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A Banach spaces-based fully-mixed finite element method for the stationary chemotaxis-Navier-Stokes problem*

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Abstract

In this paper we introduce and analyze a Banach spaces-based approach yielding a fully-mixed finite element method for numerically solving the stationary chemotaxis-Navier-Stokes problem. This is a nonlinear coupled model representing the biological process given by the cell movement, driven by either an internal or an external chemical signal, within an incompressible fluid. In addition to the velocity and pressure of the fluid, the velocity gradient and the Bernouilli-type stress tensor are introduced as further unknowns, which allows to eliminate the pressure from the equations and compute it afterwards via a postprocessing formula. In turn, besides the cell density and the chemical signal concentration, the pseudostress associated with the former and the gradient of the latter are introduced as auxiliary unknowns as well. The resulting continuous formulation, posed in suitable Banach spaces, consists of a coupled system of three saddle point-type problems, each one of them perturbed with trilinear forms that depend on data and the unknowns of the other two. The well-posedness of it is analyzed by means of a fixed-point strategy, so that the classical Banach theorem, along with the Babuška-Brezzi theory in Banach spaces, allow to conclude, under a smallness assumption on the data, the existence of a unique solution. Adopting an analogue approach for the associated Galerkin scheme, and under suitable hypotheses on arbitrary finite element subspaces employed, we apply the Brouwer and Banach theorems to show existence and then uniqueness of the discrete solution. General a priori error estimates, including those for the postprocessed pressure, are also derived. Next, a specific set of finite element subspaces satisfying the required stability conditions is introduced, which, given an integer $k \geqslant 0$, is defined in terms of Raviart-Thomas spaces of order k and piecewise polynomials of degree $\leqslant k$ only. The respective rates of convergence of the resulting Galerkin method are then provided. Finally, several numerical experiments confirming the latter and illustrating the good performance of the method, are reported.

Key words: Chemotaxis, Navier-Stokes, Banach framework, Babuška-Brezzi, fixed-point, mixed finite element methods, a priori error analysis.

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1 Introduction

Chemotaxis refers to the active and directed movement of cells triggered by a chemical stimulus in their surrounding microenvironment. From the development of multicellular organisms, to blood vessel formation, to immune system function, to cancer growth and metastasis, chemotaxis plays an essential role in many different biological processes [31]. The study of this phenomenon has particularly allowed valuable insights for basic research, drug discovery to decrease or inhibit certain infectious diseases and has ignited much hope for new prognostic tools and therapeutic interventions in oncology [23, 32]. From the mathematical point of view, the well-known Keller-Segel system and their variations [1, 27] are the simplest models for describing this phenomenon, which only relate the cell density and the concentration of the chemical signal, neglecting any interplay with further components. However, in many contexts, cell migration may influence the motion of a surrounding fluid through buoyant forces due to differences in densities, and vice versa the fluid-driven transport of cells and signal may substantially affect the overall behavior [13, 34]. In this regard, and for understanding the chemotaxis systems interaction with liquid environments, several models have been studied (see, e.g. [4, 26, 28, 33, 36, 37] and the references therein), which couple the Keller-Seguel equations to a Navier-Stokes system. These works include models describing chemo-repulsion, chemo-attraction, the presence of either a signal production mechanism or a singular sensitivity, double-chemotaxis, among others. In particular, theoretical results on existence and uniqueness of solutions to the unsteady chemotaxis–Navier-Stokes system when the chemical signal is consumed by the organisms, case we focus in this work, are found in [25, 35, 36].

Regarding the numerical solvability, a wide variety of techniques have been constructed so far to simulate the chemotaxis–fluid interaction [8, 12, 14, 29, 30]. These references include a combined finite volume-nonconforming finite element method [30], a high-resolution vorticity-based hybrid finite-volume finite-difference discretization [8], a splitting-type Navier-Stokes solver for a realistic three-dimensional setting [29] and an upwind finite element technique in two dimensions [12]. Other numerical techniques for close models can be found in the references of the aforementioned works. In turn, and up to our knowledge, [14] is the only work in which a finite element method for approximating the solutions of the full chemotaxis–Navier–Stokes system is proposed and analyzed, including corresponding optimal errors estimates. More precisely, an equivalent model in Hilbert spaces is proposed in [14] by using a splitting technique based on the introduction of the chemical concentration gradient as an extra unknown, allowing to control the strong regularity required by the model, which is one of the main difficulties appearing throughout the respective numerical analysis.

On the other hand, it is well-known that when dealing with problems involving couplings and nonlinearities, the introduction of additional variables, that is the use of mixed methods, yields the corresponding variational settings to be properly posed in terms of Banach spaces. This has become particularly frequent in recent years for a wide family of models (see, e.g. [2, 6, 7, 10, 11, 20, 21] and the references therein), whose resulting mixed formulations show mainly saddle-point, twofold saddle-point, or perturbed saddle-point structures. One of the advantages of keeping this functional framework, in addition to avoiding the incorporation of further redundant Galerkin-type penalty terms, as it has been usual, for instance, for diverse augmented schemes, lies on the fact that the sought variables belong to the natural Banach spaces that are originated after carrying out the respective testing and integration by parts procedures. Furthermore, the above not only allows to develop numerical schemes that are conservative but also to compute additional physically relevant variables that might be introduced into the formulation or by employing postprocessing formulae in terms of the discrete solution. Nevertheless, no mixed methods with these features seem to be available in the literature so far to solve the chemotaxis–Navier–Stokes model, which certainly constitutes a gap in the field.

According to the previous discussion, and in order, on one hand, to fill the aforementioned gap, and on the other hand, to continue extending the applicability of Banach spaces-based approaches to study the continuous and discrete well-posedness of nonlinear coupled problems in fluid mechanics, our present purpose is to introduce and analyze a continuous Banach framework yielding a fully-mixed finite element method for the stationary Chemotaxis-Navier-Stokes model. The work is organized as follows. The rest of this section first collects some preliminary notations, definitions, and results to be utilized throughout the paper, and then describes the model of interest. In particular, the auxiliary unknowns are introduced here. In Section 2 we derive the fully-mixed variational formulation of the problem by splitting the analysis according to the there equations forming the coupled model. Suitable integration by parts formulae jointly with the Cauchy-Schwarz and Hölder inequalities are crucial for determining the right Lebesgue and related spaces to which the unknowns and corresponding test functions are required to belong. In Section 3, a fixed-point strategy is adopted to analyze the solvability of the continuous formulation. The Babuška-Brezzi theory in Banach spaces is employed to study the corresponding uncoupled problems, and then the classical Banach theorem is applied to conclude the existence of a unique solution. An analogue fixed-point approach to that of Section 3 is utilized in Section 4 to study the well-posedness of the associated Galerkin scheme. Under suitable stability conditions on the finite element subspaces employed, existence and uniqueness of solution are proved by applying the Brouwer and Banach theorems along with the discrete Babuška-Brezzi theory. Specific finite element subspaces satisfying those assumptions are then introduced in Section 5, and the rates of convergence of the resulting discrete scheme are also established there. Several numerical examples confirming these theoretical findings and illustrating the good performance of the method, are presented in Section 6. Finally, further properties of the Raviart-Thomas interpolator to be employed in Section 5, are proved in Appendix A.

1.1 Preliminaries

Throughout the paper Ω is a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, whose outward unit normal at its boundary Γ is denoted $\mathbf{n}$. Standard notation will be adopted for Lebesgue spaces $\mathrm{L}^t(\Omega)$, with $t \in [1,+\infty)$, and Sobolev spaces $\mathrm{W}^{\ell,t}(\Omega)$ and $\mathrm{W}_0^{\ell,t}(\Omega)$, with $\ell \geqslant 0$, whose corresponding norms and seminorms, either for the scalar, vector, or tensorial version, are denoted by $\|\cdot\|_{0,t;\Omega}$, $\|\cdot\|_{\ell,t;\Omega}$, and $|\cdot|_{\ell,t;\Omega}$, respectively. Note that $\mathrm{W}^{0,t}(\Omega) = \mathrm{L}^t(\Omega)$, and that when $t = 2$, we simply write $\mathrm{H}^\ell(\Omega)$ instead of $\mathrm{W}^{\ell,2}(\Omega)$, with its norm and seminorm denoted by $\|\cdot\|_{\ell,\Omega}$ and $|\cdot|_{\ell,\Omega}$, respectively. Now, letting $t,\, t' \in (1,+\infty)$ conjugate to each other, that is such that $1/t + 1/t' = 1$, we let $\mathrm{W}^{1/t',t}(\Gamma)$ and $\mathrm{W}^{-1/t',t'}(\Gamma)$ be the trace space of $\mathrm{W}^{1,t}(\Omega)$ and its dual, respectively, and denote the duality pairing between them by $\langle\cdot,\cdot\rangle$. In particular, when $t = t' = 2$, we simply write $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$ instead of $\mathrm{W}^{1/2,2}(\Gamma)$ and $\mathrm{W}^{-1/2,2}(\Gamma)$, respectively. Also, given any generic scalar functional space M, we let $\mathbf{M}$ and $\mathbb{M}$ be its vector and tensorial counterparts. Furthermore, for any vector fields $\mathbf{v} = (v_i)_{i=1,n}$ and $\mathbf{w} = (w_i)_{i=1,n}$, we set the gradient, divergence, and tensor product operators, as

$$\nabla\mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,n}, \qquad \operatorname{div}(\mathbf{v}) := \sum_{j=1}^{n} \frac{\partial v_j}{\partial x_j}, \qquad \text{and} \quad \mathbf{v}\otimes\mathbf{w} := (v_i w_j)_{i,j=1,n}\,.$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,n}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,n}$, we let $\mathbf{div}(\boldsymbol{\tau})$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})_{i,j=1,n}\,, \qquad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{n} \tau_{ii}, \qquad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{n} \tau_{ij}\zeta_{ij}\,,$$

$$\text{and} \qquad \boldsymbol{\tau}^{\mathtt{d}} := \boldsymbol{\tau} - \frac{1}{n}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I}\,,$$

where $\mathbb{I}$ stands for the identity tensor of $\mathbb{R} := \mathrm{R}^{n\times n}$. On the other hand, for each $t,\, j \in [1,+\infty)$ such that $t \geqslant j$, we introduce the Banach spaces

$$\mathbf{H}(\mathrm{div}_t;\Omega) \;:=\; \Big\{\boldsymbol{\tau}\in \mathbf{L}^2(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^t(\Omega)\Big\}\,, \tag{1.1}$$

$$\mathbb{H}(\mathbf{div}_t;\Omega) \;:=\; \Big\{\boldsymbol{\tau}\in \mathbb{L}^2(\Omega): \quad \mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)\Big\}\,, \tag{1.2}$$

and

$$\mathbf{H}^t(\mathrm{div}_j;\Omega) \;:=\; \Big\{\boldsymbol{\tau}\in \mathbf{L}^t(\Omega): \quad \mathrm{div}(\boldsymbol{\tau}) \in \mathrm{L}^j(\Omega)\Big\}\,, \tag{1.3}$$

which are endowed with the natural norms

$$\|\boldsymbol{\tau}\|_{\mathrm{div}_t;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in \mathbf{H}(\mathrm{div}_t;\Omega)\,, \tag{1.4}$$

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_t;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{0,t;\Omega} \qquad \forall\,\boldsymbol{\tau}\in \mathbb{H}(\mathbf{div}_t;\Omega)\,, \tag{1.5}$$

and

$$\|\boldsymbol{\tau}\|_{t,\mathrm{div}_j;\Omega} \;:=\; \|\boldsymbol{\tau}\|_{0,t;\Omega} \,+\, \|\mathrm{div}(\boldsymbol{\tau})\|_{0,j;\Omega} \qquad \forall\,\boldsymbol{\tau}\in \mathbf{H}^t(\mathrm{div}_j;\Omega)\,. \tag{1.6}$$

Then, we recall that, proceeding as in [18, eq. (1.43), Section 1.3.4] (see also [6, Section 4.1] and [10, Section 3.1]), one can prove that for each $t \geqslant \frac{2n}{n+2}$ there holds

$$\langle \boldsymbol{\tau}\cdot\mathbf{n}, v\rangle \;=\; \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v \,+\, v\,\mathrm{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v)\in \mathbf{H}(\mathrm{div}_t;\Omega)\times \mathrm{H}^1(\Omega)\,, \tag{1.7}$$

and analogously

$$\langle \boldsymbol{\tau}\,\mathbf{n}, \boldsymbol{v}\rangle \;=\; \int_\Omega \Big\{\boldsymbol{\tau}:\nabla \boldsymbol{v} \,+\, \boldsymbol{v}\cdot\mathbf{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},\boldsymbol{v})\in \mathbb{H}(\mathbf{div}_t;\Omega)\times \mathbf{H}^1(\Omega)\,, \tag{1.8}$$

where $\langle\cdot,\cdot\rangle$ denotes in (1.7) (resp. (1.8)) the duality pairing between $\mathrm{H}^{1/2}(\Gamma)$ (resp. $\mathbf{H}^{1/2}(\Gamma)$) and $\mathrm{H}^{-1/2}(\Gamma)$ (resp. $\mathbf{H}^{-1/2}(\Gamma)$). In turn, given $t,\, t' \in (1,+\infty)$ conjugate to each other, there also holds (cf. [16, Corollary B.57])

$$\langle \boldsymbol{\tau}\cdot\mathbf{n}, v\rangle \;=\; \int_\Omega \Big\{\boldsymbol{\tau}\cdot\nabla v \,+\, v\,\mathrm{div}(\boldsymbol{\tau})\Big\} \qquad \forall\,(\boldsymbol{\tau},v)\in \mathbf{H}^t(\mathrm{div}_t;\Omega)\times \mathrm{W}^{1,t'}(\Omega)\,, \tag{1.9}$$

where $\langle\cdot,\cdot\rangle$ stands for the duality pairing between $\mathrm{W}^{-1/t,t}(\Gamma)$ and $\mathrm{W}^{1/t,t'}(\Gamma)$.

1.2 The model problem

The stationary chemotaxis-Navier-Stokes problem consists of finding the velocity vector field $\mathbf{u}$ and the pressure scalar field p of an incompressible fluid occupying the region Ω, along with the additional scalar fields given by the cell density η, and the chemical signal concentration φ, satisfying the following system of coupled partial differential equations:

$$\begin{aligned}
-\,\nu\,\Delta\mathbf{u} \,+\, \lambda\,(\nabla\mathbf{u})\,\mathbf{u} \,+\, \nabla p \,-\, \eta\,\nabla f \;&=\; \mathbf{f} && \text{in} \quad \Omega\,,\\
\mathrm{div}(\mathbf{u}) \;&=\; 0 && \text{in} \quad \Omega\,,\\
\int_\Omega p \;&=\; 0\,, &&\\
-\,k_\eta\Delta\eta + \mu\,\mathrm{div}\big(\eta\nabla\varphi\big) \,+\, \mathbf{u}\cdot\nabla\eta \;&=\; f_\eta && \text{in} \quad \Omega\,,\\
-\,k_\varphi\Delta\varphi \,+\, \gamma\eta\varphi \,+\, \mathbf{u}\cdot\nabla\varphi \;&=\; f_\varphi && \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_D\,, \quad \eta \,=\, \eta_D \quad \text{and} \quad \varphi \;&=\; \varphi_D && \text{on} \quad \Gamma\,,
\end{aligned} \tag{1.10}$$

where f, $\mathbf{f}$, f_η, and f_φ are given functions belonging to suitable spaces to be indicated later on, whereas ν, λ, κ_η, κ_φ, μ, and γ are positive constants representing the fluid viscosity, the fluid density, the cell diffusion constant, the chemical diffusion constant, the chemotactic coefficient, and the consumption rate of the chemical signal, respectively. In turn, $\mathbf{u}_D$, η_D, and φ_D are corresponding Dirichlet data belonging to suitable spaces as well to be specified throughout the analysis. Meanwhile, we observe here that, due to the incompressibility of the fluid (cf. second equation of (1.10)), $\mathbf{u}_D$ must satisfy the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} \,=\, 0\,. \tag{1.11}$$

Next, in order to derive a fully-mixed formulation of (1.10) in Section 2, we first adopt the approach from [11] (see also [10]) and introduce the velocity gradient and the Bernoulli-type stress tensor as further unknowns, that is

$$\mathbf{t} \,:=\, \nabla\mathbf{u} \quad \text{in} \quad \Omega \qquad \text{and} \qquad \boldsymbol{\sigma} \,:=\, \nu\,\mathbf{t} \,-\, \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u}) \,-\, p\,\mathbb{I} \quad \text{in} \quad \Omega\,, \tag{1.12}$$

so that the second equation of (1.12) is considered from now on as the constitutive law of the fluid. Then, noting that $\mathbf{div}(\mathbf{u}\otimes\mathbf{u}) = (\nabla\mathbf{u})\mathbf{u} = \mathbf{t}\mathbf{u}$, which follows from the fact that $\mathrm{div}(\mathbf{u}) \,=\, 0$, we find that the first equation of (1.10) can be rewritten as

$$-\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{\lambda}{2}\,\mathbf{t}\mathbf{u} \,-\, \eta\,\nabla f \,=\, \mathbf{f} \quad \text{in} \quad \Omega\,.$$

In turn, it is straightforward to prove, taking matrix trace and the deviatoric part of the aforementioned constitutive equation, that the latter and the incompressibility condition, which can also be stated as the identity $\mathrm{tr}(\mathbf{t}) = 0$, are equivalent to

$$\begin{aligned} \boldsymbol{\sigma}^{\mathrm{d}} \,&=\, \nu\,\mathbf{t} \,-\, \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \quad \text{in} \quad \Omega \quad \text{and} \\ p \,&=\, -\,\frac{1}{n}\mathrm{tr}\Big(\boldsymbol{\sigma} \,+\, \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \quad \text{in} \quad \Omega\,, \end{aligned} \tag{1.13}$$

and thus the pressure can be eliminated from the system and computed afterwards in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as indicated in the foregoing equation. As a consequence, the third equation of (1.10), which constitutes a uniqueness condition for p, is rewritten as

$$\int_\Omega \mathrm{tr}\Big(\boldsymbol{\sigma} \,+\, \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \,=\, 0\,.$$

On the other hand, for the cell density and chemical signal concentration equations, we proceed similarly and define the auxiliary unknowns

$$\widetilde{\boldsymbol{\sigma}} \,:=\, \nabla\eta \,-\, \kappa_\eta^{-1}\,\mu\,\eta\,\nabla\varphi \,-\, \kappa_\eta^{-1}\,\eta\,\mathbf{u} \quad \text{in} \quad \Omega \qquad \text{and} \qquad \mathbf{p} \,:=\, \nabla\varphi \quad \text{in} \quad \Omega\,,$$

and observe that the fourth and fifth equations of (1.10) become, respectively,

$$\mathrm{div}(\widetilde{\boldsymbol{\sigma}}) \,=\, -\kappa_\eta^{-1}\,f_\eta \quad \text{in} \quad \Omega\,,$$

and

$$\mathrm{div}(\mathbf{p}) \,-\, \kappa_\varphi^{-1}\,\gamma\,\eta\varphi \,-\, \kappa_\varphi^{-1}\,\mathbf{u}\cdot\mathbf{p} \,=\, -\kappa_\varphi^{-1}\,f_\varphi \quad \text{in} \quad \Omega\,.$$

Note that $\widetilde{\boldsymbol{\sigma}}$ can be seen as the pseudostress associated with the cell density equation. Summarizing, (1.10) can be equivalently reformulated as: Find $\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, \widetilde{\boldsymbol{\sigma}}, \eta, \mathbf{p}$ and φ in proper spaces to be introduced below, such that

$$
\begin{aligned}
\mathbf{t} \,&=\, \nabla\mathbf{u} && \text{in} \quad \Omega\,,\\
-\boldsymbol{\sigma}^{\mathrm{d}} \,+\, \nu\,\mathbf{t} \,-\, \frac{\lambda}{2}\,(\mathbf{u}\otimes\mathbf{u})^{\mathrm{d}} \,&=\, 0 && \text{in} \quad \Omega\,,\\
-\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{\lambda}{2}\,\mathbf{t}\,\mathbf{u} \,&=\, \eta\,\nabla f \,+\, \mathbf{f} && \text{in} \quad \Omega\,,\\
\int_\Omega \operatorname{tr}\Big(\boldsymbol{\sigma} \,+\, \frac{\lambda}{2}(\mathbf{u}\otimes\mathbf{u})\Big) \,&=\, 0\,, && \\
\widetilde{\boldsymbol{\sigma}} \,-\, \nabla\eta \,+\, \kappa_\eta^{-1}\,\mu\,\eta\,\mathbf{p} \,+\, \kappa_\eta^{-1}\,\eta\,\mathbf{u} \,&=\, 0 && \text{in} \quad \Omega\,,\\
\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \,&=\, -\kappa_\eta^{-1}\,f_\eta && \text{in} \quad \Omega\,,\\
\mathbf{p} \,&=\, \nabla\varphi && \text{in} \quad \Omega\,,\\
\operatorname{div}(\mathbf{p}) \,-\, k_\varphi^{-1}\,\gamma\,\eta\,\varphi \,-\, k_\varphi^{-1}\,\mathbf{u}\cdot\mathbf{p} \,&=\, -\kappa_\varphi^{-1}\,f_\varphi && \text{in} \quad \Omega\,,\\
\mathbf{u} \,=\, \mathbf{u}_D, \quad \eta \,=\, \eta_D \quad \text{and} \quad \varphi \,&=\, \varphi_D && \text{on} \quad \Gamma\,.
\end{aligned}
\tag{1.14}
$$

2 The fully-mixed formulation

In this section we derive a Banach spaces-based fully-mixed formulation of (1.14). The integration by parts formulae provided by (1.7) - (1.9), along with the Cauchy-Schwarz and Hölder inequalities, play a key role in this derivation. The corresponding analysis is split in the following three subsections, which correspond to the Navier-Stokes equations (first to fourth rows of (1.14)), the cell density equations (fifth and sixth rows of (1.14)), and the chemical signal concentration equations (seventh and eighth rows of (1.14)), respectively.

2.1 The Navier-Stokes equations

We begin by seeking originally $\mathbf{u} \in \mathbf{H}^1(\Omega)$, which requires to assume that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$. Then, a straightforward application of (1.8) with $t \geqslant \frac{2n}{n+2}$ and $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)$, gives

$$
\int_\Omega \boldsymbol{\tau} : \nabla\mathbf{u} \,=\, -\int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,+\, \langle\boldsymbol{\tau}\,\mathbf{n}, \mathbf{u}_D\rangle\,,
$$

and hence the corresponding testing of the first equation of (1.14) becomes

$$
\int_\Omega \boldsymbol{\tau} : \mathbf{t} \,+\, \int_\Omega \mathbf{u}\cdot\mathbf{div}(\boldsymbol{\tau}) \,=\, \langle\boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\,\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_t;\Omega)\,. \tag{2.1}
$$

It is clear, thanks to Cauchy-Schwarz's inequality, that the first term of (2.1) makes sense for $\mathbf{t} \in \mathbb{L}^2(\Omega)$, so that according to its free trace property, we look for this unknown in the space

$$
\mathbb{L}^2_{\mathtt{tr}}(\Omega) \,:=\, \Big\{\mathbf{s} \in \mathbb{L}^2(\Omega): \quad \operatorname{tr}(\mathbf{s}) \,=\, 0\Big\}\,. \tag{2.2}
$$

In addition, knowing that $\mathbf{div}(\boldsymbol{\tau}) \in \mathbf{L}^t(\Omega)$, and using Hölder's inequality, we deduce from the second term of (2.1) that, instead of $\mathbf{H}^1(\Omega)$, it would suffice to look for $\mathbf{u}$ in $\mathbf{L}^{t'}(\Omega)$, where t' is the conjugate of t. Nevertheless, testing the second equation of (1.14) against tensors in $\mathbb{L}^2_{\mathtt{tr}}(\Omega)$, we formally get

$$
-\int_\Omega \boldsymbol{\sigma} : \mathbf{s} \,+\, \nu\int_\Omega \mathbf{t} : \mathbf{s} \,-\, \frac{\lambda}{2}\int_\Omega (\mathbf{u}\otimes\mathbf{u}) : \mathbf{s} \,=\, 0 \qquad \forall\,\mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega)\,, \tag{2.3}
$$

from which, employing the Cauchy-Schwarz and Hölder inequalities, we deduce that its third term makes sense for $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and hence from now we chose $t' = 4$, which yields $t = 4/3$. Needless to say, the first term in (2.3) is finite if $\boldsymbol{\sigma} \in \mathbb{L}^2(\Omega)$, and thus, aiming to use the same space for this unknown and its test functions $\boldsymbol{\tau}$, we seek $\boldsymbol{\sigma}$ in $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ as well. In this way, knowing now that $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{L}^{4/3}(\Omega)$, we test the third equation of (1.14) against the vector functions in $\mathbf{L}^4(\Omega)$, which yields

$$-\int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\sigma}) \,+\, \frac{\lambda}{2}\int_\Omega \mathbf{t}\mathbf{u}\cdot\mathbf{v} \,=\, \int_\Omega \eta\,\nabla f\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\mathbf{v}\in\mathbf{L}^4(\Omega)\,. \tag{2.4}$$

Note here, thanks again to the aforementioned inequalities and the already established spaces for $\mathbf{t}$, $\mathbf{u}$, and $\mathbf{v}$, that the first, second, and fourth terms of (2.4) are well-defined, the latter if the datum $\mathbf{f}$ belongs to $\mathbf{L}^{4/3}(\Omega)$, which is assumed from now on. Regarding the third one, which will depend on where to look for η, and where to assume the datum f, we will refer to it in Section 2.2. We now consider the decomposition

$$\mathbb{H}(\mathbf{div}_{4/3};\Omega) \,=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \oplus \mathrm{R}\,\mathbb{I}\,, \tag{2.5}$$

where

$$\mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \,:=\, \left\{\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega): \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\right\}, \tag{2.6}$$

and observe, in particular, that the unknown $\boldsymbol{\sigma}$ can be uniquely decomposed, according to (2.5) and the mean value condition given by the fourth equation of (1.14), as $\boldsymbol{\sigma} \,=\, \boldsymbol{\sigma}_0 + c_0\mathbb{I}$, where

$$\boldsymbol{\sigma}_0 \,\in\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega) \qquad \text{and} \qquad c_0 \,:=\, \frac{1}{n\,|\Omega|}\int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) \,=\, -\frac{\lambda}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}\otimes\mathbf{u})\,. \tag{2.7}$$

In this way, similarly as for the pressure, the constant c_0 can be computed once the velocity is known, and hence it only remains to obtain $\boldsymbol{\sigma}_0$. In this regard, we notice that (2.3) and (2.4) remain unchanged if $\boldsymbol{\sigma}$ is replaced by $\boldsymbol{\sigma}_0$. In addition, thanks to the fact that $\mathbf{t}$ is sought in $\mathbb{L}^2_{\mathrm{tr}}(\Omega)$, and using the compatibility condition (1.11), we realize that testing (2.1) against $\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{4/3};\Omega)$ is equivalent to doing it against $\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$. Consequently, bearing in mind the foregoing discussion, introducing the notations

$$\vec{\mathbf{u}} \,=\, (\mathbf{u},\mathbf{t})\,, \quad \vec{\mathbf{v}} \,=\, (\mathbf{v},\mathbf{s})\,, \quad \vec{\mathbf{w}} \,=\, (\mathbf{w},\boldsymbol{\vartheta}) \,\in\, \mathbf{H} \,:=\, \mathbf{L}^4(\Omega)\times\mathbb{L}^2_{\mathrm{tr}}(\Omega)\,, \quad \text{and} \quad \mathbf{Q} \,:=\, \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)\,,$$

redenoting from now on $\boldsymbol{\sigma}_0$ as simply $\boldsymbol{\sigma}\in\mathbf{Q}$, and suitably gathering (2.1), (2.3), and (2.4), we arrive at the following mixed formulation for the Navier-Stokes equations: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in\mathbf{H}\times\mathbf{Q}$ such that

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; \mathbf{F}_\eta(\vec{\mathbf{v}}) \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{H}\,,\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\,\boldsymbol{\tau}\in\mathbf{Q}\,, \end{aligned} \tag{2.8}$$

where, given $\mathbf{z}\in\mathbf{L}^4(\Omega)$, the bilinear forms $\mathbf{a}:\mathbf{H}\times\mathbf{H}\to\mathrm{R}$, $\mathbf{b}:\mathbf{H}\times\mathbf{Q}\to\mathrm{R}$, and $\mathbf{c}(\mathbf{z};\cdot,\cdot):\mathbf{H}\times\mathbf{H}\to\mathrm{R}$, are defined as

$$\mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}}) \,:=\, \nu\int_\Omega \boldsymbol{\vartheta}:\mathbf{s} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{2.9}$$

$$\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) \,:=\, -\int_\Omega \boldsymbol{\tau}:\mathbf{s} \,-\, \int_\Omega \mathbf{v}\cdot\mathbf{div}(\boldsymbol{\tau}) \qquad \forall\,(\vec{\mathbf{v}},\boldsymbol{\tau})\in\mathbf{H}\times\mathbf{Q}\,, \tag{2.10}$$

and

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) \,:=\, \frac{\lambda}{2}\left\{\int_\Omega \boldsymbol{\vartheta}\,\mathbf{z}\cdot\mathbf{v} \,-\, \int_\Omega (\mathbf{w}\otimes\mathbf{z}):\mathbf{s}\right\} \qquad \forall\,\vec{\mathbf{w}},\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{2.11}$$

whereas, given χ in the same space where η will be sought, the linear functionals $\mathbf{F}_\chi:\mathbf{H}\to\mathrm{R}$ and $\mathbf{G}:\mathbf{Q}\to\mathrm{R}$ are given by

$$\mathbf{F}_\chi(\vec{\mathbf{v}}) \,:=\, \int_\Omega \chi\,\nabla f\cdot\mathbf{v} \,+\, \int_\Omega \mathbf{f}\cdot\mathbf{v} \qquad \forall\,\vec{\mathbf{v}}\in\mathbf{H}\,, \tag{2.12}$$

and

$$\mathbf{G}(\boldsymbol{\tau}) \,:=\, -\langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle_\Gamma \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \tag{2.13}$$

Next, it is easily seen that $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}(\mathbf{z};\cdot,\cdot)$, and $\mathbf{G}$ are bounded. In fact, endowing $\mathbf{H}$ and $\mathbf{Q}$ with the norms

$$\|\vec{\mathbf{v}}\|_{\mathbf{H}} \,:=\, \|\mathbf{v}\|_{0,4;\Omega} \,+\, \|\mathbf{s}\|_{0,\Omega} \quad \forall\, \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{H}\,, \quad \|\boldsymbol{\tau}\|_{\mathbf{Q}} \,:=\, \|\boldsymbol{\tau}\|_{\mathbf{div}_{4/3};\Omega} \quad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \tag{2.14}$$

applying the Cauchy-Schwarz and Hölder inequalities, and invoking (1.8) along with the continuous injection $\mathbf{i}_4 : \mathbf{H}^1(\Omega) \to \mathbf{L}^4(\Omega)$, we find that there exists positive constants, denoted and given as

$$\|\mathbf{a}\| \,:=\, \nu\,, \qquad \|\mathbf{b}\| \,:=\, 1\,, \qquad \|\mathbf{c}\| \,:=\, \frac{\lambda}{2}\,, \qquad \text{and} \quad \|\mathbf{G}\| \,:=\, \big(1 + \|\mathbf{i}_4\|\big)\, \|\mathbf{u}_D\|_{1/2,\Gamma}\,, \tag{2.15}$$

such that

$$|\mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}})| \,\leqslant\, \|\mathbf{a}\|\, \|\vec{\mathbf{w}}\|_{\mathbf{H}}\, \|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{2.16}$$

$$|\mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau})| \,\leqslant\, \|\mathbf{b}\|\, \|\vec{\mathbf{v}}\|_{\mathbf{H}}\, \|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\, (\vec{\mathbf{v}},\boldsymbol{\tau}) \in \mathbf{H}\times\mathbf{Q},\,, \tag{2.17}$$

$$|\mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}})| \,\leqslant\, \|\mathbf{c}\|\, \|\mathbf{z}\|_{0,4;\Omega}\, \|\vec{\mathbf{w}}\|_{\mathbf{H}}\, \|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{2.18}$$

and

$$|\mathbf{G}(\boldsymbol{\tau})| \,\leqslant\, \|\mathbf{G}\|\, \|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \tag{2.19}$$

In addition, simple algebraic computations show that

$$\mathbf{c}(\mathbf{z};\vec{\mathbf{v}},\vec{\mathbf{v}}) \,=\, 0 \qquad \forall\, \mathbf{z} \in \mathbf{L}^4(\Omega)\,, \quad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,. \tag{2.20}$$

Regarding $\mathbf{F}_\chi$ (cf. (2.12)), and as already commented for its first term, we remark that its well-definedness will be concluded below at the end of Section 2.2.

2.2 The cell density equations

Testing the fifth equation of (1.14) against functions $\widetilde{\boldsymbol{\tau}} \in \mathbf{L}^2(\Omega)$, we formally obtain

$$\int_\Omega \widetilde{\boldsymbol{\sigma}}\cdot\widetilde{\boldsymbol{\tau}} \,-\, \int_\Omega \nabla\eta\cdot\widetilde{\boldsymbol{\tau}} \,+\, \kappa_\eta^{-1}\,\mu \int_\Omega \eta\,\mathbf{p}\cdot\widetilde{\boldsymbol{\tau}} \,+\, \kappa_\eta^{-1} \int_\Omega \eta\,\mathbf{u}\cdot\widetilde{\boldsymbol{\tau}} \,=\, 0\,, \tag{2.21}$$

from which we observe that the first and second terms of (2.21) are finite if $\widetilde{\boldsymbol{\sigma}} \in \mathbf{L}^2(\Omega)$ and $\eta \in \mathrm{H}^1_0(\Omega)$, respectively. In turn, using the Cauchy-Schwarz and Hölder inequalities, we find that for all $l,\, j \in (1,+\infty)$ conjugate to each other, there hold

$$\left|\int_\Omega \eta\,\mathbf{p}\cdot\widetilde{\boldsymbol{\tau}}\right| \,\leqslant\, \|\eta\|_{0,2l;\Omega}\, \|\mathbf{p}\|_{0,2j;\Omega}\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega} \tag{2.22}$$

and

$$\left|\int_\Omega \eta\,\mathbf{u}\cdot\widetilde{\boldsymbol{\tau}}\right| \,\leqslant\, \|\eta\|_{0,2l;\Omega}\, \|\mathbf{u}\|_{0,2j;\Omega}\, \|\widetilde{\boldsymbol{\tau}}\|_{0,\Omega}\,, \tag{2.23}$$

from which we deduce that the third and fourth terms of (2.21) make sense for $\eta \in \mathrm{L}^{2l}(\Omega)$, $\mathbf{p} \in \mathbf{L}^{2j}(\Omega)$, and $\mathbf{u} \in \mathbf{L}^{2j}(\Omega)$. However, since we already know from Section 2.1 that $\mathbf{u}$ will be sought in $\mathbf{L}^4(\Omega)$, we have to impose here that $2j \leqslant 4$. On the other hand, in order to be able to apply (1.7) to $\widetilde{\boldsymbol{\tau}}$ and η, so that we obtain

$$\int_\Omega \nabla\eta\cdot\widetilde{\boldsymbol{\tau}} \,=\, -\int_\Omega \eta\,\mathrm{div}(\widetilde{\boldsymbol{\tau}}) \,+\, \langle \widetilde{\boldsymbol{\tau}}\cdot\mathbf{n}, \eta_D\rangle_\Gamma\,, \tag{2.24}$$

with $\tilde{\boldsymbol{\tau}} \cdot \mathbf{n} \in \mathrm{H}^{-1/2}(\Gamma)$ and $\langle \cdot, \cdot \rangle$ denoting the duality pairing between $\mathrm{H}^{-1/2}(\Gamma)$ and $\mathrm{H}^{1/2}(\Gamma)$, it suffices to assume that $\operatorname{div}(\tilde{\boldsymbol{\tau}}) \in \mathrm{L}^{(2l)'}(\Omega)$, where $(2l)' := \frac{2l}{2l-1}$ is the conjugate of $2l$, the datum η_D belongs to $\mathrm{H}^{1/2}(\Gamma)$, and $\mathrm{H}^1(\Omega)$ is continuously embedded in $\mathrm{L}^{2l}(\Omega)$. The later is guaranteed for $2l \in [1, +\infty)$ when $n = 2$, which is always satisfied, and for $2l \in [1, 6]$ when $n = 3$ (cf. [16, Corollary B.43]).

On the other hand, in order to utilize later on a result on the $\mathrm{W}^{1,2j}(\Omega)$-solvability of a Poisson equation, which will be required to establish a continuous inf-sup condition, and according to the result detailed in [19, Theorem 3.2] (see also [24, Theorems 1.1 and 1.3]), we need that $4/3 \leqslant 2j \leqslant 4$ when $n = 2$, and $3/2 \leqslant 2j \leqslant 3$ when $n = 3$. Note that these constraints are compatible with the previous requirement that $2j \leqslant 4$. Now, since the respective lower bounds are already satisfied, we just look at the upper ones, and readily observe that for $n = 2$, $2j = \dfrac{2l}{l-1} \leqslant 4$ if and only if $2l \geqslant 4$, whereas for $n = 3$, $2j = \dfrac{2l}{l-1} \leqslant 3$ if and only if $2l \geqslant 6$. Thus, intersecting the above with the previous restrictions on $2l$, we find that when $n = 2$ we require $4 \leqslant 2l$, and when $n = 3$ the only possible choice is $2l = 6$. Therefore, denoting

$$(r, s) := (2j, (2j)'), \quad \text{and} \quad (\rho, \varrho) := (2l, (2l)'),$$

we conclude from the foregoing discussion that the feasible ranges for r, s, ρ, ϱ, j and l, are given by

$$\begin{cases} r \in (2,4] \quad \text{and} \quad s \in [4/3, 2) & \text{if } n = 2, \\ r = 3 \quad \text{and} \quad s = 3/2 & \text{if } n = 3, \end{cases} \qquad \begin{cases} \rho \in [4, +\infty) \quad \text{and} \quad \varrho \in (1, 4/3] & \text{if } n = 2, \\ \rho = 6 \quad \text{and} \quad \varrho = 6/5 & \text{if } n = 3, \end{cases} \tag{2.25}$$

and

$$\begin{cases} j \in (1,2] \quad \text{and} \quad l \in [2, +\infty) & \text{if } n = 2, \\ j = 3/2 \quad \text{and} \quad l = 3 & \text{if } n = 3. \end{cases} \tag{2.26}$$

Needless to say, once j (or its conjugate l) is chosen according to the indicated range, then r and ρ, and their respective conjugates s and ϱ, are fixed. For instance, taking for $n = 2$, $j = l = 2$ yields $r = \rho = 4$ and $s = \varrho = 4/3$.

Hence, in terms of these indexes, we look for $\eta \in \mathrm{L}^\rho(\Omega)$ and $\mathbf{p} \in \mathbf{L}^r(\Omega)$, whereas the test function $\tilde{\boldsymbol{\tau}} \in \mathbf{L}^2(\Omega)$ is such that $\operatorname{div}(\tilde{\boldsymbol{\tau}}) \in \mathrm{L}^\varrho(\Omega)$. In this way, replacing the resulting expression from (2.24) into (2.21), and taking into account the definition (1.1), we arrive at

$$\int_\Omega \tilde{\boldsymbol{\sigma}} \cdot \tilde{\boldsymbol{\tau}} + \int_\Omega \eta \operatorname{div}(\tilde{\boldsymbol{\tau}}) + \tilde{c}_{\mathbf{u},\mathbf{p}}(\tilde{\boldsymbol{\tau}}, \eta) = \tilde{\mathrm{F}}(\tilde{\boldsymbol{\tau}}) \qquad \forall\, \tilde{\boldsymbol{\tau}} \in \mathbf{H}(\operatorname{div}_\varrho; \Omega),$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $\mathbf{q} \in \mathbf{L}^r(\Omega)$, $\tilde{c}_{\mathbf{z},\mathbf{q}} : \mathbf{H}(\operatorname{div}_\varrho; \Omega) \times \mathrm{L}^\rho(\Omega) \to \mathrm{R}$ is the bilinear form given by

$$\tilde{c}_{\mathbf{z},\mathbf{q}}(\tilde{\boldsymbol{\tau}}, \xi) := \kappa_\eta^{-1} \mu \int_\Omega \xi\, \mathbf{q} \cdot \tilde{\boldsymbol{\tau}} + \kappa_\eta^{-1} \int_\Omega \xi\, \mathbf{z} \cdot \tilde{\boldsymbol{\tau}} \qquad \forall\, (\tilde{\boldsymbol{\tau}}, \xi) \in \mathbf{H}(\operatorname{div}_\varrho; \Omega) \times \mathrm{L}^\rho(\Omega), \tag{2.27}$$

and $\tilde{\mathrm{F}} : \mathbf{H}(\operatorname{div}_\varrho; \Omega) \to \mathrm{R}$ is the linear functional defined as

$$\tilde{\mathrm{F}}(\tilde{\boldsymbol{\tau}}) := \langle \tilde{\boldsymbol{\tau}} \cdot \mathbf{n}, \eta_D \rangle_\Gamma \qquad \forall\, \tilde{\boldsymbol{\tau}} \in \mathbf{H}(\operatorname{div}_\varrho; \Omega). \tag{2.28}$$

In turn, testing now the sixth equation of (1.14) against $\xi \in \mathrm{L}^\rho(\Omega)$, which implicitly impose the unknown $\tilde{\boldsymbol{\sigma}}$ to live in $\mathbf{H}(\operatorname{div}_\varrho; \Omega)$, and assuming that the datum f_η belongs to $\mathrm{L}^\varrho(\Omega)$, we obtain

$$\int_\Omega \xi \operatorname{div}(\tilde{\boldsymbol{\sigma}}) = \tilde{\mathrm{G}}(\xi) \qquad \forall\, \xi \in \mathrm{L}^\rho(\Omega),$$

where $\widetilde{\mathrm{G}} : \mathrm{L}^\rho(\Omega) \to \mathrm{R}$ is the functional given by

$$\widetilde{\mathrm{G}}(\xi) \,:=\, -\kappa_\eta^{-1} \int_\Omega f_\eta\, \xi \qquad \forall\, \xi \in \mathrm{L}^\rho(\Omega)\,. \tag{2.29}$$

In this way, given $\mathbf{p} \in \mathbf{L}^r(\Omega)$ and $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and denoting the spaces

$$\mathrm{H} \,:=\, \mathbf{H}(\mathrm{div}_\varrho;\Omega) \quad \text{and} \quad \mathrm{Q} \,:=\, \mathrm{L}^\rho(\Omega)\,, \tag{2.30}$$

the mixed formulation for the cell density equation reduces to: Find $(\widetilde{\boldsymbol{\sigma}}, \eta) \in \mathrm{H} \times \mathrm{Q}$ such that

$$\begin{aligned} \widetilde{a}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \eta) \,+\, \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}}, \eta) &\,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \xi) &\,=\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\, \xi \in \mathrm{Q}\,, \end{aligned} \tag{2.31}$$

where $\widetilde{a} : \mathrm{H} \times \mathrm{H} \to \mathrm{R}$ and $\widetilde{b} : \mathrm{H} \times \mathrm{Q} \to \mathrm{R}$ are the bilinear forms defined as

$$\widetilde{a}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}}) \,:=\, \int_\Omega \widetilde{\boldsymbol{\zeta}} \cdot \widetilde{\boldsymbol{\tau}} \qquad \forall\, \widetilde{\boldsymbol{\zeta}},\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{2.32}$$

and

$$\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \xi) \,:=\, \int_\Omega \xi\, \mathrm{div}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H} \times \mathrm{Q}\,. \tag{2.33}$$

It is easily seen that $\widetilde{a}$, $\widetilde{b}$, $\widetilde{c}_{\mathbf{z},\mathbf{q}}$, $\widetilde{\mathrm{F}}$, and $\widetilde{\mathrm{G}}$ are bounded with the corresponding norms given by $\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} := \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{div}_\varrho;\Omega}$ for all $\widetilde{\boldsymbol{\tau}} \in \mathrm{H}$, and $\|\xi\|_{\mathrm{Q}} := \|\xi\|_{0,\rho;\Omega}$ for all $\xi \in \mathrm{Q}$. Indeed, applying the Hölder and Cauchy-Schwarz inequalities, invoking the bounds provided by (2.22) and (2.23), along with the fact that $\|\cdot\|_{0,r;\Omega} \leqslant |\Omega|^{(4-r)/4r}\, \|\cdot\|_{0,4;\Omega}$ for $\widetilde{c}_{\mathbf{z},\mathbf{q}}$, and proceeding similarly to $\mathbf{G}$ (cf. (2.15), (2.19)) for $\widetilde{\mathrm{F}}$, besides letting $\mathrm{i}_\rho : \mathrm{H}^1(\Omega) \to \mathrm{L}^\rho(\Omega)$ be the respective continuous injection, we deduce the existence of positive constants, denoted and given as

$$\begin{aligned} &\|\widetilde{a}\| \,:=\, 1\,, \qquad \|\widetilde{b}\| \,:=\, 1\,, \qquad \|\widetilde{c}\| \,:=\, \kappa_\eta^{-1} \max\big\{\mu, |\Omega|^{(4-r)/4r}\big\}\,, \\ &\|\widetilde{\mathrm{F}}\| \,:=\, \big(1 + \|\mathrm{i}_\rho\|\big)\, \|\eta_D\|_{1/2,\Gamma}\,, \qquad \text{and} \qquad \|\widetilde{\mathrm{G}}\| \,:=\, \kappa_\eta^{-1}\, \|f_\eta\|_{0,\varrho;\Omega}\,, \end{aligned} \tag{2.34}$$

such that

$$|\widetilde{a}(\widetilde{\boldsymbol{\zeta}}, \widetilde{\boldsymbol{\tau}})| \,\leqslant\, \|\widetilde{a}\|\, \|\widetilde{\boldsymbol{\zeta}}\|_{\mathrm{H}}\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\zeta}},\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{2.35}$$

$$|\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \xi)| \,\leqslant\, \|\widetilde{b}\|\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H} \times \mathrm{Q}\,, \tag{2.36}$$

$$|\widetilde{c}_{\mathbf{z},\mathbf{q}}(\widetilde{\boldsymbol{\tau}}, \xi)| \,\leqslant\, \|\widetilde{c}\|\, \big(\|\mathbf{z}\|_{0,4;O} + \|\mathbf{q}\|_{0,r;\Omega}\big)\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H} \times \mathrm{Q}\,, \tag{2.37}$$

$$|\widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}})| \,\leqslant\, \|\widetilde{\mathrm{F}}\|\, \|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \tag{2.38}$$

and

$$|\widetilde{\mathrm{G}}(\xi)| \,\leqslant\, \|\widetilde{\mathrm{G}}\|\, \|\xi\|_{\mathrm{Q}} \qquad \forall\, \xi \in \mathrm{Q}\,. \tag{2.39}$$

Finally, knowing that η will be sought in $\mathrm{L}^\rho(\Omega)$, we consider $\chi \in \mathrm{L}^\rho(\Omega)$, proceed similarly to the derivation of (2.22) and (2.23), and use that $\|\cdot\|_{0,\Omega} \leqslant |\Omega|^{1/4}\, \|\cdot\|_{0,4;\Omega}$, to bound the first term defining $\mathbf{F}_\chi$ (cf. (2.12)) as

$$\left| \int_\Omega \chi\, \nabla f \cdot \mathbf{v} \right| \,\leqslant\, |\Omega|^{1/4}\, \|\chi\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega}\, \|\mathbf{v}\|_{0,4;\Omega} \qquad \forall\, \mathbf{v} \in \mathbf{L}^4(\Omega)\,, \tag{2.40}$$

which requires to assume from now on that $\nabla f \in \mathbf{L}^r(\Omega)$. Then, bearing in mind the definition of $\mathbf{F}_\chi$ (cf. (2.12)) and the foregoing estimate, and setting the constant

$$\|\mathbf{F}\| \,:=\, \max\big\{1, |\Omega|^{1/4}\big\}\,, \tag{2.41}$$

we readily conclude that

$$|\mathbf{F}_\chi(\vec{\mathbf{v}})| \,\leqslant\, \|\mathbf{F}\|\, \big(\|\chi\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega}\big)\, \|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{2.42}$$

thus confirming the announced well-definedness and boundedness of $\mathbf{F}_\chi$.

2.3 The chemical signal concentration equations

Knowing already that $\mathbf{p} \in \mathbf{L}^r(\Omega)$, the seventh equation of (1.14) suggests to look originally for φ in $\mathrm{W}^{1,r}(\Omega)$. In this way, testing that equation against $\mathbf{q} \in \mathbf{H}^s(\mathrm{div}_s;\Omega)$ (cf. (1.3)), and then employing (1.9) and the Dirichlet boundary condition for φ, we obtain

$$\int_\Omega \mathbf{p}\cdot\mathbf{q} \,+\, \int_\Omega \varphi\,\mathrm{div}(\mathbf{q}) \,=\, \langle \mathbf{q}\cdot\mathbf{n}, \varphi_D\rangle_\Gamma\,, \tag{2.43}$$

which requires to assume that $\varphi_D \in \mathrm{W}^{1/s,r}(\Gamma)$. It follows from (2.43) that it suffices to seek the concentration φ of the chemical signal in the space $\mathrm{L}^r(\Omega)$. In turn, testing the eighth equation of (1.14) against an arbitrary function ϕ belonging to a space to be determined, we formally get

$$\int_\Omega \phi\,\mathrm{div}(\mathbf{p}) \,-\, \kappa_\varphi^{-1}\gamma\int_\Omega \eta\,\varphi\,\phi \,-\, \kappa_\varphi^{-1}\int_\Omega \mathbf{u}\cdot\mathbf{p}\,\phi \,=\, -\kappa_\varphi^{-1}\int_\Omega f_\varphi\,\phi\,. \tag{2.44}$$

Next, given the same $l,\, j \in (1,+\infty)$ conjugate to each other as before, and proceeding similarly to the derivation of (2.22) and (2.23), we find that

$$\left|\int_\Omega \eta\,\varphi\,\phi\right| \,\leqslant\, \|\eta\|_{0,2j;\Omega}\,\|\varphi\|_{0,2j;\Omega}\,\|\phi\|_{0,l;\Omega} \,=\, \|\eta\|_{0,r;\Omega}\,\|\varphi\|_{0,r;\Omega}\,\|\phi\|_{0,l;\Omega} \tag{2.45}$$

and

$$\left|\int_\Omega \mathbf{u}\cdot\mathbf{p}\,\phi\right| \,\leqslant\, \|\mathbf{u}\|_{0,2j;\Omega}\,\|\mathbf{p}\|_{0,2j;\Omega}\,\|\phi\|_{0,l;\Omega} \,=\, \|\mathbf{u}\|_{0,r;\Omega}\,\|\mathbf{p}\|_{0,r;\Omega}\,\|\phi\|_{0,l;\Omega}\,, \tag{2.46}$$

whence, recalling from (2.25) that $r \leqslant 4 \leqslant \rho$, we deduce that the second and third terms of (2.44) make sense for $\eta \in \mathrm{L}^\rho(\Omega)$, $\varphi \in \mathrm{L}^r(\Omega)$, $\phi \in \mathrm{L}^l(\Omega)$, $\mathbf{u} \in \mathbf{L}^4(\Omega)$, and $\mathbf{p} \in \mathbf{L}^r(\Omega)$. In addition, in order for the first and fourth terms to be well-defined, we need that both $\mathrm{div}(\mathbf{p})$ and the datum f_φ belong to $\mathrm{L}^j(\Omega)$, which yields, in particular, to look for $\mathbf{p}$ in $\mathbf{H}^r(\mathrm{div}_j;\Omega)$ (cf. (1.3)).

According to the foregoing discussion, we now set the Banach spaces

$$X_2 \,:=\, \mathbf{H}^r(\mathrm{div}_j;\Omega)\,,\quad X_1 \,:=\, \mathbf{H}^s(\mathrm{div}_s;\Omega)\,,\quad M_1 \,:=\, \mathrm{L}^r(\Omega)\,,\quad \text{and}\quad M_2 \,:=\, \mathrm{L}^l(\Omega)\,, \tag{2.47}$$

so that, given $\mathbf{u} \in \mathbf{L}^4(\Omega)$ and $\eta \in \mathrm{L}^\rho(\Omega)$, the mixed formulation for the chemical signal concentration equation reduces to: Find $(\mathbf{p},\varphi) \in X_2 \times M_1$ such that

$$\begin{aligned} a(\mathbf{p},\mathbf{q}) \,+\, b_1(\mathbf{q},\varphi) \,&=\, \mathrm{F}(\mathbf{q}) \quad \forall\,\mathbf{q} \in X_1\,,\\ b_2(\mathbf{p},\phi) \,-\, c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\phi\big) \,&=\, \mathrm{G}(\phi) \quad \forall\,\phi \in M_2\,, \end{aligned} \tag{2.48}$$

where, given $\mathbf{z} \in \mathbf{L}^4(\Omega)$ and $\chi \in \mathrm{L}^\rho(\Omega)$, the bilinear forms $a : X_2 \times X_1 \to \mathrm{R}$, $b_i : X_i \times M_i \to \mathrm{R}$, $i \in \{1,2\}$, and $c_{\mathbf{z},\chi} : \big(X_2 \times M_1\big) \times M_2 \to \mathrm{R}$, are defined as

$$a(\mathbf{r},\mathbf{q}) \,:=\, \int_\Omega \mathbf{r}\cdot\mathbf{q} \qquad \forall\,(\mathbf{r},\mathbf{q}) \in X_2 \times M_1\,, \tag{2.49}$$

$$b_i(\mathbf{q},\phi) \,:=\, \int_\Omega \phi\,\mathrm{div}(\mathbf{q}) \qquad \forall\,(\mathbf{q},\phi) \in X_i \times M_i\,, \tag{2.50}$$

and

$$c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big) \,:=\, \kappa_\varphi^{-1}\int_\Omega \mathbf{z}\cdot\mathbf{r}\,\phi \,+\, \kappa_\varphi^{-1}\gamma\int_\Omega \chi\,\psi\,\phi \qquad \forall\,\big((\mathbf{r},\psi),\phi\big) \in \big(X_2 \times M_1\big) \times M_2\,, \tag{2.51}$$

whereas the linear functionals $\mathrm{F} : X_1 \to \mathrm{R}$ and $\mathrm{G} : M_2 \to \mathrm{R}$ are given by

$$\mathrm{F}(\mathbf{q}) \,:=\, \langle \mathbf{q}\cdot\mathbf{n}, \varphi_D\rangle_\Gamma \qquad \forall\,\mathbf{q} \in X_1\,, \tag{2.52}$$

and

$$\mathrm{G}(\phi) \,:=\, -\kappa_\varphi^{-1}\int_\Omega f_\varphi\,\phi \qquad \forall\,\phi\in M_2\,. \tag{2.53}$$

Next, it is straightforward to see that the bilinear forms a, b_i, $i\in\{1,2\}$, and $c_{\mathbf{z},\chi}$, as well as the functionals F and G, are all bounded. In fact, applying Hölder's inequality, appealing to the bounds given by (2.45) and (2.46), and making use of the fact that $\|\cdot\|_{0,r;\Omega}\leqslant|\Omega|^{(4-r)/4r}\,\|\cdot\|_{0,4;\Omega}$ and $\|\cdot\|_{0,r;\Omega}\leqslant|\Omega|^{(\rho-r)/\rho r}\,\|\cdot\|_{0,\rho;\Omega}$ for $c_{\mathbf{z},\chi}$, we find that there exist positive constants, given by

$$\begin{gathered}\|a\| \,:=\, 1\,,\qquad \|b_i\| \,:=\, 1\,\big(i\in\{1,2\}\big)\,,\qquad \|c\| \,:=\, \kappa_\varphi^{-1}\,\max\big\{|\Omega|^{(4-r)/4r},\gamma\,|\Omega|^{(\rho-r)/\rho r}\big\}\,,\\ \text{and}\qquad \|\mathrm{G}\| \,=\, \kappa_\varphi^{-1}\,\|f_\varphi\|_{0,j;\Omega}\,,\end{gathered} \tag{2.54}$$

such that

$$|a(\mathbf{r},\mathbf{q})| \,\leqslant\, \|a\|\,\|\mathbf{r}\|_{X_2}\,\|\mathbf{q}\|_{M_1} \qquad \forall\,(\mathbf{r},\mathbf{q})\in X_2\times M_1\,, \tag{2.55}$$

$$|b_i(\mathbf{q},\phi)| \,\leqslant\, \|b_i\|\,\|\mathbf{q}\|_{X_i}\,\|\phi\|_{M_i} \qquad \forall\,(\mathbf{q},\phi)\in X_i\times M_i\,, \tag{2.56}$$

$$\begin{gathered}|c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big)| \,\leqslant\, \|c\|\,\big(\|\mathbf{z}\|_{0,4;\Omega} \,+\, \|\chi\|_{0,\rho;\Omega}\big)\,\|(\mathbf{r},\psi)\|_{X_2\times M_1}\,\|\phi\|_{M_2}\\ \forall\,\big((\mathbf{r},\psi),\phi\big)\in\big(X_2\times M_1\big)\times M_2\,,\end{gathered} \tag{2.57}$$

and

$$|\mathrm{G}(\phi)| \,\leqslant\, \|\mathrm{G}\|\,\|\phi\|_{M_2} \qquad \forall\,\phi\in M_2\,. \tag{2.58}$$

In turn, for the boundedness F we first observe, thanks to [16, Lemma A.36] and the surjectivity of the trace operator mapping $\mathrm{W}^{1,r}(\Omega)$ onto $\mathrm{W}^{1/s,r}(\Gamma)$, that there exists a fixed constant $C_r>0$ such that for each $\varphi\in\mathrm{W}^{1/s,r}(\Gamma)$ there exists $v\in\mathrm{W}^{1,r}(\Omega)$ satisfying $v|_\Gamma=\varphi$ and

$$\|v\|_{1,r;\Omega} := \|v\|_{0,r;\Omega} + \|\nabla v\|_{0,r;\Omega} \,\leqslant\, C_r\,\|\varphi\|_{1/s,r;\Gamma}\,.$$

In particular, denoting by $v_D\in\mathrm{W}^{1,r}(\Omega)$ a corresponding function for $\varphi_D\in\mathrm{W}^{1/s,r}(\Gamma)$, applying (1.9) to $(t,t')=(s,r)$ and $(\boldsymbol{\tau},v)=(\mathbf{q},v_D)$, and then using Hölder's inequality, we deduce that

$$|\mathrm{F}(\mathbf{q})| \,\leqslant\, \|\mathrm{F}\|\,\|\mathbf{q}\|_{X_1} \qquad \forall\,\mathbf{q}\in X_1\,, \tag{2.59}$$

with the constant

$$\|\mathrm{F}\| \,:=\, C_r\,\|\varphi_D\|_{1/s,r;\Gamma}\,. \tag{2.60}$$

As a consequence of the analysis developed in Sections 2.1 and 2.2, and the present Section 2.3, and under the assumption that the data belong to the indicated spaces, namely $\nabla f\in\mathbf{L}^r(\Omega)$, $\mathbf{f}\in\mathbf{L}^{4/3}(\Omega)$, $f_\eta\in\mathrm{L}^\varrho(\Omega)$, $f_\varphi\in\mathrm{L}^j(\Omega)$, $\mathbf{u}_D\in\mathbf{H}^{1/2}(\Gamma)$, $\eta_D\in\mathrm{H}^{1/2}(\Gamma)$, and $\varphi_D\in\mathrm{W}^{1/s,r}(\Gamma)$, we conclude that the fully-mixed formulation of the chemotaxis-Navier-Stokes problem (1.14) can be summarized by gathering (2.8), (2.31) and (2.48), that is: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma})\in\mathbf{H}\times\mathbf{Q}$, $(\widetilde{\boldsymbol{\sigma}},\eta)\in\mathrm{H}\times\mathrm{Q}$, and $(\mathbf{p},\varphi)\in X_2\times M_1$, such that

$$\begin{aligned}\mathbf{a}(\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\vec{\mathbf{v}}) \,+\, \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) \;&=\; \mathbf{F}_\eta(\vec{\mathbf{v}}) &&\forall\,\vec{\mathbf{v}}\in\mathbf{H}\,,\\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) \;&=\; \mathbf{G}(\boldsymbol{\tau}) &&\forall\,\boldsymbol{\tau}\in\mathbf{Q}\,,\\ \widetilde{a}(\widetilde{\boldsymbol{\sigma}},\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}},\eta) \,+\, \widetilde{c}_{\mathbf{u},\mathbf{p}}(\widetilde{\boldsymbol{\tau}},\eta) \;&=\; \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) &&\forall\,\widetilde{\boldsymbol{\tau}}\in\mathrm{H}\,,\\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}},\xi) \;&=\; \widetilde{\mathrm{G}}(\xi) &&\forall\,\xi\in\mathrm{Q}\,,\\ a(\mathbf{p},\mathbf{q}) \,+\, b_1(\mathbf{q},\varphi) \;&=\; \mathrm{F}(\mathbf{q}) &&\forall\,\mathbf{q}\in X_1\,,\\ b_2(\mathbf{p},\phi) \,-\, c_{\mathbf{u},\eta}\big((\mathbf{p},\varphi),\phi\big) \;&=\; \mathrm{G}(\phi) &&\forall\,\phi\in M_2\,.\end{aligned} \tag{2.61}$$

3 The continuous solvability analysis

In this section we proceed similarly as in [10] and [20] (see also [2], [6], [21], and some of the references therein) and adopt a fixed-point strategy to analyze the solvability of (2.61).

3.1 The fixed-point approach

We begin by rewriting (2.61) as an equivalent fixed point equation. To this end, we first let $\mathbf{S} : \mathbf{L}^4(\Omega) \times \mathrm{Q} \to \mathbf{L}^4(\Omega)$ be the operator defined by

$$\mathbf{S}(\mathbf{z}, \chi) := \mathbf{u} \qquad \forall\, (\mathbf{z}, \chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}\,, \tag{3.1}$$

where $(\vec{\mathbf{u}}, \boldsymbol{\sigma}) = \big((\mathbf{u}, \mathbf{t}), \boldsymbol{\sigma}\big) \in \mathbf{H} \times \mathbf{Q}$ is the unique solution (to be confirmed below) of problem (2.8) (equivalently, the first and second rows of (2.61)) when $\mathbf{c}(\mathbf{u}; \cdot, \cdot)$ and $\mathbf{F}_\eta$ are replaced by $\mathbf{c}(\mathbf{z}; \cdot, \cdot)$ and $\mathbf{F}_\chi$, respectively, that is

$$\begin{aligned} \mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{v}}) \,+\, \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{v}}) \,+\, \mathbf{b}(\vec{\mathbf{v}}, \boldsymbol{\sigma}) &= \mathbf{F}_\chi(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{u}}, \boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \end{aligned} \tag{3.2}$$

Similarly, we let $\widetilde{\mathrm{S}} : \mathbf{L}^4(\Omega) \times X_2 \to \mathrm{Q}$ be the operator given by

$$\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r}) := \eta \qquad \forall\, (\mathbf{z}, \mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2\,, \tag{3.3}$$

where $(\widetilde{\boldsymbol{\sigma}}, \eta) \in \mathrm{H} \times \mathrm{Q}$ is the unique solution (to be confirmed below) of problem (2.31) (equivalently, the third and fourth rows of (2.61)) when $\widetilde{c}_{\mathbf{u},\mathbf{p}}$ is replaced by $\widetilde{c}_{\mathbf{z},\mathbf{r}}$, that is

$$\begin{aligned} \widetilde{a}(\widetilde{\boldsymbol{\sigma}}, \widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}}, \eta) \,+\, \widetilde{c}_{\mathbf{z},\mathbf{r}}(\widetilde{\boldsymbol{\tau}}, \eta) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \mathrm{H}\,, \\ \widetilde{b}(\widetilde{\boldsymbol{\sigma}}, \xi) &= \widetilde{\mathrm{G}}(\xi) \qquad \forall\, \xi \in \mathrm{Q}\,. \end{aligned} \tag{3.4}$$

In turn, we let $\mathbf{T} : \mathbf{L}^4(\Omega) \times \mathrm{Q} \to X_2$ be the operator given by

$$\mathbf{T}(\mathbf{z}, \chi) := \mathbf{p} \quad \forall\, (\mathbf{z}, \chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}\,, \tag{3.5}$$

where $(\mathbf{p}, \varphi) \in X_2 \times M_1$ is the unique solution (to be confirmed below) of problem (2.48) (equivalently, the fifth and sixth rows of (2.61)) when $c_{\mathbf{u},\eta}$ is replaced by $c_{\mathbf{z},\chi}$, that is

$$\begin{aligned} a(\mathbf{p}, \mathbf{q}) \,+\, b_1(\mathbf{q}, \varphi) &= \mathrm{F}(\mathbf{q}) \qquad \forall\, \mathbf{q} \in X_1\,, \\ b_2(\mathbf{p}, \phi) \,-\, c_{\mathbf{z},\chi}\big((\mathbf{p}, \varphi), \phi\big) &= \mathrm{G}(\phi) \qquad \forall\, \phi \in M_2\,. \end{aligned} \tag{3.6}$$

Thus, defining the operator $\boldsymbol{\Xi} : \mathbf{L}^4(\Omega) \times X_2 \to \mathbf{L}^4(\Omega) \times X_2$ as

$$\boldsymbol{\Xi}(\mathbf{z}, \mathbf{r}) := \Big(\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big), \mathbf{T}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\Big) \qquad \forall\, (\mathbf{z}, \mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2\,, \tag{3.7}$$

we realize that solving (2.61) is equivalent to seeking a fixed point of $\boldsymbol{\Xi}$, that is: Find $(\mathbf{u}, \mathbf{p}) \in \mathbf{L}^4(\Omega) \times X_2$ such that

$$\boldsymbol{\Xi}(\mathbf{u}, \mathbf{p}) = (\mathbf{u}, \mathbf{p})\,. \tag{3.8}$$

3.2 Well-posedness of the uncoupled problems

We now employ the Babuska-Brezzi theory in Banach spaces (cf. [3, Theorem 2.1, Corollary 2.1, Section 2.1] for the general case, and [16, Theorem 2.34] for a particular one), and the Banach-Nečas-Babuška Theorem (also known as the generalized Lax-Milgram Lemma) (cf. [16, Theorem 2.6]), to establish the well-posedness of the problems (3.2), (3.4), and (3.6), defining the operators $\mathbf{S}$, $\widetilde{\mathrm{S}}$, and $\mathbf{T}$, respectively.

3.2.1 Well-definedness of operator $\mathbf{S}$

Here we apply [16, Theorem 2.34] to prove that problem (3.2) is well-posed (equivalently, that $\mathbf{S}$ is well-defined). In this regard, it is important to stress that the structure of (3.2) is the same of the problem stated in [10, eq. (3.23)], and hence, several results and techniques from there will be employed in what follows. Indeed, given $(\mathbf{z},\chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}$, we proceed as in [10, Section 3.3], and introduce first the bilinear form $\mathcal{A}_{\mathbf{z}} : \mathbf{H} \times \mathbf{H} \to \mathrm{R}$ defined by

$$\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{w}},\vec{\mathbf{v}}) := \mathbf{a}(\vec{\mathbf{w}},\vec{\mathbf{v}}) + \mathbf{c}(\mathbf{z};\vec{\mathbf{w}},\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{3.9}$$

so that problem (3.2) can be rewritten as: Find $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H} \times \mathbf{Q}$ such that

$$\begin{aligned} \mathcal{A}_{\mathbf{z}}(\vec{\mathbf{u}},\vec{\mathbf{v}}) + \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\sigma}) &= \mathbf{F}_{\chi}(\vec{\mathbf{v}}) \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{H}\,, \\ \mathbf{b}(\vec{\mathbf{u}},\boldsymbol{\tau}) &= \mathbf{G}(\boldsymbol{\tau}) \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,. \end{aligned} \tag{3.10}$$

Now, we let $\mathbf{V}$ be the kernel of the operator induced by the bilinear form $\mathbf{b}$ (cf. (2.10)), that is

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{H} : \quad \mathbf{b}(\vec{\mathbf{v}},\boldsymbol{\tau}) = 0 \quad \forall\, \boldsymbol{\tau} \in \mathbf{Q} \Big\}\,,$$

which, exactly as [10, eq. (3.34)], reduces to

$$\mathbf{V} := \Big\{ \vec{\mathbf{v}} := (\mathbf{v},\mathbf{s}) \in \mathbf{H} : \quad \nabla \mathbf{v} = \mathbf{s} \quad \text{and} \quad \mathbf{v} \in \mathbf{H}^1_0(\Omega) \Big\}\,. \tag{3.11}$$

Then, letting c_P be the positive constant yielding the Friedrichs-Poincaré inequality, which states that $|\mathbf{v}|^2_{1,\Omega} \geqslant c_P \|\mathbf{v}\|^2_{1,\Omega}$ for all $\mathbf{v} \in \mathbf{H}^1_0(\Omega)$, denoting by $\mathbf{i}_4$ the continuous injection of $\mathbf{H}^1(\Omega)$ into $\mathbf{L}^4(\Omega)$, bearing in mind (3.9) and (2.20), and proceeding analogously to the proof of [10, eq. (3.41), Lemma 3.2], we find that

$$\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{v}},\vec{\mathbf{v}}) = \mathbf{a}(\vec{\mathbf{v}},\vec{\mathbf{v}}) \geqslant \boldsymbol{\alpha}\, \|\vec{\mathbf{v}}\|^2_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{v}} \in \mathbf{V}\,, \tag{3.12}$$

with $\boldsymbol{\alpha} := \frac{\nu}{2} \min\big\{1, \frac{c_P}{\|\mathbf{i}_4\|^2}\big\}$, which gives the $\mathbf{V}$-ellipticity of $\mathcal{A}_{\mathbf{z}}$. Thus, it is easily seen, thanks to (3.12), that $\mathcal{A}_{\mathbf{z}}$ satisfies the hypothesis specified in [16, Theorem 2.34, eq. (2.28)] with the constant $\boldsymbol{\alpha}$ defined above. In addition, it follows from (3.9), along with (2.15), (2.16), and (2.18), that there holds

$$|\mathcal{A}_{\mathbf{z}}(\vec{\mathbf{w}},\vec{\mathbf{v}})| \leqslant \|\mathcal{A}_{\mathbf{z}}\|\, \|\vec{\mathbf{w}}\|_{\mathbf{H}}\, \|\vec{\mathbf{v}}\|_{\mathbf{H}} \qquad \forall\, \vec{\mathbf{w}},\, \vec{\mathbf{v}} \in \mathbf{H}\,, \tag{3.13}$$

with the constant

$$\|\mathcal{A}_{\mathbf{z}}\| := \|\mathbf{a}\| + \|\mathbf{c}\|\, \|\mathbf{z}\|_{0,4;\Omega} = \nu + \frac{\lambda}{2}\, \|\mathbf{z}\|\,, \tag{3.14}$$

which says that $\mathcal{A}$ is bounded.

In turn, using that for each $t \geqslant \frac{2n}{n+2}$ there exists a constant $C_t > 0$, depending only on Ω, such that

$$C_t\, \|\boldsymbol{\tau}\|^2_{0,\Omega} \leqslant \|\boldsymbol{\tau}^{\mathrm{d}}\|^2_{0,\Omega} + \|\mathbf{div}(\boldsymbol{\tau})\|^2_{0,t;\Omega} \qquad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_t;\Omega)\,, \tag{3.15}$$

which follows from a slight modification of the proof of [18, Lemma 2.3], one can prove the continuous inf-sup condition for the bilinear form $\mathbf{b}$. More precisely, employing (3.15) with $t = 4/3$, it is shown in [10, Lemma 3.3, eq. (3.44)] that there exists a positive constant $\boldsymbol{\beta}$, depending only on $C_{4/3}$, such that

$$\sup_{\substack{\vec{\mathbf{v}} \in \mathbf{H} \\ \vec{\mathbf{v}} \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}},\vec{\boldsymbol{\tau}})}{\|\vec{\mathbf{v}}\|_{\mathbf{H}}} \geqslant \boldsymbol{\beta}\, \|\boldsymbol{\tau}\|_{\mathbf{Q}} \qquad \forall\, \boldsymbol{\tau} \in \mathbf{Q}\,, \tag{3.16}$$

whence the bilinear form $\mathbf{b}$ satisfies the hypothesis indicated in [16, Theorem 2.34, eq. (2.29)].

We are now in position to confirm that the operator $\mathbf{S}$ is well-defined.

Lemma 3.1. *For each* $(\mathbf{z},\chi)\in\mathbf{L}^4(\Omega)\times\mathrm{Q}$ *there exists a unique* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big)\in\mathbf{H}\times\mathbf{Q}$ *solution of* (3.10) (*equivalently* (3.2)), *and hence one can define* $\mathbf{S}(\mathbf{z},\chi) := \mathbf{u}\in\mathbf{L}^4(\Omega)$. *Moreover, there exists a positive constant* $C_{\mathbf{S}}$, *depending only on* $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}$, *and* $\boldsymbol{\beta}$, *such that*

$$\begin{aligned}\|\mathbf{S}(\mathbf{z},\chi)\|_{0,4;\Omega} \,&=\, \|\mathbf{u}\|_{0,4;\Omega} \,\leqslant\, \|\vec{\mathbf{u}}\|_{\mathbf{H}}\\ &\leqslant\, C_{\mathbf{S}}\Big\{\|\chi\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}.\end{aligned} \tag{3.17}$$

Proof. Having previously established that $\mathcal{A}_{\mathbf{z}}$ and $\mathbf{b}$ satisfy [16, eqs. (2.28) and (2.29)], and knowing that $\mathcal{A}_{\mathbf{z}}$, $\mathbf{b}$, $\mathbf{F}_\chi$, and $\mathbf{G}$ are all bounded, a straightforward application of [16, Theorem 2.34] confirms the existence of a unique $(\vec{\mathbf{u}},\boldsymbol{\sigma}) := \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big)\in\mathbf{H}\times\mathbf{Q}$ solution of (3.10). In addition, the corresponding a priori estimate in [16, Theorem 2.34, eq. (2.30)] yields

$$\|\vec{\mathbf{u}}\|_{\mathbf{H}} \,\leqslant\, \frac{1}{\boldsymbol{\alpha}}\,\|\mathbf{F}_\chi\| \,+\, \frac{1}{\boldsymbol{\beta}}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}\|\,. \tag{3.18}$$

Then, noting from (2.41) and (2.42) that

$$\|\mathbf{F}_\chi\| \,\leqslant\, \max\big\{1,|\Omega|^{1/4}\big\}\,\big(\|\chi\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega}\big)\,, \tag{3.19}$$

invoking the expressions for $\|\mathbf{G}\|$ and $\|\mathcal{A}_{\mathbf{z}}\|$ provided in (2.15) and (3.14), respectively, and performing some minor algebraic manipulations, we readily derive from (3.18) the required inequality (3.17). □

Regarding the a priori estimate for the component $\boldsymbol{\sigma}$ of the unique solution of (3.10), which will be used later on, we recall that the second inequality in [16, Theorem 2.34, eq. (2.30)] gives

$$\|\boldsymbol{\sigma}\|_{\mathbf{Q}} \,\leqslant\, \frac{1}{\boldsymbol{\beta}}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{F}_\chi\| \,+\, \frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\beta}^2}\left(1+\frac{\|\mathcal{A}_{\mathbf{z}}\|}{\boldsymbol{\alpha}}\right)\|\mathbf{G}\|\,,$$

which, proceeding similarly to the derivation of (3.17), yields

$$\begin{aligned}\|\boldsymbol{\sigma}\|_{\mathbf{Q}} \,=\, \|\boldsymbol{\sigma}\|_{\mathbf{div}_{4/3};\Omega} \,\leqslant\, \bar{C}_{\mathbf{S}}\,\big(1 \,+\, \|\mathbf{z}\|_{0,4;\Omega}\big)\,\Big\{&\|\chi\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\\ &+\,\|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\},\end{aligned} \tag{3.20}$$

where $\bar{C}_{\mathbf{S}}$ is a positive constant depending as well on $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}$, and $\boldsymbol{\beta}$.

3.2.2 Well-definedness of operator $\widetilde{\mathrm{S}}$

In this section we make use of [16, Theorems 2.34 and 2.6] to show that (3.4) is well-posed (equivalently, that $\widetilde{\mathrm{S}}$ is well-defined). To this end, and similarly to Section 3.2.1, we notice that, given $(\mathbf{z},\mathbf{r})\in\mathbf{L}^4(\Omega)\times X_2$, the structure of (3.4) is analogous to that of the problem specified in [20, eq. (2.33), Section 2.3], so that some results and techniques from its corresponding analysis are employed below. In particular, following the approach from [20, Section 2.4.3], we first apply [16, Theorem 2.34] to a perturbation of (3.4), and then employ [16, Theorem 2.6] to conclude that the whole problem (3.4) is well-posed. More precisely, we let $\widetilde{A} : (\mathrm{H}\times\mathrm{Q})\times(\mathrm{H}\times\mathrm{Q})\to\mathrm{R}$ be the bounded bilinear form arising from (3.4) after adding the left hand sides of its equations, but without including $\widetilde{c}_{\mathbf{z},\mathbf{r}}$, that is

$$\widetilde{A}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi)) \,:=\, \widetilde{a}(\widetilde{\boldsymbol{\zeta}},\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\tau}},\chi) \,+\, \widetilde{b}(\widetilde{\boldsymbol{\zeta}},\xi) \tag{3.21}$$

for all $(\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi)\in\mathrm{H}\times\mathrm{Q}$, and show next that $\widetilde{A}$ satisfies a global continuous inf-sup condition. Note that, being $\widetilde{A}$ symmetric, the latter will be valid with respect to any of its components. We also remark that the boundedness of $\widetilde{A}$ follows from those of $\widetilde{a}$ and $\widetilde{b}$ (cf. (2.34), (2.35), and (2.36)).

Since establishing the aforementioned property for $\widetilde{A}$ is equivalent to proving that the bilinear forms $\widetilde{a}$ and $\widetilde{b}$ satisfy the hypotheses of [16, Theorem 2.34], we proceed with the latter in what follows. We begin by letting $\widetilde{\mathrm{V}}$ be the null space of the operator induced by the bilinear form $\widetilde{b}$, that is

$$\widetilde{\mathrm{V}} := \Big\{ \widetilde{\boldsymbol{\tau}} \in \mathrm{H} : \quad b(\widetilde{\boldsymbol{\tau}}, \xi) \,=\, 0 \quad \forall\, \xi \in \mathrm{Q} \Big\},$$

which, according to the definitions of $\widetilde{b}$ (cf. (2.33)) and the spaces H and Q (cf. (2.30)), yields

$$\widetilde{\mathrm{V}} := \Big\{ \widetilde{\boldsymbol{\tau}} \in \mathrm{H} : \quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}) \,=\, 0 \Big\}. \tag{3.22}$$

Then, it is straightforward to see from the definitions of $\widetilde{a}$ (cf. (2.32)) and the norm of $\mathrm{H} := \mathbf{H}(\mathrm{div}_\varrho; \Omega)$ (cf. (1.4)) that there holds

$$\widetilde{a}(\widetilde{\boldsymbol{\tau}}, \widetilde{\boldsymbol{\tau}}) \,=\, \|\widetilde{\boldsymbol{\tau}}^2\|_{\mathrm{H}} \qquad \forall\, \widetilde{\boldsymbol{\tau}} \in \widetilde{\mathrm{V}}, \tag{3.23}$$

from which one easily deduces that $\widetilde{a}$ satisfies the hypotheses given by [16, Theorem 2.34, eq. (2.28)] with the constant $\widetilde{\alpha} = 1$.

Furthermore, since the continuous inf-sup condition for $\widetilde{b}$ has already been established (see, e.g. [6, Lemma 2.1], [20, Lemma 2.9], and also [21, Lemma 3.5] for a closely related result), we provide next only the main details of its corresponding proof. In fact, given $\xi \in \mathrm{Q} := \mathrm{L}^\rho(\Omega)$, we note from (2.25) that $\rho > 2$, introduce $\xi_\varrho := |\xi|^{\rho-2}\,\xi$, and observe that

$$\xi_\varrho \in \mathrm{L}^\varrho(\Omega) \qquad \text{and} \qquad \int_\Omega \xi\,\xi_\varrho \,=\, \|\xi\|_{0,\rho;\Omega}\,\|\xi_\varrho\|_{0,\varrho;\Omega}\,. \tag{3.24}$$

Then, letting $w \in \mathrm{H}^1_0(\Omega)$ be the unique weak solution of $\Delta\, w = -\xi_\varrho$ in Ω, $w = 0$ on Γ, for which there holds $\|w\|_{1,\Omega} \leqslant \frac{\|i_\rho\|}{c_P}\,\|\xi_\varrho\|_{0,\varrho;\Omega}$, where c_P is the constant yielding the Friedrichs-Poincaré inequality, and i_ρ is the continuous injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^\rho(\Omega)$, we define $\widetilde{\boldsymbol{\zeta}} := -\nabla w \in \mathbf{L}^2(\Omega)$ and notice that $\mathrm{div}(\widetilde{\boldsymbol{\zeta}}) = \xi_\varrho$, so that $\widetilde{\boldsymbol{\zeta}} \in \mathrm{H} := \mathbf{H}(\mathrm{div}_\varrho; \Omega)$. In this way, bounding by below with $\widetilde{\boldsymbol{\tau}} = \widetilde{\boldsymbol{\zeta}}$, and using the above identities and estimates, we arrive at

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}} \in \mathrm{H} \\ \widetilde{\boldsymbol{\tau}} \neq \mathbf{0}}} \frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}, \xi)}{\|\widetilde{\boldsymbol{\tau}}\|_{\mathrm{H}}} \,\geqslant\, \widetilde{\beta}\,\|\xi\|_{\mathrm{Q}}\,, \tag{3.25}$$

with $\widetilde{\beta} := \big(1 + \frac{\|i_\rho\|}{c_P}\big)^{-1}$.

Consequently, thanks to (3.23) and (3.25), the hypotheses of [16, Theorem 2.34] are satisfied, and hence the a priori estimates given by [16, Theorem 2.34, eq. (2.30)] imply the existence of a positive constant $\alpha_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}$, $\widetilde{\beta}$, and $\|\widetilde{a}\|$, such that

$$\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q} \\ (\widetilde{\boldsymbol{\tau}},\xi) \neq 0}} \frac{\widetilde{A}((\widetilde{\boldsymbol{\zeta}}, \chi), (\widetilde{\boldsymbol{\tau}}, \xi))}{\|(\widetilde{\boldsymbol{\tau}}, \xi)\|_{\mathrm{H}\times\mathrm{Q}}} \,\geqslant\, \alpha_{\widetilde{\mathrm{S}}}\,\|(\widetilde{\boldsymbol{\zeta}}, \chi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\,(\widetilde{\boldsymbol{\zeta}}, \chi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.26}$$

Next, we let $\widetilde{A}_{\mathbf{z},\mathbf{r}} : \big(\mathrm{H}\times\mathrm{Q}\big) \times \big(\mathrm{H}\times\mathrm{Q}\big) \to \mathrm{R}$ be the bounded bilinear form that results after adding the full left hand sides of the equations of (3.4), that is

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}}, \chi), (\widetilde{\boldsymbol{\tau}}, \xi)) \,:=\, \widetilde{A}((\widetilde{\boldsymbol{\zeta}}, \chi), (\widetilde{\boldsymbol{\tau}}, \xi)) \,+\, \widetilde{c}_{\mathbf{z},\mathbf{r}}(\widetilde{\boldsymbol{\tau}}, \chi) \qquad \forall\,(\widetilde{\boldsymbol{\zeta}}, \chi),\,(\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H}\times\mathrm{Q}\,, \tag{3.27}$$

whence problem (3.4) can be rewritten, equivalently, as: Find $(\widetilde{\boldsymbol{\sigma}}, \eta) \in \mathrm{H}\times\mathrm{Q}$ such that

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\sigma}}, \eta), (\widetilde{\boldsymbol{\tau}}, \xi)) \,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\,(\widetilde{\boldsymbol{\tau}}, \xi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.28}$$

We remark that the boundedness of $\widetilde{A}$ and $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (2.37)) implies the same property for $\widetilde{A}_{\mathbf{z},\mathbf{r}}$. In turn, it follows from (3.26), (3.27), and the boundedness of $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (2.34) and (2.37)), that for each $(\widetilde{\boldsymbol{\zeta}}, \chi) \in \mathrm{H} \times \mathrm{Q}$ there holds

$$\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q} \\ (\widetilde{\boldsymbol{\tau}},\xi)\neq\mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \alpha_{\widetilde{\mathrm{S}}}\, \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}} - \|\widetilde{c}\| \left(\|\mathbf{z}\|_{0,4;O} + \|\mathbf{r}\|_{0,r;\Omega}\right) \|\chi\|_{\mathrm{Q}}$$

$$\geqslant \left\{\alpha_{\widetilde{\mathrm{S}}} - \|\widetilde{c}\| \left(\|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{0,r;\Omega}\right)\right\} \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}}\,,$$

and thus, under the assumption that

$$\|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{0,r;\Omega} \leqslant \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|}\,, \tag{3.29}$$

we arrive at

$$\sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in \mathrm{H}\times\mathrm{Q} \\ (\widetilde{\boldsymbol{\tau}},\xi)\neq\mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \frac{\alpha_{\widetilde{\mathrm{S}}}}{2}\, \|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\zeta}},\chi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.30}$$

Analogously, noting that $\widetilde{A}$ is symmetric, proceeding as before, and under the same assumption (3.29), we obtain

$$\sup_{\substack{(\widetilde{\boldsymbol{\zeta}},\chi)\in \mathrm{H}\times\mathrm{Q} \\ (\widetilde{\boldsymbol{\zeta}},\chi)\neq\mathbf{0}}} \frac{\widetilde{A}_{\mathbf{z},\mathbf{r}}((\widetilde{\boldsymbol{\zeta}},\chi),(\widetilde{\boldsymbol{\tau}},\xi))}{\|(\widetilde{\boldsymbol{\zeta}},\chi)\|_{\mathrm{H}\times\mathrm{Q}}} \geqslant \frac{\alpha_{\widetilde{\mathrm{S}}}}{2}\, \|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}} \qquad \forall\, (\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.31}$$

According to the foregoing analysis, the well-definedness of $\widetilde{\mathrm{S}}$ is established as follows.

Lemma 3.2. *For each $(\mathbf{z},\mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2$ satisfying (3.29) there exists a unique $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$ solution of (3.28) (equivalently (3.4)), and hence one can define $\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) := \eta \in \mathrm{Q}$. Moreover, there exists a positive constant $C_{\widetilde{\mathrm{S}}}$, depending only on $\alpha_{\widetilde{\mathrm{S}}}$, $\|\mathrm{i}_\rho\|$, and κ_η, such that*

$$\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\|_{\mathrm{Q}} = \|\eta\|_{0,\rho;\Omega} \leqslant \|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant C_{\widetilde{\mathrm{S}}} \left\{\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right\}. \tag{3.32}$$

Proof. Bearing in mind the boundedness of $\widetilde{A}_{\mathbf{z},\mathbf{r}}$, (3.30), and (3.31), a straightforward application of [16, Theorem 2.6] yields the existence of a unique solution $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$ to (3.28). In addition, the corresponding a priori estimate (cf. [16, Theorem 2.6, eq. (2.5)]) gives

$$\|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant \frac{2}{\alpha_{\widetilde{\mathrm{S}}}} \left\{\|\widetilde{\mathrm{F}}\| + \|\widetilde{\mathrm{G}}\|\right\},$$

which, along with the expressions for $\|\widetilde{\mathrm{F}}\|$ and $\|\widetilde{\mathrm{G}}\|$ provided in (2.34), lead to (3.32) with the constant $C_{\widetilde{\mathrm{S}}} := \frac{2}{\alpha_{\widetilde{\mathrm{S}}}} \max\left\{1 + \|\mathrm{i}_\rho\|, \kappa_\eta^{-1}\right\}$. □

3.2.3 Well-definedness of operator T

Our goal now is to show that (3.6) is well-posed (equivalently, that $\mathbf{T}$ is well-defined), for which we will make use of the most general Babuška-Brezzi theory in Banach spaces (cf. [3, Theorem 2.1, Corollary 2.1, Section 2.1]) and the Banach-Nečas-Babuška Theorem (cf. [16, Theorem 2.6]). To this end, and as observed for Sections 3.2.1 and 3.2.2, we notice here that, given $(\mathbf{z},\chi) \in \mathbf{L}^4(\Omega) \times \mathrm{Q}$, the structure of (3.6) is similar to a perturbation of the problem described by [20, eq. (2.32)], so that some of the techniques employed there will be adapted for our analysis below. In particular, proceeding as in [20, Section 2.4.2], we first employ [3, Theorem 2.1, Corollary 2.1, Section 2.1] to analyze part of (3.6), and then we apply [16, Theorem 2.6] to conclude the well-posedness of the whole problem. According to this, we now let $\mathbf{A} : \big(X_2 \times M_1\big) \times \big(X_1 \times M_2\big) \to \mathrm{R}$ be the bounded bilinear form arising from (3.6) after adding the left hand sides of its equations, but without including $c_{\mathbf{z},\chi}$, that is

$$\begin{aligned} \mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,&:=\, a(\mathbf{r},\mathbf{q}) \,+\, b_1(\mathbf{q},\psi) \,+\, b_2(\mathbf{r},\phi) \\ \forall\,(\mathbf{r},\psi) \in \big(X_2 \times M_1\big), &\quad \forall\,(\mathbf{q},\phi) \in \big(X_1 \times M_2\big)\,, \end{aligned} \tag{3.33}$$

and aim to prove next that $\mathbf{A}$ satisfies global continuous inf-sup conditions with respect to both its first and second component. Note that the boundedness of $\mathbf{A}$ follows from those of a, b_1 and b_2 (cf. (2.55), (2.56)).

The verification of the aforementioned properties of $\mathbf{A}$ is equivalent to establishing that the bilinear forms a, b_1, and b_2 verify the hypotheses of [3, Theorem 2.1, Section 2.1], which we address in what follows. Firstly, for each $i \in \{1,2\}$ we let K_i be the kernel of the bilinear form b_i (cf. (2.50)), that is

$$K_i \,:=\, \Big\{\mathbf{q} \in X_i : \quad b_i(\mathbf{q},\phi) \,=\, 0 \quad \forall\,\phi \,\in\, M_i\Big\}\,,$$

which, according to the definitions of X_1, X_2, M_1, and M_2 (cf. (2.47)), and b_i (cf. (2.50)), gives

$$K_1 \,=\, \Big\{\mathbf{q} \in \mathbf{H}^s(\mathrm{div}_s;\Omega) : \quad \mathrm{div}(\mathbf{q}) \,=\, 0 \quad \text{in} \quad \Omega\Big\} \tag{3.34}$$

and

$$K_2 \,=\, \Big\{\mathbf{q} \in \mathbf{H}^r(\mathrm{div}_j;\Omega) : \quad \mathrm{div}(\mathbf{q}) \,=\, 0 \quad \text{in} \quad \Omega\Big\}. \tag{3.35}$$

The following lemma introduces a suitable linear operator mapping $\mathbf{L}^t(\Omega)$ into itself for a range of t containing the one specified for s in (2.25). This result will be utilized next to establish the inf-sup conditions required by [3, Theorem 2.1] (equivalently, [3, eqs. (2.8) and (2.9)]) for our bilinear form a (cf. (2.49)).

Lemma 3.3. *Let Ω be a bounded Lipschitz-continuous domain of R^n, $n \in \{2,3\}$, and let $t, t' \in (1,+\infty)$ conjugate to each other with t (and hence t') lying in* $\begin{cases} [4/3,4] & \text{if } n=2 \\ [6/5,3] & \text{if } n=3 \end{cases}$*. Then, there exists a linear and bounded operator $D_t : \mathbf{L}^t(\Omega) \to \mathbf{L}^t(\Omega)$ such that*

$$\mathrm{div}(D_t(\mathbf{w})) \,=\, 0 \quad \text{in} \quad \Omega \qquad \forall\,\mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{3.36}$$

In addition, for each $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathrm{div}(\mathbf{z}) = 0$ in Ω, there holds

$$\int_\Omega \mathbf{z}\cdot D_t(\mathbf{w}) \,=\, \int_\Omega \mathbf{z}\cdot\mathbf{w} \qquad \forall\,\mathbf{w} \in \mathbf{L}^t(\Omega)\,. \tag{3.37}$$

Proof. It is a slight modification of the proof of [20, Lemma 2.3]. Indeed, given $\mathbf{w} \in \mathbf{L}^t(\Omega)$, with t in the range indicated, we know from the scalar version of [19, Theorem 3.2] (see also [24, Theorems 1.1 and 1.3]) that there exists a unique $u \in \mathrm{W}^{1,t}(\Omega)$ such that

$$\mathrm{div}(\nabla u + \mathbf{w}) \,=\, 0 \quad \text{in} \quad \Omega\,, \quad u \,=\, 0 \quad \text{on} \quad \partial\Omega\,,$$

and there exists a constant $C_t > 0$ such that $\|u\|_{1,t;\Omega} \,\leqslant\, C_t\,\|\mathbf{w}\|_{0,t;\Omega}$. Then, defining $D_t(\mathbf{w}) \,:=\, \nabla u + \mathbf{w}$, it is readily seen that D_t is linear and bounded, and satisfies (3.36). In turn, given $\mathbf{z} \in \mathbf{L}^{t'}(\Omega)$ such that $\mathrm{div}(\mathbf{z}) = 0$ in Ω, it is clear that $\mathbf{z} \in \mathbf{H}^{t'}(\mathrm{div}_{t'};\Omega)$, so that applying (1.9) to $\mathbf{z}$ and u, we obtain

$$\int_\Omega \mathbf{z}\cdot\nabla u \,=\, -\int_\Omega u\,\mathrm{div}(\mathbf{z}) \,+\, \langle \mathbf{z}\cdot\mathbf{n}, u\rangle \,=\, 0\,,$$

which yields (3.37) and finishes the proof. □

The following result, which makes use of Lemma 3.3, resembles [20, Lemma 2.6], which, in turn, employs [20, Lemma 2.3]. Note that the difference between Lemma 3.3 and [20, Lemma 2.3] lies on the boundary conditions involved.

Lemma 3.4. *There exists a positive constant α such that*

$$\sup_{\substack{\mathbf{q}\in K_1\\ \mathbf{q}\neq\mathbf{0}}} \frac{a(\mathbf{r},\mathbf{q})}{\|\mathbf{q}\|_{X_1}} \,\geqslant\, \alpha\,\|\mathbf{r}\|_{X_2} \qquad \forall\,\mathbf{r}\in K_2\,, \tag{3.38}$$

and

$$\sup_{\mathbf{r}\in K_2} a(\mathbf{r},\mathbf{q}) \,>\, 0 \qquad \forall\,\mathbf{q}\in K_1,\ \mathbf{q}\neq\mathbf{0}\,. \tag{3.39}$$

Proof. Given $\mathbf{r}\in K_2$ (cf. (3.35)), that is $\mathbf{r}\in\mathbf{H}^r(\mathrm{div}_j;\Omega)$ such that $\mathrm{div}(\mathbf{r}) \,=\, 0$ in Ω, and recalling from (2.25) that $r>2$, we set $\mathbf{r}_s \,:=\, |\mathbf{r}|^{r-2}\,\mathbf{r}$, and observe, similarly to (3.24), that

$$\mathbf{r}_s \,\in\, \mathbf{L}^s(\Omega) \qquad \text{and} \qquad \int_\Omega \mathbf{r}\cdot\mathbf{r}_s \,=\, \|\mathbf{r}\|_{0,r;\Omega}\,\|\mathbf{r}_s\|_{0,s;\Omega}\,. \tag{3.40}$$

Then, noting from (2.25) that s does belong to the range required by Lemma 3.3, an application of this result to $t = s$ yields $D_s(\mathbf{r}_s)\in K_1$, and hence, using (3.37), the identity given in (3.40), and the boundedness of D_s, we find that

$$\sup_{\substack{\mathbf{q}\in K_1\\ \mathbf{q}\neq\mathbf{0}}} \frac{a(\mathbf{r},\mathbf{q})}{\|\mathbf{q}\|_{X_1}} \,\geqslant\, \frac{a(\mathbf{r},D_s(\mathbf{r}_s))}{\|D_s(\mathbf{r}_s)\|_{X_1}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{r}\cdot D_s(\mathbf{r}_s)}{\|D_s(\mathbf{r}_s)\|_{0,s;\Omega}} \,=\, \frac{\displaystyle\int_\Omega \mathbf{r}\cdot\mathbf{r}_s}{\|D_s(\mathbf{r}_s)\|_{0,s;\Omega}} \,\geqslant\, \frac{1}{\|D_s\|}\,\|\mathbf{r}\|_{0,r;\Omega}\,,$$

which proves (3.38) with $\alpha \,=\, \dfrac{1}{\|D_s\|}$. In turn, we now take $\mathbf{q}\,\in\,K_1$ (cf. (3.34)), $\mathbf{q}\,\neq\,\mathbf{0}$, define $\mathbf{q}_r := \begin{cases} |\mathbf{q}|^{s-2}\,\mathbf{q} & \text{if } \mathbf{q}\neq\mathbf{0} \\ \mathbf{0} & \text{if } \mathbf{q}=\mathbf{0} \end{cases}$, and observe, similarly to (3.24) and (3.40), that

$$\mathbf{q}_r \,\in\, \mathbf{L}^r(\Omega) \qquad \text{and} \qquad \int_\Omega \mathbf{q}\cdot\mathbf{q}_r \,=\, \|\mathbf{q}\|_{0,s;\Omega}^s\,. \tag{3.41}$$

In this way, noting from Lemma 3.3 that $D_r(\mathbf{q}_r)\in K_2$ (cf. (3.35)), and using (3.37) and the identity in (3.41), we obtain

$$\sup_{\mathbf{r}\in K_2} a(\mathbf{r},\mathbf{q}) \,\geqslant\, \int_\Omega D_r(\mathbf{q}_r)\cdot\mathbf{q} \,=\, \int_\Omega \mathbf{q}_r\cdot\mathbf{q} \,=\, \|\mathbf{q}\|_{0,s;\Omega}^s \,>\, 0\,,$$

which shows (3.39) and finishes the proof of the lemma. □

We stress here that, belonging the index r as well (cf. (2.25)) to the range required by Lemma 3.3, we can proceed analogously to the proof of Lemma 3.4 to conclude that the inequalities (3.38) and (3.39) remain valid if the roles of X_2 and X_1 (and hence of K_2 and K_1) are exchanged. More precisely, we have the following result.

Lemma 3.5. *There exists a positive constant α such that*

$$\sup_{\substack{\mathbf{r}\in K_2\\ \mathbf{r}\neq\mathbf{0}}} \frac{a(\mathbf{r},\mathbf{q})}{\|\mathbf{r}\|_{X_2}} \,\geqslant\, \alpha\,\|\mathbf{q}\|_{X_1} \qquad \forall\,\mathbf{q}\in K_1\,, \tag{3.42}$$

and

$$\sup_{\mathbf{q}\in K_1} a(\mathbf{r},\mathbf{q}) \,>\, 0 \qquad \forall\,\mathbf{r}\in K_2,\ \mathbf{r}\neq\mathbf{0}\,. \tag{3.43}$$

The continuous inf-sup conditions for the bilinear forms b_i, $i \in \{1,2\}$, which resemble, though with relevant differences, the results given by [20, Lemma 2.7], are established in the following lemma.

Lemma 3.6. *For each $i \in \{1,2\}$ there exists a positive constant β_i such that*

$$\sup_{\substack{\mathbf{q}\in X_i\\ \mathbf{q}\neq\mathbf{0}}} \frac{b_i(\mathbf{q},\phi)}{\|\mathbf{q}\|_{X_i}} \,\geqslant\, \beta_i\,\|\phi\|_{M_i} \qquad \forall\,\phi\in M_i\,. \tag{3.44}$$

Proof. For the case $i = 1$, in which $X_i = \mathbf{H}^s(\mathrm{div}_s;\Omega)$ and $M_i = \mathrm{L}^r(\Omega)$, with r and s conjugate to each other (cf. (2.25)), the present proof proceeds similarly to that of [20, Lemma 2.7], except for the fact that the boundary conditions of the auxiliary problems utilized are homogeneous Dirichlet and Neumann, respectively. We omit further details and refer to [20, Lemma 2.7]. On the other hand, for the case $i = 2$, in which $X_i = \mathbf{H}^r(\mathrm{div}_j;\Omega)$ and $M_i = \mathrm{L}^l(\Omega)$, with j and l conjugate to each other (cf. (2.26)), we first let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar{\Omega}$. Then, given $\phi\in M_2 = \mathrm{L}^l(\Omega)$, we recall from (2.26) that $l \geqslant 2$, set $\phi_j := |\phi|^{l-2}\,\phi$, and observe, as before, that

$$\phi_j \,\in\, \mathrm{L}^j(\Omega) \qquad \text{and} \qquad \int_\Omega \phi\,\phi_j \,=\, \|\phi\|_{0,l;\Omega}\,\|\phi_j\|_{0,j;\Omega}\,. \tag{3.45}$$

Next, we define $g := \begin{cases} \phi_j & \text{in} \quad \Omega\,,\\ 0 & \text{in} \quad \mathcal{O}\backslash\bar{\Omega}\,. \end{cases}$, which clearly belongs to $\mathrm{L}^j(\mathcal{O})$, and deduce, applying [17, Corollary 1] to $j\in(1,2]$ (cf. (2.26)), that there exists a unique $z \in \mathrm{W}^{1,j}_0(\mathcal{O}) \cap \mathrm{W}^{2,j}(\mathcal{O})$ such that

$$\Delta z \,=\, g \qquad \text{in} \quad \mathcal{O}\,, \quad z \,=\, 0 \quad \text{on} \quad \partial\mathcal{O}\,,$$

and

$$\|z\|_{2,j;\mathcal{O}} \,\leqslant\, C_j\,\|g\|_{0,j;\mathcal{O}} \,=\, C_j\,\|\phi_j\|_{0,j;\Omega}\,,$$

with a positive constant C_j depending only on j and $\mathcal{O}$. Thus, letting $\bar{\mathbf{q}} := \nabla z|_\Omega \in \mathrm{W}^{1,j}(\Omega)$, it follows that $\mathrm{div}(\bar{\mathbf{q}}) = \phi_j$ in Ω, whereas using the continuous embedding $i_{j,r}$ from $\mathrm{W}^{1,j}(\Omega)$ into $\mathrm{L}^r(\Omega)$, which is valid (cf. [16, Corollary B.43]) for the ranges of r and j specified in (2.25) and (2.26), respectively, we get

$$\|\bar{\mathbf{q}}\|_{0,r;\Omega} \,\leqslant\, \|i_{j,r}\|\,\|\bar{\mathbf{q}}\|_{1,j;\Omega} \,\leqslant\, \|i_{j,r}\|\,\|z\|_{2,j;\mathcal{O}} \,\leqslant\, \|i_{j,r}\|\,C_j\,\|\phi_j\|_{0,j;\Omega}\,.$$

In this way, we conclude that $\bar{\mathbf{q}} \in X_2 := \mathbf{H}^r(\mathrm{div}_j;\Omega)$, and that

$$\|\bar{\mathbf{q}}\|_{X_2} \,=\, \|\bar{\mathbf{q}}\|_{0,r;\Omega} \,+\, \|\mathrm{div}(\bar{\mathbf{q}})\|_{0,j;\Omega} \,\leqslant\, \big(1 \,+\, \|i_{j,r}\|\,C_j\big)\,\|\phi_j\|_{0,j;\Omega}\,,$$

whence, using the identity in (3.45) as well, we find that

$$\sup_{\substack{\mathbf{q}\in X_2\\ \mathbf{q}\neq\mathbf{0}}} \frac{b_2(\mathbf{q},\phi)}{\|\mathbf{q}\|_{X_2}} \,\geqslant\, \frac{b_2(\bar{\mathbf{q}},\phi)}{\|\bar{\mathbf{q}}\|_{X_2}} \,\geqslant\, \frac{1}{\big(1 \,+\, \|i_{j,r}\|\,C_j\big)}\,\frac{\displaystyle\int_\Omega \phi\,\phi_j}{\|\phi_j\|_{0,j;\Omega}} \,=\, \frac{1}{\big(1 \,+\, \|i_{j,r}\|\,C_j\big)}\,\|\phi\|_{0,l;\Omega}\,,$$

which proves (3.44) with $\beta_2 \,=\, \big(1 \,+\, \|i_{j,r}\|\,C_j\big)^{-1}$. □

According to Lemmas 3.4 and 3.6 (equivalently, Lemmas 3.5 and 3.6), the required hypotheses of [3, Theorem 2.1, Section 2.1] are satisfied, and hence the a priori estimates provided by [3, Corollary 2.1, Section 2.1] imply the existence of a positive constant $\alpha_{\mathbf{T}}$, depending only on α, β_1, β_2, and $\|a\|$, such that

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\ (\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \,\geqslant\, \alpha_{\mathbf{T}}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1} \qquad \forall\,(\mathbf{r},\psi)\in X_2\times M_1\,, \tag{3.46}$$

and

$$\sup_{\substack{(\mathbf{r},\psi)\in X_2\times M_1\\(\mathbf{r},\psi)\neq\mathbf{0}}} \frac{\mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{r},\psi)\|_{X_2\times M_1}} \,\geqslant\, \alpha_{\mathbf{T}}\,\|(\mathbf{q},\phi)\|_{X_1\times M_2} \qquad \forall\,(\mathbf{q},\phi)\in X_1\times M_2\,. \tag{3.47}$$

Now, we let $\mathbf{A}_{\mathbf{z},\chi} : (X_2\times M_1)\times(X_1\times M_2)\to \mathrm{R}$ be the bounded bilinear form arising from (3.6) after adding the full left hand sides of its equations, that is

$$\begin{gathered}\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,:=\, \mathbf{A}((\mathbf{r},\psi),(\mathbf{q},\phi)) \,-\, c_{\mathbf{z},\chi}\big((\mathbf{r},\psi),\phi\big)\\ \forall\,(\mathbf{r},\psi)\in\big(X_2\times M_1\big),\quad \forall\,(\mathbf{q},\phi)\in\big(X_1\times M_2\big)\,,\end{gathered} \tag{3.48}$$

and realize that (3.6) can be rewritten, equivalently, as: Find $(\mathbf{p},\varphi)\in X_2\times M_1$ such that

$$\mathbf{A}_{\mathbf{z},\chi}((\mathbf{p},\varphi),(\mathbf{q},\phi)) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi)\in X_1\times M_2\,. \tag{3.49}$$

Note that the boundedness of $\mathbf{A}$ and $c_{\mathbf{z},\chi}$ (cf. (2.57)) guarantees that $\mathbf{A}_{\mathbf{z},\chi}$ is bounded as well. Thus, bearing in mind (3.48), and employing (3.46) and (2.57), we find that for each $(\mathbf{r},\psi)\in X_2\times M_1$ there holds

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\(\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \,\geqslant\, \Big\{\alpha_{\mathbf{T}} \,-\, \|c\|\,\big(\|\mathbf{z}\|_{0,4;\Omega} \,+\, \|\chi\|_{0,\rho;\Omega}\big)\Big\}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1}\,, \tag{3.50}$$

and then, under the assumption that

$$\|\mathbf{z}\|_{0,4;\Omega} + \|\chi\|_{0,\rho;\Omega} \,\leqslant\, \frac{\alpha_{\mathbf{T}}}{2\,\|c\|}\,, \tag{3.51}$$

we arrive at

$$\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\(\mathbf{q},\phi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}} \,\geqslant\, \frac{\alpha_{\mathbf{T}}}{2}\,\|(\mathbf{r},\psi)\|_{X_2\times M_1} \qquad \forall\,(\mathbf{r},\psi)\,\in\, X_2\times M_1\,. \tag{3.52}$$

Similarly, but employing now (3.47) instead of (3.46), and under the same assumption (3.51), we obtain

$$\sup_{\substack{(\mathbf{r},\psi)\in X_2\times M_1\\(\mathbf{r},\psi)\neq\mathbf{0}}} \frac{\mathbf{A}_{\mathbf{z},\chi}((\mathbf{r},\psi),(\mathbf{q},\phi))}{\|(\mathbf{r},\psi)\|_{X_2\times M_1}} \,\geqslant\, \frac{\alpha_{\mathbf{T}}}{2}\,\|(\mathbf{q},\phi)\|_{X_1\times M_2} \qquad \forall\,(\mathbf{q},\phi)\,\in\, X_1\times M_2\,. \tag{3.53}$$

We are now in position to establish the well-definedness of $\mathbf{T}$.

Lemma 3.7. *For each $(\mathbf{z},\chi)\in \mathbf{L}^4(\Omega)\times \mathrm{Q}$ satisfying (3.51), there exists a unique $(\mathbf{p},\varphi)\in X_2\times M_1$ solution of (3.49) (equivalently (3.6)), and hence one can define $\mathbf{T}(\mathbf{z},\chi) := \mathbf{p}\in X_2$. Moreover, there exists a positive constant $C_{\mathbf{T}}$, depending only on $\alpha_{\mathbf{T}}$, C_r, and κ_φ, such that*

$$\|\mathbf{T}(\mathbf{z},\chi)\|_{X_2} \,=\, \|\mathbf{p}\|_{X_2} \,\leqslant\, \|(\mathbf{p},\varphi)\|_{X_2\times M_1} \,\leqslant\, C_{\mathbf{T}}\Big\{\|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\}\,. \tag{3.54}$$

Proof. Thanks to the boundedness of $\mathbf{A}_{\mathbf{z},\chi}$, and the global inf-sup conditions (3.52) and (3.53), a direct application of [16, Theorem 2.6] provides the existence of a unique solution $(\mathbf{p},\varphi)\in X_2\times M_1$ to (3.49). Moreover, the corresponding a priori estimate (cf. [16, Theorem 2.6, eq. (2.5)]) yields

$$\|(\mathbf{p},\varphi)\|_{X_2\times M_1} \,\leqslant\, \frac{2}{\alpha_{\mathbf{T}}}\Big\{\|\mathrm{F}\| \,+\, \|\mathrm{G}\|\Big\}\,,$$

which, together with the expressions for $\|\mathrm{F}\|$ and $\|\mathrm{G}\|$ given in (2.60) and (2.54), imply (3.54) with $C_{\mathbf{T}} := \frac{2}{\alpha_{\mathbf{T}}}\max\big\{C_r,\kappa_\varphi^{-1}\big\}$. □

3.3 Solvability analysis of the fixed-point equation

Knowing that the operators $\mathbf{S}$, $\widetilde{\mathrm{S}}$, $\mathbf{T}$ and hence $\boldsymbol{\Xi}$ as well, are well defined, in this section we address the solvability of the fixed point equation (3.7). To this end, in what follows we aim to verify the hypotheses of the respective Banach Theorem. We begin the analysis by establishing sufficient conditions under which $\boldsymbol{\Xi}$ maps a closed ball of $\mathbf{L}^4(\Omega) \times X_2$ into itself. Indeed, given a radius δ to be explicitly defined later on, we first set

$$W_\delta := \Big\{ (\mathbf{z}, \mathbf{r}) \in \mathbf{L}^4(\Omega) \times X_2 : \quad \|(\mathbf{z}, \mathbf{r})\| := \|\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}\|_{X_2} \leqslant \delta \Big\} . \tag{3.55}$$

Then, given $(\mathbf{z}, \mathbf{r}) \in W_\delta$, we have from the a priori estimate for $\mathbf{S}$ (cf. (3.17) in Lemma 3.1) that

$$\begin{aligned} &\|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} \\ &\quad \leqslant C_{\mathbf{S}} \Big\{ \|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{0,\rho;\Omega} \, \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} , \end{aligned} \tag{3.56}$$

from which, using the corresponding estimate for $\widetilde{\mathrm{S}}$ (cf. (3.32), Lemma 3.2), and assuming (cf. (3.29))

$$\|\mathbf{z}\|_{0,4;O} + \|\mathbf{r}\|_{0,r;\Omega} \leqslant \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|} , \tag{3.57}$$

we get

$$\begin{aligned} \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} \leqslant C_{\mathbf{S}} \Big\{ & C_{\widetilde{\mathrm{S}}} \left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \right) \|\nabla f\|_{0,r;\Omega} \\ & + \|\mathbf{f}\|_{0,4/3;\Omega} + \big(1 + \|\mathbf{z}\|_{0,4;\Omega}\big) \, \|\mathbf{u}_D\|_{1/2,\Gamma} \Big\} . \end{aligned} \tag{3.58}$$

Furthermore, supposing now that (cf. (3.51))

$$\|\mathbf{z}\|_{0,4;O} + \|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{\mathrm{Q}} \leqslant \frac{\alpha_{\mathbf{T}}}{2\,\|c\|} , \tag{3.59}$$

the a priori estimate for $\mathbf{T}$ (cf. (3.54) in Lemma 3.7) gives

$$\|\mathbf{T}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{X_2} \leqslant C_{\mathbf{T}} \Big\{ \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega} \Big\} . \tag{3.60}$$

Regarding (3.57), we observe that it is satisfied if we consider δ such that $\delta \leqslant \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|}$. In turn, noting that certainly $\|\mathbf{z}\|_{0,4;\Omega} \leqslant \delta$, and according to the estimate for $\|\widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\|_{\mathrm{Q}}$ (cf. (3.32)), we deduce that a sufficient condition for (3.59) is given by the assumptions

$$\delta \leqslant \frac{\alpha_{\mathbf{T}}}{4\,\|c\|} \quad \text{and} \quad C_{\widetilde{\mathrm{S}}} \left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \right) \leqslant \frac{\alpha_{\mathbf{T}}}{4\,\|c\|} . \tag{3.61}$$

In this way, defining

$$\delta := \min \left\{ \frac{\alpha_{\widetilde{\mathrm{S}}}}{2\,\|\widetilde{c}\|} , \frac{\alpha_{\mathbf{T}}}{4\,\|c\|} \right\} , \tag{3.62}$$

(3.57) and (3.59) are satisfied, whence (3.58) and (3.60) are valid, and thus, assuming the second inequality in (3.61), and recalling that $\|\boldsymbol{\Xi}(\mathbf{z}, \mathbf{r})\| := \|\mathbf{S}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{0,4;\Omega} + \|\mathbf{T}\big(\mathbf{z}, \widetilde{\mathrm{S}}(\mathbf{z}, \mathbf{r})\big)\|_{X_2}$, we obtain

$$\begin{aligned} \|\boldsymbol{\Xi}(\mathbf{z}, \mathbf{r})\| \leqslant C(\delta) \Big\{ & \big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega} \big) \, \|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} \\ & + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega} \Big\} , \end{aligned} \tag{3.63}$$

where $C(\delta)$ is a positive constant depending explicitly on $C_{\mathbf{S}}$, $C_{\widetilde{\mathrm{S}}}$, $(1 + \delta)$, and $C_{\mathbf{T}}$.

We have then proved the following result.

Lemma 3.8. *Assume that the data are sufficiently small so that*

$$C_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right) \,\leqslant\, \frac{\alpha_{\mathbf{T}}}{4\,\|c\|}\,, \tag{3.64}$$

and

$$\begin{aligned} C(\delta)\Big\{&\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \\ +\;\; &\|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\} \,\leqslant\, \delta\,. \end{aligned} \tag{3.65}$$

Then, $\mathbf{\Xi}(W_\delta) \,\subseteq\, W_\delta$.

We now aim to show that the operator $\mathbf{\Xi}$ is Lipschitz-continuous, for which, according to its definition (cf. (3.7)), it suffices to show that $\mathbf{S}$, $\widetilde{\mathrm{S}}$ and $\mathbf{T}$ satisfy suitable continuity properties. We begin with the corresponding result for $\mathbf{S}$.

Lemma 3.9. *There exists a positive constant* $L_{\mathbf{S}}$*, depending on* $\boldsymbol{\alpha}$*,* $|\Omega|$*, and* $\|\mathbf{c}\|$*, such that*

$$\begin{aligned} &\|\mathbf{S}(\mathbf{z},\chi) \,-\, \mathbf{S}(\mathbf{z}_0,\chi_0)\|_{\mathbf{H}} \\ &\qquad\leqslant\, L_{\mathbf{S}}\Big\{\|\nabla f\|_{0,r;\Omega}\,\|\chi \,-\, \chi_0\|_{0,\rho;\Omega} \,+\, \boldsymbol{\mathcal{F}}(\mathbf{z}_0,\chi_0)\,\|\mathbf{z} \,-\, \mathbf{z}_0\|_{0,4;\Omega}\Big\} \end{aligned} \tag{3.66}$$

for all $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times \mathrm{Q}$*, where*

$$\boldsymbol{\mathcal{F}}(\mathbf{z}_0,\chi_0) \,:=\, C_{\mathbf{S}}\Big\{\|\chi_0\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \left(1 + \|\mathbf{z}_0\|_{0,4;\Omega}\right)\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,. \tag{3.67}$$

Proof. Given $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times \mathrm{Q}$, we let $\mathbf{S}(\mathbf{z},\chi) := \mathbf{u} \in \mathrm{L}^4(\Omega)$ and $\mathbf{S}(\mathbf{z}_0,\chi_0) := \mathbf{u}_0 \in \mathrm{L}^4(\Omega)$, where $(\vec{\mathbf{u}},\boldsymbol{\sigma}) = \big((\mathbf{u},\mathbf{t}),\boldsymbol{\sigma}\big) \in \mathbf{H}\times\mathbf{Q}$ and $(\vec{\mathbf{u}}_0,\boldsymbol{\sigma}_0) = \big((\mathbf{u}_0,\mathbf{t}_0),\boldsymbol{\sigma}_0\big) \in \mathbf{H}\times\mathbf{Q}$ are the respective solutions of (3.2). It follows from the corresponding second equations of (3.2) that $\vec{\mathbf{u}} - \vec{\mathbf{u}}_0 \in \mathbf{V}$ (cf. (3.11)), and then the $\mathbf{V}-$ellipticity of $\mathbf{a}$ (cf. (3.12)) gives

$$\boldsymbol{\alpha}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}^2 \,\leqslant\, \mathbf{a}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,. \tag{3.68}$$

In turn, applying the corresponding first equations of (3.2) to $\vec{\mathbf{v}} = \vec{\mathbf{u}} - \vec{\mathbf{u}}_0$, we obtain

$$\mathbf{a}(\vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{F}_\chi(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,, \tag{3.69}$$

and

$$\mathbf{a}(\vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}_0; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{F}_{\chi_0}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,, \tag{3.70}$$

so that, subtracting (3.70) from (3.69), and using, thanks to the bilinearity of $\mathbf{c}(\mathbf{z};\cdot,\cdot)$ and (2.20), that

$$\mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \mathbf{c}(\mathbf{z}; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,,$$

we find

$$\mathbf{a}(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,=\, \big(\mathbf{F}_\chi - \mathbf{F}_{\chi_0}\big)(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,+\, \mathbf{c}(\mathbf{z}_0 - \mathbf{z}; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0)\,. \tag{3.71}$$

Next, utilizing (2.40), we get

$$\begin{aligned} \big(\mathbf{F}_\chi - \mathbf{F}_{\chi_0}\big)(\vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,&=\, \int_\Omega (\chi - \chi_0)\,\nabla f\cdot(\mathbf{u} - \mathbf{u}_0) \\ &\leqslant\, |\Omega|^{1/4}\,\|\chi - \chi_0\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}\,, \end{aligned} \tag{3.72}$$

whereas the boundedness property of $\mathbf{c}$ (cf. (2.18)) yields

$$\mathbf{c}(\mathbf{z}_0 - \mathbf{z}; \vec{\mathbf{u}}_0, \vec{\mathbf{u}} - \vec{\mathbf{u}}_0) \,\leqslant\, \|\mathbf{c}\|\,\|\mathbf{z} - \mathbf{z}_0\|_{0,4;\Omega}\,\|\vec{\mathbf{u}}_0\|_{\mathbf{H}}\,\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}\,. \tag{3.73}$$

Finally, employing (3.72) and (3.73) in (3.71), replacing the resulting estimate back into (3.68), simplifying by $\|\vec{\mathbf{u}} - \vec{\mathbf{u}}_0\|_{\mathbf{H}}$, and bounding $\|\vec{\mathbf{u}}_0\|_{\mathbf{H}}$ by the corresponding upper bound in (3.17), we arrive at the required inequality (3.66) with $L_{\mathbf{S}} \,:=\, \boldsymbol{\alpha}^{-1}\,\max\big\{|\Omega|^{1/4}, \|\mathbf{c}\|\big\}$. □

The continuity of $\widetilde{\mathrm{S}}$ is addressed next. More precisely, we have the following result.

Lemma 3.10. *There exists a positive constant $L_{\widetilde{\mathrm{S}}}$, depending only on $\|\widetilde{c}\|$, $\alpha_{\widetilde{\mathrm{S}}}$, and $C_{\widetilde{\mathrm{S}}}$, such that*

$$\begin{aligned} &\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) \,-\, \widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{\mathrm{Q}} \\ &\qquad \leqslant\, L_{\widetilde{\mathrm{S}}}\left\{\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\right\}\|(\mathbf{z},\mathbf{r}) \,-\, (\mathbf{z}_0,\mathbf{r}_0)\| \end{aligned} \tag{3.74}$$

for all $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in \mathbf{L}^4(\Omega)\times X_2$ satisfying (3.29).

Proof. Given $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in \mathbf{L}^4(\Omega)\times X_2$, we let $\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r}) := \eta \in \mathrm{Q}$ and $\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0) := \eta_0 \in \mathrm{Q}$, where $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$ and $(\widetilde{\boldsymbol{\sigma}}_0,\eta_0) \in \mathrm{H}\times\mathrm{Q}$ are the respective solutions of (3.4), equivalently (3.28), that is

$$\widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q}\,, \tag{3.75}$$

and

$$\widetilde{A}_{\mathbf{z}_0,\mathbf{r}_0}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}) \,+\, \widetilde{\mathrm{G}}(\xi) \qquad \forall\,(\widetilde{\boldsymbol{\tau}},\xi) \in \mathrm{H}\times\mathrm{Q}\,. \tag{3.76}$$

It follows from the foregoing identities and the definitions of $\widetilde{A}_{\mathbf{z},\mathbf{r}}$ (cf. (3.27)) and $\widetilde{c}_{\mathbf{z},\mathbf{q}}$ (cf. (2.27)) that

$$\begin{aligned} \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta)-(\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,&=\, \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}},\eta),(\widetilde{\boldsymbol{\tau}},\xi)\big) - \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \\ =\, \widetilde{A}_{\mathbf{z}_0,\mathbf{r}_0}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) &- \widetilde{A}_{\mathbf{z},\mathbf{r}}\big((\widetilde{\boldsymbol{\sigma}}_0,\eta_0),(\widetilde{\boldsymbol{\tau}},\xi)\big) \,=\, \widetilde{c}_{\mathbf{z}_0-\mathbf{z},\mathbf{r}_0-\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\eta_0)\,, \end{aligned} \tag{3.77}$$

and hence, applying the global inf-sup condition (3.30) to $(\widetilde{\boldsymbol{\sigma}},\eta)-(\widetilde{\boldsymbol{\sigma}}_0,\eta_0)$, and employing (3.77) and the boundedness of $\widetilde{c}_{\mathbf{z},\mathbf{r}}$ (cf. (2.37)), we find that

$$\begin{aligned} \|(\widetilde{\boldsymbol{\sigma}},\eta)-(\widetilde{\boldsymbol{\sigma}}_0,\eta_0)\|_{\mathrm{H}\times\mathrm{Q}} \,&\leqslant\, \frac{2}{\alpha_{\widetilde{\mathrm{S}}}} \sup_{\substack{(\widetilde{\boldsymbol{\tau}},\xi)\in\mathrm{H}\times\mathrm{Q}\\ (\widetilde{\boldsymbol{\tau}},\xi)\neq\mathbf{0}}} \frac{\widetilde{c}_{\mathbf{z}_0-\mathbf{z},\mathbf{r}_0-\mathbf{r}}(\widetilde{\boldsymbol{\tau}},\eta_0)}{\|(\widetilde{\boldsymbol{\tau}},\xi)\|_{\mathrm{H}\times\mathrm{Q}}} \\ &\leqslant\, \frac{2\,\|\widetilde{c}\|}{\alpha_{\widetilde{\mathrm{S}}}}\,\|\eta_0\|_{\mathrm{Q}}\left\{\|\mathbf{z}-\mathbf{z}\|_{0,4;\Omega} + \|\mathbf{r}-\mathbf{r}_0\|_{0,r;\Omega}\right\}, \end{aligned}$$

which, together with the a priori estimate (3.32) for $\|\eta_0\|_{\mathrm{Q}}$, yields (3.74) with $L_{\widetilde{\mathrm{S}}} := 2\,\|\widetilde{c}\|\,\alpha_{\widetilde{\mathrm{S}}}^{-1}\,C_{\widetilde{\mathrm{S}}}$. $\square$

It remains to establish the continuity of $\mathbf{T}$, which is the purpose of the following lemma.

Lemma 3.11. *There exists a positive constant $L_{\mathbf{T}}$, depending only on $\|c\|$, $\alpha_{\mathbf{T}}$, and $C_{\mathbf{T}}$, such that*

$$\begin{aligned} &\|\mathbf{T}(\mathbf{z},\chi) \,-\, \mathbf{T}(\mathbf{z}_0,\chi_0)\|_{X_2} \\ &\qquad \leqslant\, L_{\mathbf{T}}\left\{\|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\right\}\|(\mathbf{z},\chi) \,-\, (\mathbf{z}_0,\chi_0)\| \end{aligned} \tag{3.78}$$

for all $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times\mathrm{Q}$ satisfying (3.51).

Proof. Given $(\mathbf{z},\chi)$, $(\mathbf{z}_0,\chi_0) \in \mathbf{L}^4(\Omega)\times\mathrm{Q}$ as indicated, we proceed similarly to the proof of Lemma 3.10 and let $\mathbf{T}(\mathbf{z},\chi) := \mathbf{p} \in X_2$ and $\mathbf{T}(\mathbf{z}_0,\chi_0) := \mathbf{p}_0 \in X_2$, where $(\mathbf{p},\varphi) \in X_2\times M_1$ and $(\mathbf{p}_0,\varphi_0) \in X_2\times M_1$ are the respective solutions of (3.6), equivalently (3.49), that is

$$\mathbf{A}_{\mathbf{z},\chi}\big((\mathbf{p},\varphi),(\mathbf{q},\phi)\big) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi) \in X_1\times M_2\,, \tag{3.79}$$

and

$$\mathbf{A}_{\mathbf{z}_0,\chi_0}\big((\mathbf{p}_0,\varphi_0),(\mathbf{q},\phi)\big) \,=\, \mathrm{F}(\mathbf{q}) \,+\, \mathrm{G}(\phi) \qquad \forall\,(\mathbf{q},\phi) \in X_1\times M_2\,. \tag{3.80}$$

Next, proceeding analogously to the derivation of (3.77), we deduce from the identities (3.79) and (3.80), along with the definitions of $\mathbf{A}_{\mathbf{z},\chi}$ (cf. (3.48)) and $c_{\mathbf{z},\chi}$ (cf. (2.51)) that

$$\mathbf{A}_{\mathbf{z},\chi}\big((\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0),(\mathbf{q},\phi)\big) \;=\; c_{\mathbf{z}-\mathbf{z}_0,\chi-\chi_0}\big((\mathbf{p}_0,\varphi_0),\phi\big)\,, \tag{3.81}$$

and thus, applying the global inf-sup condition (3.52) to $(\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0)$, and making use of (3.81) and the boundedness of $c_{\mathbf{z},\chi}$ (cf. (2.57)), we get

$$\begin{aligned}\|(\mathbf{p},\varphi)-(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1} \;&\leqslant\; \frac{2}{\alpha_{\mathbf{T}}}\sup_{\substack{(\mathbf{q},\phi)\in X_1\times M_2\\(\mathbf{q},\phi)\neq\mathbf{0}}}\frac{c_{\mathbf{z}-\mathbf{z}_0,\chi-\chi_0}\big((\mathbf{p}_0,\varphi_0),\phi\big)}{\|(\mathbf{q},\phi)\|_{X_1\times M_2}}\\ &\leqslant\; \frac{2\,\|c\|}{\alpha_{\mathbf{T}}}\,\|(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1}\Big\{\|\mathbf{z}-\mathbf{z}_0\|_{0,4;\Omega} \,+\, \|\chi-\chi_0\|_{0,\rho;\Omega}\Big\}\,,\end{aligned}$$

which, together with the a priori estimate (3.54) for $\|(\mathbf{p}_0,\varphi_0)\|_{X_2\times M_1}$, yields (3.78) with $L_{\mathbf{T}} := 2\,\|c\|\,\alpha_{\mathbf{T}}^{-1}\,C_{\mathbf{T}}$. □

Having proved Lemmas 3.9, 3.10 and 3.11, we now aim to establish the continuity property of the fixed point operator $\mathbf{\Xi}$ in the closed ball W_δ (cf. (3.55)). Indeed, given $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0)\in W_\delta$, we first observe from the definition of $\mathbf{\Xi}$ (cf. (3.7)) that

$$\begin{aligned}\|\mathbf{\Xi}(\mathbf{z},\mathbf{r})-\mathbf{\Xi}(\mathbf{z}_0,\mathbf{r}_0)\| \;&=\; \|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega}\\ &\quad+\,\|\mathbf{T}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{T}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{X_2}\,.\end{aligned}\tag{3.82}$$

Then, employing the continuity properties of $\mathbf{S}$ (cf. Lemma 3.9, (3.66)) and $\widetilde{\mathrm{S}}$ (cf. Lemma 3.10, (3.74)), we find that

$$\begin{aligned}&\|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega}\\ &\quad\leqslant\; L_{\mathbf{S}}\Big\{\|\nabla f\|_{0,r;\Omega}\,\|\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\,-\,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{0,\rho;\Omega} \,+\, \boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\,\|\mathbf{z}\,-\,\mathbf{z}_0\|_{0,4;\Omega}\Big\}\\ &\quad\leqslant\; L_{\mathbf{S}}\Big\{L_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega} + \boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\Big\}\,\|(\mathbf{z},\mathbf{r})-(\mathbf{z}_0,\mathbf{r}_0)\|\end{aligned}\tag{3.83}$$

whereas (3.67) and the a priori estimate of $\widetilde{\mathrm{S}}$ (cf. (3.32)) gives

$$\begin{aligned}&\boldsymbol{\mathcal{F}}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\\ &\quad\leqslant\; C_{\mathbf{S}}\Big\{\|\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}_0\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\\ &\quad\leqslant\; C_{\mathbf{S}}\Big\{C_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\right)\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}_0\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.\end{aligned}\tag{3.84}$$

In this way, replacing the bound from (3.84) into (3.83), and using that $\|\mathbf{z}_0\|_{0,4;\Omega}\leqslant\delta$, we deduce the existence of a positive constant $L_{\mathbf{\Xi},\mathbf{S}}$, depending only on $L_{\mathbf{S}}$, $L_{\widetilde{\mathrm{S}}}$, $C_{\mathbf{S}}$, $C_{\widetilde{\mathrm{S}}}$, and δ, such that

$$\begin{aligned}\|\mathbf{S}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{S}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{0,4;\Omega} \;&\leqslant\; L_{\mathbf{\Xi},\mathbf{S}}\Big\{\big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big)\,\|\nabla f\|_{0,r;\Omega}\\ &\quad+\quad \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,\|(\mathbf{z},\mathbf{r})-(\mathbf{z}_0,\mathbf{r}_0)\|\,.\end{aligned}\tag{3.85}$$

In turn, proceeding similarly as before, but applying now the continuity properties of $\mathbf{T}$ (cf. Lemma 3.11, (3.78)) and $\widetilde{\mathrm{S}}$ (cf. Lemma 3.10, (3.74)), we arrive at

$$\begin{aligned}&\|\mathbf{T}\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)-\mathbf{T}\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)\|_{X_2}\\ &\quad\leqslant\; L_{\mathbf{\Xi},\mathbf{T}}\left(1 \,+\, \|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\right)\Big\{\|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\Big\}\,\|(\mathbf{z},\mathbf{r})-(\mathbf{z}_0,\mathbf{r}_0)\|\,,\end{aligned}\tag{3.86}$$

where $L_{\Xi,\mathbf{T}}$ is a positive constant depending only on $L_{\mathbf{T}}$ and $L_{\widetilde{\mathrm{S}}}$.

Defining $L_{\Xi} := \max\left\{L_{\Xi,\mathbf{S}}, L_{\Xi,\mathbf{T}}\right\}$, we summarize the above discussion in the following result.

Lemma 3.12. *Assume* (3.64), *that is*

$$C_{\widetilde{\mathrm{S}}}\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right) \leqslant \frac{\alpha_{\mathbf{T}}}{4\,\|c\|}\,.$$

Then, there holds

$$\begin{aligned}\|\Xi(\mathbf{z},\mathbf{r}) - \Xi(\mathbf{z}_0,\mathbf{r}_0)\| \leqslant L_{\Xi}\Big\{&\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right)\left(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\right)\\ &+ \ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\}\|(\mathbf{z},\mathbf{r}) - (\mathbf{z}_0,\mathbf{r}_0)\|\,,\end{aligned} \tag{3.87}$$

for all $(\mathbf{z},\mathbf{r})$, $(\mathbf{z}_0,\mathbf{r}_0) \in W_\delta$.

Proof. We first stress that (3.64) is assumed here to ensure that both $\big(\mathbf{z},\widetilde{\mathrm{S}}(\mathbf{z},\mathbf{r})\big)$ and $\big(\mathbf{z}_0,\widetilde{\mathrm{S}}(\mathbf{z}_0,\mathbf{r}_0)\big)$ verify the hypothesis (3.51), which is required by the definition of $\mathbf{T}$ and its continuity property. Then, it is readily seen that (3.87) follows directly from (3.82), (3.85), and (3.86) □

The main result of this section is hence stated as follows.

Theorem 3.13. *Assume that the data are sufficiently small so that* (3.64) *and* (3.65) *hold. In addition, suppose that*

$$\begin{aligned}L_{\Xi}\Big\{&\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right)\left(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\right)\\ &+ \ \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} < 1\,.\end{aligned} \tag{3.88}$$

Then, the operator Ξ *has a unique fixed point* $(\mathbf{u},\mathbf{p}) \in W_\delta$. *Equivalently, the coupled problem* (2.61) *has a unique solution* $(\vec{\mathbf{u}},\boldsymbol{\sigma}) \in \mathbf{H}\times\mathbf{Q}$, $(\widetilde{\boldsymbol{\sigma}},\eta) \in \mathrm{H}\times\mathrm{Q}$, *and* $(\mathbf{p},\varphi) \in X_2 \times M_1$, *with* $(\mathbf{u},\mathbf{p}) \in W_\delta$. *Moreover, there hold the following a priori estimates*

$$\begin{aligned}\|(\vec{\mathbf{u}},\boldsymbol{\sigma})\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}\Big\{\|\nabla f\|_{0,r;\Omega}\left(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\right) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,,\\ \|(\widetilde{\boldsymbol{\sigma}},\eta)\|_{\mathrm{H}\times\mathrm{Q}} &\leqslant C_{\widetilde{\mathrm{S}}}\Big\{\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\Big\}\,,\\ \|(\mathbf{p},\varphi)\|_{X_2\times M_1} &\leqslant C_{\mathbf{T}}\Big\{\|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\}\,,\end{aligned}$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}$ *is a positive constant depending only on* $C_{\mathbf{S}}$, $\bar{C}_{\mathbf{S}}$, $C_{\widetilde{\mathrm{S}}}$, *and* δ.

Proof. We begin by recalling from Lemma 3.8 that (3.64) and (3.65) guarantee that Ξ maps W_δ into itself. Hence in virtue of the equivalence between (2.61) and (3.8), and bearing in mind the Lipschitz-continuity of Ξ (cf. Lemma 3.12) and the hypothesis (3.88), a straightforward application of the Banach fixed point Theorem implies the existence of a unique solution $(\mathbf{u},\mathbf{p}) \in W_\delta$ of (2.61). In addition, the a priori estimates follow straightforwardly from (3.17), (3.20), (3.32) and (3.54), and bounding $\|\mathbf{u}\|_{0,4;\Omega}$ by δ. □

4 The Galerkin scheme

In this section we introduce the Galerkin scheme of the fully-mixed formulation (2.61), analyse its solvability by employing a discrete version of the fixed point strategy introduced in Section 3.1, and develop the corresponding a priori error analysis.

4.1 Preliminaries

We begin by considering arbitrary finite element subspaces $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, H_h, Q_h, $X_{2,h}$, $M_{1,h}$, $X_{1,h}$ and $M_{2,h}$ of the spaces $\mathbf{L}^4(\Omega)$, $\mathbb{L}^2_{\mathbf{tr}}(\Omega)$, $\mathbb{H}(\mathbf{div}_{4/3};\Omega)$, H, Q, X_2, M_1, X_1, and M_2, respectively. Hereafter, h stands for both the sub-index of each foregoing subspace and the size of a regular triangulation $\mathcal{T}_h$ of $\bar{\Omega}$ made up of triangles K (when $n = 2$) or tetrahedra K (when $n = 3$) of diameter h_K, that is $h := \max\big\{h_K : \quad K \in \mathcal{T}_h\big\}$. Specific finite element subspaces satisfying the stability conditions to be introduced along the analysis will be provided later on in Section 5. Then, defining the spaces

$$\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}, \quad \mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega),$$

and setting the notations

$$\vec{\mathbf{u}}_h := (\mathbf{u}_h, \mathbf{t}_h), \quad \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{H}_h,$$

the Galerkin scheme associated with (2.61) reads: Find $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, and $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$, such that

$$\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{u}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\eta_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h,\\
\mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h,\\
\widetilde{a}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) + \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}_h) && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h,\\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \xi_h) &= \widetilde{\mathrm{G}}(\xi_h) && \forall\, \xi_h \in \mathrm{Q}_h,\\
a(\mathbf{p}_h, \mathbf{q}_h) + b_1(\mathbf{q}_h, \varphi_h) &= \mathrm{F}(\mathbf{q}_h) && \forall\, \mathbf{q}_h \in X_{1,h},\\
b_2(\mathbf{p}_h, \phi_h) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h, \varphi_h), \phi_h\big) &= \mathrm{G}(\phi_h) && \forall\, \phi_h \in M_{2,h}.
\end{aligned} \tag{4.1}$$

Throughout the rest of this section, we adopt the discrete analogue of the fixed point strategy introduced in Section 3.1 to analyse the solvability of (4.1). According to it, we now let $\mathbf{S}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h \to \mathbf{H}_h^{\mathbf{u}}$ be the operator defined by

$$\mathbf{S}_h(\mathbf{z}_h, \chi_h) := \mathbf{u}_h \qquad \forall\, (\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h, \tag{4.2}$$

where $(\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h) = \big((\mathbf{u}_h, \mathbf{t}_h), \boldsymbol{\sigma}_h\big) \in \mathbf{H}_h \times \mathbf{Q}_h$ is the unique solution (to be confirmed) of the first and second rows of (4.1) when $\mathbf{c}(\mathbf{u}_h; \cdot, \cdot)$ and $\mathbf{F}_{\eta_h}$ are replaced by $\mathbf{c}(\mathbf{z}_h; \cdot, \cdot)$ and $\mathbf{F}_{\chi_h}$, respectively, that is

$$\begin{aligned}
\mathbf{a}(\vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{c}(\mathbf{z}_h; \vec{\mathbf{u}}_h, \vec{\mathbf{v}}_h) + \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\sigma}_h) &= \mathbf{F}_{\chi_h}(\vec{\mathbf{v}}_h) && \forall\, \vec{\mathbf{v}}_h \in \mathbf{H}_h,\\
\mathbf{b}(\vec{\mathbf{u}}_h, \boldsymbol{\tau}_h) &= \mathbf{G}(\boldsymbol{\tau}_h) && \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h.
\end{aligned} \tag{4.3}$$

In turn, we let $\widetilde{\mathrm{S}}_h : \mathbf{H}_h^{\mathbf{u}} \times X_{2,h} \to \mathrm{Q}_h$ be the operator given by

$$\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h) := \eta_h \qquad \forall\, (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}, \tag{4.4}$$

where $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ is the unique solution (to be confirmed) of the third and fourth rows of (4.1) when $\widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}$ is replaced by $\widetilde{c}_{\mathbf{z}_h,\mathbf{r}_h}$, that is

$$\begin{aligned}
\widetilde{a}(\widetilde{\boldsymbol{\sigma}}_h, \widetilde{\boldsymbol{\tau}}_h) + \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) + \widetilde{c}_{\mathbf{z}_h,\mathbf{r}_h}(\widetilde{\boldsymbol{\tau}}_h, \eta_h) &= \widetilde{\mathrm{F}}(\widetilde{\boldsymbol{\tau}}_h) && \forall\, \widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h,\\
\widetilde{b}(\widetilde{\boldsymbol{\sigma}}_h, \xi_h) &= \widetilde{\mathrm{G}}(\xi_h) && \forall\, \xi_h \in \mathrm{Q}_h.
\end{aligned} \tag{4.5}$$

Similarly, we let $\mathbf{T}_h : \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h \to X_{2,h}$ be the operator given by

$$\mathbf{T}_h(\mathbf{z}_h, \chi_h) := \mathbf{p}_h \quad \forall\, (\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h, \tag{4.6}$$

where $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$ is the unique solution (to be confirmed) of the fifth and sixth rows of (4.1) when $c_{\mathbf{u}_h,\eta_h}$ is replaced by $c_{\mathbf{z}_h,\chi_h}$, that is

$$\begin{aligned} a(\mathbf{p}_h, \mathbf{q}_h) \,+\, b_1(\mathbf{q}_h, \varphi_h) &\;=\; \mathrm{F}(\mathbf{q}_h) \qquad \forall\, \mathbf{q}_h \in X_{1,h}\,, \\ b_2(\mathbf{p}_h, \phi_h) \,-\, c_{\mathbf{z}_h,\chi_h}\big((\mathbf{p}_h, \varphi_h), \phi_h\big) &\;=\; \mathrm{G}(\phi_h) \qquad \forall\, \phi_h \in M_{2,h}\,. \end{aligned} \tag{4.7}$$

Finally, we define $\boldsymbol{\Xi}_h : \mathbf{H}^{\mathbf{u}}_h \times X_{2,h} \to \mathbf{H}^{\mathbf{u}}_h \times X_{2,h}$ as

$$\boldsymbol{\Xi}_h(\mathbf{z}_h, \mathbf{r}_h) \,:=\, \Big(\mathbf{S}_h\big(\mathbf{z}_h, \widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\big), \mathbf{T}_h\big(\mathbf{z}_h, \widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\big)\Big) \qquad \forall\, (\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}^{\mathbf{u}}_h \times X_{2,h}\,, \tag{4.8}$$

and notice that solving (4.1) is equivalent to seeking a fixed point of $\boldsymbol{\Xi}_h$, that is: Find $(\mathbf{u}_h, \mathbf{p}_h) \in \mathbf{H}^{\mathbf{u}}_h \times X_{2,h}$ such that

$$\boldsymbol{\Xi}_h(\mathbf{u}_h, \mathbf{p}_h) \,=\, (\mathbf{u}_h, \mathbf{p}_h)\,. \tag{4.9}$$

4.2 Discrete solvability analysis

Similarly to the approach from Section 3, here we establish the well-posedness of the discrete system (4.1) by studying the equivalent fixed-point equation (4.9). More precisely, being the respective analyses fully analogous to those developed in Sections 3.2 and 3.3, in what follows we basically collect the corresponding results and, eventually, discuss some details of the respective proofs.

We begin by stating next that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and $\mathbf{T}_h$ are well-defined, equivalently, that the problems (4.3), (4.5), and (4.7) are well-posed. Certainly, instead of [3, Theorem 2.1, Corollary 2.1, Section 2.1], [16, Theorem 2.34], and [16, Theorem 2.6], we now resort to the respective discrete versions given by [3, Corollary 2.2, Section 2.2], [16, Proposition 2.42], and [16, Theorem 2.22]. To this end, we need to introduce general hypotheses on the finite element subspaces to be utilized in (4.1), and later on in Section 5 we will introduce specific examples of the latter satisfying them. According to the above, and in order to address first the well-definedness of $\mathbf{S}$, we assume that

(H.1) there exists a positive constant $\boldsymbol{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|_{\mathbf{H}}} \,\geqslant\, \boldsymbol{\beta}_{\mathtt{d}}\, \|\boldsymbol{\tau}_h\|_{\mathbf{Q}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h\,.$$

In addition, we let $\mathbf{V}_h$ be the discrete kernel of the bilinear form $\mathbf{b}$, that is

$$\mathbf{V}_h \,:=\, \Big\{ \vec{\mathbf{v}}_h \in \mathbf{H}_h : \quad \mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h) \,=\, 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h \Big\}\,, \tag{4.10}$$

and suppose that

(H.2) there exists a positive constant $C_{\mathtt{d}}$, independent of h, such that

$$\|\mathbf{s}_h\|_{0,\Omega} \,\geqslant\, C_{\mathtt{d}}\, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \,\in\, \mathbf{V}_h\,.$$

Then, given $\mathbf{z}_h \in \mathbf{H}^{\mathbf{u}}_h$, it readily follows from the definitions of $\mathcal{A}_{\mathbf{z}_h}$ (cf. (3.9)) and $\mathbf{a}$ (cf. (2.9)), the identity (2.20), and the assumption **(H.2)**, that

$$\mathcal{A}_{\mathbf{z}_h}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,=\, \mathbf{a}(\vec{\mathbf{v}}_h, \vec{\mathbf{v}}_h) \,=\, \nu\, \|\mathbf{s}_h\|^2_{0,\Omega} \,\geqslant\, \frac{\nu}{2}\, C^2_{\mathtt{d}}\, \|\mathbf{v}_h\|^2_{0,4;\Omega} + \frac{\nu}{2}\, \|\mathbf{s}_h\|^2_{0,\Omega} \quad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \,\in\, \mathbf{V}_h\,, \tag{4.11}$$

which proves the $\mathbf{V}_h$-ellipticity of $\mathcal{A}_{\mathbf{z}_h}$ with constant $\boldsymbol{\alpha}_{\mathtt{d}} \,:=\, \frac{\nu}{2} \min\big\{C^2_{\mathtt{d}}, 1\big\}$. Thus, the discrete analogue of Lemma 3.1 reads as follows.

Lemma 4.1. *For each* $(\mathbf{z}_h,\chi_h)\in \mathbf{H}_h^{\mathbf{u}}\times \mathrm{Q}_h$ *there exists a unique* $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) := \big((\mathbf{u}_h,\mathbf{t}_h),\boldsymbol{\sigma}_h\big)\in \mathbf{H}_h\times\mathbf{Q}_h$ *solution of* (4.3)*, and hence one can define* $\mathbf{S}_h(\mathbf{z}_h,\chi_h) := \mathbf{u}_h\in\mathbf{H}_h^{\mathbf{u}}$*. Moreover, there exists a positive constant* $C_{\mathbf{S},\mathtt{d}}$*, depending only on* $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}_{\mathtt{d}}$*, and* $\boldsymbol{\beta}_{\mathtt{d}}$*, such that*

$$\begin{aligned}\|\mathbf{S}_h(\mathbf{z}_h,\chi_h)\|_{0,4;\Omega} \,&=\, \|\mathbf{u}_h\|_{0,4;\Omega} \,\leqslant\, \|\vec{\mathbf{u}}_h\|_{\mathbf{H}}\\ &\leqslant\, C_{\mathbf{S},\mathtt{d}}\Big\{\|\chi_h\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega} \,+\, \|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \big(1+\|\mathbf{z}_h\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,.\end{aligned}\tag{4.12}$$

Proof. Having the discrete inf-sup condition for $\mathbf{b}$ (cf. **(H.1)**) and the $\mathbf{V}_h$-ellipticity of $\mathcal{A}_{\mathbf{z}_h}$ for each $\mathbf{z}_h\in\mathbf{H}_h^{\mathbf{u}}$ (cf. (4.11)), the existence of a unique solution to (4.3) is a straightforward application of [16, Proposition 2.42], whereas the a priori estimate (4.12) follows from [16, eq. (2.30)]. □

We remark here that the discrete analogue of (3.20) reads

$$\begin{aligned}\|\boldsymbol{\sigma}_h\|_{\mathbf{Q}} \,=\, \|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leqslant\, \bar{C}_{\mathbf{S},\mathtt{d}}\,\big(1\,+\,\|\mathbf{z}_h\|_{0,4;\Omega}\big)\,\Big\{&\|\chi_h\|_{0,\rho;\Omega}\,\|\nabla f\|_{0,r;\Omega}\\ &+\,\|\mathbf{f}\|_{0,4/3;\Omega}\,+\,\big(1+\|\mathbf{z}_h\|_{0,4;\Omega}\big)\,\|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}\,,\end{aligned}\tag{4.13}$$

where $\bar{C}_{\mathbf{S},\mathtt{d}}$ is a positive constant depending as well on $|\Omega|$, $\|\mathbf{i}_4\|$, ν, λ, $\boldsymbol{\alpha}_{\mathtt{d}}$, and $\boldsymbol{\beta}_{\mathtt{d}}$.

In turn, for the well-definedness of $\widetilde{\mathrm{S}}_h$, we now look at the discrete kernel of $\widetilde{b}$, that is

$$\widetilde{\mathrm{V}}_h \,:=\, \Big\{\widetilde{\boldsymbol{\tau}}_h\in\mathrm{H}_h:\quad \widetilde{b}(\widetilde{\boldsymbol{\tau}}_h,\xi_h)\,=\,0\quad\forall\,\xi_h\in\mathrm{Q}_h\Big\}\,,\tag{4.14}$$

and suppose that

(H.3) there holds $\mathrm{div}\big(\mathrm{H}_h\big)\,\subseteq\,\mathrm{Q}_h$,

(H.4) there exists a positive constant $\widetilde{\beta}_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\widetilde{\boldsymbol{\tau}}_h\in\mathrm{H}_h\\ \widetilde{\boldsymbol{\tau}}_h\neq 0}}\frac{\widetilde{b}(\widetilde{\boldsymbol{\tau}}_h,\xi_h)}{\|\widetilde{\boldsymbol{\tau}}_h\|_{\mathrm{H}}}\,\geqslant\,\widetilde{\beta}_{\mathtt{d}}\,\|\xi_h\|_{\mathrm{Q}}\qquad\forall\,\xi_h\in\mathrm{Q}_h\,.$$

Bearing in mind the definition of $\widetilde{b}$ (cf. (2.33)), and employing **(H.3)**, we deduce from (4.14) that

$$\widetilde{\mathrm{V}}_h \,=\, \Big\{\widetilde{\boldsymbol{\tau}}_h\in\mathrm{H}_h:\quad \mathrm{div}(\widetilde{\boldsymbol{\tau}}_h)\,=\,0\Big\}\,,$$

which yields the discrete analogue of (3.23), and hence the $\widetilde{\mathrm{V}}_h$-ellipticity of $\widetilde{a}$ (cf. (2.32)) with constant $\widetilde{\alpha}_{\mathtt{d}}=1$. This fact together with **(H.4)** guarantee, thanks to [16, Proposition 2.42], the discrete global inf-sup condition for $\widetilde{A}$ (cf. (3.21)) with a positive constant $\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}$ depending only on $\widetilde{\alpha}_{\mathtt{d}}$, $\widetilde{\beta}_{\mathtt{d}}$, and $\|\widetilde{a}\|$, and thus the same property is transferred to $\widetilde{A}_{\mathbf{z}_h,\mathbf{r}_h}$ (cf. (3.27)) for each $(\mathbf{z}_h,\mathbf{r}_h)\in\mathbf{H}_h^{\mathbf{u}}\times X_{2,h}$ satisfying the discrete version of (3.29), that is

$$\|\mathbf{z}_h\|_{0,4;\Omega}+\|\mathbf{r}_h\|_{0,r;\Omega}\,\leqslant\,\frac{\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}}{2\,\|\widetilde{c}\|}\,.\tag{4.15}$$

In this way, the well-definedness of $\widetilde{\mathrm{S}}_h$ is established by the following lemma.

Lemma 4.2. *For each* $(\mathbf{z}_h, \mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h}$ *verifying* (4.15)*, there exists a unique* $(\widetilde{\boldsymbol{\sigma}}_h, \eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$ *solution of* (4.5)*, and hence one can define* $\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h) := \eta_h \in \mathrm{Q}_h$*. Moreover, there exists a positive constant* $C_{\widetilde{\mathrm{S}},\mathtt{d}}$*, depending only on* $\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}$*,* $\|\mathrm{i}_\rho\|$*, and* κ_η*, such that*

$$\|\widetilde{\mathrm{S}}_h(\mathbf{z}_h, \mathbf{r}_h)\|_{\mathrm{Q}} \;=\; \|\eta_h\|_{0,\rho;\Omega} \;\leqslant\; \|(\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \;\leqslant\; C_{\widetilde{\mathrm{S}},\mathtt{d}} \left\{ \|\eta_D\|_{1/2,\Gamma} \;+\; \|f_\eta\|_{0,\varrho;\Omega} \right\}. \tag{4.16}$$

Proof. It is a direct application of [16, Theorem 2.22]. □

Furthermore, the well-definedness of $\mathbf{T}_h$ requires the introduction of the discrete kernels of b_1 and b_2, namely

$$K_{1,h} \;:=\; \left\{ \mathbf{q}_h \in X_{1,h} : \quad b_1(\mathbf{q}_h, \phi_h) \;=\; 0 \quad \forall\, \phi_h \in M_{1,h} \right\},$$

and

$$K_{2,h} \;:=\; \left\{ \mathbf{q}_h \in X_{2,h} : \quad b_2(\mathbf{q}_h, \phi_h) \;=\; 0 \quad \forall\, \phi_h \in M_{2,h} \right\},$$

and the following hypotheses:

(H.5) there exists a positive constant $\alpha_{\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\mathbf{q}_h \in K_{1,h} \\ \mathbf{q}_h \neq 0}} \frac{a(\mathbf{r}_h, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{X_1}} \;\geqslant\; \alpha_{\mathtt{d}}\, \|\mathbf{r}_h\|_{X_2} \qquad \forall\, \mathbf{r}_h \in K_{2,h}\,, \quad \text{and}$$

$$\sup_{\mathbf{r}_h \in \mathcal{K}_{2,h}} a(\mathbf{r}_h, \mathbf{q}_h) \;>\; 0 \qquad \forall\, \mathbf{q}_h \in K_{1,h},\; \mathbf{q}_h \;\neq\; \mathbf{0}\,,$$

(H.6) for each $i \in \{1, 2\}$ there exists a positive constant $\beta_{i,\mathtt{d}}$, independent of h, such that

$$\sup_{\substack{\mathbf{q}_h \in X_{i,h} \\ \mathbf{q}_h \neq 0}} \frac{b_i(\mathbf{q}_h, \phi_h)}{\|\mathbf{q}_h\|_{X_i}} \;\geqslant\; \beta_{i,\mathtt{d}}\, \|\phi_h\|_{M_i} \qquad \forall\, \phi_h \in M_{i,h}\,.$$

Thanks to **(H.5)** and **(H.6)**, a straightforward application of [3, Corollary 2.2, Section 2.2] implies the discrete global inf-sup condition for $\mathbf{A}$ (cf. (3.33)) with a positive constant $\alpha_{\mathbf{T},\mathtt{d}}$ depending only on $\alpha_{\mathtt{d}}$, $\beta_{1,\mathtt{d}}$, $\beta_{2,\mathtt{d}}$ and $\|a\|$, and hence the same property is shared by $\mathbf{A}_{\mathbf{z}_h,\chi_h}$ (cf. (3.48)) for each $(\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h$ satisfying the discrete version of (3.51), that is

$$\|\mathbf{z}_h\|_{0,4;\Omega} + \|\chi_h\|_{0,\rho;\Omega} \;\leqslant\; \frac{\alpha_{\mathbf{T},\mathtt{d}}}{2\,\|c\|}\,. \tag{4.17}$$

In this way, the well-definedness of $\mathbf{T}_h$ is stated as follows.

Lemma 4.3. *For each* $(\mathbf{z}_h, \chi_h) \in \mathbf{H}_h^{\mathbf{u}} \times \mathrm{Q}_h$ *verifying* (4.17)*, there exists a unique* $(\mathbf{p}_h, \varphi_h) \in X_{2,h} \times M_{1,h}$ *solution of* (4.7)*, and hence one can define* $\mathbf{T}_h(\mathbf{z}_h, \chi_h) := \mathbf{p}_h \in X_{2,h}$*. Moreover, there exists a positive constant* $C_{\mathbf{T},\mathtt{d}}$*, depending only on* $\alpha_{\mathbf{T},\mathtt{d}}$*,* C_r*, and* κ_φ*, such that*

$$\|\mathbf{T}_h(\mathbf{z}_h, \chi_h)\|_{X_2} \;=\; \|\mathbf{p}_h\|_{X_2} \;\leqslant\; \|(\mathbf{p}_h, \varphi_h)\|_{X_2 \times M_1} \;\leqslant\; C_{\mathbf{T},\mathtt{d}} \left\{ \|\varphi_D\|_{1/s,r;\Gamma} \;+\; \|f_\varphi\|_{0,j;\Omega} \right\}. \tag{4.18}$$

Proof. Similarly to the proof of Lemma 4.2, it reduces to a simple application of [16, Theorem 2.22]. □

Having established that the discrete operators $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, $\mathbf{T}_h$, and hence $\boldsymbol{\Xi}_h$ (under the constraint imposed by (4.17)), are all well defined, we now proceed as in Section 3.3 to address the solvability of the corresponding fixed point equation (4.9). Indeed, letting $\delta_{\mathtt{d}}$ be the discrete version of (3.62), that is

$$\delta_{\mathtt{d}} := \min\left\{\frac{\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}}{2\,\|\widetilde{c}\|}, \frac{\alpha_{\mathbf{T},\mathtt{d}}}{4\,\|c\|}\right\}, \tag{4.19}$$

we first introduce the ball

$$W_{\delta_{\mathtt{d}}} := \Big\{(\mathbf{z}_h,\mathbf{r}_h) \in \mathbf{H}_h^{\mathbf{u}} \times X_{2,h} : \quad \|(\mathbf{z}_h,\mathbf{r}_h)\| := \|\mathbf{z}_h\|_{0,4;\Omega} + \|\mathbf{r}_h\|_{X_2} \leqslant \delta_{\mathtt{d}}\Big\}. \tag{4.20}$$

Then, analogously to the derivation of Lemma 3.8 (cf. beginning of Section 3.3), we deduce that $\boldsymbol{\Xi}_h$ maps $W_{\delta_{\mathtt{d}}}$ into itself under the discrete versions of (3.64) and (3.65), which read exactly as those, except that the constants $C_{\widetilde{\mathrm{S}}}$, $\alpha_{\mathbf{T}}$, and $C(\delta)$, and the radius δ utilized there are replaced by $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, $\alpha_{\mathbf{T},\mathtt{d}}$, $C_{\mathtt{d}}(\delta)$, and $\delta_{\mathtt{d}}$, respectively, where, similarly to $C(\delta)$, $C_{\mathtt{d}}(\delta)$ depends explicitly on $C_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, $(1+\delta)$, and $C_{\mathbf{T},\mathtt{d}}$. Moreover, following analogue arguments to those employed in the proofs of Lemmas 3.9, 3.10, and 3.11, we are able to prove the continuity properties of $\mathbf{S}_h$, $\widetilde{\mathrm{S}}_h$, and $\mathbf{T}_h$, that is the discrete versions of (3.66), (3.74), and (3.78), which are the same as the latter, but instead of $L_{\mathbf{S}}$, $L_{\widetilde{\mathrm{S}}}$, and $L_{\mathbf{T}}$, the resulting constants are given by

$$L_{\mathbf{S},\mathtt{d}} := \boldsymbol{\alpha}_{\mathtt{d}}^{-1}\,\max\big\{|\Omega|^{1/4}, \|\mathbf{c}\|\big\}, \quad L_{\widetilde{\mathrm{S}},\mathtt{d}} := 2\,\|\widetilde{c}\|\,\alpha_{\widetilde{\mathrm{S}},\mathtt{d}}^{-1}\,C_{\widetilde{\mathrm{S}},\mathtt{d}}, \quad \text{and} \quad L_{\mathbf{T},\mathtt{d}} := 2\,\|c\|\,\alpha_{\mathbf{T},\mathtt{d}}^{-1}\,C_{\mathbf{T},\mathtt{d}},$$

respectively. Hence, proceeding analogously to the derivation of (3.85), (3.86), and the consequent Lemma 3.12, we are able to show that, under the discrete version of (3.64), there holds

$$\begin{aligned}
&\|\boldsymbol{\Xi}_h(\mathbf{z}_h,\mathbf{r}_h) - \boldsymbol{\Xi}_h(\mathbf{z}_{0,h},\mathbf{r}_{0,h})\| \\
&\quad \leqslant L_{\boldsymbol{\Xi},\mathtt{d}}\Big\{\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\
&\quad + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\}\|(\mathbf{z}_h,\mathbf{r}_h) - (\mathbf{z}_{0,h},\mathbf{r}_{0,h})\|,
\end{aligned} \tag{4.21}$$

for all $(\mathbf{z}_h,\mathbf{r}_h)$, $(\mathbf{z}_{0,h},\mathbf{r}_{0,h}) \in W_{\delta_{\mathtt{d}}}$, where $L_{\boldsymbol{\Xi},\mathtt{d}}$ is a positive constant depending only on $L_{\mathbf{S},\mathtt{d}}$, $L_{\widetilde{\mathrm{S}},\mathtt{d}}$, $L_{\mathbf{T},\mathtt{d}}$, $C_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, and δ.

According to the above, the main result of this section is established as follows.

Theorem 4.4. *Assume that the data are sufficiently small so that the discrete versions of (3.64) and (3.65) hold, that is*

$$C_{\widetilde{\mathrm{S}},\mathtt{d}}\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) \leqslant \frac{\alpha_{\mathbf{T},\mathtt{d}}}{4\,\|c\|}, \tag{4.22}$$

and

$$\begin{aligned}
C_{\mathtt{d}}(\delta)\Big\{&\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\,\|\nabla f\|_{0,r;\Omega} + \|\mathbf{f}\|_{0,4/3;\Omega} \\
&+ \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \leqslant \delta_{\mathtt{d}}.
\end{aligned} \tag{4.23}$$

Then, the operator $\boldsymbol{\Xi}_h$ has a fixed point $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_{\mathtt{d}}}$. Equivalently, the coupled problem (4.1) has a solution $(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h) \in \mathbf{H}_h \times \mathbf{Q}_h$, $(\widetilde{\boldsymbol{\sigma}}_h,\eta_h) \in \mathrm{H}_h \times \mathrm{Q}_h$, and $(\mathbf{p}_h,\varphi_h) \in X_{2,h} \times M_{1,h}$, with $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_{\mathtt{d}}}$. Moreover, there hold the following a priori estimates

$$\begin{aligned}
\|(\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} &\leqslant C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}\Big\{\|\nabla f\|_{0,r;\Omega}\big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big) + \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\}, \\
\|(\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} &\leqslant C_{\widetilde{\mathrm{S}},\mathtt{d}}\Big\{\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\Big\}, \\
\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1} &\leqslant C_{\mathbf{T},\mathtt{d}}\Big\{\|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\},
\end{aligned}$$

where $C_{\vec{\mathbf{u}},\boldsymbol{\sigma},\mathtt{d}}$ is a positive constant depending only on $C_{\mathbf{S},\mathtt{d}}$, $\bar{C}_{\mathbf{S},\mathtt{d}}$, $C_{\widetilde{\mathrm{S}},\mathtt{d}}$, and $\delta_{\mathtt{d}}$. Furthermore, under the additional assumption

$$\begin{aligned} L_{\boldsymbol{\Xi},\mathtt{d}} \Big\{ & \big(\|\eta_D\|_{1/2,\Gamma} + \|f_\eta\|_{0,\varrho;\Omega}\big)\big(\|\nabla f\|_{0,r;\Omega} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\big) \\ & + \quad \|\mathbf{f}\|_{0,4/3;\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\varphi_D\|_{1/s,r;\Gamma} + \|f_\varphi\|_{0,j;\Omega}\Big\} \,<\, 1\,, \end{aligned} \tag{4.24}$$

the aforementioned solutions of (4.9) and (4.1) are unique.

Proof. As previously mentioned, (4.22) and (4.23) guarantee that $\boldsymbol{\Xi}_h$ maps $W_{\delta_{\mathtt{d}}}$ into itself. Then, knowing from (4.21) that $\boldsymbol{\Xi}_h : W_{\delta_{\mathtt{d}}} \to W_{\delta_{\mathtt{d}}}$ is continuous, a straightforward application of Brouwer's theorem (cf. [9, Theorem 9.9-2]) implies the existence of solution of (4.9), and hence of (4.1). In turn, under the further hypotheses (4.24), the Banach fixed-point theorem yields the respective uniqueness of solution. Finally, in any case, the a priori estimates are consequences of (4.12), (4.13), (4.16) and (4.18), and the fact that $\|\mathbf{u}_h\|_{0,4;\Omega} \leqslant \delta_{\mathtt{d}}$. □

4.3 A priori error analysis

In this section we derive an a priori error estimate for the Galerkin scheme (4.1) with arbitrary finite element subspaces satisfying the hypotheses introduced in Section 4.2. More precisely, we are interested in establishing the Céa estimate for the error

$$\mathtt{E} \,:=\, \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \,+\, \|(\mathbf{p},\varphi) - (\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}\,, \tag{4.25}$$

where $\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),(\widetilde{\boldsymbol{\sigma}},\eta),(\mathbf{p},\varphi)\big) \in (\mathbf{H}\times\mathbf{Q})\times(\mathrm{H}\times\mathrm{Q})\times(X_2\times M_1)$ is the unique solution of (2.61) with $(\mathbf{u},\mathbf{p}) \in W_\delta$ (cf. (3.55)), and $\big((\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h),(\widetilde{\boldsymbol{\sigma}}_h,\eta_h),(\mathbf{p}_h,\varphi_h)\big) \in (\mathbf{H}_h\times\mathbf{Q}_h)\times(\mathrm{H}_h\times\mathrm{Q}_h)\times(X_{2,h}\times M_{1,h})$ is a solution of (4.1) with $(\mathbf{u}_h,\mathbf{p}_h) \in W_{\delta_{\mathtt{d}}}$ (cf. (4.20)). To this end, we consider the pairs of associated continuous and discrete formulations arising from (2.61) and (4.1) once the latter are split according to the three equations forming the full model. In what follows, given a subspace Z_h of a generic Banach space $(Z,\|\cdot\|_Z)$, we set for each $z \in Z$

$$\mathrm{dist}(z,Z_h) \,:=\, \inf_{z_h\in Z_h} \|z \,-\, z_h\|_Z\,. \tag{4.26}$$

We begin by applying the Strang estimate provided by [10, Lemma 6.1], whose proof is a simple modification of that of [18, Theorem 2.6], to the context given by the first two rows of (2.61) and (4.1). As a consequence, we deduce the existence of a positive constant $\widehat{C}_{\mathbf{S}}$, depending only on $\boldsymbol{\alpha}_{\mathtt{d}}$ $\boldsymbol{\beta}_{\mathtt{d}}$, $\|\mathbf{a}\|$, $\|\mathbf{b}\|$, $\|\mathbf{c}\|$, δ, and $\delta_{\mathtt{d}}$, such that

$$\begin{aligned} \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) \,-\, (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, \widehat{C}_{\mathbf{S}} \Big\{ & \mathrm{dist}(\vec{\mathbf{u}},\mathbf{H}_h) \,+\, \mathrm{dist}(\boldsymbol{\sigma},\mathbf{Q}_h) \\ & + \|\mathbf{F}_\eta - \mathbf{F}_{\eta_h}\|_{\mathbf{H}_h'} \,+\, \|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'}\Big\}\,. \end{aligned} \tag{4.27}$$

Then, using the boundedness properties of $\mathbf{F}_\eta$ (cf. (2.40) and (3.72)) and $\mathbf{c}$ (cf. (2.18) and (3.73)), we readily obtain

$$\|\mathbf{F}_\eta - \mathbf{F}_{\eta_h}\|_{\mathbf{H}_h'} \,\leqslant\, |\Omega|^{1/4}\, \|\eta - \eta_h\|_{0,\rho;\Omega}\, \|\nabla f\|_{0,r;\Omega}\,,$$

and

$$\|\mathbf{c}(\mathbf{u};\vec{\mathbf{u}},\cdot) - \mathbf{c}(\mathbf{u}_h;\vec{\mathbf{u}},\cdot)\|_{\mathbf{H}_h'} \,\leqslant\, \|\mathbf{c}\|\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\, \|\vec{\mathbf{u}}\|_{\mathbf{H}}\,,$$

which, replaced back in (4.27), give

$$\begin{aligned} \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) \,-\, (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,\leqslant\, & \widehat{C}_{\mathbf{S}} \Big\{\mathrm{dist}(\vec{\mathbf{u}},\mathbf{H}_h) \,+\, \mathrm{dist}(\boldsymbol{\sigma},\mathbf{Q}_h)\Big\} \\ & + \bar{C}_{\mathbf{S}} \Big\{\|\nabla f\|_{0,r;\Omega}\, \|\eta - \eta_h\|_{0,\rho;\Omega} \,+\, \|\vec{\mathbf{u}}\|_{\mathbf{H}}\, \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}\Big\}\,, \end{aligned} \tag{4.28}$$

where $\bar{C}_{\mathbf{S}} := \widehat{C}_{\mathbf{S}} \max\big\{|\Omega|^{1/4}, \|\mathbf{c}\|\big\}$.

Next, we apply the Strang a priori error estimate from [3, Proposition 2.1, Corollary 2.3, and Theorem 2.3] to the context given by the third and fourth rows of (2.61) and (4.1), in which each term involving $\widetilde{c}$ is considered as part of the respective functional on the right hand side. In this way, we deduce the existence of a positive constant $\widehat{C}_{\widetilde{\mathrm{S}}}$, depending only on $\widetilde{\alpha}_{\mathbf{d}}$, $\widetilde{\beta}_{\mathbf{d}}$, $\|\widetilde{a}\|$ and $\|\widetilde{b}\|$, such that

$$\|(\widetilde{\boldsymbol{\sigma}}, \eta) - (\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \leqslant \widehat{C}_{\widetilde{\mathrm{S}}} \Big\{ \mathrm{dist}(\widetilde{\boldsymbol{\sigma}}, \mathrm{H}_h) + \mathrm{dist}(\eta, \mathrm{Q}_h) + \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot, \eta) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot, \eta_h)\|_{\mathrm{H}'_h} \Big\}. \tag{4.29}$$

In turn, subtracting and adding η_h to the second component of $\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot, \eta)$, making use of the triangle inequality, bearing in mind the definition of $\widetilde{c}_{\mathbf{z},\mathbf{q}}$ (cf. (2.27)), and employing its boundedness property (cf. (2.37)), we find that

$$\begin{aligned} \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot, \eta) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot, \eta_h)\|_{\mathrm{H}'_h} &\leqslant \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot, \eta - \eta_h)\|_{\mathrm{H}'_h} + \|\widetilde{c}_{\mathbf{u},\mathbf{p}}(\cdot, \eta_h) - \widetilde{c}_{\mathbf{u}_h,\mathbf{p}_h}(\cdot, \eta_h)\|_{\mathrm{H}'_h} \\ &\leqslant \|\widetilde{c}\| \Big\{ \big(\|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{p}\|_{0,r;\Omega}\big) \|\eta - \eta_h\|_{\mathrm{Q}} + \|\eta_h\|_{\mathrm{Q}} \big(\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{0,r;\Omega}\big) \Big\}, \end{aligned}$$

which, along with (4.29), yield

$$\begin{aligned} \|(\widetilde{\boldsymbol{\sigma}}, \eta) - (\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H}\times\mathrm{Q}} &\leqslant \widehat{C}_{\widetilde{\mathrm{S}}} \Big\{ \mathrm{dist}(\widetilde{\boldsymbol{\sigma}}, \mathrm{H}_h) + \mathrm{dist}(\eta, \mathrm{Q}_h) \Big\} \\ &+ \bar{C}_{\widetilde{\mathrm{S}}} \Big\{ \big(\|\mathbf{u}\|_{0,4;\Omega} + \|\mathbf{p}\|_{0,r;\Omega}\big) \|\eta - \eta_h\|_{\mathrm{Q}} + \|\eta_h\|_{\mathrm{Q}} \big(\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\mathbf{p} - \mathbf{p}_h\|_{0,r;\Omega}\big) \Big\}, \end{aligned} \tag{4.30}$$

where $\bar{C}_{\widetilde{\mathrm{S}}} := \widehat{C}_{\widetilde{\mathrm{S}}} \|\widetilde{c}\|$.

Furthermore, we proceed analogously to the previous case for the context given by the fifth and sixth rows of (2.61) and (4.1), that is, we consider each term involving c as part of the respective functional on the right hand side, and then apply the Strang a priori error estimate from [3, Proposition 2.1, Corollary 2.3, and Theorem 2.3]. As a result of it we obtain

$$\begin{aligned} &\|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \\ &\quad \leqslant \widehat{C}_{\mathbf{T}} \Big\{ \mathrm{dist}(\mathbf{p}, X_{2,h}) + \mathrm{dist}(\varphi, M_{1,h}) + \|c_{\mathbf{u},\eta}\big((\mathbf{p}, \varphi), \cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h, \varphi_h), \cdot\big)\|_{M'_{2,h}} \Big\}, \end{aligned} \tag{4.31}$$

where $\widehat{C}_{\mathbf{T}}$ is a positive constant depending only on $\alpha_{\mathbf{d}}$, $\beta_{1,\mathbf{d}}$, $\beta_{2,\mathbf{d}}$, $\|a\|$, $\|b_1\|$, and $\|b_2\|$. Now, in order to estimate the consistency error term of (4.31), we subtract and add $(\mathbf{p}_h, \varphi_h)$ in the first component of $c_{\mathbf{u},\eta}\big((\mathbf{p}, \varphi), \cdot\big)$, employ triangle inequality, and invoke the definition of $c_{\mathbf{z},\chi}$ (cf. (2.51)) and its boundedness property (cf. (2.57)), to arrive at

$$\begin{aligned} &\|c_{\mathbf{u},\eta}\big((\mathbf{p}, \varphi), \cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h, \varphi_h), \cdot\big)\|_{M'_{2,h}} \\ &\quad \leqslant \|c_{\mathbf{u},\eta}\big((\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h), \cdot\big)\|_{M'_{2,h}} + \|c_{\mathbf{u},\eta}\big((\mathbf{p}_h, \varphi_h), \cdot\big) - c_{\mathbf{u}_h,\eta_h}\big((\mathbf{p}_h, \varphi_h), \cdot\big)\|_{M'_{2,h}} \\ &\quad \leqslant \|c\| \Big\{ \big(\|\mathbf{u}\|_{0,4;\Omega} + \|\eta\|_{\mathrm{Q}}\big) \|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \\ &\quad + \big(\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\eta - \eta_h\|_{\mathrm{Q}}\big) \|(\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \Big\}, \end{aligned}$$

which, jointly with (4.31), imply

$$\begin{aligned} \|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} &\leqslant \widehat{C}_{\mathbf{T}} \Big\{ \mathrm{dist}(\mathbf{p}, X_{2,h}) + \mathrm{dist}(\varphi, M_{1,h}) \Big\} \\ &+ \bar{C}_{\mathbf{T}} \Big\{ \big(\|\mathbf{u}\|_{0,4;\Omega} + \|\eta\|_{\mathrm{Q}}\big) \|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \\ &+ \|(\mathbf{p}_h, \varphi_h)\|_{X_2\times M_1} \big(\|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega} + \|\eta - \eta_h\|_{\mathrm{Q}}\big) \Big\}, \end{aligned} \tag{4.32}$$

with $\bar{C}_{\mathbf{T}} := \widehat{C}_{\mathbf{T}}\,\|c\|$.

Consequently, adding the inequalities (4.28), (4.30), and (4.32), denoting $\widehat{C} := \max\{\widehat{C}_{\mathbf{S}}, \widehat{C}_{\widetilde{\mathrm{S}}}, \widehat{C}_{\mathbf{T}}\}$, employing the bounds for $\|\vec{\mathbf{u}}\|_{\mathbf{H}}$, $\|\mathbf{p}\|_{X_2}$, $\|\eta\|_{\mathrm{Q}}$, $\|\eta_h\|_{\mathrm{Q}}$, and $\|(\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1}$ provided by Theorems 3.13 and 4.4, and performing some algebraic manipulations, we find, in terms of the notations introduced in (4.25) and (4.26), that

$$
\begin{aligned}
\mathtt{E} \,\leqslant\, & \widehat{C}\left\{\mathrm{dist}\big((\vec{\mathbf{u}},\boldsymbol{\sigma}),\mathbf{H}_h\times\mathbf{Q}_h\big) \,+\, \mathrm{dist}\big((\widetilde{\boldsymbol{\sigma}},\eta),\mathrm{H}_h\times\mathrm{Q}_h\big) \,+\, \mathrm{dist}\big((\mathbf{p},\varphi),X_{2,h}\times M_{1,h}\big)\right\} \\
& +\quad \widehat{C}_0\left\{\big(1+\|\nabla f\|_{0,r;\Omega}\big)\big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \,+\, \|\nabla f\|_{0,r;\Omega}\right. \\
& +\quad \left.\|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\right\}\mathtt{E}\,,
\end{aligned}
\tag{4.33}
$$

where $\widehat{C}_0$ is a positive constant depending on $\bar{C}_{\mathbf{S}}$, $\bar{C}_{\widetilde{\mathrm{S}}}$, $\bar{C}_{\mathbf{T}}$, $C_{\vec{\mathbf{u}},\boldsymbol{\sigma}}$, $C_{\widetilde{\mathrm{S}}}$, $C_{\mathbf{T}}$, $C_{\widetilde{\mathrm{S}},\mathbf{d}}$, and $C_{\mathbf{T},\mathbf{d}}$.

We are now in a position to establish the announced Céa estimate.

Theorem 4.5. *In addition to the hypotheses of Theorems 3.13 and 4.4, assume that*

$$
\begin{aligned}
& \widehat{C}_0\left\{\big(1+\|\nabla f\|_{0,r;\Omega}\big)\big(\|\eta_D\|_{1/2,\Gamma} \,+\, \|f_\eta\|_{0,\varrho;\Omega}\big) \,+\, \|\nabla f\|_{0,r;\Omega}\right. \\
& +\quad \left.\|\mathbf{f}\|_{0,4/3;\Omega} \,+\, \|\mathbf{u}_D\|_{1/2,\Gamma} \,+\, \|\varphi_D\|_{1/s,r;\Gamma} \,+\, \|f_\varphi\|_{0,j;\Omega}\right\} \,\leqslant\, \frac{1}{2}\,.
\end{aligned}
\tag{4.34}
$$

Then, denoting $\bar{C} := 2\,\widehat{C}$, there holds

$$
\begin{aligned}
& \|(\vec{\mathbf{u}},\boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h,\boldsymbol{\sigma}_h)\|_{\mathbf{H}\times\mathbf{Q}} \,+\, \|(\widetilde{\boldsymbol{\sigma}},\eta) - (\widetilde{\boldsymbol{\sigma}}_h,\eta_h)\|_{\mathrm{H}\times\mathrm{Q}} \,+\, \|(\mathbf{p},\varphi) - (\mathbf{p}_h,\varphi_h)\|_{X_2\times M_1} \\
& \quad \leqslant\, \bar{C}\left\{\mathrm{dist}(\vec{\mathbf{u}},\mathbf{H}_h) + \mathrm{dist}(\boldsymbol{\sigma},\mathbf{Q}_h) + \mathrm{dist}(\widetilde{\boldsymbol{\sigma}},\mathrm{H}_h) + \mathrm{dist}(\eta,\mathrm{Q}_h) + \mathrm{dist}(\mathbf{p},X_{2,h}) + \mathrm{dist}(\varphi,M_{1,h})\right\}.
\end{aligned}
$$

Proof. It follows straightforwardly from (4.33). □

We end the section with the a priori estimate for $\|p-p_h\|_{0,\Omega}$, where p_h is the discrete pressure suggested by the postprocessing formula given by the second identity in (1.13), which, according to (2.7), becomes

$$
p_h \,=\, -\frac{1}{n}\mathrm{tr}\Big(\boldsymbol{\sigma}_h + c_h\,\mathbb{I} \,+\, \frac{\lambda}{2}(\mathbf{u}_h\otimes\mathbf{u}_h)\Big)\,,
\tag{4.35}
$$

with

$$
c_h \,:=\, -\frac{\lambda}{2\,n\,|\Omega|}\int_\Omega \mathrm{tr}(\mathbf{u}_h\otimes\mathbf{u}_h)\,.
\tag{4.36}
$$

Then, applying Cauchy-Schwarz's inequality, performing some algebraic manipulations, and employing the a priori bounds for $\|\mathbf{u}\|_{0,4;\Omega}$ and $\|\mathbf{u}_h\|_{0,4;\Omega}$, we deduce the existence of a positive constant C, depending on data, but independent of h, such that

$$
\|p-p_h\|_{0,\Omega} \,\leqslant\, C\left\{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\|_{0,\Omega} \,+\, \|\mathbf{u}-\mathbf{u}_h\|_{0,4;\Omega}\right\}.
\tag{4.37}
$$

5 Specific finite element subspaces

We now define specific finite element subspaces satisfying the conditions (**H.1**) - (**H.6**) that were introduced in Section 4.2, and provide the rates of convergence of the resulting discrete method.

5.1 Preliminaries

Bearing in mind the notations introduced at the beginning of Section 4.1, and given an integer $k \geqslant 0$ and $K \in \mathcal{T}_h$, we let $\mathrm{P}_k(K)$ be the space of polynomials of degree $\leqslant k$ defined on K, and denote its vector and tensor versions by $\mathbf{P}_k(K)$ and $\mathbb{P}_k(K)$, respectively. In addition, we let $\mathbf{RT}_k(K) = \mathbf{P}_k(K) \oplus \mathrm{P}_k(K)\mathbf{x}$ be the local Raviart-Thomas space of order k defined on K, where $\mathbf{x}$ stands for a generic vector in R^n, and denote by $\mathbb{RT}_k(K)$ its corresponding tensor counterpart. In turn, we let $\mathrm{P}_k(\mathcal{T}_h)$, $\mathbf{P}_k(\mathcal{T}_h)$, $\mathbb{P}_k(\mathcal{T}_h)$, $\mathbf{RT}_k(\mathcal{T}_h)$ and $\mathbb{RT}_k(\mathcal{T}_h)$ be the corresponding global versions of $\mathrm{P}_k(K)$, $\mathbf{P}_k(K)$, $\mathbb{P}_k(K)$, $\mathbf{RT}_k(K)$ and $\mathbb{RT}_k(K)$, respectively, that is

$$\mathrm{P}_k(\mathcal{T}_h) := \Big\{ \phi_h \in \mathrm{L}^2(\Omega) : \quad \phi_h|_K \in \mathrm{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbf{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{v}_h \in \mathbf{L}^2(\Omega) : \quad \mathbf{v}_h|_K \in \mathbf{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbb{P}_k(\mathcal{T}_h) := \Big\{ \mathbf{s}_h \in \mathbb{L}^2(\Omega) : \quad \mathbf{s}_h|_K \in \mathbb{P}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

$$\mathbf{RT}_k(\mathcal{T}_h) := \Big\{ \mathbf{q}_h \in \mathbf{H}(\mathrm{div};\Omega) : \quad \mathbf{q}_h|_K \in \mathbf{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\},$$

and

$$\mathbb{RT}_k(\mathcal{T}_h) := \Big\{ \boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div};\Omega) : \quad \boldsymbol{\tau}_h|_K \in \mathbb{RT}_k(K) \quad \forall\, K \in \mathcal{T}_h \Big\}.$$

We stress here that for each $t, s \in (1,+\infty)$ such that $t \geqslant s$, there hold $\mathrm{P}_k(\mathcal{T}_h) \subseteq \mathrm{L}^t(\Omega)$, $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}(\mathrm{div}_t;\Omega)$, $\mathbb{RT}_k(\mathcal{T}_h) \subseteq \mathbb{H}(\mathbf{div}_t;\Omega)$, and $\mathbf{RT}_k(\mathcal{T}_h) \subseteq \mathbf{H}^t(\mathrm{div}_s;\Omega)$, inclusions that are implicitly utilized below to introduce the announced specific finite element subspaces. Indeed, we now define

$$\begin{gathered}
\mathbf{H}_h^{\mathbf{u}} := \mathbf{P}_k(\mathcal{T}_h), \quad \mathbb{H}_h^{\mathbf{t}} := \mathbb{L}^2_{\mathrm{tr}}(\Omega) \cap \mathbb{P}_k(\mathcal{T}_h), \quad \mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\mathbf{t}}, \quad \mathbb{H}_h^{\boldsymbol{\sigma}} := \mathbb{RT}_k(\mathcal{T}_h), \\
\mathbf{Q}_h := \mathbb{H}_h^{\boldsymbol{\sigma}} \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega), \quad \mathrm{H}_h := \mathbf{RT}_k(\mathcal{T}_h), \quad \mathrm{Q}_h := \mathrm{P}_k(\mathcal{T}_h), \\
X_{2,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad M_{1,h} := \mathrm{P}_k(\mathcal{T}_h), \quad X_{1,h} := \mathbf{RT}_k(\mathcal{T}_h), \quad M_{2,h} := \mathrm{P}_k(\mathcal{T}_h).
\end{gathered} \tag{5.1}$$

5.2 Verification of the stability conditions

In this section we prove that the specific finite element subspaces given by (5.1) verify the assumptions **(H.1) - (H.6)**. We begin with the following lemma establishing **(H.1)** and **(H.2)**, for which we recall that the definition of the discrete kernel $\mathbf{V}_h$ of the bilinear form $\mathbf{b}$ is given in (4.10).

Lemma 5.1. *There exist positive constants $\boldsymbol{\beta}_{\mathtt{d}}$ and $C_{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\vec{\mathbf{v}}_h \in \mathbf{H}_h \\ \vec{\mathbf{v}}_h \neq \mathbf{0}}} \frac{\mathbf{b}(\vec{\mathbf{v}}_h, \boldsymbol{\tau}_h)}{\|\vec{\mathbf{v}}_h\|_{\mathbf{H}}} \geqslant \boldsymbol{\beta}_{\mathtt{d}} \, \|\boldsymbol{\tau}_h\|_{\mathbf{Q}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_h, \tag{5.2}$$

and

$$\|\mathbf{s}_h\|_{0,\Omega} \geqslant C_{\mathtt{d}} \, \|\mathbf{v}_h\|_{0,4;\Omega} \qquad \forall\, \vec{\mathbf{v}}_h := (\mathbf{v}_h, \mathbf{s}_h) \in \mathbf{V}_h. \tag{5.3}$$

Proof. We first introduce the subspace

$$\mathbf{Q}_{0,h} := \Big\{ \boldsymbol{\tau}_h \in \mathbf{Q}_h : \quad \mathbf{b}((\mathbf{v}_h, \mathbf{0}), \boldsymbol{\tau}_h) := \int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}} \Big\},$$

which, using from (5.1) that $\mathbf{div}(\mathbf{Q}_h) \subseteq \mathbf{H}_h^{\mathbf{u}}$, reduces to

$$\mathbf{Q}_{0,h} = \Big\{ \boldsymbol{\tau}_h \in \mathbf{Q}_h : \quad \mathbf{div}(\boldsymbol{\tau}_h) = 0 \quad \text{in} \quad \Omega \Big\}.$$

Next, we proceed as in [2, Lemma 4.2] and apply the abstract equivalence result provided by [10, Lemma 5.1] to the setting $X = \mathbf{H}_h^{\mathbf{u}}$, $Y = Y_1 = \mathbb{H}_h^{\mathbf{t}}$, $Y_2 = \{\mathbf{0}\}$, $V = \mathbf{V}_h$, $Z = \mathbf{Q}_h$, and $Z_0 = \mathbf{Q}_{0,h}$, where X, Y, Y_1, Y_2, V, Z, and Z_0 correspond to the notations employed in [10, Lemma 5.1]. As a consequence of it, we deduce that (5.2) and (5.3) are jointly equivalent to the existence of positive constants $\boldsymbol{\beta}_1$ and $\boldsymbol{\beta}_2$, independent of h, such that there hold

$$\sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{Q}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\mathbf{b}((\mathbf{v}_h, 0), \boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{Q}}} = \sup_{\substack{\boldsymbol{\tau}_h \in \mathbf{Q}_h \\ \boldsymbol{\tau}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{v}_h \cdot \mathbf{div}(\boldsymbol{\tau}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{Q}}} \geqslant \boldsymbol{\beta}_1 \|\mathbf{v}_h\|_{0,4;\Omega} \quad \forall\, \mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}, \tag{5.4}$$

and

$$\sup_{\substack{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{\mathbf{b}((\mathbf{0}, \mathbf{s}_h), \boldsymbol{\tau}_h)}{\|\mathbf{s}_h\|_{0,\Omega}} = \sup_{\substack{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}} \\ \mathbf{s}_h \neq \mathbf{0}}} \frac{\displaystyle\int_\Omega \mathbf{s}_h : \boldsymbol{\tau}_h}{\|\mathbf{s}_h\|_{0,\Omega}} \geqslant \boldsymbol{\beta}_2 \|\boldsymbol{\tau}_h\|_{\mathbf{Q}} \quad \forall\, \boldsymbol{\tau}_h \in \mathbf{Q}_{0,h}. \tag{5.5}$$

Regarding (5.4), we stress that this result was already established in [10, Lemma 5.5]. In turn, for the proof of (5.5), we first recall from [18, proof of Theorem 3.3] that, being $\mathbf{Q}_h \subseteq \mathbb{RT}_k(\mathcal{T}_h)$, there holds $\mathbf{Q}_{0,h} \subseteq \mathbb{P}_k(\mathcal{T}_h)$. In this way, given $\boldsymbol{\tau}_h \in \mathbf{Q}_{0,h}$, it is clear that $\boldsymbol{\tau}_h^{\mathbf{d}} \in \mathbb{H}_h^{\mathbf{t}}$, and hence bounding below the supremum in (5.5) with $\mathbf{s}_h := \boldsymbol{\tau}_h^{\mathbf{d}}$, and employing (3.15) for $t = 4/3$, gives the required inequality with $\boldsymbol{\beta}_2 := C_{4/3}^{1/2}$. □

Now, as far as **(H.3)** and **(H.4)** are concerned, we observe from (5.1) that $\mathrm{div}(\mathrm{H}_h) \subseteq \mathrm{Q}_h$, which confirms the former hypothesis, whereas the latter is proved in [20, Lemma 4.8].

On the other hand, in order to address the verification of **(H.5)** and **(H.6)**, we first notice from (5.1) that $\mathrm{div}(X_{i,h}) \subseteq M_{i,h}$ for all $i \in \{1,2\}$. Thus, being the pairs $(X_{2,h}, M_{2,h})$ and $(X_{1,h}, M_{1,h})$ algebraically equal, the corresponding discrete kernels of the bilinear forms b_1 and b_2 (cf. (2.50)) coincide as well, and it is easily seen that they become the space

$$K_h^k := \Big\{ \mathbf{q}_h \in \mathbf{RT}_k(\mathcal{T}_h) : \quad \mathrm{div}(\mathbf{q}_h) = 0 \quad \text{in} \quad \Omega \Big\}. \tag{5.6}$$

In turn, we let $\Theta_h^k : \mathbf{L}^1(\Omega) \to K_h^k$ be the projector defined for each $\mathbf{r} \in \mathbf{L}^1(\Omega)$ as the unique $\Theta_h^k(\mathbf{r}) \in K_h^k$ satisfying

$$\int_\Omega \Theta_h^k(\mathbf{r}) \cdot \mathbf{q}_h = \int_\Omega \mathbf{r} \cdot \mathbf{q}_h \quad \forall\, \mathbf{q}_h \in K_h^k. \tag{5.7}$$

Then, we recall from [15, Theorem 3.1] (see also [20, Lemma 4.2] for a slight variant of it), that in the 2D case, given $t \in (1, +\infty)$ and an integer $k \geqslant 0$, there exist positive constants C_t^k and $\bar{C}_t^k$, independent of h, such that, defining

$$c_t^k := \begin{cases} C_t^k & \text{if } \Omega \text{ is convex}, \\ \bar{C}_t^k \{-\log(h)\}^{|1-2/t|} & \text{if } \Omega \text{ is non-convex and } k = 0, \\ \bar{C}_t^k & \text{if } \Omega \text{ is non-convex and } k \geqslant 1 \end{cases}$$

there holds

$$\|\Theta_h^k(\mathbf{r})\|_{0,t;\Omega} \leqslant c_t^k \|\mathbf{r}\|_{0,t;\Omega} \qquad \forall\, \mathbf{r} \in \widetilde{\mathbf{H}}^t(\mathrm{div}_j;\Omega), \tag{5.8}$$

where

$$\widetilde{\mathbf{H}}^t(\mathrm{div}_j;\Omega) := \Big\{ \mathbf{r} \in \mathbf{H}^t(\mathrm{div}_j;\Omega) : \quad \mathrm{div}(\mathbf{r}) = 0 \quad \text{in} \quad \Omega \Big\}.$$

We stress here that only when Ω is non-convex and $k = 0$, c_t^k depends on h, though in a very harmless manner. In fact, the term $\{-\log(h)\}^{|1-2/t|}$ grows very slowly when h approaches 0, and thus it remains reasonably bounded for very small values of the mesh size. In particular, taking $t = 3/2$, which lies in the range for s (cf. (2.25)), index with which (5.8) will be applied below, we observe that for $h \geqslant 10^{-10}$ there holds $\{-\log(h)\}^{|1-2/t|} = \{-\log(h)\}^{1/3} < 3$. In addition, we remark that whether the boundedness property (5.8) is satisfied or not in 3D is still an open problem, and hence the hypothesis **(H.5)**, to be established next by using (5.8), constitutes the only aspect of the analysis of the present section that is not valid in 3D. All the other stability conditions hold in both 2D and 3D.

Lemma 5.2. *There exists a positive constant $\alpha_{\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\mathbf{q}_h \in K_h^k \\ \mathbf{q}_h \neq \mathbf{0}}} \frac{a(\mathbf{r}_h, \mathbf{q}_h)}{\|\mathbf{q}_h\|_{X_1}} \geqslant \alpha_{\mathtt{d}} \|\mathbf{r}_h\|_{X_2} \qquad \forall\, \mathbf{r}_h \in K_h^k, \tag{5.9}$$

and

$$\sup_{\mathbf{r}_h \in K_h^k} a(\mathbf{r}_h, \mathbf{q}_h) > 0 \qquad \forall\, \mathbf{q}_h \in K_h^k,\ \mathbf{q}_h \neq \mathbf{0}. \tag{5.10}$$

Proof. It proceeds analogously to the proof of [20, Lemma 4.3]. Indeed, given $\mathbf{r}_h \in K_h^k$ (cf. (5.6)), $\mathbf{r}_h \neq \mathbf{0}$, one first defines $\mathbf{r}_{h,s} := |\mathbf{r}_h|^{r-2}\,\mathbf{r}_h$, which belongs to $\mathbf{L}^s(\Omega)$. Note from (2.25) that $r > 2$. Next, bounding below the supremum in (5.9) with $\mathbf{q}_h := \Theta_h^k\big(D_s(\mathbf{r}_{h,s})\big) \in K_h^k$, and then employing (5.7), (3.37) (cf. Lemma 3.3), and the boundedness of Θ_h^k (cf. (5.8)) and D_s (cf. Lemma 3.3), we arrive at (5.9) with $\alpha_{\mathtt{d}} := \big(c_s^k\,\|D_s\|\big)^{-1}$. A similar procedure is applied to derive (5.10). We omit further details and refer to the proof of [20, Lemma 4.3]. □

We now employ the notations and results from the Appendix (cf. Section A) to prove **(H.6)**, that is the discrete inf-sup conditions for the bilinear forms b_i, $i \in \{1, 2\}$. Actually, being the proof for $i = 1$ a slight modification of that for [20, Lemma 4.5], we omit its details and just focus on the case $i = 2$.

Lemma 5.3. *There exists a positive constant $\beta_{2,\mathtt{d}}$, independent of h, such that*

$$\sup_{\substack{\mathbf{q}_h \in X_{2,h} \\ \mathbf{q}_h \neq \mathbf{0}}} \frac{b_2(\mathbf{q}_h, \phi_h)}{\|\mathbf{q}_h\|_{X_2}} \geqslant \beta_{2,\mathtt{d}} \|\phi_h\|_{M_2} \qquad \forall\, \phi_h \in M_{2,h}. \tag{5.11}$$

Proof. Given $\phi_h \in M_{2,h}$, we set $\phi_{h,j} := |\phi_h|^{l-2}\,\phi_h$, which belongs to $\mathrm{L}^j(\Omega)$, and notice that

$$\int_\Omega \phi_{h,j}\,\phi_h = \|\phi_{h,j}\|_{0,j;\Omega}\,\|\phi_h\|_{0,l;\Omega}. \tag{5.12}$$

Note from (2.26) that $l \geqslant 2$. Also, we let $\mathcal{O}$ be a bounded convex polygonal domain containing $\bar{\Omega}$, and set

$$g := \begin{cases} \phi_{h,j} & \text{in} \quad \Omega, \\ 0 & \text{in} \quad \mathcal{O}\backslash\overline{\Omega}. \end{cases}$$

It is clear that $g \in \mathrm{L}^j(\mathcal{O})$ and $\|g\|_{0,j;\mathcal{O}} = \|\phi_{h,j}\|_{0,j;\Omega}$. Then, applying the elliptic regularity result provided in [17, Corollary 1], we deduce that there exists a unique $z \in \mathrm{W}^{2,j}(\mathcal{O}) \cap \mathrm{W}_0^{1,j}(\mathcal{O})$ such that: $\Delta z = g$ in $\mathcal{O}$, $z = 0$ on $\partial\mathcal{O}$, and there exists a positive constant $C_{\mathtt{reg}}$, depending only on $\mathcal{O}$, such that

$$\|z\|_{2,j;\mathcal{O}} \leqslant C_{\mathtt{reg}}\,\|g\|_{0,j;\mathcal{O}} = C_{\mathtt{reg}}\,\|\phi_{h,j}\|_{0,j;\Omega}. \tag{5.13}$$

Thus, defining $\mathbf{r} := \nabla z|_\Omega \in \mathrm{W}^{1,j}(\Omega)$, we observe that $\mathrm{div}(\mathbf{r}) = \phi_{h,j}$ in Ω, and, using (5.13), there holds

$$\|\mathbf{r}\|_{1,j;\Omega} \leqslant \|z\|_{2,j;\mathcal{O}} \leqslant C_{\mathtt{reg}}\,\|\phi_{h,j}\|_{0,j;\Omega}. \tag{5.14}$$

In addition, letting $\mathbf{r}_h$ be the global Raviart-Thomas interpolant of $\mathbf{r}$, that is $\mathbf{r}_h := \Pi_h^k(\mathbf{r})$, and employing (A.1), we find that

$$\operatorname{div}(\mathbf{r}_h) \,=\, \operatorname{div}\big(\Pi_h^k(\mathbf{r})\big) \,=\, \mathcal{P}_h^k\big(\operatorname{div}(\mathbf{r})\big) \,=\, \mathcal{P}_h^k(\phi_{h,j})\,, \tag{5.15}$$

so that, thanks to the stability estimate (A.5), it follows that

$$\|\operatorname{div}(\mathbf{r}_h)\|_{0,j;\Omega} \,\leqslant\, C_{\mathcal{P}}\,\|\phi_{h,j}\|_{0,j;\Omega}\,. \tag{5.16}$$

In turn, noting from (2.25) and (2.26) that $j < r \leqslant \dfrac{nj}{n-j}$, Lemma A.3 and (5.14) yield

$$\|\mathbf{r}_h\|_{0,r;\Omega} \,=\, \|\Pi_h^k(\mathbf{r})\|_{0,r;\Omega} \,\leqslant\, C_\Pi\,\|\mathbf{r}\|_{1,j;\Omega} \,\leqslant\, C_\Pi\,C_{\mathtt{reg}}\,\|\phi_{h,j}\|_{0,j;\Omega}\,,$$

which, jointly with (5.16), imply

$$\|\mathbf{r}_h\|_{X_2} \,=\, \|\mathbf{r}_h\|_{0,r;\Omega} \,+\, \|\operatorname{div}(\mathbf{r}_h)\|_{0,j;\Omega} \,\leqslant\, \big(C_{\mathcal{P}} \,+\, C_\Pi\,C_{\mathtt{reg}}\big)\,\|\phi_{h,j}\|_{0,j;\Omega}\,. \tag{5.17}$$

Finally, bounding below the supremum in (5.11) with $\mathbf{r}_h \in X_{2,h}$, and using (5.15), (A.2), (5.12), and (5.17), we conclude the required discrete inf-sup condition for b_2 with $\beta_{2,\mathtt{d}} := \big(C_{\mathcal{P}} \,+\, C_\Pi\,C_{\mathtt{reg}}\big)^{-1}$. $\square$

5.3 The rates of convergence

In this section we provide the rates of convergence of the Galerkin scheme (4.1) with the specific finite element subspaces introduced in Section 5.1. To this end, we first collect the approximation properties of the latter. Indeed, it is easily seen from (A.3) and its corresponding vector and tensorial versions, along with interpolation estimates of Sobolev spaces, that those of $\mathbf{H}_h^{\mathbf{u}}$, $\mathbb{H}_h^{\mathbf{t}}$, Q_h, and $M_{1,h}$, are given as follows

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{v} \in \mathbf{W}^{l,4}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{v}, \mathbf{H}_h^{\mathbf{u}}\big) \,:=\, \inf_{\mathbf{v}_h \in \mathbf{H}_h^{\mathbf{u}}} \|\mathbf{v} \,-\, \mathbf{v}_h\|_{0,4;\Omega} \,\leqslant\, C\,h^l\,\|\mathbf{v}\|_{l,4;\Omega}\,,$$

$\big(\mathbf{AP}_h^{\mathbf{t}}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\mathbf{s} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{s}, \mathbb{H}_h^{\mathbf{t}}\big) \,:=\, \inf_{\mathbf{s}_h \in \mathbb{H}_h^{\mathbf{t}}} \|\mathbf{s} \,-\, \mathbf{s}_h\|_{0,\Omega} \,\leqslant\, C\,h^l\,\|\mathbf{s}\|_{l,\Omega}\,,$$

$\big(\mathbf{AP}_h^{\eta}\big)$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\xi \in \mathrm{W}^{l,\rho}(\Omega)$, there holds

$$\operatorname{dist}\big(\xi, \mathrm{Q}_h\big) \,:=\, \inf_{\xi_h \in \mathrm{Q}_h} \|\xi \,-\, \xi_h\|_{0,\rho;\Omega} \,\leqslant\, C\,h^l\,\|\xi\|_{l,\rho;\Omega}\,,$$

$(\mathbf{AP}_{\mathbf{h}}^{\phi})$ there exists a positive constant C, independent of h, such that for each $l \in [0, k+1]$, and for each $\psi \in \mathrm{W}^{l,r}(\Omega)$, there holds

$$\operatorname{dist}\big(\psi, M_{1,h}\big) \,:=\, \inf_{\psi_h \in M_{1,h}} \|\psi \,-\, \psi_h\|_{0,r;\Omega} \,\leqslant\, C\,h^l\,\|\mathbf{r}\|_{l,r;\Omega}\,.$$

In turn, from [20, eq. (4.6), Section 4.1] and its tensorial version, along with interpolation estimates of Sobolev spaces as well, we obtain the approximation properties of $\mathbf{Q}_h$ and H_h, which reduce to

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$ with $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, there holds

$$\operatorname{dist}\big(\boldsymbol{\tau}, \mathbf{Q}_h\big) \,:=\, \inf_{\boldsymbol{\tau}_h \in \mathbf{Q}_h} \|\boldsymbol{\tau} \,-\, \boldsymbol{\tau}_h\|_{\mathbf{div}_{4/3};\Omega} \,\leqslant\, C\, h^l \left\{ \|\boldsymbol{\tau}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\tau})\|_{l,4/3;\Omega} \right\},$$

$\left(\mathbf{AP}_h^{\widetilde{\boldsymbol{\sigma}}}\right)$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\widetilde{\boldsymbol{\tau}} \in \mathbf{H}^l(\Omega)$ with $\operatorname{div}(\widetilde{\boldsymbol{\tau}}) \in \mathrm{W}^{l,\varrho}(\Omega)$, there holds

$$\operatorname{dist}\big(\widetilde{\boldsymbol{\tau}}, \mathrm{H}_h\big) \,:=\, \inf_{\widetilde{\boldsymbol{\tau}}_h \in \mathrm{H}_h} \|\widetilde{\boldsymbol{\tau}} \,-\, \widetilde{\boldsymbol{\tau}}_h\|_{\operatorname{div}_\varrho;\Omega} \,\leqslant\, C\, h^l \left\{ \|\widetilde{\boldsymbol{\tau}}\|_{l,\Omega} \,+\, \|\operatorname{div}(\widetilde{\boldsymbol{\tau}})\|_{l,\varrho;\Omega} \right\}.$$

Finally, that of $X_{2,h}$, which follows from Lemma A.2 and (A.4) (with $m = 0$), and applying again interpolation estimates of Sobolev spaces, becomes

$(\mathbf{AP}_{\mathbf{h}}^{\mathbf{p}})$ there exists a positive constant C, independent of h, such that for each $l \in [1, k+1]$, and for each $\mathbf{q} \in \mathbf{W}^{l,r}(\Omega)$ with $\operatorname{div}(\mathbf{q}) \in \mathrm{W}^{l,j}(\Omega)$, there holds

$$\operatorname{dist}\big(\mathbf{q}, X_{2,h}\big) \,:=\, \inf_{\mathbf{q}_h \in X_{2,h}} \|\mathbf{q} - \mathbf{q}_h\|_{r,\operatorname{div}_j;\Omega} \,\leqslant\, C\, h^l \left\{ \|\mathbf{q}\|_{l,r;\Omega} \,+\, \|\operatorname{div}(\mathbf{q})\|_{l,j;\Omega} \right\}.$$

Hence, we can state the following main theorem.

Theorem 5.4. *Let $\big((\vec{\mathbf{u}}, \boldsymbol{\sigma}), (\widetilde{\boldsymbol{\sigma}}, \eta), (\mathbf{p}, \varphi)\big) \in (\mathbf{H} \times \mathbf{Q}) \times (\mathrm{H} \times \mathrm{Q}) \times (X_2 \times M_1)$ be the unique solution of (2.61) with $(\mathbf{u}, \mathbf{p}) \in W_\delta$ (cf. (3.55)), and let $\big((\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h), (\widetilde{\boldsymbol{\sigma}}_h, \eta_h), (\mathbf{p}_h, \varphi_h)\big) \in (\mathbf{H}_h \times \mathbf{Q}_h) \times (\mathrm{H}_h \times \mathrm{Q}_h) \times (X_{2,h} \times M_{1,h})$ be a solution of (4.1) with $(\mathbf{u}_h, \mathbf{p}_h) \in W_{\delta_{\mathrm{d}}}$ (cf. (4.20)), which is guaranteed by Theorems 3.13 and 4.4, respectively. In turn, let p and p_h given by (1.13) and (4.35), respectively. Assume the hypotheses of Theorem 4.5, and that there exists $l \in [1, k+1]$ such that $\mathbf{u} \in \mathbf{W}^{l,4}(\Omega)$, $\mathbf{t} \in \mathbb{H}^l(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^l(\Omega) \cap \mathbb{H}_0(\mathbf{div}_{4/3};\Omega)$, $\mathbf{div}(\boldsymbol{\sigma}) \in \mathbf{W}^{l,4/3}(\Omega)$, $\widetilde{\boldsymbol{\sigma}} \in \mathbf{H}^l(\Omega)$, $\operatorname{div}(\widetilde{\boldsymbol{\sigma}}) \in \mathrm{W}^{l,\varrho}(\Omega)$, $\eta \in \mathrm{W}^{l,\rho}(\Omega)$, $\mathbf{p} \in \mathbf{W}^{l,r}(\Omega)$, $\operatorname{div}(\mathbf{p}) \in \mathrm{W}^{l,j}(\Omega)$, and $\varphi \in \mathrm{W}^{l,r}(\Omega)$. Then, there exists a positive constant C, independent of h, such that*

$$\begin{aligned}
&\|(\vec{\mathbf{u}}, \boldsymbol{\sigma}) - (\vec{\mathbf{u}}_h, \boldsymbol{\sigma}_h)\|_{\mathbf{H} \times \mathbf{Q}} \,+\, \|(\widetilde{\boldsymbol{\sigma}}, \eta) - (\widetilde{\boldsymbol{\sigma}}_h, \eta_h)\|_{\mathrm{H} \times \mathrm{Q}} \,+\, \|(\mathbf{p}, \varphi) - (\mathbf{p}_h, \varphi_h)\|_{X_2 \times M_1} \,+\, \|p - p_h\|_{0,\Omega} \\
&\qquad \leqslant C\, h^l \Big\{ \|\mathbf{u}\|_{l,4;\Omega} \,+\, \|\mathbf{t}\|_{l,\Omega} \,+\, \|\boldsymbol{\sigma}\|_{l,\Omega} \,+\, \|\mathbf{div}(\boldsymbol{\sigma})\|_{l,4/3;\Omega} \,+\, \|\widetilde{\boldsymbol{\sigma}}\|_{l,\Omega} \\
&\qquad + \|\operatorname{div}(\widetilde{\boldsymbol{\sigma}})\|_{l,\varrho;\Omega} \,+\, \|\eta\|_{l,\rho;\Omega} \,+\, \|\mathbf{p}\|_{l,r;\Omega} \,+\, \|\operatorname{div}(\mathbf{p})\|_{l,j;\Omega} \,+\, \|\varphi\|_{l,r;\Omega} \Big\}.
\end{aligned}$$

Proof. It follows straightforwardly from Theorem 4.5, (4.37), and the above approximation properties. □

6 Numerical results

In this section we present three examples illustrating the performance of the fully-mixed finite element method (4.1) on a set of quasi-uniform triangulations of the respective domains, and considering the finite element subspaces defined by (5.1) (cf. Section 5.1). In what follows, we refer to the corresponding sets of finite element subspaces generated by $k = 0$ and $k = 1$, as simply $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ and $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{RT}_1 - \mathrm{P}_1 - \mathbf{RT}_1 - \mathrm{P}_1$, respectively. The implementation of the numerical method is based on a `FreeFem++` code [22]. A Newton–Raphson algorithm with a fixed tolerance $\mathtt{tol} = 1\mathrm{E}-6$ is used for the resolution of the nonlinear problem (4.1). As usual, the iterative method is finished when the relative error between two consecutive iterations of the complete coefficient vector, namely $\mathbf{coeff}^m$ and $\mathbf{coeff}^{m+1}$, is sufficiently small, that is,

$$\frac{\|\mathbf{coeff}^{m+1} - \mathbf{coeff}^m\|}{\|\mathbf{coeff}^{m+1}\|} \leqslant \mathsf{tol}\,,$$

where $\|\cdot\|$ stands for the usual Euclidean norm in $\mathrm{R}^{\mathtt{DOF}}$ with $\mathtt{DOF}$ denoting the total number of degrees of freedom defining the finite element subspaces $\mathbf{H}_h^{\mathbf{u}}, \mathbb{H}_h^{\mathbf{t}}, \mathbb{H}_h^{\boldsymbol{\sigma}}, \mathrm{H}_h, \mathrm{Q}_h, X_{2,h}$, and $M_{1,h}$ (cf. (5.1)).

We now introduce some additional notation. The individual errors are denoted by:

$$\mathsf{e}(\mathbf{u}) := \|\mathbf{u} - \mathbf{u}_h\|_{0,4;\Omega}, \quad \mathsf{e}(\mathbf{t}) := \|\mathbf{t} - \mathbf{t}_h\|_{0,\Omega}, \quad \mathsf{e}(\boldsymbol{\sigma}) := \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h\|_{\mathbf{div}_{4/3};\Omega}, \quad \mathsf{e}(p) := \|p - p_h\|_{0,\Omega},$$

$$\mathsf{e}(\widetilde{\boldsymbol{\sigma}}) := \|\widetilde{\boldsymbol{\sigma}} - \widetilde{\boldsymbol{\sigma}}_h\|_{\mathrm{div}_\varrho;\Omega}, \quad \mathsf{e}(\eta) := \|\eta - \eta_h\|_{0,\rho;\Omega}, \quad \mathsf{e}(\mathbf{p}) := \|\mathbf{p} - \mathbf{p}_h\|_{r,\mathrm{div}_j;\Omega}, \quad \mathsf{e}(\varphi) := \|\varphi - \varphi_h\|_{0,r;\Omega},$$

where ϱ, ρ, r and j are described in (2.25)–(2.26), and will be specified in the examples below. Next, as usual, for each $\star \in \{\mathbf{u}, \mathbf{t}, \boldsymbol{\sigma}, p, \widetilde{\boldsymbol{\sigma}}, \eta, \mathbf{p}, \varphi\}$ we let $\mathsf{r}(\star)$ be the experimental rate of convergence given by $\mathsf{r}(\star) := \log\big(\mathsf{e}(\star)/\widehat{\mathsf{e}}(\star)\big)/\log(h/\widehat{h})$, where h and $\widehat{h}$ denote two consecutive meshsizes with errors e and $\widehat{\mathsf{e}}$, respectively.

The examples to be considered in this section are described next. In the first two examples, for the sake of simplicity, we take $\nu = 1$, $\lambda = 1$, $\kappa_\eta = 1$, $\mu = 1$, $\kappa_\varphi = 1$, and $\gamma = 1$. In addition, the mean value of $\mathrm{tr}(\boldsymbol{\sigma}_h)$ over Ω is fixed via a Lagrange multiplier strategy (adding one row and one column to the matrix system that solves (4.3) for $\mathbf{u}_h, \mathbf{t}_h$, and $\boldsymbol{\sigma}_h$).

Example 1: Convergence against smooth exact solutions in a 2D domain

In this test we corroborate the rates of convergence in a two-dimensional domain. The domain is the square $\Omega = (-1,1)^2$. We choose $j = l = 2$, whence the remaining parameters become $r = \rho = 4$ and $\varrho = 4/3$ (cf. (2.25)–(2.26)). In turn, we consider the given function $f(x_1,x_2) = \sin(x_1 + x_2)$, and choose the data $\mathbf{f}, f_\eta, f_\varphi$ (cf. (1.14)) such that the exact solution is given by

$$\mathbf{u}(x_1,x_2) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2) \\ -\cos(\pi x_1)\sin(\pi x_2) \end{pmatrix}, \quad p(x_1,x_2) = \cos(\pi x_1)\exp(x_2)\,,$$
$$\eta(x_1,x_2) = 0.5 + 0.5\cos(x_1 x_2)\,, \quad \text{and} \quad \varphi(x_1,x_2) = 0.1 + 0.3\exp(x_1 x_2)\,.$$

The model problem is then complemented with the appropriate Dirichlet boundary conditions. Tables 6.1 and 6.2 show the convergence history for a sequence of quasi-uniform mesh refinements, including the number of Newton iterations. Notice that we are able not only to approximate the original unknowns but also the pressure field through the formula (4.35). The results confirm that the optimal rates of convergence $\mathcal{O}(h^{k+1})$ predicted by Theorem 5.4 are attained for $k = 0, 1$. The Newton method exhibits a behavior independent of the meshsize, converging in six iterations in all cases.

Example 2: Convergence against smooth exact solutions in a 3D domain

In the second example we consider the cube domain $\Omega = (0,1)^3$ and the only possible choice of parameters in 3D, that is $j = 3/2$, $r = 3$, $\rho = 6$, and $\varrho = 6/5$ (cf. (2.25)–(2.26)). The solution is given by

$$\mathbf{u}(x_1,x_2,x_3) = \begin{pmatrix} \sin(\pi x_1)\cos(\pi x_2)\cos(\pi x_3) \\ -2\cos(\pi x_1)\sin(\pi x_2)\cos(\pi x_3) \\ \cos(\pi x_1)\cos(\pi x_2)\sin(\pi x_3) \end{pmatrix}, \quad p(x_1,x_2,x_3) = \cos(\pi x_1)\exp(x_2 + x_3)\,,$$
$$\eta(x_1,x_2,x_3) = 0.5 + 0.5\cos(x_1 x_2 x_3)\,, \quad \text{and} \quad \varphi(x_1,x_2,x_3) = 0.1 + 0.3\exp(x_1 x_2 x_3)\,.$$

Similarly to the first example, we consider $f(x_1, x_2, x_3) = \sin(x_1 + x_2 + x_3)$, whereas the data $\mathbf{f}, f_\eta, f_\varphi$ are computed from (1.14) using the above solution. The convergence history for a set of quasi-uniform mesh refinements using $k = 0$ is shown in Table 6.3. Again, the mixed finite element method converges optimally with order $\mathcal{O}(h)$, as it was proved by Theorem 5.4. In addition, some components of the numerical solution are displayed in Figure 6.1, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation with meshsize $h = 0.0643$ and $63,888$ tetrahedral elements (actually representing $1,483,944$ `DOF`). The numerical results suggest that perhaps only technical difficulties stop us of proving (5.8) for the 3D framework.

Example 3: Movement of cells guided by the concentration of a chemical signal

In the last example, we replicate the one from [14, Test1, Section 7]. More precisely, we consider the rectangle domain $\Omega = (0,2) \times (0,1)$, and the unsteady version of the problem (1.14) with physical parameters $\nu = 10, \lambda = 1, \kappa_\eta = 4, \mu = 8, \kappa_\varphi = 1, \gamma = 6$, data $f(x_1, x_2) = -1000\, x_2$, $\mathbf{f} = 0, f_\eta = 0, f_\varphi = 0$, boundary conditions $\mathbf{u} = 0$ on Γ, $\widetilde{\boldsymbol{\sigma}} \cdot \mathbf{n} = 0$ on Γ, $\mathbf{p} \cdot \mathbf{n} = 0$ on Γ, and initial conditions

$$\mathbf{u}_0 = 0, \quad \eta_0 = \sum_{i=1}^{3} 70 \exp(-8(x_1 - s_i)^2 - 10(x_2 - 1)^2), \quad \varphi_0 = 30 \exp(-5(x_1 - 1)^2 - 5(x_2 - 0.5)^2),$$

where $s_1 = 0.2, s_2 = 0.5$ and $s_3 = 1.2$. We employ a suitable backward Euler time discretization, with time step $\Delta t = 10^{-5}$ and final time $T = 5 \times 10^{-3}$. We observe that at each time step we are solving a slight adaptation of the stationary problem (4.1). In Figure 6.2, we display the computed magnitude of the velocity, and the cell density and chemical signal concentration fields, which were built using the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation on a mesh with meshsize $h = 0.0298$ and $18,566$ triangle elements (actually representing $242,126$ `DOF`). Similarly to [14], the cells are in two clusters in the upper part of the domain at time $T = 10^{-5}$, and then they begin to orient their movement in the direction of greater concentration of the chemical signal (the center of the domain) as we can see at time $T = 10^{-3}$, where the organisms tend to agglomerate in the center of the rectangle. This interesting behavior occurs because the chemotaxis/cross-diffusion term is the dominant one in the initial times. However, as time progresses, the chemical signal is consumed, which causes that the cross-diffusion loses strength, and the self-diffusion of the cells begins to dominate, and therefore they begin to distribute themselves homogeneously over the domain. At final time $T = 5 \times 10^{-3}$ the cells move towards the bottom of the domain, which is due to the external force $\nabla f = (0, -1000)$. In addition, some changes in the velocity field are evidenced, influenced by the movement of the cells.

A Further properties of the Raviart-Thomas interpolator

We begin by introducing for all $t,\ s \in (1, +\infty)$ such that $t \geqslant s$, the space

$$\mathbf{H}_s^t := \left\{ \boldsymbol{\tau} \in \mathbf{H}^t(\mathbf{div}_s; \Omega) : \quad \boldsymbol{\tau}|_K \in \mathbf{W}^{1,s}(K) \quad \forall\, K \in \mathcal{T}_h \right\},$$

and let $\Pi_h^k : \mathbf{H}_s^t \to \mathbf{RT}_k(\mathcal{T}_h)$ be the global Raviart-Thomas interpolation operator (cf. [5, Section 2.5]). Then, we recall from [5, Proposition 2.5.2 and eq. (2.5.27)] that the commuting diagram property states that

$$\operatorname{div}\big(\Pi_h^k(\mathbf{q})\big) \;=\; \mathcal{P}_h^k\big(\operatorname{div}(\mathbf{q})\big) \qquad \forall\, \mathbf{q} \in \mathbf{H}_s^t\,, \tag{A.1}$$

where $\mathcal{P}_h^k : \mathrm{L}^1(\Omega) \to \mathrm{P}_k(\mathcal{T}_h)$ is the projector defined analogously to (5.7), that is, given $\phi \in \mathrm{L}^1(\Omega)$, $\mathcal{P}_h^k(\phi)$ is the unique element in $\mathrm{P}_k(\mathcal{T}_h)$ satisfying

$$\int_\Omega \mathcal{P}_h^k(\phi)\, \psi_h \;=\; \int_\Omega \phi\, \psi_h \qquad \forall\, \psi_h \in \mathrm{P}_k(\mathcal{T}_h)\,. \tag{A.2}$$

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
500	0.7454	6.08E-01	–	3.60E-00	–	2.14E+01	–	1.50E-00	–
2170	0.3667	2.87E-01	1.056	1.76E-00	1.014	9.75E-00	1.109	5.84E-01	1.325
8032	0.1971	1.50E-01	1.048	9.09E-01	1.061	5.05E-00	1.060	3.08E-01	1.033
31508	0.1036	7.43E-02	1.092	4.60E-01	1.057	2.52E-00	1.079	1.52E-01	1.102
126066	0.0554	3.76E-02	1.083	2.29E-01	1.113	1.27E-00	1.099	7.63E-02	1.095
509350	0.0284	1.87E-02	1.049	1.13E-01	1.057	6.28E-01	1.053	3.70E-02	1.085

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
1.13E-00	–	4.20E-02	–	3.55E-01	–	6.47E-02	–	6
5.42E-01	1.033	1.98E-02	1.058	1.83E-01	0.934	3.22E-02	0.984	6
2.90E-01	1.011	1.09E-02	0.960	9.68E-02	1.028	1.77E-02	0.965	6
1.45E-01	1.078	5.71E-03	1.009	4.92E-02	1.052	9.35E-03	0.991	6
7.30E-02	1.093	2.88E-03	1.089	2.50E-02	1.082	4.89E-03	1.035	6
3.60E-02	1.059	1.44E-03	1.046	1.21E-02	1.088	2.37E-03	1.087	6

Table 6.1: Example 1, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation of the chemotaxis-Navier–Stokes model.

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
1540	0.7454	1.93E-01	–	9.99E-01	–	5.79E-00	–	3.47E-01	–
6770	0.3667	3.74E-02	2.310	2.04E-01	2.241	1.25E-00	2.164	6.63E-02	2.332
25184	0.1971	9.84E-03	2.153	5.47E-02	2.119	3.32E-01	2.134	1.66E-02	2.233
99076	0.1036	2.46E-03	2.156	1.36E-02	2.168	8.38E-02	2.139	4.06E-03	2.187
397002	0.0554	6.11E-04	2.220	3.46E-03	2.177	2.09E-02	2.212	1.05E-03	2.160
1605230	0.0284	1.48E-04	2.124	8.51E-04	2.105	5.11E-03	2.114	2.59E-04	2.100

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
3.12E-01	–	8.27E-03	–	4.27E-02	–	7.64E-03	–	6
6.63E-02	2.185	1.41E-03	2.491	1.02E-02	2.021	1.81E-03	2.026	6
1.77E-02	2.131	3.85E-04	2.095	2.93E-03	2.005	5.18E-04	2.018	6
4.44E-03	2.148	1.12E-04	1.923	7.90E-04	2.039	1.48E-04	1.946	6
1.10E-03	2.218	2.71E-05	2.261	2.07E-04	2.132	3.95E-05	2.109	6
2.71E-04	2.106	6.43E-06	2.155	4.87E-05	2.172	9.42E-06	2.150	6

Table 6.2: Example 1, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_1 - \mathbb{P}_1 - \mathbb{RT}_1 - \mathbf{RT}_1 - \mathrm{P}_1 - \mathbf{RT}_1 - \mathrm{P}_1$ approximation of the chemotaxis-Navier–Stokes model.

In turn, employing the $\mathrm{W}^{m,t}$ version of the Deny-Lions Lemma (cf. [16, Lemma B.67]) with integer $m \geqslant 0$ and $t \in (1, +\infty)$, along with the associated scaling estimates (cf. [16, Lemma 1.101]) and the regularity of $\{\mathcal{T}_h\}_{h>0}$, we deduce the existence of positive constants C_1, C_2, independent of h, such that for integers l and m verifying $0 \leqslant l \leqslant k+1$ and $0 \leqslant m \leqslant l$, there hold

$$|\phi \,-\, \mathcal{P}_k(\phi)|_{m,s;\Omega} \,\leqslant\, C_1\, h^{l-m}\, |\phi|_{l,s;\Omega} \tag{A.3}$$

for all $\phi \in \mathrm{W}^{l,s}(\Omega)$, and

$$|\mathrm{div}(\mathbf{q}) \,-\, \mathrm{div}(\Pi_h^k(\mathbf{q}))|_{m,s;\Omega} \,\leqslant\, C_2\, h^{l-m}\, |\mathrm{div}(\mathbf{q})|_{l,s;\Omega} \tag{A.4}$$

DOF	h	e($\mathbf{u}$)	r($\mathbf{u}$)	e($\mathbf{t}$)	r($\mathbf{t}$)	e($\boldsymbol{\sigma}$)	r($\boldsymbol{\sigma}$)	e(p)	r(p)
1224	0.7071	5.74E-01	–	2.63E-00	–	1.50E+01	–	1.18E-00	–
9312	0.3536	3.02E-01	0.927	1.44E-00	0.872	8.00E-00	0.911	6.46E-01	0.874
72576	0.1768	1.55E-01	0.961	7.41E-01	0.955	4.03E-00	0.989	3.00E-01	1.110
384552	0.1010	8.90E-02	0.990	4.27E-01	0.982	2.29E-00	1.007	1.54E-01	1.185
1483944	0.0643	5.68E-02	0.997	2.73E-01	0.992	1.46E-00	1.007	9.17E-02	1.152

e($\widetilde{\boldsymbol{\sigma}}$)	r($\widetilde{\boldsymbol{\sigma}}$)	e(η)	r(η)	e($\mathbf{p}$)	r($\mathbf{p}$)	e(φ)	r(φ)	iter
6.11E-01	–	3.90E-02	–	2.13E-01	–	4.52E-02	–	5
3.48E-01	0.811	2.34E-02	0.734	1.12E-01	0.929	2.37E-02	0.930	5
1.83E-01	0.927	1.22E-02	0.945	5.66E-02	0.985	1.20E-02	0.982	5
1.06E-01	0.974	6.98E-03	0.995	3.24E-02	0.997	6.87E-03	0.995	5
6.79E-02	0.989	4.44E-03	1.001	2.06E-02	0.999	4.38E-03	0.998	5

Table 6.3: Example 2, Number of degrees of freedom, meshsizes, errors, rates of convergence, and number of Newton iterations for the fully-mixed $\mathbf{P}_0 - \mathbb{P}_0 - \mathbb{RT}_0 - \mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{RT}_0 - \mathrm{P}_0$ approximation of the chemotaxis-Navier–Stokes model.

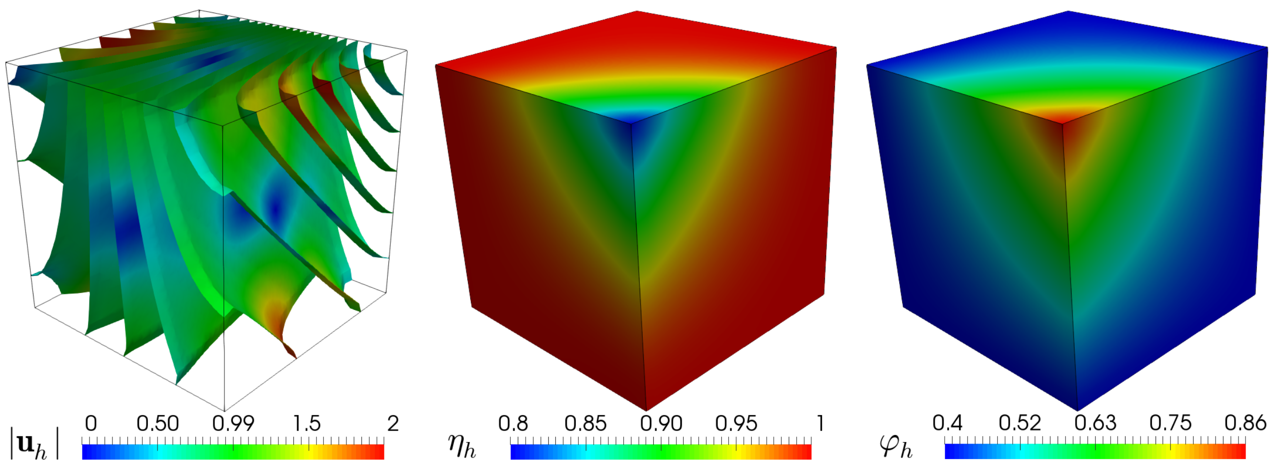

Figure 6.1: Example 2, Computed magnitude of the velocity, cell density field and chemical signal concentration field.

$\forall\, \mathbf{q} \in \mathbf{W}^{1,s}(\Omega)$ with $\mathrm{div}(\mathbf{q}) \in \mathrm{W}^{l,s}(\Omega)$. Note that (A.4) follows from (A.1) and a direct application of (A.3) to $\phi = \mathrm{div}(\mathbf{q})$. In turn, taking in particular $m = l = 0$ in (A.3), we deduce the stability of $\mathcal{P}_h^k$ with respect to $\|\cdot\|_{0,s;\Omega}$, that is the existence of a positive constant $C_{\mathcal{P}}$, independent of h, such that

$$\|\mathcal{P}_h^k(\phi)\|_{0,s;\Omega} \leqslant C_{\mathcal{P}}\, \|\phi\|_{0,s;\Omega} \quad \forall\, \phi \in \mathrm{L}^s(\Omega)\,. \tag{A.5}$$

In what follows we prove additional approximation properties of Π_h^k. To this end, we now denote the reference element of $\mathcal{T}_h$ by $\widehat{K}$, so that, given $K \in \mathcal{T}_h$, we let $F_K : \widehat{K} \to K$ be the bijective affine mapping defined by $F_K(\widehat{\mathbf{x}}) := B_K\,\widehat{\mathbf{x}} + b_K \;\; \forall\, \widehat{\mathbf{x}} \in \widehat{K}$, with $B_K \in \mathrm{R}^{n\times n}$ invertible and $b_K \in \mathrm{R}^n$. Then, the scaling properties via Piola's transformation between $\mathbf{W}^{m,t}(K)$ and $\mathbf{W}^{m,t}(\widehat{K})$, with m a non-negative integer and $t \in (1,+\infty)$, establish the existence of positive constants $\widehat{C}_{\mathtt{P}}$ and $C_{\mathtt{P}}$, such that for each $K \in \mathcal{T}_h$ there hold

$$|\widehat{\mathbf{q}}|_{m,t;\widehat{K}} \leqslant \widehat{C}_{\mathtt{P}}\, \|B_K\|^m\, \|B_K^{-1}\|\, |\det(B_K)|^{1-1/t}\, |\mathbf{q}|_{m,t;K} \qquad \forall\, \mathbf{q} \in \mathbf{W}^{m,t}(K)\,, \tag{A.6}$$

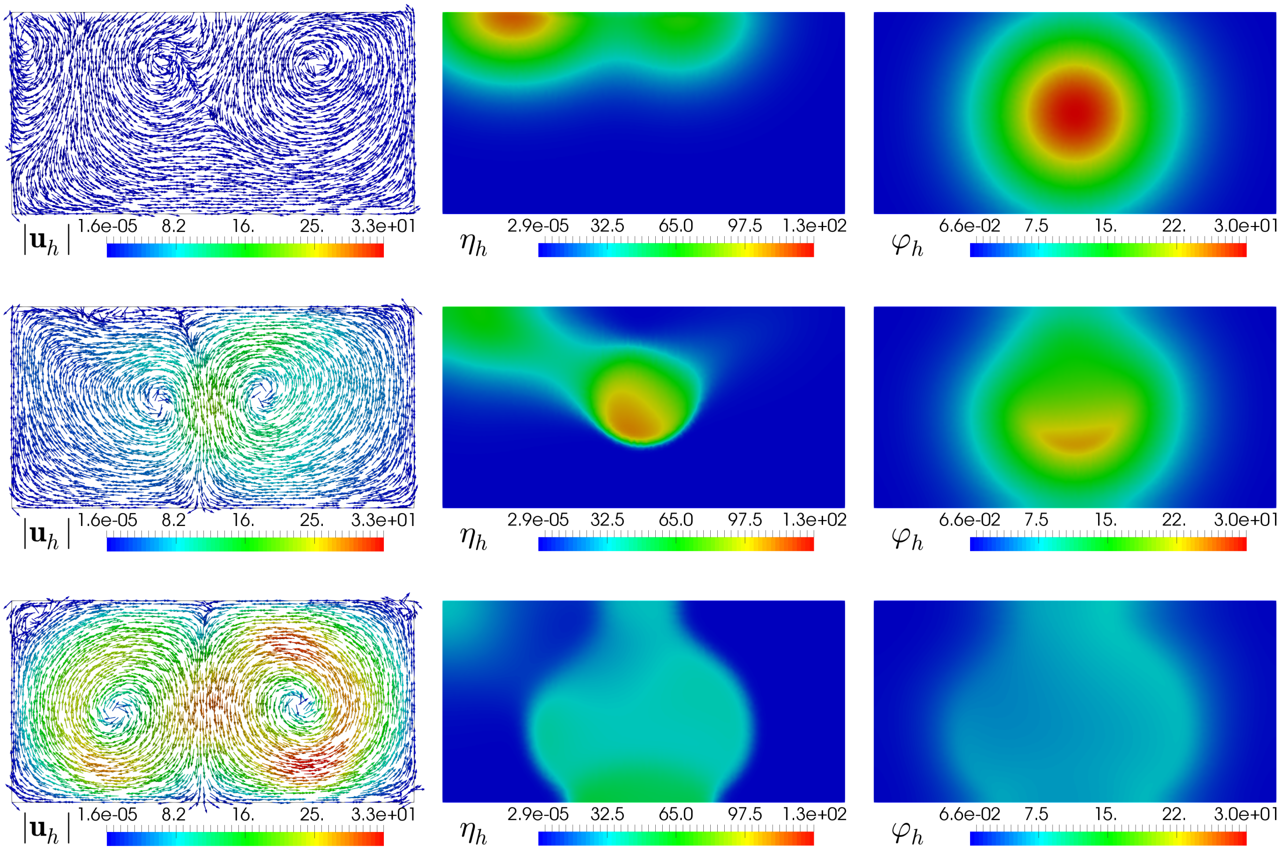

Figure 6.2: Example 3, Computed magnitude of the velocity, cell density field and chemical signal concentration field at time $T = 10^{-5}$ (top plots), at time $T = 10^{-3}$ (middle plots), and at time $T = 5 \times 10^{-3}$ (bottom plots).

and

$$|\mathbf{q}|_{m,t;K} \leqslant C_{\mathtt{P}} \, \|B_K^{-1}\|^m \, \|B_K\| \, |\det(B_K)|^{1/t-1} \, |\widehat{\mathbf{q}}|_{m,t;\widehat{K}} \qquad \forall \, \widehat{\mathbf{q}} \in \mathbf{W}^{m,t}(\widehat{K}) \,. \tag{A.7}$$

Then, letting $\Pi_K^k : \mathbf{W}^{1,s}(K) \to \mathbf{RT}_k(K)$ be the local Raviart-Thomas interpolator for each $K \in \mathcal{T}_h$, and letting $\Pi_{\widehat{K}}^k$ be the corresponding operator for $\widehat{K}$, we have the following approximation property.

Lemma A.1. *Let k and l be integers such that $1 \leqslant l \leqslant k+1$, and let t and s such that $1 \leqslant t \leqslant \frac{ns}{n-s}$ if $s < n$, or $s \leqslant t < +\infty$ if $s = n$. Then, there exists a positive constant C, depending only on $\widehat{K}$, $\Pi_{\widehat{K}}^k$, k, n, t, and s, such that*

$$\|\mathbf{q} \, - \, \Pi_K^k(\mathbf{q})\|_{0,t;K} \, \leqslant \, C \, h_K^{l+\frac{n}{t}-\frac{n}{s}} \, |\mathbf{q}|_{l,s;K} \qquad \forall \, \mathbf{q} \in \mathbf{W}^{l,s}(K) \,. \tag{A.8}$$

Proof. Given $\mathbf{q} \in \mathbf{W}^{l,s}(K)$, we use (A.7) with $m = 0$ to obtain

$$\|\mathbf{q} - \Pi_K^k(\mathbf{q})\|_{0,t;K} \, \leqslant \, C_{\mathtt{P}} \, \|B_K\| \, |\det B_K|^{1/t-1} \, |\widehat{\mathbf{q}} - \Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})|_{0,t;\widehat{K}} \,,$$

which, thanks to the continuous embedding of $\mathbf{W}^{1,s}(\widehat{K})$ in $\mathbf{L}^t(\widehat{K})$ for the indicated ranges of s and t, yields

$$\|\mathbf{q} - \Pi_K^k(\mathbf{q})\|_{0,t;K} \, \leqslant \, C \, \|B_K\| \, |\det(B_K)|^{1/t-1} \, \|\widehat{\mathbf{q}} - \Pi_{\widehat{K}}^k(\widehat{\mathbf{q}})\|_{1,s;\widehat{K}} \,. \tag{A.9}$$

Next, since $\Pi^k_{\widehat{K}}(\widehat{\mathbf{q}}) = \widehat{\mathbf{q}} \ \forall\, \widehat{\mathbf{q}} \in \mathbf{RT}_k(\widehat{K})$, and there holds $\mathbf{P}_{l-1}(\widehat{K}) \subseteq \mathbf{P}_k(\widehat{K}) \subseteq \mathbf{RT}_k(\widehat{K})$, the Bramble-Hilbert Lemma implies that

$$\|\widehat{\mathbf{q}} - \Pi^k_{\widehat{K}}(\widehat{\mathbf{q}})\|_{m,s;\widehat{K}} \leqslant C\, |\widehat{\mathbf{q}}|_{l,s;\widehat{K}} \quad \text{for } 0 \leqslant m \leqslant l,$$

and hence, using in particular the above with $m = 1$ we deduce

$$\|\widehat{\mathbf{q}} - \Pi^k_{\widehat{K}}(\widehat{\mathbf{q}})\|_{1,s;\widehat{K}} \leqslant C\, |\widehat{\mathbf{q}}|_{l,s;\widehat{K}} \,. \tag{A.10}$$

In this way, replacing (A.10) into (A.9), and then employing (A.6), it follows that

$$\begin{aligned}\|\mathbf{q} - \Pi^k_K(\mathbf{q})\|_{0,t;K} &\leqslant C\, \|B_K\|\, |\det(B_K)|^{1/t-1}\, |\widehat{\mathbf{q}}|_{l,s;\widehat{K}} \\ &\leqslant C\, \widehat{C}_{\mathrm{P}}\, \|B_K\|^{l+1}\, \|B_K^{-1}\|\, |\det(B_K)|^{1/t-1/s}\, |\mathbf{q}|_{l,s;K}\,,\end{aligned}$$

from which, using that $\|B_K\| \leqslant C\, h_K$, $\|B_K^{-1}\| \leqslant C\, h_K^{-1}$, and $|\det(B_K)| \cong h_K^n$, we arrive at (A.8) and end the proof. □

The extension of Lemma A.1 to the global Raviart-Thomas interpolator Π^k_h is stated next.

Lemma A.2. *Let k and l be integers such that $1 \leqslant l \leqslant k+1$, and let t and s such that $1 \leqslant t \leqslant \frac{ns}{n-s}$ if $s < n$, or $s \leqslant t < +\infty$ if $s = n$. Then, with the same constant C from (A.8), there holds*

$$\|\mathbf{q} - \Pi^k_h(\mathbf{q})\|_{0,t;\Omega} \leqslant C\, h^{l+\frac{n}{t}-\frac{n}{s}}\, |\mathbf{q}|_{l,s;\Omega} \qquad \forall\, \mathbf{q} \in \mathbf{W}^{l,s}(\Omega)\,.$$

Proof. Given $\mathbf{q} \in \mathbf{W}^{l,s}(\Omega)$, it suffices to see that

$$\|\mathbf{q} - \Pi^k_h(\mathbf{q})\|_{0,t;\Omega} = \Big\{ \sum_{K\in\mathcal{T}_h} \|\mathbf{q} - \Pi^k_K(\mathbf{q})\|^t_{0,t;K} \Big\}^{1/t} = \Big\{ \Big(\sum_{K\in\mathcal{T}_h} \|\mathbf{q} - \Pi^k_K(\mathbf{q})\|^t_{0,t;K} \Big)^{s/t} \Big\}^{1/s},$$

and then apply the sub-additivity property with exponent $\frac{s}{t} \in (0,1]$, and Lemma A.1. □

Finally, a simple corollary of Lemma A.2 reads as follows.

Lemma A.3. *Let k and l be integers such that $1 \leqslant l \leqslant k+1$, and let t and s such that $1 \leqslant t \leqslant \frac{ns}{n-s}$ if $s < n$, or $s \leqslant t < +\infty$ if $s = n$. Then, there exists $C_\Pi > 0$, depending only on C, $|\Omega|$, n, t, and s, such that*

$$\|\Pi^k_h(\mathbf{q})\|_{0,t;\Omega} \leqslant C_\Pi\, \|\mathbf{q}\|_{1,s;\Omega} \qquad \forall\, \mathbf{q} \in \mathbf{W}^{1,s}(\Omega)\,. \tag{A.11}$$

Proof. Given $\mathbf{q} \in \mathbf{W}^{1,s}(\Omega)$, the embedding $\mathbf{i}_{s,t} : \mathbf{W}^{1,s}(\Omega) \to \mathbf{L}^t(\Omega)$ and Lemma A.2 (with $l = 1$) imply

$$\|\Pi^k_h(\mathbf{q})\|_{0,t;\Omega} \leqslant \|\mathbf{q}\|_{0,t;\Omega} + \|\mathbf{q} - \Pi^k_h(\mathbf{q})\|_{0,t;\Omega} \leqslant \|\mathbf{i}_{s,t}\|\, \|\mathbf{q}\|_{1,s;\Omega} + C\, |\Omega|^{1+\frac{n}{t}-\frac{n}{s}}\, \|\mathbf{q}\|_{1,s;\Omega}\,,$$

which yields (A.11) with $C_\Pi := \|\mathbf{i}_{s,t}\| + C\, |\Omega|^{1+\frac{n}{t}-\frac{n}{s}}$. □

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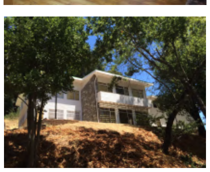
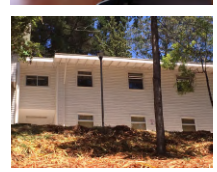

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