of the Nitsche parameter, than the variant $\theta = 1$, see [61]. Concerning the skew-symmetric variant $\theta = -1$, the well-posedness of the discrete formulation and the optimal convergence are preserved irrespectively of the value of the Nitsche parameter. Lately, various extensions of the method proposed in [17, 21] have been carried out (see for instance [15] or [46] for overviews, as well as [10] for the link between Nitsche and the augmented Lagrangian). Particularly, an extension to Tresca's friction has been studied in [13]. Optimal convergence in $H^1(\Omega)$-norm has been established as well, without any assumption other than usual Sobolev regularity. An extension to (static) Coulomb's friction has been formulated in [61] and tested numerically using a generalized Newton algorithm. Later on, the existence of solutions for Coulomb friction has been studied in [19, 20], and application to fracture mechanics has been considered in [4, 5]. The case of contact between two elastic bodies is addressed in [22, 30, 55]. In [30] Nitsche's method is combined with a cut-FEM / fictitious domain discretization, in the small deformations framework. In [22, 55] an unbiased variant implements the frictional contact between two elastic bodies without making any difference between master and slave contact surfaces. The contact condition is the same on each surface. This is an advantage for treatment of self-contact or multi-body contact. Residual-based *a posteriori* error estimates are presented in [16]. Upper and lower bounds are proved under a saturation assumption, for frictionless contact and Tresca friction, and the performance of the error estimates is investigated numerically for frictionless contact.

In this paper we focus on different friction models, namely Tresca friction and Coulomb friction, discretized with Lagrange FEM and Nitsche's method. We consider both unilateral and bilateral contact settings. For these discrete friction problems, we provide residual based *a posteriori* error estimates. Moreover, for $\theta = 0$, and following some ideas of [25], we prove an upper bound for the dual norm of the corresponding residual, without any extra regularity assumption or saturation assumption, and valid for both Tresca and Coulomb friction. Numerical experiments allow to assess the performance of the error estimates. For Tresca friction discretized with Nitsche-FEM, a residual based *a posteriori* error estimator has already been suggested in [16] (see the Appendix page 951). Its reliability, under a saturation assumption, and its efficiency have been established. Nevertheless, it has never been implemented and assessed numerically. For Coulomb friction with Nitsche, no *a posteriori* error estimates exist to the best of our knowledge.

To put our work in perspective, let us mention that, in a recent contribution [35], another kind of residual *a posteriori* error estimate has been designed for a stabilized mixed method close to the symmetric variant of Nitsche's method. Notably, thanks to additional terms, the authors of [35] manage to prove an upper bound for this estimator without any saturation assumption. For frictionless (Signorini) contact, such a technique has already been presented in [34] (in addition to an *a priori* error bound under minimal regularity assumption). In the same manner, an upper bound for an *a posteriori* error estimator based on equilibrated fluxes has been proven, without saturation assumption and for the scalar Signorini problem [11]. Still for an equilibrated fluxes estimator and frictionless contact in elasticity, an upper bound has been established with a saturation assumption in [25]. As well, for Tresca friction discretized with the Virtual Element Method (VEM), a new *a posteriori* estimate has been designed in [24]. For Coulomb friction, the upper and lower bounds for residual *a posteriori* error estimates are difficult to establish, see for instance [44] and references therein for mixed methods. See also [47] for an estimator for Coulomb friction based on equilibrated fluxes. In fact, very few error estimators with mathematical justification of robustness or efficiency have been proposed for Coulomb friction.

This paper is outlined as follows. In Section 2 we introduce a frictional contact problem and its Nitsche finite element approximation. Section 3 details the residual *a posteriori* error estimate. Section 4 presents an upper bound for the dual norm of the residual and the incomplete variant of Nitsche. Section 5 reports different numerical experiments with uniform and adaptive refinement, for contact with Tresca and Coulomb friction, in order to assess the performance of the estimator in practice.

Let us introduce some useful notations. In what follows, bold letters like $\mathbf{u}, \mathbf{v}$, indicate vector or tensor valued quantities, while the capital ones (e.g., $\mathbf{V}, \mathbf{K} \ldots$) represent functional sets or operators involving vector fields.

As usual, we denote by $H^s(\cdot)$, $s \in \mathbb{R}$, the Sobolev spaces of real-valued functions, and $H^s(\cdot;\mathbb{R}^d)$, $s \in \mathbb{R}$, $d \in \mathbb{N}^*$ the Sobolev spaces of vector-valued functions in $\mathbb{R}^d$ (see [1]). For D a domain, the standard scalar product (resp. norm) of $H^s(D;\mathbb{R}^d)$ is denoted by $(\cdot,\cdot)_{s,D}$ (resp. $\|\cdot\|_{s,D}$) and we keep the same notation for all the values of d. The letter C stands for a generic constant, independent of the discretization parameters.

2 Setting and Nitsche-FEM

In this section we first present the frictional contact problems under consideration and then their approximation with Nitsche-FEM.

2.1 Frictional contact

We consider an elastic body whose reference configuration is represented by the domain Ω in $\mathbb{R}^d$ with $d = 2$ or $d = 3$. Small strain assumption is made, as well as plane strain when $d = 2$. The boundary $\partial\Omega$ of Ω is polygonal or polyhedral and we partition $\partial\Omega$ in three nonoverlapping parts Γ_D, Γ_N and the (potential) contact/friction boundary Γ_C, with $\text{meas}(\Gamma_D) > 0$ and $\text{meas}(\Gamma_C) > 0$. The contact/friction boundary is supposed to be a straight line segment when $d = 2$ or a planar polygon when $d = 3$ to simplify. The normal unit outward vector on $\partial\Omega$ is denoted $\mathbf{n}$. The body is clamped on Γ_D for the sake of simplicity. It is subjected to volume forces $\mathbf{f} \in L^2(\Omega;\mathbb{R}^d)$ and to surface loads $\mathbf{f}_N \in L^2(\Gamma_N;\mathbb{R}^d)$.

The contact problem under consideration consists in finding the displacement field $\mathbf{u} : \Omega \to \mathbb{R}^d$ verifying the equations and conditions (1)–(2)–(3):

$$\begin{aligned} \mathbf{div}\,\boldsymbol{\sigma}(\mathbf{u}) + \mathbf{f} = \mathbf{0} \qquad & \text{in } \Omega, \qquad & \boldsymbol{\sigma}(\mathbf{u}) = \mathbf{C}\,\boldsymbol{\varepsilon}(\mathbf{u}) \qquad & \text{in } \Omega, \\ \mathbf{u} = \mathbf{0} \qquad & \text{on } \Gamma_D, \qquad & \boldsymbol{\sigma}(\mathbf{u})\mathbf{n} = \mathbf{f}_N \qquad & \text{on } \Gamma_N, \end{aligned} \tag{1}$$

where $\boldsymbol{\sigma} = (\sigma_{ij})$, $1 \le i,j \le d$, stands for the stress tensor field and $\mathbf{div}$ denotes the divergence operator of tensor valued functions. The notation $\boldsymbol{\varepsilon}(\mathbf{v}) = (\boldsymbol{\nabla}\mathbf{v} + \boldsymbol{\nabla}\mathbf{v}^T)/2$ represents the linearized strain tensor field and $\mathbf{C}$ is the fourth order symmetric elasticity tensor having the usual uniform ellipticity and boundedness property. For any displacement field $\mathbf{v}$ and for any density of surface forces $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$ defined on $\partial\Omega$ we adopt the following decomposition into normal and tangential components,

$$\mathbf{v} = v_{\mathbf{n}}\mathbf{n} + \mathbf{v}_{\mathbf{t}} \quad \text{and} \quad \boldsymbol{\sigma}(\mathbf{v})\mathbf{n} = \sigma_{\mathbf{n}}(\mathbf{v})\mathbf{n} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}).$$

where $v_{\mathbf{n}} := \mathbf{v}\cdot\mathbf{n}$ and $\mathbf{v}_{\mathbf{t}} := \mathbf{v} - v_{\mathbf{n}}\mathbf{n}$ (the same applies for the vector $\boldsymbol{\sigma}(\mathbf{v})\mathbf{n}$). The contact conditions on Γ_C are formulated as:

$$\sigma_{\mathbf{n}} = -[u_{\mathbf{n}} - \sigma_{\mathbf{n}}]_C \tag{2}$$

where $[\cdot]_C$ is a projection operator onto a convex subset of $\mathbb{R}$. Two special cases will be considered:

1. Unilateral contact conditions, for which $[\cdot]_C$ is the positive part operator:

$$[x]_C = \frac{1}{2}(x + |x|)$$

for $x \in \mathbb{R}$. Then condition (2) is equivalent to the Signorini conditions

$$u_n \le 0, \qquad \sigma_n(\mathbf{u}) \le 0, \qquad \sigma_n(\mathbf{u})\,u_n = 0.$$

2. Bilateral contact conditions, for which $[\cdot]_C$ is the identity:

$$[x]_C = x$$

for $x \in \mathbb{R}$. Then condition (2) is the generalized Dirichlet condition:

$$u_{\mathbf{n}} = 0.$$

The friction conditions on Γ_C are given by

$$\begin{cases} |\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \;\le\; S(\mathbf{u}) & \text{if } \mathbf{u}_{\mathbf{t}} = \mathbf{0}, \quad (i) \\ \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \;=\, -\,S(\mathbf{u})\dfrac{\mathbf{u}_{\mathbf{t}}}{|\mathbf{u}_{\mathbf{t}}|} & \text{otherwise}, \quad (ii) \end{cases} \tag{3}$$

where $|\cdot|$ stands for the euclidean norm in $\mathbb{R}^{d-1}$. The above formulation encompasses both Tresca and Coulomb friction models. Indeed, for Tresca, we set $S(\mathbf{u}) = s_T$ where $s_T \in L^2(\Gamma_C)$, $s \ge 0$ is a given threshold. For Coulomb friction, we set $S(\mathbf{u}) = -\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u})$, where $\mathscr{F} \ge 0$ is the friction coefficient. Note that conditions (3)–(i) and (3)–(ii) imply that $|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| \le S(\mathbf{u})$ in all cases, and that if $|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})| < S(\mathbf{u})$, we must have $\mathbf{u}_{\mathbf{t}} = \mathbf{0}$.

For Coulomb friction, conditions (3) mean that, at a given contact point on Γ_C, sliding can not occur while the magnitude of the tangential stress $|\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})|$ is strictly below the threshold $-\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u})$. When the threshold $-\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u})$ is reached, sliding may happen, in a direction opposite to $\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})$ (see, e.g., [49, Chapter 10]). Remark that this static form of Coulomb friction is an adaptation of the quasi-static (or dynamic) Coulomb's law, in which the tangential velocity $\dot{\mathbf{u}}_{\mathbf{t}}$ plays the same role as the displacement $\mathbf{u}_{\mathbf{t}}$ (see *e.g.*, [3, 26, 28]). A formulation such as (3) is obtained for instance when the quasi-static Coulomb's law is discretized with a time-marching scheme (see, e.g., [65]).

Remark 2.1. *In the Tresca friction model ($S(\mathbf{u}) = s_T$ in (3)), it is assumed that the amplitude of the normal friction threshold is known (i.e., $\mathscr{F}|\sigma_{\mathbf{n}}(\mathbf{u})| = s_T$, see, e.g., [49, Section 10.3]). The introduction of the Tresca friction model is rather motivated by its mathematical simplicity than by physical reasons, though it can be of use in special situations. For instance when a thin belt/tape is pressed against an elastic body, with a known pressure (see, e.g., [49, Chapter 10]), or for persistent contact between solids with high intensity of contact pressures, such as it may occur in earth sciences (see, e.g., [60]). Moreover, the Tresca friction model can be useful, for instance, when Coulomb friction is approximated iteratively (see, e.g., [39] and [51, Section 9, Theorem 7]).*

We introduce the Hilbert space $\mathbf{V}$ and the set $\mathbf{K}$ of admissible displacements which satisfy the condition on the contact zone Γ_C:

$$\mathbf{V} := \left\{\mathbf{v} \in H^1(\Omega;\mathbb{R}^d) \ : \mathbf{v} = \mathbf{0} \text{ on } \Gamma_D\right\}, \quad \mathbf{K} := \{\mathbf{v} \in \mathbf{V} \ : \ [v_{\mathbf{n}}]_C = 0 \text{ on } \Gamma_C\}.$$

Define

$$a(\mathbf{u},\mathbf{v}) := \int_\Omega \boldsymbol{\sigma}(\mathbf{u}) : \boldsymbol{\varepsilon}(\mathbf{v})\, \mathrm{d}\Omega, \qquad L(\mathbf{v}) := \int_\Omega \mathbf{f}\cdot\mathbf{v}\, \mathrm{d}\Omega + \int_{\Gamma_N} \mathbf{f}_N \cdot \mathbf{v}\, \mathrm{d}\Gamma, \qquad j(\mathbf{u};\mathbf{v}) := \int_{\Gamma_C} S(\mathbf{u})|\mathbf{v}_{\mathbf{t}}|\, \mathrm{d}\Gamma,$$

for any $\mathbf{u}$ and $\mathbf{v}$ in $\mathbf{V}$, and $\mathbf{u}$ regular enough.

The weak formulation of Problem (1)–(2)–(3) as a (quasi-)variational inequality is (see, *e.g.*, [26]):

$$\left\{\begin{array}{l} \text{Find } \mathbf{u} \in \mathbf{K} \text{ (regular enough) such that:} \\ a(\mathbf{u},\mathbf{v}-\mathbf{u}) + j(\mathbf{u};\mathbf{v}) - j(\mathbf{u};\mathbf{u}) \geq L(\mathbf{v}-\mathbf{u}), \qquad \forall\, \mathbf{v} \in \mathbf{K}. \end{array}\right. \tag{4}$$

In the case of Tresca friction ($S(\mathbf{u}) = s$), the above weak formulation is a variational inequality of the second kind that admits a unique solution (see, *e.g.*, [32, Theorem 5.1, Remark 5.2, p.69]).

Contact with Coulomb friction in elastostatics remains a difficult problem, with still some open issues in its mathematical analysis, both for the continuous and the discrete problems. In the continuous case, there is indeed no complete characterization of existence and uniqueness when the friction coefficient is varied (see, *e.g.*, [27, 28, 56] for existence results when the friction coefficient is small). Moreover it can be proved that uniqueness is lost in some configurations and multiple solutions can be obtained, see, *e.g.*, [3, 43].

2.2 The Nitsche-based finite element method

Let $\mathbf{V}^h \subset \mathbf{V}$ be a family of finite dimensional vector spaces (see [9, 23, 29]) indexed by h coming from a family $\mathcal{T}^h$ of triangulations of the domain Ω ($h = \max_{T\in\mathcal{T}^h} h_T$ where h_T is the diameter of T). We suppose that the family of triangulations is regular, *i.e.*, there exists $\sigma > 0$ such that $\forall T \in \mathcal{T}^h, h_T/\rho_T \leq \sigma$ where ρ_T denotes the radius of the inscribed ball in T. Furthermore, we suppose that this family is conformal to the subdivision of the boundary into Γ_D, Γ_N and Γ_C (*i.e.*, a face of an element $T \in \mathcal{T}^h$ is not allowed to have simultaneous non-empty intersection with more than one part of the subdivision). We choose a standard Lagrange finite element method of degree k with $k = 1$ or $k = 2$, *i.e.*:

$$\mathbf{V}^h := \left\{\mathbf{v}^h \in \mathscr{C}^0(\overline{\Omega};\mathbb{R}^d) : \mathbf{v}^h{}_{|\,T} \in \mathbb{P}_k(T;\mathbb{R}^d), \forall T \in \mathcal{T}^h, \mathbf{v}^h = \mathbf{0} \text{ on } \Gamma_D\right\}. \tag{5}$$

However, the analysis would be similar for any $\mathscr{C}^0$-conforming finite element method.

Moreover, for any $\alpha \in \mathbb{R}^+$, we introduce the notation $[\cdot]_\alpha$ for the orthogonal projection onto $\mathscr{B}(\mathbf{0},\alpha) \subset \mathbb{R}^{d-1}$, where $\mathscr{B}(\mathbf{0},\alpha)$ is the closed ball centered at the origin $\mathbf{0}$ and of radius α. This operation can be defined analytically, for $\mathbf{x} \in \mathbb{R}^{d-1}$ by:

$$[\mathbf{x}]_\alpha = \left\{\begin{array}{cl} \mathbf{x} & \text{if } |\mathbf{x}| \leq \alpha, \\ \alpha\frac{\mathbf{x}}{|\mathbf{x}|} & \text{otherwise.} \end{array}\right.$$

The next result has been pointed out earlier in [2] (see as well [13, 17, 20] for detailed formal proofs).

Proposition 2.2. *Let γ be a positive function defined on Γ_C. The contact with friction conditions* (2)–(3) *can be reformulated as follows:*

$$\sigma_{\mathbf{n}}(\mathbf{u}) = -\left[\gamma\, u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})\right]_C, \tag{6}$$

$$\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) = -\left[\gamma\, \mathbf{u}_{\mathbf{t}} - \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})\right]_{S(\mathbf{u})}. \tag{7}$$

Remember that for Tresca friction, we have $S(\mathbf{u}) = s_T$ where $s_T \in L^2(\Gamma_C)$, $s \geq 0$ is a given threshold. For Coulomb friction, we can use (6) to have two different expressions of the solution-dependent threshold:

$$S(\mathbf{u}) = -\mathscr{F}\sigma_{\mathbf{n}}(\mathbf{u}) = \mathscr{F}\left[\gamma\, u_{\mathbf{n}} - \sigma_{\mathbf{n}}(\mathbf{u})\right]_C.$$

Let now $\theta \in \mathbb{R}$ be a fixed parameter and γ a positive function defined on Γ_C. The Nitsche method for frictional contact is derived as follows [15, 46]. Let $\mathbf{u}$ be the solution to the frictional contact problem in its strong form (1)–(2)–(3). We assume that $\mathbf{u}$ is sufficiently regular so that all the following calculations make sense. From the Green formula and equation (1), we get for every $\mathbf{v} \in \mathbf{K}$:

$$a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_C} \sigma_{\mathbf{n}}(\mathbf{u})\, v_{\mathbf{n}}\, \mathrm{d}\Gamma - \int_{\Gamma_C} \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot \mathbf{v}_{\mathbf{t}}\, \mathrm{d}\Gamma = L(\mathbf{v}).$$

Note that

$$\begin{aligned} v_{\mathbf{n}} &= \frac{1}{\gamma}(\gamma v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v})) + \frac{\theta}{\gamma}\sigma_{\mathbf{n}}(\mathbf{v}), \\ \mathbf{v}_{\mathbf{t}} &= \frac{1}{\gamma}(\gamma \mathbf{v}_{\mathbf{t}} - \theta\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v})) + \frac{\theta}{\gamma}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}). \end{aligned}$$

So:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \sigma_{\mathbf{n}}(\mathbf{u})\, \sigma_{\mathbf{n}}(\mathbf{v})\, \mathrm{d}\Gamma - \int_{\Gamma_C} \frac{1}{\gamma}\sigma_{\mathbf{n}}(\mathbf{u})\, (\gamma v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v}))\, \mathrm{d}\Gamma & \\ - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v})\, \mathrm{d}\Gamma - \int_{\Gamma_C} \frac{1}{\gamma}\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot (\gamma\mathbf{v}_{\mathbf{t}} - \theta\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}))\, \mathrm{d}\Gamma &= L(\mathbf{v}). \end{aligned}$$

Finally, using conditions (6) and (7), we obtain:

$$\begin{aligned} a(\mathbf{u},\mathbf{v}) - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \sigma_{\mathbf{n}}(\mathbf{u})\, \sigma_{\mathbf{n}}(\mathbf{v})\, \mathrm{d}\Gamma + \int_{\Gamma_C} \frac{1}{\gamma}\left[\gamma\, u_{\mathbf{n}} - \sigma_n(\mathbf{u})\right]_C\, (\gamma v_{\mathbf{n}} - \theta\sigma_{\mathbf{n}}(\mathbf{v}))\, \mathrm{d}\Gamma & \\ - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}) \cdot \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v})\, \mathrm{d}\Gamma + \int_{\Gamma_C} \frac{1}{\gamma}\left[\gamma\, \mathbf{u}_{\mathbf{t}} - \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u})\right]_{S(\mathbf{u})} \cdot (\gamma\mathbf{v}_{\mathbf{t}} - \theta\boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{v}))\, \mathrm{d}\Gamma &= L(\mathbf{v}). \end{aligned} \tag{8}$$

Formula (8) is the starting point of the Nitsche-based formulation. We remark that it may have no sense at the continuous level if $\mathbf{u}$ lacks of regularity (the only assumption $\mathbf{u} \in \mathbf{V}$ is not sufficient to justify the above calculations). Nevertheless we consider in what follows that γ is a positive piecewise constant function on the contact and friction interface Γ_C which satisfies

$$\gamma|_{T \cap \Gamma_C} = \frac{\gamma_0}{h_T}, \tag{9}$$

for every T that has a non-empty intersection of dimension $d-1$ with Γ_C, and where γ_0 is a positive given constant (the Nitsche parameter). Note that the value of γ on element intersections has no influence. Let us introduce the discrete linear operators

$$\mathrm{P}^{\mathbf{n}}_{\theta,\gamma} : \begin{array}{ccc} \mathbf{V}^h & \to & L^2(\Gamma_C) \\ \mathbf{v}^h & \mapsto & \gamma v^h_n - \theta\sigma_{\mathbf{n}}(\mathbf{v}^h) \end{array} \quad \text{and} \quad \mathbf{P}^{\mathbf{t}}_{\theta,\gamma} : \begin{array}{ccc} \mathbf{V}^h & \to & (L^2(\Gamma_C))^{d-1} \\ \mathbf{v}^h & \mapsto & \gamma\mathbf{v}^h_{\mathbf{t}} - \theta\boldsymbol{\sigma}_t(\mathbf{v}^h) \end{array}.$$

Define as well the bilinear form:

$$A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) := a(\mathbf{u}^h,\mathbf{v}^h) - \int_{\Gamma_C} \frac{\theta}{\gamma}\, \boldsymbol{\sigma}(\mathbf{u}^h)\mathbf{n} \cdot \boldsymbol{\sigma}(\mathbf{v}^h)\mathbf{n}\, \mathrm{d}\Gamma.$$

The Nitsche-based method for frictional contact then reads:

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{V}^h \text{ such that:} \\ A_{\theta\gamma}(\mathbf{u}^h,\mathbf{v}^h) + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_C \mathrm{P}^{\mathbf{n}}_{\theta,\gamma}(\mathbf{v}^h)\, \mathrm{d}\Gamma \\ + \displaystyle\int_{\Gamma_C} \frac{1}{\gamma}\left[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)\right]_{S^h(\mathbf{u}^h)} \cdot \mathbf{P}^{\mathbf{t}}_{\theta,\gamma}(\mathbf{v}^h)\, \mathrm{d}\Gamma = L(\mathbf{v}^h), \qquad \forall\, \mathbf{v}^h \in \mathbf{V}^h. \end{cases} \tag{10}$$

For Tresca friction we take simply:

$$S^h(\mathbf{u}^h) = s_T.$$

For Coulomb friction we preferably take:

$$S^h(\mathbf{u}^h) = \left(\mathscr{F}\left[\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)\right]_C\right).$$

For unilateral contact, it ensures that the threshold for the projection operator remains nonnegative. The parameter θ can be set to some particular values, namely:

1. for $\theta = 1$ we recover a symmetric method.
2. for $\theta = 0$ we recover a simple method close to penalty and to the augmented Lagrangian.
3. for $\theta = -1$ the skew-symmetric method is well-posed irrespectively of the value of $\gamma_0 > 0$.

Existence and uniqueness of solutions to the discrete problem (10) has been studied in [13] for Tresca friction: in this case well-posedness is ensured provided that γ_0 is large enough ($\gamma_0 > 0$ for $\theta = 1$). For Coulomb friction, a fixed point argument allows to assess the existence of solutions to (10) for γ_0 large enough, and every value of the friction coefficient. Uniqueness can be recovered provided a restrictive condition, for friction small enough, and dependent of the mesh size, see [20] for details. In [13] an optimal *a priori* error bound for Tresca friction in the natural norm has been derived, for 2D and 3D problems, and for linear and quadratic FEM.

3 Residual-based *a posteriori* error estimate

An explicit residual-based *a posteriori* error estimate can be derived for Problem (10), that has been proposed in [16] for frictionless contact and Tresca friction. We extend it here to the Coulomb friction model. This *a posteriori* error estimate has its origin in [6] (see also, e.g., [64] for linear elasticity). We introduce standard notations for this purpose:

- We define E_h the set of edges/faces of the triangulation and define $E_h^{int} := \{E \in E_h : E \subset \Omega\}$ as the set of interior edges/faces of $\mathcal{T}^h$. We denote by $E_h^N := \{E \in E_h : E \subset \Gamma_N\}$ the set of Neumann edges/faces and similarly $E_h^C := \{E \in E_h : E \subset \Gamma_C\}$ is the set of contact edges/faces.
- For an element T, we denote by E_T the set of edges/faces of T and according to the above notation, we set $E_T^{int} := E_T \cap E_h^{int}$, $E_T^N := E_T \cap E_h^N$, $E_T^C := E_T \cap E_h^C$.
- For an edge/face E of an element T, introduce $\boldsymbol{\nu}_{T,E}$ the unit outward normal vector to T along E. Furthermore, for each edge/face E, we fix one of the two normal vectors and denote it by $\boldsymbol{\nu}_E$. The jump of some vector valued function $\mathbf{v}$ across an edge/face $E \in E_h^{int}$ at a point $\mathbf{y} \in E$ is defined as

$$[\![\mathbf{v}]\!]_E(\mathbf{y}) := \lim_{\alpha \to 0^+} \mathbf{v}(\mathbf{y} + \alpha \boldsymbol{\nu}_E) - \mathbf{v}(\mathbf{y} - \alpha \boldsymbol{\nu}_E).$$

- Let ω_T be the union of all elements having a nonempty intersection with T. Similarly for a node $\boldsymbol{x}$ and an edge/face E, let $\omega_x := \cup_{T : x \in T} T$ and $\omega_E := \cup_{x \in \overline{E}} \omega_x$.
- $\mathbf{f}_T$ (resp. $\mathbf{f}_{N,E}$) is a computable quantity that approximates $\mathbf{f}$ on the element $T \in \mathcal{T}^h$ (resp. $\mathbf{f}_N$ on the edge $E \in E_h^N$).

The *a posteriori* error estimator is defined as follows.

Definition 3.1. *For the unified formulation* (10) *the local error estimators* η_T *and the the global estimator* η *are defined by*

$$\begin{aligned}
\eta_T &:= \left(\sum_{i=1}^{4} \eta_{iT}^2 \right)^{1/2}, \\
\eta_{1T} &:= h_T \| \mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^h) \; + \; \mathbf{f}_T \|_{0,T}, \\
\eta_{2T} &:= h_T^{1/2} \left(\sum_{E \in E_T^{int} \cup E_T^N} \| J_{E,n}(\mathbf{u}^h) \|_{0,E}^2 \right)^{1/2}, \\
\eta_{3T} &:= h_T^{1/2} \left(\sum_{E \in E_T^C} \left\| \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h) + \left[\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}^h) \right]_{S^h(\mathbf{u}^h)} \right\|_{0,E}^2 \right)^{1/2}, \\
\eta_{4T} &:= h_T^{1/2} \left(\sum_{E \in E_T^C} \left\| \sigma_{\mathbf{n}}(\mathbf{u}^h) + \left[\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{u}^h) \right]_C \right\|_{0,E}^2 \right)^{1/2}, \\
\eta &:= \left(\sum_{T \in \mathcal{T}^h} \eta_T^2 \right)^{1/2},
\end{aligned}$$

where $J_{E,n}(\mathbf{u}^h)$ *means the constraint jump of* $\mathbf{u}^h$ *in the normal direction, i.e.,*

$$J_{E,n}(\mathbf{u}^h) \quad := \quad \begin{cases} [\![\boldsymbol{\sigma}(\mathbf{u}^h)\boldsymbol{\nu}_E]\!]_E, & \forall E \in E_h^{int}, \\ \boldsymbol{\sigma}(\mathbf{u}^h)\boldsymbol{\nu}_E - \mathbf{f}_{N,E}, & \forall E \in E_h^N. \end{cases} \tag{11}$$

The local and global approximation terms are given by

$$\zeta_T := \left(h_T^2 \sum_{T' \subset \omega_T} \|\mathbf{f} - \mathbf{f}_{T'}\|_{0,T'}^2 + h_E \sum_{E \subset E_T^N} \|\mathbf{f}_N - \mathbf{f}_{N,E}\|_{0,E}^2 \right)^{1/2},$$
$$\zeta := \left(\sum_{T \in \mathcal{T}^h} \zeta_T^2 \right)^{1/2}.$$

4 An upper bound for the residual

In [16] a global upper bound, under a saturation assumption, and for regularity $H^s(\Omega)$, $s > 3/2$, of the solution $\mathbf{u}$, have been proven, for frictionless contact and also for Tresca friction. Local optimal lower bounds have been established too.

We can try here to follow the idea of a recent work from Di Pietro et al [25] and to derive an upper bound for the dual norm of the residual without any saturation assumption and with solely a regularity $H^1(\Omega)$ for $\mathbf{u}$. This can be done for 2D and 3D problems, linear and quadratic finite elements, and above all for both friction models: Tresca friction and Coulomb friction. As in [25], we focus on the incomplete version of Nitsche and set $\theta = 0$.

The residual $\mathcal{R}$ can be viewed as an application

$$\mathcal{R} : \mathbf{V}^h \to \mathbf{V}^\star$$

where $\mathbf{V}^\star$ is the topological dual of $\mathbf{V}$. This application is defined, for $\mathbf{w}^h \in \mathbf{V}^h$ as:

$$\begin{cases} \langle \mathcal{R}(\mathbf{w}^h), \mathbf{v} \rangle \\ := L(\mathbf{v}) - a(\mathbf{w}^h, \mathbf{v}) - \int_{\Gamma_C} [\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{w}^h)]_C v_{\mathbf{n}} \, \mathrm{d}\Gamma - \int_{\Gamma_C} [\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{w}^h)]_{S^h(\mathbf{u}^h)} \cdot \mathbf{v}_{\mathbf{t}} \, \mathrm{d}\Gamma, \qquad \forall\, \mathbf{v} \in \mathbf{V}. \end{cases} \tag{12}$$

We see here why it is not convenient to define a residual for $\theta \neq 0$: in this situation the Nitsche boundary terms may have no meaning in $L^2(\Gamma_C)$, since we suppose only $\mathbf{v} \in H^1(\Omega; \mathbb{R}^d)$. The dual norm of the residual is then defined as

$$\|\mathcal{R}(\mathbf{w}^h)\|_* := \sup_{\mathbf{v} \in \mathbf{V}} \frac{\langle \mathcal{R}(\mathbf{w}^h), \mathbf{v} \rangle}{\|\mathbf{v}\|_{1,\Omega}}.$$

The following result holds:

Proposition 4.1. *Let us suppose that $\theta = 0$, $\gamma_0 > 0$ large enough and let $\mathbf{u}^h \in \mathbf{V}^h \subset \mathbf{V}$ be a solution to Nitsche formulation* (10)*. Then the residual defined by* (12) *is bounded by the a posteriori error estimate and the approximation terms of Definition 3.1 as follows*

$$\|\mathcal{R}(\mathbf{u}^h)\|_* \le C\,(\eta + \zeta)$$

where the constant $C > 0$ is independent of h.

Proof. Let $\mathbf{u}^h \in \mathbf{V}^h \subset \mathbf{V}$ be a solution to Nitsche formulation (10), let $\mathbf{v} \in \mathbf{V}$ be a test function, and let us write first

$$\langle \mathcal{R}(\mathbf{u}^h), \mathbf{v} \rangle = L(\mathbf{v}) - a(\mathbf{u}^h, \mathbf{v}) - \int_{\Gamma_C} [\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{u}^h)]_C v_{\mathbf{n}} \, \mathrm{d}\Gamma - \int_{\Gamma_C} [\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} \cdot \mathbf{v}_{\mathbf{t}} \, \mathrm{d}\Gamma.$$

Since $\mathbf{u}^h$ solves (10), then there holds:

$$\langle \mathcal{R}(\mathbf{u}^h), \mathbf{v}^h \rangle = 0,$$

for every test function $\mathbf{v}^h \in \mathbf{V}^h$. So there holds also

$$\begin{aligned} & \langle \mathcal{R}(\mathbf{u}^h), \mathbf{v} \rangle \\ = \; & \langle \mathcal{R}(\mathbf{u}^h), \mathbf{v} \rangle - \langle \mathcal{R}(\mathbf{u}^h), \mathbf{v}^h \rangle = \langle \mathcal{R}(\mathbf{u}^h), \mathbf{v} - \mathbf{v}^h \rangle \\ = \; & L(\mathbf{v} - \mathbf{v}^h) - a(\mathbf{u}^h, \mathbf{v} - \mathbf{v}^h) - \int_{\Gamma_C} [\mathrm{P}_{1,\gamma}^{\mathbf{n}}(\mathbf{u}^h)]_C (v_{\mathbf{n}} - v_{\mathbf{n}}^h) \, \mathrm{d}\Gamma - \int_{\Gamma_C} [\mathbf{P}_{1,\gamma}^{\mathbf{t}}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} \cdot (\mathbf{v}_{\mathbf{t}} - \mathbf{v}_{\mathbf{t}}^h) \, \mathrm{d}\Gamma. \end{aligned}$$

Now we can transform the above expression by integrating by parts on each triangle T, using the definition of $J_{E,n}(\mathbf{u}^h)$ and splitting up the integrals on Γ_C into normal and tangential components. As a result, we get:

$$\begin{aligned}
\langle \mathcal{R}(\mathbf{u}^h), \mathbf{v}\rangle \quad = \quad & \sum_{T\in\mathcal{T}_h} \int_T (\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^h) \;+\; \mathbf{f}) \cdot (\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \\
& - \sum_{E\in E_h^C} \int_E \left([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)]_C + \sigma_{\mathbf{n}}(\mathbf{u}^h)\right) (v_{\mathbf{n}} - v^h_{\mathbf{n}})\, \mathrm{d}\Gamma \\
& - \sum_{E\in E_h^C} \int_E \left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\right) \cdot (\mathbf{v}_{\mathbf{t}} - \mathbf{v}^h_{\mathbf{t}})\, \mathrm{d}\Gamma \\
& - \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\mathbf{u}^h)\cdot(\mathbf{v}-\mathbf{v}^h)\ \ \mathrm{d}\Gamma \\
& + \sum_{E\in E_h^N} \int_E (\mathbf{f}_N - \mathbf{f}_{N,E})\cdot (\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma. \qquad (13)
\end{aligned}$$

We now need to estimate each term of this right-hand side. For that purpose, we take

$$\mathbf{v}^h = \mathcal{SZ}^h(\mathbf{v}), \qquad (14)$$

where $\mathcal{SZ}^h$ is the quasi-interpolation operator of Scott & Zhang, see [62] and [29].
We start with the integral term on simplices T. Cauchy-Schwarz's inequality implies

$$\sum_{T\in\mathcal{T}_h} \int_T (\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^h) \;+\; \mathbf{f}) \cdot (\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \le \sum_{T\in\mathcal{T}_h} \|\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^h) \;+\; \mathbf{f}\|_{0,T} \|\mathbf{v}-\mathbf{v}^h\|_{0,T},$$

and it suffices to estimate $\|\mathbf{v}-\mathbf{v}^h\|_{0,T}$ for any simplex T. From the definition of $\mathbf{v}^h$ and the approximation properties of $\mathcal{SZ}^h$, we get:

$$\|\mathbf{v}-\mathbf{v}^h\|_{0,T} = \|\mathbf{v}-\mathcal{SZ}^h\mathbf{v}\|_{0,T} \le C h_T \|\mathbf{v}\|_{1,\omega_T}.$$

As a consequence

$$\left| \int_\Omega (\mathbf{div}\, \boldsymbol{\sigma}(\mathbf{u}^h) \;+\; \mathbf{f}) \cdot (\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \right| \le C(\eta+\zeta)\|\mathbf{v}\|_{1,\Omega}.$$

For the interior and Neumann boundary terms in (13), the application of Cauchy-Schwarz's inequality leads to

$$\left| \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\mathbf{u}^h)\cdot(\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \right| \le \sum_{E\in E_h^{int}\cup E_h^N} \|J_{E,n}(\mathbf{u}^h)\|_{0,E}\|\mathbf{v}-\mathbf{v}^h\|_{0,E}.$$

Using once again the approximation properties of $\mathcal{SZ}^h$, we get:

$$\|\mathbf{v}-\mathbf{v}^h\|_{0,E} = \|\mathbf{v}-\mathcal{SZ}^h\mathbf{v}\|_{0,E} \lesssim h_E^{1/2}\|\mathbf{v}\|_{1,\omega_E},$$

and we deduce that

$$\left| \sum_{E\in E_h^{int}\cup E_h^N} \int_E J_{E,n}(\mathbf{u}^h)\cdot(\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \right| \le C\eta\|\mathbf{v}\|_{1,\Omega}.$$

Moreover we can bound

$$\left| \sum_{E\in E_h^N} \int_E (\mathbf{f}_N - \mathbf{f}_{N,E})\cdot (\mathbf{v}-\mathbf{v}^h)\, \mathrm{d}\Gamma \right| \le C\zeta\|\mathbf{v}\|_{1,\Omega}.$$

Let us focus now on the friction term:

$$\left| \sum_{E\in E_h^C} \int_E \left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\right) \cdot (\mathbf{v}_{\mathbf{t}} - \mathbf{v}^h_{\mathbf{t}})\, \mathrm{d}\Gamma \right| \le \sum_{E\in E_h^C} \|[\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\|_{0,E}\|\mathbf{v}-\mathbf{v}^h\|_{0,E}.$$

Still we use the quasi-interpolation properties of $\mathcal{SZ}^h$ and then we bound

$$\left| \sum_{E\in E_h^C} \int_E \left([\mathbf{P}^{\mathbf{t}}_{1,\gamma}(\mathbf{u}^h)]_{S^h(\mathbf{u}^h)} + \boldsymbol{\sigma}_{\mathbf{t}}(\mathbf{u}^h)\right) \cdot (\mathbf{v}_{\mathbf{t}} - \mathbf{v}^h_{\mathbf{t}})\, \mathrm{d}\Gamma \right| \le C\eta\|\mathbf{v}\|_{1,\Omega}.$$

We do exactly the same for the contact term:

$$\left| \sum_{E\in E_h^C} \int_E \Big([\mathrm{P}^{\mathbf{n}}_{1,\gamma}(\mathbf{u}^h)]_C + \sigma_{\mathbf{n}}(\mathbf{u}^h) \Big) \cdot (v_{\mathbf{n}} - v^h_{\mathbf{n}})\ \mathrm{d}\Gamma \right| \le C\eta \|\mathbf{v}\|_{1,\Omega}.$$

We combine all the previous results and get finally

$$\langle \mathcal{R}(\mathbf{u}^h), \mathbf{v}\rangle \le C(\eta + \zeta)\|\mathbf{v}\|_{1,\Omega}.$$

By definition of the dual norm, we conclude that

$$\|\mathcal{R}(\mathbf{u}^h)\|_* \le C(\eta + \zeta).$$

and this ends the proof. □

Remark 4.2. *In the frictionless case, one can recover from the above result an upper bound on the natural norm of the discretization error ($\|u - u_h\|_{1,\Omega}$), under a saturation assumption and with enough regularity on the continuous solution* $\mathbf{u}$*, see [25]. This is also the case for Tresca friction, and one then recovers the upper bound stated in [16], in the case $\theta = 0$. For Coulomb friction, the obtention of an upper bound on the natural norm of the discretization error remains an open issue.*

5 Numerical experiments

Contact and friction problems can be solved by using a generalized (semi-smooth) Newton method, which means that Problem (10) is derived with respect to $\mathbf{u}^h$ to obtain the tangent system. The term "generalized Newton's method" comes from the fact that the operators such as $[\cdot]_C$ and $[\cdot]_{S(\mathbf{u})}$ are not Gâteaux-differentiable at some specific points. However, no special treatment is considered. If a point of non-differentiability is encountered, the tangent system corresponding to one of the two alternatives ($x < 0$ or $x > 0$ for $[\cdot]_C$) is chosen arbitrarily. Note that, for frictionless contact, the situation where the solution is non-differentiable at an integration point is very rare and corresponds to what is called a "grazing contact" (both $u_{\mathbf{n}} = 0$ and $\sigma_{\mathbf{n}} = 0$).

Problem (10) is implemented within the open source finite element library FEniCS [54]. We took advantage of the capabilities of FEniCS in terms of automatic symbolic differentiation: the weak form (10) is written directly, and we did not need to specify explicitely the tangent problem.

First we present a first numerical test with Tresca friction and a manufactured solution, from [14]. Another test, still for Tresca, is from a classical benchmark, originally presented in [8]. The last numerical experiment is about Coulomb friction and comes from [53].

5.1 A manufactured solution

We first carry out some tests on the problem with a manufactured solution, presented in [14]. In this example we consider $\Omega := (0,1)\times(0,1)$, $\Gamma_C := (0,1)\times\{0\}$, $\Gamma_D := \partial\Omega \setminus \Gamma_C$. The Lamé coefficients are $\lambda = 1000$ and $\mu = 2$. The rhs $\mathbf{f}$ and the Dirichlet boundary condition $\mathbf{u}_D$ are such that the exact solution is given by:

$$\mathbf{u} := (u_1, u_2), \quad \text{where} \quad u_1(x,y) := \left(1 + \frac{1}{1+\lambda}\right) x\, e^{x+y}, \quad u_2(x,y) := \left(-1 + \frac{1}{1+\lambda}\right) y\, e^{x+y}.$$

The friction thresold s, defined on Γ_C, is given by

$$s(x) := \mu \left(1 + \frac{1}{1+\lambda}\right) x\, e^{x}.$$

Note that this setting is slightly different from Section 2, since there is no Neumann boundary, and a nonhomogeneous Dirichlet boundary condition. Lagrange finite elements of order $\mathbb{P}_k$ are used ($k = 1, 2$). Nitsche's parameter is set as $\gamma_0 = \lambda$ when $\theta = -1$ and $\theta = 0$. For the symmetric variant $\theta = 1$, it is set as $\gamma_0 = 4\lambda$ for $k = 1$ and $\gamma_0 = 6\lambda$ for $k = 2$.

5.1.1 Uniform refinement

Problem (10) is solved on a sequence of 7 structured meshes, with uniform refinement: the mesh size is divided by 2 each time. This allows to check the convergence properties of the error estimate.

For the skew-symmetric variant ($\theta = -1$), results are presented in Table 1, Table 2, Table 3 and Table 4. For Table 1 (order $k = 1$) and Table 3 (order $k = 2$), we detail the error in the L^2-norm, H^1 semi-norm and norm and the value of the residual error estimate η. The empirical convergence rates are denoted by r_2, r_{s1}, r_1 and

r_η, for the L^2-norm, the H^1 semi-norm, the H^1 norm and η, respectively. The efficiency index is denoted by E and defined, as usual, as:

$$E := \frac{\eta}{\|\mathbf{u} - \mathbf{u}_h\|_{1,\Omega}}.$$

As it can be seen in Table 1 for order $k = 1$, the errors in L^2 and H^1 norm are reduced with a rate higher than expected. This slight superconvergence phenomenon is unexplained. The a posteriori error estimate η decreases with the expected rate of approximately 1. The efficiency index E is large.

For order $k = 2$ we can observe from Table 3 that similar conclusions hold than for $k = 1$, still with a slight superconvergence for the a priori error rates. Note however, that the convergence rate of the residual estimator η is slightly below the theoretical rate of 2. The efficiency index is still large.

For Table 2 (order $k = 1$) and Table 4 (order $k = 2$), we report the value of the different components of η and their convergence rates. As in [16], we observe that the components η_1 and η_2 (residual and jumps on the edges) that are, roughly speaking, related to the elasticity equations, are predominent, and converge with the expected rate of approximately 1, for linear finite elements. The components related to Tresca friction and to unilateral contact, respectively η_3 and η_4, are of smaller magnitude, and converge faster, with a rate around 1.5. We have no explanation for this phenomenon. Remark also that the estimator related to Tresca friction is even smaller than the estimator for contact. For order $k = 2$, we can draw similar conclusions from Table 4. Moreover we remark that for η_1 and η_2 the convergence order is below 2, whereas the components η_3 and η_4 for friction and contact converge much faster, with rates around 2.5. Remark also that the component related to Tresca friction is smaller than the one for contact. These results are in agreement with the previous ones for frictionless contact, reported in [16].

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.190873	–	1.729627	–	1.740127	–	2286.60	–	1314.0
0.157	0.160036	0.25	1.480559	0.22	1.489183	0.22	1209.89	0.91	812.4
0.078	0.080972	0.98	0.894973	0.72	0.898628	0.72	635.38	0.92	707.0
0.039	0.033845	1.25	0.437086	1.03	0.438394	1.03	334.63	0.92	763.3
0.019	0.011733	1.52	0.175056	1.32	0.175449	1.32	174.46	0.93	994.3
0.009	0.003487	1.75	0.060809	1.52	0.060909	1.52	89.39	0.96	1467.7
0.004	0.000940	1.89	0.018999	1.67	0.019022	1.67	45.21	0.98	2376.9

Table 1: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial degree $k = 1$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = -1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	1352.65	–	1843.12	–	1.683	–	42.243	–	2286.60	–
0.157	680.34	0.99	1000.26	0.88	0.742	1.18	20.934	1.01	1209.89	0.91
0.078	340.44	0.99	536.43	0.89	0.184	2.00	7.414	1.49	635.38	0.92
0.039	170.25	0.99	288.06	0.89	0.060	1.59	2.815	1.39	334.63	0.92
0.019	85.13	0.99	152.28	0.91	0.026	1.18	1.056	1.41	174.46	0.93
0.009	42.56	0.99	78.61	0.95	0.009	1.51	0.390	1.43	89.39	0.96
0.004	21.28	0.99	39.89	0.97	0.002	1.75	0.142	1.45	45.21	0.98

Table 2: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial degree $k = 1$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = -1$.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.009256	–	0.190779	–	0.191004	–	122.42	–	640.9
0.157	0.001242	2.89	0.055794	1.77	0.055808	1.77	26.89	2.18	481.8
0.078	0.000162	2.94	0.016672	1.74	0.016673	1.74	7.54	1.83	452.3
0.039	0.000021	2.97	0.004534	1.87	0.004534	1.87	2.33	1.69	515.2
0.019	0.000002	3.18	0.001023	2.14	0.001023	2.14	0.70	1.72	690.7
0.009	0.000000	3.41	0.000198	2.36	0.000198	2.36	0.19	1.83	1000.4
0.004	0.000000	3.50	0.000036	2.45	0.000036	2.45	0.05	1.91	1452.6

Table 3: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = -1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	116.60	–	37.28	–	0.055925	–	1.533677	–	122.42	–
0.157	25.37	2.19	8.88	2.06	0.020291	1.46	0.406656	1.91	26.89	2.18
0.078	7.20	1.81	2.24	1.98	0.003910	2.37	0.086384	2.23	7.54	1.83
0.039	2.21	1.70	0.74	1.59	0.000838	2.22	0.015668	2.46	2.33	1.69
0.019	0.65	1.75	0.25	1.52	0.000146	2.52	0.002847	2.46	0.70	1.72
0.009	0.18	1.85	0.07	1.70	0.000021	2.77	0.000526	2.43	0.19	1.83
0.004	0.04	1.93	0.02	1.85	0.000003	2.79	0.000096	2.45	0.05	1.91

Table 4: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = -1$.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.198267	–	1.731564	–	1.742878	–	2274.84	–	1305.22
0.157	0.160284	0.30	1.474357	0.23	1.483044	0.23	1208.74	0.91	815.04
0.078	0.081131	0.98	0.890005	0.72	0.893695	0.73	634.99	0.92	710.53
0.039	0.033877	1.25	0.435946	1.02	0.437260	1.03	334.49	0.92	764.98
0.019	0.011734	1.52	0.174749	1.31	0.175143	1.31	174.43	0.93	995.93
0.009	0.003487	1.75	0.060711	1.52	0.060811	1.52	89.38	0.96	1469.91
0.004	0.000940	1.89	0.018970	1.67	0.018994	1.67	45.21	0.98	2380.42

Table 5: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = 0$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	1352.65	–	1825.32	–	1.763916	–	115.923480	–	2274.84	–
0.157	680.34	0.99	997.98	0.87	0.732652	1.26	47.342641	1.29	1208.74	0.91
0.078	340.44	0.99	535.70	0.89	0.163662	2.16	18.499760	1.35	634.99	0.92
0.039	170.25	0.99	287.83	0.89	0.049425	1.72	7.065624	1.38	334.49	0.92
0.019	85.13	0.99	152.22	0.91	0.020210	1.29	2.609322	1.43	174.43	0.93
0.009	42.56	0.99	78.59	0.95	0.006700	1.59	0.948872	1.45	89.38	0.96
0.004	21.28	0.99	39.88	0.97	0.001919	1.80	0.342196	1.47	45.21	0.98

Table 6: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = 0$.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.009667	–	0.191831	–	0.192074	–	124.93	–	650.47
0.157	0.001273	2.92	0.055813	1.78	0.055827	1.78	26.98	2.21	483.38
0.078	0.000165	2.94	0.016654	1.74	0.016655	1.74	7.55	1.83	453.35
0.039	0.000021	2.96	0.004530	1.87	0.004530	1.87	2.33	1.69	515.95
0.019	0.000002	3.16	0.001023	2.14	0.001023	2.14	0.70	1.72	691.21
0.009	0.000000	3.35	0.000198	2.36	0.000198	2.36	0.19	1.83	1000.83
0.004	0.000000	3.38	0.000036	2.45	0.000036	2.45	0.05	1.91	1452.91

Table 7: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = 0$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	118.99	–	36.93	–	0.070056	–	9.276328	–	124.93	–
0.157	25.51	2.22	8.69	2.08	0.021140	1.72	1.248022	2.89	26.98	2.21
0.078	7.21	1.82	2.22	1.96	0.003731	2.50	0.218437	2.51	7.55	1.83
0.039	2.21	1.70	0.74	1.58	0.000799	2.22	0.042261	2.36	2.33	1.69
0.019	0.65	1.75	0.25	1.51	0.000138	2.52	0.008389	2.33	0.70	1.72
0.009	0.18	1.85	0.07	1.70	0.000020	2.76	0.001601	2.38	0.19	1.83
0.004	0.04	1.93	0.02	1.85	0.000003	2.78	0.000294	2.44	0.05	1.91

Table 8: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = \lambda$ and $\theta = 0$.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.196380	–	1.719563	–	1.730740	–	2306.40	–	1332.61
0.157	0.158883	0.30	1.455742	0.24	1.464386	0.24	1213.42	0.92	828.62
0.078	0.080869	0.97	0.884652	0.71	0.888341	0.72	635.71	0.93	715.62
0.039	0.033718	1.26	0.434487	1.02	0.435793	1.02	334.61	0.92	767.82
0.019	0.011686	1.52	0.174424	1.31	0.174815	1.31	174.44	0.93	997.88
0.009	0.003476	1.74	0.060645	1.52	0.060744	1.52	89.38	0.96	1471.55
0.004	0.000937	1.89	0.018957	1.67	0.018980	1.67	45.21	0.98	2382.16

Table 9: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 4\lambda$ and $\theta = 1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	1349.43	–	1856.37	–	1.862072	–	228.896586	–	2306.40	–
0.157	679.94	0.98	1001.31	0.89	0.682431	1.44	86.362286	1.40	1213.42	0.92
0.078	340.39	0.99	535.91	0.90	0.126684	2.42	32.581834	1.40	635.71	0.93
0.039	170.25	0.99	287.81	0.89	0.037613	1.75	11.864399	1.45	334.61	0.92
0.019	85.13	0.99	152.20	0.91	0.014332	1.39	4.211949	1.49	174.44	0.93
0.009	42.56	0.99	78.58	0.95	0.004429	1.69	1.493011	1.49	89.38	0.96
0.004	21.28	0.99	39.88	0.97	0.001218	1.86	0.530160	1.49	45.21	0.98

Table 10: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 4\lambda$ and $\theta = 1$.

h	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u}-\boldsymbol{u}_h\|_{1,\Omega}$	r_1	η	r_{eta}	E
0.314	0.024923	–	0.278289	–	0.279403	–	153.61	–	549.79
0.157	0.005953	2.06	0.080538	1.78	0.080757	1.79	28.79	2.41	356.56
0.078	0.001476	2.01	0.021749	1.88	0.021799	1.88	7.74	1.89	355.44
0.039	0.000370	1.99	0.005696	1.93	0.005708	1.93	2.37	1.70	415.87
0.019	0.000093	1.99	0.001339	2.08	0.001342	2.08	0.71	1.73	531.84
0.009	0.000023	1.99	0.000293	2.19	0.000294	2.18	0.19	1.84	677.35
0.004	0.000006	2.00	0.000065	2.17	0.000065	2.17	0.05	1.92	806.11

Table 11: Problem with a manufactured solution. Errors for a uniform refinement. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = 6\lambda$ and $\theta = 1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r_η
0.314	141.37	–	51.50	–	0.133297	–	30.925213	–	153.61	–
0.157	26.60	2.40	10.30	2.32	0.026320	2.34	3.915698	2.98	28.79	2.41
0.078	7.32	1.86	2.43	2.07	0.005198	2.34	0.687062	2.51	7.74	1.89
0.039	2.23	1.70	0.77	1.66	0.001231	2.07	0.161259	2.09	2.37	1.70
0.019	0.66	1.75	0.26	1.55	0.000237	2.37	0.039925	2.01	0.71	1.73
0.009	0.18	1.86	0.07	1.72	0.000036	2.73	0.008839	2.17	0.19	1.84
0.004	0.04	1.93	0.02	1.86	0.000005	2.90	0.001740	2.34	0.05	1.92

Table 12: Problem with a manufactured solution. Components of the estimator η and their respective convergence order. Polynomial order $k = 2$. Nitsche parameters $\gamma_0 = 6\lambda$ and $\theta = 1$.

5.1.2 Adaptive refinement

We now provide numerical results when mesh adaptation driven by the local components of η is carried out. Figure 1 depicts the initial mesh and the final adapted mesh. We check that the refinement occurs in the expected region at the top right corner of the square. On Figure 2 we observe that the final adapted solution is undistinguishable from the exact solution, contrarily to the solution on the initial mesh.

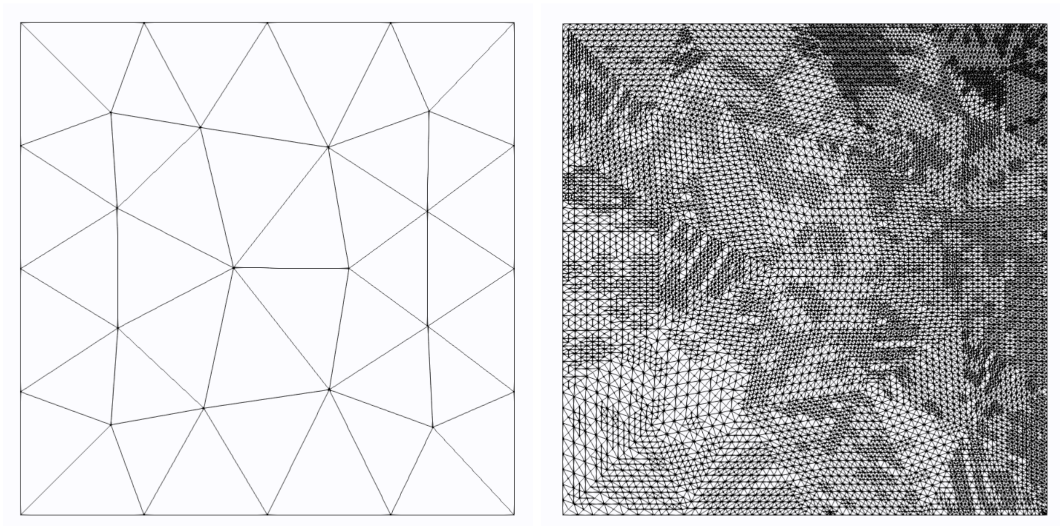

Figure 1: Problem with a manufactured solution. Mesh refinement. Initial mesh (left) and final adapted mesh (right). Polynomial order $k = 1$ and symmetric variant ($\theta = 1$).

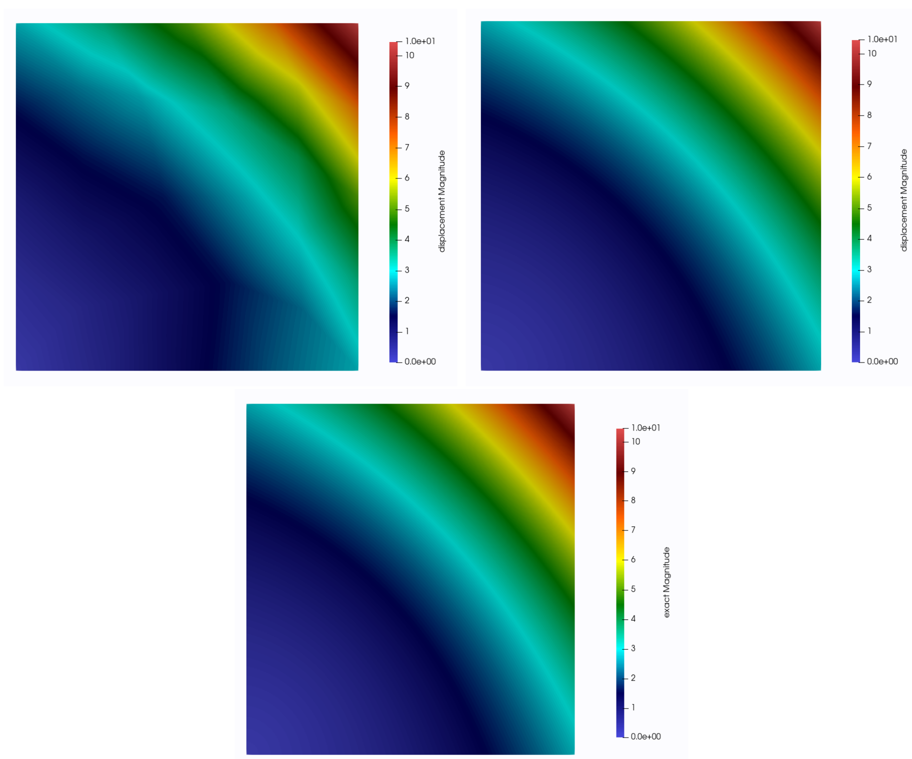

Figure 2: Problem with a manufactured solution. Solution for the initial mesh (left), for final adapted mesh (center), and exact solution (right). Polynomial order $k = 1$ and symmetric variant ($\theta = 1$).

5.2 A test case from Bostan & Han

We reproduce here the test case described in Bostan & Han [8] for Tresca friction with stick and slip transition. In this example, we set $\Omega := (0,8) \times (0,4)$, $\Gamma_D := (0,8) \times \{4\}$, $\Gamma_N := \{0\} \times (0,4) \cup \{8\} \times (0,4)$. The contact boundary is located at the bottom: $\Gamma_C :=]0,8[\times \{0\}$. The Lamé coefficients are $\lambda := 576.9$ and $\mu := 384.6$. The source term is $\boldsymbol{f} := \mathbf{0}$. The Dirichlet and Neumann boundary conditions are given, respectively, by:

$$\boldsymbol{u}_D := \mathbf{0}, \qquad \text{and} \qquad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \boldsymbol{g}, \quad \text{with} \quad \boldsymbol{g} = (400,0)^T.$$

The friction thresold s, defined on Γ_C, is constant, and given by

$$s(x) := 150.$$

Note that in this case we do not dispose of an analytical solution. Figure 3 depicts the initial and the final adapted meshes. Figure 4 depicts the final solution and the deformed adapted mesh. As expected, refinement occurs at corners, and, more interestingly, near the transition point between slip and stick. The results are in good agreement with [8].

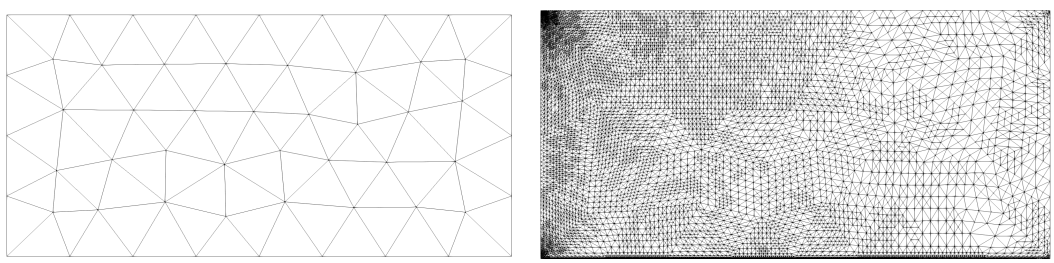

Figure 3: Test from Bostan & Han. Initial mesh (left) and final adapted mesh (right). Polynomial order $k = 1$. Skew-symmetric Nitsche method ($\theta = -1$).

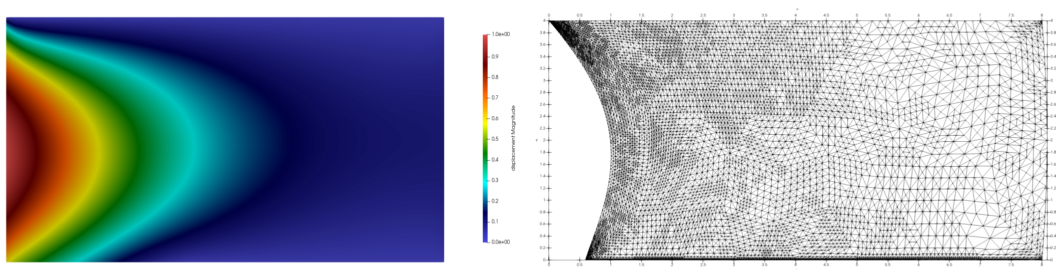

Figure 4: Test from Bostan & Han. Solution obtained using the final adapted mesh (left) and deformed final adapted mesh (right). Polynomial order $k = 1$. Skew-symmetric Nitsche method ($\theta = -1$).

5.3 Coulomb friction

Finally, we reproduce the test case from [44, 53] to test the behavior of the residual error estimate η in the case of Coulomb friction. In this example, we set $\Omega := (0,1) \times (0,1)$ with $\Gamma_D := \{0\} \times (0,1)$, $\Gamma_C := \{1\} \times (0,1)$ and $\Gamma_N := (0,1) \times (\{0\} \cup \{1\})$. The Young modulus and Poisson ratio are respectively $E_Y := 10^6$ and $\nu := 0.3$. The rhs source term is given by $\boldsymbol{f} := (0, -76518)$. The Dirichlet and Neumann boundary conditions are given, respectively, by:

$$\boldsymbol{u}_D := \mathbf{0}, \qquad \text{and} \qquad \boldsymbol{\sigma}(\boldsymbol{u})\boldsymbol{n} = \mathbf{0}$$

The friction coefficient $\mathscr{F}$, defined on Γ_C, is $\mathscr{F} := 0.2$. Note that in this case we do not dispose of an analytical solution yet we will use as reference a $\mathbb{P}_1$ solution computed on a highly uniform refined mesh of $800 \cdot 000$ triangles. The computed solution, obtained with the symmetric Nitsche method and linear Lagrange finite elements is depicted Figure 5, Figure 6 and Figure 7. The results are in good agreement with those of [53] obtained with a mixed method.

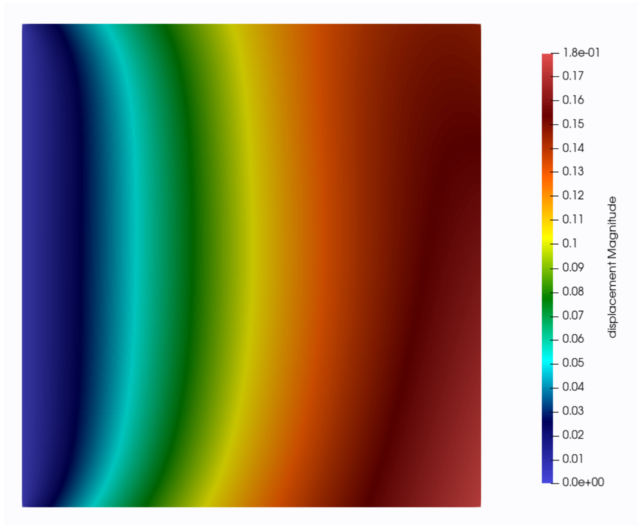

Figure 5: Coulomb friction. Magnitude of the computed solution. Polynomial order $k = 1$ and symmetric Nitsche ($\theta = 1$).

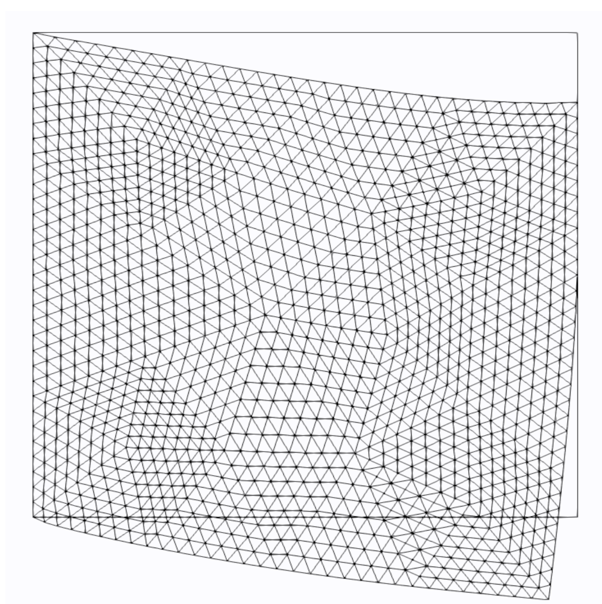

Figure 6: Coulomb friction. Deformed mesh obtained using the computed solution. Polynomial order $k = 1$ and symmetric Nitsche ($\theta = 1$).

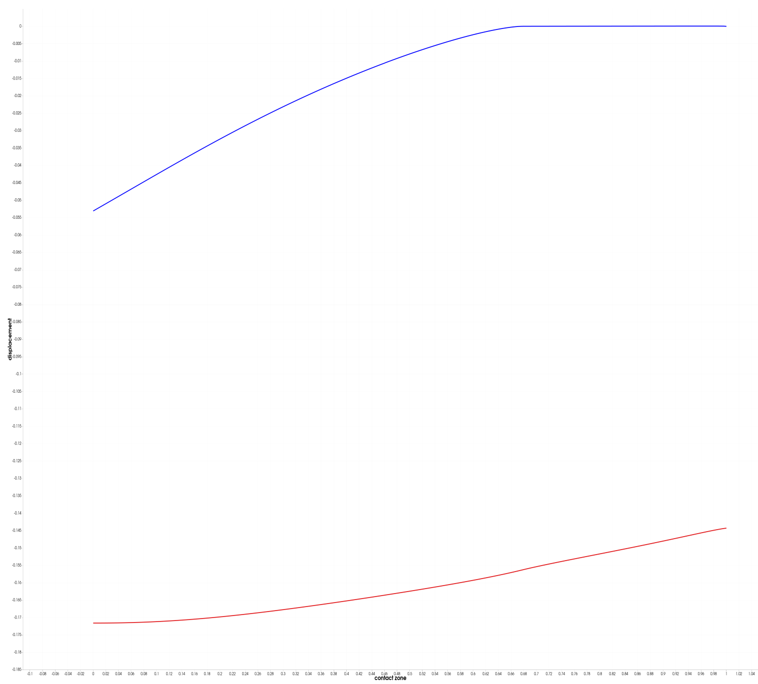

Figure 7: Coulomb friction. Normal displacement (blue) and tangential displacement (red) on the contact boundary Γ_C. Polynomial order $k = 1$ and symmetric Nitsche ($\theta = 1$).

Table 13 presents the a priori error in L^2 and H^1 norms with their respective convergence rates, for the skew-symmetric variant ($\theta = -1$). The computed rates are slightly suboptimal (around 1.6 in the L^2 norm and 0.7 in the H^1 norm, in comparison with the expected optimal rates of 2 and 1, respectively). Note that, for Coulomb friction, the best theoretical a priori error bounds are in $\mathcal{O}(h^{\frac{1}{2}})$ [45], for mixed methods, but it is not known still if these theoretical rates can be improved or not. For Nitsche method, there is no theoretical a priori error estimate available for the moment. The limitation in the convergence rate may come from the limited regularity of the solution. For comparison purpose, rates of 1.12 and 0.95 were obtained in [19], but for a different test case with a solution expected to be smoother.

The different components of the estimator are reported Table 14. Suboptimal rates can be observed, and a global convergence rate of approximately 0.7 can be deduced for the overall estimator as well as for the components η_2 associated with jumps on internal and Neumann edges. Notably the components η_3 and η_4 associated with friction and contact have a low, suboptimal rate of 0.5. This is in contrast with previous results for Tresca friction, were superconvergence was observed. Moreover, in this situation the relative magnitude of the components associated with contact and friction is high. The convergence rates are close to those of the a priori error in the H^1 norm, and the efficiency remains poor. Remark finally that the rate of 0.7 is in agreement with the rate reported in [44] and [53] for a mixed method and another error estimator. It seems to indicate a limited regularity of the continuous solution.

Results for the incomplete variant $\theta = 0$ are reported Table 15 and Table 16. Conclusions similar as in the previous case $\theta = -1$ can be drawn, except that better overall convergence rates can be observed, with, for instance, a rate slightly above 0.8 for the H^1-norm. Surprisingly, the components of the estimator η_3 and η_4 associated with friction and contact converge with rates much better than in the case $\theta = -1$, with even a superconvergence for η_3. Results for the symmetric variant $\theta = 1$ are reported Table 17 and Table 18, and are very similar to those obtained for the skew-symmetric variants. Finally, we carried out tests for larger values of the friction coefficient $\mathscr{F} = 0.4, 0.6, 0.8$. We do not report the details but the same conclusions hold as for the case $\mathscr{F} = 0.2$. Particularly, the difference of convergence rates for the components η_3 and η_4, for the specific value $\theta = 0$ in comparison to the variants $\theta = -1, 1$ is persisting.

h	$\Vert\boldsymbol{u}-\boldsymbol{u}_h\Vert_{0,\Omega}$	r_2	$\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{1,\Omega}$	r_{s1}	$\Vert\boldsymbol{u}-\boldsymbol{u}_h\Vert_{1,\Omega}$	r_1
0.353	0.010331	–	0.029722	–	0.031466	–
0.176	0.004172	1.30	0.014970	0.98	0.015540	1.01
0.088	0.001375	1.60	0.007838	0.93	0.007957	0.96
0.044	0.000468	1.55	0.004608	0.76	0.004632	0.78
0.022	0.000151	1.62	0.002833	0.70	0.002837	0.70
0.011	0.000049	1.61	0.001751	0.69	0.001751	0.69
0.005	0.000016	1.62	0.001051	0.73	0.001051	0.73

Table 13: Coulomb friction. Errors for a uniform refinement and their respective convergence orders. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = -1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r
0.353	27053.19	–	61579.44	–	5257.79	–	11427.64	–	68426.16	–
0.176	13526.59	1.00	42257.54	0.54	2783.30	0.91	7965.37	0.52	45164.84	0.59
0.088	6763.29	1.00	25874.90	0.70	1308.64	1.08	6127.21	0.37	27468.31	0.71
0.044	3381.64	1.00	15384.52	0.75	655.60	0.99	4374.37	0.48	16361.06	0.74
0.022	1690.82	1.00	9190.32	0.74	406.23	0.69	3091.68	0.50	9851.11	0.73
0.011	845.41	1.00	5568.05	0.72	271.77	0.57	2179.60	0.50	6045.04	0.70
0.005	422.70	1.00	3430.46	0.69	189.68	0.51	1538.49	0.50	3788.10	0.67

Table 14: Coulomb friction. Components of the error estimator η for a uniform refinement and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = -1$.

h	$\Vert\boldsymbol{u}-\boldsymbol{u}_h\Vert_{0,\Omega}$	r_2	$\vert\boldsymbol{u}-\boldsymbol{u}_h\vert_{1,\Omega}$	r_{s1}	$\Vert\boldsymbol{u}-\boldsymbol{u}_h\Vert_{1,\Omega}$	r_1
0.353	0.010154	–	0.028469	–	0.030225	–
0.176	0.004236	1.26	0.013869	1.03	0.014502	1.05
0.088	0.001490	1.50	0.006825	1.02	0.006986	1.05
0.044	0.000523	1.51	0.003757	0.86	0.003793	0.88
0.022	0.000177	1.56	0.002153	0.80	0.002160	0.81
0.011	0.000059	1.58	0.001239	0.79	0.001240	0.80
0.005	0.000017	1.75	0.000688	0.84	0.000689	0.84

Table 15: Coulomb friction. Errors for a uniform refinement and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = 0$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r
0.353	27053.19	–	60948.11	–	6567.94	–	10762.72	–	67863.99	–
0.176	13526.59	1.00	40792.63	0.57	2755.17	1.25	2498.26	2.10	43137.45	0.65
0.088	6763.29	1.00	24828.46	0.71	1063.29	1.37	629.45	1.98	25762.79	0.74
0.044	3381.64	1.00	14611.70	0.76	385.29	1.46	334.65	0.91	15006.59	0.77
0.022	1690.82	1.00	8537.63	0.77	149.61	1.36	247.77	0.43	8708.26	0.78
0.011	845.41	1.00	5016.40	0.76	55.82	1.42	138.35	0.84	5089.32	0.77
0.005	422.70	1.00	2970.76	0.75	22.24	1.32	66.85	1.04	3001.51	0.76

Table 16: Components of the error estimator η for a uniform refinement and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = 0$.

h	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,\Omega}$	r_2	$\lvert\boldsymbol{u} - \boldsymbol{u}_h\rvert_{1,\Omega}$	r_{s1}	$\|\boldsymbol{u} - \boldsymbol{u}_h\|_{1,\Omega}$	r_1
0.353	0.010008	–	0.028979	–	0.030658	–
0.176	0.004328	1.20	0.015067	0.94	0.015677	0.96
0.088	0.001600	1.43	0.007955	0.92	0.008114	0.95
0.044	0.000577	1.47	0.004657	0.77	0.004693	0.78
0.022	0.000202	1.51	0.002856	0.70	0.002863	0.71
0.011	0.000068	1.57	0.001767	0.69	0.001769	0.69
0.005	0.000018	1.90	0.001065	0.73	0.001066	0.73

Table 17: Coulomb friction. Errors for a uniform refinement and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = 1$.

h	η_1	r_1	η_2	r_2	η_3	r_3	η_4	r_4	η	r
0.353	27053.19	–	63388.12	–	8399.56	–	15577.33	—	71155.71	–
0.176	13526.59	1.00	42236.24	0.58	3446.07	1.28	10013.60	0.63	45596.23	0.64
0.088	6763.29	1.00	25992.38	0.70	1557.03	1.14	6491.10	0.62	27674.99	0.72
0.044	3381.64	1.00	15545.69	0.74	788.78	0.98	4487.18	0.53	16548.75	0.74
0.022	1690.82	1.00	9299.40	0.74	484.59	0.70	3170.96	0.50	9981.36	0.72
0.011	845.41	1.00	5640.33	0.72	324.71	0.57	2244.53	0.49	6137.70	0.70
0.005	422.70	1.00	3480.77	0.69	225.48	0.52	1588.37	0.49	3855.93	0.677

Table 18: Components of the error estimator η for a uniform refinement and their respective convergence order. Polynomial order $k = 1$. Nitsche parameters $\gamma_0 = 10\lambda$ and $\theta = 1$.

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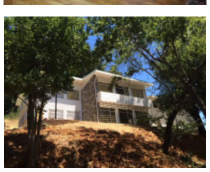

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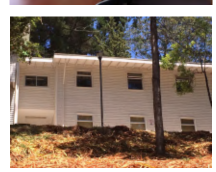

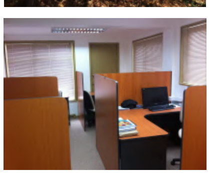

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