UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Centro de Investigación en Ingeniería Matemática

A stabilized finite element method for the Stokes–Temperature coupled problem

Rodolfo Araya, Cristian Cárcamo,
Abner Poza

PREPRINT 2022-27

SERIE DE PRE-PUBLICACIONES

A stabilized finite element method for the Stokes–Temperature coupled problem

Rodolfo Araya[a,*], Cristian Cárcamo[a], Abner H. Poza[b]

[a]*Departamento de Ingeniería Matemática & CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile*
[b]*Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile*

Abstract

In this work, we introduce and analyze a new stabilized finite element scheme for the Stokes–Temperature coupled problem. This new scheme allows equal order of interpolation to approximate the quantities of interest, i.e. velocity, pressure, temperature, and stress. We analyze an equivalent variational formulation of the coupled problem inspired by the ideas proposed in [3]. The existence of the discrete solution is proved, decoupling the proposed stabilized scheme and using the help of continuous dependence results and Brouwer's theorem under the standard assumption of sufficiently small data. Optimal convergence is proved under classic regularity assumptions of the solution. Finally, we present some numerical examples to show the quality of our scheme, in particular, we compare our results with those coming from a standard reference in geosciences described in [38].

Keywords: Coupled Stokes–Temperature problem, stabilized finite element method, a priori error analysis.

1. Introduction

The thermal structure of subduction zones [36] is an interesting problem that plays a central role in the definition of friction zones between continental and ocean crust. In particular, the relation between the temperature distribution at the interior of the crust and the generation of phenomena such as earthquaks or volcano eruptions are still open problems. This relation explains the interest, over the last decade, of the geophysicists, geologists, and practitioners to apply different numerical techniques to compute the distribution of the temperature field [38, 40, 43].

The mantle dynamics can be described as the thermal convection of an incompressible Boussinesq fluid with an infinite Prandtl number. In that respect, we can mention, for example, the works [1, 7, 13, 24, 33], and the references therein, concerning the numerical computation of an approximated solution. A different approach, to study the dynamics of subduction zones, is to consider the flow of a fluid modeled by the Stokes equation whose viscosity depends on temperature, which is given by a transport equation with a convective term defined by the fluid velocity (for details, see [27, 39]). In a different context, this type of coupling also appears in sedimentation-consolidation of particles processes, where the variable of interest is the local solids concentration instead of temperature (see [2, 3, 20] and the references therein).

For the coupling of Stokes-Temperature problems, we can consider finite element approximations based on stable spaces, as in [22], where a discrete scheme based on continuous and discontinuous piecewise polynomial spaces was analyzed. Also, we mention the works [3, 20] where the authors introduce an augmented mixed-primal dual formulation, and where the numerical analysis is carried out using stable finite element spaces of the Raviart–Thomas type. When the spaces of approximations are not stable, it is known that new

*Corresponding author
Email addresses: rodolfo.araya@udec.cl (Rodolfo Araya), ccarcamo@udec.cl (Cristian Cárcamo), apoza@ucsc.cl (Abner H. Poza)

Preprint submitted to Elsevier *October 5, 2022*

terms should be added to Galerkin's formulation to get stability, these kind of methods are known in the literature as stabilized methods. A small list of classical stabilized schemes can be SUPG (streamline-upwind-Petrov-Galerkin), SDFEM (streamline diffusion), or GLS (Galerkin-least-squares) methods (see, for instance, [14, 15, 25, 37]). When projection terms of the residual type are added to the stabilized formulation we have, for example, [10] for the Stokes equations, [26] for the Darcy equations, and [4] for the Navier–Stokes equations. In the case of not residual-based stabilization, we can mention [6, 11, 12, 17, 21], which is clearly an incomplete list. Finally, when the viscosity of the fluid is not constant, as in viscoelastic fluids, we can mention [9, 18, 19, 32, 41, 42] and the references therein.

The purpose of this paper is to present and analyze a new stabilized finite element method to approximate the Stokes equation coupled with the convection-diffusion equation allowing the use of equal order of interpolation for each variable. Our work is based on similar arguments as in [3, 20] for the existence of a solution of the continuous formulation, in the sense of proposing a nonlinear variational formulation that is well posed using, a decoupled problem, and Banach's and Schauder's fixed point theorems with a compactness results from Rellich–Kondrashov. The uniqueness of the continuous solution is stated under the standard assumption of sufficiently small data and an additional regularity hypothesis of the continuous solution.

In our approach, the discrete formulation is based on a stabilized scheme for the Stokes, inspired by the scheme proposed and analyzed in [25], and for the transport equation, we consider a variation of the scheme proposed in [30]. In both methodologies, residual mesh-dependent terms of the momentum equations are added, which allows the stabilized method proposed to be consistent. Again, based on the assumption of small data, the existence of the discrete solution can be stated using Brouwer's fixed point theorem. Optimal convergence of the proposed scheme is proved using standard nonlinear finite element results (for a similar result of a stabilized scheme applied to a nonlinear Darcy equation, see [5]).

This work is organized as follows: in Section 2 we introduce the model problem and some preliminary results that we will use in the sequel. In Section 3 we present the continuous variational problem and show that it is well-possed. The stabilized numerical scheme is presented in Section 4 where results concerning the existence of the solution are introduced. In Section 5 the convergence of the discrete stabilized scheme is introduced jointly with an a priori error analysis. Finally, in Section 6 we present some numerical experiments to assess the quality of our new scheme, using both analytical solutions and a solution coming from the benchmark experiment introduced in [38].

2. Model problem and preliminary results

Let $\Omega \subset \mathbb{R}^d$, $d \in \{2, 3\}$, be an open, bounded domain with Lipschitz continuous boundary $\partial\Omega$. We will use standard notation for Lebesgue spaces $L^q(\Omega)$, with norm $\|\cdot\|_{0,q,\Omega}$, for $q > 2$, and $\|\cdot\|_{0,\Omega}$ for $q = 2$ and inner product $(\cdot,\cdot)$, and Sobolev spaces $H^m(\Omega)$, with norm $\|\cdot\|_{m,\Omega}$ and semi-norm $|\cdot|_{m,\Omega}$.

Inspired by a model for the dynamics of the thermal structure of subduction zones [38], we consider the Stokes–Temperature coupled problem given by: *Find the velocity $\boldsymbol{u}$, pressure p, stress $\boldsymbol{\sigma}$ and temperature ϕ such that*

$$
(\mathrm{P}) \quad \left\{
\begin{array}{rll}
\boldsymbol{\sigma} & = 2\mu(\phi)\,\boldsymbol{\varepsilon}(\boldsymbol{u}) & \text{in } \Omega,\\
-\nabla\cdot\boldsymbol{\sigma} + \nabla p & = \alpha\phi\boldsymbol{f} & \text{in } \Omega,\\
\nabla\cdot\boldsymbol{u} & = 0 & \text{in } \Omega,\\
\boldsymbol{u} & = \boldsymbol{0} & \text{on } \partial\Omega,\\
-k\,\Delta\phi + \nabla\phi\cdot\boldsymbol{u} & = g & \text{in } \Omega,\\
\phi & = 0 & \text{on } \partial\Omega,
\end{array}
\right.
$$

where $\boldsymbol{\varepsilon}(\boldsymbol{u}) := \dfrac{1}{2}\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^t\right)$ is the strain rate tensor, μ is the effective dynamic viscosity and $k \in \mathbb{R}^+$ is the thermal conductivity. Additionally, we suppose that there exist positive constants $\mu_{\min}$, $\mu_{\max}$ and C_{Lips}, such that

$$
0 < \mu_{\min} \le \mu(s) \le \mu_{\max} \qquad \forall s \in \mathbb{R}, \qquad |\mu(r) - \mu(s)| \le C_{\text{Lips}}\,|r - s| \qquad \forall r, s \in \mathbb{R}. \tag{2.1}
$$

Finally, we assume that $\boldsymbol{f} \in L^\infty(\Omega)^d$, $g \in L^2(\Omega)$ and α, the thermal Rayleigh number, is a positive constant.

To introduce a variational formulation of problem (P), we need the following Hilbert spaces: $\boldsymbol{H} := H_0^1(\Omega)^d$, $Q := L_0^2(\Omega)$, $\boldsymbol{R} := \{\boldsymbol{\tau} \in L^2(\Omega)^{d\times d} : \boldsymbol{\tau}^T = \boldsymbol{\tau}\}$, and $V := H_0^1(\Omega)$. Thus, a variational formulation associated to (P) is given by: *Find* $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$ *such that*

$$B((\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi), (\boldsymbol{v}, q, \boldsymbol{\tau}, \psi)) = F(\boldsymbol{v}, q, \boldsymbol{\tau}, \psi), \tag{2.2}$$

for all $(\boldsymbol{v}, q, \boldsymbol{\tau}, \psi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$, where

$$\begin{aligned} B((\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi), (\boldsymbol{v}, q, \boldsymbol{\tau}, \psi)) := &\left(\frac{\boldsymbol{\sigma}}{2\mu(\phi)}, \boldsymbol{\tau}\right) - (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\tau}) + (q, \nabla\cdot\boldsymbol{u}) - (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v})) \\ &+ (p, \nabla\cdot\boldsymbol{v}) + k\,(\nabla\phi, \nabla\psi) + (\nabla\phi\cdot\boldsymbol{u}, \psi), \end{aligned}$$

for all $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi), (\boldsymbol{v}, q, \boldsymbol{\tau}, \psi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$ and

$$F(\boldsymbol{v}, q, \boldsymbol{\tau}, \psi) := -\,(\alpha\phi\boldsymbol{f}, \boldsymbol{v}) + (g, \psi),$$

for all $(\boldsymbol{v}, q, \boldsymbol{\tau}, \psi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$.

Remark 1. *If* $\boldsymbol{u}$ *is part of the solution of problem* (2.2)*, then* $\nabla\cdot\boldsymbol{u} = 0$*. In fact, if we take* $\boldsymbol{v} = \boldsymbol{0}$*,* $\boldsymbol{\tau} = \boldsymbol{0}$ *and* $\psi = 0$ *in* (2.2)*, we obtain that* $(q, \nabla\cdot\boldsymbol{u}) = 0$ *for all* $q \in L_0^2(\Omega)$*. Now, using that* $\boldsymbol{u} = \boldsymbol{0}$ *on* $\partial\Omega$ *we get that* $\nabla\cdot\boldsymbol{u} = 0$*.*

In the sequel we will need the following results:

Lemma 2.1 (Korn)**.** *There exist a positive constant* C_K*, depending on* Ω*, such that*

$$\|\boldsymbol{v}\|_{1,\Omega} \le C_K\, \|\boldsymbol{\varepsilon}(\boldsymbol{v})\|_{0,\Omega},$$

for all $\boldsymbol{v} \in \boldsymbol{H}$.

Lemma 2.2 (Poincaré)**.** *There exists a positive constant* C_P*, depending on* Ω*, such that*

$$\|\psi\|_{1,\Omega} \le C_P\, |\psi|_{1,\Omega},$$

for all $\psi \in V$*.*

Lemma 2.3 (Rellich–Kondrashov)**.** *For* $q \ge 1$*, if* $d = 2$*, or* $1 \le q < 6$*, if* $d = 3$*, we have the compact inclusion* $H^1(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$*, thus there exists a positive constant* C_q *such that*

$$\|v\|_{0,q,\Omega} \le C_q\, \|v\|_{1,\Omega} \qquad \forall v \in H^1(\Omega).$$

Proof. See, for instance, [34, Theorem 1.3.5]. □

Theorem 2.4 (Banach)**.** *Let* (X, d) *be a complete metric space and* $f : X \to X$ *be a contractive operator, i.e. there exists* $\lambda \in (0,1)$ *such that* $d(f(x), f(y)) \le \lambda\, d(x,y)$ *for any* $x, y \in X$*. Then* f *has a unique fixed point.*

Proof. See [8]. □

Theorem 2.5 (Schauder)**.** *Let* W *be a closed and convex subset of a Banach space* X *and let* $f : W \to W$ *be a continuous function such that* $\overline{f(W)}$ *is compact. Then* f *has at least one fixed point.*

Proof. See [35]. □

Theorem 2.6 (Brouwer)**.** *Let* W *be a compact and convex subset of a finite-dimensional Banach space* V*, and let* $f : W \to W$ *a continuous function. Then* f *has at least one fixed point.*

Proof. See [16]. □

Over the space $\boldsymbol{H} \times Q \times \boldsymbol{R}$ we will use the following norm

$$|||(\boldsymbol{v}, q, \boldsymbol{\tau})||| := \left\{\|\boldsymbol{v}\|_{1,\Omega}^2 + \|q\|_{0,\Omega}^2 + \|\boldsymbol{\tau}\|_{0,\Omega}^2\right\}^{1/2} \quad \forall(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}.$$

From now on, C and C_k will denote positive constants independent of the mesh size h, but possibly depending on physical parameters.

3. Equivalent variational formulation

In this section we will introduce and analyze a variational formulation equivalent to (2.2). To this end, we need to define the following variational problems: Given $\psi \in V$: *Find* $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$ *such that*

$$B_\psi((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})) = F_\psi(\boldsymbol{v}, q, \boldsymbol{\tau}) \qquad \forall(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}, \tag{3.1}$$

where $B_\psi : (\boldsymbol{H} \times Q \times \boldsymbol{R}) \times (\boldsymbol{H} \times Q \times \boldsymbol{R}) \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$B_\psi((\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau})) := \left(\frac{\boldsymbol{\sigma}}{2\mu(\psi)}, \boldsymbol{\tau}\right) - (\boldsymbol{\varepsilon}(\boldsymbol{u}), \boldsymbol{\tau}) + (q, \nabla \cdot \boldsymbol{u}) - (\boldsymbol{\sigma}, \boldsymbol{\varepsilon}(\boldsymbol{v})) + (p, \nabla \cdot \boldsymbol{v}),$$

for all $(\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$, and $F_\psi : \boldsymbol{H} \times Q \times \boldsymbol{R} \longrightarrow \mathbb{R}$ is the linear functional defined by

$$F_\psi(\boldsymbol{v}, q, \boldsymbol{\tau}) := -(\alpha\psi \boldsymbol{f}, \boldsymbol{v}),$$

for all $(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$.
On the other hand, given $\boldsymbol{w} \in \boldsymbol{H}$, with $\nabla \cdot \boldsymbol{w} = 0$, we define the problem: *Find* $\varphi \in V$ *such that*

$$A_{\boldsymbol{w}}(\varphi, \psi) = G(\psi) \qquad \forall\psi \in V, \tag{3.2}$$

where $A_{\boldsymbol{w}} : V \times V \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$A_{\boldsymbol{w}}(\phi, \psi) := k\,(\nabla\phi, \nabla\psi) + (\nabla\phi \cdot \boldsymbol{w}, \psi),$$

for all $\phi, \psi \in V$, and $G : V \longrightarrow \mathbb{R}$ is the linear functional defined by

$$G(\psi) := (g, \psi),$$

for all $\psi \in V$.

Note that the nonlinear scheme (2.2) can be rewritten as: *Find* $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$ *such that*

$$B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau})) = F_\phi(\boldsymbol{v}, q, \boldsymbol{\tau}) \qquad \forall(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}, \tag{3.3}$$

$$A_{\boldsymbol{u}}(\phi, \psi) = G(\psi) \qquad \forall\psi \in V. \tag{3.4}$$

3.1. Well possedness of the variational formulation

In this section, based on the arguments introduced in [3], we prove the existence and uniqueness of the solution of variational problem (3.1), using Schauder's and Banach's fixed point theorems (see theorems 2.5 and 2.4). To this end we need to define the following operators: $\mathbb{S} : V \longrightarrow \boldsymbol{H} \times Q \times \boldsymbol{R}$, such that

$$\psi \longmapsto \mathbb{S}(\psi) = (\mathbb{S}^1(\psi), \mathbb{S}^2(\psi), \mathbb{S}^3(\psi)) := (\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}),$$

where $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$ is the solution of (3.1). Let $\tilde{\boldsymbol{H}} := \{\boldsymbol{w} \in \boldsymbol{H} \ : \ \nabla \cdot \boldsymbol{w} = 0\}$, we define the operator $\mathbb{M} : \tilde{\boldsymbol{H}} \longrightarrow V$, by

$$\boldsymbol{w} \longmapsto \mathbb{M}(\boldsymbol{w}) = \varphi,$$

where $\varphi \in V$ is the solution of (3.2).
Finally, we define the operator $\mathbb{T} : V \longrightarrow V$ by

$$\mathbb{T}(\varphi) := \mathbb{M}(\mathbb{S}^1(\varphi)), \tag{3.5}$$

for all $\varphi \in V$. In this way, problem (3.3)–(3.4) can be written as the following fixed point problem: *Find $\phi \in V$ such that*

$$\mathbb{T}(\phi) = \phi. \tag{3.6}$$

Remark 2. *All these operators are well defined as we will see in lemmas 3.1 and 3.2. Furthermore, since problem* (2.2) *is equivalent to the fixed-point problem* (3.6)*, our goal will be to prove that the operator* $\mathbb{T}$ *satisfies the hypotheses of Schauder's and Banach's theorems.*

Lemma 3.1. *Problem* (3.1) *has a unique solution* $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$. *Moreover, there is a positive constant* $C_{\mathbb{S}}$ *such that*

$$|||\mathbb{S}(\psi)||| = |||(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}})||| \leq C_{\mathbb{S}} \, \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\psi\|_{1,\Omega} \qquad \forall \psi \in V. \tag{3.7}$$

Proof. Defining the following bounded linear operators:

$$\begin{aligned}
\boldsymbol{A}_1 : \boldsymbol{R} \to \boldsymbol{R}', &\qquad \langle \boldsymbol{A}_1 \boldsymbol{\sigma}, \boldsymbol{\tau} \rangle := \left(\frac{\boldsymbol{\sigma}}{2\mu(\phi)}, \boldsymbol{\tau} \right), \\
\boldsymbol{B}_1 : \boldsymbol{R} \to \boldsymbol{H}', &\qquad \langle \boldsymbol{B}_1 \boldsymbol{\tau}, \boldsymbol{v} \rangle := -(\boldsymbol{\varepsilon}(\boldsymbol{v}), \boldsymbol{\tau}), \\
\boldsymbol{B} : \boldsymbol{H} \to Q', &\qquad \langle \boldsymbol{B} \boldsymbol{v}, q \rangle := (\nabla \cdot \boldsymbol{v}, q), \\
H_\psi : \boldsymbol{H} \to \mathbb{R}, &\qquad \langle H_\psi, \boldsymbol{v} \rangle := -(\alpha \psi \boldsymbol{f}, \boldsymbol{v}),
\end{aligned}$$

problem (3.1) can be expressed as the following system:

$$\begin{pmatrix} \boldsymbol{A}_1 & \boldsymbol{B}_1^T & \boldsymbol{0} \\ \boldsymbol{B}_1 & \boldsymbol{0} & \boldsymbol{B}^T \\ \boldsymbol{0} & \boldsymbol{B} & \boldsymbol{0} \end{pmatrix} \begin{pmatrix} \tilde{\boldsymbol{\sigma}} \\ \tilde{\boldsymbol{u}} \\ \tilde{p} \end{pmatrix} = \begin{pmatrix} \boldsymbol{0} \\ H_\psi \\ \boldsymbol{0} \end{pmatrix}. \tag{3.8}$$

Note that it is clear that

$$\langle \boldsymbol{A}_1 \boldsymbol{\tau}, \boldsymbol{\tau} \rangle \geq \frac{1}{2\mu_{max}} \|\boldsymbol{\tau}\|_{0,\Omega}^2 \qquad \forall \boldsymbol{\tau} \in \boldsymbol{R}, \tag{3.9}$$

and

$$\sup_{\substack{\boldsymbol{\tau} \in \boldsymbol{R} \\ \boldsymbol{\tau} \neq \boldsymbol{0}}} \frac{\langle \boldsymbol{B}_1 \boldsymbol{\tau}, \boldsymbol{v} \rangle}{\|\boldsymbol{\tau}\|_{0,\Omega}} \geq \frac{1}{C_K} \|\boldsymbol{v}\|_{1,\Omega} \qquad \forall \boldsymbol{v} \in \boldsymbol{H}. \tag{3.10}$$

Finally, it is well known (see [29]) that there exists $\beta > 0$, such that

$$\sup_{\substack{\boldsymbol{v} \in \boldsymbol{H} \\ \boldsymbol{v} \neq \boldsymbol{0}}} \frac{\langle \boldsymbol{B} \boldsymbol{v}, q \rangle}{\|\boldsymbol{v}\|_{1,\Omega}} \geq \beta \, \|q\|_{0,\Omega} \qquad \forall q \in Q. \tag{3.11}$$

Thus, using (3.9), (3.10), (3.11) and [28, Theorem 2.1], we have that problem (3.1) has a unique solution $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$, and there exists a positive constant $C_{\mathbb{S}}$ such that

$$|||(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}})||| \leq C_{\mathbb{S}} \, \|H_\psi\|_{\boldsymbol{H}'}. \tag{3.12}$$

□

Remark 3. *Note that* (3.12) *is equivalent to the following global inf–sup condition*

$$|||(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}})||| \leq C_{\mathbb{S}} \sup_{\substack{(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \\ (\boldsymbol{v}, q, \boldsymbol{\tau}) \neq (\boldsymbol{0}, 0, \boldsymbol{0})}} \frac{B_\psi((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau}))}{|||(\boldsymbol{v}, q, \boldsymbol{\tau})|||}, \tag{3.13}$$

for all $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$.

Lemma 3.2. *Given* $\boldsymbol{w} \in \tilde{\boldsymbol{H}}$*, problem* (3.2) *has a unique solution* $\varphi \in V$*. Additionally, there exists a positive constant* $C_{\mathbb{M}}$*, such that*

$$\|\mathbb{M}(\boldsymbol{w})\|_{1,\Omega} = \|\varphi\|_{1,\Omega} \le C_{\mathbb{M}} \, \|g\|_{0,\Omega}. \tag{3.14}$$

Proof. The proof is a direct application of the Lax–Milgram's lemma using the fact that $A_{\boldsymbol{w}}$ is a coercive bilinear form, i.e.

$$\frac{\kappa}{C_P^2} \|\varphi\|_{1,\Omega}^2 \le A_{\boldsymbol{w}}(\varphi, \varphi) \qquad \forall \varphi \in V. \tag{3.15}$$

□

Lemma 3.3. *Let* W *the subset of* V *defined by*

$$W := \{\varphi \in V : \ \|\varphi\|_{1,\Omega} \le r\},$$

where $r := C_{\mathbb{M}} \, \|g\|_{0,\Omega}$*, then* $\mathbb{T}(W) \subseteq W$*.*

Proof. The proof is a direct consequence of the definition of operator $\mathbb{T}$, given in (3.5), and (3.14). □

Lemma 3.4. *There exists a positive constant* C_1*, such that for all* $\phi, \psi \in V$*, we have*

$$|\!|\!|\mathbb{S}(\phi) - \mathbb{S}(\psi)|\!|\!| \le C_1 \, \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\phi - \psi\|_{0,\Omega} + \|\mathbb{S}^3(\psi)\|_{0,4,\Omega} \|\phi - \psi\|_{0,4,\Omega} \right\}.$$

Proof. Let $\phi, \psi \in V$ given, we define $(\boldsymbol{u}, p, \boldsymbol{\sigma}) := \mathbb{S}(\phi)$ and $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) := \mathbb{S}(\psi)$ by

$$\begin{aligned} B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau})) &= F_\phi(\boldsymbol{v}, q, \boldsymbol{\tau}), \\ B_\psi((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})) &= F_\psi(\boldsymbol{v}, q, \boldsymbol{\tau}), \end{aligned}$$

for all $(\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q \times \boldsymbol{R}$. Then, we have

$$\begin{aligned} B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}) - (\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})) &= B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau})) - B_\phi((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})) \\ &= (F_\phi - F_\psi)(\boldsymbol{v}, q, \boldsymbol{\tau}) + (B_\psi - B_\phi)((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})). \end{aligned}$$

Now, using the definition of F_ϕ and F_ψ, we obtain

$$(F_\phi - F_\psi)(\boldsymbol{v}, q, \boldsymbol{\tau}) = (\alpha(\phi - \psi)\boldsymbol{f}, \boldsymbol{v}) \ \le \alpha \, \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\phi - \psi\|_{0,\Omega} \, \|\boldsymbol{v}\|_{0,\Omega}. \tag{3.16}$$

On the other hand, using Hölder's inequality and (2.1), we have that

$$\begin{aligned} (B_\psi - B_\phi)((\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau})) &= \left(\left(\frac{1}{2\mu(\psi)} - \frac{1}{2\mu(\phi)} \right) \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau} \right) \\ &= \left(\left(\frac{\mu(\phi) - \mu(\psi)}{2\mu(\psi)\mu(\phi)} \right) \tilde{\boldsymbol{\sigma}}, \boldsymbol{\tau} \right) \\ &\le \frac{C_{\mathrm{Lips}}}{2\mu_{\max}^2} \|\phi - \psi\|_{0,4,\Omega} \, \|\tilde{\boldsymbol{\sigma}}\|_{0,4,\Omega} \|\boldsymbol{\tau}\|_{0,\Omega}. \end{aligned} \tag{3.17}$$

Thus, using (3.16) and (3.17), we obtain that

$$\frac{B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}) - (\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}), (\boldsymbol{v}, q, \boldsymbol{\tau}))}{|\!|\!|(\boldsymbol{v}, q, \boldsymbol{\tau})|\!|\!|} \le \ \alpha \, \|\boldsymbol{f}\|_{\infty,\Omega} \, \|\phi - \psi\|_{0,\Omega} + \frac{C_{\mathrm{Lips}}}{2\mu_{\max}^2} \, \|\mathbb{S}^3(\psi)\|_{0,4,\Omega} \|\phi - \psi\|_{0,4,\Omega}. \tag{3.18}$$

The result follows from (3.13), (3.18) and Lemma 3.1. □

Lemma 3.5. *There exists a positive constant* C_2*, such that for all* $\boldsymbol{u}, \tilde{\boldsymbol{u}} \in \tilde{\boldsymbol{H}}$*, we have*

$$\|\mathbb{M}(\boldsymbol{u}) - \mathbb{M}(\tilde{\boldsymbol{u}})\|_{1,\Omega} \le C_2 \, \|g\|_{0,\Omega} \, \|\boldsymbol{u} - \tilde{\boldsymbol{u}}\|_{1,\Omega}.$$

Proof. Defining $\phi := \mathbb{M}(\boldsymbol{u})$ and $\tilde{\phi} := \mathbb{M}(\tilde{\boldsymbol{u}})$, we have that for all $\psi \in \boldsymbol{V}$

$$A_{\boldsymbol{u}}(\phi,\psi) = G(\psi), \qquad A_{\tilde{\boldsymbol{u}}}(\tilde{\phi},\psi) = G(\psi).$$

Then, from (3.15), Lemma 3.2, and Lemma 2.3, with $q = 4$, we get

$$\begin{aligned}
\frac{\kappa}{C_P^2}\,\|\phi-\tilde{\phi}\|_{1,\Omega}^2 \le A_{\boldsymbol{u}}(\phi-\tilde{\phi},\phi-\tilde{\phi}) &= A_{\boldsymbol{u}}(\phi,\phi-\tilde{\phi}) - A_{\boldsymbol{u}}(\tilde{\phi},\phi-\tilde{\phi})\\
&= G(\phi-\tilde{\phi}) - G(\phi-\tilde{\phi}) + A_{\tilde{\boldsymbol{u}}}(\tilde{\phi},\phi-\tilde{\phi}) - A_{\boldsymbol{u}}(\tilde{\phi},\phi-\tilde{\phi})\\
&= (\nabla\tilde{\phi}\cdot(\tilde{\boldsymbol{u}}-\boldsymbol{u}),\phi-\tilde{\phi})\\
&\le |\tilde{\phi}|_{1,\Omega}\,\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{0,4,\Omega}\,\|\phi-\tilde{\phi}\|_{0,4,\Omega}\\
&\le C_q^2\,|\tilde{\phi}|_{1,\Omega}\,\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{1,\Omega}\,\|\phi-\tilde{\phi}\|_{1,\Omega}\\
&\le C_q^2\,C_{\mathbb{M}}\,\|g\|_{0,\Omega}\,\|\boldsymbol{u}-\tilde{\boldsymbol{u}}\|_{1,\Omega}\,\|\phi-\tilde{\phi}\|_{1,\Omega},
\end{aligned}$$

and the result follows. □

Lemma 3.6. *There exists a positive constant C_3, such that for all $\phi,\tilde{\phi}\in V$ there holds*

$$\|\mathbb{T}(\phi)-\mathbb{T}(\tilde{\phi})\|_{1,\Omega} \le C_3\,\|g\|_{0,\Omega}\left\{\|\boldsymbol{f}\|_{\infty,\Omega}\,\|\phi-\tilde{\phi}\|_{0,\Omega} + \|\mathbb{S}^3(\tilde{\phi})\|_{0,4,\Omega}\,\|\phi-\tilde{\phi}\|_{0,4,\Omega}\right\}. \tag{3.19}$$

Proof. The result is evident from the definition of $\mathbb{T}$ and lemmas 3.4 and 3.5. □

In the sequel we will assume that problem (3.1), used to define operator $\mathbb{S}$, has an extra regularity in the sense that $\mathbb{S}(\psi) = (\boldsymbol{u},p,\boldsymbol{\sigma}) \in (\boldsymbol{H}\cap H^2(\Omega)^d)\times(Q\cap L^4(\Omega))\times(\boldsymbol{R}\cap L^4(\Omega)^{d\times d})$, and

$$\|\boldsymbol{u}\|_{2,\Omega} + \|p\|_{0,4,\Omega} + \|\boldsymbol{\sigma}\|_{0,4,\Omega} \le C_{\mathbb{S}}\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|\psi\|_{1,\Omega}. \tag{3.20}$$

The next result stated the existence of the solution of problem (2.2), or equivalently (3.3)–(3.4).

Theorem 3.7. *Let W and $r>0$ as in Lemma* 3.3. Under the regularity assumption (3.20), the variational problem (2.2) has at least one solution $(\boldsymbol{u},p,\boldsymbol{\sigma},\phi)\in\boldsymbol{H}\times Q\times\boldsymbol{R}\times V$, with $\phi\in W$, and there holds

$$\|\phi\|_{1,\Omega} \le C_{\mathbb{M}}\,\|g\|_{0,\Omega}, \tag{3.21}$$

and

$$\|(\boldsymbol{u},p,\boldsymbol{\sigma})\| \le C_{\mathbb{S}}\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\|g\|_{0,\Omega}. \tag{3.22}$$

Proof. From lemmas 3.3 and 3.6, and using the fact that $H^1(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$ with $q = 6$, we have that $\mathbb{T}: W \longrightarrow W$ is continuous. On the other hand, let $\{\varphi_n\}_{n\in\mathbb{N}}$ be a sequence in W, which is clearly a bounded sequence in $H_0^1(\Omega)$, then there exists a subsequence $\{\varphi_{n_k}\}_{k\in\mathbb{N}}$ of $\{\varphi_n\}_{n\in\mathbb{N}}$ and an element $\varphi\in H_0^1(\Omega)$ such $\varphi_{n_k} \overset{w}{\to} \varphi$. Now, using again that $H^1(\Omega) \overset{c}{\hookrightarrow} L^q(\Omega)$ for $q = 2$ and $q = 4$, we have that $\varphi_{n_k} \longrightarrow \varphi$ in $L^2(\Omega)$ and $L^4(\Omega)$, respectively. Thus, using Lemma 3.6 we have that $T(\varphi_{n_k}) \longrightarrow T(\varphi)$, which prove that $\overline{T(W)}$ is compact. Now, using Schauder's fixed point Theorem (see Theorem 2.5) we get that there exists $(\boldsymbol{u},p,\boldsymbol{\sigma},\phi)\in\boldsymbol{H}\times Q\times\boldsymbol{R}\times V$ solution of (2.2). Finally, estimates (3.21) and (3.22) are a direct consequence of lemmas 3.1 and 3.2. □

Theorem 3.8. *Let W and $r>0$ as in Lemma 3.3. Assume that the regularity assumption* (3.20) *holds and that the data satisfy*

$$C_3\,\|g\|_{0,\Omega}\,\|\boldsymbol{f}\|_{\infty,\Omega}\,\{1 + C_{\mathbb{S}}\,C_q\,r\} < 1. \tag{3.23}$$

Then the variational problem (2.2) *has a unique solution* $(\boldsymbol{u},p,\boldsymbol{\sigma},\phi)\in\boldsymbol{H}\times Q\times\boldsymbol{R}\times V$.

Proof. Using lemmas 2.3, with $q = 4$, and 3.6, together with the regularity assumption (3.20), we have, for all $\phi, \tilde{\phi} \in W \subseteq V$, that

$$\begin{aligned} \|\mathbb{T}(\phi) - \mathbb{T}(\tilde{\phi})\|_{1,\Omega} &\leq C_3 \|g\|_{0,\Omega} \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} \|\phi - \tilde{\phi}\|_{0,\Omega} + \|\mathbb{S}^3(\tilde{\phi})\|_{0,4,\Omega} \|\phi - \tilde{\phi}\|_{0,4,\Omega} \right\} \\ &\leq C_3 \|g\|_{0,\Omega} \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} + C_q \|\mathbb{S}^3(\tilde{\phi})\|_{0,4,\Omega} \right\} \|\phi - \tilde{\phi}\|_{1,\Omega} \\ &\leq C_3 \|g\|_{0,\Omega} \|\boldsymbol{f}\|_{\infty,\Omega} \{1 + C_q C_{\mathbb{S}} r\} \|\phi - \tilde{\phi}\|_{1,\Omega}. \end{aligned}$$

This last inequality together with condition (3.23) prove that $\mathbb{T} : W \to W$ is a contraction, thus, using Banach's point fixed theorem (see Theorem 2.4), the result follows. □

4. A stabilized finite element method

From now on, we denote by $\{\mathcal{T}_h\}_{h>0}$ a regular family of triangulations of $\bar{\Omega}$ composed by simplexes. For a $\mathcal{T}_h$ we will denote by K the elements of the triangulation. As usual h_K means the diameter of K and $h := \max_{K \in \mathcal{T}_h} h_K$. Also we introduce the following finite element subspaces of $\boldsymbol{H}$, Q, $\boldsymbol{R}$, and V, respectively:

$$\begin{aligned} \boldsymbol{H}_h &:= \left\{ \boldsymbol{v} \in C(\overline{\Omega})^d \ : \ \boldsymbol{v}|_K \in \mathbb{P}_l(K)^d, \quad \forall K \in \mathcal{T}_h \right\} \cap \boldsymbol{H}, \\ Q_h &:= \left\{ q \in C(\overline{\Omega}) \ : \ q|_K \in \mathbb{P}_l(K), \quad \forall K \in \mathcal{T}_h \right\} \cap Q, \\ \boldsymbol{R}_h &:= \left\{ \boldsymbol{\tau} \in C(\overline{\Omega})^{d\times d} \ : \ \boldsymbol{\tau}|_K \in \mathbb{P}_l(K)^{d\times d}, \quad \forall K \in \mathcal{T}_h \right\} \cap \boldsymbol{R}, \\ V_h &:= \left\{ \boldsymbol{v} \in C(\overline{\Omega}) \ : \ \boldsymbol{v}|_K \in \mathbb{P}_l(K), \quad \forall K \in \mathcal{T}_h \right\} \cap V, \end{aligned}$$

with $l \geq 1$, where $\mathbb{P}_l$ stands for the space of polynomials of total degree less or equal to l.

The discrete stabilized scheme analyzed in this work is given by: *Find* $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$ *such that*

$$B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h)) = F_{\mathrm{stab}}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h), \tag{4.1}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$, where

$$\begin{aligned} &B_{\mathrm{stab}}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h)) := \left(\frac{\boldsymbol{\sigma}_h}{2\mu(\phi_h)}, \boldsymbol{\tau}_h \right) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h) + (q_h, \nabla \cdot \boldsymbol{u}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) \\ &+ (p_h, \nabla \cdot \boldsymbol{v}_h) - \beta \left(2\mu(\phi_h) \left(\frac{\boldsymbol{\sigma}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{u}_h) \right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h) \right) \right) \\ &+ \sum_{K \in \mathcal{T}_h} m_l h_K^2 \left(\nabla p_h - \nabla \cdot \boldsymbol{\sigma}_h, \frac{1}{8\mu(\phi_h)} (\nabla q_h - \nabla \cdot \boldsymbol{\tau}_h) \right)_K + k\, (\nabla \phi_h, \nabla \psi_h) + (\nabla \phi_h \cdot \boldsymbol{u}_h, \psi_h) \\ &+ \gamma \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{k} (-k\, \Delta \phi_h + \nabla \phi_h \cdot \boldsymbol{u}_h, k\, \Delta \psi_h + \nabla \psi_h \cdot \boldsymbol{u}_h)_K, \end{aligned}$$

for all $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$ and

$$\begin{aligned} F_{\mathrm{stab}}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h) &:= -\, (\alpha \phi_h \boldsymbol{f}, \boldsymbol{v}_h) + \sum_{K \in \mathcal{T}_h} m_l h_K^2 \left(\alpha \phi_h \boldsymbol{f}, \frac{1}{8\mu(\phi_h)} (\nabla q_h - \nabla \cdot \boldsymbol{\tau}_h) \right)_K \\ &\quad + (g, \psi_h) + \gamma \sum_{K \in \mathcal{T}_h} \frac{h_K^2}{k} (g, k\, \Delta \psi_h + \nabla \psi_h \cdot \boldsymbol{u}_h)_K, \end{aligned}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h, \psi_h) \in \boldsymbol{H}_h \times Q_h \times V_h$, where β and γ are positive constants, and $m_l := \min \left\{ \frac{1}{3}, 2\, C_l \right\}$ with C_l the constant appearing in the following inverse inequality

$$C_l \sum_{K \in \mathcal{T}_h} h_K^2 \|\nabla \cdot \boldsymbol{\sigma}_h\|_{0,K}^2 \leq \|\boldsymbol{\sigma}_h\|_{0,\Omega}^2, \tag{4.2}$$

for all $\boldsymbol{\sigma}_h \in \boldsymbol{R}_h$.

Remark 4. *The nonlinear discrete scheme* (4.1) *is inspired by the stabilized method proposed and analyzed in [25], for the Stokes equation, and in [30], for the transport equation. The analysis in these works is based on the fact that the viscosity is constant and in the case of the transport equation it is considered that the convective term also is constant. For this reason, the arguments that will be used in the next sections to prove the existence of a discrete solution, and the convergence of the discrete scheme, are different from the ones found in these references.*

4.1. Equivalent discrete formulation

In the sequel we will use the following standard results:

Lemma 4.1. *There are positive constants $\tilde{C}$, $C_{\inf}$ and C_{inv}, independent of h, such that*

$$\begin{aligned} \|\boldsymbol{v}_h\|_{l,p,K} &\leq \tilde{C}\, h_K^{m-l+d(1/p-1/q)}\, \|\boldsymbol{v}_h\|_{m,q,K}, && (4.3)\\ \|\boldsymbol{v}_h\|_{\infty,K} &\leq C_{\inf}\, h_K^{-1/2} |\boldsymbol{v}_h|_{1,\Omega}, && (4.4)\\ h_K\, |\boldsymbol{v}_h|_{1,K} &\leq C_{\mathrm{inv}}\, \|\boldsymbol{v}_h\|_{0,K}, && (4.5) \end{aligned}$$

for all $\boldsymbol{v}_h \in \boldsymbol{H}_h$, where $0 \leq m \leq l$ and $1 \leq p, q \leq \infty$.

Proof. See [23, Lemma 1.138]. □

In some places we will use the Lagrange interpolation (see [23] for details) operator in its vectorial, tensorial and scalar versions which we denote in the same way. For instance, in the vectorial case we have $\boldsymbol{I}_h : \boldsymbol{H} \cap H^{l+1}(\Omega)^d \longrightarrow \boldsymbol{H}_h$. In all cases we have an equivalent result to the following vectorial result

Lemma 4.2. *There exist a positive constant C, independent of h, such that*

$$\|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}\|_{0,\Omega} + h|\boldsymbol{u} - \boldsymbol{I}_h\boldsymbol{u}|_{1,\Omega} \leq Ch^{k+1}\, |\boldsymbol{u}|_{k+1,\Omega}, \tag{4.6}$$

for all $\boldsymbol{u} \in \boldsymbol{H} \cap H^{l+1}(\Omega)^d$, with $1 \leq k \leq l$.

Proof. See [23, Lemma 1.111]. □

Note that the discrete scheme (4.1) can be written as follows: *Find $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$ such that*

$$\begin{aligned} B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) &= F_{\phi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \qquad \forall (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h, && (4.7)\\ A_{\boldsymbol{u}_h}(\phi_h, \psi_h) &= G_{\boldsymbol{u}_h}(\psi_h) \qquad \forall \psi_h \in V_h, && (4.8) \end{aligned}$$

where, given $\phi_h \in V_h$, $B_{\phi_h} : (\boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h) \times (\boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h) \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) &:= \left(\frac{\boldsymbol{\sigma}_h}{2\mu(\phi_h)}, \boldsymbol{\tau}_h\right) - (\boldsymbol{\varepsilon}(\boldsymbol{u}_h), \boldsymbol{\tau}_h) + (q_h, \nabla\cdot\boldsymbol{u}_h) - (\boldsymbol{\sigma}_h, \boldsymbol{\varepsilon}(\boldsymbol{v}_h))\\ &+ (p_h, \nabla\cdot\boldsymbol{v}_h) - \beta\left(2\mu(\phi_h)\left(\frac{\boldsymbol{\sigma}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{u}_h)\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right)\\ &+ \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\nabla p_h - \nabla\cdot\boldsymbol{\sigma}_h, \frac{1}{8\mu(\phi_h)}(\nabla q_h - \nabla\cdot\tau_h)\right)_K, \end{aligned}$$

for all $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$, and $F_{\phi_h} : \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \longrightarrow \mathbb{R}$ is the linear functional defined by

$$F_{\phi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) := -(\alpha\phi_h\boldsymbol{f}, \boldsymbol{v}_h) + \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha\phi_h\boldsymbol{f}, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K,$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$. Moreover, given $\boldsymbol{w}_h \in \boldsymbol{H}_h$, $A_{\boldsymbol{w}_h} : V \times V \longrightarrow \mathbb{R}$ is the bilinear form defined by

$$\begin{aligned} A_{\boldsymbol{w}_h}(\phi_h, \psi_h) &:= k\,(\nabla\phi_h, \nabla\psi_h) + (\nabla\phi_h \cdot \boldsymbol{w}_h, \psi_h) \\ &\quad + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(-k\,\Delta\phi_h + \nabla\phi_h \cdot \boldsymbol{w}_h, k\,\Delta\psi_h + \nabla\psi_h \cdot \boldsymbol{w}_h\right)_K, \end{aligned}$$

for all $\phi_h, \psi_h \in V_h$, and $G_{\boldsymbol{w}_h} : V \longrightarrow \mathbb{R}$ is the linear functional defined by

$$G_{\boldsymbol{w}_h}(\psi_h) := (g, \psi_h) + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(g, k\,\Delta\psi_h + \nabla\psi_h \cdot \boldsymbol{w}_h\right)_K,$$

for all $\psi_h \in V_h$.

As we did in the continuous case, we need to define the following two variational problems: Given $\phi_h \in V_h$: *Find* $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$ *such that*

$$B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) = F_{\phi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h), \tag{4.9}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$.

Also, given $\boldsymbol{w}_h \in \boldsymbol{H}_h$, we define the problem: *Find* $\phi_h \in V_h$ *such that*

$$A_{\boldsymbol{w}_h}(\phi_h, \psi_h) = G_{\boldsymbol{w}_h}(\psi_h), \tag{4.10}$$

for all $\psi_H \in V_h$.

To use similar arguments than those developed in Section 3, we will define the following discrete operators: $\mathbb{S}_h : V_h \longrightarrow \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$, such that

$$\phi_h \longmapsto \mathbb{S}_h(\phi_h) = (\mathbb{S}_h^1(\phi_h), \mathbb{S}_h^2(\phi_h), \mathbb{S}_h^3(\phi_h)) = (\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h),$$

where $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$ is solution of (4.9) and $\mathbb{M}_h : \boldsymbol{H}_h \longrightarrow V_h$, such that

$$\boldsymbol{w}_h \longmapsto \mathbb{M}_h(\boldsymbol{w}_h) = \phi_h,$$

where $\phi_h \in V_h$ is solution of (4.10). Thus, we define the operator $\mathbb{T}_h : V_h \longrightarrow V_h$ by

$$\mathbb{T}_h(\phi_h) := \mathbb{M}_h(\mathbb{S}_h^1(\phi_h)),$$

for all $\phi_h \in V_h$. In this way, the discrete scheme (4.7)–(4.8) can be written as follows: *Find* $\phi_h \in V_h$ *such that*

$$\mathbb{T}_h(\phi_h) = \phi_h. \tag{4.11}$$

4.2. Well–posedness of the uncoupled problems

We define the following mesh–dependent norm over $\boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$

$$|||(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)|||_h := \left\{ \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} h_K^2\, |q_h|_{1,K}^2 + \|\boldsymbol{\tau}_h\|_{0,\Omega}^2 \right\}^{1/2},$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$.

Lemma 4.3. *Assume that the stabilized parameter* β *satisfies the condition*

$$\beta < \frac{1}{2}\,\frac{\mu_{min}^2}{\mu_{max}^2}. \tag{4.12}$$

Then, given $\phi_h \in V_h$*, problem* (4.9) *has a unique solution* $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$*. Moreover, there exists a positive constant* $C_{\mathbb{S}_h}$*, independent of* h*, such that*

$$|||\mathbb{S}_h(\phi_h)|||_h = |||(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)|||_h \le C_{\mathbb{S}_h}\, \|\boldsymbol{f}\|_{\infty,\Omega}\, \|\phi_h\|_{1,\Omega}.$$

Proof. By definition of B_{ϕ_h}, we have

$$
\begin{aligned}
&B_{\phi_h}((\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \\
=&\left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)}, \boldsymbol{\tau}_h\right) - \beta\left(2\mu(\phi_h)\left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} + \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
&+ \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h, \frac{1}{8\mu(\phi_h)}(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K \\
\geq&\frac{1}{2\mu_{\max}}\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 - 2\beta\,\mu_{\max}\left\{\left\|\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)}\right\|_{0,\Omega}^2 - \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2\right\} + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K}^2 \\
\geq&\frac{1}{2\mu_{\max}}\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 - \frac{\beta\,\mu_{\max}}{2\mu_{\min}^2}\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K}^2 \\
\geq&\frac{1}{2}\left(\frac{1}{\mu_{\max}} - \frac{\beta\,\mu_{\max}}{\mu_{\min}^2}\right)\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \\
&\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\left\{|q_h|_{1,K}^2 + \|\nabla\cdot\boldsymbol{\tau}_h\|_{0,K}^2\right\} - \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}2\,|q_h|_{1,K}\|\nabla\cdot\boldsymbol{\tau}_h\|_{0,K}. \qquad (4.13)
\end{aligned}
$$

Using Young inequality with $\delta = 3$, inverse inequality (4.2) and the condition (4.12), we obtain

$$
\begin{aligned}
&B_{\phi_h}((\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \geq \frac{1}{2}\left(\frac{1}{\mu_{\max}} - \frac{\beta\,\mu_{\max}}{\mu_{\min}^2}\right)\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \\
&\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\left(1 - \frac{1}{\delta}\right)|q_h|_{1,K}^2 + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}(1-\delta)\,\|\nabla\cdot\boldsymbol{\tau}_h\|_{0,K}^2 \\
&\geq \frac{1}{2}\left(\frac{1}{\mu_{\max}} - \frac{\beta\,\mu_{\max}}{\mu_{\min}^2}\right)\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\left(1 - \frac{1}{\delta}\right)|q_h|_{1,K}^2 + \frac{1}{8\mu_{\max}}(1-\delta)\,\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 \\
&\geq \frac{1}{2}\left(\frac{1}{\mu_{\max}} - \frac{\beta\,\mu_{\max}}{\mu_{\min}^2} + \frac{1}{4\mu_{\max}}(1-\delta)\right)\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\max}}\left(1 - \frac{1}{\delta}\right)|q_h|_{1,K}^2 \\
&= \left(\frac{\mu_{\min}^2 - 2\beta\,\mu_{\max}^2}{4\mu_{\min}^2\mu_{\max}}\right)\|\boldsymbol{\tau}_h\|_{0,\Omega}^2 + 2\beta\,\mu_{\max}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{12\mu_{\max}}|q_h|_{1,K}^2 \\
&\geq\; C_B\,|||(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)|||_h^2, \qquad (4.14)
\end{aligned}
$$

where C_B is a positive constant independent of h, thus problem (4.9) has a unique solution $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$. Moreover, from (4.2) (4.9), (4.14), Cauchy-Schwarz inequality, and Lemma 2.1, we get

$$
\begin{aligned}
&C_B\,|||(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)|||_h^2 \leq B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)) = F_{\phi_h}(-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \\
&= (\alpha\phi_h\boldsymbol{f}, \boldsymbol{u}_h) + \sum_{K\in\mathcal{T}_h} m_l h_K^2\left(\alpha\phi_h\boldsymbol{f}, \frac{1}{8\mu(\phi_h)}(\nabla p_h - \nabla\cdot\boldsymbol{\sigma}_h)\right)_K \\
&\leq \alpha\,\|\phi_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\|\boldsymbol{u}_h\|_{0,\Omega} + \alpha\,\|\phi_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\min}}\{|p_h|_{1,K} + \|\nabla\cdot\boldsymbol{\sigma}_h\|_{0,K}\} \\
&\leq \alpha\,\|\phi_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\left\{\|\boldsymbol{u}_h\|_{0,\Omega} + \frac{m_l}{8\mu_{\min}}\sum_{K\in\mathcal{T}_h} h_K^2|p_h|_{1,K} + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\min}}\|\nabla\cdot\boldsymbol{\sigma}_h\|_{0,K}\right\} \\
&\leq \alpha\,\|\phi_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\left\{C_K\,\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h)\|_{0,\Omega} + \frac{m_l}{8\mu_{\min}}|\Omega|\sum_{K\in\mathcal{T}_h} h_K|p_h|_{1,K} + \frac{m_l}{8\sqrt{C_l}\mu_{\min}}|\Omega|\sum_{K\in\mathcal{T}_h}\sqrt{C_l}h_K\,\|\nabla\cdot\boldsymbol{\sigma}_h\|_{0,K}\right\} \\
&\leq\; \alpha\,\max\{C_K, \frac{m_l}{8\mu_{\min}}|\Omega|, \frac{m_l}{8\sqrt{C_l}\mu_{\min}}|\Omega|\}\,\|\phi_h\|_{0,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\,|||(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)|||_h,
\end{aligned}
$$

which implies estimate (3.7). □

Lemma 4.4. *Assume that the stabilization parameter γ satisfies $\gamma < 1/\tilde{C}^2$, with $\tilde{C}$ the constant appearing in* (4.3)*. If $\boldsymbol{w}_h \in \boldsymbol{H}_h$ is such that*

$$|\boldsymbol{w}_h|_{1,\Omega} < \frac{k}{2C_q^2\, C_P} \left(1 - \gamma\, \tilde{C}^2\right), \tag{4.15}$$

then problem (4.10) *has a unique solution $\phi_h \in V_h$. Additionally, there exists a positive constant $C_{\mathbb{M}_h}$, independent of h, such that*

$$\|\mathbb{M}_h(\boldsymbol{w}_h)\|_{1,\Omega} \le C_{\mathbb{M}_h}\, \|g\|_{0,\Omega}. \tag{4.16}$$

Proof. From definition of the bilinear form $A_{\boldsymbol{w}_h}(\cdot,\cdot)$, Cauchy–Schwarz and inverse inequalities, we have

$$\begin{aligned} A_{\boldsymbol{w}_h}(\psi_h,\psi_h) &= k\,|\psi_h|_{1,\Omega}^2 + (\nabla\psi_h\cdot\boldsymbol{w}_h,\psi_h) + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\left\{-k^2\,\|\Delta\psi_h\|_{0,K}^2 + \|\nabla\psi_h\cdot\boldsymbol{w}_h\|_{0,K}^2\right\} \\ &\ge k\,|\psi_h|_{1,\Omega}^2 - \gamma \sum_{K\in\mathcal{T}_h} k\,\tilde{C}^2\,\|\psi_h\|_{1,K}^2 + (\nabla\psi_h\cdot\boldsymbol{w}_h,\psi_h) + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\|\nabla\psi_h\cdot\boldsymbol{w}_h\|_{0,K}^2 \\ &\ge k\left(1-\gamma\,\tilde{C}^2\right)\,\|\psi_h\|_{1,\Omega}^2 + (\nabla\psi_h\cdot\boldsymbol{w}_h,\psi_h). \end{aligned} \tag{4.17}$$

Now, using Lemma 2.3, with $q = 4$, we get

$$(\nabla\psi_h\cdot\boldsymbol{w}_h,\psi_h) \ge -|\psi_h|_{1,\Omega}\|\boldsymbol{w}_h\|_{0,4,\Omega}\|\psi_h\|_{0,4,\Omega} \ge -C_q^2\,C_P\,\|\psi_h\|_{1,\Omega}^2|\boldsymbol{w}_h|_{1,\Omega}.$$

Inserting this into (4.17), and using (4.15) and Hölder inequality, we can conclude that

$$A_{\boldsymbol{w}_h}(\psi_h,\psi_h) \ge k\left(1-\gamma\,\tilde{C}^2 - \frac{C_q^2\,C_P}{k}\,|\boldsymbol{w}_h|_{1,\Omega}\right)\,\|\psi_h\|_{1,\Omega}^2 > \frac{k}{2}\,\left(1-\gamma\,\tilde{C}^2\right)\,\|\psi_h\|_{1,\Omega}^2 =: C_A\,\|\psi_h\|_{1,\Omega}^2, \tag{4.18}$$

which proves the solvability of problem (4.10). For the continuous dependence result (4.16), we use the last inequality, (4.10) and (4.15), to get

$$\begin{aligned} & C_A\,\|\phi_h\|_{1,\Omega}^2 \le A_{\boldsymbol{w}_h}(\phi_h,\phi_h) \\ &\le \|g\|_{0,\Omega}\|\phi_h\|_{0,\Omega} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\|g\|_{0,K}\|k\,\Delta\phi_h + \nabla\phi_h\cdot\boldsymbol{w}_h\|_{0,K} \\ &\le \|g\|_{0,\Omega}\|\phi_h\|_{0,\Omega} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\|g\|_{0,K}\|k\,\Delta\phi_h\|_{0,K} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\|g\|_{0,K}\|\nabla\phi_h\cdot\boldsymbol{w}_h\|_{0,K} \\ &\le \|g\|_{0,\Omega}\|\phi_h\|_{0,\Omega} + \gamma\,\tilde{C} \sum_{K\in\mathcal{T}_h} h_K\|g\|_{0,K}\|\phi_h\|_{1,K} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k}\|g\|_{0,K}|\phi_h|_{1,K}\|\boldsymbol{w}_h\|_{\infty,K} \\ &\le \|g\|_{0,\Omega}\|\phi_h\|_{0,\Omega} + \gamma\,\tilde{C} \sum_{K\in\mathcal{T}_h} h_K\|g\|_{0,K}\|\phi_h\|_{1,K} + \gamma\,C_{\text{inf}} \sum_{K\in\mathcal{T}_h} \frac{h_K^{3/2}}{k}\|g\|_{0,K}|\phi_h|_{1,K}|\boldsymbol{w}_h|_{1,\Omega} \\ &\le \|g\|_{0,\Omega}\|\phi_h\|_{0,\Omega} + \gamma\,\tilde{C}\,|\Omega|\,\|g\|_{0,\Omega}\|\phi_h\|_{1,\Omega} + \gamma\,C_{\text{inf}}\,\frac{|\Omega|^{3/2}}{k^{3/2}}\|g\|_{0,\Omega}\|\phi_h\|_{1,\Omega}\,|\boldsymbol{w}_h|_{1,\Omega}, \end{aligned}$$

and the result follows. □

4.3. Existence of solution of equivalent discrete problem

The objective of this section is to prove the existence of a fixed point for the problem (4.11). To this end, we need the following results which will allow us to satisfy the hypotheses of Brouwer's fixed point theorem.

Lemma 4.5. *Let W_h the subset of V_h defined by*

$$W_h := \{\phi_h \in V_h : \|\phi_h\|_{1,\Omega} \leq r\},$$

where r is a positive constant, such that

$$C_K\, C_{\mathbb{S}_h}\, r\, \|\boldsymbol{f}\|_{\infty,\Omega} < \frac{k}{2C_q^2\, C_P} \left(1 - \gamma\, \tilde{C}^2\right). \tag{4.19}$$

If we assume that the datum $g \in L^2(\Omega)$ satisfies the following condition:

$$C_{\mathbb{M}_h}\, \|g\|_{0,\Omega} \leq r, \tag{4.20}$$

and the stabilization parameter β satisfies (4.12)*, then $\mathbb{T}_h(W_h) \subseteq W_h$.*

Proof. Let $\phi_h \in W_h$. By Lemma 4.3, we get

$$\|\!|\mathbb{S}_h(\phi_h)|\!\|_h \leq C_{\mathbb{S}_h}\, \|\boldsymbol{f}\|_{\infty,\Omega}\, \|\phi_h\|_{1,\Omega},$$

and thus, by Korn inequality and (4.19) we can conclude that

$$\left|\mathbb{S}_h^1(\phi_h)\right|_{1,\Omega} \leq C_K \left\|\boldsymbol{\varepsilon}\left(\mathbb{S}_h^1(\phi_h)\right)\right\|_{0,\Omega} \leq C_K\, \|\!|\mathbb{S}_h(\phi_h)|\!\|_h < \frac{k}{2C_q^2\, C_P} \left(1 - \gamma\, \tilde{C}^2\right),$$

and we have the hypothesis of Lemma 4.4. Next, from definition of $\mathbb{T}_h$, (4.16) and (4.20), we have

$$\|\mathbb{T}_h(\phi_h)\| = \|\mathbb{M}_h(\mathbb{S}_h^1(\phi_h))\| \leq C_{\mathbb{M}_h}\, \|g\|_{0,\Omega} \leq r,$$

and therefore $\mathbb{T}_h(\phi_h) \in W_h$. □

Lemma 4.6. *Assume the hypothesis of Lemma 4.3. Then, there exists a positive constant C_4, independent of h, such that*

$$\|\!|\mathbb{S}_h(\phi_h) - \mathbb{S}_h(\psi_h)|\!\|_h \leq C_4 \left\{ \|\boldsymbol{f}\|_{\infty,\Omega}\, (1 + \|\psi_h\|_{1,\Omega}) + \left\|\mathbb{S}_h^3(\psi_h)\right\|_{0,4,\Omega} \right\} \|\phi_h - \psi_h\|_{1,\Omega},$$

for all $\phi_h, \psi_h \in V_h$.

Proof. Let ϕ_h, ψ_h be two arbitrary elements of V_h, and define $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) := \mathbb{S}_h(\phi_h)$ and $(\tilde{\boldsymbol{u}}, \tilde{p}, \tilde{\boldsymbol{\sigma}}) := \mathbb{S}_h(\psi_h)$ by

$$\begin{aligned} B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) &= F_{\phi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h), \\ B_{\psi_h}((\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) &= F_{\psi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h), \end{aligned}$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$. It is evident that,

$$F_{\psi_h}\Big((-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big) = B_{\psi_h}\Big((\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big).$$

Then from this and (4.14), we have

$$
\begin{aligned}
&C_B \,|\!|\!|(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)|\!|\!|_h^2 \le B_{\phi_h}\Big((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big)\\
&= B_{\phi_h}\Big((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big) - B_{\phi_h}\Big((\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big)\\
&= (F_{\phi_h} - F_{\psi_h})\Big((-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big) + (B_{\psi_h} - B_{\phi_h})\Big((\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h), (-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big).
\end{aligned}
\tag{4.21}
$$

Now, using the definition of F_{ϕ_h} and F_{ψ_h}, we get

$$
\begin{aligned}
&(F_{\phi_h} - F_{\psi_h})\Big((-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big) = (\alpha(\phi_h - \psi_h)\boldsymbol{f}, \boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h)\\
&+ \frac{1}{8}\sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha\left(\frac{\phi_h}{\mu(\phi_h)} - \frac{\psi_h}{\mu(\psi_h)}\right)\boldsymbol{f}, \nabla(p_h - \tilde{p}_h) - \nabla\cdot(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h)\right)_K.
\end{aligned}
\tag{4.22}
$$

We can notice by the hypothesis on μ (cf. (2.1)) that

$$
\begin{aligned}
\left|\frac{\phi_h}{\mu(\phi_h)} - \frac{\psi_h}{\mu(\psi_h)}\right| &= \left|\frac{\mu(\psi_h)\phi_h - \mu(\phi_h)\psi_h}{\mu(\phi_h)\mu(\psi_h)}\right|\\
&= \left|\frac{\mu(\psi_h)(\phi_h - \psi_h) - (\mu(\phi_h) - \mu(\psi_h))\psi_h}{\mu(\phi_h)\mu(\psi_h)}\right|\\
&\le \left|\frac{\phi_h - \psi_h}{\mu(\phi_h)}\right| + \left|\frac{(\mu(\phi_h) - \mu(\psi_h))\psi_h}{\mu(\phi_h)\mu(\psi_h)}\right|\\
&\le \frac{|\phi_h - \psi_h|}{\mu_{\min}} + C_{\text{Lips}}\frac{|\phi_h - \psi_h|\,|\psi_h|}{\mu_{\min}^2}\\
&\le \frac{|\phi_h - \psi_h|}{\mu_{\min}}\left(1 + C_{\text{Lips}}\frac{|\psi_h|}{\mu_{\min}}\right).
\end{aligned}
\tag{4.23}
$$

Next, using Hölder's inequality, (4.22) and (4.23), we have

$$
\begin{aligned}
&(F_{\phi_h} - F_{\psi_h})\Big((-\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (-\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)\Big)\\
&\le \|\alpha(\phi_h - \psi_h)\boldsymbol{f}\|_{0,\Omega}\|\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h\|_{0,\Omega} +\\
&\quad \frac{1}{8}\sum_{K\in\mathcal{T}_h}\alpha m_l h_K^2 \left\|\left(\frac{\phi_h}{\mu(\phi_h)} - \frac{\psi_h}{\mu(\psi_h)}\right)\boldsymbol{f}\right\|_{0,K}\|\nabla(p_h - \tilde{p}_h) - \nabla\cdot(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\\
&\le C_P\, C_K \alpha\, |\phi_h - \psi_h|_{1,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h)\|_{0,\Omega} +\\
&\quad \frac{1}{8}\sum_{K\in\mathcal{T}_h}\alpha m_l h_K^2 \left\|\frac{(\phi_h - \psi_h)}{\mu_{\min}}\left(1 + C_{\text{Lips}}\frac{|\psi_h|}{\mu_{\min}}\right)\boldsymbol{f}\right\|_{0,K}\|\nabla(p_h - \tilde{p}_h) - \nabla\cdot(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\\
&\le C_P\, C_K \alpha |\phi_h - \psi_h|_{1,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega}\|\boldsymbol{\varepsilon}(\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h)\|_{0,\Omega} +\\
&\quad \frac{\alpha}{8}\left\{\frac{\|\phi_h - \psi_h\|_{0,\Omega}}{\mu_{\min}}\|\boldsymbol{f}\|_{\infty,\Omega} + C_{\text{Lips}}\|\phi_h - \psi_h\|_{0,\Omega}\frac{\|\psi_h\|_{0,\Omega}}{\mu_{\min}^2}\|\boldsymbol{f}\|_{\infty,\Omega}\right\}\sum_{K\in\mathcal{T}_h} m_l h_K^2\, \|\nabla(p_h - \tilde{p}_h) - \nabla\cdot(\boldsymbol{\sigma}_h - \tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\\
&\le C\left\{|\phi_h - \psi_h|_{1,\Omega}\,\|\boldsymbol{f}\|_{\infty,\Omega} + |\phi_h - \psi_h|_{1,\Omega}\,|\psi_h|_{1,\Omega}\,\|\boldsymbol{f}\|_{\infty,\Omega}\right\}|\!|\!|(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)|\!|\!|_h\\
&\le C\,|\phi_h - \psi_h|_{1,\Omega}\,\|\boldsymbol{f}\|_{\infty,\Omega}\,(1 + |\psi_h|_{1,\Omega})\,|\!|\!|(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) - (\tilde{\boldsymbol{u}}_h, \tilde{p}_h, \tilde{\boldsymbol{\sigma}}_h)|\!|\!|_h.
\end{aligned}
\tag{4.24}
$$

On the other hand, using Lemma 2.3, with $q=4$, and Lemma 4.1, we obtain

$$
\begin{aligned}
&(B_{\psi_h}-B_{\phi_h})\Big((\tilde{\boldsymbol{u}}_h,\tilde{p}_h,\tilde{\boldsymbol{\sigma}}_h),(-\boldsymbol{u}_h,p_h,\boldsymbol{\sigma}_h)-(-\tilde{\boldsymbol{u}}_h,\tilde{p}_h,\tilde{\boldsymbol{\sigma}}_h)\Big)\\
&=\frac{1+\alpha}{2}\left(\tilde{\boldsymbol{\sigma}}_h\left[\frac{1}{\mu(\psi_h)}-\frac{1}{\mu(\phi_h)}\right],\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\right)\\
&\quad+\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8}\left(\left[\frac{1}{\mu(\psi_h)}-\frac{1}{\mu(\phi_h)}\right](\nabla\tilde{p}_h-\nabla\cdot\tilde{\boldsymbol{\sigma}}_h),\nabla(\tilde{p}-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)\right)_K\\
&=\frac{1+\alpha}{2}\left(\tilde{\boldsymbol{\sigma}}_h\left[\frac{\mu(\phi_h)-\mu(\psi_h)}{\mu(\phi_h)\mu(\psi_h)}\right],\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\right)\\
&\quad+\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8}\left(\left[\frac{\mu(\phi_h)-\mu(\psi_h)}{\mu(\phi_h)\mu(\psi_h)}\right](\nabla\tilde{p}_h-\nabla\cdot\tilde{\boldsymbol{\sigma}}_h),\nabla(\tilde{p}_h-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)\right)_K\\
&\leq C_{\mathrm{Lips}}\frac{1+\alpha}{2\,\mu_{\min}^2}\,\|\tilde{\boldsymbol{\sigma}}_h\|_{0,4,\Omega}\,\|\phi_h-\psi_h\|_{0,4,\Omega}\|\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\\
&\quad+\sum_{K\in\mathcal{T}_h}\frac{C_{\mathrm{Lips}}\,m_l h_K^2}{8\mu_{\min}^2}\,\|\phi_h-\psi_h\|_{\infty,K}\,\|\nabla\tilde{p}_h-\nabla\cdot\tilde{\boldsymbol{\sigma}}_h\|_{0,K}\|\nabla(\tilde{p}_h-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\Bigg\}\\
&\leq C\Bigg\{\|\tilde{\boldsymbol{\sigma}}_h\|_{0,4,\Omega}\,|\phi_h-\psi_h|_{1,\Omega}\|\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\\
&\quad+\sum_{K\in\mathcal{T}_h}h_K^2\,\|\phi_h-\psi_h\|_{\infty,K}\,\|\nabla\tilde{p}_h-\nabla\cdot\tilde{\boldsymbol{\sigma}}_h\|_{0,K}\|\nabla(\tilde{p}_h-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\Bigg\}\\
&\leq C\Bigg\{\|\tilde{\boldsymbol{\sigma}}_h\|_{0,4,\Omega}\,|\phi_h-\psi_h|_{1,\Omega}\|\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\\
&\quad+h^{1/2}\,|\phi_h-\psi_h|_{1,\Omega}\sum_{K\in\mathcal{T}_h}h_K\,\left(\|\nabla\tilde{p}_h\|_{0,K}+\|\nabla\cdot\tilde{\boldsymbol{\sigma}}_h\|_{0,K}\right)\|\nabla(\tilde{p}_h-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)\|_{0,K}\Bigg\}\\
&\leq C\Bigg\{\|\tilde{\boldsymbol{\sigma}}_h\|_{0,4,\Omega}\,|\phi_h-\psi_h|_{1,\Omega}\|\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega}\\
&\quad+h^{1/2}\,|\phi_h-\psi_h|_{1,\Omega}\,\|\!|\mathbb{S}_h(\psi_h)|\!\|_h\sum_{K\in\mathcal{T}_h}h_K\,|\nabla(\tilde{p}_h-p_h)-\nabla\cdot(\boldsymbol{\sigma}_h-\tilde{\boldsymbol{\sigma}}_h)|_{0,K}\Bigg\}\\
&\leq C\Bigg\{|\Omega|^{1/2}\,\|\!|\mathbb{S}_h(\psi_h)|\!\|_h+\|\tilde{\boldsymbol{\sigma}}_h\|_{0,4,\Omega}\Bigg\}\,|\phi_h-\psi_h|_{1,\Omega}\,\|\!|(\boldsymbol{u}_h,p_h,\boldsymbol{\sigma}_h)-(\tilde{\boldsymbol{u}}_h,\tilde{p}_h,\tilde{\boldsymbol{\sigma}}_h)|\!\|_h. \qquad (4.25)
\end{aligned}
$$

Now, using Lemma 4.3 and replacing (4.24) and (4.25) in (4.21), the result follows. □

Lemma 4.7. *Let $\boldsymbol{u}_h,\tilde{\boldsymbol{u}}_h$ be two elements in $\boldsymbol{H}_h$ satisfying the condition* (4.15) *of Lemma 4.4. Then, there exists a positive constant C_5, independent of h, such that*

$$\|\mathbb{M}_h(\boldsymbol{u}_h)-\mathbb{M}_h(\tilde{\boldsymbol{u}}_h)\|_{1,\Omega}\leq C_5\left\{\|g\|_{0,\Omega}+|\mathbb{M}_h(\tilde{\boldsymbol{u}}_h)|_{1,\Omega}\right\}|\boldsymbol{u}_h-\tilde{\boldsymbol{u}}_h|_{1,\Omega}. \qquad (4.26)$$

Proof. Defining $\phi_h:=\mathbb{M}_h(\boldsymbol{u}_h)$ and $\tilde{\phi}_h:=\mathbb{M}_h(\tilde{\boldsymbol{u}}_h)$, we get, for all $\psi_h\in\boldsymbol{V}_h$, that

$$A_{\boldsymbol{u}_h}(\phi_h,\psi_h)=G_{\boldsymbol{u}_h}(\psi_h),\qquad A_{\tilde{\boldsymbol{u}}_h}(\tilde{\phi}_h,\psi_h)=G_{\tilde{\boldsymbol{u}}_h}(\psi_h).$$

Then, from (4.18), we have that

$$
\begin{aligned}
C_A \|\phi_h - \tilde{\phi}_h\|^2 \le\; & A_{\boldsymbol{u}_h}(\phi_h - \tilde{\phi}_h, \phi_h - \tilde{\phi}_h) = A_{\boldsymbol{u}_h}(\phi_h, \phi_h - \tilde{\phi}_h) - A_{\boldsymbol{u}_h}(\tilde{\phi}_h, \phi_h - \tilde{\phi}_h) \\
= & G_{\boldsymbol{u}_h}(\phi_h - \tilde{\phi}_h) - G_{\tilde{\boldsymbol{u}}_h}(\phi_h - \tilde{\phi}_h) + A_{\tilde{\boldsymbol{u}}_h}(\tilde{\phi}_h, \phi_h - \tilde{\phi}_h) - A_{\boldsymbol{u}_h}(\tilde{\phi}_h, \phi_h - \tilde{\phi}_h).
\end{aligned} \tag{4.27}
$$

Using the inverse inequality (4.4), we get

$$
\begin{aligned}
& G_{\boldsymbol{u}_h}(\phi_h - \tilde{\phi}_h) - G_{\tilde{\boldsymbol{u}}_h}(\phi_h - \tilde{\phi}_h) \\
\le\; & \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \Big(g, k\,\Delta(\phi_h - \tilde{\phi}_h) + \nabla(\phi_h - \tilde{\phi}_h)\cdot \boldsymbol{u}_h\Big)_K - \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \Big(g, k\,\Delta(\phi_h - \tilde{\phi}_h) + \nabla(\phi_h - \tilde{\phi}_h)\cdot \tilde{\boldsymbol{u}}_h\Big)_K \\
\le\; & \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \|g\|_{0,K} |\phi_h - \tilde{\phi}_h|_{1,K} \|\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h\|_{\infty,K} \le \gamma\, C_{\inf} \sum_{K\in\mathcal{T}_h} \frac{h_K^{3/2}}{k} \|g\|_{0,K} |\phi_h - \tilde{\phi}_h|_{1,K}\, |\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h|_{1,K} \\
\le\; & C_{\inf}\, \gamma\, \frac{|\Omega|^{3/2}}{k}\, \|g\|_{0,\Omega}\, |\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h|_{1,\Omega}\, \|\phi_h - \tilde{\phi}_h\|_{1,\Omega}.
\end{aligned} \tag{4.28}
$$

On the other hand, as $\boldsymbol{u}_h, \tilde{\boldsymbol{u}}_h \in \boldsymbol{H}_h$ satisfy the condition (4.15), and using Hölder's inequality, Lemma 2.3, with $q = 4$, and Lemma 4.4, we get

$$
\begin{aligned}
& A_{\tilde{\boldsymbol{u}}_h}(\tilde{\phi}_h, \phi_h - \tilde{\phi}_h) - A_{\boldsymbol{u}_h}(\tilde{\phi}_h, \phi_h - \tilde{\phi}_h) \\
=\; & (\nabla\tilde{\phi}_h \cdot (\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h), \phi_h - \tilde{\phi}_h) + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \bigg\{ \Big(- k\,\Delta\tilde{\phi}_h, \nabla(\phi_h - \tilde{\phi}_h)\cdot(\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h)\Big)_K \\
& + \Big(\nabla\tilde{\phi}_h \cdot (\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h), k\,\Delta(\phi_h - \tilde{\phi}_h)\Big)_K + \Big(\nabla\tilde{\phi}_h \cdot \tilde{\boldsymbol{u}}_h, \nabla(\phi_h - \tilde{\phi}_h)\cdot \tilde{\boldsymbol{u}}_h\Big)_K - \Big(\nabla\tilde{\phi}_h \cdot \boldsymbol{u}_h, \nabla(\phi_h - \tilde{\phi}_h)\cdot \boldsymbol{u}_h\Big)_K \bigg\} \\
\le\; & |\tilde{\phi}_h|_{1,\Omega} \|\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{0,4,\Omega} \|\phi_h - \tilde{\phi}_h\|_{0,4,\Omega} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \bigg\{ k\,\|\Delta\tilde{\phi}_h\|_{0,K} |\phi_h - \tilde{\phi}_h|_{1,K} \|\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{\infty,K} \\
& + k\,|\tilde{\phi}_h|_{1,K} \|\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{\infty,K} \|\Delta(\phi_h - \tilde{\phi}_h)\|_{0,K} + |\tilde{\phi}_h|_{1,K} \|\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h\|_{\infty,K} |\phi_h - \tilde{\phi}_h|_{1,K} \|\tilde{\boldsymbol{u}}_h\|_{\infty,K} \\
& + |\tilde{\phi}_h|_{1,K} \|\boldsymbol{u}_h\|_{\infty,K} |\phi_h - \tilde{\phi}_h|_{1,K} \|\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h\|_{\infty,K} \bigg\} \\
\le\; & C \bigg\{ |\tilde{\phi}_h|_{1,\Omega} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} |\phi_h - \tilde{\phi}_h|_{1,\Omega} + \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \Big[h_K^{-3/2}\, k\, |\tilde{\phi}_h|_{1,K} |\phi_h - \tilde{\phi}_h|_{1,K} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} \\
& + h_K^{-3/2}\, k\, |\tilde{\phi}_h|_{1,K} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} |\phi_h - \tilde{\phi}_h|_{1,K} + h_K^{-1}\, |\tilde{\phi}_h|_{1,K} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} |\phi_h - \tilde{\phi}_h|_{1,K} |\tilde{\boldsymbol{u}}_h|_{1,\Omega} \\
& + h_K^{-1}\, |\tilde{\phi}_h|_{1,K} |\boldsymbol{u}_h|_{1,\Omega} |\phi_h - \tilde{\phi}_h|_{1,K} |\boldsymbol{u}_h - \tilde{\boldsymbol{u}}_h|_{1,\Omega} \Big] \bigg\} \\
\le\; & C \left\{ 1 + \frac{1}{k} \Big[|\boldsymbol{u}_h|_{1,\Omega} + |\tilde{\boldsymbol{u}}_h|_{1,\Omega} \Big] \right\} |\tilde{\phi}_h|_{1,\Omega} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} |\phi_h - \tilde{\phi}_h|_{1,\Omega} \\
\le\; & C \left\{ 1 + \frac{1}{k} \Big[|\boldsymbol{u}_h|_{1,\Omega} + |\tilde{\boldsymbol{u}}_h|_{1,\Omega} \Big] \right\} |\tilde{\phi}_h|_{1,\Omega} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} \|\phi_h - \tilde{\phi}_h\|_{1,\Omega} \\
\le\; & C \left\{ 1 + \frac{1}{C_q^2\, C_P} \Big(1 - \gamma\, \tilde{C}^2\Big) \right\} |\tilde{\phi}_h|_{1,\Omega} |\tilde{\boldsymbol{u}}_h - \boldsymbol{u}_h|_{1,\Omega} \|\phi_h - \tilde{\phi}_h\|_{1,\Omega}.
\end{aligned} \tag{4.29}
$$

Finally, from (4.27), (4.28) and (4.29), we conclude (4.26). □

Lemma 4.8. *Let $W_h := \{\phi_h \in V_h : \|\phi_h\|_{1,\Omega} \le r\}$, where $r > 0$ and the data $g \in L^2(\Omega)$ satisfy the hypothesis of Lemma 4.5. If the stabilization parameters satisfy the conditions of lemmas 4.3 and 4.4, respectively, then there exists a positive constant C_6, independent of h, such that, for all $\phi_h, \tilde{\phi}_h \in W_h$ there holds*

$$\|\mathbb{T}_h(\phi_h) - \mathbb{T}_h(\tilde{\phi}_h)\|_{1,\Omega} \le C_6 \|g\|_{0,\Omega} \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} + \|\mathbb{S}_h^3(\tilde{\phi}_h)\|_{0,4,\Omega} \right\} \|\phi_h - \tilde{\phi}_h\|_{1,\Omega}. \tag{4.30}$$

Proof. The result is a consequence of the definition of $\mathbb{T}_h$ and lemmas 4.4, 4.6 and 4.7. □

Theorem 4.9. *Let $W_h := \{\phi_h \in V_h : \|\phi_h\|_{1,\Omega} \le r\}$, where $r > 0$ and the data $g \in L^2(\Omega)$ satisfy the conditions of Lemma 4.5. If the stabilization parameters satisfy the conditions of lemmas 4.3 and 4.4, respectively, then the stabilized scheme* (4.1) *has at least one solution $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$, with $\phi_h \in W_h$, and there holds*

$$\|\phi_h\| \le C_{\mathbb{M}_h} \|g\|_{0,\Omega},$$

and

$$|||(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h)|||_h \le C_{\mathbb{S}_h} \|\boldsymbol{f}\|_{\infty,\Omega} \|\phi_h\|.$$

Proof. From Lemma 4.8, $\mathbb{T}_h : W_h \longrightarrow W_h$ is continuous, and, by the Brouwer fixed point Theorem (see Theorem 2.6), it has at least one fixed point. □

Remark 5. *Usually, to prove the uniqueness of the solution of* (4.1) *Banach's fixed point theorem is used. In our case, due to the presence of the term $\|\mathbb{S}_h^3(\tilde{\phi}_h)\|_{0,4,\Omega}$ in* (4.30)*, it is not possible to state that $\mathbb{T}_h$ is a contraction on W_h. Thus we only have the existence of the solution of problem* (4.1).

5. Convergence of the stabilized scheme

In this section, we present an a priori error analysis for the stabilized finite element scheme (4.1). We consider $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in \boldsymbol{H} \times Q \times \boldsymbol{R} \times V$, with $\phi \in W$, and $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$, with $\phi_h \in W_h$, such that

$$B_\phi((\boldsymbol{u}, p, \boldsymbol{\sigma}), (\boldsymbol{v}, q, \boldsymbol{\tau})) = F_\phi(\boldsymbol{v}, q, \boldsymbol{\tau}) \qquad \forall (\boldsymbol{v}, q, \boldsymbol{\tau}) \in \boldsymbol{H} \times Q, \tag{5.1}$$

$$B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) = F_{\phi_h}(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \qquad \forall (\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h, \tag{5.2}$$

and

$$A_{\boldsymbol{u}}(\phi, \psi) = G(\psi) \qquad \forall \psi \in V, \tag{5.3}$$

$$A_{\boldsymbol{u}_h}(\phi_h, \psi_h) = G_{\boldsymbol{u}_h}(\psi_h) \qquad \forall \psi_h \in V_h. \tag{5.4}$$

Lemma 5.1. *Assume that $r > 0$ and $g \in L^2(\Omega)$ satisfy conditions* (4.19) *and* (4.20)*, respectively. Let $(\boldsymbol{u}, \phi) \in (H^{l+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^{l+1}(\Omega)^d \cap V)$, and $(\boldsymbol{u}_h, \phi_h) \in \boldsymbol{H}_h \times V_h$ be solutions of* (5.1), (5.3) *and* (5.2), (5.4)*, respectively, with $\phi \in W$ and $\phi_h \in W_h$. Then there exist a positive constant C_7, independent of h, such that for all $\psi_h \in V_h$, we have*

$$A_{\boldsymbol{u}_h}(\phi_h - \phi, \psi_h) \le C_7\, h\, \|\phi\|_{2,\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \|\psi_h\|_{1,\Omega}. \tag{5.5}$$

Proof. Let $\boldsymbol{u}_h \in \boldsymbol{H}_h$ be the solution of (5.2). Then, from (5.3)-(5.4), Cauchy-Schwarz and Hölder inequalities, we get

$$\begin{aligned}
A_{\boldsymbol{u}_h}(\phi_h - \phi, \psi_h) &= A_{\boldsymbol{u}_h}(\phi_h, \psi_h) - A_{\boldsymbol{u}_h}(\phi, \psi_h) \\
&= G_{\boldsymbol{u}_h}(\psi_h) - G(\psi_h) - \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(-k\,\Delta\phi + \nabla\phi\cdot\boldsymbol{u}_h, k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K \\
&= \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(g + k\,\Delta\phi - \nabla\phi\cdot\boldsymbol{u}_h, k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K \\
&= \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(g + k\,\Delta\phi - \nabla\phi\cdot\boldsymbol{u}, k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K + \\
&\quad \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(\nabla\phi\cdot\boldsymbol{u} - \nabla\phi\cdot\boldsymbol{u}_h, k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K \\
&= \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left(\nabla\phi\cdot(\boldsymbol{u} - \boldsymbol{u}_h), k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K \\
&\leq \gamma \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \|\nabla\phi\|_{0,4,K} \|\boldsymbol{u} - \boldsymbol{u}_h\|_{0,4,K} \|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\leq \gamma\, C_q^2\, \|\phi\|_{2,\Omega} |\boldsymbol{u} - \boldsymbol{u}_h|_{1,\Omega} \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\leq \gamma\, C_K\, C_q^2\, \|\phi\|_{2,\Omega} \|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K}. \qquad (5.6)
\end{aligned}$$

Next, using Lemma 4.1, we get that

$$\begin{aligned}
\sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} &\leq \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left\{k\,\|\Delta\psi_h\|_{0,K} + \|\nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K}\right\} \\
&\leq \sum_{K\in\mathcal{T}_h} \frac{h_K^2}{k} \left\{\tilde{C}\,k\,h_K^{-1}|\psi_h|_{1,K} + C_{\inf}\,h_K^{-1/2}\,|\psi_h|_{1,K}\,|\boldsymbol{u}_h|_{1,\Omega}\right\} \\
&\leq \sum_{K\in\mathcal{T}_h} \left\{\tilde{C}\,h_K|\psi_h|_{1,K} + \frac{h_K^{3/2}}{k} C_{\inf}\,|\psi_h|_{1,K}\,|\boldsymbol{u}_h|_{1,\Omega}\right\} \\
&\leq \left\{\tilde{C}\,h|\psi_h|_{1,\Omega} + \frac{h^{3/2}}{k} C_{\inf}\,|\psi_h|_{1,\Omega}\,|\boldsymbol{u}_h|_{1,\Omega}\right\} \\
&= h\left\{\tilde{C} + \frac{h^{1/2}}{k} C_{\inf}\,|\boldsymbol{u}_h|_{1,\Omega}\right\} \|\psi_h\|_{1,\Omega} \qquad (5.7) \\
&\leq h\left\{\tilde{C} + \frac{h^{1/2}}{2C_q^2\,C_P} C_{\inf}\left(1 - \gamma\,\tilde{C}^2\right)\right\} \|\psi_h\|_{1,\Omega}. \qquad (5.8)
\end{aligned}$$

Finally, from (5.6) and (5.8), we conclude (5.5). □

Lemma 5.2. *Assume that $r > 0$ and $g \in L^2(\Omega)$ satisfy conditions* (4.19) *and* (4.20)*, respectively. Let $(\boldsymbol{u}, \phi) \in (H^{l+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^{l+1}(\Omega)^d \cap V)$, and $(\boldsymbol{u}_h, \phi_h) \in \boldsymbol{H}_h \times V_h$ be solutions of* (5.1), (5.3) *and* (5.2), (5.4)*, respectively, with $\phi \in W$ and $\phi_h \in W_h$. Then, there exist two positive constants C_8 and C_9, independent of h, such that*

$$\|\phi - \phi_h\|_{1,\Omega} \leq C_8\,h\,\|\phi\|_{2,\Omega}\,\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + C_9\,h^l\,|\phi|_{l+1,\Omega}. \qquad (5.9)$$

Proof. We use the standard notation for the interpolation error $\eta^\phi := \phi - I_h\phi$. Then, using Cauchy-Schwarz and Hölder inequalities and (5.7), we get, for all $\psi_h \in V_h$

$$
\begin{aligned}
A_{\boldsymbol{u}_h}(\eta^\phi, \psi_h) &= k\,(\nabla\eta^\phi, \nabla\psi_h) + (\nabla\eta^\phi\cdot\boldsymbol{u}_h, \psi_h) + \gamma\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{k}\left(-k\,\Delta\eta^\phi + \nabla\eta^\phi\cdot\boldsymbol{u}_h, k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\right)_K \\
&\le k\,|\eta^\phi|_{1,\Omega}|\psi_h|_{1,\Omega} + |\eta^\phi|_{1,\Omega}\|\boldsymbol{u}_h\|_{0,4,\Omega}\|\psi_h\|_{0,4,\Omega} + \\
&\quad \gamma\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{k}\|-k\,\Delta\eta^\phi + \nabla\eta^\phi\cdot\boldsymbol{u}_h\|_{0,K}\|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\le k\,|\eta^\phi|_{1,\Omega}|\psi_h|_{1,\Omega} + C_q^2\,|\eta^\phi|_{1,\Omega}|\boldsymbol{u}_h|_{1,\Omega}|\psi_h|_{1,\Omega} + \\
&\quad \gamma\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{k}\|-k\,\Delta\eta^\phi + \nabla\eta^\phi\cdot\boldsymbol{u}_h\|_{0,K}\|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\le k\,|\eta^\phi|_{1,\Omega}|\psi_h|_{1,\Omega} + C_q^2\,|\eta^\phi|_{1,\Omega}|\boldsymbol{u}_h|_{1,\Omega}|\psi_h|_{1,\Omega} + \\
&\quad \gamma\sum_{K\in\mathcal{T}_h}\frac{h_K^2}{k}\left[k\,\|\eta^\phi\|_{2,K} + |\eta^\phi|_{1,K}\,\|\boldsymbol{u}_h\|_{\infty,K}\right]\|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\le (k + C_q^2\,|\boldsymbol{u}_h|_{1,\Omega})\,|\eta^\phi|_{1,\Omega}|\psi_h|_{1,\Omega} + \\
&\quad \gamma\left[kh\,\|\eta^\phi\|_{2,\Omega} + C_{\inf}h^{1/2}|\eta^\phi|_{1,\Omega}\,|\boldsymbol{u}_h|_{1,\Omega}\right]\sum_{K\in\mathcal{T}_h}\frac{h_K}{k}\|k\,\Delta\psi_h + \nabla\psi_h\cdot\boldsymbol{u}_h\|_{0,K} \\
&\le \left(k + C_q^2\,|\boldsymbol{u}_h|_{1,\Omega}\right)\,|\eta^\phi|_{1,\Omega}\|\psi_h\|_{1,\Omega} + \\
&\quad \gamma\left[kh\,\|\eta^\phi\|_{2,\Omega} + C_{\inf}h^{1/2}|\eta^\phi|_{1,\Omega}\,|\boldsymbol{u}_h|_{1,\Omega}\right]\left\{\tilde{C} + \frac{h^{1/2}}{k}C_{\inf}\,|\boldsymbol{u}_h|_{1,\Omega}\right\}\|\psi_h\|_{1,\Omega} \\
&\le \left(k + C_q^2\,|\boldsymbol{u}_h|_{1,\Omega} + \gamma\,C_{\inf}h^{1/2}\,|\boldsymbol{u}_h|_{1,\Omega}\left\{\tilde{C} + \frac{h^{1/2}}{k}C_{\inf}\,|\boldsymbol{u}_h|_{1,\Omega}\right\}\right)|\eta^\phi|_{1,\Omega}\|\psi_h\|_{1,\Omega} + \\
&\quad \gamma\,k\,h\,\|\eta^\phi\|_{2,\Omega}\left\{\tilde{C} + \frac{h^{1/2}}{k}C_{\inf}\,|\boldsymbol{u}_h|_{1,\Omega}\right\}\|\psi_h\|_{1,\Omega} \\
&\le C\,h^l\,|\phi|_{l+1,\Omega}\|\psi_h\|_{1,\Omega}.
\end{aligned} \tag{5.10}
$$

Now, let $e_h^\phi := \phi_h - I_h\phi$, then by Lemma 5.1, (4.18) and (5.10), we have

$$
\begin{aligned}
C_A\,\|e_h^\phi\|_{1,\Omega}^2 \le A_{\boldsymbol{u}_h}(e_h^\phi, e_h^\phi) &= A_{\boldsymbol{u}_h}(\phi_h - \phi, e_h^\phi) + A_{\boldsymbol{u}_h}(\eta^\phi, e_h^\phi) \\
&\le C_7\,h\,\|\phi\|_{2,\Omega}\,\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega}\,\|e_h^\phi\|_{1,\Omega} + C\,h^l\,|\phi|_{l+1,\Omega}\|e_h^\phi\|_{1,\Omega},
\end{aligned}
$$

and dividing by $C_A\,\|e_h^\phi\|_{1,\Omega}$, we arrive at

$$
\|e_h^\phi\|_{1,\Omega} \le \frac{C_7}{C_A}\,h\,\|\phi\|_{2,\Omega}\,\|\boldsymbol{\varepsilon}(\boldsymbol{u} - \boldsymbol{u}_h)\|_{0,\Omega} + \frac{C}{C_A}\,h^l\,|\phi|_{l+1,\Omega}. \tag{5.11}
$$

Finally, using triangle inequality, interpolation properties and (5.11), the result follows. □

Lemma 5.3. *Let $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in (H^{l+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^l(\Omega) \cap Q) \times (H^l(\Omega)^{d\times d} \cap \boldsymbol{R}) \times (H^{l+1}(\Omega)^d \cap V)$, and $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$ be the solutions of (5.1), (5.3) and (5.2),(5.4), respectively, with $\phi \in W$ and $\phi_h \in W_h$. Then, for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$, there exists a positive constant C_{10}, independent of h, such that*

$$
\frac{B_{\phi_h}((\boldsymbol{u}_h - \boldsymbol{u}, p_h - p, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h))}{\|(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)\|_h} \le C_{10}\left\{\|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}\right\}\|\phi - \phi_h\|_{1,\Omega}.
$$

Proof. Let $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$ be the solution of (5.2). Then from (5.3)-(5.4), Cauchy-Schwarz and Hölder inequalities, we get

$$
\begin{aligned}
& B_{\phi_h}((\boldsymbol{u}_h - \boldsymbol{u}, p_h - p, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \\
= & B_{\phi_h}((\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) - B_{\phi_h}((\boldsymbol{u}, p, \boldsymbol{\sigma}), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \\
= & F_{\phi_h}(-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) - B_{\phi_h}((\boldsymbol{u}, p, \boldsymbol{\sigma}), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \\
= & \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha\phi_h \boldsymbol{f}, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K + \beta\left(2\mu(\phi_h)\left(\frac{\boldsymbol{\sigma}}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{u})\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
& - \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\nabla p - \nabla\cdot\boldsymbol{\sigma}, \frac{1}{8\mu(\phi_h)}(\nabla q_h - \nabla\cdot\tau_h)\right)_K \\
= & \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha\phi_h \boldsymbol{f} + \nabla\cdot\boldsymbol{\sigma} - \nabla p, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K + \beta\left(2\mu(\phi_h)\left(\frac{\boldsymbol{\sigma}}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{u})\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
= & \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha(\phi_h - \phi)\boldsymbol{f}, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K + \beta\left(2\mu(\phi_h)\left(\frac{\boldsymbol{\sigma}}{2\mu(\phi_h)} - \frac{\boldsymbol{\sigma}}{2\mu(\phi)}\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
= & \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha(\phi_h - \phi)\boldsymbol{f}, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K + \beta\left(\mu(\phi_h)\,\boldsymbol{\sigma}\left(\frac{\mu(\phi) - \mu(\phi_h)}{\mu(\phi_h)\mu(\phi)}\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
= & \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\alpha(\phi_h - \phi)\boldsymbol{f}, \frac{1}{8\mu(\phi_h)}\,(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K + \beta\left(\boldsymbol{\sigma}\left(\frac{\mu(\phi) - \mu(\phi_h)}{\mu(\phi)}\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
\le & \sum_{K\in\mathcal{T}_h} \frac{m_l h_K^2}{8\mu_{\min}}\,\alpha\|\phi_h - \phi\|_{0,K}\|\boldsymbol{f}\|_{\infty,K}\,\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K} + \frac{\beta C_{\text{Lips}}}{\mu_{\min}}\,\|\boldsymbol{\sigma}\|_{0,4,\Omega}\|\phi - \phi_h\|_{0,4,\Omega}\left\|\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right\|_{0,\Omega} \\
\le & \alpha C_q^2\, C_P|\phi_h - \phi|_{1,\Omega}\|\boldsymbol{f}\|_{\infty,\Omega} \sum_{K\in\mathcal{T}_h} \frac{m_l h_K^2}{8\mu_{\min}}\,\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K} + \frac{\beta C_{\text{Lips}}}{\mu_{\min}} C_q^2\,\|\boldsymbol{\sigma}\|_{1,\Omega}|\phi - \phi_h|_{1,\Omega}\left\|\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right\|_{0,\Omega} \\
\le & C_q^2\,\|\phi_h - \phi\|_{1,\Omega}\left\{C_P\alpha\|\boldsymbol{f}\|_{\infty,\Omega}\sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\min}}\,\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K} + \frac{\beta C_{\text{Lips}}}{\mu_{\min}}\,\|\boldsymbol{\sigma}\|_{1,\Omega}\left\|\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right\|_{0,\Omega}\right\} \\
\le & C_q^2\,\|\phi_h - \phi\|_{1,\Omega}\left\{C_P\alpha\|\boldsymbol{f}\|_{\infty,\Omega}\sum_{K\in\mathcal{T}_h}\frac{m_l h_K}{8\mu_{\min}}\left[h_K\,|q_h|_{1,K} + \frac{1}{C_l^{1/2}}\|\boldsymbol{\tau}_h\|_{0,K}\right] + \frac{\beta C_{\text{Lips}}}{\mu_{\min}}\,\|\boldsymbol{\sigma}\|_{1,\Omega}\left\{\frac{\|\boldsymbol{\tau}_h\|_{0,\Omega}}{2\mu_{\min}} + \|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega}\right\}\right\} \\
\le & C\,C_q^2\,\|\phi_h - \phi\|_{1,\Omega}\left\{C_P\alpha\|\boldsymbol{f}\|_{\infty,\Omega}\,\frac{m_l h}{8\mu_{\min}} + \frac{\beta C_{\text{Lips}}}{\mu_{\min}}\,\|\boldsymbol{\sigma}\|_{1,\Omega}\right\}|||(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)|||_h,
\end{aligned}
$$

which concludes the proof. □

Lemma 5.4. *Let $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in (H^{l+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^l(\Omega) \cap Q) \times (H^l(\Omega)^{d\times d} \cap \boldsymbol{R}) \times (H^{l+1}(\Omega)^d \cap V)$, and $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times W_h$ be the solutions of (5.1), (5.3) and (5.2),(5.4), respectively, with $\phi \in W$ and $\phi_h \in W_h$. Then there exist two positive constants C_{11} and C_{12}, independent of h, such that*

$$
|||(\boldsymbol{u}_h - \boldsymbol{u}, p_h - p, \boldsymbol{\sigma}_h - \boldsymbol{\sigma})|||_h \le C_{11}\left\{\|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}\right\}\|\phi_h - \phi\|_{1,\Omega} + C_{12}\,h^l\left\{|\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega}\right\}. \quad (5.12)
$$

Proof. We use again a standard notation for the interpolations error $\eta^{\boldsymbol{u}} := \boldsymbol{u} - I_h\boldsymbol{u}$, $\eta^p := p - I_h p$, and

$\eta^{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - I_h\boldsymbol{\sigma}$, to obtain

$$
\begin{aligned}
& B_{\phi_h}((\eta^{\boldsymbol{u}}, \eta^p, \eta^{\boldsymbol{\sigma}}), (-\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)) \\
=& \left(\frac{\eta^{\boldsymbol{\sigma}}}{2\mu(\phi_h)}, \boldsymbol{\tau}_h\right) - (\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}}), \boldsymbol{\tau}_h) + (q_h, \nabla\cdot\eta^{\boldsymbol{u}}) + (\eta^{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) \\
& - (\eta^p, \nabla\cdot\boldsymbol{v}_h) - \beta\left(2\mu(\phi_h)\left(\frac{\eta^{\boldsymbol{\sigma}}}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}})\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} + \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
& + \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\nabla\eta^p - \nabla\cdot\eta^{\boldsymbol{\sigma}}, \frac{1}{8\mu(\phi_h)}(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K \\
=& \left(\frac{\eta^{\boldsymbol{\sigma}}}{2\mu(\phi_h)}, \boldsymbol{\tau}_h\right) - (\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}}), \boldsymbol{\tau}_h) - (\nabla q_h, \eta^{\boldsymbol{u}}) + (\eta^{\boldsymbol{\sigma}}, \boldsymbol{\varepsilon}(\boldsymbol{v}_h)) \\
& - (\eta^p, \nabla\cdot\boldsymbol{v}_h) - \beta\left(2\mu(\phi_h)\left(\frac{\eta^{\boldsymbol{\sigma}}}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}})\right), \left(\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} + \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right)\right) \\
& + \sum_{K\in\mathcal{T}_h} m_l h_K^2 \left(\nabla\eta^p - \nabla\cdot\eta^{\boldsymbol{\sigma}}, \frac{1}{8\mu(\phi_h)}(\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h)\right)_K \\
\le& \frac{1}{2\mu_{\min}}\|\eta^{\boldsymbol{\sigma}}\|_{0,\Omega}\|\boldsymbol{\tau}_h\|_{0,\Omega} + \|\boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}})\|_{0,\Omega}\|\boldsymbol{\tau}_h\|_{0,\Omega} + |q_h|_{1,\Omega}\|\eta^{\boldsymbol{u}}\|_{0,\Omega} + \|\eta^{\boldsymbol{\sigma}}\|_{0,\Omega}\|\boldsymbol{\varepsilon}(\boldsymbol{v}_h)\|_{0,\Omega} \\
& + \|\eta^p\|_{0,\Omega}|\boldsymbol{v}_h|_{1,\Omega} + 2\beta\,\mu_{\max}\left\|\frac{\eta^{\boldsymbol{\sigma}}}{2\mu(\phi_h)} - \boldsymbol{\varepsilon}(\eta^{\boldsymbol{u}})\right\|_{0,\Omega}\left\|\frac{\boldsymbol{\tau}_h}{2\mu(\phi_h)} + \boldsymbol{\varepsilon}(\boldsymbol{v}_h)\right\|_{0,\Omega} \\
& + \sum_{K\in\mathcal{T}_h}\frac{m_l h_K^2}{8\mu_{\min}}\|\nabla\eta^p - \nabla\cdot\eta^{\boldsymbol{\sigma}}\|_{0,K}\,\|\nabla q_h - \nabla\cdot\boldsymbol{\tau}_h\|_{0,K} \\
\le& C\,h^l\left\{|\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega}\right\}|||(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h)|||_h,
\end{aligned} \tag{5.13}
$$

for all $(\boldsymbol{v}_h, q_h, \boldsymbol{\tau}_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h$. Let $e_h^{\boldsymbol{u}} := \boldsymbol{u}_h - I_h\boldsymbol{u}$, $e_h^p := p_h - I_h p$, $e_h^{\boldsymbol{\sigma}} := \boldsymbol{\sigma}_h - I_h\boldsymbol{\sigma}$. Next, from (4.14), (5.12) and Lemma 5.3, we can see that

$$
\begin{aligned}
& C_B\,|||(e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})|||_h^2 \le B_{\phi_h}((e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}}), (-e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})) \\
&= B_{\phi_h}((\boldsymbol{u}_h - \boldsymbol{u}, p_h - p, \boldsymbol{\sigma}_h - \boldsymbol{\sigma}), (-e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})) - B_{\phi_h}((\eta^{\boldsymbol{u}}, \eta^p, \eta^{\boldsymbol{\sigma}}), (-e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})) \\
&\le \left[C_{10}\,\|\phi_h - \phi\|_{1,\Omega}\left\{\|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}\right\} + C_7\,h^l\left\{|\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega}\right\}\right]|||(e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})|||_h,
\end{aligned}
$$

thus

$$
|||(e_h^{\boldsymbol{u}}, e_h^p, e_h^{\boldsymbol{\sigma}})|||_h \le \frac{C_{10}}{C_B}\|\phi_h - \phi\|_{1,\Omega}\left\{\|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega}\right\} + \frac{C_7}{C_B}h^l\left\{|\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega}\right\}. \tag{5.14}
$$

Finally, using (5.14), triangle inequality, and the inequality

$$
|||(\eta^{\boldsymbol{u}}, \eta^p, \eta^{\boldsymbol{\sigma}})|||_h \le C_8\,h^l\left\{|\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega}\right\},
$$

the result follows. □

Theorem 5.5. *Let* $(\boldsymbol{u}, p, \boldsymbol{\sigma}, \phi) \in (H^{l+1}(\Omega)^d \cap \boldsymbol{H}) \times (H^l(\Omega) \cap Q) \times (H^l(\Omega)^{d\times d} \cap \boldsymbol{R}) \times (H^{l+1}(\Omega)^d \cap V)$*, and* $(\boldsymbol{u}_h, p_h, \boldsymbol{\sigma}_h, \phi_h) \in \boldsymbol{H}_h \times Q_h \times \boldsymbol{R}_h \times V_h$ *be the solutions of* (5.1), (5.3) *and* (5.2), (5.4)*, respectively, with*

$\phi \in W$ and $\phi_h \in W_h$. Then, there exists a positive constant C_{13}, independent of h, and a positive number h_0 such that for all $h < h_0$ we have that

$$\|\!|(\boldsymbol{u}-\boldsymbol{u}_h, p-p_h, \boldsymbol{\sigma}-\boldsymbol{\sigma}_h)|\!\|_h + \|\phi-\phi_h\|_{1,\Omega} \le C_{13}\, h^l \left\{ |\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega} + \|\phi\|_{l+1,\Omega} \right\}.$$

Proof. Using lemmas 5.2 and 5.4 we get

$$\begin{aligned}\|\!|(\boldsymbol{u}_h-\boldsymbol{u}, p_h-p, \boldsymbol{\sigma}_h-\boldsymbol{\sigma})|\!\|_h &\le C_{11} \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega} \right\} \|\phi_h-\phi\|_{1,\Omega} + C_{12}\, h^l \left\{ |\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega} \right\} \\ &\le C_{11} \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega} \right\} \left\{ C_8\, h\, \|\phi\|_{2,\Omega}\, \|\boldsymbol{\varepsilon}(\boldsymbol{u}-\boldsymbol{u}_h)\|_{0,\Omega} + C_9\, h^l\, |\phi|_{l+1,\Omega} \right\} + C_{12}\, h^l \left\{ |\boldsymbol{u}|_{l+1,\Omega} + |p|_{l,\Omega} + \|\boldsymbol{\sigma}\|_{l,\Omega} \right\},\end{aligned}$$

now, taking h such that $C_8\, C_{11}\, h \left\{ \|\boldsymbol{f}\|_{\infty,\Omega} + \|\boldsymbol{\sigma}\|_{1,\Omega} \right\} \|\phi\|_{2,\Omega} < 1$, the result follows. □

6. Numerical examples

In this section, we focus on showing convergence rates for the proposed scheme with different analytical solutions and assess the performance of the stabilized method for a benchmark coming from the geosciences. All calculations have been obtained using the open source finite element library `FEniCS` [31] . Furthermore, the stabilization parameters employed to implement the proposed method are $\beta = 1/2$, $\gamma = 1/24$ and $m_l = 1/3$.

6.1. Example 1: Analytical solution in two dimensions

In the first example, inspired by a test case introduced in [3], we choose $\Omega = (0,1)^2$, and the source terms such that the exact solution of problem (P) is given by

$$\boldsymbol{u}(x,y) = \begin{pmatrix} \sin(2\pi x)\cos(2\pi y) \\ -\cos(2\pi x)\sin(2\pi y) \end{pmatrix}, \quad p(x,y) = \sin(2\pi x)\sin(2\pi y), \quad \phi(x,y) = 15 - 15\exp(-x(x-1)y(y-1)),$$

while the physical parameters are $\mu(r) = \dfrac{1}{(1-0.5r)^2}$, $\kappa = 1$, and $\alpha = 1$. Note that $0 \le \phi \le 1$ in Ω, and that $|\mu'(r)| \le 8$ for $0 < r < 1$, thus μ is a Lipschitz function.

In Figure 1, we present the exact and approximated solutions in a mesh with 32,768 elements. Note that there is a remarkable agreement between both solutions.

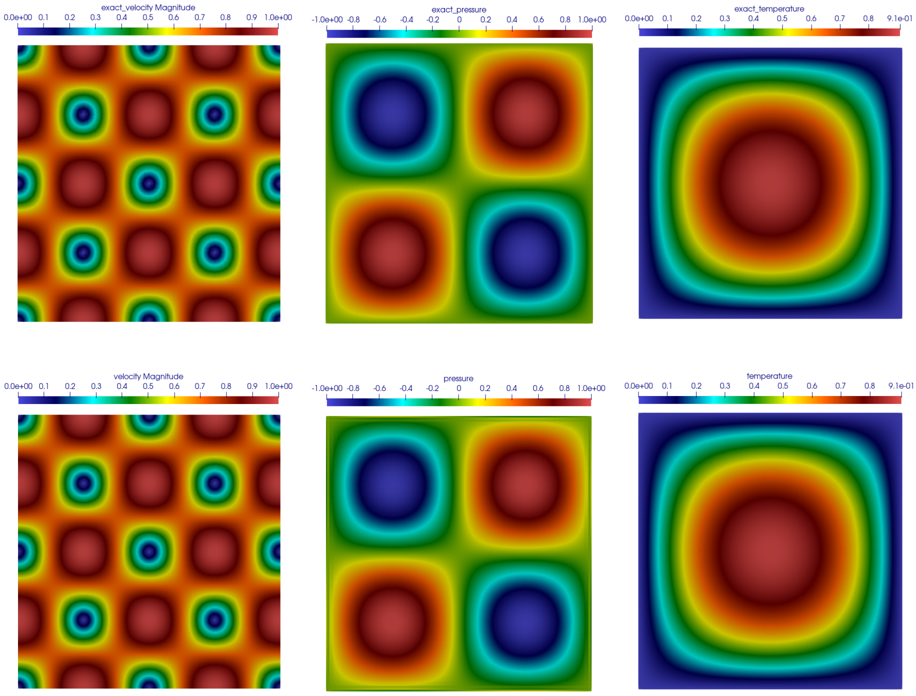


Figure 1: Isolines of exact solutions (top) and computed solutions (bottom) using $\mathbb{P}_1$ finite element spaces for each variable, in a mesh with 32,768 elements; velocity magnitude (left), pressure (medium) and temperature (right).

Next, in Figure 2 we show the convergence orders which are obtained using a sequence of uniform refined meshes. Note the agreement between the convergence results with those predicted by the theory in Theorem 5.5.

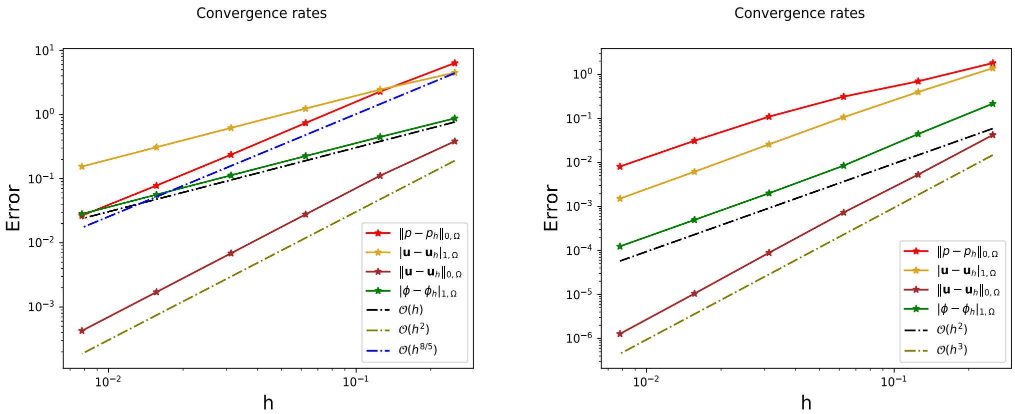


Figure 2: Convergence plots obtained using $\mathbb{P}_1$ (left) and $\mathbb{P}_2$ (right) polynomial spaces.

6.2. Example 2: A benchmark for a subduction zone

The second example has been taken from [38] and the geometry consists of a kinematic slab driving flow in the viscous mantle wedge below a rigid overriding plate. Spite of this test case do not fulfilled all the conditions of our framework, we emphasize that our goal is to check a good working of our stabilized method on a realistic problem. In this case $\Omega = (0, 660) \times (0, 600)$ which represents a subduction zone in two dimensions. The physical parameters to define this problem are given in Table 1

Table 1: Physical parameters and nomenclature.

Quantity	**Symbol**	**Reference value**
Velocity of the slab	$\boldsymbol{u}_D$	5 cm/yr
Temperature on right-middle boundary	ϕ_M	1573 K
Temperature on surface	ϕ_S	273 K
Density	ρ	3300 kg/m^3
Heat capacity	c_p	1250 J/kg K
Thermal conductivity	k	3 W/mK
Powerlaw for dislocation creep	n	3.5
Activation energy (dislocation creep)	E	540 kJ/mol
Pre-exponential constant (dislocation creep)	A	28968.6 Pa s$^{1/n}$
Maximum viscosity	$\hat{\mu}_{max}$	10^{26} Pa s
Gas constant	R	8.3145 J/mol K
Age in seconds	t_{50}	50 Myr

The boundary conditions are represented in Figure 3 where the plate is 50 km deep, the interface between the slab and wedge has a 45° tilt and the right-middle boundary, where ϕ_M is prescribed as boundary condition, is 200 km deep.

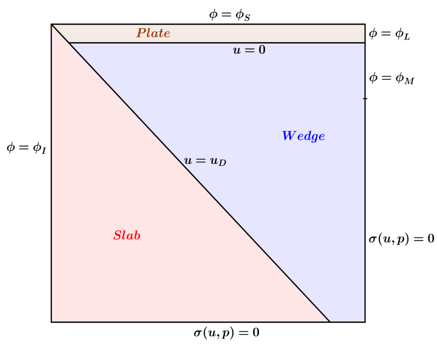


Figure 3: Domain and boundary conditions.

The boundary functions ϕ_L and ϕ_I are defined by

$$\phi_L(x, y) := \frac{(\phi_S - \phi_M)}{5 \times 10^4} y + \phi_S, \quad \phi_I(x, y) := \phi_S + (\phi_M - \phi_S)\,\mathrm{erf}\left(\frac{y}{2\sqrt{\kappa\, t_{50}}}\right).$$

In addition, the viscosity of olivine deforming by dislocation creep, assuming zero activation volume, is given by

$$\hat{\mu} := A \exp\left(\frac{E}{nR\phi}\dot{\epsilon}^{(1-n)/n}\right),$$

where $\dot{\epsilon} := \left[\frac{1}{2}\varepsilon(\boldsymbol{u}) : \varepsilon(\boldsymbol{u})\right]^{1/2}$ is the second invariant of the strain-rate tensor. Finally, the source terms are given by $\boldsymbol{f} := (0, 0)$ and $g := 0$, respectively.

Such as is explained in [38], to avoid some numerical artifacts it is convenient to truncate the viscosity $\hat{\mu}$ at a fixed value $\hat{\mu}_{max}$. Hence, we will use the following modified viscosity

$$\mu := \left(\frac{1}{\hat{\mu}} + \frac{1}{\hat{\mu}_{max}} \right)^{-1}.$$

In Figure 4 we present the results obtained with our scheme using $\mathbb{P}_1$ finite element spaces for each variable in a mesh with $59,442$ elements. A qualitative comparison between our results and those of [38] is presented in Figure 5. Note that both results are quite similar.

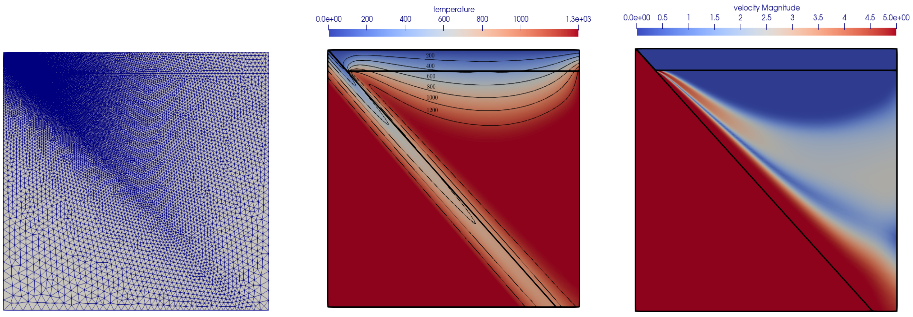


Figure 4: Mesh (left), temperature (middle) and magnitude of the velocity field (right).

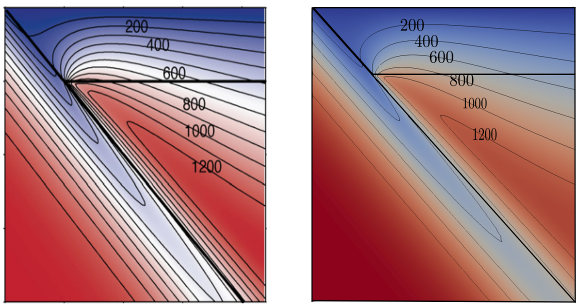


Figure 5: Close up of the model with dislocation creep. Benchmark results introduced in [38] (left) and results using the proposed stabilized scheme (right).

In addition, following [38], we compared the computed values of the temperature field ϕ_h, as discrete values ϕ_{ij}, on an equidistant grid with 6 km spacing, which generate a 111×101 matrix stored row-wise starting in the top left corner of the domain Ω. The results of interest are: $\phi_{11,11}$, the temperature at the coordinates $(60, -60)$; ϕ_{slab} which is the l^2-norm of the slab-wedge interface temperature between 0 and 210 km depth and ϕ_{wedge}, the l^2- norm of the temperature at the triangular part of the wedge's tip, between 54 and 120 km depth

$$\|\phi_{slab}\| = \left(\frac{1}{36} \sum_{i=1}^{36} \phi_{ii}^2 \right)^{1/2} \qquad \text{and} \qquad \|\phi_{wedge}\| = \left(\frac{1}{78} \sum_{i=10}^{21} \sum_{j=10}^{i} \phi_{ii}^2 \right)^{1/2}.$$

The results obtained using our scheme are presented in Table 2, and compared with those introduced in [38]. There, we can observe that our results are in the range of accepted results if we compare them with those presented in Table 2 of [38].

Table 2: Comparative benchmarks results.

Code	$T_{11,11}$	$\|T_{slab}\|$	$\|T_{wedge}\|$
LDEO	550.17	593.48	994.11
NTU	551.60	608.85	984.08
PGC	582.65	604.51	998.71
UM	583.36	605.11	1000.01
VT	574.84	603.80	995.24
WHOI	583.11	604.96	1000.05
Present Method	546.75	598.71	988.03

6.3. *Example 3: Analytical solution in three dimensions*

In this example the computational domain is $\Omega = (0,1)^3$, and the source terms $\boldsymbol{f}$ and g are chosen such that the exact solution of problem (P) is given by

$$\boldsymbol{u}(x,y,z) = \frac{1}{2}(-y^2, z^2, x^2), \quad p(x,y,z) = -0.125 + xyz, \quad \text{and} \quad \phi(x,y,z) = 0.1 - 0.3e^{-xyz},$$

where the velocity as well the pressure have been taken from [5]. The viscosity is given by $\mu(r) = 0.5e^{0.1/r}$ and the thermal conductivity used is $k = 1$. Analogously to Example 1, it is possible to observe that $-0.2 \leq \phi \leq -0.1$, and thus $|\mu'(r)| \leq 1.83$ for $r \in (-0.2\,, -0.1)$, and therefore μ is a Lipschitz continuous function.

The obtained solutions are collected in Figure 6. The numerical results reported here confirm the good performance of the stabilized finite element scheme (4.1).

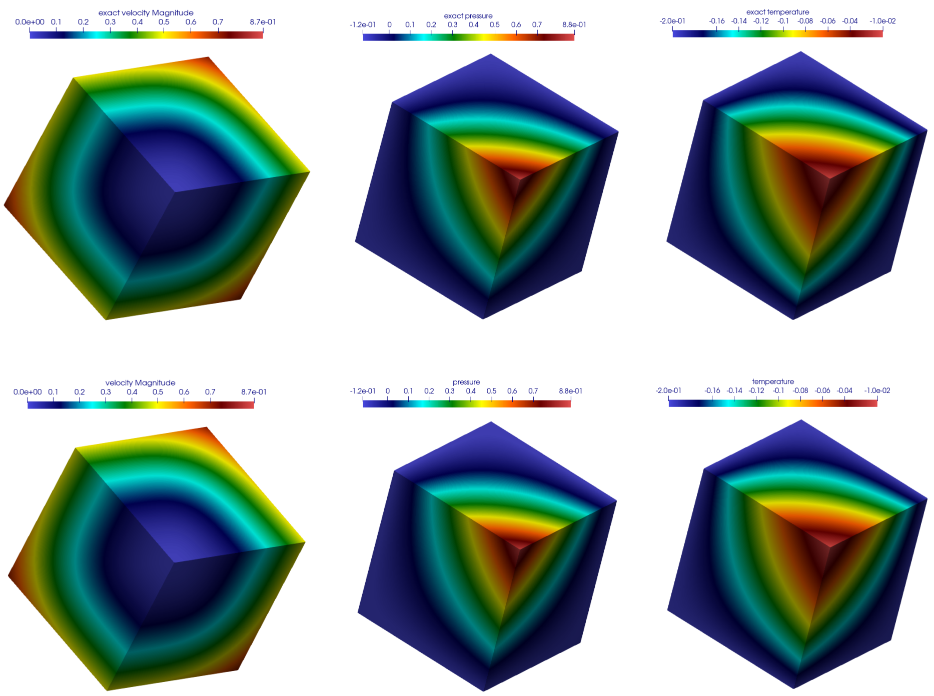


Figure 6: Isovalues of the exact solutions (top) and computed solutions (bottom) using $\mathbb{P}_1$ finite element spaces for each variable, in a mesh with 24,576 elements; velocity (left), pressure (medium) and temperature (right).

Figure 7 displays convergence curves for the problem. These results confirm the theoretical results, where the error decays with the expected order.

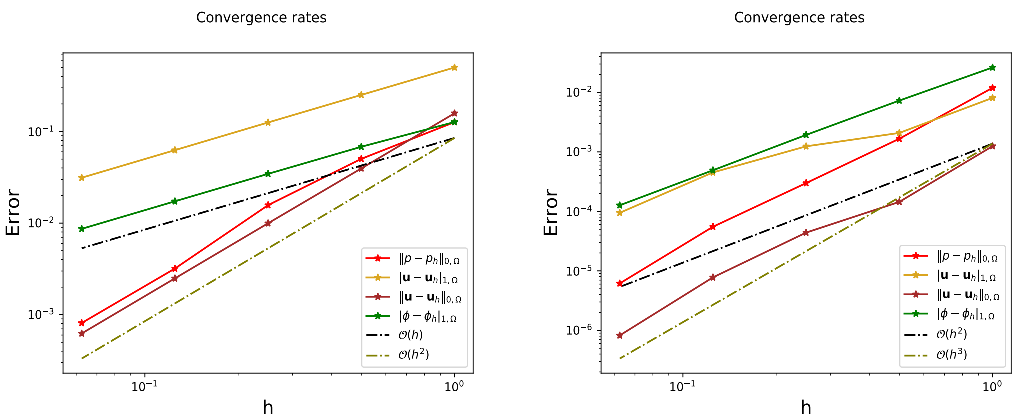


Figure 7: Convergence plots obtained using $\mathbb{P}_1$ (left) and $\mathbb{P}_2$ (right) polynomial spaces.

Conclusions

A new stabilized mixed method for the coupling of Stokes and temperature equations, with a fluid viscosity depending on the temperature, is proposed, analyzed, and tested numerically. This new scheme, which adds stress as a new unknown, allows equal order of interpolation for all the unknowns defining the stabilized method, which is a characteristic of interest of the practitioners. Using fixed point theorems we prove the existence of a solution, both in the continuous as in the discrete cases. The scheme has been assessed numerically with 2D and 3D tests, corroborating the theoretical a priori results and showing effectiveness in solving realistic problems.

Acknowledgements

The first author was partially supported by ANID-Chile through the projects: Centro de Modelamiento Matemático (FB210005) of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal, Anillo ACT1118 and Fondecyt Regular No 1211649. The second author was partially supported by ANID-Chile through the project Centro de Modelamiento Matemático (FB210005) of the PIA Program: Concurso Apoyo a Centros Científicos y Tecnológicos de Excelencia con Financiamiento Basal. Finally, the third author was partially supported by Dirección de Investigación of the Universidad Católica de la Santísima Concepción through project DIREG 05/2022 and Fondecyt Regular No 1211649.

References

[1] Allendes, A., Barrenechea, G.R., Naranjo, C., 2018. A divergence-free low-order stabilized finite element method for a generalized steady state Boussinesq problem. Comput. methods Appl. Mech. Engrg. 340, 90–120. doi:`10.1016/j.cma.2018.05.020`.

[2] Alvarez, M., Gatica, G., Ruiz-Baier, R., 2016. A posteriori error analysis for a viscous flow-transport problem. ESAIM Math. Model. Numer. Anal. 50, 1789–1816. doi:`10.1051/m2an/2016007`.

[3] Alvarez, M., Gatica, G.N., Ruiz–Baier, R., 2015. An augmented mixed-primal finite element method for a coupled flow-transport problem. ESAIM: M2AN 49, 1399–1427. doi:`10.1051/m2an/2015015`.

[4] Araya, R., Barrenechea, G., Poza, A., Valentin, F., 2012. Convergence analysis of a residual local projection finite element method for the Navier-Stokes equations. SIAM J. Numer. Anal. 50, 669–699. doi:`10.1137/110829283`.

[5] Araya, R., Cárcamo, C., Poza, A., 2021. An adaptive stabilized finite element method for the Darcy's equations with pressure dependent viscosities. Comput. methods Appl. Mech. Engrg. 387, 114100. doi:`10.1016/j.cma.2021.114100`.

[6] Araya, R., Poza, A., Valentin, F., 2016. A low-order local projection method for the incompressible Navier-Stokes equations in two- and three-dimensions. IMA J. Numer. Anal. 36, 267–295. doi:10.1093/imanum/drv004.
[7] Aytekin, C., Songül, K., 2011. A projection-based stabilized finite element method for steady-state natural convection problem. J. Math. Anal. Appl. 381, 469–484. doi:10.1016/j.jmaa.2011.02.020.
[8] Banach, S., 1922. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund. Math. 3, 133–181. doi:10.4064/fm-3-1-133-181.
[9] Barrenechea, G., Castillo, E., Codina, R., 2019. Time-dependent semidiscrete analysis of the viscoelastic fluid flow problem using a variational multiscale stabilized formulation. IMA J. Numer. Anal. 39, 792–819. doi:10.1093/imanum/dry018.
[10] Barrenechea, G., Valentin, F., 2010. Consistent local projection stabilized finite element methods. SIAM J. Numer. Anal. 48, 1801–1825. doi:10.1137/090753334.
[11] Barrenechea, G., Valentin, F., 2011. Beyond pressure stabilization: a low-order local projection method for the Oseen equation. Internat. J. Numer. Methods Engrg. 86, 801–815. doi:10.1002/nme.3075.
[12] Becker, R., Braack, M., 2001. A finite element pressure gradient stabilization for the Stokes equations based on local projections. Calcolo 38, 173–199. doi:10.1007/s10092-001-8180-4.
[13] Bernardi, C., Métivet, B., Pernaud-Thomas, B., 1995. Couplage des équations de Navier-Stokes et de la chaleur : le modèle et son approximation par éléments finis. ESAIM: M2AN 29, 871–921. doi:10.1051/m2an/1995290708711.
[14] Bochev, P., Gunzburger, M.D., 2009. Least-squares finite element methods. volume 166 of *Applied Mathematical Sciences*. Springer Verlag, New York. doi:10.1007/b13382.
[15] Brooks, A., Hughes, T.R., 1982. Streamline upwind/Petrov-Galerkin formulations for convection dominated flows with particular emphasis on the incompressible Navier-Stokes equations. Comput. Methods Appl. Mech. Engrg. 32, 199–259. doi:10.1016/0045-7825(82)90071-8.
[16] Brouwer, L.E.J., 1911. Über abbildung von mannigfaltigkeiten. Math. Ann. 71, 97–115. doi:10.1007/BF01456931.
[17] Burman, E., Fernández, M.A., Hansbo, P., 2006. Continuous interior penalty finite element method for Oseen's equations. SIAM J. Numer. Anal. 44, 1248–1274. doi:10.1137/040617686.
[18] Castillo, E., Codina, R., 2014. Variational multi-scale stabilized formulations for the stationary three-field incompressible viscoelastic flow problem. Comput. Methods Appl. Mech. Engrg. 279, 579–605. doi:10.1016/j.cma.2014.07.006.
[19] Castillo, E., Codina, R., 2017. Finite element approximation of the viscoelastic flow problem: a non-residual based stabilized formulation. Comput. & Fluids 142, 72–78. doi:10.1016/j.compfluid.2016.07.012.
[20] Caucao, S., Gatica, G. N., Oyarzúa, R., 2018. Analysis of an augmented fully-mixed formulation for the coupling of the Stokes and heat equations. ESAIM: M2AN 52, 1947–1980. doi:10.1051/m2an/2018027.
[21] Codina, R., 2008. Analysis of a stabilized finite element approximation of the Oseen equations using orthogonal subscales. Appl. Numer. Math. 58, 264–283. doi:10.1016/j.apnum.2006.11.011.
[22] Cox, C., Lee, H., Szurley, D., 2007. Finite element approximation of the non-isothermal Stokes-Oldroyd equations. Int. J. Numer. Anal. Model. 4, 425–440.
[23] Ern, A., Guermond, J.L., 2004. Theory and practice of finite elements. volume 159 of *Applied Mathematical Sciences*. Springer-Verlag, New York. doi:10.1007/978-1-4757-4355-5.
[24] Farhloul, M., Nicaise, S., Paquet, L., 2000. A mixed formulation of boussinesq equations: Analysis of nonsingular solutions. Math. Comp. 69, 965–986. doi:10.1090/S0025-5718-00-01186-8.
[25] Franca, L., Frey, S., Hughes, T.R., 1992. Stabilized finite element methods. I. Application to the advective-diffusive model. Comput. Methods Appl. Mech. Engrg. 95, 253–276. doi:10.1016/0045-7825(92)90143-8.
[26] Franca, L., Harder, C., Valentin, F., 2009. On a residual local projection method for the Darcy equation. C. R. Math. Acad. Sci. Paris 347, 1105–1110. doi:10.1016/j.crma.2009.06.016.
[27] Furukawa, Y., 1993. Depth of the decoupling plate interface and thermal structure under arcs. J. Geophys. Res. 98, 20005–20013. doi:10.1029/93JB02020.
[28] Gatica, G.N., Heuer, N., Meddahi, S., 2003. On the numerical analysis of nonlinear twofold saddle point problems. IMA J. Numer. Anal. 23, 301–330. doi:10.1093/imanum/23.2.301.
[29] Girault, V., Raviart, P., 1986. Finite element methods for Navier–Stokes equations: Theory and algorithms. Springer Series in Computational Mathematics, Springer-Verlag Berlin Heidelberg. doi:10.1007/978-3-642-61623-5.
[30] Hughes, T.R., Franca, L., Hulbert, G.M., 1989. A new finite element formulation for computational fluid dynamics. VIII. The Galerkin/least-squares method for advective-diffusive equations. Comput. Methods Appl. Mech. Engrg. 73, 173–189. doi:10.1016/0045-7825(89)90111-4.
[31] Logg, A., Mardal, K., Wells, G., 2012. Automated solution of differential equations by the finite element method. The FEniCS book. volume 84 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag Berlin Heidelberg. doi:10.1007/978-3-642-23099-8.
[32] Moreno, L., Codina, R., Baiges, J., 2021. Numerical simulation of non-isothermal viscoelastic fluid flows using a VMS stabilized finite element formulation. J. Non-Newton. Fluid Mech. 296, Paper No. 104640, 18. doi:10.1016/j.jnnfm.2021.104640.
[33] Pérez, C.E., Thomas, J.M., Blancher, S., Creff, R., 2008. The steady Navier–Stokes/energy system with temperature-dependent viscosity—part 2: The discrete problem and numerical experiments. Int. J. Numer. Methods Fluids 56, 91–114. doi:10.1002/fld.1572.
[34] Quarteroni, A., Valli, A., 2008. Numerical Approximation of Partial Differential Equations. volume 23 of *Springer Series in Computational Mathematics*. Springer Berlin, Heidelberg. doi:10.1007/978-3-540-85268-1.
[35] Schauder, J., 1930. Der fixpunktsatz in funktionalraümen. Stud. Math. 2, 171–180. doi:10.4064/sm-2-1-171-180.
[36] Schubert, G., Turcotte, D.L., Olson, P., 2001. Mantle Convection in the Earth and Planets. Cambridge University Press. doi:10.1017/CBO9780511612879.

[37] Tobiska, L., Verfürth, R., 1996. Analysis of a streamline diffusion finite element method for the Stokes and Navier-Stokes equations. SIAM J. Numer. Anal. 33, 107–127. doi:10.1137/0733007.

[38] van Keken, P.E., Currie, C., King, S.D., Behn, M.D., Cagnioncle, A., He, J., Katz, R.F., Lin, S.C., Parmentier, E.M., Spiegelman, M., Wang, K., 2008. A community benchmark for subduction zone modeling. Phys. Earth Planet. Inter. 171, 187–197. doi:10.1016/j.pepi.2008.04.015.

[39] van Keken, P.E., Kiefer, B., Peacock, S.M., 2002. High-resolution models of subduction zones: Implications for mineral dehydration reactions and the transport of water into the deep mantle. Geochem. Geophys. Geosyst. 3, 1056. doi:10.1029/2001GC000256.

[40] van Keken, P.E., King, S.D., Schmeling, H., Christensen, U.R., Neumeister, D., Doin, M.P., 1997. A comparison of methods for the modeling of thermochemical convection. J. Geophys. Res. 102, 22477–22495. doi:10.1029/97JB01353.

[41] Varchanis, S., Syrakos, A., Dimakopoulos, Y., Tsamopoulos, J., 2020. PEGAFEM-V: a new Petrov-Galerkin finite element method for free surface viscoelastic flows. J. Non-Newton. Fluid Mech. 284, 104365, 17. doi:10.1016/j.jnnfm.2020.104365.

[42] Venkatesan, J., Ganesan, S., 2020. Finite element computations of viscoelastic two-phase flows using local projection stabilization. Internat. J. Numer. Methods Fluids 92, 825–854. doi:10.1002/fld.4808.

[43] Vynnytska, L., Rognes, M., Clark, S., 2013. Benchmarking FEniCS for mantle convection simulations. Comput. and Geosci. 50, 95–105. doi:10.1016/j.cageo.2012.05.012.

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2022

2022-16 Sergio Caucao, Eligio Colmenares, Gabriel N. Gatica, Cristian Inzunza: *A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem*

2022-17 Felipe Lepe, David Mora, Gonzalo Rivera, Iván Velásquez: *A posteriori virtual element method for the acoustic vibration problem*

2022-18 Franz Chouly: *A review on some discrete variational techniques for the approximation of essential boundary conditions*

2022-19 Veronica Anaya, Ruben Caraballo, Sergio Caucao, Luis F. Gatica, Ricardo Ruiz-Baier, Ivan Yotov: *A vorticity-based mixed formulation for the unsteady Brinkman-Forchheimer equations*

2022-20 Olga Barrera, Stéphane P. A. Bordas, Raphaël Bulle, Franz Chouly, Jack S. Hale: *An a posteriori error estimator for the spectral fractional power of the Laplacian*

2022-21 Stéphane P. A. Bordas, Raphaël Bulle, Franz Chouly, Jack S. Hale, Alexei Lozinski: *Hierarchical a posteriori error estimation of Bank–Weiser type in the Fenics project*

2022-22 Franz Chouly, Patrick Hild, Vanessa Lleras, Yves Renard: *Nitsche method for contact with Coulomb friction: existence results for the static and dynamic finite element formulations*

2022-23 Franz Chouly, Patrick Hild: *On a finite element approximation for the elastoplastic torsion problem*

2022-24 Raimund Bürger, Stefan Diehl, María Carmen Martí, Yolanda Vásquez: *A difference scheme for a triangular system of conservation laws with discontinuous flux modeling three-phase flows*

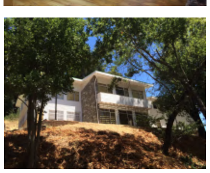
2022-25 Raimund Bürger, Harold D. Contreras, Luis M. Villada: *A Hilliges-Weidlich-type scheme for a one-dimensional scalar conservation law with nonlocal flux*

2022-26 Claudio I. Correa, Gabriel N. Gatica, Ricardo Ruiz-Baier: *New mixed finite element methods for the coupled Stokes and Poisson-Nernst-Planck equations in Banach spaces*

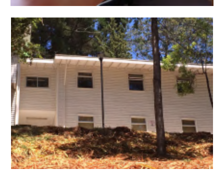
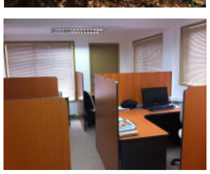
2022-27 Rodolfo Araya, Cristian Cárcamo, Abner Poza: *A stabilized finite element method for the Stokes–Temperature coupled problem*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



CENTRO DE INVESTIGACIÓN EN
INGENIERÍA MATEMÁTICA (CI^{2}MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl