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#### A HILLIGES-WEIDLICH-TYPE SCHEME FOR A ONE-DIMENSIONAL SCALAR CONSERVATION LAW WITH NONLOCAL FLUX

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ABSTRACT. The simulation model proposed in [M. Hilliges and W. Weidlich. A phenomenological model for dynamic traffic flow in networks. Transportation Research Part B: Methodological, 29(6):407-431, 1995] can be understood as a simple method for approximating solutions of scalar conservation laws whose flux is of density times velocity type, where the density and velocity factors are evaluated on neighboring cells. The resulting scheme is monotone and converges to the unique entropy solution of the underlying problem. The same idea is applied to devise a numerical scheme for a class of one-dimensional scalar conservation laws with nonlocal flux and initial and boundary conditions. Uniqueness of entropy solutions to the nonlocal model follows the from Lipschitz continuous dependence of a solution on initial and boundary data. By various uniform estimates, namely a maximum principle and bounded variation estimates, along with a discrete entropy inequality, the sequence of approximate solutions is shown to converge to an entropy weak solution of the nonlocal problem. The improved accuracy of the proposed scheme in comparison to schemes based on the Lax-Friedrichs flux is illustrated by numerical examples. A second-order scheme based on MUSCL methods is presented.

#### 1. Introduction.

1.1. **Scope.** Nonlocal conservation laws model various phenomena, such as the dynamics of crowds [8–10], vehicular traffic [7, 18], supply chains [3], granular materials [1], and sedimentation phenomena [5]. These nonlocal equations are given by expressions of the type

$$\partial_t \rho + \operatorname{div}_{\boldsymbol{x}} F(\boldsymbol{x}, t, \rho, W) = 0, \quad t > 0, \quad \boldsymbol{x} \in \mathbb{R}^d, \ d \ge 1,$$

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where  $\rho = \rho(\mathbf{x}, t) \in \mathbb{R}^N$ ,  $N \geq 1$  is the vector of the conserved quantities and the variable  $W = W(\mathbf{x}, t, \rho)$  depends on an integral evaluation of  $\rho$ . The aim of this work is to propose an approach for a rigorous treatment of boundary conditions in the case of a spatially one-dimensional nonlocal problems, through development of new numerical schemes that are more accurate and less diffusive. The strategies that we employ are inspired by the results obtained in [5,6,17]. Particularly, we propose to study a simplification of the problem studied in [5], we adopt the treatment of the boundary conditions proposed in [17] and we present a numerical scheme based on local one studied in [6, 19]. Our proposed scheme takes advantage of the form in which the flow is written, namely density  $\rho$  times a local decreasing factor  $g(\rho)$  times a nonlocal convolution term  $V(x, t) = (\omega * v(\rho))$ , where v is a given velocity function and  $\omega$  is a convolution kernel such that the governing conservation law becomes

$$\partial_t + \partial_x \big( \rho g(\rho) V(x, t) \big) = 0. \tag{1.1}$$

In the case of a standard (local) conservation law, captured by setting V = const., the above-mentioned approach results in a monotone scheme [6], so it is possible to invoke standard arguments to prove its convergence to an entropy solution. This idea is extended herein to the nonlocal equation (1.1), although we emphasize that that the resulting scheme is *not* monotone.

1.2. Related work. There are many works about existence and uniqueness results for nonlocal equations, see e.g. [2, 14, 15, 17] for the scalar case in one space dimension. In these papers a first-order Lax-Friedrichs (LxF)-type numerical scheme is used to approximate the problem and to prove the existence of solutions and the nonlocal term is considered as a convolution between a kernel function and the unknown (mean downstream density approach). LxF-type schemes are the most common approach used to solve nonlocal conservation laws because they are easy to implement and due to their monotonicity, they make it possible to numerically analyze nonlocal flux problems. Their well-known main disadvantage is, however, their large amount of numerical diffusion that smears out sharp features of the exact solution. To reduce this phenomenon, Friedrich et al. [16] proposed a Godunov-type numerical scheme where the nonlocal term is considered as a convolution between a kernel function and the velocity of unknown (mean downstream velocity approach). We adopt this idea about the convolution to propose and develop our model and computations. A well-known early analysis of initial-boundary value problems (IBVP) for conservation laws is due to Bardos et al. [4], existence, uniqueness and continuous dependence of the solution on initial data in the case of zero boundary data are prove. These results were extended to more general but smooth boundary data by Colombo and Rossi [11]. Rossi [21] studied an IBVP for a general scalar balance law in one space dimension. Under rather general assumptions on the flux and source functions, the author proves the well-posedness of the problem and stability of its solutions with respect to variations in the flux and the source terms. For both results, the initial and boundary data are required to be bounded functions with bounded total variation. In [14] a global well-posedness result for a class of weak entropy solutions of bounded variations of scalar conservation laws with nonlocal flux on bounded domains is established under suitable regularity assumptions on the flux function. The nonlocal operator is the standard convolution product. The existence of solutions is obtained by proving the convergence of an adapted LxF

algorithm. Lipschitz continuous dependence from initial and boundary data is derived applying Kružkovs doubling of variables technique. In [17] Goatin and Rossi study the same problem as Filippis and Goatin [14], but with a different approach, namely following the treatment of the boundary conditions proposed by Colombo and Rossi in [12] where a particular multi-dimensional system of conservation laws in bounded domains with zero boundary conditions was considered. More specifically, the nonlocal operator in the flux function is not a mere convolution product, but it is assumed to be 'aware' of boundaries and by introducing an adapted LxF algorithm, various estimates on the approximate solutions that allow to prove the existence of solutions to the original IBVP are introduced. Uniqueness was derived from the Lipschitz continuous dependence on initial and boundary data, which is proved exploiting results available for the local problem.

1.3. Outline of the paper. This work is organized as follows: In Section 2 we present the considered class of nonlocal conservation laws and the assumptions needed on studied problem as well as the main result of this paper, whose proof is postponed to end of Section 3. Lipschitz continuous dependence of solutions to the studied problem on initial and boundary data is proved in Section 3. Afterwards, in Section 4 we introduce the numerical scheme and derive some of its important properties such as the maximum principle, **BV**, and  $L^1$  Lipschitz continuity in time estimates. These imply convergence of the scheme proposed, which in turn covers the existence part of the well-posedness of the governing model. Throughout the paper we address the new scheme as a "HW scheme" according to the proponent of the original idea (Hilliges and Weidlich [19]), and in Section 5 we provide a second-order version of a HW-type numerical scheme. Finally, In Section 6 we present some numerical examples, analysing the  $L^1$ -error of the approximate solutions of studied problem computed with different schemes. Appendix A collects some estimates necessary throughout the paper.

2. Initial-boundary value problem. We consider a particular initial-boundary value problem which is a version of a nonlocal model of sedimentation proposed in [5]. Our model has the following structure:

$$\partial_t \rho + \partial_x (f(\rho)V(x,t)) = 0, \quad (x,t) \in ]a, b[\times \mathbb{R}^+,$$
  

$$\rho(x,0) = \rho_0(x), \quad x \in ]a, b[,$$
  

$$\rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), \quad t \in \mathbb{R}^+.$$
(2.1)

where

$$f(\rho) := \rho g(\rho), \tag{2.2}$$

$$V(x,t) := \left(\omega * v(\rho)\right)(x,t) = \frac{1}{W(x)} \int_a^b v\left(\rho(y,t)\right) \omega(y-x) \,\mathrm{d}y \tag{2.3}$$

with  $W(x) := \int_a^b \omega(y-x) \, \mathrm{d}y$  for a suitable convolution kernel  $\omega$ .

**Remark 2.1.** The particular combination of local and nonlocal evaluations of  $\rho$  present in (2.2), (2.3) can be motivated by following the discussion of [5, Sect. 1.2] for a model of sedimentation. Namely, if we assume that the nonlocal model describes the volume fraction of solids  $\rho \in [0,1]$  within a solid-fluid two-phase flow system, then the solid and fluid conservation equations in differential form are  $\partial_t \rho + \partial_x (\rho v_s) = 0$  and  $\partial_t (1 - \rho) + \partial_x ((1 - \rho)v_f) = 0$ , where  $v_s$  and  $v_f$  are the solid and fluid phase velocities and x is the vertical spatial coordinate. One then

defines the volume average velocity of the mixture  $q := \rho v_s + (1-\rho)v_f$  and the solidfluid relative velocity  $v_r = v_s - v_f$ . Now for the particular case of batch settling in a closed column, we have q = 0 for all x and t, and then  $\rho v_s = \rho(1-\rho)v_r$ , so that the unique PDE to be solved is

$$\partial_t \rho + \partial_x \left( \rho (1 - \rho) v_{\mathbf{r}} \right) = 0, \tag{2.4}$$

where  $v_r$  is specified by some constitutive function. This scenario corresponds to (2.1)-(2.3) if we choose  $g(\rho) = 1 - \rho$  and assume that  $v_r$  is given through the nonlocal convolution

$$v_{\mathbf{r}} = v_{\mathbf{r}}(x,t) = (\omega * v(\rho))(x,t), \qquad (2.5)$$

where  $\rho \mapsto v(\rho)$  is a given, in general nonlinear function. In other words, the local and nonlocal evaluations of  $\rho$  in (2.1) arise from the combination of properly defined volume fractions in mixture theory with the constitutive assumption (2.5). The standard local evaluation  $v_r(x,t) = v(\rho(x,t))$  corresponds to the well-known kinematic sedimentation model, while utilizing  $v(\omega * \rho)(x,t)$  instead of  $(\omega * v(\rho))(x,t)$ is the model alternative explored in [5].

**Assumptions 2.1.** The initial-boundary value problem (2.1) is studied under the following assumptions:

- (i) The initial function satisfies  $\rho_0 \in BV(I; \mathbb{R}^+)$ , where  $I := ]a, b] \subseteq \mathbb{R}^+$ .
- (ii) The function g satisfies  $g \in C^2([0,1]; \mathbb{R}^+_0)$ ,  $g'(\rho) \leq 0$  for  $\rho \in [0,1]$ , and g(1) = 0.
- (iii) The function v satisfies  $v \in C^2([0,1]; \mathbb{R}^+)$ ,  $v'(\rho) \leq 0$  for  $\rho \in [0,1]$ , and  $0 = v(1) \leq v(\rho) \leq v(0) = 1$ .
- (iv) The convolution kernel  $\omega$  satisfies  $\omega \in (C^2 \cap W^{2,1} \cap W^{2,\infty})(\mathbb{R};\mathbb{R})$  such that  $\int_{\mathbb{R}} \omega(y) \, \mathrm{d}y = 1$ , and there exists  $K_{\omega} > 0$  such that for all  $x \in I$ ,  $W(x) = \int_{a}^{b} \omega(y-x) \, \mathrm{d}y \geq K_{\omega}$ .

In what follows, we denote  $\|\cdot\|_{L^{\infty}([0,1])} := \|\cdot\|_{\infty}$ . The weak entropy solution of problem (2.1) is defined, as in [14,17], in the following sense:

**Definition 2.1.** A function  $\rho \in (L^1 \cap L^\infty \cap BV)(]a, b[\times \mathbb{R}^+; \mathbb{R})$  is an entropy weak solution if for all  $\varphi \in C^1_c(\mathbb{R}^2; \mathbb{R}^+)$  and  $k \in \mathbb{R}$ ,

$$\int_{0}^{\infty} \int_{a}^{b} \left( |\rho - k| \varphi_{t} + \operatorname{sgn}(\rho - k) (f(\rho) - f(k)) V \varphi_{x} - \operatorname{sgn}(\rho - k) f(k) V_{x} \varphi \right) dx dt$$
  
+ 
$$\int_{a}^{b} |\rho_{0}(x) - k| \varphi(x, 0) dx$$
  
+ 
$$\int_{0}^{\infty} \operatorname{sgn}(\rho_{a} - \kappa) (f(\rho(a+, t)) - f(\kappa)) V(a, t) \varphi(a, t) dt$$
  
- 
$$\int_{0}^{\infty} \operatorname{sgn}(\rho_{b} - \kappa) (f(\kappa) - f(\rho(b-, t))) V(b, t) \varphi(b, t) dt \ge 0.$$

**Definition 2.2.** A function  $\rho \in L^{\infty}(]a, b[\times \mathbb{R}^+; [0, 1])$  is an entropy weak solution to problem (2.1) if, for all  $\varphi \in C^1_c(\mathbb{R}^2, \mathbb{R}^+)$  and  $k \in \mathbb{R}$ ,

$$\int_0^\infty \int_a^b \left( (\rho - k)^{\pm} \partial_t \varphi(x, t) + \operatorname{sgn}^{\pm} (\rho - k) \left( f(\rho) - f(k) \right) V(x, t) \partial_x \varphi(x, t) - \operatorname{sgn}^{\pm} (\rho - k) f(k) \varphi(x, t) \partial_x V(x, t) \right) \mathrm{d}x \, \mathrm{d}t + \int_a^b \left( \rho_0(x) - k \right)^{\pm} \varphi(x, 0) \, \mathrm{d}x$$

$$+ L\left(\int_0^\infty \left(\rho_a(t) - k\right)^{\pm} \varphi(a, t) \, \mathrm{d}t + \int_0^\infty \left(\rho_b(t) - k\right)^{\pm} \varphi(b, t) \, \mathrm{d}t\right) \ge 0,$$

where

$$L := \|v\|_{\infty} (\|g\|_{\infty} + \|g'\|_{\infty}).$$
(2.6)

Here we have used the notation  $s^+ := \max\{s, 0\}, s^- := \max\{-s, 0\}$ , and

$$\operatorname{sgn}^+(s) := \begin{cases} 1 & \text{if } s > 0, \\ 0 & \text{if } s \le 0, \end{cases} \quad \operatorname{sgn}^-(s) := \begin{cases} 0 & \text{if } s \ge 0, \\ -1 & \text{if } s < 0. \end{cases}$$

Definition 2.1 will be useful in the existence proof, while Definition 2.2 will be used in the uniqueness proof. We need to remark that in the frame of functions in  $L^{\infty}$ , Definition 2.2 implies Definition 2.1, for more details see [20]. In the rest of the paper, we will denote  $\mathcal{I}(r,s) = [\min\{r,s\}, \max\{r,s\}]$ , for any  $r, s \in \mathbb{R}$ .

Our main result concerning the new model is given by the following theorem, which states the well-posedness of the problem.

**Theorem 2.1** (Well-posedness). Let  $\rho_0 \in BV(I; \mathbb{R}^+)$ ,  $\rho_a, \rho_b \in BV(\mathbb{R}^+; [0, 1])$  and Assumptions 2.1 be in effect. Then, for all T > 0, the Cauchy problem (2.1) admits a unique entropy weak solution  $\rho \in (L^1 \cap L^\infty \cap BV)(I \times [0, T]; \mathbb{R}^+)$  in the sense of the Definitions 2.1 and 2.2. Moreover, the following estimates hold: for any  $t \in [0, T]$ ,

$$0 \le \rho(x,t) \le 1 \quad \text{for all } x \in I, \tag{2.7}$$

$$\left\|\rho(\cdot,t)\right\|_{L^1(I)} \le \mathcal{R}_1,\tag{2.8}$$

$$\operatorname{TV}(\rho(\cdot, t); I) \leq e^{t\mathcal{T}_{1}} \Big( \operatorname{TV}(\rho_{0}; I) + \operatorname{TV}(\rho_{a}; ]0, T[) + |\rho_{0}(a+) - \rho_{a}(0+)|$$
  
 
$$+ \operatorname{TV}(\rho_{b}; ]0, T[) + |\rho_{0}(b-) - \rho_{b}(0+)| \Big) + \frac{\mathcal{T}_{2}(t)}{\mathcal{T}_{1}(t)} (e^{t\mathcal{T}_{1}} - 1),$$
 (2.9)

and for  $\tau > 0$ ,

$$\left\|\rho(.,t)-\rho(.,t-\tau)\right\|_{L^{1}(I)} \leq \tau \left(\mathcal{C}_{t}(t)+L\left(\mathrm{TV}(\rho_{a};]t-\tau,t[)+\mathrm{TV}(\rho_{a};]t-\tau,t[)\right)\right)$$

where L is defined by (2.6) and

$$\mathcal{R}_{1}(t) := c \|\rho_{0}\|_{L^{1}(I)} + L(\|\rho_{a}\|_{L^{1}([0,t])} + \|\rho_{b}\|_{L^{1}([0,t])}),$$
  

$$\mathcal{T}_{1} := \mathcal{L}(\|g\|_{\infty} + \|g'\|_{\infty}),$$
  

$$\mathcal{L} := 2K_{\omega}^{-1}\|v\|_{\infty}\|\omega'\|_{L^{1}(\mathbb{R})},$$
  

$$\mathcal{T}_{2} := (\mathcal{W}\mathcal{R}_{1}(t) + 2\mathcal{L})\|g\|_{\infty},$$
  
(2.10)

$$\mathcal{W} := 2K_{\omega}^{-1} \|v\|_{\infty} \|\omega''\|_{L^{1}(\mathbb{R})} + 4K_{\omega}^{-2} \|\omega'\|_{L^{1}(I)}^{2} \|v\|_{\infty},$$

$$\mathcal{C}_{t}(t) := \|v\|_{\infty} (\|g'\|_{\infty} + \|g\|_{\infty}) \mathcal{C}_{x}(t) + \|g\|_{\infty} \mathcal{LR}_{1}(t).$$
(2.11)

The proof consists of two parts: existence and uniqueness of entropy solutions. While uniqueness follows from the Lipschitz continuous dependence of weak entropy solutions on initial and boundary data, existence is based on a construction of a converging sequence of approximate solutions defined by a numerical scheme. 3. Uniqueness of entropy solutions. One part of the proof of Theorem 2.1 is to show uniqueness of weak entropy solutions for the model (2.1). Therefore, we prove the Lipschitz continuous dependence of weak entropy solutions with respect to initial and boundary data. Here, we follow [17]. We define V(x,t) by (2.3) and analogously U(x,t) by replacing  $\rho$  in (2.3) by another function  $\sigma$ . Furthermore, we let r(x,t,u) := ug(u)V(x,t) and h(x,t,u) := ug(u)U(x,t). Observe that by the definition of V and U,

$$V(x,t) \le \|v\|_{\infty}, \quad U(x,t) \le \|v\|_{\infty},$$

furthermore, we have the following estimates derived in Appendix A:

$$\left|\partial_{x}V(x,t)\right| \le 2K_{\omega}^{-1} \|v\|_{\infty} \|\omega'\|_{L^{1}(I)} =: \mathcal{P}_{1}, \tag{3.1}$$

$$\begin{aligned} \left|\partial_{xx}^{2}V\right| &\leq K_{\omega}^{-2} \|v\|_{\infty} \|\omega''\|_{L^{1}(I)} \|\omega\|_{L^{1}(I)} + \mathcal{P}_{1}K_{\omega}^{-2} \|\omega\|_{L^{1}(I)} \\ &+ \mathcal{P}_{1}K_{\omega}^{-1} + K_{\omega}^{-1} \|v\|_{\infty} \|\omega''\|_{L^{1}(I)} =: \mathcal{P}_{2}, \end{aligned}$$
(3.2)

$$|V(x,t) - U(x,t)| \le \mathcal{P}_3 \int_a^b |\rho(y,t) - \sigma(y,t)| \, \mathrm{d}y, \quad \mathcal{P}_3 := K_{\omega}^{-1} \|\omega\|_{L^{\infty}(\mathbb{R})} \|v'\|_{\infty},$$
(3.3)

$$\begin{aligned} \left|\partial_x V(x,t) - \partial_x U(x,t)\right| &\leq \mathcal{M} \int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \mathrm{d}y, \\ \mathcal{M} &:= K_\omega^{-2} \|\omega'\|_{L^1(\mathbb{R})} \|v'\|_\infty \|\omega\|_{L^\infty(\mathbb{R})} + K_\omega^{-1} \|v'\|_\infty \|\omega'\|_{L^\infty(\mathbb{R})}. \end{aligned}$$
(3.4)

In order to obtain the desired estimate, we first consider the local initial-boundary value problem

$$\partial_t \phi + \partial_x r(x, t, \phi) = 0, \quad (x, t) \in I \times ]0, T[, \phi(x, 0) = \sigma_0(x), \quad x \in I; \quad \phi(a, t) = \sigma_a(t), \quad \phi(b, t) = \sigma_b(t), \quad t \in ]0, T[.$$
 (3.5)

By Assumptions 2.1,  $r \in C^2(I \times [0, T] \times \mathbb{R}; \mathbb{R})$  and  $\partial_u r \in L^{\infty}(I \times [0, T] \times \mathbb{R}; \mathbb{R})$  and by estimation (3.1),  $\partial_{xu}^2 r(x, t, u) < \infty$ . Thus, we may apply Theorem 2.4 of [21] to deduce that problem (3.5) admits a unique solution in  $(L^{\infty} \cap BV)(I \times ]0, T[; \mathbb{R})$ which satisfies, for all  $t \in [0, T[, 0 \le \phi(x, t) \le 1$  for all  $x \in I$  and

$$TV(\phi(t)) \le e^{tC_2(t)} (TV(\sigma_0) + TV(\sigma_a; ]0, t[) + |\sigma_0(a+) - \sigma_a(0+)| + TV(\sigma_b; ]0, t[) + |\sigma_0(b-) - \sigma_b(0+)| + \mathcal{K}t),$$
(3.6)

where

$$C_2(t) := \mathcal{P}_1(\|g\|_{\infty} + \|g'\|_{\infty}), \quad \mathcal{K} := 2((\mathcal{P}_1 + \mathcal{P}_2)\|g\|_{\infty} + \mathcal{P}_1\|g'\|_{\infty}).$$

Assume that  $\rho$  is a solution to the IBVP

$$\begin{aligned} \partial_t \rho + \partial_x r(x,t,\rho) &= 0, \quad (x,t) \in I \times ]0, T[, \\ \rho(x,0) &= \rho_0(x), \quad x \in I; \quad \rho(a,t) = \rho_a(t), \quad \rho(b,t) = \rho_b(t), \quad t \in ]0, T[ \end{aligned}$$

and that  $\sigma$  is a solution to the analogous IBVP

$$\begin{split} \partial_t \sigma + \partial_x h(x,t,\sigma) &= 0, \quad (x,t) \in I \times ]0, T[, \\ \sigma(x,0) &= \sigma_0(x), \quad x \in I; \quad \sigma(a,t) = \sigma_a(t), \quad \sigma(b,t) = \sigma_b(t), \quad t \in ]0, T[. \end{split}$$

Therefore, for t > 0 we compute

$$\left\|\rho(\cdot,t) - \sigma(\cdot,t)\right\|_{L^{1}(I)} \le \left\|\rho(\cdot,t) - \phi(\cdot,t)\right\|_{L^{1}(I)} + \left\|\phi(\cdot,t) - \sigma(\cdot,t)\right\|_{L^{1}(I)},$$
(3.7)

where the first term on the right-hand side of (3.7) evaluates the distance between solutions to IBVPs with the same flux function, but different initial and boundary data. Then, we can apply Proposition 3.7 of [21] to get

$$\begin{aligned} &\|\rho(\cdot,t) - \phi(\cdot,t)\|_{L^{1}(I)} \\ &\leq \|\rho_{0} - \sigma_{0}\|_{L^{1}(I)} + L\big(\|\rho_{a} - \sigma_{a}\|_{L^{1}([0,t])} + \|\rho_{b} - \sigma_{b}\|_{L^{1}([0,t])}\big) =: A(t). \end{aligned}$$

Now, the second term on the right-hand side of (3.7) evaluates the distance between solutions to IBVPs with different flux functions, but same initial and boundary data. Therefore, we apply Theorem 2.6 of [21] to obtain

$$\begin{split} \|\phi(\cdot,t) - \sigma(\cdot,t)\|_{L^{1}(I)} \\ &\leq \int_{0}^{t} \int_{a}^{b} \|\partial_{x}(r-h)(x,s,\cdot)\|_{L^{\infty}(\mathcal{U})} \,\mathrm{d}x \,\mathrm{d}s \\ &+ \int_{0}^{t} \|\partial_{u}(r-h)(\cdot,s,\cdot)\|_{L^{\infty}(I\times\mathcal{U})} \min\{\mathrm{TV}(\sigma(\cdot,s)),\mathrm{TV}(\phi(\cdot,s))\} \,\mathrm{d}s \\ &+ 2 \int_{0}^{t} \|(r-h)(a,s,\cdot)\|_{L^{\infty}(\mathcal{U})} \,\mathrm{d}s + 2 \int_{0}^{t} \|(r-h)(b,s,\cdot)\|_{L^{\infty}(\mathcal{U})} \,\mathrm{d}s, \end{split}$$
(3.8)

where

$$\begin{aligned} \mathcal{U} &:= [-\max\{\|\pi(s)\|_{L^{\infty}(I)}, \|\sigma(s)\|_{L^{\infty}(I)}\}, \max\{\|\pi(s)\|_{L^{\infty}(I)}, \|\sigma(s)\|_{L^{\infty}(I)}\}] \\ &= [-1, 1]. \end{aligned}$$

Next, we estimate all terms appearing in (3.8). First of all, by Theorem 2.1,

$$\operatorname{TV}(\sigma(\cdot, t)) \leq e^{t\mathcal{T}_{1}(t)} \Big( \operatorname{TV}(\sigma_{0}; I) + \operatorname{TV}(\sigma_{a}; (0, t)) + |\sigma_{0}(a+) - \sigma_{a}(0+)| \\ + \operatorname{TV}(\sigma_{b}; (0, t)) + |\sigma_{0}(b-) - \sigma_{b}(0+)| \Big) + \frac{\mathcal{T}_{2}(t)}{\mathcal{T}_{1}(t)} (e^{t\mathcal{T}_{1}(t)} - 1)$$

$$(3.9)$$

with

$$\mathcal{T}_{1}(t) := \mathcal{L}(\|g\|_{\infty} + \|g'\|_{\infty}), \quad \mathcal{T}_{2}(t) := (\mathcal{WS}_{1}(t) + 2\mathcal{L})\|g\|_{\infty},$$
$$\mathcal{S}_{1}(t) := \|\sigma_{0}\|_{L^{1}(I)} + L(\|\sigma_{a}\|_{L^{1}([0,t])} + \|\sigma_{b}\|_{L^{1}([0,t])}).$$

Thus, by (3.6) and (3.9), we get

$$\min\{ \operatorname{TV}(\sigma(\cdot, s)), \operatorname{TV}(\phi(\cdot, s)) \}$$

$$\leq e^{t\mathcal{T}_{3}(t)} \Big( \operatorname{TV}(\sigma_{0}; I) + \operatorname{TV}(\sigma_{a}; (0, t)) + |\sigma_{0}(a+) - \sigma_{a}(0+)|$$

$$+ \operatorname{TV}(\sigma_{b}; (0, t)) + |\sigma_{0}(b-) - \sigma_{b}(0+)| \Big)$$

$$+ \min\{ \mathcal{K}te^{\mathcal{C}_{2}(t)t}, (\mathcal{T}_{2}(t)/\mathcal{T}_{1}(t))(e^{t\mathcal{T}_{1}} - 1) \} =: \mathcal{T}_{4}(t).$$

$$(3.10)$$

To handle the first term on the right-hand side of (3.8), we use the estimate

$$\left|\partial_x(r-h)(x,t,u)\right| = \left|ug(u)\partial_x(V-U)\right| \le C|u|\mathcal{M}\int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \mathrm{d}y,$$

which implies

$$\left\|\partial_x(r-h)(x,s,\cdot)\right\|_{L^{\infty}(\mathcal{U})} \le C\mathcal{M}\int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \,\mathrm{d}y.$$
(3.11)

Next, in view of  $\partial_u(r-h)(x,t,u) = \partial_u(ug(u))(V-U)$  we get

$$\begin{aligned} \left\| \partial_{u}(r-h)(\cdot,s,\cdot) \right\|_{L^{\infty}(I\times\mathcal{U})} &\leq \left\| \partial_{u}(ug(u)) \right\|_{\infty} \mathcal{P}_{3} \int_{a}^{b} \left| \rho(y,s) - \sigma(y,s) \right| \mathrm{d}y \\ &\leq \left( \|g\|_{\infty} + \|g'\|_{\infty} \right) \mathcal{P}_{3} \int_{a}^{b} \left| \rho(y,s) - \sigma(y,s) \right| \mathrm{d}y. \end{aligned}$$

$$(3.12)$$

The third integral on the right-hand side of (3.8) is estimated by considering that

$$\left| (r-h)(a,t,u) \right| = \left| ug(u)(V-U) \right| \le C |u| \mathcal{P}_3 \int_a^b \left| \rho(y,t) - \sigma(y,t) \right| \mathrm{d}y,$$

hence

$$\left\| (r-h)(a,s,\cdot) \right\|_{L^{\infty}([-1,1])} \le C\mathcal{P}_3 \int_a^b \left| \rho(y,t) - \sigma(y,t) \right| \mathrm{d}y;$$
(3.13)

the fourth integral is treated similarly. Finally, combining (3.10) to (3.13) we get

$$\left\|\phi(\cdot,t) - \sigma(\cdot,t)\right\|_{L^{1}(I)} \le B(t) \int_{0}^{t} \int_{a}^{b} \left|\rho(y,t) - \sigma(y,t)\right| \mathrm{d}y \,\mathrm{d}s,\tag{3.14}$$

where we define

$$B(t) = C\mathcal{M} + \mathcal{P}_3\Big(\big(\|g\|_{\infty} + \|g'\|_{\infty}\big)\mathcal{T}_4(t) + 4C\Big).$$
(3.15)

Inserting A(t) and (3.14) into (3.7) yields

$$\|\rho(\cdot,t) - \sigma(\cdot,t)\|_{L^{1}(I)} \le A(t) + B(t) \int_{0}^{t} \|\rho(\cdot,s) - \sigma(\cdot,s)\|_{L^{1}(I)}, \mathrm{d}s,$$

so by an application of Gronwall's lemma we arrive at the estimate

$$\begin{aligned} \left\| \rho(\cdot,t) - \sigma(\cdot,t) \right\|_{L^{1}(I)} &\leq A(t) + \int_{0}^{t} A(s)B(s) \exp\left(\int_{s}^{t} B(\tau) \,\mathrm{d}\tau\right) \mathrm{d}s \\ &\leq A(t) + B(t) \int_{0}^{t} A(s) \mathrm{e}^{B(t)(t-s)} \,\mathrm{d}s \leq A(t) \left(1 + B(t)t \mathrm{e}^{B(t)t}\right). \end{aligned}$$

Consequently, we have proven the following lemma.

**Lemma 3.1** (Lipschitz continuous dependence on initial and boundary data). If Assumptions (2.1) are in effect and  $\rho$  and  $\sigma$  are two entropy solutions to (2.1) with initial data  $\rho_0$ ,  $\sigma_0 \in \mathbf{BV}(I; \mathbb{R}^+)$  and  $\rho_a, \rho_b, \sigma_a, \sigma_b \in \mathbf{BV}(]0, T[; [0, 1])$ , then the estimate

$$\begin{aligned} \left\| \rho(\cdot, T) - \sigma(\cdot, T) \right\|_{L^{1}(I)} \\ &\leq \left( \| \rho_{0} - \sigma_{0} \| + L \big( \| \rho_{a} - \sigma_{a} \|_{L^{1}([0,T])} + \| \rho_{b} - \sigma_{b} \|_{L^{1}([0,T])} \big) \right) \\ &\times \left( 1 + B(T) T e^{B(T)T} \right) \end{aligned}$$
(3.16)

holds for any T > 0, where B(T) is defined in (3.15).

4. Existence of solutions. The proof of existence of solutions consists of several steps that are developed in this section. We construct a sequence of approximate solutions to (2.1) and derive the compactness estimates necessary to prove its convergence by Helly's theorem. We then show that the limit function is a weak entropy solution to the IBVP (2.1).

4.1. Numerical scheme. Fix T > 0, we take a space step  $\Delta x = (b - a)/M$  with  $M \in \mathbb{N}$  and a time step  $\Delta t$  that is subject to a CFL condition specified later, and we set  $\lambda = \Delta t/\Delta x$ . We denote the center of the cells by  $x_j := a + (j - 1/2)\Delta x$  for  $j = 1, \ldots, M$ , and  $x_{j+1/2} = a + j\Delta x$ ,  $j = 0, \ldots, M$  are the cells interfaces. Moreover, we set  $N_T = \lfloor T\Delta t \rfloor$  and, for  $n = 0, \ldots, N_T$  let  $t^n = n\Delta t$  be the time mesh. The initial datum and the boundary data are approximated as

$$\rho_j^0 := \frac{1}{\Delta x} \int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x) \, \mathrm{d}x, \quad j = 1, \dots, M;$$
  
$$\rho_a^n := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \rho_a(t) \, \mathrm{d}t, \quad \rho_b^n := \frac{1}{\Delta t} \int_{t^n}^{t^{n+1}} \rho_b(t) \, \mathrm{d}t \quad n = 0, \dots, N_T - 1,$$

furthermore, we set  $\rho_0^n := \rho_a$  and  $\rho_{M+1}^n := \rho_b$ . For  $n = 0, \ldots, N_T - 1$ , we set  $\omega^k := \omega((k-1/2)\Delta x)$  for  $k \in \mathbb{Z}$  and define

$$W_{j+1/2} := \Delta x \sum_{k=1}^{M} \omega^{k-j}, \quad V_{j+1/2}^n := \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho_k) \quad \text{for } j = 0, \dots, M.$$

We define a piecewise constant approximate solution  $\rho_{\Delta}(x,t)$  to (2.1) as

$$\rho_{\Delta}(x,t) = \rho_j^n \quad \text{for } t \in [t^n, t^{n+1}], \ x \in [x_{j-1/2}, x_{j+1/2}],$$
(4.1)  
where  $n = 0, \dots, N_T - 1, \ j = 1, \dots, M$ , through the numerical scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda \left( F_{j+1/2}^n(\rho_j^n, \rho_{j+1}^n) - F_{j-1/2}^n(\rho_{j-1}^n, \rho_j^n) \right), \quad j = 1, \dots, M,$$
(4.2)

where a nonlocal version of the monotone numerical flux proposed in [19] and also used in [6] is employed, namely

$$F_{j+1/2}^n(u,w) = ug(w)V_{j+1/2}^n.$$
(4.3)

Next, we study the properties of the numerical scheme (4.2)-(4.3). Particularly, we going to prove that the sequence of approximate solutions  $\rho_{\Delta}(x,t)$  satisfies the assumptions of Hellys compactness theorem.

#### 4.2. Uniform bounds of numerical solutions.

Lemma 4.1 (Maximum principle). If Assumptions 2.1 and the CFL condition

$$\lambda \|v\|_{\infty} \left( \|g\|_{\infty} + \|g'\|_{\infty} \right) \le 1 \tag{4.4}$$

hold, then if  $\rho_0(x) \in [0,1]$  for  $x \in I$ , the approximate solution satisfies

$$0 \leq \rho_j^n \leq 1$$
 for all  $j = 1, \ldots, M$  and  $n = 1, \ldots, N_T$ .

*Proof.* We assume that  $0 \le \rho_j^n \le 1$  for  $j = 1, \ldots, M$ . From (4.2) we have

$$\rho_j^{n+1} = \rho_j^n - \lambda \left( \rho_j^n g(\rho_{j+1}^n) V_{j+1/2}^n - \rho_{j-1}^n g(\rho_j^n) V_{j-1/2}^n \right)$$
  
$$\leq \rho_j^n + \lambda \rho_{j-1}^n g(\rho_j^n) V_{j-1/2}^n \leq \rho_j^n + \lambda g(\rho_j^n) V_{j-1/2}^n =: G(\rho_j^n).$$

In view of  $G'(\rho) = 1 + \lambda g'(\rho) V_{j-1/2}^n$ , under the CFL condition (4.4), G is a nondecreasing function of  $\rho$ . Thus

$$\max_{\rho_j^n \in [0,1]} G(\rho_j^n) = G(1) = 1,$$

which implies that  $\rho_j^{n+1} \leq 1$ . Returning to (4.2), we obtain that

$$\rho_{j}^{n+1} \ge \rho_{j}^{n} - \lambda \rho_{j}^{n} g(\rho_{j+1}^{n}) V_{j+1/2}^{n} = \left(1 - \lambda g(\rho_{j+1}^{n}) V_{j+1/2}^{n}\right) \rho_{j}^{n}, \quad j = 1, \dots, M.$$
  
Consequently, if (4.4) is in effect, then  $\rho_{j}^{n+1} \ge 0.$ 

**Lemma 4.2** ( $L^1$  bound). Let Assumptions 2.1 and the CFL condition (4.4) hold. If  $\rho_0 \in L^{\infty}(I; [0, 1])$  and  $\rho_a, \rho_b \in L^{\infty}(\mathbb{R}^+; [0, 1])$ , then, for all t > 0,  $\rho_{\Delta}$  satisfies

$$\left\|\rho_{\Delta}(\cdot,t)\right\|_{L^{1}(I)} \leq \|\rho_{0}\|_{L^{1}(I)} + L\left(\|\rho_{a}\|_{L^{1}([0,t])} + \|\rho_{b}\|_{L^{1}([0,t])}\right) =: \mathcal{C}_{1}(t),$$
(4.5)

where L is defined in (2.6).

*Proof.* Lemma 4.1 (for n = 0, ..., N) and the assumption g(1) = 0 imply

$$\begin{split} &\|\rho_{\Delta}(\cdot,t^{n+1})\|_{L^{1}(I)} \\ &= \Delta x \left(\rho_{1}^{n} + \dots + \rho_{M}^{n}\right) + \Delta t \left(\rho_{a}^{n}g(\rho_{1}^{n})V_{1/2} - \rho_{M}^{n}g(\rho_{b}^{n})V_{M+1/2}^{n}\right) \\ &= \|\rho_{\Delta}(\cdot,t^{n})\|_{L^{1}(I)} + \Delta t \left(\rho_{a}^{n}g(\rho_{1}^{n})V_{1/2}^{n} + \rho_{M}^{n}(g(1) - g(\rho_{b}^{n}))V_{M+1/2}^{n}\right) \\ &= \|\rho_{\Delta}(\cdot,t^{n})\|_{L^{1}(I)} + \Delta t \rho_{a}^{n}g(\rho_{1}^{n})V_{1/2}^{n} + \Delta t \rho_{M}^{n}g'(\zeta_{j}^{n})(1 - \rho_{b}^{n})V_{M+1/2}^{n}, \end{split}$$

where  $\zeta_i^n \in \mathcal{I}(\rho_b^n, 1)$ . Now, using item (ii) of Assumptions 2.1 and the nonnegativity of  $\rho_a^n$  and  $\rho_b^n$  we have

$$\|\boldsymbol{\rho}^{n+1}\|_{L^{1}(I)} \leq \|\rho^{n}\|_{L^{1}(I)} + \Delta t \|v\|_{\infty} (\|g\|_{\infty} + \|g'\|) (\rho_{a}^{n} + \rho_{b}^{n}).$$

An iterative argument yields the desired estimate (4.5).

**Lemma 4.3** (**BV** estimate in space). Let Assumptions 2.1 hold,  $\rho_0 \in BV(I; \mathbb{R}^+)$ ,  $\rho_a, \rho_b \in BV(\mathbb{R}^+, [0, 1])$  and let  $\rho_{\Delta}$  be given by (4.2). If the CFL condition (4.4) holds, then for all  $n = 1, ..., N_T$  the discrete space BV estimate

$$\sum_{j=0}^{M} \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \le C_{x}(t^{n})$$
(4.6)

is satisfied, where we define the time-dependent bound

$$\mathcal{C}_{x}(t^{n}) := e^{\mathcal{K}_{1}t^{n}} \left( \sum_{j=0}^{M} \left| \rho_{j+1}^{0} - \rho_{j}^{0} \right| + \sum_{m=1}^{n} \left| \rho_{a}^{m} - \rho_{a}^{m-1} \right| + \sum_{m=1}^{n} \left| \rho_{b}^{m} - \rho_{b}^{m-1} \right| \right) + \mathcal{K}_{1}^{-1} \mathcal{K}_{2} \left( e^{\mathcal{K}_{1}t^{n}} - 1 \right)$$

$$(4.7)$$

and  $\mathcal{K}_1$  and  $\mathcal{K}_2$  are defined as

$$\mathcal{K}_1 := \mathcal{L}\big(\|g'\|_{\infty} + \|g\|_{\infty}\big), \quad \mathcal{K}_2 := \big(\mathcal{WC}_1(t) + 2\mathcal{L}\big)\|g\|_{\infty}.$$
(4.8)

*Proof.* Subtracting two versions of (4.2) from each other, we obtain

$$\rho_{j+1}^{n+1} - \rho_j^{n+1} = \mathcal{A}_j^n - \lambda \mathcal{B}_j^n,$$

where

$$\begin{split} \mathcal{A}_{j}^{n} &:= \rho_{j+1}^{n} - \rho_{j}^{n} - \lambda \left( \rho_{j+1}^{n} g(\rho_{j+2}^{n}) V_{j+3/2}^{n} - \rho_{j}^{n} g(\rho_{j+1}^{n}) V_{j+1/2}^{n} - \rho_{j}^{n} g(\rho_{j+1}^{n}) V_{j+3/2}^{n} \right. \\ &+ \rho_{j-1}^{n} g(\rho_{j}^{n}) V_{j+1/2}^{n} \right), \\ \mathcal{B}_{j}^{n} &:= \rho_{j}^{n} g(\rho_{j+1}^{n}) V_{j+3/2}^{n} - \rho_{j}^{n} g(\rho_{j+1}^{n}) V_{j+1/2}^{n} + \rho_{j-1}^{n} g(\rho_{j}^{n}) V_{j-1/2}^{n} - \rho_{j-1}^{n} g(\rho_{j}^{n}) V_{j+1/2}^{n}. \end{split}$$
A straightforward computation reveals that  $\mathcal{A}_{j}^{n}$  can be written in the form

A straightforward computation reveals that  $\mathcal{A}_{j}^{n}$  can be written in the form

$$\mathcal{A}_{j}^{n} = \left(1 - \lambda \left(g(\rho_{j+1}^{n})V_{j+3/2}^{n} - \rho_{j}^{n}g'(\xi_{j+1/2}^{n})V_{j+1/2}^{n}\right)\right)(\rho_{j+1}^{n} - \rho_{j}^{n}) \\ - \lambda \rho_{j+1}^{n}g'(\xi_{j+3/2}^{n})V_{j+3/2}^{n}(\rho_{j+2}^{n} - \rho_{j+1}^{n}) + \lambda g(\rho_{j}^{n})V_{j+1/2}^{n}(\rho_{j}^{n} - \rho_{j-1}^{n}),$$

$$(4.9)$$

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where  $\xi_{j+3/2}^n \in \mathcal{I}(\rho_{j+1}^n, \rho_{j+2}^n)$ . By the CFL condition (4.4), the first term in the right-hand side of (4.9) is positive, thus summing over  $j \in \{1, \ldots, M-1\}$  yields

$$\sum_{j=1}^{M-1} \left| \mathcal{A}_{j}^{n} \right| \leq \sum_{j=1}^{M-1} \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| + \lambda g(\rho_{1}^{n}) V_{3/2}^{n} |\rho_{1}^{n} - \rho_{a}^{n}| - \lambda g(\rho_{M}^{n}) V_{M+1/2}^{n} |\rho_{M}^{n} - \rho_{M-1}^{n}| - \lambda \rho_{M}^{n} g'(\xi_{M+1/2}^{n}) V_{M+1/2}^{n} |\rho_{b}^{n} - \rho_{M}^{n}| + \lambda \rho_{1}^{n} g'(\xi_{3/2}^{n}) V_{3/2}^{n} |\rho_{2}^{n} - \rho_{1}^{n}|.$$

$$(4.10)$$

On the other hand,

$$\begin{aligned} \mathcal{B}_{j}^{n} &= -\rho_{j}^{n} \big( V_{j+1/2}^{n} - V_{j+3/2}^{n} \big) g'(\xi_{j+1/2}^{n}) (\rho_{j+1}^{n} - \rho_{j}^{n}) \\ &+ g(\rho_{j}^{n}) \big( V_{j+1/2}^{n} - V_{j-1/2}^{n} \big) (\rho_{j}^{n} - \rho_{j-1}^{n}) + \rho_{j}^{n} g(\rho_{j}^{n}) \big( V_{j+3/2}^{n} - 2V_{j+1/2}^{n} + V_{j-1/2}^{n} \big). \end{aligned}$$

Taking absolute values and summing over  $j \in \{1, \ldots, M-1\}$  we have

$$\begin{split} \lambda \sum_{j=1}^{M-1} \left| \mathcal{B}_{j}^{n} \right| &\leq -\lambda \sum_{j=1}^{M-1} \rho_{j}^{n} g'(\xi_{j+1/2}^{n}) \left| V_{j+3/2}^{n} - V_{j+1/2}^{n} \right| \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \\ &+ \lambda \sum_{j=1}^{M-1} g(\rho_{j}^{n}) \left| V_{j-1/2}^{n} - V_{j+1/2}^{n} \right| \left| \rho_{j}^{n} - \rho_{j-1}^{n} \right| \\ &+ \lambda \sum_{j=1}^{M-1} \rho_{j}^{n} g(\rho_{j}^{n}) \left| V_{j+3/2}^{n} - 2V_{j+1/2}^{n} + V_{j-1/2}^{n} \right| \\ &= -\lambda \sum_{j=1}^{M-1} \left( \rho_{j}^{n} g'(\xi_{j+1/2}^{n}) - g(\rho_{j+1}^{n}) \right) \left| V_{j+3/2}^{n} - V_{j+1/2}^{n} \right| \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \\ &+ \lambda \sum_{j=1}^{M-1} \rho_{j}^{n} g(\rho_{j}^{n}) \left| V_{j+3/2}^{n} - 2V_{j+1/2}^{n} + V_{j-1/2}^{n} \right| \\ &+ \lambda g(\rho_{1}^{n}) \left| V_{3/2}^{n} - V_{1/2}^{n} \right| \left| \rho_{1}^{n} - \rho_{n}^{n} \right| \\ &- \lambda g(\rho_{M}^{n}) \left| V_{M+1/2}^{n} - V_{M-1/2}^{n} \right| \left| \rho_{M}^{n} - \rho_{M-1}^{n} \right|. \end{split}$$

By using the following estimations (which are proved in Appendix A)

$$\left|V_{j+3/2}^{n} - V_{j+1/2}^{n}\right| \le \mathcal{L}\Delta x, \quad \left|V_{j+3/2}^{n} - 2V_{j+1/2}^{n} + V_{j-1/2}^{n}\right| \le \Delta x^{2}\mathcal{W}, \tag{4.11}$$

we obtain

$$\begin{split} \lambda \sum_{j=1}^{M-1} \left| \mathcal{B}_{j}^{n} \right| &\leq -\lambda \mathcal{L} \Delta x \sum_{j=1}^{M-1} \left( \rho_{j}^{n} g'(\xi_{j+1/2}^{n}) - g(\rho_{j+1}^{n}) \right) \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \\ &+ \lambda \Delta x^{2} \mathcal{W} \sum_{j=1}^{M-1} \rho_{j}^{n} g(\rho_{j}^{n}) + \lambda g(\rho_{1}^{n}) |V_{3/2}^{n} - V_{1/2}^{n}| |\rho_{1}^{n} - \rho_{a}^{n}| \\ &- \lambda g(\rho_{M}^{n}) |V_{M+1/2}^{n} - V_{M-1/2}^{n}| |\rho_{M}^{n} - \rho_{M-1}^{n}| \\ &\leq -\mathcal{L} \Delta t \sum_{j=1}^{M-1} \left( \rho_{j}^{n} g'(\xi_{j+1/2}^{n}) - g(\rho_{j+1}^{n}) \right) \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \\ &+ \Delta t \mathcal{W} \|g\|_{\infty} \|\rho\|_{L^{1}(I)} + \mathcal{L} \Delta t g(\rho_{1}^{n}) |\rho_{1}^{n} - \rho_{a}^{n}|. \end{split}$$

$$(4.12)$$

We now deal with the boundary terms, first for the left boundary term. By the definition of the scheme (4.2),

$$\begin{split} \rho_1^{n+1} - \rho_a^{n+1} &= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda \Big( \big( \rho_1^n g(\rho_2^n) - \rho_1^n g(\rho_1^n) \big) V_{3/2}^n \\ &+ \big( \rho_1^n g(\rho_1^n) - \rho_a^n g(\rho_1^n) \big) V_{3/2}^n + \rho_a^n g(\rho_1^n) (V_{3/2}^n - V_{1/2}^n) \Big) \\ &= \rho_1^n - \rho_a^n + \rho_a^n - \rho_a^{n+1} - \lambda \Big( \rho_1^n g'(\xi_{3/2}^n) V_{3/2}^n (\rho_2^n - \rho_1^n) \\ &+ (\rho_1^n - \rho_a^n) g(\rho_1^n) V_{3/2}^n + \rho_a^n g(\rho_1^n) (V_{3/2}^n - V_{1/2}^n) \Big) \\ &= \rho_a^n - \rho_a^{n+1} + \big( 1 - \lambda g(\rho_1^n) V_{3/2}^n \big) \big( \rho_1^n - \rho_a^n) - \lambda \rho_1^n g'(\xi_{3/2}^n) (\rho_2^n - \rho_1^n) \\ &- \lambda \rho_a^n g(\rho_1^n) \big( V_{3/2}^n - V_{1/2}^n \big). \end{split}$$

Taking absolute values, invoking (4.4) and using (4.11) we obtain

$$\left| \rho_1^{n+1} - \rho_a^{n+1} \right| \le \left| \rho_a^n - \rho_a^{n+1} \right| + \left( 1 - \lambda g(\rho_1^n) V_{3/2}^n \right) \left| \rho_1^n - \rho_a^n \right| - \lambda \rho_1^n g'(\xi_{3/2}^n) V_{3/2}^n \left| \rho_2^n - \rho_1^n \right| + \mathcal{L} \Delta t \rho_a^n g(\rho_1^n).$$

$$(4.13)$$

An analogous discussion of the other boundary term yields

$$\begin{aligned} \left| \rho_b^{n+1} - \rho_M^{n+1} \right| &\leq \left| \rho_b^n - \rho_b^{n+1} \right| + (1 + \lambda \rho_M^n g'(\xi_{M+1/2}^n) V_{M+1/2}^n) \left| \rho_M^n - \rho_b^n \right| \\ &+ \lambda g(\rho_M^n) V_{M+1/2}^n \left| \rho_M^n - \rho_{M-1}^n \right| + \Delta t \rho_{M-1}^n g(\rho_M^n) \mathcal{L}. \end{aligned}$$
(4.14)

Finally, collecting the estimates (4.10), (4.12), (4.13) and (4.14) we arrive at

$$\begin{split} &\sum_{j=0}^{M} \left| \rho_{j+1}^{n+1} - \rho_{j}^{n+1} \right| \\ &\leq \left| \rho_{a}^{n} - \rho_{a}^{n+1} \right| + \sum_{j=0}^{M} \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| - \mathcal{L}\Delta t \sum_{j=1}^{M-1} \left( \rho_{j}^{n} g'(\xi_{j+1/2}^{n}) - g(\rho_{j+1}^{n}) \right) \right| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \\ &+ \Delta t \mathcal{W} \| g \|_{\infty} \| \rho \|_{L^{1}(I)} + \Delta t \mathcal{L} g(\rho_{1}^{n}) | \rho_{1}^{n} - \rho_{a}^{n} | + | \rho_{b}^{n} - \rho_{b}^{n+1} | \\ &+ \Delta t \mathcal{L} (\rho_{a}^{n} g(\rho_{1}^{n}) + \rho_{M-1}^{n} g(\rho_{M}^{n})) \\ &\leq \left| \rho_{a}^{n} - \rho_{a}^{n+1} \right| + \left( 1 + \Delta t \mathcal{L} (\| g' \|_{\infty} + \| g \|_{\infty}) \right) \sum_{j=0}^{M} |\rho_{j+1}^{n} - \rho_{j}^{n} | \\ &+ \Delta t \mathcal{W} \| g \|_{\infty} \| \rho \|_{L^{1}(I)} + | \rho_{b}^{n} - \rho_{b}^{n+1} | + 2\Delta t \mathcal{L} \| g \|_{\infty} \\ &= \left| \rho_{a}^{n} - \rho_{a}^{n+1} \right| + \left( 1 + \Delta t \mathcal{K}_{1} \right) \sum_{j=0}^{M} |\rho_{j+1}^{n} - \rho_{j}^{n}| + |\rho_{b}^{n} - \rho_{b}^{n+1}| + \Delta t \mathcal{K}_{2}, \end{split}$$

with  $\mathcal{L}$ ,  $\mathcal{W}$ ,  $\mathcal{K}_1$  and  $\mathcal{K}_2$  defined as in (2.10), (2.11), and (4.8). The previous estimate implies (4.6)-(4.7) by standard arguments.

**Lemma 4.4** (*BV* estimate in space and time). Let  $\rho_0 \in BV(I; \mathbb{R}^+)$  and  $\rho_a$ ,  $\rho_b \in BV(\mathbb{R}^+; [0, 1])$ . If Assumptions 2.1 and the CFL condition (4.4) hold, then for all  $n = 0, \ldots, N_T$ , the estimate

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M} \Delta t \left| \rho_{j+1}^m - \rho_j^m \right| + \sum_{m=0}^{n-1} \sum_{j=0}^{M+1} \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| \le \mathcal{C}_{xt}(t^n)$$
(4.15)

holds, where

$$\mathcal{C}_{xt}(t^n) = t^n \mathcal{C}_x(t^n) + \mathcal{C}_t(t^n) + \Delta x \big( \operatorname{TV}(\rho_a; [0, T]) + \operatorname{TV}(\rho_b; [0, T]) \big).$$

*Proof.* By Lemma 4.3 we have

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M} \Delta t \left| \rho_{j+1}^m - \rho_j^m \right| \leq n \Delta t \mathcal{C}_x(n \Delta t).$$
(4.16)

By the definition of numerical scheme (4.2), for  $m \in \{0, \dots, n-1\}$  and  $j \in \{1, \dots, M\}$  we get

$$\begin{split} \left| \rho_{j}^{m+1} - \rho_{j}^{m} \right| &= \left| \lambda \rho_{j}^{m} g'(\xi_{j+1/2}^{m}) V_{j+1/2}^{m}(\rho_{j+1}^{m} - \rho_{j}^{m}) + \lambda \rho_{j}^{m} g(\rho_{j}^{m}) (V_{j+1/2}^{m} - V_{j-1/2}^{m}) \right. \\ &+ \lambda g(\rho_{j}^{m}) V_{j-1/2}^{m}(\rho_{j}^{m} - \rho_{j}^{m}) \Big| \\ &\leq \lambda \|g'\|_{\infty} \|v\|_{\infty} \left| \rho_{j+1}^{m} - \rho_{j}^{m} \right| + \lambda \|g\|_{\infty} \mathcal{L} \Delta x \rho_{j}^{m} \\ &+ \lambda \|g\|_{\infty} \|v\|_{\infty} \left| \rho_{j}^{m} - \rho_{j-1}^{m} \right|. \end{split}$$

Multiplying the last inequality by  $\Delta x$  and summing for j from 1 to M we get

$$\begin{split} &\sum_{j=1}^{M} \Delta x \left| \rho_{j}^{m+1} - \rho_{j}^{m} \right| \\ &\leq \Delta t \|v\|_{\infty} \left( \|g'\|_{\infty} + \|g\|_{\infty} \right) \sum_{j=0}^{M} \left| \rho_{j+1}^{m} - \rho_{j}^{m} \right| + \Delta t \|g\|_{\infty} \mathcal{L} \|\rho^{m}\|_{L^{1}(I)}. \end{split}$$

Lemmas 4.2 and 4.3 now imply that

$$\sum_{j=1}^{M} \Delta x \left| \rho_j^{m+1} - \rho_j^{m} \right| \le \Delta t \|v\|_{\infty} \left( \|g'\|_{\infty} + \|g\|_{\infty} \right) \mathcal{C}_x(m\Delta t)$$
$$+ \Delta t \|g\|_{\infty} \mathcal{L} \mathcal{C}_1(m\Delta t) = \Delta t \mathcal{C}_t(m\Delta t),$$

where we define

$$\mathcal{C}_t(\tau) := \|v\|_{L^{\infty}([0,1])} \big( \|g'\|_{\infty} + \|g\|_{\infty} \big) \mathcal{C}_x(\tau) + \|g\|_{\infty} \mathcal{L}\mathcal{C}_1(\tau).$$

In particular,

$$\sum_{j=0}^{M+1} \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| = \Delta x \left| \rho_a^{m+1} - \rho_a^m \right| + \Delta x \left| \rho_b^{m+1} - \rho_b^m \right| + \sum_{j=1}^M \Delta x \left| \rho_j^{m+1} - \rho_j^m \right|$$
  
$$\leq \Delta x \left| \rho_a^{m+1} - \rho_a^m \right| + \Delta x \left| \rho_b^{m+1} - \rho_b^m \right| + \Delta t \mathcal{C}_t(m\Delta t),$$
(4.17)

which, summed over  $m = 0, \ldots, n - 1$ , yields

$$\sum_{m=0}^{n-1} \sum_{j=0}^{M+1} \Delta x \left| \rho_j^{m+1} - \rho_j^m \right| \le \Delta x \sum_{m=0}^{n-1} \left( \left| \rho_a^{m+1} - \rho_a^m \right| + \left| \rho_b^{m+1} - \rho_b^m \right| \right) + n \Delta t \mathcal{C}_t(n \Delta t).$$
(4.18)

Summing (4.16) and (4.18) we get the desired estimate (4.15).

4.3. Convergence analysis. Lemmas 4.1 and 4.4 allow us to apply Helly's compactness theorem that ensures the existence of a subsequence of  $\rho_{\Delta}$ , still denoted by  $\rho_{\Delta}$ , that converges in  $L^1$  to a function  $\rho \in L^{\infty}(I \times [0, T])$ , for all T > 0. Now we need to prove that this limit function is indeed an entropy weak solution to (2.1) in the sense of Definition 2.2. First we will show that the approximate solutions obtained by the scheme (4.2) satisfies a discrete entropy inequality. To this end, for  $j = 1, \ldots, M, n = 0, \ldots, N_T - 1$ , and  $k \in \mathbb{R}$ , we define

$$\begin{split} H_{j}^{n}(u,w,z) &:= w - \lambda \big( F_{j+1/2}^{n}(w,z) - F_{j-1/2}^{n}(u,w) \big), \\ G_{j+1/2}^{n,k}(u,w) &:= F_{j+1/2}^{n}(u \lor k, w \lor k) - F_{j+1/2}^{n}(k,k), \\ L_{j+1/2}^{n,k}(u,w) &:= F_{j+1/2}^{n}(k,k) - F_{j+1/2}^{n}(u \land k, w \land k), \end{split}$$

where  $w \wedge z := \min\{w, z\}$ ,  $w \vee z := \max\{w, z\}$ , and  $F_{j+1/2}^n(u, w)$  is defined as in (4.3). Observe that, due to the definition of the scheme,

$$\rho_j^{n+1} = H_j^n \left( \rho_{j-1}^n, \rho_j^n, \rho_{j+1}^n \right),$$

and we also recall the equivalence  $(s-k)^+ = s \vee k - k$  and  $(s-k)^- = k - s \wedge k$ .

**Lemma 4.5** (Discrete entropy inequalities). If Assumptions 2.1 and the CFL condition (4.4) are in effect, then the approximate solution  $\rho_{\Delta}$  in (4.1) satisfies the discrete entropy inequalities

$$\begin{pmatrix} \rho_j^{n+1} - k \end{pmatrix}^+ - (\rho_j^n - k)^+ + \lambda \begin{pmatrix} G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n) \end{pmatrix} + \lambda \operatorname{sgn}^+ (\rho_j^{n+1} - k) f(k) \begin{pmatrix} V_{j+1/2}^n - V_{j-1/2}^n \end{pmatrix} \le 0 \quad \text{and} \qquad (4.19) \begin{pmatrix} \rho_j^{n+1} - k \end{pmatrix}^- - (\rho_j^n - k)^- + \lambda \begin{pmatrix} L_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - L_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n) \end{pmatrix}$$

$$+\lambda \operatorname{sgn}^{-}(\rho_{j}^{n+1}-k)f(k) \left(V_{j+1/2}^{n}-V_{j-1/2}^{n}\right) \leq 0,$$
(4.20)

for j = 1, ..., M,  $n = 0, ..., N_T - 1$  and  $k \in \mathbb{R}$ .

*Proof.* By the CFL condition (4.4), the map  $(u, w, z) \mapsto H_i^n(u, w, z)$  satisfies

$$\begin{aligned} \partial_u H_j^n(u, w, z) &= \lambda g(w) V_{j-/2}^n \ge 0, \\ \partial_w H_j^n(u, w, z) &= 1 - \lambda \left( g(z) V_{j+1/2}^n - ug'(w) V_{j-1/2}^n \right) \ge 0, \\ \partial_z H_j^n(u, w, z) &= -\lambda w g'(z) V_{j+1/2}^n \ge 0. \end{aligned}$$
(4.21)

Notice that

$$H_j^n(k,k,k) = k - \lambda f(k) \left( V_{j+1/2}^n - V_{j-1/2}^n \right).$$

The monotonicity properties (4.21) imply that

$$\begin{split} H_{j}^{n}(\rho_{j-1}^{n} \vee k, \rho_{j}^{n} \vee k, \rho_{j+1}^{n} \vee k) &- H_{j}^{n}(k, k, k) \\ \geq H_{j}^{n}(\rho_{j-1}^{n}, \rho_{j}^{n}, \rho_{j+1}^{n}) \vee H_{j}^{n}(k, k, k) - H_{j}^{n}(k, k, k) \\ &= \left(H_{j}^{n}(\rho_{j-1}^{n}, \rho_{j}^{n}, \rho_{j+1}^{n}) - H_{j}^{n}(k, k, k)\right)^{+} = \left(\rho_{j}^{n+1} - k + \lambda f(k) \left(V_{j+1/2}^{n} - V_{j-1/2}^{n}\right)\right)^{+}, \\ \text{moreover, we also have} \end{split}$$

 $\begin{aligned} H_{j}^{n}(\rho_{j-1}^{n} \lor k, \rho_{j}^{n} \lor k, \rho_{j+1}^{n} \lor k) &- H_{j}^{n}(k, k, k) \\ &= (\rho_{j}^{n} \lor k) - \lambda \left( F_{j+1/2}^{n}(\rho_{j}^{n} \lor k, \rho_{j+1}^{n} \lor k) - F_{j-1/2}^{n}(\rho_{j-1}^{n} \lor k, \rho_{j}^{n} \lor k) \right) \right) \\ &- \left( k - \lambda \left( F_{j+1/2}^{n}(k, k) - F_{j-1/2}^{n}(k, k) \right) \right) \\ &= \left( \rho_{j}^{n} \lor k \right) - k - \lambda \left( F_{j+1/2}^{n}(\rho_{j}^{n} \lor k, \rho_{j+1}^{n} \lor k) - F_{j-1/2}^{n}(\rho_{j-1}^{n} \lor k, \rho_{j}^{n} \lor k) \right) \end{aligned}$ 

$$- F_{j+1/2}^{n}(k,k) + F_{j-1/2}^{n}(k,k) \Big)$$
  
=  $(\rho_{j}^{n} - k)^{+} - \lambda \Big( G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j+1}^{n}) - G_{j-1/2}^{n,k}(\rho_{j-1}^{n},\rho_{j}^{n}) \Big),$ 

hence

$$\begin{split} & \left(\rho_{j}^{n}-k\right)^{+}-\lambda\left(G_{j+1/2}^{n,k}\left(\rho_{j}^{n},\rho_{j+1}^{n}\right)-G_{j-1/2}^{n,k}\left(\rho_{j-1}^{n},\rho_{j}^{n}\right)\right)\\ &\geq\left(\rho_{j}^{n+1}-k+\lambda f(k)\left(V_{j+1/2}^{n}-V_{j-1/2}^{n}\right)\right)^{+}\\ &=\mathrm{sgn}^{+}\left(\rho_{j}^{n+1}-k+\lambda f(k)\left(V_{j+1/2}^{n}-V_{j-1/2}^{n}\right)\right)\left(\rho_{j}^{n+1}-k+\lambda f(k)\left(V_{j+1/2}^{n}-V_{j-1/2}^{n}\right)\right)\\ &\geq\left(\rho_{j}^{n+1}-k\right)+\lambda\,\mathrm{sgn}^{+}\left(\rho_{j}^{n+1}-k\right)f(k)\left(V_{j+1/2}^{n}-V_{j-1/2}^{n}\right), \end{split}$$

which proves (4.19), while (4.20) is proven in an entirely analogous way.

**Remark 4.1.** Lemma 4.5 and its proof mimic standard arguments known for monotone schemes of local conservation laws [13], although the scheme is not monotone in the proper sense since the argument (4.21) suppresses the presence of  $\rho_{j-1}^n$ ,  $\rho_j^n$ and  $\rho_{j+1}^n$  within  $V_{j-1/2}^n$  and  $V_{j+1/2}^n$ .

**Lemma 4.6.** Let  $\rho_0 \in BV(]a, b[; \mathbb{R}^+)$ ,  $\rho_a, \rho_b \in BV(\mathbb{R}^+; \mathbb{R}^+)$ , and Assumptions 2.1 and the CFL condition (4.4) be in effect. Then the piecewise constant approximate solutions  $\rho_{\Delta}$  in (4.1) resulting from the HW scheme (4.2) converge, as  $\Delta x \to 0$ , towards an entropy weak solution of initial boundary value problem (2.1).

*Proof.* Adding and subtracting  $\lambda G_{j+1/2}^{n,k}(\rho_j^n,\rho_j^n)$  we may rewrite (4.19) as

$$0 \ge \left(\rho_{j}^{n+1} - k\right)^{+} - \left(\rho_{j}^{n} - k\right)^{+} + \lambda \left(G_{j+1/2}^{n,k}(\rho_{j}^{n}, \rho_{j+1}^{n}) - G_{j+1/2}^{n,k}(\rho_{j}^{n}, \rho_{j}^{n})\right) \\ + \lambda \left(G_{j+1/2}^{n,k}(\rho_{j}^{n}, \rho_{j}^{n}) - G_{j-1/2}^{n,k}(\rho_{j-1}^{n}, \rho_{j}^{n})\right) \\ + \lambda \operatorname{sgn}^{+} \left(\rho_{j}^{n+1} - k\right) f(k) \left(V_{j+1/2}^{n} - V_{j-1/2}^{n}\right).$$

Let  $\varphi \in C_c^1([0,T] \times I; \mathbb{R}^+)$  for some T > 0. Multiplying the inequality above by  $\Delta x \varphi(x_j, t^n)$  and summing over  $j = 1, \ldots, M$  and  $n \in \mathbb{N}$  yields

$$0 \ge T_1 + T_2 + T_3 + T_4$$

where we define the terms

$$\begin{split} T_1 &:= \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} \left( (\rho_j^{n+1} - k)^+ - (\rho_j^n - k)^+ \right) \varphi(x_j, t^n), \\ T_2 &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \left( G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) \right) \varphi(x_j, t^n), \\ T_3 &:= -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \left( G_{j-1/2}^{n,k}(\rho_{j-1}^n, \rho_j^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) \right) \varphi(x_j, t^n), \\ T_4 &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \operatorname{sgn}^+ \left( \rho_j^{n+1} - k \right) f(k) \left( V_{j+1/2}^n - V_{j-1/2}^n \right) \varphi(x_j, t^n). \end{split}$$

Summing by parts, we obtain

$$T_1 = \Delta x \sum_{n=1}^{\infty} \sum_{j=1}^{M} (\rho_j^n - k)^+ \varphi(x_j, t^{n-1}) - \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} (\rho_j^n - k)^+ \varphi(x_j, t^n)$$

$$= -\Delta x \sum_{j=1}^{M} (\rho_j^0 - k)^+ \varphi(x_j, 0) - \Delta x \Delta t \sum_{n=1}^{\infty} \sum_{j=1}^{M} (\rho_j^n - k)^+ \frac{\varphi(x_j, t^n) - \varphi(x_j, t^{n-1})}{\Delta t},$$

and by the dominated convergence theorem,

$$T_1 \xrightarrow{\Delta x \to 0^+} -\int_a^b (\rho_0(x) - k)^+ \varphi(x_j, 0) \, \mathrm{d}x - \int_0^\infty \int_a^b (\rho(x, t) - k)^+ \partial_t \varphi(x, t) \, \mathrm{d}x \, \mathrm{d}t.$$

Again by the dominated convergence theorem,

$$T_4 = \Delta t \Delta x \sum_{n=0}^{\infty} \sum_{j=1}^{M} \operatorname{sgn}^+ \left(\rho_j^{n+1} - k\right) f(k) \frac{V_{j+1/2}^n - V_{j-1/2}^n}{\Delta x} \varphi(x_j, t^n)$$
$$\xrightarrow{\Delta x \to 0^+} \int_0^\infty \int_a^b \operatorname{sgn}^+ \left(\rho(x, t) - k\right) f(k) (\partial_x V) \varphi(x, t) \, \mathrm{d}x \, \mathrm{d}t.$$

Concerning  $T_2$  and  $T_3$ , we get

$$T_{2} + T_{3} = \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M} \left( G_{j+1/2}^{n,k} \left( \rho_{j}^{n}, \rho_{j+1}^{n} \right) - G_{j+1/2}^{n,k} \left( \rho_{j}^{n}, \rho_{j}^{n} \right) \right) \varphi(x_{j}, t^{n})$$
$$- \Delta t \sum_{n=0}^{\infty} \sum_{j=0}^{M-1} \left( G_{j+1/2}^{n,k} \left( \rho_{j}^{n}, \rho_{j+1}^{n} \right) - G_{j+3/2}^{n,k} \left( \rho_{j+1}^{n}, \rho_{j+1}^{n} \right) \right) \varphi(x_{j+1}, t^{n})$$
$$= T_{20} + T_{30} = T_{23},$$

where we define

$$\begin{split} T_{20} &:= \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \Bigl( \bigl( G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j+1}^{n}) - G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j}^{n}) \bigr) \varphi(x_{j},t^{n}) \\ &- \bigl( G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j+1}^{n}) - G_{j+3/2}^{n,k}(\rho_{j+1}^{n},\rho_{j+1}^{n}) \bigr) \varphi(x_{j+1},t^{n}) \Bigr), \\ T_{30} &:= \Delta t \sum_{n=0}^{\infty} \Bigl( \bigl( G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{b}^{n}) - G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n}) \bigr) \varphi(x_{M},t^{n}) \\ &- \bigl( G_{1/2}^{n,k}(\rho_{a}^{n},\rho_{1}^{n}) - G_{3/2}^{n,k}(\rho_{1}^{n},\rho_{1}^{n}) \bigr) \varphi(x_{1},t^{n}) \Bigr). \end{split}$$

Now we define

$$S := -\Delta x \Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{M} G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) \frac{\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n)}{\Delta x}$$

$$- L\Delta t \sum_{n=0}^{+\infty} \left( (\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n) \right).$$
(4.22)

Since

$$\begin{aligned} G_{j+1/2}^{n,k}(\rho_j^n,\rho_j^n) &= F_{j+1/2}^n(\rho_j^n \lor k,\rho_j^n \lor k) - F_{j+1/2}^n(k,k) \\ &= \left(f(\rho_j^n \lor k) - f(k)\right)V_{j+1/2}^n = \operatorname{sgn}^+(\rho_j^n - k)\left(f(\rho_j^n) - f(k)\right)V_{j+1/2}^n, \end{aligned}$$

it follows that

$$S \xrightarrow{\Delta x \to 0^+} -\int_0^{+\infty} \int_a^b \operatorname{sgn}^+(\rho(x,t)-k) \big( f(\rho_j^n) - f(k) \big) V \partial_x \varphi(x,t) \, \mathrm{d}x \, \mathrm{d}t - L \Big( \int_0^{+\infty} (\rho_a(t)-k)^+ \varphi(a,t) \, \mathrm{d}t + \int_0^{+\infty} (\rho_b(t)-k)^+ \varphi(b,t) \, \mathrm{d}t \Big).$$

Let us rewrite S (4.22) as follows

$$S = -\Delta t \sum_{n=0}^{+\infty} \sum_{j=1}^{M} G_{j+1/2}^{n,k} (\rho_j^n, \rho_j^n) (\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n))$$
  
$$- L\Delta t \sum_{n=0}^{+\infty} \left( (\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n)) \right)$$
  
$$= -\Delta t \sum_{n=0}^{+\infty} \left( \sum_{j=1}^{M} G_{j+1/2}^{n,k} (\rho_j^n, \rho_j^n) \varphi(x_{j+1}, t^n) - \sum_{j=0}^{M-1} G_{j+3/2}^{n,k} (\rho_{j+1}^n, \rho_{j+1}^n) \varphi(x_{j+1}, t^n) \right)$$
  
$$- L\Delta t \sum_{n=0}^{+\infty} \left( (\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n) \right) = S_{20} + S_{30},$$

where we define

$$S_{20} := -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \left\{ G_{j+1/2}^{n,k} (\rho_j^n, \rho_j^n) - G_{j+3/2}^{n,k} (\rho_{j+1}^n, \rho_{j+1}^n) \right\} \varphi(x_{j+1}, t^n),$$
  

$$S_{30} := -\Delta t \sum_{n=0}^{+\infty} \left( G_{M+1/2}^{n,k} (\rho_M^n, \rho_M^n) \varphi(x_{M+1}, t^n) - G_{3/2}^{n,k} (\rho_1^n, \rho_1^n) \varphi(x_1, t^n) \right)$$
  

$$- L\Delta t \sum_{n=0}^{+\infty} \left( (\rho_a^n - k)^+ \varphi(a, t^n) + (\rho_b^n - k)^+ \varphi(b, t^n) \right).$$

Adding and subtracting  $G_{j+1/2}^{n,k}(\rho_j^n,\rho_j^n)$ , we may rewrite  $S_{20}$  as

$$S_{20} = -\Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \left( G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n) \right) \varphi(x_{j+1}, t^n) - \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \left( G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+3/2}^{n,k}(\rho_{j+1}^n, \rho_{j+1}^n) \right) \varphi(x_{j+1}, t^n).$$

We evaluate now the distance between  $T_{20}$  and  $S_{20}$ :

$$|T_{20} - S_{20}| \le \Delta t \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} |G_{j+1/2}^{n,k}(\rho_j^n, \rho_{j+1}^n) - G_{j+1/2}^{n,k}(\rho_j^n, \rho_j^n)| |\varphi(x_{j+1}, t^n) - \varphi(x_j, t^n)|,$$

since

$$\begin{split} \left| G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j+1}^{n}) - G_{j+1/2}^{n,k}(\rho_{j}^{n},\rho_{j}^{n}) \right| \\ &= \left| F_{j+1/2}^{n}(\rho_{j}^{n} \lor k,\rho_{j+1}^{n} \lor k) - F_{j+1/2}^{n}(\rho_{j}^{n} \lor k,\rho_{j}^{n} \lor k) \right| \\ &= \left| (\rho_{j}^{n} \lor k) \left( g(\rho_{j+1}^{n} \lor k) - g(\rho_{j}^{n} \lor k) \right) V_{j+1/2}^{n} \right| \\ &= \left| (\rho_{j}^{n} \lor k) g'(\eta_{j+1/2}^{n}) \left( (\rho_{j+1}^{n} \lor k) - (\rho_{j}^{n} \lor k) \right) V_{j+1/2}^{n} \right| \\ &\leq \|v\|_{\infty} \|g'\|_{\infty} \left| (\rho_{j}^{n} \lor k) \left( (\rho_{j+1}^{n} \lor k) - (\rho_{j}^{n} \lor k) \right) \right| \\ &\leq \|v\|_{\infty} \|g'\|_{\infty} \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right| \leq L \left| \rho_{j+1}^{n} - \rho_{j}^{n} \right|, \end{split}$$

in light of the uniform BV estimate (4.6) we deduce that

$$\begin{aligned} \left| T_{20} - S_{20} \right| &\leq L \Delta x \Delta t \| \partial_x \varphi \|_{\infty} \sum_{n=0}^{\infty} \sum_{j=1}^{M-1} \left| \rho_{j+1}^n - \rho_j^n \right| \\ &\leq L \Delta x T \| \partial_x \varphi \|_{\infty} \max_{0 \leq n \leq T/\Delta t} \operatorname{TV} \left( \rho_\Delta(\cdot, t^n) \right) = \mathcal{O}(\Delta x). \end{aligned}$$

$$\tag{4.23}$$

Furthermore, we obtain

$$S_{30} - T_{30} = -\Delta t \sum_{n=0}^{\infty} \left( G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n})\varphi(x_{M+1},t^{n}) - G_{3/2}^{n,k}(\rho_{1}^{n},\rho_{1}^{n})\varphi(x_{1},t^{n}) \right) - L\Delta t \sum_{n=0}^{\infty} \left( (\rho_{a}^{n}-k)^{+}\varphi(a,t^{n}) + (\rho_{b}^{n}-k)^{+}\varphi(b,t^{n}) \right) - \Delta t \sum_{n=0}^{\infty} \left( \left( G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{b}^{n}) - G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n}) \right) \varphi(x_{M},t^{n}) - \left( G_{1/2}^{n,k}(\rho_{a}^{n},\rho_{1}^{n}) - G_{3/2}^{n,k}(\rho_{1}^{n},\rho_{1}^{n}) \right) \varphi(x_{1},t^{n}) \right),$$

which we can write as  $S_{30} - T_{30} = R_1 + R_2 + R_3$  with

$$R_{1} := \Delta t \sum_{n=0}^{\infty} \left( G_{1/2}^{n,k}(\rho_{a}^{n},\rho_{1}^{n})\varphi(x_{1},t^{n}) - L(\rho_{a}^{n}-k)^{+}\varphi(a,t^{n}) \right),$$

$$R_{2} := -\Delta t \sum_{n=0}^{\infty} \left( L(\rho_{b}^{n}-k)^{+}\varphi(b,t^{n}) + G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{b}^{n})\varphi(x_{M},t^{n}) \right),$$

$$R_{3} := -\Delta t \sum_{n=0}^{\infty} G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n}) \left( \varphi(x_{M+1},t^{n}) - \varphi(x_{M},t^{n}) \right).$$

Observe that

$$\begin{aligned} \partial_u F_{j+1/2}^n(u,z) &= \partial_u \left( ug(z) V_{j+1/2}^n \right) = g(z) V_{j+1/2}^n \ge 0, \\ \partial_z F_{j+1/2}^n(u,z) &= \partial_z \left( ug(z) V_{j+1/2}^n \right) = ug'(z) V_{j+1/2}^n \le 0, \end{aligned}$$

meaning that the numerical flux is increasing with respect to the first variable and the decreasing with respect to the second one. Consequently,

$$\begin{split} G^{n,k}_{j+1/2}(u,z) &= F^n_{j+1/2}(u \lor k, z \lor k) - F^n_{j+1/2}(k,k) \\ &\geq F^n_{j+1/2}(k,z \lor k) - F^n_{j+1/2}(k,k) = (kg(z \lor k) - kg(k))V^n_{j+1/2} \\ &= kg'(\nu^n_{j+1/2})((z \lor k) - k)V^n_{j+1/2} \ge - \|v\|_{\infty}\|g'\|_{\infty}(z-k)^+ \\ &\geq -L(z-k)^+, \\ G^{n,k}_{j+1/2}(u,z) &= F^n_{j+1/2}(u \lor k, z \lor k) - F^n_{j+1/2}(k,k) \\ &\leq F^n_{j+1/2}(u \lor k,k) - F^n_{j+1/2}(k,k) = (u \lor k)g(k)V^n_{j+1/2} - kg(k)V^n_{j+1/2} \\ &= \left((u \lor k) - k\right)g(k)V^n_{j+1/2} \le \|v\|_{\infty}\|g\|_{\infty}(u-k)^+ \le L(u-k)^+, \end{split}$$

hence

$$R_1 = \Delta t \sum_{n=0}^{\infty} G_{1/2}^{n,k}(\rho_a^n, \rho_1^n) \big(\varphi(x_1, t^n) - \varphi(a, t^n)\big)$$

$$\begin{split} &+ \Delta t \sum_{n=0}^{\infty} \left( G_{1/2}^{n,k}(\rho_{a}^{n},\rho_{1}^{n}) - L(\rho_{a}^{n}-k)^{+} \right) \varphi(a,t^{n}) \\ &\leq LT\Delta x \|\partial_{x}\varphi\|_{\infty} \sup_{0 \leq n \leq T/\Delta t} (\rho_{a}^{n}-k)^{+} + L\Delta t \sum_{n=0}^{\infty} \left( (\rho_{a}^{n}-k)^{+} - (\rho_{a}^{n}-k)^{+} \right) \varphi(a,t^{n}) \\ &\leq T\Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|\rho_{a}\|_{L^{\infty}([0,t])} = \mathcal{O}(\Delta x), \\ R_{2} &= -\Delta t \sum_{n=0}^{\infty} \left( L(\rho_{b}^{n}-k)^{+} + G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{b}^{n}) \right) \varphi(b,t^{n}) \\ &- \Delta t \sum_{n=0}^{\infty} G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{b}^{n}) \left( \varphi(x_{M},t^{n}) - \varphi(b,t^{n}) \right) \\ &\leq -L\Delta t \sum_{n=0}^{\infty} \left( (\rho_{b}^{n}-k)^{+} - (\rho_{b}^{n}-k)^{+} \right) \varphi(b,t^{n}) \\ &+ LT\Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \sup_{0 \leq n \leq T/\Delta t} (\rho_{b}^{n}-k)^{+} \\ &\leq LT\Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|\rho_{b}\|_{L^{\infty}([0,t])} = \mathcal{O}(\Delta x), \\ R_{3} &\leq \Delta t \left| \sum_{n=0}^{\infty} G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n}) \left( \varphi(x_{M+1},t^{n}) - \varphi(x_{M},t^{n}) \right) \right| \\ &\leq \Delta t\Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \sum_{n=0}^{+\infty} |G_{M+1/2}^{n,k}(\rho_{M}^{n},\rho_{M}^{n})|. \end{split}$$

Taking into account that

$$\begin{split} G_{M+1/2}^{n,k}(\rho_M^n,\rho_M^n) &= F_{M+1/2}^n(\rho_M^n \lor k,\rho_M^n \lor k) - F_{M+1/2}^n(k,k) \\ &= (\rho_M^n \lor k)g(\rho_M^n \lor k)V_{M+1/2}^n - kg(k)V_{M+1/2}^n \\ &= \left((\rho_M^n \lor k)(g(\rho_M^n \lor k) - g(k)) + (\rho_M^n \lor k - k)g(k)\right)V_{M+1/2}^n \\ &= \left((\rho_M^n \lor k)g'(\eta_{j+1/2}^n)(\rho_M^n \lor k - k) + (\rho_M^n \lor k - k)g(k)\right)V_{M+1/2}^n \\ &= \left((\rho_M^n \lor k)g'(\eta_{j+1/2}^n) + g(k)\right)(\rho_M^n \lor k - k)V_{M+1/2}^n, \end{split}$$

we get

$$\begin{aligned} R_{3} &= \Delta t \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \sum_{n=0}^{+\infty} \left| \left( (\rho_{M}^{n} \lor k)g'(\eta_{j+1/2}^{n}) + g(k) \right) (\rho_{M}^{n} \lor k - k) V_{M+1/2}^{n} \right| \\ &\leq \Delta t \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \sum_{n=0}^{+\infty} \left| \left( g'(\eta_{j+1/2}^{n}) + g(k) \right) (\rho_{M}^{n} \lor k - k) V_{M+1/2}^{n} \right| \\ &\leq L \Delta t \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|v\|_{\infty} \sum_{n=0}^{\infty} \left| \rho_{M}^{n} \lor k - k \right| \\ &= L \Delta t \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|v\|_{\infty} \sum_{n=0}^{\infty} (\rho_{M}^{n} - k)^{+} \\ &\leq LT \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|v\|_{\infty} \sup_{0 \leq n \leq T/\Delta t} \|\rho^{n}\|_{\infty} \leq LT \Delta x \|\partial_{x}\varphi\|_{L^{\infty}} \|v\|_{\infty} = \mathcal{O}(\Delta x), \end{aligned}$$

thanks to the maximum principle estimate. Hence,  $S_{30} - T_{30} \leq \mathcal{O}(\Delta x)$ , so that we finally get

$$0 \ge T_1 + T_2 + T_3 + T_4 = T_1 + T_4 + T_{23} \pm S = T_1 + T_4 + S - \mathcal{O}(\Delta x).$$

This concludes the proof.

Proof of Theorem 2.1. The existence of solutions to problem (2.1) follows from the results of Section 4. The uniqueness is ensured by the Lipschitz continuous dependence of solutions to (2.1) on initial and boundary data, see Section 3. The estimates on the solution to (2.1) are obtained from the corresponding discrete estimates passing to the limit. In particular, the  $L^1$  bound follows from (4.2), the Maximum principle from (4.1), the total variation bound from (4.3) and the Lipschitz continuity in time from (4.17), since  $\Delta x = \Delta t/\lambda$  and taking  $\lambda = 1/L$ .

5. A second-order scheme. The scheme (4.2), (4.3) is only first-order accurate. We propose here a second-order accuracy scheme, constructed using MUSCL-type variable extrapolation and Runge-Kutta temporal differencing. To implement it, we approximate  $\rho(x, t^n)$  by a piecewise linear functions in each cell, i.e.  $\hat{\rho}_j(x, t^n) = \rho_j^n + \sigma_j^n(x - x_j)$ , where the slopes  $\sigma_j^n$  are calculated via the generalized minmod limiter, i.e.

$$\sigma_j^n = \frac{1}{\Delta x} \operatorname{minmod} \left( \vartheta(\rho_j^n - \rho_{j-1}^n), \frac{1}{2} (\rho_{j+1}^n - \rho_{j-1}^n), \vartheta(\rho_{j+1}^n - \rho_j^n) \right),$$

where  $\vartheta \in [1, 2]$  and

$$\operatorname{minmod}(a, b, c) := \begin{cases} \operatorname{sgn}(a) \min\{|a|, |b|, |c|\} & \text{if } \operatorname{sgn}(c) = \operatorname{sgn}(b) = \operatorname{sgn}(a), \\ 0 & \text{otherwise.} \end{cases}$$

This extrapolation enables one to define left and right values at the cell interfaces respectively by

$$\rho_{j+1/2}^{\mathrm{L}} := \hat{\rho}_j \left( x_j + \frac{\Delta x}{2}, t^n \right) = \rho_j^n + \sigma_j^n \frac{\Delta x}{2},$$
$$\rho_{j-1/2}^{\mathrm{R}} := \hat{\rho}_j \left( x_j - \frac{\Delta x}{2}, t^n \right) = \rho_j^n - \sigma_j^n \frac{\Delta x}{2}.$$

and

$$\begin{split} \hat{V}_{j+1/2}^{n} &= \frac{1}{\hat{W}_{j+1/2}} \int_{a}^{b} v(\hat{\rho}(y,t)) \omega(y - x_{j+1/2}) \, \mathrm{d}y \\ &= \frac{1}{\hat{W}_{j+1/2}} \sum_{k=1}^{M} \int_{x_{k-1/2}}^{x_{k+1/2}} v(\hat{\rho}_{k}(y,t)) \omega(y - x_{j+1/2}) \, \mathrm{d}y \\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \int_{-1}^{1} v\left(\hat{\rho}_{k}(\frac{\Delta x}{2}y + x_{k},t^{n})\right) \omega\left(\frac{\Delta x}{2}y + (k - j + 1/2)\Delta x\right) \, \mathrm{d}y \\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \sum_{e=1}^{N_{G}} p_{e} v\left(\hat{\rho}_{k}(\frac{\Delta x}{2}y_{e} + x_{k},t^{n})\right) \omega\left(\frac{\Delta x}{2}y_{e} + (k - j + 1/2)\Delta x\right) \\ &= \frac{\Delta x}{2\hat{W}_{j+1/2}} \sum_{k=1}^{M} \sum_{e=1}^{N_{G}} p_{e} v\left(\rho_{k}^{n} + \frac{\Delta x}{2}y_{e}\sigma_{k}^{n}\right) \omega\left(\frac{\Delta x}{2}y_{e} + (k - j + 1/2)\Delta x\right) \end{split}$$

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where  $\hat{W}_{j+1/2} = \int_a^b \omega(y - x_{j+1/2}) dy$  is computed in exact form, and  $y_e$  are the Gauss-Lobatto-Quadrature points. The MUSCL version of the numerical flux reads

$$F_{j+1/2}^n := \rho_{j+1/2}^{\mathrm{L}} g(\rho_{j+1/2}^{\mathrm{R}}) \hat{V}_{j+1/2}^n.$$

To achieve formal second-order accuracy also in time, we use second-order Runge-Kutta (RK) time stepping. More precisely, if we write our scheme with first-order Euler time differences and second-order spatial differences formally as

$$\rho_j^{n+1} = \rho_j^n - \lambda L_j(\rho^n) := \rho_j^n - \lambda \left( F_{j+1/2}^n - F_{j-1/2}^n \right), \tag{5.1}$$

then the RK version takes the two-step form

$$\rho_j^1 = \rho_j^n - \lambda L_j(\rho^n); \quad \rho_j^{n+1} = \frac{1}{2}(\rho_j^n + \rho_j^1) - \frac{\lambda}{2}L_j(\rho_j^1).$$
(5.2)

6. Numerical examples. In this section we solve (2.1) by using the numerical scheme (4.2) on the x-interval I = [0,1] with suitable boundary conditions and  $t \in [0,T]$ , with T specified later. In the numerical examples we consider the equation (2.1) with  $g(\rho) = 1 - \rho$  and  $v(\rho) = (1 - \rho)^4$ , where we recall that  $f(\rho) = \rho g(\rho)$  and  $V(x,t) = (\omega * v(\rho))(x,t)$ , where the kernel function  $\omega(x)$  is specified later in each case. For numerical experiments the interval I is subdivided into M subintervals of length  $\Delta x = 1/M$ , and the time step is computed taking account the CFL condition (4.4), and for each numerical experiment, we specify the initial and boundary conditions.

6.1. **Example 1.** In this example we compare numerical approximations obtained by scheme (4.2) with those generated by an adapted LxF-type scheme proposed by Goatin and Rossi in [17], starting from the initial and boundary conditions

$$\rho_0(x) = 0.2$$
 for  $x \in I$ ;  $\rho_a(t) = 0.1$ ,  $\rho_b(t) = 0.5$  for  $t > 0$ .

Here we employ the symmetric kernel function

$$\omega(y) = \omega_1(y) := \frac{3}{4\eta} \left( 1 - \frac{y^2}{\eta^2} \right) \chi_{[-\eta,\eta]}(y)$$
(6.1)

with  $\eta = 0.05$ . In Figure 1 we display the numerical approximations for M = 800 at simulated times T = 2 and T = 8 and compare them with the reference solution which is computed with the LxF scheme with  $M_{\text{ref}} = 12800$ . We observe better accuracy for the proposed scheme. This property also becomes apparent in Table 1 where we show the corresponding approximate  $L^1$  errors for discretizations  $M = 100 \times 2^l$ ,  $l = 0, 1, \ldots, 4$  and respective experimental orders of convergence (E.O.C.). We observe that the approximate  $L^1$  errors decrease as the grid is refined and E.O.C. assumes values close to one, in agreement with the formal first order of acccuracy of the scheme.

6.2. Example 2. We now compare the dynamics in the solution of model (2.1) by using various kernel functions. We consider the symmetric kernel  $\omega_1$  (6.1) as in Example 1 along with a non-symmetric kernel

$$\omega_2(y) := \frac{20}{\eta} \left( \frac{5y}{\eta} + \frac{1}{2} \right) \exp\left( -\frac{10y}{\eta} - 1 \right) \chi_{\left[ -\frac{\eta}{10}, \eta \right]}(y) \tag{6.2}$$

and the anisotropic discontinuous kernels

$$\omega_3(y) := \frac{1}{\eta} \chi_{[0,\eta]}(y), \quad \omega_4(y) := \frac{3}{\eta^3} (\eta - y)^2 \chi_{[0,\eta]}(y), \tag{6.3}$$

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FIGURE 1. Example 1: numerical approximations obtained with HW and LxF numerical flux with M = 800 and symmetric kernel  $\omega_1(y)$  with  $\eta = 0.05$  at simulated times (a) T = 2, (b) T = 8.

TABLE 1. Example 1: approximate  $L^1$ -error  $e_M(u)$  and E.O.C. for the LxF and HW numerical fluxes with  $\Delta x = 1/M$  and symmetric kernel (6.1) with  $\eta = 0.05$  at simulated times T = 2 and T = 8.

	T=2				T = 8			
	LxF		HW		LxF		HW	
M	$e_M( ho_\Delta)$	E.O.C.	$e_M( ho_\Delta)$	E.O.C.				
100	1.71e-01	_	1.02e-01	_	6.34e-01		5.28e-01	_
200	1.11e-01	0.63	5.42 e- 02	0.92	5.96e-01	0.89	5.80e-01	-0.14
400	4.64 e- 02	1.25	2.74e-02	0.98	4.89e-01	0.28	3.69e-01	0.65
800	2.02e-02	1.20	1.39e-02	0.98	2.86e-01	0.78	1.19e-01	1.63
1600	9.58e-03	1.07	6.84e-03	1.03	1.11e-01	1.37	4.18e-02	1.51

where in all cases we choose  $\eta = 0.2$ . In Figure 2 we display the different kernel functions. The initial and boundary conditions are given by

$$\rho_0(x) = 0.1$$
 for  $x \in I$ ;  $\rho_a(t) = 0.1$ ,  $\rho_b(t) = 1$  for  $t > 0$ .

In Figure 3 we display numerical approximations with  $\Delta x = 1/400$  at times T = 10and T = 100. We can evidence that the dynamics of the solution is different for each kernel functions; by using  $\omega_1$  we observe that numerical solution goes faster to a stationary state solution than for other kernel functions, in which this stationary state is not observed for enough large simulation time. Regarding the kernel  $\omega_2$ we can see the formation of some oscillations. Now, for the kernel  $\omega_3$  we can see the formation of some layers on the numerical solution and that the period of these layers are proportional to  $\eta$ . Finally, for  $\omega_4$  we can observe a numerical solution more smooth than in the previous solutions.

6.3. **Example 3.** The aim of the present example is to investigate the behavior of numerical solutions considering the kernel functions  $\omega_1$ ,  $\omega_2$ , and  $\omega_3$  as well as for



FIGURE 2. Example 2: kernel functions  $\omega(x) = \omega_i(x)$ ,  $i = 1, \ldots, 4$ , given by (6.1), (6.2) and (6.3) with  $\eta = 0.2$ .

two different values of the parameter  $\eta$ , namely  $\eta = 0.1$  and  $\eta = 0.025$ . The initial and boundary conditions are

$$\rho_0(x) = 0.5 \quad \text{for } x \in I; \quad \rho_a(t) = 0, \quad \rho_b(t) = 1 \quad \text{for } t > 0,$$

which leaves zero flow conditions at boundary, i.e.  $f_{1/2} = f_{M+1/2} = 0$ . Numerical approximations are computed at simulated times T = 2, T = 7 and T = 15 with discretization M = 400. In Figure 4, first we observe that numerical solutions for the nonlocal problem (2.1) get closer to the solution of the local problem as  $\eta$  takes a smaller value, but  $\omega_1$  and  $\omega_2$  make it slower due to the presence of the oscillations that the numerical solutions present when we use these functions.

6.4. Example 4: Error test for second order scheme. We consider the problem (2.1), with a smooth initial datum

$$\rho_0(x) = 0.9 \exp\left(-70(x - 0.4)^2\right) \quad \text{for } x \in [0, 1], \tag{6.4}$$

with boundary conditions  $\rho_a = \rho_b = 0$ , and with the symmetric kernel function  $\omega_1$ with  $\eta = 0.2$ . In Fig. 5 we display the numerical approximations obtained with the second order scheme (5.1)–(5.2), computed with M = 100 at T = 0.1. The reference solution is computed with M = 6400. As expected, the numerical solutions obtained with second order version of the HW scheme capture the reference solution better than the first order one. In Table 2 we compute the  $L^1$ –error and E.O.C. We recover the correct order of accuracy for the second order HW scheme. Instead, we obtain just first order accuracy for HW scheme (4.2). We also can observe for scheme (4.2) that the  $L^1$ –error for each level of refinement is bigger than the error of the second order version scheme.

6.5. Conclusions. In this paper we extend to its nonlocal version a numerical scheme presented in [6,19] where we take advantages of the form in which the flow is written,  $\rho v(\rho)V(x,t)$ , where  $v(\rho)$  is a positive and non-increasing function, and V(x,t) is a positive function containing the nonlocal terms. The resulting scheme is monotone, and we have proved maximum principal,  $L^1$ -bound, and BV estimations, also, a discrete entropy inequality in order to prove the well-possedness of an 1-D and nonlocal IBVP. In future work, we aim to investigate how to extend this numerical



FIGURE 3. Example 2: dynamics of the model (2.1) for various kernel functions ( $\omega_1$ ,  $\omega_2$ ,  $\omega_3$  or  $\omega_4$ ). Numerical solutions with M = 800 at simulated times T = 10 and T = 100.

TABLE 2. Example 4: approximate  $L^1$  errors  $e_M(\rho)$  and E.O.C for the first and second order version of the HW scheme with  $\Delta x = 1/M$ , at T = 0.1.

-										
	T = 0.1									
	HW first-c	order version	HW second-order version							
M	$e_M( ho_\Delta)$	E.O.C.	$e_M(\rho_\Delta)$	E.O.C.						
100	8.71e-03	-	3.20e-04	_						
200	4.60e-03	9.19e-01	8.63e-05	1.89						
400	2.38e-03	9.54e-01	2.24e-05	1.95						
800	1.21e-03	9.77e-01	5.53e-06	2.02						

scheme in applications such as crowd dynamics models where the function V(x,t) is a vector field containing the preference directions and nonlocal corrections terms.



FIGURE 4. Example 3: numerical solutions of (2.1) for M = 400 at indicated simulated times with (left)  $\eta = 0.1$  and (right)  $\eta = 0.025$ .

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FIGURE 5. Example 4: comparison of the numerical solutions at T = 0.1 corresponding to initial condition (6.4), computed with  $1/\Delta x = 100$  and kernel function  $\omega_1$ .

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Appendix A: Technical estimates. In this appendix we show the computations for some estimates used in the proofs of above sections. First, we prove estimates (4.11) in the proof of Lemma 4.3. We denote  $y_k := (k - 1/2)\Delta x$ , for  $k \in \mathbb{Z}$  and then we compute

$$\begin{split} \left| V_{j+3/2}^{n} - V_{j+1/2}^{n} \right| \\ &= \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} \omega^{k-j-1} v(\rho_{k}^{n}) - \frac{\Delta x}{W_{j+1/2}} \sum_{k=1}^{M} \omega^{k-j} v(\rho_{k}^{n}) \right| \\ &= \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho_{k}^{n}) + \Delta x \left( \frac{1}{W_{j+3/2}} - \frac{1}{W_{j+1/2}} \right) \sum_{k=1}^{M} \omega^{k-j} v(\rho_{k}^{n}) \right| \\ &\leq \left| \frac{\Delta x}{W_{j+3/2}} \sum_{k=1}^{M} (\omega^{k-j-1} - \omega^{k-j}) v(\rho_{k}^{n}) \right| \\ &+ \left| \frac{\Delta x^{2}}{W_{j+3/2}W_{j+1/2}} \left( \sum_{k=1}^{M} \omega^{k-j} v(\rho_{k}^{n}) \right) \sum_{k=1}^{M} (\omega^{k-j} - \omega^{k-j-1}) \right| \\ &\leq \frac{\Delta x \|v\|_{\infty}}{|W_{j+3/2}|} \sum_{k=1}^{M} |\omega^{k-j-1} - \omega^{k-j}| \\ &+ \frac{\Delta x \|v\|_{\infty}}{W_{j+3/2}W_{j+1/2}} \left| \Delta x \left( \sum_{k=1}^{M} \omega^{k-j} \right) \sum_{k=1}^{M} (\omega^{k-j} - \omega^{k-j-1}) \right| \\ &\leq \frac{\Delta x}{K_{\omega}} \|v\|_{\infty} \left( \sum_{k=1}^{M} \left| \int_{y_{k-j-1}}^{y_{k-j}} \omega'(y) \mathrm{d}y \right| + \sum_{k=1}^{M} |\omega^{k-j} - \omega^{k-j-1}| \right) \leq \mathcal{L}\Delta x \end{split}$$

with  $\mathcal{L} = 2K_{\omega}^{-1} \|v\|_{\infty} \|\omega'\|_{L^1(\mathbb{R})}$ . Now following closely [17], we compute

$$\begin{aligned} \left| V_{j+3/2}^n - 2V_{j+1/2}^n + V_{j-1/2}^n \right| \\ &= \left| \Delta x \sum_{k=1}^M v(\rho_k^n) \left( \frac{\omega^{k-j-1}}{W_{j+3/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} - \frac{\omega^{k-j}}{W_{j+1/2}} + \frac{\omega^{k-j+1}}{W_{j-1/2}} \right) \right| \end{aligned}$$

$$\begin{split} &\leq \left| \frac{\Delta x^3}{W_{j+3/2}W_{j+1/2}} \left( \left( \sum_{k=1}^M v(\rho_k^n) \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y) \, \mathrm{d}y \right) \sum_{l=1}^M \omega^{l-j} \right. \\ &\quad - \left( \sum_{k=1}^M v(\rho_k^n) \omega^{k-j} \right) \sum_{l=1}^M \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y) \, \mathrm{d}y \right) \right| \\ &\quad + \left| \frac{2\Delta x^4 ||\omega'||_{L^1}}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left( \left( \sum_{k=1}^M v(\rho_k^n) \omega^{k-j} \right) \sum_{l=1}^M \omega'(\xi_{l-j+1/2}) \right) \right. \\ &\quad - \left( \sum_{k=1}^M v(\rho_k^n) \omega'(\xi_{k-j+1/2}) \right) \sum_{l=1}^M \omega^{l-j} \right) \right| \\ &\leq \left| \frac{\Delta x^2 ||v||_{\infty}}{W_{j+3/2}W_{j+1/2}} \left( \sum_{k=1}^M \int_{\xi_{k-j-1/2}}^{\xi_{k-j+1/2}} \omega''(y) \, \mathrm{d}y \right) \Delta x \sum_{l=1}^M \omega^{l-j} \right| \\ &\quad + \left| \frac{\Delta x^2 ||v||_{\infty}}{W_{j+3/2}W_{j+1/2}} \left( \Delta x \sum_{k=1}^M \omega^{k-j} \right) \sum_{l=1}^M \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y) \, \mathrm{d}y \right| \\ &\quad + \left| \frac{2\Delta x^2 ||u'||_{L^1(I)} ||v||_{\infty}}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left( \Delta x \sum_{k=1}^M \omega^{k-j} \right) \Delta x \sum_{l=1}^M \omega'(\xi_{l-j+1/2}) \right| \\ &\quad + \left| \frac{2\Delta x^2 ||u'||_{L^1(I)} ||v||_{\infty}}{W_{j+3/2}W_{j+1/2}W_{j-1/2}} \left( \Delta x \sum_{k=1}^M \omega^{k-j} \right) \Delta x \sum_{l=1}^M \omega^{l-j} \right| \\ &\leq \Delta x^2 ||v||_{\infty} \left( \left| \frac{1}{W_{j+3/2}} \sum_{k=1}^M \int_{\xi_{l-j-1/2}}^{\xi_{l-j+1/2}} \omega''(y) \, \mathrm{d}y \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{l-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}{W_{j+3/2}W_{j-1/2}} \Delta x \sum_{k=1}^M \omega'(\xi_{k-j+1/2}) \right| \\ &\quad + \left| \frac{2 \|\omega'||_{L^1(I)}}}{W_{\omega} + 2 K_{\omega}^{-2} \Delta x^2 \|\omega'||_{L^1(I)}^2 \|v||_{\infty} \le \Delta x^2 W, \end{aligned} \right\}$$

where we set  $\mathcal{W} := 2K_{\omega}^{-1} \|v\|_{\infty} \|\omega''\|_{L^1(\mathbb{R})} + 4K_{\omega}^{-2} \|\omega'\|_{L^1(I)}^2 \|v\|_{\infty}$  and  $\xi_{k-j+1/2} \in ]y_{k-j-1}, y_{k-j}[$ . This concludes the proof of estimates (4.11) in the proof of Lemma 4.3. Next, we establish estimates (3.1) to (3.4) in Section 3.1. To this end we calculate

$$\begin{aligned} \left| \partial_x V(x,t) \right| \\ &= \left| -\frac{W'(x)}{(W(x))^2} \int_a^b v(\rho(y,t)) \omega(y-x) \,\mathrm{d}y - \frac{1}{W(x)} \int_a^b v(\rho(y,t)) \omega'(y-x) \,\mathrm{d}y \right| \\ &\leq 2K_\omega^{-1} \|v\|_\infty \|\omega'\|_{L^1(I)}, \end{aligned}$$

$$\begin{aligned} \left| \partial_{xx}^{2} V \right| &= \left| \frac{-(W(x))^{2} W''(x) + 2(W'(x))W(x)}{(W(x))^{4}} \int_{a}^{b} v(\rho(y,t))\omega(y-x) \, \mathrm{d}y \right. \\ &+ \frac{W'(x)}{(W(x))^{2}} \int_{a}^{b} v(\rho(y,t))\omega'(y-x) \, \mathrm{d}y \\ &+ \frac{W'(x)}{(W(x))^{2}} \int_{a}^{b} v(\rho(y,t))\omega''(y-x) \, \mathrm{d}y \right| \\ &\leq K_{\omega}^{-2} \|v\|_{\infty} \|\omega''\|_{L^{1}(I)} \|\omega\|_{L^{1}(I)} + 2K_{\omega}^{-3} \|v\|_{\infty} \|\omega'\|_{L^{1}(I)} \|\omega\|_{L^{1}(I)} \\ &+ 2K_{\omega}^{-2} \|v\|_{\infty} \|\omega''\|_{L^{1}(I)} + K_{\omega}^{-1} \|v\|_{\infty} \|\omega''\|_{L^{1}(I)}, \\ \left| V(x,t) - U(x,t) \right| &= \left| \frac{1}{W(x)} \int_{a}^{b} \left( v(\rho(y,t)) - v(\sigma(y,t)) \right) \omega(y-x) \, \mathrm{d}y \right| \\ &= \left| \frac{1}{W(x)} \int_{a}^{b} v'(\xi)(\rho-\sigma)(y,t)\omega(y-x) \, \mathrm{d}y \right| \end{aligned}$$

$$= \left| \frac{W(x)}{W(x)} \int_{a}^{b} v(\xi) (\rho - \sigma)(y, t) \omega(y - x) \, \mathrm{d}y \right|$$
  
$$\leq \frac{\|\omega\|_{\infty(\mathbb{R})} \|v'\|_{\infty}}{K_{\omega}} \int_{a}^{b} |\rho(y, t) - \sigma(y, t)| \, \mathrm{d}y,$$

$$\begin{aligned} \left|\partial_x V(x,t) - \partial_x U(x,t)\right| &= \left| -\frac{W'(x)}{(W(x))^2} \int_a^b \left(v(\rho(y,t)) - v(\sigma(y,t))\right) \omega(y-x) \, \mathrm{d}y \right| \\ &- \frac{1}{W(x)} \int_a^b \left(v(\rho(y,t)) - v(\sigma(y,t))\right) \omega'(y-x) \, \mathrm{d}y \right| \\ &= \left| -\frac{W'(x)}{(W(x))^2} \int_a^b v'(\xi)(\rho-\sigma)(y,t) \omega(y-x) \, \mathrm{d}y \right| \\ &- \frac{1}{W(x)} \int_a^b v'(\xi)(\rho-\sigma)(y,t) \omega'(y-x) \, \mathrm{d}y \right| \\ &\leq \mathcal{M} \int_a^b \left|\rho(y,t) - \sigma(y,t)\right| \, \mathrm{d}y, \end{aligned}$$

with  $\mathcal{M}$  as in (3.4). *E-mail address:* rburger@ing-mat.udec.cl

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