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Abstract

We analyze a high order hybridizable discontinuous Galerkin (HDG) method for an optimal control problem where the computational mesh does not necessarily fit the domain. The method is based on transferring the boundary data to the computational boundary by integrating the approximation of the gradient. We prove optimal order of convergence in the L^2-norm for all the variables of the state and adjoint problems, and the control variable as well. More precisely, order h^{k+1} if the local discrete spaces are constructed using polynomials of degree at most k on a triangulation of meshsize h. We present numerical experiments illustrating the performance of method.

Keywords: unfitted methods, curved domains, discontinuous Galerkin, optimal control problems

1 Introduction

Most of the numerical method for partial differential equations (PDEs) rely on a polyhedral partition of the domain of interest Ω. In applications where the boundary of the domain is not piecewise linear, a special treatment must be employed to represent, or approximate, Ω by the union of the elements of the partition and obtain an approximation of the solution of the PDE with certain degree of accuracy. In this direction, it is possible to identify two different approaches: *fitted* and *unfitted* methods. For a review we refer the reader to the introduction section in [18, 53]. Roughly speaking, in the former, the partition needs to adjusted to the boundary of Ω in such a way that the geometric error does not dominate the error of the Galerkin approximation, as it is in the case of isoparametric finite elements, for instance. This type of methods are not practical for complex geometries or domains evolving in time when high order methods are employed, since it might involve computing nonlinear mappings and consider a remeshing procedure on each time step. As an alternative, *unfitted* methods do not require the partition to "fit" the boundary of the domain, as it is in the case of immerse methods. The downside is that the geometric error in unfitted methods dominates the error when using high order Galerkin approximations.

To overcome that limitation, that is, being able to obtain high order accurate approximations and use partitions of the domain not fitted to it, the field has been quite active in recent years, especially by introducing boundary correction and extrapolation techniques, as in the shifted boundary method [37] and in the cut finite element method

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[7], among others. In addition, high-order hybrid (HHO) methods have been developed on curved meshes [3] and in polygonal unfitted meshes [6]. This work focuses on the *transfer path* method developed by [22] and analyzed in [18] in the context of hybridizable discontinuous Galerkin (HDG) methods. The main idea is to transfer the data from the curved boundary to the computational boundary through integration paths, while maintaining the high order of convergence.

During the past and present decade, hybridizable discontinuous Galerkin (HDG) methods have been extensively developed for different types of partial differential equations. For polyhedral domains, we can mention its development in diffusion equations [14, 16, 17, 31], convection-difussion equations [15, 26, 42], the wave equation [19], Stokes flow [9, 20, 27, 38], Oseen and Brinkman equations [1, 10, 28], Navier-Stokes equations [11, 41, 47], linear and nonlinear elasticity [21, 40, 55], distributed optimal control problem [12, 59], just to name a few. In addition, as mentioned before, an unfitted HDG method by employing the transfer path technique was introduced in the context of linear elliptic equations in [22], subsequently completing its theoretical development in [18]. It has been also used for solving equations, as for instance, Stokes flow [53], Oseen equations [54], the Helmholtz equation [8], convection diffusion equations [23], the Grad-Shafranov equation [49, 50], among others. Even though this transfer path method was originally developed for HDG schemes, it can be also applied to any mixed method, as long as the gradient of the primal variable is one of the unknowns [44, 45]. Moreover, it has been successfully used in interface problems where the computational interface does not match the actual interface [38, 46, 52].

On the other hand, optimal control problems governed by PDEs have numerous applications in science and engineering, such as aerodynamics [43, 51], medicine [2, 32], and mathematical finance [4, 25], among others. These types of problems have been extensively studied with finite element methods [13, 29, 30, 35, 36] and discontinuous Galerkin methods [33, 34, 57, 58]. In this paper we extend the work done in [59], proposing an HDG method to solve this optimal control problem in curved domains by employing the transferring path technique. More precisely, let Ω be a Lipschitz domain in $\mathbb{R}^d$ with $d \in \{2,3\}$ and boundary $\Gamma := \partial\Omega$ piece-wise $\mathcal{C}^2$. Given a source term $f \in L^2(\Omega)$, a target $\widetilde{y} \in L^2(\Omega)$ and $g \in H^{3/2}(\Gamma)$, we want to minimize the functional

$$J(y^*, u^*) := \frac{1}{2}\,\|y^*(u^*) - \widetilde{y}\|^2_\Omega + \frac{\alpha}{2}\,\|u^*\|^2_\Omega \tag{1.1}$$

subject to

$$-\nabla\cdot(a\,\nabla y^*) = f + u^* \qquad \text{in } \Omega\,, \tag{1.2a}$$

$$y^* = g \qquad \text{on } \Gamma\,, \tag{1.2b}$$

where $y^* := y^*(u^*)$, $Y := \{w \in H^1(\Omega), \quad w = g \text{ on } \Gamma\}$, $U := L^2(\Omega)$, $a > 0$ and $\alpha > 0$ are given diffusion and regularization parameters, respectively. We set

$$(y, u) = \arg\min J(y^*, u^*). \tag{1.3}$$

Now, given a control $u^* \in L^2(\Omega)$, the weak formulation associated to the state equation (1.2) seeks $y^*(u^*) \in Y$ such that

$$(a\nabla y^*, \nabla w)_\Omega = (f + u^*, w)_\Omega \qquad \forall\, w \in H^1_0(\Omega)\,. \tag{1.4}$$

A straightforward application of the Lax-Milgram Theorem provides existence and uniqueness of $y^* \in H^1(\Omega)$. Moreover, according to [39, Section 4.1.2], there exists a unique optimal control $u \in L^2(\Omega)$ of (1.2), and therefore $y := y(u) \in H^1(\Omega)$ is also unique. In order to characterize u, we make use of the adjoint state equation. More precisely, the state y and the control u solve the optimal control problem (1.2) if and only if there exists $(y, u, z) \in Y \times L^2(\Omega) \times H^1_0(\Omega)$ satisfying

$$(a\,\nabla y, \nabla w_1)_\Omega = (f + u, w_1)_\Omega \qquad \forall\, w_1 \in H^1_0(\Omega)\,, \tag{1.5a}$$

$$(a\,\nabla z, \nabla w_2)_\Omega = (\widetilde{y} - y, w_2)_\Omega \qquad \forall\, w_2 \in H^1_0(\Omega)\,, \tag{1.5b}$$

$$(\alpha\, u - z, r)_\Omega = 0 \qquad \forall\, r \in L^2(\Omega)\,. \tag{1.5c}$$

Since $W \subset L^2(\Omega)$, from (1.5c) we have that $u = \beta z \in H_0^1(\Omega)$ with $\beta := \alpha^{-1}$. Therefore, the problem reduces to find $(y, z) \in Y \times H_0^1(\Omega)$ such that

$$(a\,\nabla y, \nabla w_1)_\Omega \;=\; (f \,+\, \beta z, w_1)_\Omega \qquad \forall\, w_1 \in H_0^1(\Omega)\,, \tag{1.6a}$$

$$(a\,\nabla z, \nabla w_2)_\Omega \;=\; (\widetilde{y} \,-\, y, w_2)_\Omega \qquad \forall w_2 \in H_0^1(\Omega)\,. \tag{1.6b}$$

The rest of the paper is organized as follows. In Section 2, we will specify admissibility conditions on the family of computational domain, introduce the transferring patch technique and set some notation. We will also provide assumptions on the distance between Γ and the computational boundary Γ_h. The HDG scheme is presented in Section 3 and its well-posedness is proved. Subsequently, in Section 4 we provide the *a priori* error estimates of the method and present numerical experiments in Section 5. We end with concluding remarks in Section 6.

2 Preliminaries

In this section we introduce notation associated to the computational domain and to the family of paths that will allow us to transfer the boundary data from Γ to the computational boundary Γ_h. To that end, we will consider the setting specified in [48] and establish a set of assumptions under which our analysis holds.

Admissible triangulations. Given a domain Ω and a discretization parameter $h > 0$, we denote by Ω_h an open polygonal/polyhedral computational domain, with boundary Γ_h, triangulated by a simplicial mesh $\mathcal{T}_h$ of meshsize h. For a simplex K, we denote its outward unit normal by $\boldsymbol{n}_K$, writing $\boldsymbol{n}$ instead of $\boldsymbol{n}_K$ when there is no confusion. Similarly, for a facet e, we write $\boldsymbol{n}$ instead of $\boldsymbol{n}_e$ to refer to its normal vector. We also consider, by simplicity, that the triangulation does not have hanging nodes. The set of facets and boundary facets of $\mathcal{T}_h$ are denoted by $\mathcal{E}_h$ and $\mathcal{E}_h^\partial$, respectively. The set $\Omega_h^c := \Omega \setminus \overline{\Omega_h}$ refers to the non-meshed region. We say the family $\{(\Omega_h, \mathcal{T}_h)\}_{h>0}$ is *admissible* if each member $(\Omega_h, \mathcal{T}_h)$ satisfies the following conditions:

(a) $\Omega_h \subset \Omega$;

(b) $\mathcal{T}_h$ is uniformly shape-regular, that is, there exists $\gamma > 0$, independent of h, such that $h_K \le \gamma \rho_K$, where ρ_K is the radius of the largest ball contained in K and $h_K < h$ is the diameter of K;

(c) there exists bijective a mapping $\phi : \Gamma_h \to \Gamma$;

(d) for every $K \in \mathcal{T}_h$ such that $K \cap \Gamma_h \neq \varnothing$, it holds that $\max\{\mathrm{dist}(\boldsymbol{x}, \overline{\boldsymbol{x}}) : \boldsymbol{x} \in K \cap \Gamma_h \text{ and } \overline{\boldsymbol{x}} \in \Gamma\} = \mathcal{O}(h_K)$.

Moreover,

(e) for every $\epsilon > 0$ there exists a pair $(\Omega_h, \mathcal{T}_h)$ such that $\lambda(\Omega \setminus \Omega_h) < \epsilon$, where $\lambda(\cdot)$ denotes the Lebesgue measure.

Let us briefly comment on these conditions. In the event that condition (a) is not fulfilled, i.e., if Ω_h is not completely contained in Ω, the boundary data should be transferred from $\partial(\Omega_h \setminus \Omega)$ to Γ and the numerical scheme does not change. The analysis presented in this manuscript can be modified to dealt with the latter situations, but assuming the PDE is still valid outside Ω. In turn, the role of condition (c) arises from the development of the analysis, but it is not necessary for the computational implementation. On the other hand, by requiring condition (d), we are limiting the minimum size that an element with a facet on the boundary could reach, with respect to the distance between Γ_h and Γ.

Transferring paths and polynomial extrapolation. Since the problem will be solved in Ω_h, we must specify a suitable boundary data on the computational boundary Γ_h. To this end, we consider the idea proposed by [18] and transfer the boundary data g from Γ to Γ_h through transferring paths. More precisely, let $e \in \mathcal{E}_h^{\partial}$. For each $\boldsymbol{x} \in e$, we define $l(\boldsymbol{x}) := |\phi(\boldsymbol{x}) - \boldsymbol{x}|$ and denote by $\boldsymbol{t}(\boldsymbol{x})$ the unit tangent vector to the segment joining $\boldsymbol{x}$ and $\phi(\boldsymbol{x})$. We notice that, for an admissible triangulation, $l(\boldsymbol{x}) \leq C\,h$ with $C > 0$ independent of h. A line integration over this this segment, will allow us to transfer the boundary data. In fact, if a given function g_v is the trace of a function v and $\boldsymbol{q} := -\nabla v$, there holds

$$g_v(\boldsymbol{x}) \,=\, g_v(\phi(\boldsymbol{x})) \,+\, \int_0^{l(\boldsymbol{x})} \boldsymbol{q}\cdot\boldsymbol{t}(\boldsymbol{x}(s))\,ds\,, \tag{2.1}$$

with $\boldsymbol{x}(s) = (\phi(\boldsymbol{x}) - \boldsymbol{x})\,s\,/\,l(\boldsymbol{x}) + \boldsymbol{x}$, $s \in [0, l(\boldsymbol{x})]$. Now, at the discrete level, a polynomial approximation $\boldsymbol{q}_h$ of $\boldsymbol{q}$ will be available only inside Ω_h. Hence, we will extrapolate $\boldsymbol{q}_h$ to the segment $[\boldsymbol{x}, \phi(\boldsymbol{x})]$ in order to compute the above integral. This is why we define the *extension patch* as

$$K^e_{ext} := \{\boldsymbol{x} \,+\, s\,\boldsymbol{t}(\boldsymbol{x}) \,:\, 0 \,\leq\, s \,\leq\, l(\boldsymbol{x}),\, \boldsymbol{x} \in e\}\,.$$

Since ϕ is a bijection, we observe that $\Omega_h^c = \cup_{e\in\mathcal{E}_h^{\partial}} K^e_{ext}$.

On the other hand, a bijection ϕ can be constructed in several ways. For instance, a particular construction in two dimensions can be found in [22]. It is also possible to use the closest point projection as long it is unique. We further suppose that

$$\boldsymbol{t}(\boldsymbol{x}) \,=\, \boldsymbol{n} \qquad \text{for all} \quad \boldsymbol{x} \in e\,.$$

This assumption makes possible to present a simpler analysis. If this does not hold true, we can decompose $\boldsymbol{t}$ as the sum of its normal and tangential components, and the estimates that we will obtain remain valid as long as the magnitude of the latter is sufficiently small.

We denote $H_e^{\perp}$ as the largest distance of a point in K^e_{ext} to the plane determined by e and $h_e^{\perp}$ as the distance between e and the vertex of K^e opposite to e. We set $r_e \,=\, H_e^{\perp}\,/\,h_e^{\perp}$ and denote

$$R \,=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\,, \qquad R_\tau \,:=\, \max_{e\in\mathcal{E}_h^{\partial}} r_e\,\tau_e^{1/2}\,.$$

Now, given a polynomial $\boldsymbol{q}$ defined on a boundary element $K^e \in \mathcal{T}_h$ such that $e = \overline{K^e} \cap \Gamma_h$, $E(\boldsymbol{p})$ denotes its extrapolation to K^e_{ext}. In this direction, it is also useful to introduce the function

$$\Lambda^{\boldsymbol{p}}(\boldsymbol{x}) \,:=\, l^{-1}\,(\boldsymbol{x}) \int_0^{l(\boldsymbol{x})} (\boldsymbol{p}(\boldsymbol{x}) \,-\, \boldsymbol{E}(\boldsymbol{p})(\boldsymbol{x}(s)))\cdot\boldsymbol{n}\,ds \tag{2.2}$$

because it will allow us to quantify the extrapolation error on a transferring segment. It satisfies [18, section 2.3],

$$\|l^{1/2}\,\Lambda^{\boldsymbol{p}}|_{K^e}\|_e \,\leq\, \frac{1}{\sqrt{3}}\,r_e^{3/2}\,C^e_{ext}\,C^e_{inv}\,\|\boldsymbol{p}\|_{K_e} \qquad \forall\,\boldsymbol{p} \in [\mathbb{P}_k(K_e)]^d\,, \tag{2.3a}$$

$$\|l^{1/2}\,\Lambda^{\boldsymbol{p}}|_{K_e}\|_e \,\leq\, \frac{1}{\sqrt{3}}\,r_e\,\|h_e^{\perp}\,\partial_{\boldsymbol{n}}\,\boldsymbol{p}\|_{K^e_{\rm ext}} \qquad \forall\,\boldsymbol{p} \in [H^1(K^e_{\rm ext})]^d\,, \tag{2.3b}$$

where

$$C^e_{ext} := \frac{1}{\sqrt{r_e}} \sup_{\boldsymbol{\chi}\in\mathcal{V}^k} \frac{\|\boldsymbol{\chi}\cdot\boldsymbol{n}_e\|_{K^e_{ext}}}{\|\boldsymbol{\chi}\cdot\boldsymbol{n}_e\|_{K^e}} \quad \text{and} \quad C^e_{inv} := h_e^{\perp} \sup_{\boldsymbol{\chi}\in\mathcal{V}^k} \frac{\|\nabla\boldsymbol{\chi}\cdot\boldsymbol{n}_e\|_{K^e}}{\|\boldsymbol{\chi}\cdot\boldsymbol{n}_e\|_{K^e}},$$

with

$$\mathcal{V}^k := \left\{\boldsymbol{p} \in [\mathbb{P}_k(K^e_{ext} \cup K^e)]^2 \,:\, \boldsymbol{p}\cdot\boldsymbol{n}_e \neq \boldsymbol{0}\right\}.$$

The constants C^e_{ext} and C^e_{inv} are indeed independent of h, but they depend on the shape regularity constant and on the polynomial degree k [18].

Smallness assumptions. We state a set of assumption quantifying how close Γ and Γ_h must be in order to ensure well-posedness and optimal convergence of the method. Let $c = a^{-1}$. For every facet of $\mathcal{E}_h^{\partial}$, we suppose hat

(A.1) $\left(1 + \frac{1}{3}\right) r_e^3 \, (C_{ext}^e)^2 \, (C_{inv}^e)^2 \, \leq \, 1/8,$

(A.2) $r_e \, \tau \, h_e^{\perp} \, \leq \, 1/4 \,,$

(A.3) $r_e \, \leq \, C \,,$

(A.4) $2 \max_{x \in e} c \, \tau \, l(\boldsymbol{x}) \leq 1/4.$

Let us comment on the feasibility of these assumptions. Let $R \leq C_R \, h^{\delta}$, for some C_R and δ non-negative constants independent of h. For instance, if Ω is polygonal/polyhedral and coincides with Ω_h, then $C_R = 0$ and all the assumptions hold true. On the other hand, if Γ_h is a piece-wise linear interpolation of Γ, then $C_R > 0$ and $\delta = 1$ and the assumptions are valid for h small enough and τ of proportional to one. In a more general case, Ω would be embedded in a background triangulation and Ω_h is constructed by the union of the elements lying completely inside of Ω. In this latter case, $C_R > 0$ and $\delta = 0$. We observe (A.2) - (A.4) are still true, whereas the first one cannot be guaranteed. However, the numerical experiments suggest that the method is still optimal.

Sobolev space notation, mesh dependent inner-products and norms. In this paper we will make use of the usual notations for Sobolev spaces, i.e., given a domain D in $\mathbb{R}^n$ and S a Lipschitz curve ($d = 2$) or surface ($d = 3$), let s be a nonzero real number, we denote $H^s(D)$ and $H^s(S)$ by the usual definition of Sobolev spaces. The corresponding norms are denoted by $\| \cdot \|_{s,D}$ and $\| \cdot \|_{s,S}$, whereas the seminorm is denoted by $| \cdot |_{s,D}$. If $s = 0$, we just write $\| \cdot \|_D$ and $\| \cdot \|_S$ as usual. The spaces for vector valued functions will be boldfaced, for instance, $\boldsymbol{H}^s(D) := [H^s(D)]^d$ and $\boldsymbol{H}^s(S) := [H^s(S)]^d$.

For each scalar-valued function η and ζ, we define

$$(\eta, \zeta)_{\mathcal{T}_h} \; := \sum_{K \in \mathcal{T}_h} (\eta, \zeta)_K \quad \text{and} \quad \langle \eta, \zeta \rangle_{\partial \mathcal{T}_h} \; := \sum_{K \in \mathcal{T}_h} \langle \eta, \zeta \rangle_{\partial K} \,.$$

Vector-valued functions are boldfaced and, for $\boldsymbol{\eta}$ and $\boldsymbol{\zeta}$, we write

$$(\boldsymbol{\eta}, \boldsymbol{\zeta})_{\mathcal{T}_h} \; := \sum_{i=1}^{d} (\eta_i, \zeta_i)_{\mathcal{T}_h} \quad \text{and} \quad \langle \boldsymbol{\eta}, \boldsymbol{\zeta} \rangle_{\partial \mathcal{T}_h} \; := \sum_{i=1}^{d} \langle \eta_i, \zeta_i \rangle_{\partial \mathcal{T}_h} \,.$$

These inner products defined on the mesh induce the norms

$$\|\cdot\|_{\mathcal{T}_h} \; := \left(\sum_{K \in \mathcal{T}_h} \|\cdot\|_K^2 \right)^{1/2}, \quad \|\cdot\|_{\partial \mathcal{T}_h} \; := \left(\sum_{K \in \mathcal{T}_h} \|\cdot\|_{\partial K}^2 \right)^{1/2}, \quad \text{and} \quad \|\cdot\|_{\Gamma_h} \; := \left(\sum_{e \in \mathcal{E}_h^{\partial}} \|\cdot\|_e^2 \right)^{1/2}.$$

In addition, for $w > 0$, we write $\|\eta\|_{\partial \mathcal{T}_h, w} \; := \; \|w^{1/2} \, \eta\|_{\partial \mathcal{T}_h}$ and $\|\eta\|_{\Gamma_h, w} \; := \; \|w^{1/2} \, \eta\|_{\Gamma_h}$. Finally, to avoid proliferation of constants, we will write $A \lesssim B$ instead of $A \leq C \, B$, where C is a constant independent of h.

3 HDG formulation

In order to present the HDG scheme, we state the strong mixed formulation of the equation posed in the computational subdomain Ω_h, which is given by

$$c \, \boldsymbol{p} + \nabla y \, = \, 0 \quad \text{in } \Omega_h \,, \qquad (3.1a)$$

$$\nabla \cdot \boldsymbol{p} - \beta z \, = \, f \quad \text{in } \Omega_h \,, \qquad (3.1b)$$

$$y \, = \, \varphi_1 \quad \text{on } \Gamma_h \,, \qquad (3.1c)$$

$$c \, \boldsymbol{r} + \nabla z \, = \, 0 \quad \text{in } \Omega_h \,, \qquad (3.1d)$$

$$\nabla \cdot \boldsymbol{r} + y \, = \, \widetilde{y} \quad \text{in } \Omega_h \,, \qquad (3.1e)$$

$$z \, = \, \varphi_2 \quad \text{on } \Gamma_h \,, \qquad (3.1f)$$

where φ_1 and φ_2 can be obtained by integrating (3.1a) and (3.1d) along the transferring paths. More precisely, for any $\boldsymbol{x} \in e$, $e \in \mathcal{E}_h^\partial$ and $\overline{\boldsymbol{x}} \in \partial\Omega$, we deduce that (cf. (2.1))

$$\varphi_1(\boldsymbol{x}) \,=\, g(\overline{\boldsymbol{x}}) \,+\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{p}\cdot\boldsymbol{t}(\boldsymbol{x}(s))\,ds \quad \text{and} \quad \varphi_2(\boldsymbol{x}) \,=\, \int_0^{l(\boldsymbol{x})} c\,\boldsymbol{r}\cdot\boldsymbol{t}(\boldsymbol{x}(s))\,ds\,,$$

where $\boldsymbol{x}(s) \,=\, (\overline{\boldsymbol{x}} \,-\, \boldsymbol{x})\,s\,/l(\mathbf{x}) \,+\, \mathbf{x}$, $s \in [0, l(\mathbf{x})]$. The HDG method seeks an approximation $(p_h, y_h, \widehat{y}_h, r_h, z_h, \widehat{z}_h)$ of the exact solution $(p, y, y\big|_{\mathcal{E}_h}, r, z, z\big|_{\mathcal{E}_h})$ in the space $\boldsymbol{V}_h \times W_h \times M_h \times \boldsymbol{V}_h \times W_h \times M_h$ defined by

$$\boldsymbol{V}_h \,:=\, \left\{\boldsymbol{v} \in [L^2(\mathcal{T}_h)]^d \,:\, v\big|_K \in [\mathbb{P}_k(K)]^d\,,\ \forall\, K \in \mathcal{T}_h\right\}\,, \tag{3.2a}$$

$$W_h \,:=\, \left\{w \in L^2(\mathcal{T}_h) \,:\, w\big|_K \in \mathbb{P}_k(K)\,,\ \forall\, K \in \mathcal{T}_h\right\}\,, \tag{3.2b}$$

$$M_h \,:=\, \left\{\mu \in L^2(\mathcal{E}_h) \,:\, \mu\big|_e \in \mathbb{P}_k(e)\,,\ \forall e \in \mathcal{E}_h\right\}\,. \tag{3.2c}$$

such that

$$(c\,\boldsymbol{p}_h, \boldsymbol{v}_1)_{\mathcal{T}_h} \,-\, (y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \,+\, \langle\widehat{y}_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0\,, \tag{3.3a}$$

$$-\,(\boldsymbol{p}_h, \nabla w_1)_{\mathcal{T}_h} \,-\, (\beta z_h, w_1)_{\mathcal{T}_h} \,+\, \langle\widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, w_1\rangle_{\partial\mathcal{T}_h} \,=\, (f, w_1)_{\mathcal{T}_h}\,, \tag{3.3b}$$

$$(c\,\boldsymbol{r}_h, \boldsymbol{v}_2)_{\mathcal{T}_h} \,-\, (z_h, \nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} \,+\, \langle\widehat{z}_h, \boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0\,, \tag{3.3c}$$

$$-\,(\boldsymbol{r}_h, \nabla w_2)_{\mathcal{T}_h} \,+\, (y_h, w_2)_{\mathcal{T}_h} \,+\, \langle\widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, w_2\rangle_{\partial\mathcal{T}_h} \,=\, (\widetilde{y}, w_2)_{\mathcal{T}_h}\,, \tag{3.3d}$$

$$\langle\widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,=\, 0\,, \tag{3.3e}$$

$$\langle\widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \mu_2\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,=\, 0\,, \tag{3.3f}$$

$$\langle\widehat{y}_h, \mu_1\rangle_{\Gamma_h} \,=\, \langle\varphi_1^h, \mu_1\rangle_{\Gamma_h}\,, \tag{3.3g}$$

$$\langle\widehat{z}_h, \mu_2\rangle_{\Gamma_h} \,=\, \langle\varphi_2^h, \mu_2\rangle_{\Gamma_h}\,, \tag{3.3h}$$

for all $(\boldsymbol{v}_1, w_1, \mu_1, \boldsymbol{v}_2, w_2, \mu_2) \in \boldsymbol{V}_h \times W_h \times M_h \times \boldsymbol{V}_h \times W_h \times M_h$. Here,

$$\widehat{\boldsymbol{p}}_h \,=\, \boldsymbol{p}_h \,+\, \tau\,(y_h \,-\, \widehat{y}_h)\,\boldsymbol{n} \qquad \text{and} \qquad \widehat{\boldsymbol{r}}_h \,=\, \boldsymbol{r}_h \,+\, \tau\,(z_h \,-\, \widehat{z}_h)\,\boldsymbol{n}\,, \tag{3.3i}$$

τ is a non-negative stabilization parameter and, for $\boldsymbol{x} \in \mathcal{E}_h^\partial$,

$$\varphi_1^h(\boldsymbol{x}) \,:=\, g(\overline{\boldsymbol{x}}) \,+\, \int_0^{l(x)} c\,\boldsymbol{E}(\boldsymbol{p}_h)\cdot\boldsymbol{t}(\boldsymbol{x}(s))\,ds \qquad \text{and} \qquad \varphi_2^h(\boldsymbol{x}) \,:=\, \int_0^{l(x)} c\,\boldsymbol{E}(\boldsymbol{r}_h)\cdot\boldsymbol{t}(\boldsymbol{x}(s))\,ds \tag{3.3j}$$

are approximations to $\varphi_1(\boldsymbol{x})$ and $\varphi_2(\boldsymbol{x})$, resp.

We now analyze existence and uniqueness of this HDG formulation by making use of Fredholm alternative. For convenience in the notation, we define

$$E \,:=\, \alpha\,\|c^{1/2}\,\boldsymbol{p}_h\|^2_{\mathcal{T}_h} \,+\, \|c^{1/2}\,\boldsymbol{r}_h\|^2_{\mathcal{T}_h} \,+\, \alpha\,\|\tau^{1/2}\,(y_h - \widehat{y}_h)\|^2_{\partial\mathcal{T}_h} \,+\, \|\tau^{1/2}\,(z_h - \widehat{z}_h)\|^2_{\partial\mathcal{T}_h} \tag{3.4}$$

$$E_\varphi \,:=\, \alpha\,\|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\|^2_{\Gamma_h} \,+\, \|c^{-1/2}\,l^{-1/2}\,\varphi_2^h\|^2_{\Gamma_h}. \tag{3.5}$$

In other words, E quantifies the energy of the HDG solution in the computational domain Ω_h, whereas E_φ is related to the approximation of the boundary data.

Lemma 3.1. *The HDG scheme* (3.3) *is wellposed.*

Proof. Let $\widetilde{y} = 0$, $f = 0$ and $g = 0$. Testing (3.3) with $\boldsymbol{v}_1 = \boldsymbol{p}_h$, $w_1 = y_h$, $\boldsymbol{v}_2 = \boldsymbol{r}_h$ and $w_2 = z_h$; and integrating by parts in the second and fourth equations,

$$\begin{aligned}
\|c^{1/2}\,\boldsymbol{p}_h\|^2_{\mathcal{T}_h} \,-\, (y_h, \nabla\cdot\boldsymbol{p}_h)_{\mathcal{T}_h} \,+\, \langle \widehat{y}_h, \boldsymbol{p}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,,\\
(\nabla\cdot\boldsymbol{p}_h, y_h)_{\mathcal{T}_h} \,-\, \langle \boldsymbol{p}_h\cdot\boldsymbol{n}, y_h\rangle_{\partial\mathcal{T}_h} \,-\, \beta(z_h, y_h)_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, y_h\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,,\\
\|c^{1/2}\,\boldsymbol{r}_h\|^2_{\mathcal{T}_h} \,-\, (z_h, \nabla\cdot\boldsymbol{r}_h)_{\mathcal{T}_h} \,+\, \langle \widehat{z}_h, \boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,,\\
(\nabla\cdot\boldsymbol{r}_h, z_h)_{\mathcal{T}_h} \,-\, \langle \boldsymbol{r}_h\cdot\boldsymbol{n}, z_h\rangle_{\partial\mathcal{T}_h} \,+\, (y_h, z_h)_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, z_h\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,.
\end{aligned}$$

Adding the first and second equations, and the third and fourth one as well,

$$\begin{aligned}
\|c^{1/2}\,\boldsymbol{p}_h\|^2_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n} \,-\, \boldsymbol{p}_h\cdot\boldsymbol{n}, y_h\rangle_{\partial\mathcal{T}_h} \,-\, \beta\,(z_h, y_h)_{\mathcal{T}_h} \,+\, \langle \widehat{y}_h, \boldsymbol{p}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,,\\
\|c^{1/2}\,\boldsymbol{r}_h\|^2_{\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n} \,-\, \boldsymbol{r}_h\cdot\boldsymbol{n}, z_h\rangle_{\partial\mathcal{T}_h} \,+\, (y_h, z_h)_{\mathcal{T}_h} \,+\, \langle \widehat{z}_h, \boldsymbol{r}_h\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,&=\, 0\,.
\end{aligned}$$

Then, by (3.3i) we deduce that

$$\begin{aligned}
\|c^{1/2}\,\boldsymbol{p}_h\|^2_{\mathcal{T}_h} \,+\, \|\tau^{1/2}\,(y_h - \widehat{y}_h)\|^2_{\partial\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,-\, \beta\,(z_h, y_h)_{\mathcal{T}_h} \,&=\, 0\,,\\
\|c^{1/2}\,\boldsymbol{r}_h\|^2_{\mathcal{T}_h} \,+\, \|\tau^{1/2}\,(z_h - \widehat{z}_h)\|^2_{\partial\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \widehat{z}_h\rangle_{\partial\mathcal{T}_h} \,+\, (z_h, y_h)_{\mathcal{T}_h} \,&=\, 0\,,
\end{aligned}$$

Multiplying by α the first equation, adding both equations and recalling the definition in (3.4),

$$E \,+\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \widehat{z}_h\rangle_{\partial\mathcal{T}_h} \,=\, 0\,. \tag{3.6}$$

Moreover, by equations (3.3e) and (3.3g), it follows that

$$\alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h} \,=\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} \,+\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\Gamma_h} \,=\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \widehat{y}_h\rangle_{\Gamma_h} \,=\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h}\,.$$

Similarly, by equations (3.3f) and (3.3h), it follows that $\langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \widehat{z}_h\rangle_{\partial\mathcal{T}_h} \,=\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \varphi_2^h\rangle_{\Gamma_h}$, and we write (3.6) as

$$E \,+\, \alpha\,\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \widehat{\boldsymbol{r}}_h\cdot\boldsymbol{n}, \varphi_2^h\rangle_{\Gamma_h} \,=\, 0\,. \tag{3.7}$$

In turn, we can add ans subtract $\boldsymbol{p}_h(\boldsymbol{x})\cdot\boldsymbol{n}$ in the first expression of (3.3j) to write

$$\varphi_1^h(\boldsymbol{x}) \,=\, \int_0^{l(\boldsymbol{x})} c\,(\boldsymbol{E}(\boldsymbol{p}_h)\cdot\boldsymbol{n})(\boldsymbol{x}(s))\,ds \,=\, \int_0^{l(\boldsymbol{x})} c\,[\boldsymbol{E}(\boldsymbol{p}_h)(\boldsymbol{x}(s)) \,-\, \boldsymbol{p}_h(\boldsymbol{x})]\cdot\boldsymbol{n}\,ds \,+\, c\,\boldsymbol{p}_h(\boldsymbol{x})\cdot\boldsymbol{n}\,l(\boldsymbol{x})\,.$$

Then, $\boldsymbol{p}_h(\boldsymbol{x})\cdot\boldsymbol{n} \,=\, c^{-1}\,l^{-1}\,(\boldsymbol{x})\,\varphi_1^h(\boldsymbol{x}) \,+\, \Lambda^{\boldsymbol{p}_h}(\boldsymbol{x})$, where we recall the definition of $\Lambda^{\boldsymbol{p}_h}(\boldsymbol{x})$ in (2.2).

Thus, considering this identity in the first equation of (3.3i), we obtain that

$$\begin{aligned}
\langle \widehat{\boldsymbol{p}}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h} \,&=\, \langle \boldsymbol{p}_h\cdot\boldsymbol{n}, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(y_h - \widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\\
&=\, \langle c^{-1}\,l^{-1}(x)\,\varphi_1^h, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \Lambda^{\boldsymbol{p}_h}, \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau(y_h - \widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\\
&=\, \|c^{-1/2}\,l^{-1/2}\,\varphi_1^h\|^2_{\Gamma_h} \,+\, \langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau(y_h - \widehat{y}_h), \varphi_1^h\rangle_{\Gamma_h}\,.
\end{aligned}$$

Similarly, the same arguments but now applied to $\varphi_2^h(\boldsymbol{x})$ in the second expression of (3.3j) yield to

$\boldsymbol{r}_h(\boldsymbol{x})\cdot\boldsymbol{n} \,=\, c^{-1}\,l^{-1}\,(\boldsymbol{x})\,\varphi_2^h(\boldsymbol{x}) \,+\, \Lambda^{\boldsymbol{r}_h}(\boldsymbol{x})$ and also to

$$\langle \widehat{\boldsymbol{r}}_h, \varphi_2^h\rangle_{\Gamma_h} \,=\, \|c^{-1/2}\,l^{-1/2}\,\varphi_2^h\|^2_{\Gamma_h} \,+\, \langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{r}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_2^h\rangle_{\Gamma_h} \,+\, \langle \tau(z_h - \widehat{z}_h), \varphi_2^h\rangle_{\Gamma_h}\,.$$

Therefore, by considering the identities obtained for $\langle \widehat{\boldsymbol{r}}_h, \varphi_1^h\rangle_{\Gamma_h}$ and $\langle \widehat{\boldsymbol{r}}_h, \varphi_2^h\rangle_{\Gamma_h}$, and recalling the definition (3.5), we rewrite (3.6) as

$$\begin{aligned}
E \,+\, E_\varphi \,=\, &-\,\alpha\,\langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{p}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h} \,-\, \alpha\,\langle c^{1/2}\,l^{1/2}\,\tau(y_h - \widehat{y}_h), c^{-1/2}\,l^{-1/2}\,\varphi_1^h\rangle_{\Gamma_h}\\
&-\, \langle c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{r}_h}, c^{-1/2}\,l^{-1/2}\,\varphi_2^h\rangle_{\Gamma_h} \,-\, \langle c^{1/2}\,l^{1/2}\,\tau(z_h - \widehat{z}_h), c^{-1/2}\,l^{-1/2}\,\varphi_2^h\rangle_{\Gamma_h}\,.
\end{aligned} \tag{3.8}$$

We now proceed to bound the right-hand side. To that end, we consider Young's inequality to obtain

$$- \alpha \langle c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{p}_h}, c^{-1/2}\, l^{-1/2}\, \varphi_1^h \rangle_{\Gamma_h} \,-\, \alpha \langle c^{1/2}\, l^{1/2}\, \tau\,(y_h - \widehat{y}_h), c^{-1/2}\, l^{-1/2}\, \varphi_1^h \rangle_{\Gamma_h}$$
$$\leq\, 2\,\alpha\,\|c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{p}_h}\|^2_{\Gamma_h} \,+\, \frac{\alpha}{2}\,\|\tau^{1/2}\,(y_h - \widehat{y}_h)\|^2_{\Gamma_h} + \frac{\alpha}{4}\,\|c^{-1/2}\, l^{-1/2}\, \varphi_1^h\|^2_{\Gamma_h},$$

where we have considered Assumption (A.4). Similarly, we can deduce that

$$- \langle c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{r}_h}, c^{-1/2}\, l^{-1/2}\, \varphi_2^h \rangle_{\Gamma_h} \,-\, \langle c^{1/2}\, l^{1/2}\, \tau\,(z_h - \widehat{z}_h), c^{-1/2}\, l^{-1/2}\, \varphi_2^h \rangle_{\Gamma_h}$$
$$\leq\, 2\,\|c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{r}_h}\|^2_{\Gamma_h} \,+\, \frac{1}{2}\,\|\tau^{1/2}\,(z_h - \widehat{z}_h)\|^2_{\Gamma_h} + \frac{1}{4}\,\|c^{-1/2}\, l^{-1/2}\, \varphi_2^h\|^2_{\Gamma_h}.$$

Substituting these inequalities in (3.8) we obtain that

$$\frac{1}{2}\,(E \,+\, E_\varphi) \,\leq\, 2\,\alpha\,\|c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{p}_h}\|^2_{\Gamma_h} \,+\, 2\,\|c^{1/2}\, l^{1/2}\, \Lambda^{\boldsymbol{r}_h}\|^2_{\Gamma_h}\,. \tag{3.9}$$

Then, according to (2.3) and Assumption (A.1), we obtain that $E + E_\varphi \leq 0$, which implies that $\boldsymbol{p}_h = \boldsymbol{0}$, $\boldsymbol{r}_h = \boldsymbol{0}$, $\varphi_1^h = 0$, $\varphi_2^h = 0$, $y_h\big|_{\partial\mathcal{T}_h} = \widehat{y}_h\big|_{\partial\mathcal{T}_h}$, $z_h\big|_{\partial\mathcal{T}_h} = \widehat{z}_h\big|_{\partial\mathcal{T}_h}$. Moreover, by considering this information in (3.3a) and (3.3c), we have that

$$- (y_h, \nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} \,+\, \langle y_h, \boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0 \quad \forall\,\boldsymbol{v}_1 \in \boldsymbol{V}_h\,,$$
$$- (z_h, \nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} \,+\, \langle z_h, \boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \,=\, 0 \quad \forall\,\boldsymbol{v}_2 \in \boldsymbol{V}_h\,.$$

Integrating by parts, it follows that $(\nabla y_h, \boldsymbol{v}_1)_{\mathcal{T}_h} = 0 \forall\,\boldsymbol{v}_1 \in \boldsymbol{V}_h$ and $(\nabla z_h, \boldsymbol{v}_2)_{\mathcal{T}_h} = 0 \forall\,\boldsymbol{v}_2 \in \boldsymbol{V}_h$. Hence, y_h and z_h are constants but they must be zero by (3.3g) and (3.3h). □

4 Error analysis.

As it is usual in the error analysis of this types of methods, we first decompose the error as the sum of the error of the projection and the projection error. The latter will be controlled by the properties in Section 4.1, while for the former we will employ an energy argument (Section 4.2) for the mixed variables and a duality argument (Section 4.3) for the primal variables.

4.1 HDG projection.

We consider the HDG projection introduced in [16], which is a projection into the product space $\boldsymbol{V}_h \times W_h$. Given $(\boldsymbol{q}, v) \in \boldsymbol{V}_h \times W_h$, it is defined by $\Pi_h(\boldsymbol{q}, v) := (\boldsymbol{\Pi}_V \boldsymbol{q}, \Pi_W v)$, where $(\boldsymbol{\Pi}_{\boldsymbol{V}} \boldsymbol{q}, \Pi_W v)$ is the only element satisfying, for all $K \in \mathcal{T}_h$,

$$(\boldsymbol{\Pi}_{\boldsymbol{V}}\boldsymbol{q}, \boldsymbol{s})_K \,=\, (\boldsymbol{q}, \boldsymbol{s})_K \qquad \forall\,\boldsymbol{s} \in [\mathbb{P}_{k-1}(K)]^d\,, \tag{4.1a}$$

$$(\Pi_W v, t)_K \,=\, (v, t)_K \qquad \forall\,t \in \mathbb{P}_{k-1}(K)\,, \tag{4.1b}$$

$$\langle \boldsymbol{\Pi}_{\boldsymbol{V}}\boldsymbol{q}\cdot\boldsymbol{n} \,+\, \tau\,\Pi_W v, \mu\rangle_e \,=\, \langle \boldsymbol{q}\cdot\boldsymbol{n} \,+\, \tau\,v, \mu\rangle_e \qquad \forall\,\mu \in \mathbb{P}_k(e)\,,\ \forall\,e \subseteq \partial K\,. \tag{4.1c}$$

This projection is well-defined (cf. [16]). Moreover, if $\boldsymbol{q} \in \boldsymbol{H}^{l+1}(K)$ and $v \in H^{l+1}(K)$, with $l \in [0, k]$,

$$\|\boldsymbol{\Pi}_{\boldsymbol{V}}\boldsymbol{q} \,-\, \boldsymbol{q}\|_K \,\lesssim\, h_K^{l+1}\,|\boldsymbol{q}|_{l+1,K} \,+\, h_K^{l+1}\,\tau_K^*\,|v|_{l+1,K}\,, \tag{4.2a}$$

$$\|\Pi_W v \,-\, v\|_K \,\lesssim\, h_K^{l+1}\,|v|_{l+1,K} \,+\, \frac{h_K^{l+1}}{\tau_K^{max}}\,|\nabla\cdot\boldsymbol{q}|_{l+1,K}\,, \tag{4.2b}$$

where τ_K^{max} is the maximum value in ∂K and τ_K^* is the second maximum value.

In the computational boundary Γ_h and in the extrapolation region $\Omega_h^c = \cup_{e\in\mathcal{E}_h^\partial} K_{ext}^e$, the HDG projection satisfies (cf. [18, Lemma 3.8]),

$$\|(\mathbf{\Pi_V}\boldsymbol{q} - \boldsymbol{q})\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} \lesssim h^{k+1}\,|\boldsymbol{q}|_{l+1,\Omega} + h^{k+1}\,\tau|v|_{l+1,\Omega}\,, \tag{4.3a}$$

$$\|\partial_{\boldsymbol{n}}((\mathbf{\Pi_V}\boldsymbol{q} - \boldsymbol{q})\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2} \lesssim R^{1/2}\,\|\boldsymbol{I_V}\boldsymbol{q}\|_{\mathcal{T}_h} + \left(1 + R^{1/2}\right)\,h^{k+1}\,|\boldsymbol{q}|_{l+1,\Omega}\,. \tag{4.3b}$$

In addition, we will make use of the classical L^2 projection into M_h denoted by P_M. On each $K\in\mathcal{T}_h$ it satisfies (cf. [24, Lemma 1.58 and Lemma 1.59]), for $v\in H^{l+1}(K)$,

$$|v - P_M v|_{m,K} \lesssim h^{l+1-m}\,|v|_{l+1,K} \quad \forall\, m\in\{0,\dots,k\}\,, \tag{4.4a}$$

$$\|v - P_M v\|_{\partial K} \lesssim h^{l+1/2}\,|v|_{l+1,K}\,. \tag{4.4b}$$

4.2 Energy argument.

We introduce notation associated to the error, the projection of the error and the projection error, respectively:

$$\boldsymbol{e}^{\boldsymbol{p}} = \boldsymbol{p} - \boldsymbol{p}_h \quad,\quad e^y = y - y_h \quad,\quad e^{\widehat{y}} = y - \widehat{y}_h \quad, \tag{4.5a}$$

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{p}} = \mathbf{\Pi_V}\boldsymbol{p} - \boldsymbol{p}_h \quad,\quad \varepsilon_h^y = \Pi_W y - y_h \quad,\quad \varepsilon_h^{\widehat{y}} = P_M y - \widehat{y}_h, \tag{4.5b}$$

$$\boldsymbol{I_V}\boldsymbol{p} = \boldsymbol{p} - \mathbf{\Pi_V}\boldsymbol{p} \quad,\quad I_W y = y - \Pi_W y \quad. \tag{4.5c}$$

In the same way we define $\boldsymbol{e}^{\boldsymbol{r}}$, e^z, $e^{\widehat{z}}$, $\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}$, ε_h^z, $\varepsilon_h^{\widehat{z}}$, $\boldsymbol{I_V}\boldsymbol{r}$ and $I_W z$.

In order to shorten notation, we define $\mathcal{E} := \mathcal{E}_{\boldsymbol{p},y} + \mathcal{E}_{\boldsymbol{r},z}$ that takes into account the error associated to the mixed variables and the error corresponding to the stabilization term; and $\mathcal{E}_\varphi := \mathcal{E}_{\varphi_1} + \mathcal{E}_{\varphi_2}$ that measures the error in the approximation of the boundary data. More precisely,

$$\mathcal{E}_{\boldsymbol{p},y} := \alpha\,\|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}^2 + \alpha\,\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|_{\partial\mathcal{T}_h}^2, \tag{4.6a}$$

$$\mathcal{E}_{\varphi_1} := \alpha\,\|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\|_{\Gamma_h}^2, \tag{4.6b}$$

$$\mathcal{E}_{\boldsymbol{r},z} := \|c^{1/2}\,\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}\|_{\mathcal{T}_h}^2 + \|\tau^{1/2}\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|_{\partial\mathcal{T}_h}^2, \tag{4.6c}$$

$$\mathcal{E}_{\varphi_2} := \|c^{-1/2}\,l^{-1/2}\,(\varphi_2 - \varphi_2^h)\|_{\Gamma_h}^2. \tag{4.6d}$$

First of all, it is not difficult to deduce (cf. [16, 59]) that the projection of the errors satisfies the same equations as the HDG scheme (3.3), but with different right-hand sides. That is,

$$(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{p}},\boldsymbol{v}_1)_{\mathcal{T}_h} - (\varepsilon_h^y,\nabla\cdot\boldsymbol{v}_1)_{\mathcal{T}_h} + \langle\varepsilon_h^{\widehat{y}},\boldsymbol{v}_1\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\,(c\,\boldsymbol{I_V}\boldsymbol{p},\boldsymbol{v}_1)_{\mathcal{T}_h}\,, \tag{4.7a}$$

$$-\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}},\nabla w_1)_{\mathcal{T}_h} - (\beta\,\varepsilon_h^z,w_1)_{\mathcal{T}_h} + \langle\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n},w_1\rangle_{\partial\mathcal{T}_h} = (\beta\,I_W z,w_1)_{\mathcal{T}_h}\,, \tag{4.7b}$$

$$(c\,\boldsymbol{\varepsilon}_h^{\boldsymbol{r}},\boldsymbol{v}_2)_{\mathcal{T}_h} - (\varepsilon_h^z,\nabla\cdot\boldsymbol{v}_2)_{\mathcal{T}_h} + \langle\varepsilon_h^{\widehat{z}},\boldsymbol{v}_2\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} = -\,(c\,\boldsymbol{I_V}\boldsymbol{r},\boldsymbol{v}_2)_{\mathcal{T}_h}\,, \tag{4.7c}$$

$$-\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{r}},\nabla w_2)_{\mathcal{T}_h} + (\varepsilon_h^y,w_2)_{\mathcal{T}_h} + \langle\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}}\cdot\boldsymbol{n},w_2\rangle_{\partial\mathcal{T}_h} = -\,(\,I_W y,w_2)_{\mathcal{T}_h}\,, \tag{4.7d}$$

$$\langle\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n},\mu_1\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0\,, \tag{4.7e}$$

$$\langle\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}}\cdot\boldsymbol{n},\mu_2\rangle_{\partial\mathcal{T}_h\setminus\Gamma_h} = 0\,, \tag{4.7f}$$

$$\langle\varepsilon_h^{\widehat{y}},\mu_1\rangle_{\Gamma_h} = \langle\varphi_1 - \varphi_1^h,\mu_1\rangle_{\Gamma_h}\,, \tag{4.7g}$$

$$\langle\varepsilon_h^{\widehat{z}},\mu_2\rangle_{\Gamma_h} = \langle\varphi_2 - \varphi_2^h,\mu_2\rangle_{\Gamma_h} \tag{4.7h}$$

$\forall\,(\boldsymbol{v}_1,\, w_1,\, \mu_1,\, \boldsymbol{v}_2,\, w_2,\, \mu_2) \,\in\, \boldsymbol{V}_h \times W_h \times M_h \times \boldsymbol{V}_h \times W_h \times M_h$, with

$$\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n} \,=\, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \cdot \boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\,, \tag{4.8}$$

$$\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n} \,=\, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n} \,+\, \tau\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\,. \tag{4.9}$$

By mimicking the steps in the proof of Lemma 3.1, i.e, considering specific test functions, adding the equations and performing algebraic manipulations, it is possible to deduce the identities in the following lemma.

Lemma 4.1. *There holds*

$$\begin{aligned}\mathcal{E} \,+\, \alpha\,\langle \varepsilon_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \varepsilon_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \,&=\, -\,\alpha\,(c\,I_{\boldsymbol{V}}\boldsymbol{p}, \boldsymbol{\varepsilon}_h^p)_{\mathcal{T}_h} \,+\, (I_W z, \varepsilon_h^y)_{\mathcal{T}_h} \\ &\quad -\,(c\,I_{\boldsymbol{V}}\boldsymbol{r}, \boldsymbol{\varepsilon}_h^r)_{\mathcal{T}_h} \,-\, (I_W y, \varepsilon_h^z)_{\mathcal{T}_h}\,.\end{aligned} \tag{4.10}$$

Moreover, on Γ_h we have that

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \cdot \boldsymbol{n} \,=\, c^{-1}\,l^{-1}\,(\varphi_1 - \varphi_1^h) \,+\, \Lambda^{I_{\boldsymbol{V}}\boldsymbol{p}} \,+\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n}\,, \tag{4.11a}$$

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{r}} \cdot \boldsymbol{n} \,=\, c^{-1}\,l^{-1}\,(\varphi_2 - \varphi_2^h) \,+\, \Lambda^{I_{\boldsymbol{V}}\boldsymbol{r}} \,+\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}} \,-\, I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n}\,, \tag{4.11b}$$

where we recall the definition of $\Lambda^{\boldsymbol{p}}$ in (2.2).

Corollary 4.1.1. *There exists a positive constant C, independent of h such that*

$$\mathcal{E} \,+\, \mathcal{E}_\varphi \,\le\, C\,\left(\alpha\,\mathbb{T}_{\boldsymbol{p},y} \,+\, \mathbb{T}_{\boldsymbol{r},z} \,+\, \alpha\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|^2_{\mathcal{T}_h}\right)\,,$$

where, for $(\boldsymbol{q}, v) \in \{(\boldsymbol{p}, y), (\boldsymbol{r}, z)\}$,

$$\mathbb{T}_{\boldsymbol{q},v} \,:=\, \|I_{\boldsymbol{V}}\boldsymbol{q}\|^2_{\mathcal{T}_h} \,+\, \|I_W v\|^2_{\mathcal{T}_h} \,+\, R\,\|I_{\boldsymbol{V}}\boldsymbol{q}\|^2_{\Gamma_h, h^\perp} \,+\, R^2\,\|\partial_n(I_{\boldsymbol{V}}\boldsymbol{q}\cdot\boldsymbol{n})\|^2_{\Omega_h^c,(h^\perp)^2}. \tag{4.12}$$

Proof. We have from (4.8) that

$$\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,=\, \langle \boldsymbol{\varepsilon}_h^{\boldsymbol{p}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(\varepsilon_h^y - \varepsilon_h^{\hat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}\,,$$

and by (4.11a)

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} \cdot \boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,&= \|c^{-1/2}\,l^{-1/2}\,(\varphi_1 - \varphi_1^h)\|^2_{\Gamma_h} \,+\, \langle \Lambda^{I_{\boldsymbol{V}}\boldsymbol{p}}(x), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}(x), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\ &\quad -\,\langle I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(\varepsilon_h^y - \varepsilon_h^{\hat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h}\,.\end{aligned}$$

Similarly, by (4.9) and (4.11b),

$$\begin{aligned}\langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}} \cdot \boldsymbol{n}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \,&= \|c^{-1/2}\,l^{-1/2}\,(\varphi_2 - \varphi_2^h)\|^2_{\Gamma_h} \,+\, \langle \Lambda^{I_{\boldsymbol{V}}\boldsymbol{r}}(x), \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \,+\, \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}(x), \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \\ &\quad -\,\langle I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \,+\, \langle \tau\,(\varepsilon_h^z - \varepsilon_h^{\hat{z}}), \varphi_2 - \varphi_2^h\rangle_{\Gamma_h}\,.\end{aligned}$$

Then, replacing both expression in in the identity obtained in Lemma 4.1, we have that

$$\begin{aligned}\mathcal{E} \,+\, \mathcal{E}_\varphi \,=\, &-\alpha\,(c\,I_{\boldsymbol{V}}\boldsymbol{p}, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})_{\mathcal{T}_h} \,+\, (I_W z, \varepsilon_h^y)_{\mathcal{T}_h} \,-\, (c\,I_{\boldsymbol{V}}\boldsymbol{r}, \boldsymbol{\varepsilon}_h^{\boldsymbol{r}})_{\mathcal{T}_h} \,-\, (I_W y, \varepsilon_h^z)_{\mathcal{T}_h} \,-\, \alpha\,\langle \Lambda^{I_{\boldsymbol{V}}\boldsymbol{p}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \\ &-\,\alpha\,\langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} + \alpha\,\langle I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n}, \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,-\, \alpha\,\langle \tau\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}}), \varphi_1 - \varphi_1^h\rangle_{\Gamma_h} \,-\, \langle \Lambda^{I_{\boldsymbol{V}}\boldsymbol{r}}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \\ &-\,\langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} + \langle I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n}, \varphi_2 - \varphi_2^h\rangle_{\Gamma_h} \,-\, \langle \tau\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}}), \varphi_2 - \varphi_2^h\rangle_{\Gamma_h}\,.\end{aligned}$$

Thus, using the Cauchy–Schwarz and Young's inequalities, and after some algebraic manipulations

$$\begin{aligned}\frac{3}{4}\,\mathcal{E} \,+\, \frac{1}{2}\,\mathcal{E}_\varphi \le\, &2\,\alpha\,c\,\|I_{\boldsymbol{V}}\boldsymbol{p}\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|I_W z\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \,+\, 2\,c\,\|I_{\boldsymbol{V}} r\|^2 \,+\, \frac{1}{2}\,\|I_W y\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|\varepsilon_h^z\|^2_{\mathcal{T}_h} \\ &+\, 2\,\alpha\,\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{V}}\boldsymbol{p}}\|^2_{\Gamma_h} \,+\, 2\,\|c^{1/2}\,l^{1/2}\,\Lambda^{I_{\boldsymbol{V}}\boldsymbol{r}}\|^2_{\Gamma_h} \\ &+\, 2\,\alpha\,\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\|^2_{\Gamma_h} \,+\, 2\,\alpha\,c\,l\,\|I_{\boldsymbol{V}}\boldsymbol{p}\|^2_{\Gamma_h} \,+\, 2\,\alpha\,\|\tau\,c^{1/2}\,l^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|^2_{\Gamma_h} \\ &+\, 2\,\|c^{1/2}\,l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}\|^2_{\Gamma_h} \,+\, 2\,c\,l\,\|I_{\boldsymbol{V}}\boldsymbol{r}\|^2_{\Gamma_h} \,+\, 2\,\|\tau\,c^{1/2}\,l^{1/2}\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|^2_{\Gamma_h}\,.\end{aligned} \tag{4.13}$$

On the other hand, by Assumption (A.4), we note that

$$2\,\alpha\,\|\tau\, c^{1/2}\, l^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|^2_{\Gamma_h} \,+\, 2\,\|\tau\, c^{1/2}\, l^{1/2}\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|^2_{\Gamma_h} \,\leq\, \frac{\alpha}{4}\,\|\tau^{1/2}\,(\varepsilon_h^y - \varepsilon_h^{\widehat{y}})\|^2_{\partial\mathcal{T}_h} \,+\, \frac{1}{4}\,\|\tau^{1/2}\,(\varepsilon_h^z - \varepsilon_h^{\widehat{z}})\|^2_{\partial\mathcal{T}_h}\,.$$

In addition, by (2.3) and recalling that $R := \max_{e\in\mathcal{E}_h^\partial} r_e$, we have that

$$\|l^{1/2}\,\Lambda^{\boldsymbol{I_V p}}\|^2_{\Gamma_h} \,\leq\, \frac{1}{3}R^2\,\|\partial_n(\boldsymbol{I_V p}\cdot\boldsymbol{n})\|^2_{\Omega_h^c,(h^\perp)^2}$$

and

$$\|c^{1/2}\, l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\|^2_{\Gamma_h} \,\leq\, \frac{1}{3}\,\max_{e\in\mathcal{E}_h^\partial} c\, r_e^3\,(C^e_{ext}\,C^e_{int})^2\,\|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|^2_{\mathcal{T}_h} \,\leq\, \frac{c}{8}\,\|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|^2_{\mathcal{T}_h}\,,$$

where in the last step we have considered Assumption (A.1). Similar estimates can be obtained for $\|l^{1/2}\,\Lambda^{\boldsymbol{I_V r}}\|^2_{\Gamma_h}$ and $\|c^{1/2}\, l^{1/2}\,\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}\|^2_{\Gamma_h}$. From what was obtained above, applied to (4.13) and after simple algebraic operations, we deduce the following estimate

$$\begin{aligned}\frac{1}{2}\,\mathcal{E} \,+\, \frac{1}{2}\,\mathcal{E}_\varphi \,\leq\, & 2\,\alpha\, c\,\|\boldsymbol{I_V p}\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|I_W z\|^2_{\mathcal{T}_h} \,+\, 2\, c\,\|\boldsymbol{I_V r}\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|I_W y\|^2_{\mathcal{T}_h} \\ & +\, \frac{2\,\alpha\, c}{3}\,R^2\,\|\partial_n(\boldsymbol{I_V p}\cdot\boldsymbol{n})\|^2_{\Omega_h^c,(h^\perp)^2} \,+\, 2\,\alpha\, c\, R\,\|\boldsymbol{I_V p}\|^2_{\Gamma_h,h^\perp} \,+\, \frac{2\,c}{3}\,R^2\,\|\partial_n(\boldsymbol{I_V r}\cdot\boldsymbol{n})\|^2_{\Omega_h^c,(h^\perp)^2} \\ & +\, 2\, c\, R\,\|\boldsymbol{I_V r}\|^2_{\Gamma_h,h^\perp} \,+\, \frac{1}{2}\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \,+\, \frac{1}{2}\,\|\varepsilon_h^z\|^2_{\mathcal{T}_h}\,.\end{aligned}$$

and the result follows. □

In the previous result, we see that the error estimates for $\boldsymbol{e}^{\boldsymbol{p}}$ and $\boldsymbol{e}^{\boldsymbol{r}}$ depend of ε_h^y and ε_h^z. In order to bound the latter, we employ a duality argument as detail in the next section.

4.3 Duality argument.

Consider the following dual system:

$$\begin{aligned} c\,\boldsymbol{\Phi} \,+\, \nabla\Psi &= 0 \quad \text{in } \Omega\,, && (4.14a) & c\,\boldsymbol{\Phi}^{\boldsymbol{r}} \,+\, \nabla\Psi^z &= 0 \quad \text{in } \Omega\,, && (4.14d)\\ \nabla\cdot\boldsymbol{\Phi} \,+\, \beta\,\Psi^z &= \Theta_1 \quad \text{in } \Omega\,, && (4.14b) & \nabla\cdot\boldsymbol{\Phi}^{\boldsymbol{r}} \,-\, \Psi &= \Theta_2 \quad \text{in } \Omega\,, && (4.14e)\\ \Psi &= 0 \quad \text{on } \Gamma\,, && (4.14c) & \Psi^z &= 0 \quad \text{on } \Gamma\,. && (4.14f)\end{aligned}$$

To shorten notation, let $\Theta := \|\Theta_1\|_\Omega \,+\, \|\Theta_2\|_\Omega$. We assume that

$$\|\Psi\|_{2,\Omega} \,+\, \|\boldsymbol{\Phi}\|_{1,\Omega} + \|\Psi^z\|_{2,\Omega} \,+\, \|\boldsymbol{\Phi}^{\boldsymbol{r}}\|_{1,\Omega} \,\lesssim\, \Theta. \tag{4.15a}$$

This holds true, for instance, for convex polyhedral domains or when Γ is $\mathcal{C}^2$. On the other hand, we observe that (4.14) is posed on Ω, whereas the HDG method seeks the solution in Ω_h. In other words, the duality argument will involve expressions in Ω_h where we need to take into account the influence of the mismatch between Ω_h and Ω in (4.3). More precisely, we consider the following result that can be obtained similarly to the proof of [18, Lemma 5.5].

Lemma 4.2. *Under the assumptions given in Section 2 and assuming that (4.15) holds true, we have*

$$\|\Psi \,-\, P_M\Psi\|_{\Gamma,(h^\perp)^{-1}} \,\lesssim\, h\,\Theta\,, \qquad \|\Psi \,+\, c\, l\,\partial_{\boldsymbol{n}}\Psi\|_{\Gamma_h,l^{-3}} \,\lesssim\, \Theta\,,$$

$$\|\partial_{\boldsymbol{n}}\Psi \,-\, P_M\partial_{\boldsymbol{n}}\Psi\|_{\Gamma_h,l} \,\lesssim\, R\, h\,\Theta\,, \qquad \|\Psi\|_{\Gamma_h,l^{-2}} \,\lesssim\, \Theta.$$

Moreover, the same estimates hold for Ψ^z.

Lemma 4.3. *We have that*

$$\alpha\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \;=\; \alpha\,(c\,\boldsymbol{I_V p}, \boldsymbol{\Pi_V \Phi})_{\mathcal{T}_h} \;-\; \alpha\,(\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\Phi} - \boldsymbol{\Pi_V \Phi}))_{\mathcal{T}_h} \;+\; (e^z, \Pi_W \Psi)_{\mathcal{T}_h} \;+\; (\varepsilon_h^y, \Psi^z)_{\mathcal{T}_h} \;+\; \alpha\,\mathbb{S}^y\,, \tag{4.16a}$$

$$\|\varepsilon_h^z\|^2_{\mathcal{T}_h} \;=\; (c\,\boldsymbol{I_V r}, \boldsymbol{\Pi_V \Phi^r})_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, c\,(\boldsymbol{\Phi^r} - \boldsymbol{\Pi_V \Phi^r}))_{\mathcal{T}_h} \;-\; (e^y, \Pi_W \Psi^z)_{\mathcal{T}_h} \;-\; (\varepsilon_h^z, \Psi)_{\mathcal{T}_h} \;+\; \mathbb{S}^z\,, \tag{4.16b}$$

where, $\mathbb{S}^y \;=\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Psi\rangle_{\Gamma_h}$ *and* $\mathbb{S}^z \;=\; \langle \varepsilon_h^{\widehat{z}}, \boldsymbol{\Phi^r}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{r}}}\cdot\boldsymbol{n}, \Psi^z\rangle_{\Gamma_h}$.

Proof. We consider the steps in Lemma 4.6 of [59] adapted to our context.

Let, in (4.14b), $\Theta_1 \;=\; \sqrt{\alpha}\,\varepsilon_h^y$ in Ω_h and $\Theta_1 \;=\; 0$ in $\Omega\setminus\Omega_h$. By adding and subtracting convenient terms in order to generate the error of the projection $\boldsymbol{\Phi} - \boldsymbol{\Pi_V \Phi}$ and $\Psi - \Pi_W\Psi$, we have that that

$$\begin{aligned}\|\sqrt{\alpha}\varepsilon_h^y\|^2_{\mathcal{T}_h} \;=&\; (\varepsilon_h^y, \nabla\cdot\boldsymbol{\Pi_V\Phi})_{\mathcal{T}_h} \;+\; (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,\boldsymbol{\Pi_V\Phi})_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h}\\ &-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla\Pi_W\Psi)_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\Psi - \Pi_W\Psi))_{\mathcal{T}_h} \;+\; (\varepsilon_h^y, \beta\,\Psi^z)_{\mathcal{T}_h}\,.\end{aligned}$$

Then, setting $\boldsymbol{v}_1 \;=\; \boldsymbol{\Pi_V\Phi}$ in (4.7a) and $w_1 \;=\; \Pi_W\Psi$ in (4.7b) we obtain

$$\begin{aligned}\|\sqrt{\alpha}\varepsilon_h^y\|^2_{\mathcal{T}_h} \;=&\; (c\,\boldsymbol{I_V p}, \boldsymbol{\Pi_V\Phi})_{\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Pi_V\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;+\; (\beta\,I_W z, \Pi_W\Psi)_{\mathcal{T}_h} \;+\; (\beta\,\varepsilon_h^z, \Pi_W\Psi)_{\mathcal{T}_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Pi_W\Psi\rangle_{\partial\mathcal{T}_h}\\ &+\; (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\Psi - \Pi_W\Psi))_{\mathcal{T}_h} \;+\; (\varepsilon_h^y, \beta\Psi^z)_{\mathcal{T}_h} \quad (4.17)\\ =&\; (c\,\boldsymbol{I_V p}, \boldsymbol{\Pi_V\Phi})_{\mathcal{T}_h} \;+\; (\beta\,e^z, \Pi_W\Psi)_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h} \;+\; (\varepsilon_h^y, \beta\Psi^z)_{\mathcal{T}_h} \;+\; \mathbb{S}^y\,, \quad (4.18)\end{aligned}$$

where

$$\mathbb{S}^y \;:=\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Pi_V\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Pi_W\Psi\rangle_{\partial\mathcal{T}_h} \;+\; (\varepsilon_h^y, \nabla\cdot(\boldsymbol{\Phi} - \boldsymbol{\Pi_V\Phi}))_{\mathcal{T}_h} \;-\; (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, \nabla(\Psi - \Pi_W\Psi))_{\mathcal{T}_h}\,.$$

On the other hand, integrating by parts the above expression and using the HDG projection, particularly (4.1a) and (4.1b), we deduce that

$$\begin{aligned}\mathbb{S}^y \;=&\; \langle \varepsilon_h^{\widehat{y}} - \varepsilon_h^y, (\boldsymbol{\Pi_V\Phi} - \boldsymbol{\Phi})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;-\; \langle (\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})\cdot\boldsymbol{n}, \Pi_W\Psi - \Psi\rangle_{\partial\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Psi\rangle_{\partial\mathcal{T}_h}\\ =&\; \langle \varepsilon_h^{\widehat{y}} - \varepsilon_h^y, (\boldsymbol{\Pi_V\Phi} - \boldsymbol{\Phi})\cdot\boldsymbol{n}\rangle_{\partial\mathcal{T}_h} \;-\; \langle (\boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}} - \boldsymbol{\varepsilon}_h^{\boldsymbol{p}})\cdot\boldsymbol{n}, \Pi_W\Psi - \Psi\rangle_{\partial\mathcal{T}_h} \;+\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Psi\rangle_{\Gamma_h}\,,\end{aligned}$$

where in the last step we have employ (4.7e) and and the fact that $\varepsilon_h^{\widehat{y}}$ is single-valued. Moreover, by (4.8) and (4.1c),

$$\mathbb{S}^y \;=\; \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \;-\; \langle \boldsymbol{\varepsilon}_h^{\widehat{\boldsymbol{p}}}\cdot\boldsymbol{n}, \Psi\rangle_{\Gamma_h}\,,$$

which together with (4.18) implies (4.16a).

Finally, by taking $\Theta_2 \;=\; \varepsilon_h^z$ in Ω_h and $\Theta_2 \;=\; 0$ in $\Omega\setminus\Omega_h$, setting $\boldsymbol{v}_2 \;=\; \boldsymbol{\Pi_V\Phi^r}$ in (4.7c) and $w_2 \;=\; \Pi_W\Psi^z$ in (4.7d) in (4.14e), similar arguments yield to (4.16b). □

The presence of the terms $\mathbb{S}^y$ and $\mathbb{S}^z$ is due to the fact that Γ_h does not fit Γ, otherwise both would vanish by (4.14c) and (4.14f). This is why in the following lemma we re-write $\mathbb{S}^y$ and $\mathbb{S}^z$ in order to quantify explicitly the influence of the mismatch between Γ and Γ_h.

Lemma 4.4. *We have the following decomposition:* $\mathbb{S}^y \;=\; \sum_{i=1}^7 \mathbb{S}_i^y$ *and* $\mathbb{S}^z \;=\; \sum_{i=1}^7 \mathbb{S}_i^z$*, where*

$$
\begin{aligned}
\mathbb{S}_1^y &= \langle c^{-1}\,l^{-1}\,(\varphi_1-\varphi_1^h), \Psi + c\,l\,\partial_{\boldsymbol{n}}\Psi\rangle_{\Gamma_h}\,, &\qquad \mathbb{S}_1^z &= \langle c^{-1}\,l^{-1}\,(\varphi_2-\varphi_2^h), \Psi^z + c\,l\,\partial_{\boldsymbol{n}}\Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_2^y &= \langle \varphi_1-\varphi_1^h, \partial_{\boldsymbol{n}}\Psi - P_M\partial_{\boldsymbol{n}}\Psi\rangle_{\Gamma_h}\,, & \mathbb{S}_2^z &= \langle \varphi_2-\varphi_2^h, \partial_{\boldsymbol{n}}\Psi^z - P_M\partial_{\boldsymbol{n}}\Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_3^y &= \langle \Lambda^{\boldsymbol{I_V p}}, \Psi\rangle_{\Gamma_h}\,, & \mathbb{S}_3^z &= \langle \Lambda^{\boldsymbol{I_V r}}, \Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_4^y &= \langle \boldsymbol{I_V p}\cdot\boldsymbol{n}, \Psi - P_M\Psi\rangle_{\Gamma_h}\,, & \mathbb{S}_4^z &= \langle \boldsymbol{I_V r}\cdot\boldsymbol{n}, \Psi^z - P_M\Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_5^y &= -\,\langle \tau I_W y, P_M\Psi\rangle_{\Gamma_h}\,, & \mathbb{S}_5^z &= -\,\langle \tau I_W z, P_M\Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_6^y &= \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}, \Psi\rangle_{\Gamma_h}\,, & \mathbb{S}_6^z &= \langle \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}}, \Psi^z\rangle_{\Gamma_h}\,,\\
\mathbb{S}_7^y &= -\,\langle \tau\,(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}), P_M\Psi\rangle_{\Gamma_h}\,. & \mathbb{S}_7^z &= -\,\langle \tau\,(\varepsilon_h^z-\varepsilon_h^{\widehat{z}}), P_M\Psi^z\rangle_{\Gamma_h}\,.
\end{aligned}
$$

Proof. In this proof we proceed analogously to the proof of [18, Lemma 5.4]. By (4.8) and (4.11a) we note that

$$\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n} \,=\, \boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}) \,=\, c^{-1}\,l^{-1}\,(\varphi_1-\varphi_1^h) \,-\, \Lambda^{\boldsymbol{I_V p}} \,-\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, \boldsymbol{I_V p}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y-\varepsilon_h^{\widehat{y}})\,,$$

then

$$\mathbb{S}^y \,=\, \langle \varepsilon_h^{\widehat{y}}, \boldsymbol{\Phi}\cdot\boldsymbol{n}\rangle_{\Gamma_h} \,-\, \langle c^{-1}\,l^{-1}\,(\varphi_1-\varphi_1^h) \,-\, \Lambda^{\boldsymbol{I_V p}} \,-\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, \boldsymbol{I_V p}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}), \Psi\rangle_{\Gamma_h}\,.$$

By (4.7g) and since $\boldsymbol{\Phi} \,=\, -\,c^{-1}\,\nabla\Psi$ in (4.14a), it follows

$$\mathbb{S}^y \,=\, -\,\langle \varphi_1-\varphi_1^h, P_M\partial_{\boldsymbol{n}}\Psi\rangle_{\Gamma_h} \,-\, \langle c^{-1}\,l^{-1}\,(\varphi_1-\varphi_1^h) \,-\, \Lambda^{\boldsymbol{I_V p}} \,-\, \Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}} \,-\, \boldsymbol{I_V p}\cdot\boldsymbol{n} \,+\, \tau\,(\varepsilon_h^y-\varepsilon_h^{\widehat{y}}), \Psi\rangle_{\Gamma_h}\,.$$

In turn, by (4.1c), note that

$$
\begin{aligned}
\langle \boldsymbol{I_V p}\cdot\boldsymbol{n}, \Psi\rangle_{\Gamma_h} &= \langle \boldsymbol{I_V p}\cdot\boldsymbol{n}, \Psi - P_M\Psi\rangle_{\Gamma_h} \,+\, \langle \boldsymbol{I_V p}\cdot\boldsymbol{n}, P_M\Psi\rangle_{\Gamma_h}\\
&= \langle \boldsymbol{I_V p}\cdot\boldsymbol{n}, \Psi - P_M\Psi\rangle_{\Gamma_h} \,-\, \langle \tau\, I_W y, P_M\Psi\rangle_{\Gamma_h}.
\end{aligned}
$$

Therefore, the desired identity is obtained after a simple rearrangement of terms. On the other hand, the decomposition for $\mathbb{S}^z$ can be obtained analogously. □

Corollary 4.4.1. *There holds that*

$$
\begin{aligned}
|\mathbb{S}^y| \,\lesssim\, & \Big(R\,h \,+\, R^2\,h^{1/2} \,+\, R_\tau\,h\Big)\,(\mathcal{E}_{\boldsymbol{p},y}+\mathcal{E}_{\varphi_1})^{1/2}\,\Theta\\
& +\, \Big(R^{3/2}\,h^{1/2}\,\|\partial_{\boldsymbol{n}}(\boldsymbol{I_V p}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2} \,+\, h\,\|\boldsymbol{I_V p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\Big)\,\Theta \,+\, R_\tau\,h^{1/2}\,\|I_W y\|_{\Gamma_h,h^\perp}\,\Theta\,,
\end{aligned}
$$

where we recall the notation defined in (4.6). A similar estimate holds true for $|\mathbb{S}^z|$, but $(\boldsymbol{r},z,\varphi_2)$ plays the role of $(\boldsymbol{p},y,\varphi_1)$.

Proof. We will only show the first inequality since both can be treated in the same way. First of all, by Lemmas 4.4 and 4.2, we have that

$$
\begin{aligned}
|\mathbb{S}_1^y| &\lesssim \|\varphi_1-\varphi_1^h\|_{\Gamma_h,c^{-2}l}\Theta\,, &\quad |\mathbb{S}_2^y| &\lesssim R\,h\,\|\varphi_1-\varphi_1^h\|_{\Gamma_h,l^{-1}}\,\Theta\,, &\quad |\mathbb{S}_3^y| &\lesssim \|\Lambda^{\boldsymbol{I_V p}}\|_{\Gamma_h,l^2}\,\Theta\,,\\
|\mathbb{S}_4^y| &\lesssim h\,\|\boldsymbol{I_V p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\,\Theta, & |\mathbb{S}_5^y| &\lesssim \|P_M I_W y\|_{\Gamma_h,\tau^2 l^2}\,\Theta, & |\mathbb{S}_6^y| &\lesssim \|\Lambda^{\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}}\|_{\Gamma_h,l^2}\,\Theta,\\
|\mathbb{S}_7^y| &\lesssim \|\varepsilon_h^y-\varepsilon_h^{\widehat{y}}\|_{\Gamma_h,\tau^2 l^2}\,\Theta.
\end{aligned}
$$

On the other hand, let e be a side or face of Γ_h. Since $l(\boldsymbol{x}) \leq h_e^\perp\, r_e$, we obtain that

$$|\mathbb{S}_1^y| \lesssim \max_{e\in\mathcal{E}_h^\partial} c\, r_e\, h\, \|\varphi_1 - \varphi_1^h\|_{\Gamma_h,l}\, \Theta\,, \qquad |\mathbb{S}_2^y| \lesssim R\, h\, \|\varphi_1 - \varphi_1^h\|_{\Gamma_h,l^{-1}}\, \Theta\,,$$

$$|\mathbb{S}_3^y| \lesssim \max_{e\in\mathcal{E}_h^\partial} r_e^{3/2}\, h^{1/2}\, \|\partial_{\boldsymbol{n}}(\boldsymbol{I_V p}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2}\, \Theta\,, \qquad |\mathbb{S}_4^y| \lesssim h\, \|\boldsymbol{I_V p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\, \Theta,$$

$$|\mathbb{S}_5^y| \lesssim \max_{e\in\mathcal{E}_h^\partial} \tau_e\, r_e\, h^{1/2}\, \|I_W y\|_{\Gamma_h,h^\perp}\, \Theta\,, \qquad |\mathbb{S}_6^y| \lesssim \max_{e\in\mathcal{E}_h^\partial} r_e^2\, C_{ext}^e\, C_{inv}^e\, h^{1/2}\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h}\, \Theta\,,$$

$$|\mathbb{S}_7^y| \lesssim \max_{e\in\mathcal{E}_h^\partial} \tau_e^{1/2}\, r_e\, h\, \|\varepsilon_h^y - \varepsilon_h^{\widehat{y}}\|_{\Gamma_h,\tau}\, \Theta\,.$$

The result follows from the definition of $(\mathcal{E}_{\boldsymbol{p},y} + \mathcal{E}_{\varphi_1})^{1/2}$. □

We are now in position to present the main results of our work. The first one controls the L^2-norm of the projection of the error associated to the scalar variables, whereas the second one provides the corresponding estimates for the mixed variables and the boundary data.

Theorem 4.5. *We have that*

$$\begin{aligned}\Big(1 - H_1(R,h)\Big)\Big(\sqrt{\alpha}\, \|\varepsilon_h^y\|_{\mathcal{T}_h} + \|\varepsilon_h^z\|_{\mathcal{T}_h}\Big) \lesssim{}& \|I_W z\|_{\mathcal{T}_h} + \|I_W y\|_{\mathcal{T}_h} + H_2(R,h)\left(\alpha^{1/2}\,\mathbb{T}_{\boldsymbol{p},y}^{1/2} + \mathbb{T}_{\boldsymbol{r},z}^{1/2}\right)\\ &+ R_\tau\, h^{1/2}\, \left(\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}\right), \end{aligned} \tag{4.19a}$$

where

$$H_1(R,h) := h + R^2\, h^{1/2} + R_\tau\, h\,,$$

$$H_2(R,h) := h + R^{3/2}\, h^{1/2} + R_\tau\, h.$$

Moreover, if $H_1(R,h) < 1$, then

$$\begin{aligned}(\mathcal{E} + \mathcal{E}_\varphi)^{1/2} \lesssim{}& (1 + H_2(R,h))\left(\alpha^{1/2}\,\mathbb{T}_{\boldsymbol{p},y}^{1/2} + \mathbb{T}_{\boldsymbol{r},z}^{1/2}\right) + \|I_W z\|_{\mathcal{T}_h} + \|I_W y\|_{\mathcal{T}_h}\\ &+ R_\tau\, h^{1/2}\, \left(\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}\right). \end{aligned} \tag{4.19b}$$

Proof. Adding (4.16a) and (4.16b) we get

$$\begin{aligned}\alpha\, \|\varepsilon_h^y\|_{\mathcal{T}_h}^2 + \|\varepsilon_h^z\|_{\mathcal{T}_h}^2 ={}& \alpha\, (c\,\boldsymbol{I_V p}, \boldsymbol{\Pi_V \Phi})_{\mathcal{T}_h} - \alpha\, (\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}, c\,(\boldsymbol{\Phi} - \boldsymbol{\Pi_V \Phi}))_{\mathcal{T}_h} + (c\,\boldsymbol{I_V r}, \boldsymbol{\Pi_V \Phi^r})_{\mathcal{T}_h} - (\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}, c\,(\boldsymbol{\Phi^r} - \boldsymbol{\Pi_V \Phi^r}))_{\mathcal{T}_h}\\ &+ (e^z, \Pi_W \Psi)_{\mathcal{T}_h} + (\varepsilon_h^y, \Psi^z)_{\mathcal{T}_h} - (e^y, \Pi_W \Psi^z)_{\mathcal{T}_h} - (\varepsilon_h^z, \Psi)_{\mathcal{T}_h} + \alpha\, \mathbb{S}^y + \mathbb{S}^z\,. \end{aligned}$$

Let $\boldsymbol{\Phi}_h$, $\boldsymbol{\Phi}_h^{\boldsymbol{r}} \in [\mathbb{P}_{k-1}(\mathcal{T}_h)]^d$, by (4.1a), rearranging terms and recalling the definitions in (4.2), we can deduce that

$$\begin{aligned}\alpha\, \|\varepsilon_h^y\|_{\mathcal{T}_h}^2 + \|\varepsilon_h^z\|_{\mathcal{T}_h}^2 ={}& \alpha\, (c\,\boldsymbol{I_V p}, \boldsymbol{\Phi} - \boldsymbol{\Phi}_h)_{\mathcal{T}_h} - \alpha\, (\boldsymbol{p} - \boldsymbol{p}_h, c(\boldsymbol{\Phi} - \boldsymbol{\Pi_V \Phi}))_{\mathcal{T}_h} + (c\,\boldsymbol{I_V r}, \boldsymbol{\Phi^r} - \boldsymbol{\Phi}_h^{\boldsymbol{r}})_{\mathcal{T}_h}\\ &- (\boldsymbol{r} - \boldsymbol{r}_h, c(\boldsymbol{\Phi^r} - \boldsymbol{\Pi_V \Phi^r}))_{\mathcal{T}_h} + (I_W z, \Pi_W \Psi)_{\mathcal{T}_h} - (I_W y, \Pi_W \Psi^z)_{\mathcal{T}_h}\\ &+ (\varepsilon_h^y, \Psi^z - \Pi_W \Psi^z)_{\mathcal{T}_h} - (\varepsilon_h^z, \Psi - \Pi_W \Psi)_{\mathcal{T}_h} + \alpha\, \mathbb{S}^y + \mathbb{S}^z\,. \end{aligned}$$

Then, by the Cauchy-Schwarz inequality, (4.1) and (4.15), we obtain

$$\begin{aligned}
\alpha\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|^2_{\mathcal{T}_h} \,\lesssim\, & \|I_{\boldsymbol{V}}\boldsymbol{p}\|_{\mathcal{T}_h}\,\|\boldsymbol{\Phi}-\boldsymbol{\Phi}_h\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{p}-\boldsymbol{p}_h\|_{\mathcal{T}_h}\,\|\boldsymbol{\Phi}-\boldsymbol{\Pi_V}\boldsymbol{\Phi}\|_{\mathcal{T}_h} \,+\, \|I_{\boldsymbol{V}}\boldsymbol{r}\|_{\mathcal{T}_h}\,\|\boldsymbol{\Phi}^{\boldsymbol{r}}-\boldsymbol{\Phi}_h^{\boldsymbol{r}}\|_{\mathcal{T}_h} \\
& +\, \|\boldsymbol{r}-\boldsymbol{r}_h\|_{\mathcal{T}_h}\,\|\boldsymbol{\Phi}^{\boldsymbol{r}}-\boldsymbol{\Pi_V}\boldsymbol{\Phi}^{\boldsymbol{r}}\|_{\mathcal{T}_h} \,+\, \|I_W z\|_{\mathcal{T}_h}\,\|\Pi_W\Psi-\Psi\|_{\mathcal{T}_h} \,+\, \|I_W z\|_{\mathcal{T}_h}\,\|\mathbb{S}\|_{\mathcal{T}_h} \\
& +\, \|I_W y\|_{\mathcal{T}_h}\,\|\Pi_W\Psi^z-\Psi^z\|_{\mathcal{T}_h} \,+\, \|I_W y\|_{\mathcal{T}_h}\,\|\Psi^z\|_{\mathcal{T}_h} \,+\, \|\varepsilon_h^y\|_{\mathcal{T}_h}\,\|\Psi^z-\Pi_W\Psi^z\|_{\mathcal{T}_h} \\
& +\, \|\varepsilon_h^z\|_{\mathcal{T}_h}\,\|\Psi-\Pi_W\Psi\|_{\mathcal{T}_h} \,+\, \alpha\,|\mathbb{S}^y| \,+\, |\mathbb{S}^z| \\
\lesssim\, & h\left(\|I_{\boldsymbol{V}}\boldsymbol{p}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} \,+\, \|I_{\boldsymbol{V}}\boldsymbol{r}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}\|_{\mathcal{T}_h}\right)\Theta \\
& +\, \Big((1+h)\|I_W z\|_{\mathcal{T}_h} \,+\, (1+h)\,\|I_W y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^z\|_{\mathcal{T}_h}\Big)\,\Theta \,+\, \alpha\,|\mathbb{S}^y| \,+\, |\mathbb{S}^z|\,.
\end{aligned}$$

Applying Corollary 4.4.1 and considering the fact that $(1+h)$ can be bounded above by a constant.

$$\begin{aligned}
\alpha\,\|\varepsilon_h^y\|^2_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|^2_{\mathcal{T}_h} \,\lesssim\, & h\left(\|I_{\boldsymbol{V}}\boldsymbol{p}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} \,+\, \|I_{\boldsymbol{V}}\boldsymbol{r}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}\|_{\mathcal{T}_h}\right)\Theta \\
& +\, \left(\|I_W z\|_{\mathcal{T}_h} \,+\, \|I_W y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^z\|_{\mathcal{T}_h}\right)\Theta \\
& +\, \left(R\,h \,+\, R^2\,h^{1/2} \,+\, R_\tau\,h\right)\,(\mathcal{E}_{\boldsymbol{p},y}+\mathcal{E}_{\varphi_1})^{1/2}\;\Theta \\
& +\, \left(R^{3/2}\,h^{1/2}\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2} \,+\, h\,\|I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\right)\Theta \,+\, R_\tau\,h^{1/2}\,\|I_W y\|_{\Gamma_h,h^\perp}\,\Theta \\
& +\, \left(R\,h \,+\, R^2\,h^{1/2} \,+\, R_\tau\,h\right)\,(\mathcal{E}_{\boldsymbol{r},z}+\mathcal{E}_{\varphi_2})^{1/2}\;\Theta \\
& +\, \left(R^{3/2}\,h^{1/2}\,\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2} \,+\, h\,\|I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\right)\Theta \,+\, R_\tau h^{1/2}\,\|I_W z\|_{\Gamma_h,h^\perp}\,\Theta\,.
\end{aligned}$$

Now, since $\Theta_1 \;=\; \sqrt{\alpha}\,\varepsilon_h^y$ in Ω_h and $\Theta_1 \;=\; 0$ in $\Omega\setminus\Omega_h$ and $\Theta_2 \;=\; \varepsilon_h^z$ in Ω_h and $\Theta_2 \;=\; 0$ in $\Omega\setminus\Omega_h$ (cf. proof of Lemma 4.3), and recalling the definition in (4.6), a simple rearrangement of terms implies

$$\begin{aligned}
\sqrt{\alpha}\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|_{\mathcal{T}_h} \,\lesssim\, & h\left(\|I_{\boldsymbol{V}}\boldsymbol{p}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{p}}\|_{\mathcal{T}_h} \,+\, \|I_{\boldsymbol{V}}\boldsymbol{r}\|_{\mathcal{T}_h} \,+\, \|\boldsymbol{\varepsilon}_h^{\boldsymbol{r}}\|_{\mathcal{T}_h}\right) \\
& +\, \left(\|I_W z\|_{\mathcal{T}_h} \,+\, \|I_W y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^z\|_{\mathcal{T}_h}\right) \,+\, \left(R\,h \,+\, R^2\,h^{1/2} \,+\, R_\tau\,h\right)\mathcal{E}^{1/2} \\
& +\, R^{3/2}\,h^{1/2}\,\left(\|\partial_{\boldsymbol{n}}(I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2} + \|\partial_{\boldsymbol{n}}(I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n})\|_{\Omega_h^c,(h^\perp)^2}\right) \\
& +\, h\,\left(\|I_{\boldsymbol{V}}\boldsymbol{p}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp} + \|I_{\boldsymbol{V}}\boldsymbol{r}\cdot\boldsymbol{n}\|_{\Gamma_h,h^\perp}\right) \,+\, R_\tau\,h^{1/2}\,\left(\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}\right).
\end{aligned}$$

Then, by Corollary 4.1.1,

$$\begin{aligned}
\sqrt{\alpha}\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|_{\mathcal{T}_h} \,\lesssim\, & h\left(\|I_{\boldsymbol{V}}\boldsymbol{p}\|_{\mathcal{T}_h} \,+\, \|I_{\boldsymbol{V}}\boldsymbol{r}\|_{\mathcal{T}_h}\right) \\
& +\, \left(\|I_W z\|_{\mathcal{T}_h} \,+\, \|I_W y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, h\,\|\varepsilon_h^z\|_{\mathcal{T}_h}\right) \\
& +\, \left(h \,+\, R\,h \,+\, R^2\,h^{1/2} \,+\, R_\tau\,h\right)\,\left(\alpha^{1/2}\,\mathbb{T}^{1/2}_{\boldsymbol{p},y} \,+\, \mathbb{T}^{1/2}_{\boldsymbol{r},z} \,+\, \alpha^{1/2}\,\|\varepsilon_h^y\|_{\mathcal{T}_h} \,+\, \|\varepsilon_h^z\|_{\mathcal{T}_h}\right) \\
& +\, R^{3/2}\,h^{1/2}\,\left(\alpha^{1/2}\,\mathbb{T}^{1/2}_{\boldsymbol{p},y} \,+\, \mathbb{T}^{1/2}_{\boldsymbol{r},z}\right) \,+\, h\,\left(\alpha^{1/2}\,\mathbb{T}^{1/2}_{\boldsymbol{p},y} \,+\, \mathbb{T}^{1/2}_{\boldsymbol{r},z}\right) \\
& +\, R_\tau\,h^{1/2}\,\left(\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}\right).
\end{aligned}$$

Rearranging terms, recalling the definition of $H_1(R,h)$ and taking into account that $h + R\,h$ is dominated by h, and $R^2 \leq R^{3/2}$,

$$(1 - H_1(R,h))\left(\sqrt{\alpha}\,\|\varepsilon_h^y\|_{\mathcal{T}_h} + \|\varepsilon_h^z\|_{\mathcal{T}_h}\right) \lesssim \|I_W z\|_{\mathcal{T}_h} + \|I_W y\|_{\mathcal{T}_h} + \Big(h + R^{3/2}\,h^{1/2} + R_\tau\,h\Big)\Big(\alpha^{1/2}\,\mathbb{T}_{\boldsymbol{p},y}^{1/2} + \mathbb{T}_{\boldsymbol{r},z}^{1/2}\Big)$$
$$+ R_\tau\,h^{1/2}\left(\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}\right),$$

which implies (4.19a). Finally, (4.19b) follows from this estimate and Corollary 4.1.1. □

Corollary 4.5.1. *Let us assume $\boldsymbol{p}$ and $\boldsymbol{r}$ in $\boldsymbol{H}^{l+1}(\Omega)$; and y and z in $H^{l+1}(\Omega)$, with $l \in [0,k]$. Let also τ be of order one. If $H_1(R,h) < 1$, then*

$$\sqrt{\alpha}\,\|y - y_h\|_{\mathcal{T}_h} + \|z - z_h\|_{\mathcal{T}_h} \lesssim h^{l+1}\left(|\boldsymbol{p}|_{l+1,\Omega} + |\boldsymbol{r}|_{l+1,\Omega} + |z|_{l+1,\Omega} + |y|_{l+1,\Omega}\right) \tag{4.20a}$$

and

$$\sqrt{\alpha}\,\|c^{1/2}\,(\boldsymbol{p} - \boldsymbol{p}_h)\|_{\mathcal{T}_h} + \|c^{1/2}\,(\boldsymbol{r} - \boldsymbol{r}_h)\|_{\mathcal{T}_h} \lesssim h^{l+1}\left(|\boldsymbol{p}|_{l+1,\Omega} + |\boldsymbol{r}|_{l+1,\Omega} + |z|_{l+1,\Omega} + |y|_{l+1,\Omega}\right). \tag{4.20b}$$

Proof. First of all, by the definition (4.12) and the approximation properties if the HDG projection (cf. Section 4.1), we have that

$$\mathbb{T}_{\boldsymbol{p},y} \lesssim h^{l+1}\left(|\boldsymbol{p}|_{l+1,\Omega} + |y|_{l+1,\Omega}\right)$$

and a similar expression is obtained for $\mathbb{T}_{\boldsymbol{r},z}$. Moreover, $\|I_W y\|_{\mathcal{T}_h} + \|I_W z\|_{\mathcal{T}_h} \lesssim h^{l+1}\left(|y|_{l+1,\Omega} + |z|_{l+1,\Omega}\right)$. Then, since $H_2(R,h)$ and $R_\tau\,h^{1/2}$ can be bounded above by a constant independent of h, (4.19a) implies

$$\sqrt{\alpha}\,\|\varepsilon_h^y\|_{\mathcal{T}_h} + \|\varepsilon_h^z\|_{\mathcal{T}_h} \lesssim h^{l+1}\left(|\boldsymbol{p}|_{l+1,\Omega} + |\boldsymbol{r}|_{l+1,\Omega} + |z|_{l+1,\Omega} + |y|_{l+1,\Omega}\right) + \|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp}.$$

By a scaling argument and the properties of the HDG projection, we can show that

$$\|I_W y\|_{\Gamma_h,h^\perp} + \|I_W z\|_{\Gamma_h,h^\perp} \lesssim h^{l+1}\left(|y|_{l+1,\Omega} + |z|_{l+1,\Omega}\right).$$

Thus, since $y - y_h = e^y + I_W y$ and $z - z_h = e^z + I_W z$, (4.20a) follows. Finally, (4.20b) can deduced from (4.19b) by considering similar arguments. □

We end this section mentioning that error estimates over the entire domain Ω con be obtained thanks to Corollary 4.5.1 and [18, Lemma 3.7] , that is,

$$\|\boldsymbol{p} - \boldsymbol{p}_h\|_\Omega + \|\boldsymbol{r} - \boldsymbol{r}_h\|_\Omega + \|y - y_h\|_\Omega + \|z - z_h\|_\Omega \lesssim h^{l+1}.$$

5 Numerical experiments

In this section we present numerical experiments to validate the theoretical orders of convergence of the approximation provided by the HDG method in the two-dimensional case. For all the computations we consider the spaces specified in (3.2) with $k \in \{0, 1, 2, 3\}$ and the exact solutions $y = \sin(x + y)$ and $z = \exp(x + y)$. We fix $\alpha = 1$, $a = 1$ and $\tau = 1$. We consider two different domains:

Example 1: A circular domain $\Omega := \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 \leq 0.75\}$.

Example 2: a kidney-shaped domain whose boundary satisfies the equation

$$(2\,[(x + 0.5)^2 + y^2] - x - 0.5)^2 - [(x + 0.5)^2 + y^2] + 0.1 = 0.$$

According to Corollary 4.5.1, the theoretical order of convergence for the L^2-norm of the errors in all the variables is $k+1$, as long as Assumption (A.1) - (A.4) hold true. This is what we actually observe in the numerical experiments. More precisely, we have performed numerical simulations where Γ_h is a piece-wise linear interpolation of Γ by a piece-wise. In this case, the distance between Γ and Γ_h is of order h^2, R is proportional to h and therefore the set of assumptions is valid for h sufficiently small. The results (not reported here) showed the optimal convergence rate predicted by Corollary 4.5.1.

On the other hand, we do report a more interesting and practical situation where the computational domain is constructed by "embedding" Ω in a background mesh and considering Ω_h as the union of the elements lying completely inside of Ω as depicted in Figures 1 and 2. In this setting, R is of order one and we cannot guaranty (A.1) holds. However, as we observe in Tables 1 (Example 1) and Tables 3 (Example 2), the order of convergence in all the variables is still $k+1$.

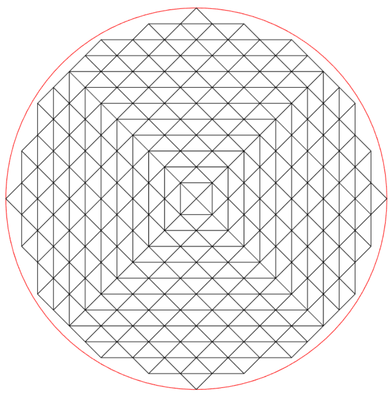

Figure 1: Representation of a circle domain.

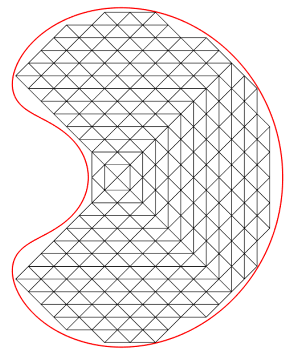

Figure 2: Representation of a kidney-shaped domain.

6 Conclusions

We have analyzed a high order HDG method for an optimal control problem governed by a second order elliptic partial differential equation in a curved domain Ω approximated by a polygonal/polyhedral subdomain. We theoretically showed that, if the distance between Γ and the computational boundary Γ_h is proportional to $h^{\delta+1}$, with $\delta > 0$, the method is well-posed and provides optimal order of convergence $k+1$ for all the variables. This convergence rate is also observed experimentally even for the case $\delta = 0$. This result is consistent with the estimates obtained in previous works on boundary value problems. To the best of our knowledge, this is the first contribution to control problems by using a boundary data transferring technique for unfitted computational domains. The optimal performance of the method, validated theoretically and experimentally, constitutes a stepping stone to deal with shape optimization problems.

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k	N	e_y	order	$e_{\boldsymbol{p}}$	order	$e_{\widehat{y}}$	order
0	16	$2.04E-02$	$-$	$8.45E-02$	$-$	$2.25E-02$	$-$
	96	$1.60E-02$	0.27	$4.44E-02$	0.72	$8.60E-03$	1.07
	400	$8.58E-03$	0.88	$2.39E-02$	0.87	$4.26E-03$	0.99
	1680	$4.75E-03$	0.82	$1.22E-02$	0.93	$1.93E-03$	1.10
	7000	$2.57E-03$	0.86	$6.27E-03$	0.94	$8.77E-04$	1.11
	28504	$1.33E-03$	0.94	$3.18E-03$	0.97	$4.21E-04$	1.05
1	16	$4.65E-03$	$-$	$1.55E-02$	$-$	$2.24E-03$	$-$
	96	$1.09E-03$	1.63	$2.56E-03$	2.01	$1.99E-04$	2.70
	400	$2.80E-04$	1.90	$6.53E-04$	1.92	$5.31E-05$	1.85
	1680	$6.97E-05$	1.94	$1.53E-04$	2.02	$1.08E-05$	2.22
	7000	$1.68E-05$	1.99	$3.42E-05$	2.10	$1.15E-06$	3.14
	28504	$4.15E-06$	1.99	$8.16E-06$	2.04	$1.25E-07$	3.16
2	16	$2.59E-04$	$-$	$3.15E-04$	$-$	$4.88E-05$	$-$
	96	$3.08E-05$	2.38	$4.82E-05$	2.10	$5.20E-06$	2.50
	400	$4.00E-06$	2.86	$8.10E-06$	2.50	$1.22E-06$	2.03
	1680	$5.15E-07$	2.86	$1.20E-06$	2.66	$1.90E-07$	2.59
	7000	$5.92E-08$	3.03	$9.85E-08$	3.50	$9.99E-09$	4.13
	28504	$7.22E-09$	3.00	$9.64E-09$	3.31	$5.00E-10$	4.27
3	16	$1.60E-05$	$-$	$7.28E-05$	$-$	$1.50E-05$	$-$
	96	$6.64E-07$	3.55	$3.38E-06$	3.43	$4.65E-07$	3.88
	400	$5.51E-08$	3.49	$2.74E-07$	3.52	$4.34E-08$	3.32
	1680	$4.06E-09$	3.63	$2.09E-08$	3.59	$3.40E-09$	3.55
	7000	$1.42E-10$	4.70	$6.95E-10$	4.77	$8.26E-11$	5.21
	28504	$7.02E-12$	4.28	$2.58E-11$	4.69	$2.24E-12$	5.14

Table 1: History of convergence history of the error in y, $\boldsymbol{p}$ and $\widehat{y}$ for the circular domain Ω.

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k	N	e_z	order	$e_{\boldsymbol{r}}$	order	$e_{\widehat{z}}$	order
0	16	$5.24E-02$	–	$2.16E-01$	–	$6.32E-02$	–
	96	$4.87E-02$	0.08	$1.14E-01$	0.71	$1.70E-02$	1.47
	400	$2.53E-02$	0.92	$5.76E-02$	0.96	$8.07E-03$	1.04
	1680	$1.36E-02$	0.86	$2.85E-02$	0.98	$3.41E-03$	1.20
	7000	$7.34E-03$	0.87	$1.43E-02$	0.97	$1.36E-03$	1.29
	28504	$3.78E-03$	0.94	$7.15E-03$	0.99	$6.13E-04$	1.14
1	16	$1.02E-02$	–	$2.20E-02$	–	$4.77E-03$	–
	96	$2.19E-03$	1.72	$4.49E-03$	1.77	$5.27E-04$	2.46
	400	$5.74E-04$	1.88	$1.23E-03$	1.82	$1.59E-04$	1.68
	1680	$1.42E-04$	1.95	$3.02E-04$	1.96	$3.64E-05$	2.06
	7000	$3.24E-05$	2.07	$6.33E-05$	2.19	$3.98E-06$	3.10
	28504	$7.80E-06$	2.03	$1.46E-05$	2.09	$4.33E-07$	3.16
2	16	$5.95E-04$	–	$1.32E-03$	–	$4.37E-04$	–
	96	$5.51E-05$	2.66	$1.59E-04$	2.36	$2.20E-05$	3.34
	400	$8.16E-06$	2.68	$2.54E-05$	2.57	$4.61E-06$	2.19
	1680	$1.10E-06$	2.79	$3.82E-06$	2.64	$6.84E-07$	2.66
	7000	$1.05E-07$	3.29	$2.94E-07$	3.59	$3.61E-08$	4.12
	28504	$1.20E-08$	3.09	$2.55E-08$	3.48	$1.84E-09$	4.24
3	16	$3.04E-05$	–	$8.54E-05$	–	$2.56E-05$	–
	96	$1.52E-06$	3.34	$6.18E-06$	2.93	$1.07E-06$	3.54
	400	$1.40E-07$	3.35	$6.11E-07$	3.24	$1.14E-07$	3.14
	1680	$1.18E-08$	3.44	$6.15E-08$	3.20	$1.04E-08$	3.34
	7000	$3.85E-10$	4.80	$2.12E-09$	4.72	$2.68E-10$	5.13
	28504	$1.73E-11$	4.42	$7.42E-11$	4.78	$7.73E-12$	5.05

Table 2: History of convergence history of the error in z, $\boldsymbol{r}$ and $\widehat{z}$ for the circular domain Ω.

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k	N	e_y	order	$e_{\boldsymbol{p}}$	order	$e_{\widehat{y}}$	order
0	22	$3.37E-02$	$-$	$8.05E-02$	$-$	$1.96E-02$	$-$
	85	$1.74E-02$	0.97	$4.75E-02$	0.78	$1.06E-02$	0.91
	370	$9.61E-03$	0.81	$2.60E-02$	0.82	$4.82E-03$	1.07
	1576	$5.21E-03$	0.85	$1.34E-02$	0.92	$2.25E-03$	1.05
	6545	$2.72E-03$	0.91	$6.89E-03$	0.93	$1.07E-03$	1.04
	26590	$1.39E-03$	0.96	$3.48E-03$	0.97	$5.28E-04$	1.01
1	22	$4.26E-03$	$-$	$9.68E-03$	$-$	$1.07E-03$	$-$
	85	$1.13E-03$	1.97	$2.72E-03$	1.88	$2.91E-04$	1.93
	370	$2.71E-04$	1.93	$6.81E-04$	1.88	$5.20E-05$	2.34
	1576	$6.66E-05$	1.94	$1.48E-04$	2.11	$7.25E-06$	2.72
	6545	$1.66E-05$	1.96	$3.35E-05$	2.08	$7.19E-07$	3.25
	26590	$4.13E-06$	1.98	$8.12E-06$	2.02	$9.28E-08$	2.92
2	22	$2.51E-04$	$-$	$3.03E-04$	$-$	$4.97E-05$	$-$
	85	$3.54E-05$	2.90	$6.87E-05$	2.20	$1.78E-05$	1.52
	370	$3.94E-06$	2.98	$7.71E-06$	2.97	$1.10E-06$	3.79
	1576	$4.75E-07$	2.92	$8.10E-07$	3.11	$9.57E-08$	3.37
	6545	$5.75E-08$	2.97	$8.66E-08$	3.14	$5.43E-09$	4.03
	26590	$7.13E-09$	2.98	$9.72E-09$	3.12	$4.27E-10$	3.63
3	22	$9.54E-06$	$-$	$2.67E-05$	$-$	$6.00E-06$	$-$
	85	$1.24E-06$	3.02	$4.66E-06$	2.58	$1.11E-06$	2.50
	370	$5.68E-08$	4.19	$4.26E-07$	3.25	$4.86E-08$	4.25
	1576	$2.58E-09$	4.27	$1.77E-08$	4.39	$1.91E-09$	4.47
	6545	$1.14E-10$	4.38	$6.33E-10$	4.68	$4.34E-11$	5.31
	26590	$6.81E-12$	4.02	$2.87E-11$	4.41	$1.65E-12$	4.67

Table 3: History of convergence history of the error in y, $\boldsymbol{p}$ and $\widehat{y}$ for the kidney-shaped domain Ω.

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k	N	e_z	order	$e_{\boldsymbol{r}}$	order	$e_{\widehat{z}}$	order
0	22	$1.13E-01$	$-$	$2.38E-01$	$-$	$3.10E-02$	$-$
	85	$5.68E-02$	1.02	$1.28E-01$	0.92	$1.65E-02$	0.94
	370	$3.32E-02$	0.73	$6.60E-02$	0.90	$5.63E-03$	1.46
	1576	$1.75E-02$	0.88	$3.26E-02$	0.97	$2.13E-03$	1.34
	6545	$9.10E-03$	0.92	$1.63E-02$	0.97	$8.30E-04$	1.32
	26590	$4.61E-03$	0.97	$8.15E-03$	0.99	$3.71E-04$	1.15
1	22	$1.07E-02$	$-$	$2.07E-02$	$-$	$3.85E-03$	$-$
	85	$3.02E-03$	1.87	$6.87E-03$	1.63	$1.35E-03$	1.55
	370	$6.30E-04$	2.13	$1.34E-03$	2.22	$1.28E-04$	3.20
	1576	$1.52E-04$	1.96	$3.07E-04$	2.03	$2.21E-05$	2.42
	6545	$3.66E-05$	2.00	$6.98E-05$	2.08	$2.45E-06$	3.09
	26590	$9.04E-06$	1.99	$1.69E-05$	2.03	$4.08E-07$	2.56
2	22	$5.54E-04$	$-$	$1.22E-03$	$-$	$2.71E-04$	$-$
	85	$1.03E-04$	2.49	$3.20E-04$	1.98	$7.79E-05$	1.85
	370	$8.51E-06$	3.39	$2.54E-05$	3.45	$3.85E-06$	4.09
	1576	$1.00E-06$	2.95	$2.65E-06$	3.12	$3.72E-07$	3.23
	6545	$1.13E-07$	3.07	$2.59E-07$	3.27	$1.89E-08$	4.18
	26590	$1.39E-08$	2.99	$2.80E-08$	3.17	$1.69E-09$	3.44
3	22	$2.55E-05$	$-$	$7.12E-05$	$-$	$1.75E-05$	$-$
	85	$4.05E-06$	2.72	$1.60E-05$	2.21	$3.77E-06$	2.27
	370	$1.15E-07$	4.85	$6.64E-07$	4.33	$8.25E-08$	5.20
	1576	$6.58E-09$	3.94	$3.34E-08$	4.13	$4.37E-09$	4.06
	6545	$2.99E-10$	4.34	$1.51E-09$	4.35	$1.14E-10$	5.12
	26590	$1.86E-11$	3.97	$7.50E-11$	4.28	$6.62E-12$	4.06

Table 4: History of convergence history of the error in z, $\boldsymbol{r}$ and $\widehat{z}$ for the kidney-shaped domain Ω.

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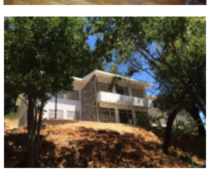

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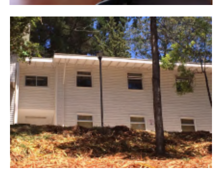
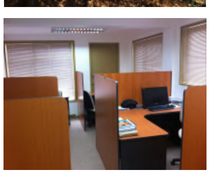

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