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Synchronizing Boolean networks asynchronously

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Abstract

The *asynchronous automaton* associated with a Boolean network $f : \{0,1\}^n \to \{0,1\}^n$, considered in many applications, is the finite deterministic automaton where the set of states is $\{0,1\}^n$, the alphabet is $[n]$, and the action of letter i on a state x consists in either switching the ith component if $f_i(x) \neq x_i$ or doing nothing otherwise. These actions are extended to words in the natural way. A word is then *synchronizing* if the result of its action is the same for every state. In this paper, we ask for the existence of synchronizing words, and their minimal length, for a basic class of Boolean networks called and-or-nets: given an arc-signed digraph G on $[n]$, we say that f is an *and-or-net* on G if, for every $i \in [n]$, there is a such that, for all state x, $f_i(x) = a$ if and only if $x_j = a$ ($x_j \neq a$) for every positive (negative) arc from j to i; so if $a = 1$ ($a = 0$) then f_i is a conjunction (disjunction) of positive or negative literals. Our main result is that if G is strongly connected and has no positive cycles, then either every and-or-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$, much smaller than the bound $(2^n-1)^2$ given by the well known Černý's conjecture, or G is a cycle and no and-or-net on G has a synchronizing word. This contrasts with the following complexity result: it is coNP-hard to decide if every and-or-net on G has a synchronizing word, even if G is strongly connected or has no positive cycles.

1 Introduction

A *Boolean network* (BN) is a finite dynamical system usually defined by a function

$$f : \{0,1\}^n \to \{0,1\}^n, \qquad x = (x_1, \dots, x_n) \mapsto f(x) = (f_1(x), \dots, f_n(x)).$$

BNs have many applications. In particular, since the seminal papers of McCulloch and Pitts [23], Hopfield [18], Kauffman [19, 20] and Thomas [30, 31], they are omnipresent in the modeling of neural and gene networks (see [7, 22] for reviews). They are also essential tools in computer science, see [2, 14, 9, 10, 15] for instance.

The "network" terminology comes from the fact that the *interaction digraph* of f is often considered as the main parameter of f: the vertex set is $[n] = \{1, \dots, n\}$ and there is an arc from j to i if f_i depends on input j. The *signed interaction digraph* provides useful additional information about interactions, and is commonly consider in the context of gene networks: the vertex set is $[n]$ and there is a positive (negative) arc from j to i if there is $x, y \in \{0,1\}^n$ that only differ in $x_j < y_j$ such that $f_i(y) - f_i(x)$ is positive (negative). Note that the presence of both a positive and a negative arc from one vertex to another is allowed. If G is the signed interaction digraph f then we say that f is a BN *on* G.

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From a dynamical point of view, the successive iterations of f describe the so called synchronous dynamics: if x^t is the state of the system at time t, then $x^{t+1} = f(x^t)$ is the state of the system at the next time. Hence, all components are updated in parallel at each time step. However, when BNs are used as models of natural systems, such as gene networks, synchronicity can be an issue. This led researchers to consider the (fully) asynchronous dynamics, where one component is updated at each time step (see e.g. [33, 31, 32, 1]). This asynchronous dynamics can be described by the paths of a deterministic finite automaton called *asynchronous automaton* of f: the set of states is $\{0,1\}^n$, the alphabet is $[n]$ and the action of letter i on a state x, denoted $f^i(x)$, is the state obtained from x by updating the component i only, that is, $f^i(x) = (x_1, \dots, f_i(x), \dots, x_n)$. These actions are extended to any word $w = i_1, \dots, i_\ell$ over the alphabet $[n]$ in the natural way, by setting $f^w = f^{i_\ell} \circ f^{i_{\ell-1}} \circ \cdots \circ f^{i_1}$.

In this paper, we study synchronizing properties of this asynchronous automaton, as proposed in [6]. A word w over $[n]$ *synchronizes* f if it synchronizes its asynchronous automaton, that is, if f^w is a constant function. If w synchronizes f then w is a *synchronizing word* for f, and if f admits a synchronizing word then f is *synchronizing*. The central open problem concerning synchronization is the famous Černý's conjecture [11, 12], which says that any synchronizing deterministic automaton with q states has a synchronizing word of length at most $(q-1)^2$. This conjecture has been proved for several classes of automata, but the best general bounds are only cubic in q, see [36] for a review.

A common research direction concerning BNs tries to deduce from a signed digraph G on $[n]$ the dynamical properties of the BNs f on G. An influential result is this direction is the following [3]: *(i)* if G has no negative cycles, then f has at least one fixed point, and *(ii)* if G has no positive cycles, then f has at most one fixed point. It is then natural to follow this line of research, and ask what can be said on synchronizing properties under these hypothesis.

First, suppose that G has no negative cycles. If f has at least two fixed points then it is not synchronizing. Otherwise, by *(i)*, f has a unique fixed point and it is not difficult to show that f has a synchronizing word of length n (see Proposition 8 in Appendix A). This completely solves the case where G has no negative cycles.

Second, suppose that G has no positive cycles. By *(ii)*, f has at most one fixed point. If f has indeed a fixed point, then one can show, as previously, that f has a synchronizing word of length n (see Proposition 9 in Appendix A). However, what happens if f has no fixed points? Here some difficulties come and our main result provides a partial answer.

Suppose that G has no positive cycles and is, in addition, strongly connected and non-trivial (that is, contains at least one arc). Then f has no fixed points [3], which is the interesting case mentioned above, thus these additional assumptions are natural. It is still difficult to understand synchronizing properties, but our main result, Theorem 1 below, gives a clear picture when f belongs to a well studied class of BNs, called and-or-nets (see e.g. [16, 4, 13, 25, 35, 28, 34, 5] and the references therein for studies about this class of BNs).

We say that f is an *and-or-net* if, for every $i \in [n]$, the component f_i of f is a conjunction or a disjunction of positive or negative literals, that is, there is $a \in \{0,1\}$ such that, for every state x, we have $f_i(x) = a$ if and only if $x_j = a$ for all positive arcs of G from j to i and $x_j \neq a$ for all negative arcs of G from j to i (so if $a = 1$ then f_i is a conjunction, and if $a = 0$ then f_i is a disjunction). We can now state our main result.

Theorem 1. *Let G be a strongly connected signed digraph on $[n]$ without positive cycles. Either*

- *G is a cycle and no BN on G is synchronizing, or*
- *every and-or-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$.*

Since $\sqrt{5}+1<4$, the bound $10(\sqrt{5}+1)^n$ is sub-quadratic according to the number 2^n of states, and thus much smaller than the bound $(2^n-1)^2$ predicted by Černý's conjecture.

The difficult part is the second case. For that, the main tool is the following lemma, which should be of independent interest since it holds for every BNs, and not only for and-or-nets. It says that if G is non-trivial, strongly connected and without positive cycles, then no components are definitively fixed in the asynchronous dynamics.

Lemma 1. *Let G be a non-trivial strongly connected signed digraph on $[n]$ without positive cycles, and let f be a BN on G. For all $i\in[n]$ and $x\in\{0,1\}^n$, we have $f^w(x)_i\neq x_i$ for some word w.*

Let G be as in Theorem 1 and non-trivial. To obtain the bound $10(\sqrt{5}+1)^n$, we prove that if f is an and-or-net on G, the word w in the previous lemma is of length at most $F(n+2)$, the $(n+2)$th Fibonacci number. From that, we next prove that if G is not a cycle, then for every states x,y there is a word w of length at most $3F(n+4)$ such that $f^w(x)=f^w(y)$. This immediately implies that f has a synchronizing word of length at most $3F(n+4)(2^n-1)$ and an easy computation shows that this is less than $10(\sqrt{5}+1)^n$. However, we will show that at least one and-or-net on G can be synchronized much more quickly, with a word of sub-linear length according to the number of states.

Theorem 2. *Let G be a strongly connected signed digraph on $[n]$ without positive cycles, which is not a cycle. At least one and-or-net on G has a synchronizing word of length at most $5n(\sqrt{2})^n$.*

Concerning complexity issues, by Theorem 1, if G is strongly connected and has no positive cycles, then one can decide in linear time if every and-or-net on G is synchronizing. This contrasts with the following results, which shows that if one of the two hypotheses made on G (strongly connected, no positive cycles) is removed, the decision problem becomes much more harder.

Theorem 3. *Let G be a signed digraph on $[n]$. If G is not strongly connected or has a positive cycle, then it is coNP-hard to decide if every and-or-net on G is synchronizing (even if G has maximum in-degree at most 2 and a vertex meeting every cycle).*

Finally, the following (easy) property shows that the second case of Theorem 1 cannot be generalized to all the BNs on G. We say that G is *simple* if it does not contains both a positive and negative arc from one vertex to another (if G is strongly connected and has no positive cycles then it is necessarily simple).

Proposition 1. *If G is a simple signed digraph without vertices of in-degree 0 or 2, then at least one BN on G is not synchronizing.*

This proposition also shows that conjunctions and disjunctions are in some sense necessary, since if G is simple and i is a vertex of in-degree at most 2, then f_i is either a conjunction or a disjunction for every BN f on G.

The paper is organized as follows. In Section 2, basic definitions and results are given; Proposition 1 is proved there. The proofs of Theorems 1, 2 and 3 are given in Sections 3, 4 and 5, respectively. Some conclusions and perspectives are given in Section 6. In Appendix A, the two results mentioned before the statement of Theorem 1 are proved (see Propositions 8 and 9 and in Appendix A).

2 Preliminaries

2.1 Digraphs and signed digraphs

A *digraph* is a couple $G = (V, E)$ where V is a set of vertices and $E \subseteq V^2$ is a set of arcs. Given $I \subseteq V$, the subgraph of G induced by I is denoted $G[I]$, and $G \setminus I$ means $G[V \setminus I]$. A *strongly connected component* (*strong component* for short) of a digraph G is an induced subgraph which is *strongly connected* (*strong* for short) and maximal for this property. A strong component $G[I]$ is *initial* if G has no arc from $V \setminus I$ to I, and *terminal* if G has no arc from I to $V \setminus I$. A *source* is a vertex of in-degree zero. A digraph is *trivial* if it has a unique vertex and no arc.

A *signed digraph* G is a couple (V, E) where $E \subseteq V^2 \times \{-1, 1\}$. If $(j, i, s) \in E$ then G has an arc from j to i of sign s; we also say that j is an in-neighbor of i of sign s and that i is an out-neighbor of j of sign s. We say that G is *simple* if it has not both a positive arc and a negative arc from one vertex to another, and *full-positive* if all its arcs are positive. A subgraph of G is a signed digraph (V', E') with $V' \subseteq V$ and $E' \subseteq E$. Cycles and paths of G are regarded as simple subgraphs (and thus have no repeated vertices). The *sign of a cycle or a path* of G is the product of the signs of its arcs. The underlying (unsigned) digraph of G has vertex set V and an arc from j to i if G has a positive or a negative arc from j to i. Every graph concept made on G that does not involved signs are tacitly made on its underlying digraph. For instance, G is strongly connected if its underlying digraph is.

2.2 Configurations and words

A *configuration* on a finite set V is a map x from V to $\{0, 1\}$, and for $i \in V$ we write x_i instead of $x(i)$. The set of configurations on V is denoted $\{0, 1\}^V$. Given $I \subseteq V$, we denote by x_I the restriction of x to I. We denote by e_I the configuration on V defined by $(e_I)_i = 1$ for all $i \in I$ and $(e_I)_i = 0$ for all $i \in V \setminus I$. If $i \in V$, we write e_i instead of $e_{\{i\}}$. If x, y are configurations on V then $x + y$ is the configuration z on V such that $z_i = x_i + y_i$ for all $i \in V$, where the sum is modulo two. Hence $x + e_i$ is the configuration obtained from x by flipping component i. We denote by $\mathbf{0}$ and $\mathbf{1}$ the all-zero and all-one configurations.

A word w over V is a finite sequence of elements in V; its length is denoted $|w|$ and the set of letters that appear in w is denoted $\{w\}$. The concatenation of two words u and v is denoted u, v or uv. The empty word, of length 0, is denoted ϵ.

2.3 Boolean networks

A *Boolean network* (BN) with component set V is a function $f : \{0, 1\}^V \to \{0, 1\}^V$. For $i \in V$, we denote by f_i the map from $\{0, 1\}^V$ to $\{0, 1\}$ defined by $f_i(x) = f(x)_i$ for all configurations x on V. For $a \in \{0, 1\}$, we say that f_i is the *a-constant function* if $f_i(x) = a$ for all $x \in \{0, 1\}^V$.

The *signed interaction digraph* of f is the signed digraph G with vertex set V such that, for all $i, j \in V$, there is a positive (negative) arc from j to i is there exists a configuration x on V with $x_j = 0$ such that $f_i(x + e_j) - f_i(x)$ is positive (negative). Given a signed digraph G, a BN *on* G is a BN whose signed interaction digraph is G.

Given a vertex i in G, an *i-unstable* configuration in G is a configuration $x \in \{0, 1\}^V$ such that $x_j \neq x_i$ for every positive in-neighbor j of i in G and $x_j = x_i$ for every negative in-neighbor j of i in G. Almost all results that gives relationships between G and the dynamical properties of the BNs on G use, at some point, the following property; the proof is easy and omitted.

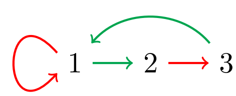
Proposition 2. *If i is a vertex of G of in-degree at least one, and x is an i-unstable configuration in G, then $f_i(x) \neq x_i$ for every BN f on G.*

Let f be a BN with component set V and G its signed interaction digraph. For $i \in V$, we say that f_i is a *conjunction* if, for all configurations x on V, we have $f_i(x) = 1$ if and only if $x_j = 1$ for all positive in-neighbors j of i in G and $x_j = 0$ for all the negative in-neighbors j of i in G. Similarly, we say that f_i is a *disjunction* if, for all configurations x on V, we have $f_i(x) = 0$ if and only if $x_j = 0$ for all positive in-neighbors j of i in G and $x_j = 1$ for all the negative in-neighbors j of i in G. We say that f is an *and-or-net* if f_i is a conjunction or a disjunction for all $i \in V$. We say that f is an *and-net* (*or-net*) if f_i is a conjunction (disjunction) for all $i \in V$. If G is simple and has maximum in-degree at most 2, then every BN f on G is an and-or-net.

For all $i \in V$, we denote by f^i the BN with component set V defined as follows: for all configurations x on V, we have $f^i(x)_i = f(x)_i$ and $f^i(x)_j = x_j$ for all $j \in V \setminus \{i\}$. Given a word $w = i_1, \dots, i_\ell$ over V, we set $f^w = f^{i_\ell} \circ f^{i_{\ell-1}} \circ \cdots \circ f^{i_1}$. For convenience, f^ϵ, where ϵ is the empty word, is the identity on $\{0,1\}^V$. Also, if w is a word and $i \in \{w\} \setminus V$ then f^i is the identity on V. In this way, f^w is well defined for any word w. We say that w is a *synchronizing word* for f if f^w is a constant function, and f is *synchronizing* if it has at least one synchronizing word.

The functions f^i define a deterministic finite automaton, called *asynchronous automaton* of f, in a natural way: the alphabet is V, the set of states is $\{0,1\}^V$ and the transition function $\delta : \{0,1\}^V \times [n] \to \{0,1\}^V$ is defined by $\delta(x,i) = f^i(x)$ for all $(x,i) \in \{0,1\}^V \times [n]$. So f is synchronizing if and only if its asynchronous automaton is synchronizing in the usual sense. The *state diagram* of the asynchronous automaton of f is then the labelled digraph $\Gamma(f)$ with vertex set $\{0,1\}^V$ and with an arc from x to $f^i(x)$ labelled by i for all $(x,i) \in \{0,1\}^V \times [n]$. It is a very classical model for the dynamics of gene networks [30, 31]. In this context, $\Gamma(f)$ is called the asynchronous transition graph of f.

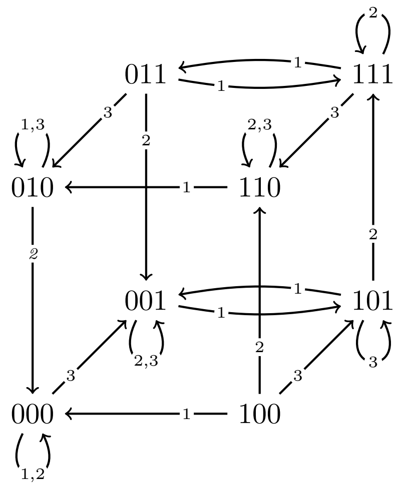
Example 1. *Let G be the following signed digraph, where green arcs are positive and red arcs are negative (this convention is used throughout the paper):*



It is simple and has exactly two cycles, both negative. Let f be the and-net on G, that is: $f_1(x) = \overline{x_1} \wedge x_3$ and $f_2(x) = x_1$ and $f_3(x) = \overline{x_2}$ for all configurations x on $[3]$. We have the following tables:

x	$f(x)$	x	$f^1(x)$	x	$f^2(x)$	x	$f^3(x)$
000	001	000	000	000	000	000	001
001	101	001	101	001	001	001	001
010	000	010	010	010	000	010	010
011	100	011	111	011	001	011	010
100	011	100	000	100	110	100	101
101	011	101	001	101	111	101	101
110	010	110	010	110	110	110	110
111	010	111	011	111	111	111	110

The state diagram of the asynchronous automaton of f is the following:



f is synchronizing, since $w = 231123$ is a synchronizing word for f. Indeed, $f^w(x) = 001$ for all configurations x on $[3]$, as shown below:

$$
\begin{array}{ccccccccccccc}
000 & & & & & & & & & & & & \\
001 & & & & & & & & & & & & \\
010 & & 000 & & & & & & & & & & \\
011 & \xrightarrow{f^2} & 010 & \xrightarrow{f^3} & 001 & \xrightarrow{f^1} & 101 & \xrightarrow{f^1} & 001 & \xrightarrow{f^2} & 001 & \xrightarrow{f^3} & 001 \\
100 & & 110 & & 110 & & 010 & & 010 & & 000 & & \\
101 & & 111 & & & & & & & & & & \\
110 & & & & & & & & & & & & \\
111 & & & & & & & & & & & &
\end{array}
$$

Let h be the or-net on G, that is: $h_1(x) = \overline{x_1} \vee x_3$ and $h_2(x) = x_1$ and $h_3(x) = \overline{x_2}$ for all configurations x on $[3]$. h is synchronizing, since one can check that $w = 231123$ is also a synchronizing word for h: we have $h^w(x) = 110$ for all configurations x on $[3]$. Since there are only two BNs on G (namely f and h), $w = 231123$ is a synchronizing word for every BN on G.

We now prove Proposition 1 given in the introduction, that we restate.

Proposition 1. *Suppose that G is a simple signed digraph without vertices of in-degree 0 or 2. Then at least one BN on G is not synchronizing.*

Proof. Let V be the vertex set of G. Given an arc of G from j to i, we set $a_{ji} = 0$ if this arc is positive and $a_{ji} = 1$ otherwise. For all $i \in V$, let d_i be the in-degree of i in G, and suppose that $d_i \neq 0, 2$. Let f be the BN with component set V defined as follows. Given a configuration x on V, let $h_i(x)$ be the number of in-neighbors j of i with $x_j + a_{ji} = 1$. For each vertex i, we fix an in-neighbor $\phi(i)$ of i and we define f_i as follows: for all configurations x on V,

$$
f_i(x) = \begin{cases} 1 & \text{if } h_i(x) > d_i/2, \\ x_{\phi(i)} + a_{\phi(i)i} & \text{if } h_i(x) = d_i/2, \\ 0 & \text{if } h_i(x) < d_i/2. \end{cases}
$$

Let H be the signed interaction digraph of f, and let us prove that $H = G$. One easily check that H is a subgraph of G. To prove the converse, suppose that G has an arc from j to i. If d_i is odd there is x such that $x_j + a_{ji} = 0$ and $h_i(x) = (d_i - 1)/2$. Thus $f_i(x) < f_i(x + e_j)$ so

H has an arc from j to i which is positive if $a_{ji} = 0$ and negative otherwise. If d_i is even then, since $d_i \geq 4$, there is x such that $x_j + a_{ji} = x_{\phi(i)} + a_{\phi(i)i} = 0$ and $h_i(x) = d_i/2$. We deduce that $f_i(x) < f_i(x+e_j)$ so H has an arc from j to i which is positive if $a_{ji} = 0$ and negative otherwise. Consequently, $H = G$, that is, f is a BN on G.

We now prove that f is not synchronizing. Let x, y be opposite configurations, that is, $x_i \neq y_i$ for all $i \in V$. We have $h_i(y) = d_i - h_i(x)$, thus if $h_i(x) \neq d_i/2$ then $f_i(x) \neq f_i(y)$. Otherwise, $h_i(x) = h_i(y) = d_i/2$ thus $f_i(x) = x_{\phi(i)} + a_{\phi(i)i} \neq y_{\phi(i)} + a_{\phi(i)i} = f_i(y)$. So $f_i(x) \neq f_i(y)$ in every case. Consequently, $f^i(x)$ and $f^i(y)$ are opposite. We deduce that, for any word w, $f^w(x)$ and $f^w(y)$ are opposite, and thus f is not synchronizing. $\square$

2.4 Switches

Let $G = (V, E)$ be a signed digraph. Let $I \subseteq V$ and, for all $i \in V$, let $\sigma_I(i) = 1$ if $i \in I$ and $\sigma_I(i) = -1$ if otherwise. The *I-switch* of G is the signed digraph $G^I = (V, E^I)$ with $E^I = \{(j, i, \sigma_I(j) \cdot s \cdot \sigma_I(i)) \mid (j, i, s) \in E\}$; note that $G^I = G^{V\setminus I}$ and $(G^I)^I = G$. We say that G is *switch equivalent* to H if $H = G^I$ for some $I \subseteq V$. Obviously, G and G^I have the same underlying digraph. Note also that C is a cycle in G if and only if C^I is a cycle in G^I, and C and C^I have the same sign. Thus if G has no positive (negative) cycles then every switch of G has no positive (negative) cycles: this property is invariant by switch. The *symmetric version* of G is the signed digraph $G^s = (V, E^s)$ where $E^s = E \cup \{(i, j, s) \mid (j, i, s) \in E\}$. A basic result concerning switch is the following adaptation of Harary's theorem [17].

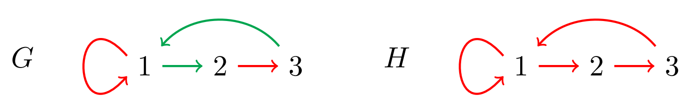
Proposition 3. *A signed digraph G is switch equivalent to a full-positive signed digraph if and only if G^s has no negative cycles.*

There is an analogue of the switch operation for BNs. Let f be a BN with component set V and $I \subseteq V$. The *I-switch* of f is the BN h with component set V defined by $h(x) = f(x+e_I)+e_I$ for all configurations x on V; note that if h is the I-switch of f then f is the I-switch of h. The analogy comes from first point of the following easy property.

Proposition 4. *If h is the I-switch of f, then:*

- *the signed interaction digraph of h is the I-switch of the signed interaction digraph of f;*
- *if f is an and-or-net, then h is an and-or-net;*
- *for any word w, h^w is the I-switch of f^w.*

Example 2. *Let G and H be the following signed digraphs:*



Then H is the $\{2, 3\}$-switch of G. We deduce from the first point of Proposition 4 that every BN on H is the I-switch of a BN on G. As shown in Example 1, $w = 231123$ is a synchronizing word for every BN on G, and we deduce from the third point of Proposition 4 that w is also a synchronizing word for every BN on H.

3 Proof of Theorem 1

In all this section, G is a signed digraph with vertex set V, and $n = |V|$.

3.1 Initial cycles

An initial strong component of G which is isomorphic to a cycle is an *initial cycle* of G. The first case in Theorem 1 is a consequence of the following easy property.

Lemma 2. *If G has an initial cycle, then no BN on G is synchronizing.*

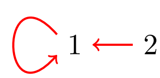
Proof. Suppose that G has an initial cycle with vertex set I, and let f be a BN on G. Each vertex $i \in I$ has a unique in-neighbor in G, say $\phi(i)$, and $\phi(i) \in I$. Then one can easily check that $f_i(x) = x_{\phi(i)} + a_i$ for all configurations x on V, where $a_i = 0$ if the arc from $\phi(i)$ to i is positive, and $a_i = 1$ otherwise. Let x and y be I-opposite configurations, that is, $x_i \neq y_i$ for all $i \in I$. Then, for all $i \in I$, we have $f_i(x) = x_{\phi(i)} + a_i \neq y_{\phi(i)} + a_i = f_i(y)$, thus $f^i(x)$ and $f^i(y)$ are I-opposite configurations. We deduce that $f^w(x)$ and $f^w(y)$ are I-opposite configurations for any word w, and thus f is not synchronizing. $\square$

3.2 Homogeneity

Let us say that G is *and-or-synchronizing* if every and-or-net on G is synchronizing. It remains to prove the second case in Theorem 1, and the difficult part is to show that if G is strong, has no positive cycles, and is not a cycle, then G is and-or-synchronizing. We proceed by induction on the number of vertices and, for inductive purpose, it is convenient to prove something stronger, relaxing the strong connectivity which is difficult to handle during inductive proofs.

If G is no longer strongly connected, then Lemma 2 shows that the condition "is not a cycle" must be replaced by "has no initial cycles". So suppose that G has no positive cycles and no initial cycles. Then G is not necessarily and-or-synchronizing, as shown below, so additional assumptions are needed.

Example 3. *Let G be the following signed digraph:*

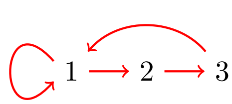


Let f be the and-net on G: $f_1(x) = \overline{x_1} \wedge \overline{x_2}$ and $f_2(x) = 1$ for all configurations x on $[2]$. For $X = \{01, 11\}$ we have $f^1(X) = X$ and $f^2(X) = X$. Hence, for any word w we have $f^w(X) = X$ thus f is not synchronizing.

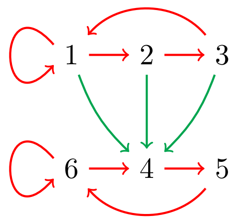
In the previous example, the source can be fixed to the state 1, and the network then behaves as an "isolated" negative cycle of length one (since when $x_2 = 1$ we have $f_1(x) = \overline{x_1}$) which is not synchronizing (Lemma 2).

To prevent this type of phenomenon, we can assume that G has, in addition, no sources. But this is not enough, as shown by the next example. Actually, it shows that if G has no positive cycles, no sources and no initial cycles, then G is not necessarily and-or-synchronizing, even if each strong component of G is and-or-synchronizing. Thus we have to impose some additional conditions about the connections between the strong components of G.

Example 4. *Let H be the following signed digraph, which is strong, has no positive cycles and which is not a cycle:*



By the second case of Theorem 1, H is and-or-synchronizing (see Example 2). Let G be the following signed digraph, which has no positive cycles, no sources, and no initial cycles:



The two strong components of G are and-or-synchronizing, since there are isomorphic to H. However, G is not and-or-synchronizing. Indeed, let f be the and-or-net such that all components of f are conjunctions, excepted f_6 which is a disjunction. Let X be the set of configurations x on $[6]$ with $x_i = 0$ for some $i \in [3]$, $x_4 = 0$ and $x_5 = 1$. One can easily check that $f^i(X) \subseteq X$ for every $i \in [6]$. Furthermore, given $x \in X$, since $x_5 = 1$ we have $f_6(x) = \overline{x_5} \vee \overline{x_6} = \overline{x_6}$. Consequently, given $x, y \in X$ with $x_6 \neq y_6$, we have $f^i(x), f^i(y) \in X$ and $f^i_6(x) \neq f^i_6(y)$ for every $i \in [6]$, and we deduce that $f^w_6(x) \neq f^w_6(y)$ for any word w. Hence f is not synchronizing.

To express the additional conditions between the strong components of G we need some definitions. We say that a path P of G of is *forward* if P has at least one arc and the two vertices of the last arc belong to distinct strong components. Given a vertex $i \in V$, we say that G is *i-homogenous* if each strong component of G contains a vertex j such that all the forward paths from j to i have the same sign (we obviously, if there is no forward paths from j to i, then all the forward paths from j to i have the same sign). We say that G is *homogenous* if it is i-homogenous for every $i \in V$.

For instance, if G is strong, then there is no forward paths, and G is trivially homogenous, so homogeneity is indeed a relaxation of strong connectivity. Also, the 6-vertex signed digraph of the previous example (which is not and-or-synchronizing) is not homogenous since, for $i = 1, 2, 3$, it has a forward positive path and a forward negative path from i to 4.

It appears that homogeneity works and is well adapted for a proof by induction. So our aim is now to prove the following theorem, which generalizes the second case of Theorem 1.

Theorem 4. *Suppose that G is homogenous and has no positive cycles, no sources and no initial cycles. Then every and-or-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$.*

Thus Theorem 1 is a consequence of Lemma 2 (first case) and Theorem 4 (second case).

A useful observation is that the conditions of Theorem 4 are invariant by switch. Indeed, we already mentioned that the absence of positive cycles is invariant by switch, and the following observation is straightforward to check.

Lemma 3. *If G is homogenous, then every switch of G is homogenous.*

3.3 Flipping a vertex

The main tool is the following generalization of Lemma 1, obtained by replacing the strong connectivity of G by the weaker assumption that G is i-homogenous and without sources.

Lemma 4. *Suppose that G is i-homogenous and has no positive cycles and no sources. Let f be a BN on G. For every configuration x on V, there is a word w such that $f^w(x)_i \neq x_i$.*

A shortest word w with the above property is obviously of length $|w| \leq 2^{n-1}$ (the number of configurations y with $y_i = x_i$), but we think that $|w|$ is sub-linear according to the number 2^n of configurations. We obtained such a sub-linear bound when f is an and-or-net: $|w| < F(n+2)$ where $F(0), F(1), F(2) \dots$ is the Fibonacci sequence.

The word w in Lemma 4 is basically a concatenation of *canalizing words*, defined as follows. Given a BN f on G, a vertex $i \in V$ and $a \in \{0,1\}$, a *canalizing word from* (i,a) is a word w over $V \setminus i$ without repeated letters such that, for some configuration b on $\{w\}$ we have:

$$\forall x \in \{0,1\}^V, \qquad x_i = a \;\Rightarrow f^w(x)_{\{w\}} = b.$$

If w is canalizing from (i,a), the configuration b with the above property is called the *image* of w, and for each $j \in \{w\}$, we say that j is *canalized* to b_j by w. By convention, the empty word is always a canalizing word.

Example 5. *Let f be the and-net on G. Take $i \in V$ and let I be the set of positive out-neighbors of i distinct from i. For any configuration x on V, if $x_i = 0$ then $f_j(x) = 0$ for all $j \in I$. We deduce that any permutation of I is a canalizing word from $(i,0)$, which canalizes each member of I to 0. Similarly, if J is the set of negative out-neighbors of i distinct from i, then any permutation of J is a canalizing word from $(i,1)$, which canalizes each member of J to 0. We deduce that if i has at least one out-neighbor distinct from i, then there is a canalizing word from $(i,0)$ or $(i,1)$ which is of length at least one (this basic property is crucial for the bound of Lemma 5 below). Suppose now that G has a full-positive path P of length $\ell \geq 1$ with vertices $i_0, i_1, \dots, i_\ell$ in order. Then $w = i_1 i_2 \dots i_\ell$ is a canalizing word from $(i_0, 0)$ which canalizes i_k to 0 for $1 \leq k \leq \ell$.*

The following bound on the length of the word w in Lemma 4, which works for and-or-nets, is suited for a proof by induction; it implies the bound $|w| < F(n+2)$ announced above since $F(n+2-m) + m \leq F(n+2)$ for all $0 \leq m \leq n$; this property will be used many times.

Lemma 5. *Suppose that G is i-homogenous and has no positive cycles and no sources. Let f be an and-or-net on G. Let x be a configuration on V and let v be a longest canalizing word from (i, x_i). There is a word w such that $f^w(x)_i \neq x_i$ and*

$$|w| < F(n - |v| + 2) + |v|.$$

We need the following lemma. It formalizes the intuitive fact that canalizations from (i,a) operate through the paths of G starting from i.

Lemma 6. *Let f be a BN on G, $i \in V$ and $a \in \{0,1\}$. Let w be a canalizing word from (i,a) with image b, and suppose that no sources of G are in $\{w\}$. For every $j \in \{w\}$, G has a path from i to j, whose internal vertices are in $\{w\}$, which is positive if $a = b_j$ and negative otherwise.*

Proof. We proceed by induction on $|w|$. If $|w| = 0$ there is nothing to prove. Suppose that $|w| \geq 1$. Let j be the last letter of w, and let v be obtained from w by removing the last letter (thus v is the empty word if $|w| = 1$). For convenience we set $b_i = a$ and $I = \{i\} \cup \{v\}$. We also write $\sigma(0) = -1$ and $\sigma(1) = 1$. Hence, we have to prove that G has a path from i to j of sign $\sigma(b_i) \cdot \sigma(b_j)$ whose internal vertices are in $\{w\}$.

Since j is not a source, there is a configuration x on V such that $f_j(x) \neq b_j$. Let $z = x$ if $x_i = a$ and $z = x + e_i$ otherwise, so that $z_i = a$ in any case. Let $y = f^v(z)$. Since $z_i = a$ we have $f_j(y) = f^w(z)_j = b_j$ and $y_k = f^w(z)_k = b_k$ for $k \in I$. Thus $f_j(y) \neq f_j(x)$ and we deduce that j has an in-neighbor k in G of sign $(y_k - x_k) \cdot (f_j(y) - f_j(x)) = \sigma(y_k) \cdot \sigma(f_j(y)) = \sigma(y_k) \cdot \sigma(b_j)$. Since x and y can only differ in components inside I, we have $k \in I$, and we deduce that the sign of the arc from k to j is $\sigma(b_k)\cdot\sigma(b_j)$. If $k = i$ we are done: the desired path is the arc from i to j. Otherwise, $k \in \{v\}$. Since v is a canalizing word from (i, a) with image $b_{\{v\}}$, by induction, G has a path from i to k of sign $\sigma(b_i)\cdot\sigma(b_k)$ whose internal vertices are in $\{v\}$. By adding the arc from k to j to this path, we obtain a path of sign $\sigma(b_i)\cdot\sigma(b_k)\cdot\sigma(b_k)\cdot\sigma(b_j) = \sigma(b_i)\cdot\sigma(b_j)$ from i to j whose internal vertices are in $\{w\}$. This completes the induction step. □

We are now ready to prove Lemmas 4 and 5 together. The sketch is the following. Suppose that the initial state of i is 0, that is, $x_i = 0$. So we want a word w such that $f^w(x)_i = 1$, that is, a word increasing the state of i.

We first suppose that i is of out-degree zero. Since G has no sources, it has is a non-trivial initial strong component S. Since G is i-homogenous and i is of out-degree zero, there is a vertex j in S such that all the paths from j to i have the same sign, say positive. Then j can be seen as an activator of i, so in order to increase the state of i, it is preferable to put j in state 1. This is possible by induction on S, since S is j-homogenous. So we consider a word u that puts j in state 1. Then we consider a longest canalizing word v from $(j, 1)$. If i is canalized by v, then it necessarily gets the state 1, since all the paths from j to i are positive, and we are done: starting from x, the word uv increases the state of i. Otherwise, we "remove" j and the vertices canalized by v. We obtain a "subnetwork" whose interaction graph H satisfies all the hypotheses. Using induction, we get, for the subnetwork, a word w that increases the state of i and we are done: starting from x, the word uvw increases the state of i. We then prove that $|uvw| \leq F(n+2)$ when f is an and-or-net (which is the bound of the statement since i is of out-degree zero). The argument is roughly the following. The induction used to obtain u is made on S, which has at most $n-1$ vertices, and thus $|u| < F(n+1)$. The induction used to obtained w is made on H, which has $n-1-|v|$ vertices, and we easily obtain $|vw| < F(n+1)$. So if $|u| \leq 1$ then $|uvw| < F(n+2)$ as desired. If $|u| \geq 2$ then it means, and this is the key point, that there is a *non-empty* canalizing word from either $(j, 0)$ or $(j, 1)$ and, by carefully analyzing the inductive calls, we obtain $|u| \leq F(n)$ in the first case, and $|vw| \leq F(n)$ in the second. So $|uvw| < F(n) + F(n+1) = F(n+2)$ in any case.

We then suppose that i is of out-degree at least one. We consider a longest canalizing word v from $(i, 0)$. If the word vi increases the state of i then we are done since $|v|+1 < F(n-|v|+2)+|v|$. Otherwise, we "remove" the vertices canalized by v from the network, and we "remove" the outgoing arcs of i, considering that each out-neighbor of i behaves as if the state of i was permanently 0. We obtain a "subnetwork" whose interaction graph H satisfies all the hypotheses, and in which i is of out-degree zero. By the first case, we get, for the subnetwork, a word w that increases the state of i and we are done: starting from x, the word vw increases the state of i. Suppose now that f is an and-or-net. Since H has $n-|v|$ vertices we have $|w| < F(n-|v|+2)$ by the first case, so $|vw| < F(n-|v|+2)+|v|$ as desired.

We now proceed to the details.

Proof of Lemmas 4 and 5. Suppose that G is i-homogenous and has no positive cycles and no sources. Let f be a BN on G. Let x be a configuration on V. We proceed by induction on the number n of vertices in G. If $n = 1$ the result is obvious: i is the unique vertex and has a negative loop, so $f(x) \neq x$; hence we can take $w = i$ and since $1 < F(3) = 2$ we are done. So suppose that $n \geq 2$. We consider two cases.

Case 1: *i* is of out-degree zero. We have to prove that w exists, and $|w| < F(n+2)$ if f is an and-or-net (since i is of out-degree zero, the longest canalizing word from (i, x_i) is the empty word). Since i is of in-degree at least one, G contains an initial component S from which i is reachable. Since i is of out-degree zero, all the paths from S to i are forward. Since G is i-homogenous, there is a vertex j in S such that all the paths from j to i have the same sign.

Let $a = x_i + 1$ if all the paths from j to i are positive, and $a = x_i$ otherwise. Let m be the maximal size of a canalizing word from (j, x_j) with only letters in S. We first prove, using the induction hypothesis, the following.

(1) *There is a word u such that $f^u(x)_j = a$, and if f is an and-or-net then*

$$|u| < F(n - m + 1) + m \leq F(n + 1).$$

This is obvious if $x_j = a$ (we take u the empty word) so suppose that $x_j \neq a$. Let J be the vertex set of S and $I = V \setminus J$. Let h be the BN with component set J defined by $h(y_J) = f(y)_J$ for all configurations y on V such that $y_I = x_I$. Since G has no arc from I to J, one can easily check that S is the signed interaction digraph of h. Since S is strong, S is j-homogenous. Since S is a subgraph of G, it has no positive cycles, and since G has no sources, S has no sources (that is, S is non-trivial). Because S has at most $n-1$ vertices, by induction, there is a word u over J such that $h^u(x_J)_j = a$ and we can then easily check that $f^u(x)_j = a$. If f is an and-or-net then h is also an and-or-net, and since m is the maximal length of a canalizing word from (j, x_j) in h, by induction we obtain $|u| < F(|J| - m + 2) + m$. Since $|J| \leq n - 1$ this proves (1).

Let u be a shortest word as in (1). Let v be a longest canalizing word from (j, a) and b its image.

(2) *If $i \in \{v\}$ then $f^{uv}(x)_i \neq x_i$ and, if f is an and-or-net, then $|uv| < F(n + 2)$.*

Suppose that $i \in \{v\}$. By Lemma 6, G has a path P from j to i which is positive if and only if $a = b_i$. By the definition of a, we have $a \neq x_i$ if and only if P is positive. Thus $x_i \neq b_i$ and since $f^u(x)_j = a$, we have $f^{uv}(x)_i = b_i \neq x_i$. If f is an and-or-net, then $|u| < F(n+1)$ by (1) and since $|v| \leq n - 1 \leq F(n)$ we obtain $|uv| < F(n + 2)$. This proves (2).

By (2) we can suppose that $i \notin \{v\}$. Let b' be the configuration on $I = \{j\} \cup \{v\}$ such that $b'_j = a$ and $b'_{\{v\}} = b$. Note that, for all configurations y on V with $y_j = a$ we have $f^v(y)_I = b'$. In particular, since $f^u(x)_j = a$, we have $f^{uv}(x)_I = b'$.

Let h be the BN with component set $J = V \setminus I$ defined by $h(y_J) = f(y)_J$ for all configurations y on V such that $y_I = b'$. Let H be the signed interaction digraph of h.

(3) *H has no positive cycles and no sources.*

One easily check that H is a subgraph of G, and thus it has no positive cycles. Suppose, for a contradiction, that H has a source ℓ. Then it means that, for some c, we have $h_\ell(y_J) = f_\ell(y) = c$ for all configurations y on V with $y_I = b'$. But then, for all configurations y on V with $y_j = a$, we have $f^{v,\ell}(y)_I = f^v(y)_I = b'$ since $\ell \notin I$ and thus $f^{v,\ell}(y)_\ell = f_\ell(f^v(y)) = c$. Thus v, ℓ is a canalizing word from (j, a) longer than v, a contradiction. This proves (3).

(4) *H is i-homogenous.*

Suppose, for a contradiction, that H is not i-homogenous. Then it has a strong component F such that, for every vertex ℓ in F, H has both a positive path and a negative path from ℓ to i. Since G is i-homogenous, F is not a strong component of G and thus G has an arc from some vertex k not in F to some vertex ℓ in F. Let y, z be two configurations on V that only differ in component k such that $f(y)_\ell \neq f(z)_\ell$. If G has no arc from I to ℓ, then we can choose y, z in such a way that $y_I = z_I = b'$. But then $h(y_J)_\ell \neq h(z_J)_\ell$ thus H has an arc from k to ℓ, a contradiction. Hence G has an arc from some vertex $k' \in I$ to ℓ. By Lemma 6, G has a path from j to k' whose vertices are all in I, and thus it has a path P from j to ℓ whose vertices are in I excepted ℓ. Since ℓ is in F, H has a positive path P^+ from ℓ to i and a negative path P^- from ℓ to i. Then $P \cup P^+$ and $P \cup P^-$ are paths of G from j to i of opposite signs, and this contradicts our choice of j. This proves (4).

From (3) and (4) there is, by induction, a word w over J with $h^w(f^{uv}(x)_J)_i \neq x_i$. Since $f^{uv}(x)_I = b'$, and w is a word over J, we have

$$f^{uvw}(x)_J = f^w(f^{uv}(x))_J = h^w(f^{uv}(x)_J)$$

thus $f^{uvw}(x)_i \neq x_i$.

It remains to prove the upper bound on the length of uvw when f is an and-or-net. So suppose that f is an and-or-net. Then h is also an and-or-net, and since H has $n - |v| - 1$ vertices, we have $|w| < F(n - |v| + 1)$ by induction, and thus

$$|vw| < F(n - |v| + 1) + |v| \leq F(n + 1). \tag{5}$$

If $|u| \leq 1$ then $|uvw| < F(n+1) + 1 \leq F(n+2)$ as desired. So suppose that $|u| \geq 2$. Since u is as short as possible, we have $x_j \neq a$ and S has at least two vertices, so j has an out-neighbor in S distinct from j, say k. Since f_k is a conjunction or a disjunction, the single letter k is a canalizing word from (j, c) for some $c \in \{0, 1\}$. If $c = x_j$ then we deduce that $m \geq 1$, so $|u| \leq F(n)$ by (1), and with (5) we obtain $|uvw| < F(n+2)$. If $c = a$ then we deduce that $|v| \geq 1$, so $|vw| \leq F(n)$ by (5) and with (1) we obtain $|uvw| < F(n+2)$. Hence the desired bound is always obtained. This proves the first case.

Case 2: *i is of out-degree at least one.* Let $a = x_i$ and let v be a longest canalizing word from (i, a). If $f^{v,i}(x)_i \neq a$ then we are done since $|v, i| \leq n$ and $n < F(n - m + 2) + m$ for all $0 \leq m < n$. So we assume that $f^{v,i}(x)_i = a$. Let b be the image of v and $I = \{v\}$.

(1) *There is no c such that $f_i(y) = c$ for all configurations y on V with $y_i = a$ and $y_I = b$.*

Suppose, for a contradiction, that $f_i(y) = c$ for all configurations y on V with $y_i = a$ and $y_I = b$. Let y be a configuration on V with $y_i = a$ such that, for all positive (negative) in-neighbors j of i with $j \notin I$ we have $y_j \neq a$ ($y_j = a$); if $j = i$ then j is a negative in-neighbor of i since G has only negative cycles, and we have already imposed $y_j = a$. Suppose that i has a positive in-neighbor $j \in I$. Since G has only negative cycles, all the paths from i to j are negative. Hence, by Lemma 6, we have $a \neq b_j$. Thus, for *every* positive in-neighbor j of i we have $f^v(y)_j \neq a$, because $f^v(y)_j = y_j \neq a$ if $j \notin I$ and $f^v(y)_j = b_j \neq a$ otherwise. We prove similarly that, for every negative in-neighbor j of i, we have $f^v(y)_j = a$. Hence $f^v(y)$ is an i-unstable configuration in G, and since i is not a source, by Proposition 2 we have $f_i(f^v(y)) \neq a$. Since $f^v(y)_i = y_i = a$ and $f^v(y)_I = b$, we deduce that $c \neq a$. Now, since $f^v(x)_i = x_i = a$ and $f^v(x)_I = b$, we have $f_i(f^v(x)) = c \neq a$ and thus $f^{v,i}(x)_i \neq a$, which contradicts our hypothesis. This proves (1).

Given a configuration y on V we denote by $\tilde{y}$ the configuration on V such that $\tilde{y}_i = a$ and $\tilde{y}_j = y_j$ for all $j \neq i$. Let h be the BN with component set $J = V \setminus I$ such that $h(y_J) = f(\tilde{y})_J$ for all configurations y on V with $y_I = b$. Let H be the signed interaction digraph of h. Note that H is a subgraph of G in which i is of out-degree zero.

(2) *H has no positive cycles and no sources.*

Since H is a subgraph of G, it has no positive cycles. Suppose, for a contradiction, that H has a source k. Then there is c such that $h(y_J)_k = c$ for all configurations y on V. So if $y_i = a$ and $y_I = b$ then $f(y)_J = h(y_J)$ and thus $f(y)_k = h(y_J)_k = c$. For all configurations y on V with $y_i = a$, we have $f^v(y)_i = a$ and $f^v(y)_I = b$ and we deduce that $f^{v,k}(y)_k = c$. By (1) we have $k \neq i$ and since $k \notin I$ we deduce that v, k is a canalizing word from (i, a) longer than v, a contradiction. This proves (2).

(3) *H is i-homogenous.*

Suppose, for a contradiction, that H is not i-homogenous. Then it has a strong component F such that, for any vertex ℓ in F, H has both a positive path and a negative path from ℓ to i. Since G is i-homogenous, F is not a strong component of G and thus G has an arc from some vertex k not in F to some vertex ℓ in F. Let y, z be two configurations on V that only differ in component k such that $f(y)_\ell \neq f(z)_\ell$. If G has no arc from $I \cup \{i\}$ to ℓ, then we can choose y, z in such a way that $y_I = z_I = b$ and $y_i = z_i = a$. But then $h(y_J)_\ell = f(y)_\ell \neq f(z)_\ell = h(z_J)_\ell$ thus H has an arc from k to ℓ, a contradiction. Hence G has an arc from some vertex $k' \in I \cup \{i\}$ to ℓ. By Lemma 6, G has a path from i to k' whose internal vertices are in I (of length zero if $k = i$), and thus it has a path P from i to ℓ whose internal vertices are in I. Since ℓ is in F, H has a positive path P^+ from ℓ to i and a negative path P^- from ℓ to i. Then $P \cup P^+$ and $P \cup P^-$ are cycles of G of opposite signs, so G has a positive cycle, a contradiction. This proves (3).

Since i is of out-degree zero in H, we deduce from (2), (3) and the first case that there is a shortest word w over J such that $h^w(z_J)_i \neq a$ where $z = f^v(x)$. Since w is of minimal length for this property, i appears only once in w, in last position: $w = w', i$ for some word w' not containing i. Since $z_I = b$, $z_i = a$ and w' does not contain any letter in $I \cup \{i\}$, we deduce, setting $y = f^{w'}(z)$, that $y_J = h^{w'}(z_J)$. Furthermore, $y_I = b$ and $y_i = a$ so $f(y)_J = h(y_J)$. Consequently,

$$f^{vw}(x)_i = f(y)_i = h(y_J)_i = h(h^{w'}(z_J))_i = h^w(z_J)_i \neq a.$$

It remains to prove the upper bound on the length of vw when f is an and-or-net. So suppose that f is an and-or-net. Then h is also an and-or-net, and since H has $n - |v|$ vertices, we have $|w| < F(n-|v|+2)$ by the first case, and thus $|vw| < F(n-|v|+2)+|v|$ as desired. This proves the second case. □

Remark 1. *Suppose that G is strong, non-trivial, and has no positive cycles. Let f be any BN on G. It is proved in [27] that the state diagram $\Gamma(f)$ of the asynchronous automaton of f has a unique terminal strong component A. Furthermore, it is proved in [3] that f has no fixed points, and thus A contains at least 2 configurations. But Lemma 4 says something stronger: A contains at least $n+1$ configurations. Indeed, for every configuration x in A and $i \in V$ we have $f^w(x)_i \neq x_i$ for some word w, and since A is a terminal strong component of $\Gamma(f)$, we have $f^w(x) \in A$. Taking w as short as possible, i is the last letter of w, so $w = v, i$ for some word v, and $f^v(x)_i \neq f^w(x)_i$. Since A is a terminal strong component of $\Gamma(f)$, the configurations $f^v(x)$ and $f^w(x)$ are both contained in A, and thay only differ in component i. This shows that, for every $i \in V$, there are two configurations in A which differ only in component i and this implies that A contains at least $n+1$ configurations (see Appendix C for details). We conjecture that the right lower bound is $2n$.*

3.4 Synchronizing a vertex

Suppose that G is i-homogenous and has no positive cycles and no sources. Let f be the and-net on G. In this subsection, we use Lemma 5 to prove a "local" synchronization: given two configurations x, y, there is a word w synchronizing i at state 0, that is, such that $f^w(x)_i = f^w(y)_i = 0$. This will be the starting point for the "global" synchronization given in the next subsection. The argument is roughly the following. If $x_i = y_i = 0$ there is nothing to prove: we can take $w = \epsilon$. So suppose, without loss of generality, that $x_i = 1$. The key points can be explain by assuming that i has exactly two in-neighbors, say j and ℓ, both positive and distinct from i. An easy application of Lemma 5 shows that there is a word u, not containing i, such that $f^u(x)$ is an i-unstable configuration in G. Hence, setting $x' = f^u(x)$ and $y' = f^u(y)$ we have the following situation:

$$\begin{tikzcd}[row sep=small]
x'_j = 0 \arrow[dr] & & y'_j = ? \arrow[dr] & \\
& x'_i = 1 & & y'_i = ? \\
x'_\ell = 0 \arrow[ur] & & y'_\ell = ? \arrow[ur] &
\end{tikzcd}$$

Hence $f_i(x') = 0$ and if $f_i(y') = 0$ then we are done with $w = v, i$. So suppose that $f_i(y') = 1$. Since f_i is a conjunction, this implies $y'_j = y'_\ell = 1$, so we have the following situation:

$$\begin{tikzcd}[row sep=small]
x'_j = 0 \arrow[dr] & & y'_j = 1 \arrow[dr] & \\
& x'_i = 1 & & y'_i = ? \\
x'_\ell = 0 \arrow[ur] & & y'_\ell = 1 \arrow[ur] &
\end{tikzcd}$$

Setting $x'' = f^i(x')$ and $y'' = f^i(y')$ we obtain a clear situation:

$$\begin{tikzcd}[row sep=small]
x''_j = 0 \arrow[dr] & & y''_j = 1 \arrow[dr] & \\
& x''_i = 0 & & y''_i = 1 \\
x''_\ell = 0 \arrow[ur] & & y''_\ell = 1 \arrow[ur] &
\end{tikzcd}$$

By Lemma 5, there is a shortest word v such that $f^v(y'')_i = 0$. Since $y''_i = 1$, v has a shortest prefix $\tilde{v}$ whose last letter is j or ℓ. Suppose that this is ℓ. Then i and j do not appear in $\tilde{v}$ and thus, setting $x''' = f^{\tilde{v}}(x'')$ and $y''' = f^{\tilde{v}}(y'')$, we obtain the following situation:

$$\begin{tikzcd}[row sep=small]
x'''_j = 0 \arrow[dr] & & y'''_j = 1 \arrow[dr] & \\
& x'''_i = 0 & & y'''_i = 1 \\
x'''_\ell = ? \arrow[ur] & & y'''_\ell = 0 \arrow[ur] &
\end{tikzcd}$$

Since $x'''_j = y'''_\ell = 0$ we have $f_i(x''') = f_i(y''') = 0$, thus $w = u, i, \tilde{v}, i$ has the desired properties. Furthermore, from the bound given in Lemma 5, we obtain $|u|, |v| < F(n+2) - 1$ so that $|w| < 2F(n+2)$. The case where i has more than two in-neighbors, with possibly some negative, is very similar. To treat the case where i has a loop or is of in-degree one, some additional easy arguments have to be given.

We now proceed to the details, starting with the full statement of the "local" synchronization.

Lemma 7. *Suppose that G is i-homogenous and has no positive cycles and no sources. Suppose also that G has a positive path P from a vertex j to i such that j is of in-degree at least two, and all the vertices of P distinct from j are of in-degree one (if i is of in-degree at least two, this path always exists and is the trivial graph whose unique vertex is i). Let f be the and-net on G and let x, y be configurations on V. There is a word w such that $f^w(x)_i = f^w(y)_i = 0$ and*

$$|w| < 2F(n+2).$$

We need the following lemma, which shows that any configuration can be sent by some word on an i-unstable configuration.

Lemma 8. *Suppose that G is i-homogenous and has no positive cycles and no sources. Let f be an and-or-net on G, let x be a configuration on V, and let r be the maximum length of a canalizing word from (i, x_i). There is a word w, which does not contain i, such that $f^w(x)$ is an i-unstable configuration in G and*

$$|w| < F(n-r+1)+r-1.$$

Proof. Suppose that $x_i = 0$. Let h be the and-or-net on G such that $h_j = f_j$ for all $j \neq i$ and h_i is a conjunction. By Lemma 5, there is a shortest word v such that $h^v(x)_i = 1$ and $|v| < F(n-r+1)+r$ (because r does not depend on f_i). Since v is a shortest word, i appears once in w, in last position. Let w obtained from v by removing the last letter i. Then $h_i(h^w(x)) = 1$, and since h_i is a conjunction, we deduce that $h^w(x)$ is an i-unstable configuration in G. Since $i \notin \{w\}$ and $h_j = f_j$ for $j \neq i$, we have $h^w(x) = f^w(x)$. Thus w has the desired properties. If $x_i = 1$ the proof is similar, excepted that h_i is a disjunction. □

Proof of Lemma 7. If $x_i = y_i = 0$ there is nothing to prove, so suppose, without loss of generality, that $x_i = 1$. Let r be the maximal length of a canalizing word from $(i, 1)$. We prove that there is a word w such that $f^w(x)_i = f^w(y)_i = 0$ with

1. $|w| \leq 2F(n-r+2)+2r-2$ if i is of in-degree at least two,
2. $|w| \leq 2F(n+2)-1$ if i is of in-degree one.

Suppose first that i is of in-degree at least two. By Lemma 8 there is a word u, which does not contain i, such that $f^u(x)$ is an i-unstable configuration in G and

$$|u| \leq F(n-r+2)+r-2.$$

Let $x' = f^u(x)$ and $y' = f^u(y)$. Since x' is i-unstable, we have $f(x')_i = 0$. Hence if $f(y')_i = 0$ we are done by taking $w = u, i$. So suppose that $f(y')_i = 1$. Let $x'' = f^i(x')$ and $y'' = f^i(y')$. Hence we have $x''_i = 0$ and $y''_i = 1$. We consider two cases. Suppose first that i has a loop. Then it is negative thus $f(y'')_i = 0$. Let j be an in-neighbor of i distinct from i. Then $x''_j = x'_j$ and since x' is i-unstable, we have $x'_j = 0$ if j is a positive in-neighbor of i, and $x'_j = 1$ otherwise, and we deduce that $f(x'')_i = 0$ (because f_i is a conjunction). So we are done with $w = u, i, i$. For the second case, suppose that i has no loop. By Lemma 5, there is a shortest word v such that $f^v(y'')_i = 0$ and

$$|v| \leq F(n-r+2)+r-1.$$

Since $y''_i = 1$ and v is as short as possible, i is the last letter of v and at least one in-neighbor of i appears in v. Let ℓ be the first in-neighbor of i that appears v, and let $\tilde{v}$ the shortest prefix of v containing ℓ. Let $x''' = f^{\tilde{v}}(x'')$ and $y''' = f^{\tilde{v}}(y'')$. We have $y'''_\ell \neq y''_\ell$ since v is as short as possible and ℓ appears once in $\tilde{v}$. Furthermore, $y''_\ell = y'_\ell$ since $\ell \neq i$ (because i has no loop). So $y'''_\ell \neq y'_\ell$, and since $f(y')_i = 1$, we deduce that $f(y''')_i = 0$ (because f_i is a conjunction). Let j be an in-neighbor of i distinct from ℓ. Since j does not appear in $\tilde{v}$ and $j \neq i$ (because i has no loop), we have $x'''_j = x''_j = x'_j$. Since x' is an i-unstable configuration in G, we have $x'_i = 0$ if j is a positive in-neighbor of i, and $x'_i = 1$ otherwise. Consequently, $f(x''')_i = 0$ (because f_i is a conjunction). So we are done with $w = u, i, \tilde{v}, i$, observing that $|\tilde{v}| \leq |v| - 1$.

Suppose now that i is of in-degree one, and let P be as in the statement. Let s be the maximal length of a canalizing word from $(j, 1)$. Since G is i-homogenous, it is also j-homogenous and, by the first case, there is word u such that $f^u(x)_j = f^u(y)_j = 0$ and $|u| \leq 2F(n-s+2)+2s-2$. Let v be an enumeration of the vertices of $P \setminus j$ in order. Since all the vertices of $P \setminus j$ are of in-degree one, v is a canalizing word from $(j,1)$ and $(j,0)$. Thus $|v| \leq s$ and, since P is positive and $f^u(x)_j = f^u(y)_j = 0$, we obtain $f^{uv}(x)_i = f^{uv}(y)_i = 0$ (by Lemma 6). Furthermore,

$$|uv| \leq 2F(n-s+2) + 3s - 2.$$

So it is sufficient to prove that, for all $n \geq 1$ and $0 \leq s < n$,

$$2F(n-s+2) + 3s - 2 \leq 2F(n+2) - 1.$$

We proceed by induction on n. The case $n = 1$ is obvious, so suppose that $n \geq 2$. If $s = n-1$ then the inequality to prove becomes $3n \leq 2F(n+2)$ which is easy to check. So suppose that $0 \leq s \leq n-2$. Then, using the induction for the first inequality, we obtain

$$\begin{array}{rcl}
2F(n-s+2) + 3s - 2 & = & 2F((n-1) - s + 3) + 3s - 2 \\
& = & 2F((n-1) - s + 1) + 2F((n-1) - s + 2) + 3s - 2 \\
& \leq & 2F((n-1) - s + 1) + 2F((n-1) + 2) - 1 \\
& \leq & 2F(n) + 2F(n+1) - 1 \\
& = & 2F(n+2) - 1,
\end{array}$$

completing the induction. □

3.5 Synchronizing two configurations

Suppose that G is i-homogenous and has no positive cycles, no sources and no initial cycles. Let f be the and-net on G. In this subsection, we use Lemmas 5 and 7 to prove a more "global" synchronization property: given two configurations x, y, there is a word w synchronizing x and y, that is, such that $f^w(x) = f^w(y)$. As shown in the next subsection, this easily implies that every and-or-net on G is synchornizing. The argument is roughly the following. Since G has no positive cycles, it has a vertex i_1 with only negative out-neighbors. By Lemma 7, there is a word v^1 of length at most $2F(n+2)$ synchronizing i_1 at state 0. We then consider the "subnetwork" f' obtained by fixing i_1 at state 0 and removing i_1. Since i_1 has only negative out-neighbors, the signed interaction digraph of f' is $G \setminus i_1$. If $G \setminus i_1$ still satisfies all the conditions, then we can synchronize a new vertex i_2, with only negative out-neighbors, with a word v^2 of length at most $2F(n+1)$. Repeating this argument, we possible synchronize all components, obtaining a word $w = v^1 v^2 \dots$ with $f^w(x) = f^w(y)$ and $|w| \leq 2F(n+2) + 2F(n+1) + \dots + 2F(3)$. Note that $|w| \leq 2F(n+4) - 2$ since the sum of the n first Fibonacci numbers is $F(n+2) - 1$. However, this synchronizing process cannot be always completed, since $G \setminus i_1$ does not necessarily satisfy the appropriate conditions: it can have initial cycles. But these initial cycles are "controlled" by i_1 and, with fastidious arguments involving Lemmas 5, we can construct a word $\tilde{v}^1$ of length $|\tilde{v}^1| \leq 3F(n+2)$ synchronizing both i_1 and the vertices that belongs to the initial cycles of $G \setminus i_1$. From this point, we can "remove" i_1 and continue the synchronizing process without trouble. We eventually get a word $w = \tilde{v}^1 \tilde{v}^2 \dots$ such that $f^w(x) = f^w(y)$ and $|w| \leq 3F(n+4) - 3$.

Lemma 9. *Suppose that G is homogenous and has no positive cycles, no sources and no initial cycles. Let f be the and-net on G. For every configurations x, y on V, there is a word w such that $f^w(x) = f^w(y)$ and*

$$|w| \leq 3F(n+4) - 3.$$

Proof. For inductive purpose, we prove the following claim. For $0 \leq m < n$, let

$$g(n,m) = \sum_{t=m+1}^{n} 3F(t+2).$$

Claim: *Suppose that G is homogenous and has no positive cycles and no sources. Let I be the set of vertices of G that belong to an initial cycle of G. Let f be the and-net on G. For every configurations x, y on V with $x_I = y_I$, there is a word w such that $f^w(x) = f^w(y)$ and*

$$|w| \leq g(n, |I|).$$

This implies the statement since if G has no initial cycles then I is empty, thus $x_I = y_I$ is always true, and we obtain

$$|w| \leq g(n,0) = \sum_{t=1}^{n} 3F(t+2) \leq 3\sum_{t=1}^{n+2} F(t) = 3F(n+4) - 3,$$

where we use, for the last equality, the fact that $\sum_{t=1}^{n} F(t) = F(n+2) - 1$ for all $n \geq 1$, which is easy to prove by induction on n.

The claim is proved by induction on $n - |I|$, that is, the number of vertices that do not belong to an initial cycle. If $n - |I| = 0$ then $I = V$ so $x_I = y_I$ means $x = y$ and there is nothing to prove (we can take $w = \epsilon$). So suppose that $n - |I| \geq 1$. We first prove that, up to a switch, some some practical assumptions on the signs of G can be made.

(1) *We can suppose that every $i \in V \setminus I$ with in-degree one in G has a positive in-neighbor.*

Let U be the set of vertices of G of in-degree one that do not belong to I. Then $G[U]$ is a disjoint union of out-trees, say $T_1, \dots, T_k$. For $\ell \in [k]$, let U_ℓ be the vertex set of T_ℓ. By Proposition 3, there is $J^1_\ell \subseteq U_\ell$ such that the J^1_ℓ-switch of T_ℓ is full-positive. Let $J^2_\ell = U_\ell \setminus J^1_\ell$. The J^2_ℓ-switch of T_ℓ is also full-positive. Let i_ℓ be the source of T_ℓ and suppose, without loss, that $i_\ell \in J^1_\ell$. We set $J_\ell = J^1_\ell$ if the in-neighbor of i_ℓ in G is negative, and $J_\ell = J^2_\ell$ otherwise. Hence, in the J_ℓ-switch of G, the in-neighbor of each $i \in U_\ell$ is positive. Let $J = J_1 \cup \dots \cup J_k$ and let G' be the J-switch of G. Then, in G', the in-neighbor of each $i \in U$ is positive. Let f' be the J-switch of f. Since $J \subseteq U$, each vertex i of in degree at least two in G (or G') is not in J, and thus f'_i is a conjunction. So f' is the and-net of G'. By Lemma 3, G' satisfies the condition of the claim. From Proposition 4, for any word w we have $f^w(x) = f^w(y)$ if and only if $f'^w(x + e_J) = f'^w(y + e_J)$. Hence, without loss of generality, we can suppose that $G = G'$ and $f' = f$. This proves (1).

So in the following, we suppose every $i \in V \setminus I$ of in-degree one in G has a positive in-neighbor. Since $n - |I| \geq 1$, G has a terminal strong component S which is not an initial cycle. Let i be a vertex in S with only negative out-neighbors; such a vertex exists since otherwise G has a (full-)positive cycle. Let $H = G \setminus i$.

(2) *H has no positive cycles and no sources.*

Since H is a subgraph of G it has no positive cycles. If j is a source of H, then i is the unique in-neighbor of j in G and thus, by hypothesis, the arc from i to j is positive, which contradicts our choice of i. This proves (2).

(3) *H is homogenous.*

Let j be a vertex in H and suppose, for a contradiction, that H has a strong component F such that for, every vertex ℓ in F, H has both a positive and negative forward path from ℓ to j. Since G is j-homogenous, S is not a strong component of G, so G has an arc from i to some vertex ℓ in F, and thus H has a positive forward path P^+ from ℓ to j and a negative forward path P^- from ℓ to j. Since i is in a terminal strong component of G and G has a path from i to

j, G has also a path P from j to i. Since $P \setminus i$ is a path of H starting from j and P^+, P^- are forward paths of H ending in j, P^+ and P^- intersect $P \setminus i$ in j only. Thus $P^+ \cup P$ and $P^- \cup P$ are paths of G from ℓ to i with opposite signs. Since G has an arc from i to ℓ, we deduce that G has a positive cycle, a contradiction. This proves (3).

Let K be the set of vertices of H that belongs to an initial cycles of H. Since i is not in an initial cycle of G, all the initial cycles of G are initial cycles of H, so $I \subseteq K$. From the induction hypothesis, we get that two configurations that "agree" on i and K can be synchronized:

(4) *If x, y are configurations on V with $x_i = y_i = 0$ and $x_K = y_K$, then there is a word w such that $f^w(x) = f^w(y)$ and $|w| \leq g(n-1, |K|)$.*

Let x, y be configurations on V with $x_i = y_i = 0$ and $x_K = y_K$. Let h be the and-net on H. From (2), (3) and the induction hypothesis, there is a word w on $V' = V \setminus i$ such that $h^w(x_{V'}) = h^w(y_{V'})$ of length at most $g(n-1, |K|)$. Since $x_i = y_i = 0$ and i has only negative out-neighbors and $i \notin \{w\}$, we deduce that $f^w(x)_{V'} = h^w(x_{V'})$ and $f^w(y)_{V'} = h^w(y_{V'})$ and thus $f^w(x) = f^w(y)$. This proves (4).

So, by (4), to prove the claim it is sufficient to prove that, for every configurations x, y on V with $x_I = y_I$, there is a word w such that $f^w(x)_i = f^w(y)_i = 0$ and $f^w(x)_K = f^w(x)_K$. This is done in several steps.

(5) *If x, y are configurations x, y on V with $x_I = y_I$, then $f^w(x)_I = f^w(y)_I$ for any word w.*

Let x, y be two configurations on V with $x_I = y_I$. It is sufficient to prove that, for any $j \in V$, we have $f^j(x)_I = f^j(y)_I$. This is obvious if $j \notin I$. Otherwise, j has a unique in-neighbor, say k, and since $k \in I$ we have $x_k = y_k$ and we deduce that $f_j(x) = f_j(y)$ and so $f^j(x)_I = f^j(y)_I$. This proves (5).

(6) *Each initial cycle of H which is not an initial cycle of G has a unique negative arc.*

Suppose, for a contradiction, that H has an initial cycle C, which is not an initial cycle of G, such that C does not contain a unique negative arc. Since C is negative, it has at least three negative arcs. Let (j_k, i_k), $k = 1, 2, 3$, be three negative arcs in C, and let P_k be the path from i_k to i_{k+1} contained in C, where i_4 means i_1. Hence $C = P_1 \cup P_2 \cup P_3$. The three negative arcs can be chosen consecutively in C, that is, in such a way that $C = P_1 \cup P_2 \cup P_3$ and, for $k = 1, 2$, P_k has a unique negative arc, which is (j_{k+1}, i_{k+1}). Thus P_1, P_2 are negative, and so is P_3. Also, there is a negative arc from i to each of i_1, i_2, i_3. Indeed, if there is no arc from i to i_k then i_k is of in-degree one in G, so (j_k, i_k) is positive by the hypothesis resulting from (1), a contradiction. Thus (i, i_k) exists and is negative by the choice of i. Let Q be a shortest path of G from C to i (it exists since i is in a terminal strong component of G) and let j be its initial vertex. Then j belongs to $P_k \setminus i_{k+1}$ for some $1 \leq k \leq 3$. Let P'_k be the path from i_k to j contained in P_k. Since P_{k-1} is negative (P_0 means P_3), P'_k and $P_{k-1} \cup P'_k$ have distinct signs. Since G has a negative arc from i to i_k and i_{k-1} (i_0 means i_3) we deduce that G has a positive path P^+ from i to j and a negative path P^- from i to j such that P^+, P^- are internally vertex-disjoint from Q. Hence, $P^+ \cup Q$ and $P^- \cup Q$ are cycles with distinct signs, and thus G has a positive cycle, a contradiction. This proves (6).

(7) *Let C be an initial cycle of H, which is not an initial cycle of G, and let L its vertex set. If x, y are configurations on V and $x_i = 1$, then there is a word w over L such that $f^w(x)_L = f^w(y)_L = \mathbf{0}$ and $|w| \leq 2|L|$.*

By (6), C has a unique negative arc, say (j_1, i_1). By the hypothesis resulting from (1), i_1 is of in-degree at least two, so G has an arc from i to i_1, which is negative by the choice of i. Let $u = i_1 i_2 \dots i_\ell$ be an enumeration of the vertices of C in the order.

Let x be a configuration on V with $x_i = 1$. Since G has a negative arc from i to i_1, we have $f(x)_{i_1} = 0$. Since there is a positive arc from i_k to i_{k+1} for all $1 \leq k < \ell$, we deduce that $f^u(x)_L = \mathbf{0}$. Furthermore, since $i \notin \{u\}$ we have $f^u(x)_i = x_i = 1$ and the same argument shows that $f^{uu}(x)_L = \mathbf{0}$ by (a). Consequently,

$$x_i = 1 \quad \Rightarrow \quad f^u(x)_L = \mathbf{0} \quad \text{and} \quad f^{uu}(x)_L = \mathbf{0}. \tag{a}$$

Consider now a configuration x on V with $x_i = 0$. Let $x(0) = x$ and $x(k) = f^{i_k}(x(k-1))$ for $1 \leq k \leq \ell$, so that $x(\ell) = f^u(x)$. Note that $x(k)_i = 0$ for $0 \leq k \leq \ell$ since $i \notin \{u\}$. We will prove, by induction on k from 1 to ℓ, that $x(k)_{i_k} = a$, where $a = x_{i_\ell} + 1$. Since i_1 has exactly two in-neighbors, i and i_ℓ, both negative, and $x(0)_i = 0$, we have $f(x(0))_{i_1} = (x(0)_i + 1) \wedge (x(0)_{i_\ell} + 1) = 1 \wedge a = a$, and thus $x(1)_{i_1} = a$. Let $1 < k \leq \ell$. By induction, $x(k-1)_{i_{k-1}} = a$. Since there is a positive arc from i_{k-1} to i_k, and since i_k has at most one possible other in-neighbor, which is i and then negative, we have either $f(x(k-1))_{i_k} = x(k-1)_{i_{k-1}} = a$ if there is no arc from i to i_k or $f(x(k-1))_{i_k} = x(k-1)_{i_{k-1}} \wedge (x(k-1)_i + 1) = a \wedge 1 = a$. Thus $x(k)_{i_k} = a$ in both cases. This completes the induction step. Consequently, if $x_{i_\ell} = 1$ then $f^u(x)_L = \mathbf{0}$, and if $x_{i_\ell} = 0$ then $f^u(x)_L = \mathbf{1}$. So if $x_{i_\ell} = 0$ then $f^u(x)_{i_\ell} = 1$ and, since $f^u(x)_i = x(\ell)_i = 0$, the same argument shows that $f^{uu}(x)_L = \mathbf{0}$. Consequently,

$$x_i = 0 \quad \Rightarrow \quad f^u(x)_L = \mathbf{0} \quad \text{or} \quad f^{uu}(x)_L = \mathbf{0}. \tag{b}$$

Let x, y be configurations on V with $x_i = 1$. By (a) and (b) we have $f^u(x)_L = f^u(y)_L = \mathbf{0}$ or $f^{uu}(x)_L = f^{uu}(y)_L = \mathbf{0}$. This proves (7).

Let $J = K \setminus I$. So J is the set of vertices if H which belongs to an initial cycle of H which is not an initial cycle of G.

(8) *If x, y are configurations on V and $x_i = 1$, then there is a word w over J such that $f^w(x)_J = f^w(y)_J = \mathbf{0}$ and $|w| \leq 2|J|$.*

If $J = \emptyset$ there is nothing to prove. Otherwise, H has $p \geq 1$ initial cycles $C_1, \dots, C_p$ which are not initial cycles of G. For $1 \leq k \leq p$, let L_k be the vertex set of C_k, so that $J = L_1 \cup \dots \cup L_p$. By (7), for each $1 \leq k \leq p$, there is a word u^k over L_k with $f^{u^k}(x)_{L_k} = f^{u^k}(y)_{L_k} = \mathbf{0}$ and $|u^k| \leq 2|L_k|$. Since, for every distinct $r, s \in [p]$, $L_r \cap L_s = \emptyset$ and G has no arc between L_r and L_s, we deduce that $f^w(x)_J = f^w(y)_J = \mathbf{0}$ with $w = u^1, \dots, u^p$. This proves (8).

We are now in position to prove that, for every configurations x, y on V with $x_I = y_I$, there is a word w such that $f^w(x)_i = f^w(y)_i = 0$ and $f^w(x)_K = f^w(x)_K$. We consider two cases, giving (9) and (10) below. The first uses only Lemma 5 (giving a bound on $|w|$ of order $F(n+2)$), while the second uses Lemmas 5 and 7 (giving a bound on $|w|$ of order $3F(n+2)$).

(9) *Suppose that G has an arc from K to i. If x, y are configurations on V with $x_I = y_I$, then there is a word w such that $f^w(x)_i = f^w(y)_i = 0$ and $f^w(x)_K = f^w(y)_K$ with*

$$|w| \leq F(n+2) + 2|J|.$$

By Lemma 5, there is a word u over V such that $f^u(x)_i = 1$ and $|u| < F(n+2)$. So by (8) there is a word v over J such that $f^{uv}(x)_J = f^{uv}(y)_J = \mathbf{0}$ with $|v| \leq 2|J|$. We deduce from (5) that $f^{uv}(x)_K = f^{uv}(y)_K$. Suppose that G has an arc from some $j \in K$ to i. Then j belongs to an initial cycle C of H which is not an initial cycle of G, so $j \in J$. By (6), C has a unique negative arc, say (k, ℓ). By the hypothesis resulting from (1), ℓ is of in-degree at least two and thus G has an arc from i to ℓ, which is negative by the choice of i. Hence, the path from ℓ to j contained in C is full-positive and it forms, with the arc from i to ℓ and the arc from j to i, a cycle, which is negative by hypothesis. We deduce that the arc from j to i is positive. Since $f^{uv}(x)_j = f^{uv}(y)_j = 0$ we have $f_i(f^{uv}(x)) = f_i(f^{uv}(y))_i = 0$. Since $i \notin K$, we deduce that $w = uv, i$ has the desired properties. This proves (9).

(10) *Suppose that G has no arc from K to i. If x, y are configurations x, y on V with $x_I = y_I$, then there is a word w such that $f^w(x)_i = f^w(y)_i = 0$ and $f^w(x)_K = f^w(y)_K$ with*

$$|w| \leq 3F(n+2) + 2|J|.$$

By Lemma 5, there is a word u^1 over V such that $f^{u^1}(x)_i = 1$ and $|u^1| < F(n+2)$. Let P be a path of G from some vertex j of in-degree at least two to i such that all the vertices of P distinct from j are of in-degree one (if i is of in-degree at least two, then P is the path of length zero containing i); this path exists since G has no sources and i does not belong to an initial cycle of G. By the hypothesis resulting from (1), P is full-positive. So, by Lemma 7, there is a shortest word u^2 such that $f^{u^1u^2}(x)_i = f^{u^1u^2}(y)_i = 0$ with $|u^2| \leq 2F(n+2)$. Let u be obtained from u^1u^2 by removing the last letter. Since $f^{u^1}(x)_i = 1$, u^2 is not empty, and since u^2 is as short as possible we deduce that the last letter is i. So $u, i = u^1u^2$ and since u^2 is as short as possible, $f^u(x)_i = 1$ or $f^u(y)_i = 1$. Suppose that $f^u(x)_i = 1$, the other case being symmetric. By (8) there is a word v over J such that $f^{uv}(x)_J = f^{uv}(y)_J = \mathbf{0}$ and $|v| \leq 2|J|$. We deduce from (5) that $f^{uv}(x)_K = f^{uv}(y)_K$. Since G has no arc from K to i, we have $f^{uv,i}(x)_i = f^{u,i}(x)_i = 0$ and $f^{uv,i}(y)_i = f^{u,i}(y)_i = 0$. Since $i \notin K$, we deduce that $w = uv, i$ has the desired property. This proves (10).

We can now prove the claim, by combining (4) with (9) and (10). Let x, y be configurations on V with $x_I = y_I$. By (9) and (10) there is a word u such that $f^u(x)_i = f^u(y)_i = 0$ and $f^u(x)_K = f^u(y)_K$ with

$$|u| \leq 3F(n+2) + 2|J|.$$

We deduce from (4) that there is a word v such that $f^{uv}(x) = f^{uv}(y)$ with

$$|v| \leq g(n-1, |I| + |J|).$$

If $|J| = 0$ then

$$|uv| \leq 3F(n+2) + g(n-1, |I|) = g(n, |I|).$$

If $|J| \geq 1$, then using the fact that $2m \leq 3F(m+2)$ for all $m \geq 1$ to get the second inequality, we obtain

$$\begin{aligned} |uv| &\leq 3F(n+2) + 2|J| + g(n-1, |I| + |J|) \\ &= g(n, |I| + |J|) + 2|J| \\ &\leq g(n, |I| + |J|) + 3F(|I| + |J| + 2) \\ &= g(n, |I| + |J| - 1) \\ &\leq g(n, |I|). \end{aligned}$$

This proves the claim. $\square$

3.6 Global synchronization

A classical result of Černý [11] is that, if any two states of a deterministic finite automaton can be sent to the same state by some word, then this automaton is synchronizing. Here is an adaptation of this observation to our context.

Lemma 10. *Let f be a BN with component set V and suppose that, for every configurations x, y on V, there is a word u such that $f^u(x) = f^u(y)$ and $|u| \leq k$. For every non-empty subset $X \subseteq \{0,1\}^V$, there is a word w such that $|f^w(X)| = 1$ and $|w| \leq k(|X|-1)$. Taking $X = \{0,1\}^V$, we deduce that f has a synchronizing word of length at most $k(2^n - 1)$.*

Proof. We proceed by induction on $|X|$. If $|X| = 1$ the result is obvious, since we can take $w = \epsilon$. So suppose that $|X| \geq 2$. Let $x, y \in X$, distinct, and let u be a word such that $f^u(x) = f^u(y)$ and $|u| \leq k$. Then $X' = f^u(X)$ is of size at most $|X|-1$ and thus, by induction, there is a word v such that $|f^v(X')| = 1$ and $|v| \leq k(|X'| - 1)$. Setting $w = uv$ we obtain $|f^w(X)| = |f^v(X')| = 1$ and $|w| \leq k|X'| \leq k(|X| - 1)$, completing the induction. □

From this observation and Lemma 9 we deduce the following.

Lemma 11. *Suppose that G is homogenous and has no positive cycles, no sources and no initial cycles. The and-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$.*

Proof. Let f be the and-net on G. From Lemmas 10 and 9, we deduce that f has a synchronizing word of length at most $3(F(n+4)-1)(2^n-1)$. By the well known Binet's formula, we have $F(n+4) = (\varphi^{n+4} - \psi^{n+4})/\sqrt{5}$, where $\varphi = (\sqrt{5}+1)/2$ is the golden number and $\psi = 1 - \varphi$. We deduce that $F(n+4) \leq (\varphi^{n+4}/\sqrt{5}) + 1$ and thus

$$3(F(n+4)-1)(2^n-1) \leq \frac{3}{\sqrt{5}}\varphi^{n+4}(2^n-1) \leq \frac{3}{\sqrt{5}}\varphi^{n+4}2^n = \frac{3\varphi^4}{\sqrt{5}}(\sqrt{5}+1)^n.$$

Since $(3\varphi^4/\sqrt{5}) \sim 9.19... < 10$ this proves the lemma. □

Using the switch operation, we obtain Theorem 4, that we restate.

Theorem 4. *Suppose that G is homogenous and has no positive cycles, no sources and no initial cycles. Then every and-or-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$.*

Proof. Let f be an and-or-net on G, and let I be the set of $i \in V$ such that f_i is a disjunction. Let G' be the I-switch of G and f' the I-switch of f. By Proposition 4, f' is a BN on G', and one easily check that f' is actually the and-net on G'. We deduce from Lemma 3 that G' is homogenous and has no positive cycles, no sources and no initial cycles. Thus, by Lemma 11, f' has a synchronizing word w of length at most $10(\sqrt{5}+1)^n$ and, by Proposition 4, w is a synchronizing word for f. □

4 Proof of Theorem 2

Let G be a signed digraph with vertex set V, and $n = |V|$. Suppose that G satisfies the conditions of Theorem 2, that is: G is strong, has no positive cycles, and is not a cycle. These conditions are invariant by switch and so, to prove the theorem, it is sufficient to prove that the and-net on some switch of G has a synchronizing word of length at most $5n(\sqrt{2})^n$. This version is more convenient, because of the regularity of and-nets. The idea is to consider a switch G' containing a spanning full-positive out-tree, which exists since G' is strongly connected. The out-tree T "speeds up" the synchronization of the and-net f on G': for instance, denoting r the root of T and u is a topological sort of $T \setminus r$, one can easily check that $f^u(x) = f^u(y) = \mathbf{0}$ whenever

$x_r = y_r = 0$. So any configurations x, y can be synchronized quickly if there is a short word v synchronizing the root r at state 0, that is, such that $f^v(x)_r = f^v(y)_r = 0$. It appears that, with some tricky arguments, the root can by synchronized at state 0 with a word of length at most $4n$. Consequently, any two configurations can be synchronized with a word of length at most $5n$. By Černý's observation (Lemma 10), we deduce that f has a synchronizing word of length at most $5n(2^n - 1)$. To replace 2 by $\sqrt{2}$ in the bound, we need additional arguments. If G has a loop, a direct argument shows that f has a synchronizing word of length at most $7n \leq 5n(\sqrt{2})^n$. Otherwise G' has no loop and no positive cycles, and so, by a recent and difficult result of Millani, Steiner and Wiederrecht [26], there is a subset $I \subseteq V$ of size at most $n/2$ such that $G' \setminus I$ is acyclic. By classical arguments, the topological sort v of $G' \setminus I$ sends any two configurations x, y with $x_I = y_I$ on the same configuration. It follows that v sends every configuration inside a set X of at most $2^{n/2}$ configurations. Using Černý's observation, we deduce that there is a word w of length at most $5n(|X| - 1) \leq 5n(\sqrt{2})^n - 5n$ with $|f^w(X)| = 1$. Then v, w is a synchronizing word for f of length at most $5n(\sqrt{2})^n - 4n$ and we are done.

We now proceed to the details. We first show that the presence of a spanning full-positive out-tree allows a fast synchronization of two configurations. The precise statement is the following (note that it allows the presence of some positive cycles).

Lemma 12. *Suppose that G is simple and has a spanning full-positive out-tree T, where the root r has only negative in-neighbors and is of in-degree at least two. Suppose also that G has no full-positive cycle, and that each positive cycle containing r has at least four negative arcs. Let f be the and-net on G. For every configurations x, y on V there is a word w such that $f^w(x) = f^w(y)$ and $|w| \leq 5n$. Furthermore, if r has a loop, then f has a synchronizing word of length at most $7n$.*

Proof. Let u be a topological sort of $T \setminus \{r\}$. Since T is full-positive and f is an and-net, for every configuration x on V with $x_r = 0$ we have $f^u(x) = \mathbf{0}$. So for every configurations x, y on V with $x_r = y_r = 0$ the lemma holds by taking $w = u$. For the other cases, we need additional arguments. Let x be a configuration on V. We denote by $\mathbf{0}(x)$ the set of $i \in V$ with $x_i = 0$, and $\mathbf{1}(x) = V \setminus \mathbf{0}(x)$. We say that a configuration x on V is *regular* if, for all arc of T from j to i, we have $x_j \geq x_i$.

(1) *For every configuration x on V, there is a word $v(x)$ over $\mathbf{1}(x)$, without repeated letters and not containing r, such that $f^{v(x)}(x)$ is regular.*

Let T' be the subgraph of T obtained by removing r and each vertex in $\mathbf{0}(x)$. Let $v = v(x)$ be a topological sort of T'. If $v = \epsilon$ then $x = \mathbf{0}$ or $x = e_r$, thus $f^v(x) = x$ is regular. So suppose that v is not empty and suppose, for a contradiction, that $y = f^v(x)$ is not regular. Then T has an arc from j to i with $y_j < y_i$. If i is not in v, then $y_i = x_i = 0$, a contradiction. So i is in v, hence $v = v_1, i, v_2$ for some words v_1, v_2. Let $z = f^{v_1}(x)$. Since i is not in v_2, we have $f_i(z) = y_i = 1$. Since there is a positive arc from j to i, we deduce that $z_j = 1$. If j is not in v_2, then $y_j = z_j = 1$, a contradiction. So j is in v_2 and this contradicts the fact that v is a topological sort of T'. This proves (1).

Given a path P in G and a configuration x on V, we write $x_P = \mathbf{0}$ ($x_P = \mathbf{1}$) to means that $x_k = 0$ ($x_k = 1$) for all vertices k in P. The key observation is the following.

(2) *Let x be a regular configuration on V with $x_r = 1$. Let j be an in-neighbor of r. There is a word $w(x, j)$ over $\mathbf{0}(x)$, without repeated letters, such that $f^{w(x,j)}(x)_j = 1$ and such that the subgraph of G induced by $\{w(x, j)\}$ has a full-positive path from each vertex to j.*

Let $I(x)$ be the set of vertices $i \in V$ such that G has a full-positive path P from i to j with $x_P = \mathbf{0}$; we have $I(x) \subseteq \mathbf{0}(x)$. We proceed by induction on $|I(x)|$. If $|I(x)| = 0$ then $x_j = 1$ and $w(x, j)$ is the empty word. So suppose that $|I(x)| \geq 1$.

We first prove that $f_i(x) = 1$ for some $i \in I(x)$. Suppose, for a contradiction, that $f_i(x) = 0$ for all $i \in I(x)$. Let $i \in I(x)$. Since $f_i(x) = 0$, i has an in-neighbor k in G such that either $x_k = 0$ and k is a positive in-neighbor of i or $x_k = 1$ and k is a negative in-neighbor of i. Suppose first that k is a negative in-neighbor of i. Let Q be the path from r to k contained in T, which is full-positive. Since x is regular and $x_k = 1$, we have $x_Q = \mathbf{1}$. Since $i \in I(x)$, there is a full-positive path P from i to j with $x_P = \mathbf{0}$. Thus Q and P are disjoint full-positive paths, from r to k and from i to j, respectively. Since (k, i) and (j, r) are negative arcs of G, we deduce that G has a cycle with exactly two negative arcs and containing r, which contradicts the hypothesis of the statement. So $x_k = 0$ and k is a positive in-neighbor of i. If k is in P then G has a full-positive cycle, a contradiction. So k is not in P and, by adding to P the positive arc from k to i, we obtain a full-positive path P' from k to j with $x_{P'} = \mathbf{0}$. Hence, $k \in I(x)$. This proves that each vertex in $I(x)$ has a positive in-neighbor in $I(x)$. But then $G[I(x)]$ has a full-positive cycle, a contradiction.

Therefore, there is at least one $i \in I(x)$ such that $f_i(x) = 1$. Let k be the in-neighbor of i in T (it exists since $x_i < x_r$ and thus $i \neq r$). Since $f_i(x) = 1$ we have $x_k = 1$ (since T is full-positive). Let $x' = f^i(x)$. Then x' is regular (because $x'_i = x'_k = x_k = 1$). Furthermore, $I(x') \subseteq I(x)$, since $x' \geq x$, and $i \notin I(x')$ since $x'_i = 1$. Thus, by induction, there is a word $w(x', j)$ with the properties of the statement (with x' instead of x). Then $w(x, j) = i, w(x', j)$ has the desired properties. This proves (2).

For every configuration x on V and every negative in-neighbor j of r, we define $w(x, j)$ to be a word as in (2) if $x_r = 1$, and we set $w(x, j) = \epsilon$ otherwise.

Let x, y be configurations on V with $x_r = 1$ or $y_r = 1$. We will prove that there is a word $w(x, y)$ of length at most $5n$ that sends both x and y on $\mathbf{0}$, that is, $f^{w(x,y)}(x) = f^{w(x,y)}(y) = \mathbf{0}$ (for the case $x_r = y_r = 0$ it is sufficient to take $w(x, y) = u$ as already mentioned at the beginning of the proof). Let j and k be distinct in-neighbors of r, which exist and are negative by hypothesis. Since G has no full-positive cycle, either G has no full-positive paths from j to k, or G has no full-positive paths from k to j. Suppose, without loss of generality, that G has no full-positive paths from j to k.

Suppose that $x_r = y_r = 1$. Let

$$\begin{array}{llll} x^1 = f^{v(x)}(x), & x^2 = f^{w(x^1,j)}(x^1), & x^3 = f^{v(y^2)}(x^2), & x^4 = f^{w(y^3,k)}(x^3), \\ y^1 = f^{v(x)}(y), & y^2 = f^{w(x^1,j)}(y^1), & y^3 = f^{v(y^2)}(y^2), & y^4 = f^{w(y^3,k)}(y^3). \end{array}$$

By (1), x^1 is regular, and since $v(x)$ does not contain r, we have $x^1_r = x_r = 1$. So, by (2), $x^2_j = 1$. If $y^2_j = 1$ then $f_r(x^2) = f_r(y^2) = 0$ so $f^u(f^r(x^2)) = f^u(f^r(y^2)) = \mathbf{0}$. Thus

$$v(x), w(x^1, j), r, u$$

sends both x and y on $\mathbf{0}$ and is of length at most $3n - 2$. So suppose that $y^2_j = 0$. By (1), y^3 is regular, and j does not appears in $v(y^2)$ since $y^2_j = 0$. Hence, $x^3_j = x^2_j = 1$. Furthermore, r does not appear in $v(x), w(x^1, j), v(y^2)$, thus $y^3_r = y_r = 1$. Consequently, by (2), we have $y^4_k = 1$. Since G has no full-positive path from j to k, j is not contained in $w(y^3, k)$, and thus $x^4_j = x^3_j = 1$. Hence, $x^4_j = y^4_k = 1$. So $f_r(x^4) = f_r(y^4) = 0$. Consequently, $f^u(f^r(x^4)) = f^u(f^r(y^4)) = \mathbf{0}$ and we deduce that

$$v(x), w(x^1, j), v(y^2), w(y^3, k), r, u.$$

sends both x and y on $\mathbf{0}$ and is of length at most $5n-4$.

Finally, suppose that $x_r < y_r$ (the case $x_r > y_r$ is similar by symmetry). Let

$$\begin{array}{lllll} x^1 = f^u(x) & x^2 = f^{v(y^1)}(x^1) & x^3 = f^{w(y^2,j)}(x^2) & x^4 = f^r(x^3) & x^5 = f^{w(x^4,k)}(x^4) \\ y^1 = f^u(y) & y^2 = f^{v(y^1)}(y^1) & y^3 = f^{w(y^2,j)}(y^2) & y^4 = f^r(y^3) & y^5 = f^{w(x^4,k)}(y^4) \end{array}$$

We have $x^1 = \mathbf{0}$. Since r is not in $v(y^1)$ we deduce that $x^2 = \mathbf{0}$, and since r is furthermore not in u, we have $y^2_r = y_r = 1$. By (1), y^2 is regular and so by (2) we have $y^3_j = 1$, and since r is not in $w(y^2, j)$ we have $x^3 = \mathbf{0}$ because $x^2 = \mathbf{0}$. Since r has only negative in-neighbors, we have $x^4 = e_r$, which is regular, and thus by (2) we have $x^5_k = 1$. Since G has no full-positive paths from j to k, we have $j \neq r$ and j and not contained in $r, w(x^2, k)$, thus $y^4_j = y^5_j = 1$. So $x^5_k = y^5_j = 1$, and we deduce that $f_r(x^5) = f_r(y^5) = 0$. Consequently, $f^u(f^r(x^5)) = f^u(f^r(y^5)) = \mathbf{0}$ thus

$$u, v(y^1), w(y^2, j), r, w(x^4, k), r, u.$$

sends both x and y on $\mathbf{0}$ and is of length at most $5n-3$.

Thus, for every configurations x, y on V, there is a word w of length at most $5n-3$ such that $f^w(x) = f^w(y)$.

Suppose now that r has a loop. Let $z = f^u(e_r)$, let w be a word of length at most $5n-3$ such that $f^w(\mathbf{0}) = f^w(z) = \mathbf{0}$, and let us prove that u, r, u, w, which is of length at most $7n-2$, sends all the configurations on $\mathbf{0}$, and thus synchronizes f. Let x be any configuration on V, and let

$$x^1 = f^u(x), \qquad x^2 = f^r(x^1), \quad x^3 = f^u(x^2), \quad x^4 = f^w(x^3).$$

We have to prove that $x^4 = \mathbf{0}$ and by the choice of w it is sufficient to prove that x^3 is $\mathbf{0}$ or z. Suppose first that $x_r = 1$. Then $x^1_r = x_r = 1$ since r is not in u. Since r has a negative loop we have $x^2_r = 0$ and thus $x^3 = \mathbf{0}$. Suppose now that $x_r = 0$. Then $x^1 = \mathbf{0}$ and thus $x^2 = e_r$ since r has only negative in-neighbors. So $x^3 = z$ as desired. □

Observing that if G satisfies the conditions of Theorem 2 then, up to a switch, it satisfies the conditions of the previous lemma, we deduce the following.

Lemma 13. *Suppose that G is strong, has no positive cycles, and is not a cycle. There is an and-or-net f on G such that the following holds. For every configurations x, y on V, there is a word w such that $f^w(x) = f^w(y)$ and $|w| \leq 5n$. Furthermore, if G has a loop, then f has a synchronizing word of length at most $7n$.*

Proof. Since G is strong and is not a cycle, it has a vertex of in-degree at least two and any vertex with a loop is of in-degree at least two. Let r be a vertex with a loop, if it exists, and let r be any vertex of in-degree at least two otherwise. Since G strong, it has a spanning out-tree T rooted in r. By Proposition 3, there is $I \subseteq V$ such that T^I, the I-switch of T, is full-positive. Hence T^I is a spanning full-positive out-tree of G^I, the I-switch of G, rooted in r. Since G has no positive cycles, G^I has no positive cycles, thus all the in-neighbors of r are negative. Hence G^I satisfies the conditions of Lemma 12. So, denoting h the and-net on G^I, we deduce that for every configurations x, y on V, there is a word w such that $h^w(x) = h^w(y)$ and $|w| \leq 5n$. Furthermore, if G has a loop then r has a loop and we deduce that h has a synchronizing word of length at most $7n$. Let f be the I-switch of h. By Proposition 4, f is an and-or-net on G with the desired properties. □

From the previous lemma and Lemma 10, we deduce that if G satisfies the conditions of Theorem 2, then there is an and-or-net on G with a synchronizing word of length at most $5n(2^n - 1)$. To obtain the bound $5n(\sqrt{2})^n$, we need additional arguments. We first need the following easy observation, which implies a basic result of [6]: if G is acyclic, then there is a word w of length n which synchronizes every BN on G.

Lemma 14. *Let $I \subseteq V$ such that $G \setminus I$ is acyclic, and let w be a topological sort of $G \setminus I$. Let f be a BN on G. For every configurations x, y on V with $x_I = y_I$ we have $f^w(x) = f^w(y)$.*

Proof. Let x, y be configurations on V with $x_I = y_I$. Let us write $w = i_1, \dots, i_\ell$, where $\ell = |V \setminus I|$. Let $I_0 = I$ and, for $1 \leq k \leq \ell$, let $I_k = I_{k-1} \cup \{i_k\}$. We will prove, by induction on k, that

$$f^w(x)_{I_k} = f^w(y)_{I_k}. \tag{1}$$

Since $I \cap \{w\} = \emptyset$, we have $f^w(x)_I = x_I = y_I = f^w(y)_I$, thus (1) holds for $k = 0$. Suppose that $1 \leq k \leq \ell$. Let u, v such that $w = u, i_k, v$, and let

$$\begin{aligned} x^1 &= f^u(x), \qquad & x^2 &= f^{i_k}(x^1), \qquad & x^3 &= f^v(x^2), \\ y^1 &= f^u(y), \qquad & y^2 &= f^{i_k}(y^1), \qquad & y^3 &= f^v(y^2). \end{aligned}$$

By induction, $x^3_{I_{k-1}} = x^3_{I_{k-1}}$. Since $I_{k-1} \cap \{i_k, v\} = \emptyset$, this implies $x^1_{I_{k-1}} = y^1_{I_{k-1}}$. Since all the in-neighbors of i_k are in I_{k-1}, we deduce that $x^2_{i_k} = x^2_{i_k}$ and since $i_k \notin \{v\}$ this implies $x^3_{i_k} = x^3_{i_k}$. Thus $x^3_{I_k} = y^3_{I_k}$, completing the induction step. So (1) holds for all $0 \leq k \leq \ell$. Since $I_\ell = V$, we obtain $f^w(x) = f^w(y)$. □

Next, we need the following recent result, showing that signed digraphs without positive cycles are rather well structured.

Theorem 5 (Millani, Steiner and Wiederrecht [26])**.** *Suppose that G has no positive cycles and no loops. There are disjoint subsets $I, J \subseteq V$ with $I \cup J = V$ such that $G[I]$ and $G[J]$ are acyclic.*

We are now in position to prove Theorem 2, that we restate.

Theorem 2. *Let G be a strongly connected signed digraph on $[n]$ without positive cycles, which is not a cycle. At least one and-or-net on G has a synchronizing word of length at most $5n(\sqrt{2})^n$.*

Proof. If G has a loop then, by Lemma 13, there is an and-or-net on G with a synchronizing word of length at most $7n \leq 5n(\sqrt{2})^n$. So suppose that G has no loops. By Theorem 5, there is a subset $I \subseteq V$ of size at most $n/2$ such that $G \setminus I$ is acyclic. By Lemma 13, there is an and-or-net f on G such that, for every configurations x, y on V, there is a word v such that $f^v(x) = f^v(y)$ and $|v| \leq 5n$. Let u be a topological sort of $G \setminus I$, and let $X = f^u(\{0,1\}^V)$. If $|X| > 2^{|I|}$ then there are $x, y \in X$ with $x \neq y$ and $x_I = y_I$. Since $x, y \in X$, there are configurations x', y' on V with $x = f^u(x')$ and $y = f^u(y')$. Since $I \cap \{u\} = \emptyset$, we have $x'_I = x_I = y_I = y'_I$. But then, by Lemma 14, we have $f^u(x') = f^u(y')$ and thus $x = y$, a contradiction. Hence $|X| \leq 2^{|I|} \leq 2^{n/2}$. By Lemma 10, there is a word w such that $|f^w(X)| = 1$ and $|w| \leq 5n(|X| - 1)$. Thus u, w is a synchronizing word for f of length at most

$$n - |I| + 5n(|X| - 1) \leq n + 5n(2^{n/2} - 1) = 5n(\sqrt{2})^n - 4n.$$

□

5 Proof of Theorem 3

Recall that if G is a simple signed digraph with maximum in-degree at most two, then every BN on G is an and-or-net. Hence, Theorem 3 exactly says that the following two decisions problems are coNP-hard. Note that, since Robertson, Seymour and Thomas [29] and McCuaig [24] proved independently that we can decide in polynomial time if a signed digraph has a positive cycle, the conditions on the inputs of these two problems can be checked in polynomial time.

Strong-Synchronizing-Problem

- Input: A simple signed digraph G, *strongly connected*, with maximum in-degree at most two, and containing a vertex meeting every cycle.
- Question: Is every BN on G synchronizing?

Negative-Synchronizing-Problem

- Input: A simple signed digraph G, *without positive cycles*, with maximum in-degree at most two, and containing a vertex meeting every cycle.
- Question: Is every BN on G synchronizing?

We prove that these two problems are coNP-hard with reductions from 3-SAT. So consider a 3-CNF formula ψ over a set of $n \geq 2$ variables $\boldsymbol{\lambda} = \{\lambda_1, \ldots, \lambda_n\}$ consisting of $m \geq 1$ clauses $\boldsymbol{\mu} = \{\mu_1, \ldots, \mu_m\}$. To each variable λ_r is associated a positive literal λ_r^+ and a negative literal λ_r^-. The resulting sets of positive and negative literals are denoted $\boldsymbol{\lambda}^+$ and $\boldsymbol{\lambda}^-$. Each clause μ_s is a subset of $\boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$ of size three, and we write $\mu_r = \{\mu_{r,1}, \mu_{r,2}, \mu_{r,3}\}$. An *assignment* for ψ is regarded as a configuration z on $[n]$. A positive literal λ_r^+ is satisfied by z if $z_r = 1$, and a negative literal λ_r^- is satisfied by z if $z_r = 0$. A clause is satisfied by z if at least one of its literals is satisfied by z. The formula ψ is *satisfied* by z (or z is a *satisfying assignment* for ψ) if every clause in $\boldsymbol{\mu}$ is satisfied by z. We say that ψ is *satisfiable* if it has at least one satisfying assignment.

Our reductions from 3-SAT are based on the simple signed digraph H_ψ defined as follows; see Figure 1 for an illustration:

- The vertex set is $\boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^- \cup \boldsymbol{\ell} \cup \boldsymbol{\mu}' \cup \boldsymbol{\mu} \cup \boldsymbol{c}$, where $\boldsymbol{\ell} = \{\ell_0, \ldots, \ell_n\}$, $\boldsymbol{\mu}' = \{\mu'_1, \ldots, \mu'_m\}$ and $\boldsymbol{c} = \{c_1, \ldots, c_m\}$; there are thus $3n + 3m + 1$ vertices.
- The arcs are, for all $r \in [n]$ and $s \in [m]$,
  - $(\ell_{r-1}, \lambda_r^+), (\ell_{r-1}, \lambda_r^-)$,
  - $(\lambda_r^+, \ell_r), (\lambda_r^-, \ell_r)$,
  - (c_1, ℓ_0),
  - $(\mu_{s,1}, \mu'_s), (\mu_{s,2}, \mu'_s)$,$(\mu_{s,3}, \mu_s), (\mu'_s, \mu_s)$,
  - $(\mu_s, c_s), (c_{s+1}, c_s)$, where c_{m+1} means ℓ_n.
- For all $s \in [m]$, the arc (μ_s, c_s) is negative, and all the other arcs are positive.

This signed digraph H_ψ has been recently introduced in [8]. Actually, in this paper, the authors consider the signed digraph H'_ψ obtained from H_ψ by adding, for all $r \in [n]$, a new vertex i_r with a positive arc (i_r, λ_r^+) and a negative arc (i_r, λ_r^-), and then prove the following: ψ is satisfiable if and only if there is a BN on H'_ψ with at least two fixed points.

Here, we present two adaptations of this construction to the context of synchronization. For the first, let G_ψ be the signed digraph obtained from H_ψ by adding: two new vertices, q and t; a negative arc (ℓ_0, q); two positive arcs $(\ell_0, t), (q, t)$; and a positive arc (t, i) for all $i \in \boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$. An illustration is given in Figure 2. Note that G_ψ is strongly connected, simple and has maximum in-degree two (thus every BN on G_ψ is an and-or-network). Also, ℓ_0 meets every cycle. Thus G_ψ is a valide instance of the Strong-Synchronizing-Problem.

The following result shows that G_ψ as the same property that H'_ψ: ψ is satisfiable if and only if there is a BN on H'_ψ with at least two fixed points. But it gives a stronger conclusion when ψ

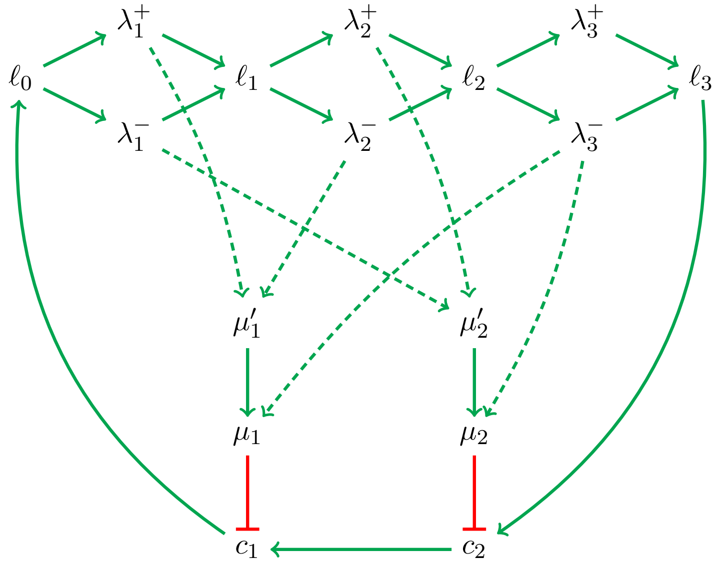


Figure 1: The signed digraph H_ψ for the 3-CNF formula $\psi = (\lambda_1 \vee \neg\lambda_2 \vee \neg\lambda_3) \wedge (\neg\lambda_1 \vee \lambda_2 \vee \neg\lambda_3)$. Using our notations, the set of variables is $\boldsymbol{\lambda} = \{\lambda_1, \lambda_2, \lambda_3\}$ and the set of clauses is $\boldsymbol{\mu} = \{\mu_1, \mu_2\}$ with $\mu_1 = \{\lambda_1^+, \lambda_2^-, \lambda_3^-\}$ and $\mu_2 = \{\lambda_1^-, \lambda_2^+, \lambda_3^-\}$. Clauses are encoded through dashed arrows.

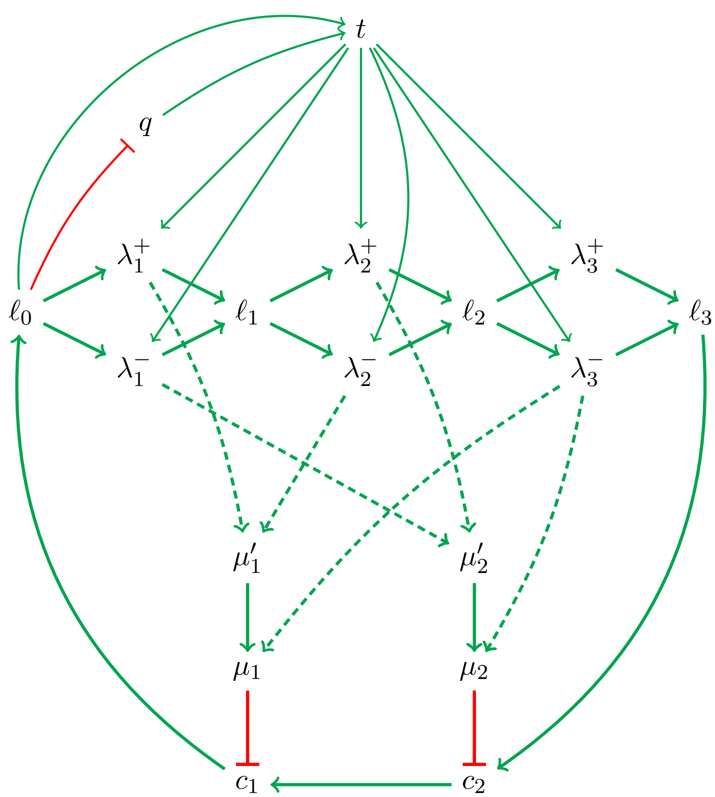


Figure 2: The signed digraph G_ψ for the 3-CNF formula of Figure 1.

is not satisfiable: every BN on G is synchronizing. Thus ψ is not satisfiable if and only if every BN on G_ψ is synchronizing. Since the SAT problem is NP-complete and G_ψ is a valid instance of the Strong-Synchronizing-Problem, we deduce that this problem is coNP-hard.

Theorem 6. *The following conditions are equivalent:*

1. *ψ is not satisfiable.*
2. *Every BN on G_ψ has at most one fixed point.*
3. *There is a word of length $12n+12m+11$ which synchronizes every BN on G_ψ.*
4. *Every BN on G_ψ is synchronizing.*

Proof. **(1 ⇒ 2)** Suppose that ψ is not satisfiable and let f be a BN on G_ψ. We will prove that f has at most one fixed point. Suppose, for a contradiction, that f has two distinct fixed points x and y. As argued in [3], G_ψ has a positive cycle C such that $x_i \neq y_i$ for every vertex i in C. Since $x_q = f_q(x) = \neg x_{\ell_0}$ and $y_q = f_q(y) = \neg y_{\ell_0}$, we have $f_t(x) = f_t(y) = 0$ if f_t is a conjunction and $f_t(x) = f_t(y) = 1$ if f_t is a disjunction, and thus $x_t = y_t$. So t is not in C. We deduce that

- C is full-positive,
- C contains all the vertices in $\boldsymbol{\ell} \cup \boldsymbol{c}$,
- C contains exactly one of λ_r^+, λ_r^- for each $r \in [n]$.

Suppose, without loss of generality, that $x_{\ell_0} < y_{\ell_0}$. We prove, by induction on s from 1 to m that $x_{c_s} < y_{c_s}$. Since c_1 is the unique in-neighbor of ℓ_0 we have $x_{c_1} < y_{c_1}$. Let $1 < s \leq m$. By induction, $x_{c_{s-1}} < y_{c_{s-1}}$. Suppose, for a contradiction, that $x_{c_s} > y_{c_s}$. If $f_{c_{s-1}}$ is a disjunction then, since $x_{c_s} = 1$, we have $f_{c_{s-1}}(x) = 1 \neq x_{c_{s-1}}$, a contradiction, and if $f_{c_{s-1}}$ is a conjunction then, since $y_{c_s} = 0$, we have $f_{c_{s-1}}(y) = 0 \neq y_{c_{s-1}}$, a contradiction. We deduce that $x_{c_s} \leq y_{c_s}$ and since the equality is not possible we obtain $x_{c_s} < y_{c_s}$. This completes the induction, and thus $x_{\mathbf{c}} = \mathbf{0}$ and $y_{\mathbf{c}} = \mathbf{1}$. Since $x_{\ell_0} \leq y_{\ell_0}$ and $x_t = y_t$, we have $x_i \leq y_i$ for $i \in \{\lambda_1^+, \lambda_1^-\}$, and thus $x_{\ell_1} \leq y_{\ell_1}$. Repeating this argument we get $x_i \leq y_i$ for every $i \in \boldsymbol{\ell} \cup \boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$. Consequently, $x_i \leq y_i$ for all $i \in \boldsymbol{\mu}'$ and thus $x_i \leq y_i$ for all $i \in \boldsymbol{\mu}$. We deduce that $x_{\mu_s} = y_{\mu_s}$ for all $s \in [m]$. Indeed, otherwise $x_{\mu_s} < y_{\mu_s}$ so $f_{c_s}(x) = 1 \neq x_{c_s}$ if f_i is a disjunction and $f_{c_s}(y) = 0 \neq y_{c_s}$ if f_i is a conjunction. Thus $x_{\mu_s} = y_{\mu_s}$. Let us prove that μ_s contains a literal i with $x_i = y_i$. Indeed, otherwise we have $x_i < y_i$ for the three literals i contained in μ_s, which implies $x_{\mu'_s} < y_{\mu'_s}$ and thus $x_{\mu_s} < y_{\mu_s}$, a contradiction. Thus each clause μ_s contains a literal which is not in C (since $x_i \neq y_i$ for all vertex i in C). Let $z \in \{0,1\}^n$ defined by $z_r = 1$ if λ_r^+ is not in C and $z_r = 0$ if λ_r^- is not in C; there is no ambiguity since exactly one of λ_r^+, λ_r^- is in C. All the literals not in C are satisfied by z, and thus each clause contains a literal satisfied by z. So ψ is satisfiable, a contradiction. Thus f has indeed at most one fixed point.

(2 ⇒ 3) Suppose that every BN on G_ψ has at most one fixed point. Let u be a topological sort of $G_\psi \setminus \{\ell_0, q, t\}$; so $|u| = 3n+3m$. The first two letters of u are λ_1^+, λ_1^- and we denote by u' the suffix of u obtained by removing these two letters; so $|u'| = 3n+3m-2$. Let V be the vertex set of G_ψ; so $|V| = 3n+3m+3$. We will prove that the word $\mathbf{w}$ defined by

$$\mathbf{w} = v, \ell_0, v, w \quad \text{where} \quad v = q, t, u \quad \text{and} \quad w = \ell_0, u, t, \ell_0, \lambda_1^+, \lambda_1^-, t, u', \ell_0, q$$

is a synchronizing word for every BN f on G_ψ. Since $|v| = |V| - 1$ and $|w| = 2|V|$, we have $|\mathbf{w}| = 4|V| - 1 = 12n + 12m + 11$ as desired.

Let f be a BN on G_ψ. For $a = 0, 1$, let f^a be the BN with component set V such that $f^a_{\ell_0}$ is the a-constant function and $f^a_i = f_i$ for every $i \neq \ell_0$. Note that, for every word w not containing

ℓ_0 and any configuration x on V, we have $f^w(x) = (f^a)^w(x)$. Let H be obtained from G_ψ by deleting (c_1, ℓ_0). Then H is the signed interaction digraph of f^a, and it is acyclic. Hence f^a has a unique fixed point; let x be the fixed point of f^0 and y the fixed point of f^1. Obviously, $x_{\ell_0} < y_{\ell_0}$. It is clear that if $f(x)_{\ell_0} < f(y)_{\ell_0}$ then x and y are fixed points of f, a contradiction. Thus $f(x)_{\ell_0} \geq f(y)_{\ell_0}$.

(1) *We have $x_q > y_q$ and, for every configuration z on V,*

- $f^v(z) = x$ *if* $z_{\ell_0} = 0$,
- $f^v(z) = y$ *if* $z_{\ell_0} = 1$,
- $f^u(y + e_{\ell_0}) = x + e_q$,
- $f^u(x + e_{\ell_0}) = y + e_q$.

We have $x_q = f_q(x) = \neg x_{\ell_0} = 1$ and $y_q = f_q(y) = \neg y_{\ell_0} = 0$. We deduce that $f_t(x) = f_t(y) = 1$ if f_t is a disjunction and $f_t(x) = f_t(y) = 0$ if f_t is a conjunction. So $x_t = y_t$ in all cases. Let z be a configuration on V. Since ℓ_0, v is a topological sort of H and x is a fixed point of f^0, we deduce from Lemma 14 that $(f^0)^{\ell^0,v}(z) = x$. So if $z_{\ell_0} = 0$ then $f^v(z) = (f^0)^v(z) = (f^0)^{\ell_0,v}(z) = x$. This proves the first item. Let $z = y + e_{\ell_0}$. Since $z_{\ell_0} = 0$ we have $f^{q,t,u}(z) = f^v(z) = x$ by the first item and since $z_t = y_t = x_t$ we deduce that $f^{q,u}(z) = x$. Since G_ψ has no arc from q to a vertex in $\{u\}$ we have $f^u(z)_i = f^{q,u}(z)_i = x_i$ for all $i \neq q$. Since $f^u(z)_q = z_q = y_q \neq x_q$ we deduce that $f^u(z) = x + e_q$. This proves the third item. We prove similarly the second and fourth items, and (1) follows.

(2) *If $f^{\ell_0,v,w}(x) = f^{\ell_0,v,w}(y)$ then* $\mathbf{w}$ *is a synchronizing word for f.*

Let z, z' be any configurations on V. If $z_{\ell_0} = z'_{\ell_0}$ then $f^v(z) = f^v(z')$ by the first two items of (1) and thus $f^{\mathbf{w}}(z) = f^{\mathbf{w}}(z')$. Suppose, without loss of generality, that $z_{\ell_0} < z'_{\ell_0}$. Then, by the first two items of (1) we have $f^v(z) = x$ and $f^v(z') = y$, and thus if $f^{\ell_0,v,w}(x) = f^{\ell_0,v,w}(y)$ we have $f^{\mathbf{w}}(z) = f^{\mathbf{w}}(z')$. This proves (2).

(3) *If $f(x)_{\ell_0} = f(y)_{\ell_0}$ then* $\mathbf{w}$ *is a synchronizing word for f.*

Suppose that $f(x)_{\ell_0} = f(y)_{\ell_0}$, that is, $f^{\ell_0}(x)_{\ell_0} = f^{\ell_0}(y)_{\ell_0}$. Then, by the first two items of (1) we have $f^{\ell_0,v}(x) = f^{\ell_0,v}(y)$, thus $f^{\ell_0,v,w}(x) = f^{\ell_0,v,w}(y)$ and by (2) $\mathbf{w}$ is a synchronizing word for f. This proves (3).

By (3) we can suppose that $f(x)_{\ell_0} \neq f(y)_{\ell_0}$ and thus $f(x)_{\ell_0} > f(y)_{\ell_0}$. Consequently, by (1) we have $f^{\ell_0,v}(x) = y$ and $f^{\ell_0,v}(y) = x$. Thus, by (2), it is sufficient to prove that $f^w(x) = f^w(y)$. Since $f(x)_{\ell_0} > f(y)_{\ell_0}$ we have $f^{\ell_0}(x) = x + e_{\ell_0}$ and $f^{\ell_0}(y) = y + e_{\ell_0}$ and from the last two items of (1) we have $f^u(x + e_{\ell_0}) = y + e_q$ and $f^u(y + e_{\ell_0}) = x + e_q$. Since $(y + e_q)_{\ell_0} = (y + e_q)_q = 1$ we have $f^t(y + e_q)_t = 1$ and since $(x + e_q)_{\ell_0} = (x + e_q)_q = 0$ we have $f^t(y + e_q)_t = 0$. Since G_ψ has no arc from $\{q, t\}$ to ℓ_0 we have $f^{t,\ell_0}(y + e_q)_{\ell_0} = f(y)_{\ell_0} = 0$ and $f^{t,\ell_0}(x + e_q)_{\ell_0} = f(x)_{\ell_0} = 1$. Thus, summing up, we have

- $f^{\ell_0,u}(x) = y + e_q$ and $f^{t,\ell_0}(y + e_q)_{\ell_0} < f^{t,\ell_0}(y + e_q)_t$,
- $f^{\ell_0,u}(y) = x + e_q$ and $f^{t,\ell_0}(x + e_q)_{\ell_0} > f^{t,\ell_0}(x + e_q)_t$.

Setting $y' = f^{t,\ell_0,\lambda_1^+,\lambda_1^-}(y + e_q)$ and $x' = f^{t,\ell_0,\lambda_1^+,\lambda_1^-}(x + e_q)$ we deduce that

- $y'_{\lambda_1^+} = x'_{\lambda_1^+} = 0$ if $f_{\lambda_1^+}$ is a conjunction,

- $y'_{\lambda_1^+} = x'_{\lambda_1^+} = 1$ if $f_{\lambda_1^+}$ is a disjunction,
- $y'_{\lambda_1^-} = x'_{\lambda_1^-} = 0$ if $f_{\lambda_1^-}$ is a conjunction,
- $y'_{\lambda_1^-} = x'_{\lambda_1^-} = 1$ if $f_{\lambda_1^-}$ is a disjunction.

Furthermore, since

- $y'_{\ell_0} = f^{t,\ell_0}(y + e_q)_{\ell_0} = 0$ and $y'_q = (y + e_q)_q = 1$,
- $x'_{\ell_0} = f^{t,\ell_0}(x + e_q)_{\ell_0} = 1$ and $x'_q = (x + e_q)_q = 0$,

we have

- $f^t(y')_t = f^t(x')_t = 0$ if f_t is a conjunction,
- $f^t(y')_t = f^t(x')_t = 1$ if f_t is a disjunction.

Hence, setting $I = \{t, \lambda_1^+, \lambda_1^-\}$, we have $f^t(y')_I = f^t(x')_I$. Since $G_\psi \setminus I$ is acyclic and u', ℓ_0, q is a corresponding topological sort, by Lemma 14 we have $f^{t,u',\ell_0,q}(y') = f^{t,u',\ell_0,q}(x')$, and thus $f^w(x) = f^w(y)$ as desired.

(3 ⇒ 4) This is obvious.

(4 ⇒ 1) Suppose that z is a satisfying assignment of ψ, and let us prove that some BN on G_ψ is not synchronizing. Let $\boldsymbol{\lambda}^1$ be the literals satisfied by z and $\boldsymbol{\lambda}^0$ the literals not satisfied by z, hence $(\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1)$ is a balanced partition of $\boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$. Let f be the and-or-net on G_ψ such that:

- for all $i \in \boldsymbol{\lambda}^1 \cup \boldsymbol{\mu} \cup \boldsymbol{\mu}' \cup \boldsymbol{c} \cup \{t\}$, f_i is a disjunction;
- for all $i \in \boldsymbol{\lambda}^0 \cup \boldsymbol{\ell}$, f_i is a conjunction.

Let $\boldsymbol{\mu}'^1$ be the vertices in $\boldsymbol{\mu}'$ with at least one in-neighbors in $\boldsymbol{\lambda}^1$ and $\boldsymbol{\mu}'^0 = \boldsymbol{\mu}' \setminus \boldsymbol{\mu}'^1$. Let $I = \boldsymbol{\lambda}^1 \cup \boldsymbol{\mu}'^1 \cup \boldsymbol{\mu}$ and $J = \boldsymbol{\lambda}^0 \cup \boldsymbol{\ell} \cup \boldsymbol{c} \cup \boldsymbol{\mu}'^0$. Let us prove that the configurations x, y on V defined as follows are fixed points of f:

- $x_q = 1$, $x_t = 1$, $x_I = \mathbf{1}$, $x_J = \mathbf{0}$,
- $y_q = 0$, $y_t = 1$, $y_I = \mathbf{1}$, $y_J = \mathbf{1}$.

(4) $f(x)_i = x_i$ *and* $f(y)_i = y_i$ *for all* $i \in \{q, t\} \cup I$.

Since $x_q \neq x_{\ell_0}$ and $y_q \neq y_{\ell_0}$, we have $f(x)_q = \neg x_{\ell_0} = x_q$ and $f(y)_q = \neg y_{\ell_0} = y_q$, and since f_t is a disjunction, we also deduce that $f_t(x) = f_t(y) = 1 = x_t = y_t$. We now prove that $f(x)_I = f(y)_I = \mathbf{1}$. If $i \in \boldsymbol{\lambda}^1 \cup \boldsymbol{\mu}'^1$, then f_i is a disjunction and i has an in-neighbor $j \in \boldsymbol{\lambda}^1 \cup \{t\}$, so $x_j = y_j = 1$ and we deduce that $f_i(x) = f_i(y) = 1$. Next, consider a clause μ_s in $\boldsymbol{\mu}$. Then f_{μ_s} is a disjunction and, since z is a satisfying assignment, μ_s contains a literal $i \in \boldsymbol{\lambda}^1$ and thus $x_i = y_i = 1$. If i is an in-neighbor of μ_s then $f_{\mu_s}(x) = f_{\mu_s}(y) = 1$. Otherwise, i is an in-neighbor of μ'_s, so $\mu'_s \in \boldsymbol{\mu}'^1$. Thus $x_{\mu'_s} = y_{\mu'_s} = 1$ and we deduce that $f_{\mu_s}(x) = f_{\mu_s}(y) = 1$. This proves (4).

(5) $f(x)_J = \mathbf{0}$.

If $i \in \boldsymbol{\mu}'^0$ then for every in-neighbor j of i we have $j \in \boldsymbol{\lambda}^0$, thus $x_j = 0$, and we deduce that $f_i(x) = 0$. If $i \in \boldsymbol{\lambda}^0 \cup \boldsymbol{\ell}$, then f_i is a conjunction and has an in-neighbor $j \in \boldsymbol{\lambda}^0 \cup \boldsymbol{\ell} \cup \boldsymbol{c}$.

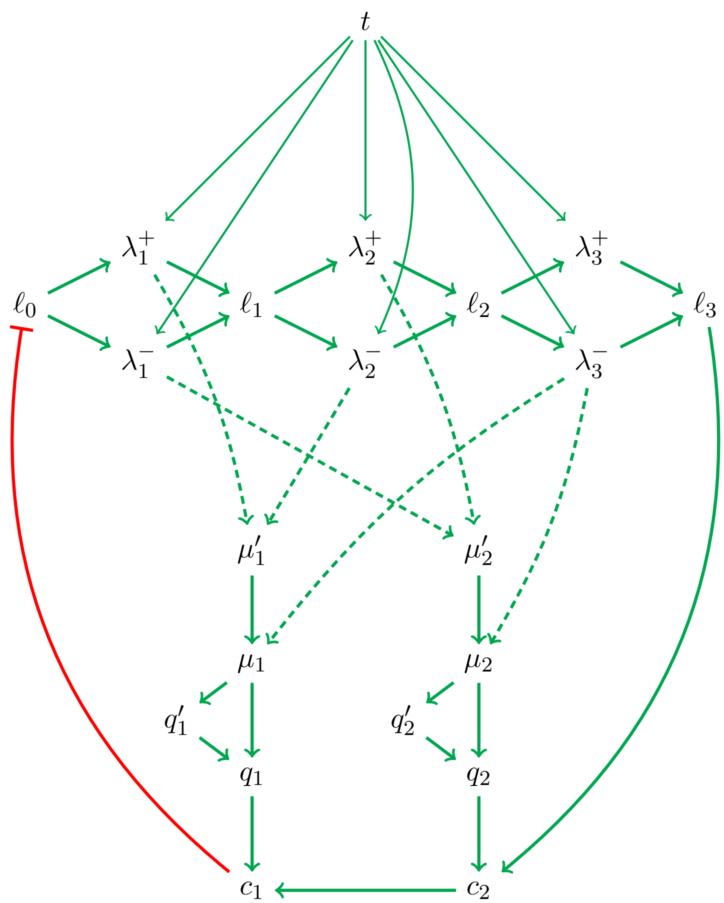


Figure 3: The signed digraph F_ψ for the 3-CNF formula of Figure 1.

Thus $x_j = 0$ we deduce that $f_i(x) = 0$. Setting $c_{m+1} = \ell_n$, for all $s \in [m]$ we have $f_{c_s}(x) = \neg x_{\mu_s} \vee x_{c_{s+1}} = \neg 1 \vee 0 = 0$. This proves (5).

(6) $f(y)_J = \mathbf{1}$.

If $i \in \boldsymbol{\lambda}^0 \cup \boldsymbol{\ell} \cup \boldsymbol{\mu}'^0$ then for every in-neighbor j of i we have $j \neq q$, thus $y_j = 1$, and we deduce that $f_i(y) = 1$. If $i \in \boldsymbol{c}$ then i has a positive in-neighbor $j \in \boldsymbol{c} \cup \{\ell_n\}$, thus $x_j = 1$, and since f_i is a disjunction, we have $f_i(x) = 1$. This proves (6).

By (4), (5) and (6), x and y are fixed points of f and thus f is not synchronizing. □

We now present our second construction. Let F_ψ be the signed digraph obtained from H_ψ by: adding a new vertex t and a positive arc (t, i) for all $i \in \boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$; making negative the arc (c_1, ℓ_0); deleting the arc (μ_s, c_s) for $s \in [m]$ and adding, for all $s \in [m]$, two new vertices q_s, q'_s and the positive arcs $(\mu_s, q'_s), (\mu_s, q_s), (q'_s, q_s)$ and (q_s, c_s). An illustration is given in Figure 3. Thus F_ψ has a unique negative arc, (c_1, ℓ_0), and since all the cycles of F_ψ contain this arc, all the cycles of F_ψ are negative (and ℓ_0 meets every cycle). Also F_ψ is simple and has maximum in-degree two, thus every BN on F_ψ is an and-or-net.

We prove below that ψ is not satisfiable if and only if every BN on F_ψ is synchronizing. Since the SAT problem is NP-complete and F_ψ is a valid instance of the Negative-Synchronizing-Problem, we deduce that this problem is coNP-hard, thus the proof of Theorem 3 is completed.

Theorem 7. *The following conditions are equivalent:*

1. *ψ is not satisfiable.*

2. *Every BN on F_ψ is synchronizing.*

Proof. **(1 ⇒ 2)** Suppose that ψ is not satisfiable and suppose, for a contradiction, that some BN f on F_ψ is not synchronizing. Since t is a source, there is $a \in \{0,1\}$ such that f_t is the a-constant function. Let u be a longest canalizing word from (t,a). Let b be the configuration on $I = \{t\} \cup \{u\}$ such that $b_t = a$ and $b_{\{u\}}$ is the image of u. Hence for all configurations x on V we have $f^{t,u}(x)_I = b$. Let V be the vertex set of F_ψ. If $I = V$ then t,u is a synchronizing word for f, a contradiction. Thus I is a strict subset of V. Let $J = V \setminus I$ and let h be the BN with component set J defined by $h(x_J) = f(x)_J$ for all configurations x on V with $x_I = b$. Let H be the signed interaction digraph of h, which is a subgraph of $F_\psi[J]$.

(1) *h is not synchronizing.*

Suppose that h has a synchronizing word v and let $w = t,u,v$. For all configurations x,y on V we have $f^{t,u}(x)_I = f^{t,u}(y)_I = b$, and since v is a word over J we deduce that $f^v(f^{t,u}(x))_J = h^v(f^{t,u}(x)_J) = h^v(f^{t,u}(y)_J) = f^v(f^{t,u}(y))_J$ and thus $f^w(x) = f^w(y)$. Hence w is a synchronizing word for f, a contradiction. This proves (1).

(2) *H has no sources.*

Suppose, for a contradiction, that H has a source i. Then there is c such that $h(x_J)_i = c$ for all configurations x on V. So if $x_I = b$ then $f(x)_J = h(y_J)$ and thus $f(x)_i = h(x_J)_i = c$. For all configurations x on V with $x_t = a$, we have $f^u(x)_I = b$ and we deduce that $f^{u,i}(x)_i = c$. Since $i \notin I$, u,i is a canalizing word from (t,a) longer than u, a contradiction. This proves (2).

(3) *H has a unique initial strong component, say S, and is homogenous.*

Let S be an initial strong component of H. Since H has no source, S contains a cycle, and since all the cycles of F_ψ contains the arc (c_1,ℓ_0), this arc is in S and S is the unique initial strong component of H. Consequently, there is a path from ℓ_0 to each vertex in H, and since all the paths of F_ψ starting from ℓ_0 are full-positive, we deduce that H is homogenous. This proves (3).

(4) *S is a cycle.*

If not then H, by (2) and (3), H has no sources and no initial cycles, and since H has only negative cycles, we deduce from Theorem 4 that h is synchronizing, and this contradicts (1). This proves (4).

(5) *If $F_\psi[J]$ has an arc from j to i and F_ψ has no arc from I to i, then H has an arc from j to i.*

Suppose that $F_\psi[J]$ has an arc from j to i and F_ψ has no arc from I to i. There is a configuration x on V such that $f_i(x) \neq f_i(x+e_j)$, and since F_ψ has no arc from I to i we can choose x in such a way that $x_I = b$. Then $h_i(x_J) = f_i(x)$ and since $(x+e_j)_I = b$ we have $h_i(x_J + e_j) = h_i((x+e_j)_J) = f_i(x+e_j)$. Hence $h_i(x_J) \neq h_i(x_J+e_j)$ thus H has an arc from j to i. This proves (5).

(6) *For all $s \in [m]$, if $\mu_s, c_s \in J$ then $(\mu_s,q_s),(q'_s,q_s),(q_s,c_s)$ are arcs of H.*

Suppose that $\mu_s, c_s \in J$. We deduce that $q'_s \in J$. Indeed, the unique in-neighbor of q'_s is μ_s, which is in J, and so from Lemma 6 we have $q'_s \in J$. So the two in-neighbors of q_s are in J and from the same lemma we deduce that $q_s \in J$. So $\mu_s, q'_s, q_s \in J$ hence F_ψ has no arc from I to q_s and by (5) we deduce that (μ_s,q_s) and (q'_s,q_s) are arcs of H. If (q_s,c_s) is not in H, we deduce from (5) that $c_{s+1} \in I$, where c_{m+1} means ℓ_n, but then c_s is a source of H and this contradicts (2). Thus (q_s,c_s) is in H. This proves (6).

(7) $\boldsymbol{\mu} \subseteq I$.

Suppose, for a contradiction, that some clause μ_s is in H, and let s be minimal for that property. If H has a cycle which does not contains c_s, then this cycle has necessarily a vertex in $\{\mu_1, \dots, \mu_{s-1}\}$, and this contradicts the choice of s. Thus every cycle of H contains c_s. Thus c_s is in S, and we deduce from (6) that $(\mu_s, q_s), (q'_s, q_s), (q_s, c_s)$ are arcs of H. Hence q_s is in S and has two in-neighbors in S. Thus S is not a cycle, and this contradicts (4). This proves (7).

By (4) and (7), S is a cycle disjoint from $\boldsymbol{\mu}$. We deduce that S contains each vertex in $\boldsymbol{\ell} \cup \boldsymbol{c}$ and exactly one of λ_r^+, λ_r^- for each $r \in [n]$. Let $z \in \{0,1\}^n$ defined by $z_r = 1$ if λ_r^+ is not in S and $z_r = 0$ if λ_r^- is not in S; there is no ambiguity since exactly one of λ_r^+, λ_r^- is in S. By (7) and Lemma 6, for each $s \in [m]$, $G[I]$ has a path from t to μ_s, and thus each clause μ_s contains a literals in I. This literal is not in C, thus it is satisfied by z. So ψ is satisfied by z, a contradiction. We deduce that if ψ is not satisfiable, then every BN on F_ψ is synchronizing.

($2 \Rightarrow 1$) Suppose that z is a satisfying assignment of ψ, and let us prove that some BN on F_ψ is not synchronizing. Let $\boldsymbol{\lambda}^1$ be the literals satisfied by z and $\boldsymbol{\lambda}^0$ the literals not satisfied by z, hence $(\boldsymbol{\lambda}^0, \boldsymbol{\lambda}^1)$ is a balanced partition of $\boldsymbol{\lambda}^+ \cup \boldsymbol{\lambda}^-$. Let f be any and-or-net on F_ψ such that:

- f_t is the 1-constant function,
- for all $i \in \boldsymbol{\lambda}^1 \cup \boldsymbol{\mu} \cup \boldsymbol{\mu}'$, f_i is a disjunction,
- for all $i \in \boldsymbol{\ell} \cup \boldsymbol{\lambda}^0 \cup \boldsymbol{c}$, f_i is a conjunction.

Let $\boldsymbol{\mu}'^1$ be the vertices in $\boldsymbol{\mu}'$ with at least one in-neighbors in $\boldsymbol{\lambda}^1$. Let $\boldsymbol{q} = \{q_1, \dots, q_m\}$ and $\boldsymbol{q}' = \{q'_1, \dots, q'_m\}$. Let $I = \{t\} \cup \boldsymbol{\lambda}^1 \cup \boldsymbol{\mu}'^1 \cup \boldsymbol{\mu} \cup \boldsymbol{q} \cup \boldsymbol{q}'$, and let X be the set of configurations x on the vertex set of F_ψ with $x_I = \mathbf{1}$.

(1) $f(X) \subseteq X$.

Let $x \in X$. We have $f_t(x) = 1$. For every $i \in \boldsymbol{\lambda}^1$, f_i is a disjunction and since $x_t = 1$ we have $f_i(x) = 1$. For every $i \in \boldsymbol{\mu}'^1$, f_i is a disjunction and i has an in-neighbor $j \in \boldsymbol{\lambda}^1$, and since $x_j = 1$ we deduce that $f_i(x) = 1$. Next, consider a clause μ_s in $\boldsymbol{\mu}$. Then f_{μ_s} is a disjunction and, since z is a satisfying assignment, μ_s contains a literal $i \in \boldsymbol{\lambda}^1$ and thus $x_i = 1$. If i is an in-neighbor of μ_s then $f_{\mu_s}(x) = 1$. Otherwise, i is an in-neighbor of μ'_s, so $\mu'_s \in \boldsymbol{\mu}'^1$. Thus $x_{\mu'_s} = 1$ and we deduce that $f_{\mu_s}(x) = 1$. Finally every $i \in \boldsymbol{q} \cup \boldsymbol{q}'$ has only in-neighbors in $\boldsymbol{q}' \cup \boldsymbol{\mu} \subseteq I$, and we deduce that $f_i(x) = 1$. Thus $f(x)_I = \mathbf{1}$, that is, $f(x) \in X$. This proves (1).

Let $J = \boldsymbol{\ell} \cup \boldsymbol{\lambda}^0 \cup \boldsymbol{c}$. So (I, J) is a partition of the vertices of F_ψ. Note $F_\psi[J]$ is a negative cycle since exactly one of λ_r^+, λ_r^- is un $\boldsymbol{\lambda}^0$ for each $r \in [n]$.

(2) *Let $i \in J$ and let j its in-neighbor in $F_\psi[J]$. For all $x, y \in X$, if $x_j \neq y_j$ then $f_i(x) \neq f_i(y)$.*

Let $x, y \in X$ with $x_j \neq y_j$. If $i = \ell_0$ then $j = c_1$ and $f_i(x) = \neg x_j \neq \neg y_j = f_i(y)$. Otherwise, j is a positive in-neighbor of i. Furthermore, i has exactly one in-neighbor $k \neq j$, which is positive and belongs to $\{t\} \cup \boldsymbol{\lambda}^1 \cup \boldsymbol{q} \subseteq I$. Hence $x_k = y_k = 1$, and since f_i is a conjunction, we obtain $f_i(x) = x_j \wedge x_k = x_j \wedge 1 = x_j$ and $f_i(y) = y_j \wedge y_k = y_j \wedge 1 = y_j$. Thus $f_i(x) \neq f_i(y)$. This proves (2).

Let x and y be J-opposite configurations in X, that is, $x, y \in X$ and $x_i \neq y_i$ for all $i \in J$. Let $i \in I$. By (1) we have $f(x), f(y) \in X$ and we deduce that $f^i(x) = x$ and $f^i(y) = y$, thus $f^i(x)$ and $f^i(y)$ are J-opposite configurations in X. Let $i \in J$. Obviously, $f^i(x), f^i(y) \in X$. Furthermore, by (2) we have $f_i(x) \neq f_i(y)$, thus $f^i(x)$ and $f^i(y)$ are J-opposite configurations in X. So for any vertex i in F_ψ, $f^i(x)$ and $f^i(y)$ are J-opposite configurations in X. This implies that $f^w(x)$ and $f^w(y)$ are J-opposite configurations in X for any word w, and thus f is not synchronizing. $\square$

6 Concluding remarks

As said in the introduction (and proved in Appendix A), if G has no negative cycles, every synchronizing BN on G has a synchronizing word of length n: this solves (in a strong form) Černý conjecture when G has no negative cycles. It would be nice to prove the conjecture when G has no positive cycles. In this paper, we were only able to do that for and-or-nets in the strong case: if G has no positive cycles and is strongly connected, every synchronizing and-or-net on G has a synchronizing word of length at most $10(\sqrt{5}+1)^n$. Another perspective is to prove the Černý conjecture for and-or-nets (independently of the interaction digraph).

We feel that the upper bound $10(\sqrt{5}+1)^n$ can be widely improved, and we were not able to find any non-trivial lower bound. Besides, it seems possible to have better expressions, using some features of G instead of the number n of vertices only. For example, let $\tau(G)$ be the minimum size of a feedback vertex set of G. Using Lemma 14, one can easily replace the bound $10(\sqrt{5}+1)^n$ by the better bound $\tau(G) + 3F(n+4)2^{\tau(G)}$. However, for the particular case $\tau(G) \leq 1$, the following gives something much better (see Appendix B for the proof).

Proposition 5. *Let G be a strongly connected signed digraph on $[n]$ without positive cycles, which is not a cycle. If $\tau(G) \leq 1$ then there is a word w of length $3n-1$ which synchronizes every and-or-net on G.*

Following the approach initiated in [6], it would be also interesting to study the length of words that synchronizes families of synchronizing BNs. The previous proposition gives an example, where the family is the set of and-or-nets on G. Let us discuss another bigger family, namely, the families $\mathcal{F}^+(n)$ of synchronizing BNs with component set $[n]$ and whose signed interaction digraphs have no negative cycles. Let $\ell^+(n)$ be the minimum length of a word synchronizing all the members of $\mathcal{F}^+(n)$. Since each member of $\mathcal{F}^+(n)$ has a synchronizing word of length n, considering the concatenation of these words, we obtain $\ell^+(n) \leq n|\mathcal{F}^+(n)|$, which is doubly exponential in n, since $\mathcal{F}^+(n)$ is. But this is far from the true value of $\ell^+(n)$, which is only quadratic in n (see Appendix A for a proof).

Proposition 6. $\ell^+(n) = n^2 - o(n^2)$.

By symmetry, it would be interesting to study the minimum length $\ell^-(n)$ of a word that synchronizes the families of synchronizing BNs with component set $[n]$ whose signed interaction digraphs have no positive cycles.

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A Synchronizing BNs with a unique fixed point

In this appendix, we prove the two results given in the introduction before the statement of Theorem 1, which are Propositions 8 and 9 below. We then prove Proposition 6 of Section 6.

The first tool is a classical result of Aracena, already mentioned.

Theorem 8 (Aracena [3])**.** *Let G be a signed digraph, and let f be a BN on G.*

1. *If G has no negative cycles, then f has at least one fixed point.*

2. *If G has no positive cycles, then f has at most one fixed point.*

3. *If G has no negative cycles and no sources, then f has at least two fixed points.*

4. *If G has no positive cycles and no sources, then f has no fixed points.*

The second tool is the following. Given two words u, v, we say that u *contains* v if v can be obtained by deleting some letters in u. For instance, aba contains aa.

Proposition 7. *Let G be signed digraph with vertex set V such that all the cycles of G have the same sign. Let f be a BN on G with a unique fixed point. There is a permutation π of V such that any word containing π synchronizes f. In particular, π synchronizes f.*

Proof. We proceed by induction on $|V|$. For $|V| = 1$, since f has a unique fixed point, f is a constant function and the result is obvious. Suppose that $|V| \geq 2$. Since all the cycles of G have the same sign and f has a unique fixed point, we deduce from Theorem 8 that G has a source, say i. Hence, f_i is the a-constant function for some $a \in \{0, 1\}$. Let $I = V \setminus \{i\}$, where V is the vertex set of G. Let h be the BN with component set I defined by $h(x_I) = f(x)_I$ for all configurations x on V with $x_i = a$. Since f has a unique fixed point, h has a unique fixed point, and since f_i is the a-constant function, for every configuration x on V and every word v we have:

$$x_i = a \;\Rightarrow\; f^v(x)_I = h^v(x_I) \text{ and } f^v(x)_i = a. \tag{1}$$

Let H be the signed interaction digraph of G. Since H is a subgraph of G, all the cycles of H have the same sign. Hence, by induction hypothesis, there is a permutation σ of I such that any word containing σ synchronizes h. Let $\pi = i, \sigma$, and let w be a word containing π. Since w contains i, there is u, v such that $w = u, i, v$ where u does not contains i. So v contains σ, thus v synchronizes h. Let x, y be any two configurations on V, and let $x' = f^{u,i}(x)$ and $y' = f^{u,i}(y)$. Since f_i is the a-constant function, we have $x'_i = y'_i = a$, and since v synchronizes h we deduce from (1) that $f^v(x')_I = h^v(x'_I) = h^v(y'_I) = f^v(y')_I$ and $f^v(x')_i = f^v(y')_i = a$. Thus $f^v(x') = f^v(y')$ and this proves that w synchronizes f. ☐

We deduce the following two propositions.

Proposition 8. *Let G be an n-vertex signed digraph without negative cycles, and let f be a BN on G. The following conditions are equivalent:*

1. *f has a unique fixed point.*

2. *f has a synchronizing word of length n.*

3. *f is synchronizing.*

Proof. **(1 ⇒ 2)** is given by Proposition 7 and **(2 ⇒ 3)** is obvious. Suppose that f is synchronizing. Then it has at most one fixed point. By Theorem 8, f has at least one fixed point. Thus f has a unique fixed point. This proves **(3 ⇒ 1)**. ☐

Proposition 9. *Let G be an n-vertex signed digraph without positive cycles, and let f be a BN on G with a fixed point. Then f has a synchronizing word of length n.*

Proof. Since f has a fixed point, by Theorem 8 it has a unique fixed point. We then deduce from Proposition 7 that f has a synchronizing word of length n. ☐

We now prove Proposition 6 of Section 6. Let us say that a word synchronizes a set of BNs if it synchronizes every BN in this set. Let $\mathcal{A}(n)$ be the set of BNs with component set $[n]$ and with an acyclic interaction digraph. A word w is *n-complete* if it contains all the permutations of $[n]$. Given a word w and a BN f with component set V, we say that w *fixes* f if $f^w(x)$ is a fixed point of f for all configurations x on V. This notion was introduced in [6], where it is proved that a word fixes every BN in $\mathcal{A}(n)$ if and only if it is n-complete. It is clear that if f has a unique fixed point, then w fixes f if and only if w synchronizes f. Also, by Theorem 8, every BN in $\mathcal{A}(n)$ has a unique fixed point. Consequently, we can restate the result of [6] mentioned above as follows.

Theorem 9 ([6])**.** *A word synchronizes $\mathcal{A}(n)$ if and only if it is n-complete.*

Let $\mathcal{F}^+(n)$ be the set of synchronizing BNs with component set $[n]$ and whose signed interaction digraphs have no negative cycles. Let $\mathcal{F}(n)$ be the set of f BN with component set $[n]$, with a unique fixed point, and such that all the cycles of the signed interaction digraph of f have the same sign. Note that $\mathcal{A}(n) \subseteq \mathcal{F}^+(n) \subseteq \mathcal{F}(n)$; the first inclusion is obvious and the second follows from Proposition 8.

Proposition 10. *The following conditions are equivalent:*

1. *w is n-complete.*
2. *w synchronizes $\mathcal{F}(n)$.*
3. *w synchronizes $\mathcal{F}^+(n)$.*
4. *w synchronizes $\mathcal{A}(n)$.*

Proof. **(1 ⇒ 2)** is given by Proposition 7. **(2 ⇒ 3)** holds since $\mathcal{F}^+(n) \subseteq \mathcal{F}(n)$. **(3 ⇒ 4)** holds since $\mathcal{A}(n) \subseteq \mathcal{F}^+(n)$. **(4 ⇒ 1)** is given by Theorem 9. □

Let $\lambda(n)$ be the shortest length of an n-complete word, and let $\ell^+(n)$ the minimum length of a word that synchronizes $\mathcal{F}^+(n)$. By the previous proposition, $\ell^+(n) = \lambda(n)$, and it is proved in [21] that $\lambda(n) = n^2 - o(n^2)$. We thus obtain Proposition 6.

B Proof of Proposition 5

We need the following two lemmas.

Lemma 15. *Let G be a strong signed digraph without positive cycles. Suppose that G has a vertex i that meets every cycle of G. Then G is switch equivalent to a signed digraph H in which all the in-coming arcs of i are negative and all the other arcs are positive.*

Proof. Let V be the vertex set of G. Let G' be obtained by changing the sign of every in-coming arc of i. Then all the cycles of G' are positive. By Proposition 3, the I-switch H' of G' is full positive for some $I \subseteq V$. Let H by obtained from H' by making negative the in-coming arcs of i. One easily check that H is the I-switch of G and thus H has the desired properties. □

Lemma 16. *Let G be a n-vertex strong signed digraph, which is not a cycle. Suppose that G has a vertex i such that: i meets every cycle; all the in-coming arcs of G are negative and all the other arcs are positive; i is of in-degree at least two. There is a word w of length $3n-1$ which synchronizes every BN f on G such that f_i is a conjunction or a disjunction.*

Proof. Let V be the vertex set of G, and let u be a topological sort of $G \setminus i$.

(1) *For every configuration x on V and prefix v of u, we have $f^v(x)_j = x_i$ for all $j \in \{v\} \cup \{i\}$.*

We proceed by induction on $|v|$. This is obvious for $|v| = 0$. Suppose that $|v| \geq 1$. Let v' be the prefix of u of length $|v| - 1$, thus $v = v', k$ for some $k \in V$. Let $y = f^{v'}(x)$. For all $j \in \{v'\} \cup \{i\}$, we have $f^v(x)_j = y_j = x_i$, where the induction is used for the second equality. Since all the in-neighbors of k are in $\{v'\} \cup \{i\}$ and positive, we deduce that $f_k(y) = x_i$, and since $f^v(x)_k = f_k(y) = x_i$ this completes the induction. This proves (1).

Let $v = \epsilon$ if i has a loop, and let v be the shortest prefix of u containing an in-neighbor of i otherwise. Note that $|v| \leq n - 1$ since $|u| = n - 1$. Let

$$w = u, i, v, i, u,$$

of length $2n + |v| \leq 3n - 1$. Let f be a BN on G such that f_i is a conjunction or a disjunction. We will prove that w synchronizes f. Let x be any configuration on V. Let y the configuration on V such that $y_j = x_i$ for all $j \in V$. Let $I = \{v\} \cup \{i\}$, which contains an in-neighbor of i, and $z = y + e_I$. By (1) we have $f^u(x) = y$. Since i has only negative in-neighbors, $f^i(y) = y + e_i$. We then deduce from (1) that $f^v(y + e_i) = y + e_I = z$. Let j be an in-neighbor of i in I. Since i if of in-degree at least two, it has an in-neighbor $k \notin I$. Then $z_j = y_j + 1 = x_i + 1$ and $z_k = y_k = x_i$. Thus $z_j \neq z_k$ and we deduce that $f(z)_i = 0$ if f_i is a conjunction, and $f(z)_i = 1$ if f_i is a disjunction. We deduce from (1) that $f^{i,u}(z) = \mathbf{0}$ if f_i is a conjunction, and $f^{i,u}(z) = \mathbf{1}$ if f_i is a disjunction. Since $f^{u,i,v}(x) = z$ we have $f^w(x) = f^{i,u}(z)$. Hence $f^w(x) = \mathbf{0}$ if f_i is a conjunction and $f^w(x) = \mathbf{1}$ if f_i is a disjunction. Thus w synchronizes f. □

We are now ready to prove Proposition 5, that we restate.

Proposition 5. *Let G be a strongly connected signed digraph on $[n]$ without positive cycles, which is not a cycle. If $\tau(G) \leq 1$ then there is a word w of length $3n - 1$ which synchronizes every and-or-net on G.*

Proof. If $\tau(G) = 0$, then, by Lemma 14, there is a word of length n which synchronizes every BN on G and we are done. Suppose that $\tau(G) = 1$. Since G is not a cycle, it has a vertex i of in-degree at least two that meets every cycle of G. By Lemma 15, G is switch equivalent to a signed digraph H which satisfies the conditions of Lemma 16. By this lemma, there is a word w of length at most $3n - 1$ which synchronizes every and-or-net on H. We then deduce from Proposition 4 that w synchronizes every and-or-net on G. □

C A result mentioned in Remark 1

Proposition 11. *Let $X \subseteq \{0, 1\}^n$ such that, for all $i \in [n]$, there is $x, y \in X$ that only differ in component i. We have $|X| \geq n + 1$.*

Proof. We proceed by induction on n. The case $n = 1$ is obvious. Suppose that $n \geq 2$ and let $I = [n - 1]$. For $a = 0, 1$, let $X^a = \{x_I \mid x \in X, x_n = a\}$. For every $i \in I$, there is $x, y \in X$ that only differ in i, so x_I and y_I only differ in i, and $x_I, y_I \in X^0 \cup X^1$. Since $X^0 \cup X^1 \subseteq \{0, 1\}^{n-1}$, by induction hypothesis,

$$n \leq |X^0 \cup X^1| = |X^0| + |X^1| - |X^0 \cap X^1| = |X| - |X^0 \cap X^1|.$$

Let $x, y \in X$ that only differ in n, say $x_n < y_n$. Then $x_I \in X^0$ and $y_I \in X^1$, and since $x_I = y_I$ we deduce that $|X^0 \cap X^1| \geq 1$ and thus $|X| \geq n + 1$. □

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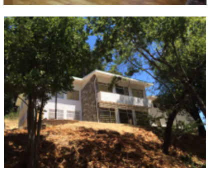

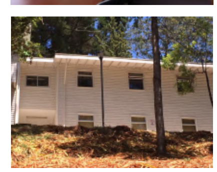

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