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ON EXISTENCE OF ENTROPY SOLUTIONS FOR 1D NONLOCAL CONSERVATION LAWS WITH SPACE-DISCONTINOUS FLUX

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Abstract. We prove the well-posedness of entropy weak solutions for a class of 1D space-discontinuous scalar conservation laws with non-local flux, describing traffic flow on roads with rough conditions. We approximate the problem through a Godunov-type numerical scheme and provide $\mathbf{L}^\infty$ and $\mathbf{BV}$ estimates for the approximate solutions. The limit model as the kernel support tends to zero is numerically investigated.

1. Introduction

Models of conservation laws with nonlocal flux describe several phenomena such as slow erosion of granular flow [3, 29], synchronization [2], sedimentation [6], crowd dynamics [16], navigation processes [4] and traffic flow [7, 10, 13, 14]. In particular, non-local traffic models describe the behaviour of drivers that adapt their velocity with respect to what happens to the cars in front of them. In this type of models, the flux function depends on a *downstream* convolution term between the density of vehicles and a kernel function with support on the negative axis. See [10] for an overview about non-local traffic models and [12] for a continuous non-local model describing the behavior of drivers on two stretches of a road with different velocities and capacities.

We are interested in the analysis of the well-posedness and the numerical approximation of solutions of nonlocal conservation laws with a single spatial discontinuity in the flux

$$\begin{cases} \partial_t \rho + \partial_x f(t,x,\rho) = 0, & (t,x) \in (0,\infty)\times\mathbb{R}, \\ \rho(0,x) = \rho_0(x), & x \in \mathbb{R}, \end{cases} \tag{1.1}$$

with

$$f(t,x,\rho) = H(-x)\ \rho\, v_l(\omega_\eta * \rho) + H(x)\ \rho\, v_r(\omega_\eta * \rho),$$

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where $H(x)$ is the Heaviside function, with which the flux $f(x,t,\rho)$ has a discontinuous at $x=0$ if the velocity functions $v_l(\rho)$ and $v_r(\rho)$ are different. We assume that the convolution term and the kernel function ω_η satisfies

$$
\begin{aligned}
&(\omega_\eta * \rho)(t,x) = \int_x^{x+\eta} \rho(t,y)\omega_\eta(y-x)dy, \qquad \eta > 0,\\
&\text{with } \omega_\eta \in \mathbf{C}^2([0,\eta],\mathbb{R}^+), \quad \omega_\eta' \leq 0, \quad \omega_\eta(\eta) = 0,
\end{aligned} \tag{1.2}
$$

and the following hypothesis hold on the velocity functions

$$
v_s(\rho) = k_s\psi(\rho), \quad s = l, r, \quad \text{and} \quad k_l < k_r, \quad \psi \in \mathbf{C}^2(\mathbb{R}),\ \text{s.t. } \psi' \leq 0. \tag{1.3}
$$

In the traffic vehicle context ρ represents the density of vehicles on the roads, ω_η is a non-increasing kernel function whose support η is proportional to the look-ahead distance of drivers, that are supposed to adapt their velocity with respect to the mean downstream traffic density. The equation in (1.1) is a non-local version of the Lightill-Whitham-Richards traffic model [19, 26, 27] with a discontinuous velocity field [15, 25].

There are many results relating to existence, uniqueness, stability and numerical approximation of weak entropy solutions of *local* conservation laws with a spatially discontinuous flux [1, 5, 8, 9, 15, 18, 20, 21, 22, 23, 24, 25]. Conversely, in the nonlocal case, a traveling waves for a traffic flow model with rough road conditions was studied in [28] and recently, in [11] the authors propose a non-local scalar space-discontinuous model to describe the traffic flow on two consecutive roads with different speed limits and they prove the well-posedness using the vanishing viscosity technique, under the hypothesis on the velocity functions

$$
v_s(\rho) = k_s(1-\rho), \quad s = l, r, \quad \text{and} \quad k_l < k_r.
$$

The aim of this paper is manifold:

- we prove the well-posedness of this non-local space-discontinuous traffic model when the maximum speed limit of the left road is less than the right one, i.e. $k_l < k_r$ for a general non-increasing speed function ψ, approximating the problem through a Godunov-type numerical scheme and proving standard compactness estimates;
- we numerically analyze the scenario with the maximum speed limit of the left road greater than the right one, i.e. $k_l > k_r$;
- we numerically study the limit model as the support of the kernel function tends to 0^+.

Following [23], we recall the following definitions of solution.

Definition 1.1. *We say that a function $\rho \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)([0,T]\times\mathbb{R}; [0,\rho_{\max}])$ is a weak solution of the initial value problem* (1.1) *if for any test function $\varphi \in \mathbf{C}^1_{\mathbf{c}}([0,T[\times\mathbb{R};\mathbb{R})$*

$$
\int_0^T \int_{\mathbb{R}} \left(\rho\partial_t\varphi + f(t,x,\rho)\partial_x\varphi\right) \mathrm{d}t\mathrm{d}x + \int_{\mathbb{R}} \rho_0(x)\varphi(0,x)\mathrm{d}x = 0.
$$

Definition 1.2. *A function* $\rho \in (\mathbf{L}^1 \cap \mathbf{L}^\infty)([0,T]\times\mathbb{R};[0,\rho_{\max}])$ *is an entropy weak solution of* (1.1)*, if for all* $c \in [0,\rho_{\max}]$*, and any test function* $\varphi \in \mathbf{C_c^1}([0,T[\times\mathbb{R};\mathbb{R}^+)$

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} |\rho-c|\,\varphi_t + \operatorname{sgn}(\rho-c)(f(t,x,\rho)-f(t,x,c))\,\varphi_x\,\mathrm{d}x\,\mathrm{d}t \\ - \int_0^T\int_{\mathbb{R}^*} \operatorname{sgn}(\rho-c) f(t,x,c)_x\varphi\,\mathrm{d}x\,\mathrm{d}t + \int_{\mathbb{R}} |\rho_0(x)-c|\,\varphi(0,x)\mathrm{d}x \\ + \int_0^T |(k_r-k_l)c\,\psi(\rho*\omega_\eta)|\,\varphi(t,0)\mathrm{d}t \geq 0.\end{aligned}$$

The paper is organized as follows. In Section 2, we present a Godunov-type numerical scheme that we use to discretize our problem. After that, in Section 3 we prove the existence and uniqueness of weak entropy solutions with $\mathbf{L}^\infty$ and $\mathbf{BV}$ bounds. Finally, in Section 4, we show some numerical tests illustrating the behaviour of solutions and investigating the limit model as the support of the kernel $\eta \to 0^+$.

2. Numerical scheme

We introduce a uniform space mesh of width Δx and a time step Δt, subject to a CFL condition, to be detailed later on. The spatial domain is discretized into uniform cells $I_j = [x_{j-1/2}, x_{j+1/2})$, where $x_{j+1/2} = x_j + \Delta x/2$ are the cell interfaces, and $x_j = j\Delta x$ the cell centers, in particular $x = 0$ where the flux function changes, falls at the midpoint of the cell $I_0 = [x_{-1/2}, x_{1/2})$. We take Δx such that $\eta = N\Delta x$ for some $N \in \mathbb{N}$. Let $t^n = n\Delta t$ be the time mesh and $\lambda = \Delta t/\Delta x$. We aim to construct a finite volume approximate solution ρ_Δ such that $\rho_\Delta(x,t) = \rho_j^n$ for $(t,x) \in [t^n, t^{n+1}[\times[x_{j-1/2}, x_{j+1/2})$. To this end, we approximate the initial datum ρ_0 with the cell averages

$$\rho_j^0 = \frac{1}{\Delta x}\int_{x_{j-1/2}}^{x_{j+1/2}} \rho_0(x)\mathrm{d}x,$$

we denote $\omega_k := \frac{1}{\Delta x}\int_{k\Delta x}^{(k+1)\Delta x}\omega(y)\mathrm{d}y$ for $k = 0,\dots,N-1$ and set the convolution term

$$R(x_{j+1/2},t^n) = (\omega_\eta * \rho^{\Delta x})(x_{j+1/2},t^n) \approx \Delta x\sum_{k=0}^{N-1}\omega_k\rho_{j+k+1}^n.$$

In this way we can define the following finite volume scheme

$$\rho_j^{n+1} = \rho_j^n - \lambda\left(F(x_{j+1/2},\rho_j^n,R_{j+1/2}^n) - F(x_{j-1/2},\rho_{j-1}^n,R_{j-1/2}^n)\right), \qquad j\in\mathbb{Z}, \tag{2.1}$$

where F is a modified Godunov numerical flux which is based on the scheme introduced in [12, 17]

$$F(x_{j+1/2},\rho,R) = \begin{cases}\rho\, v_l(R) & \text{if } x_{j+1/2} < 0,\\ \rho\, v_r(R) & \text{if } x_{j+1/2} > 0.\end{cases} \tag{2.2}$$

3. Well-posedness

Lemma 3.1. *Let hypotheses* (1.3) *hold. Given an initial datum such that* $0 \leq \rho_j^0 \leq \rho_{\max}$ *for* $j \in \mathbb{Z}$*, the finite volume scheme* (2.1)-(2.2) *is such that*

$$0 \leq \rho_j^{n+1} \leq \rho_{\max}, \qquad j \in \mathbb{Z},$$

under the CFL condition

$$\Delta t \leq \min_{s=l,r} \left\{ \frac{\Delta x}{\Delta x \omega_\eta(0) \rho_{\max} k_s \|\psi'\|_{\mathbf{L}^\infty} + k_s \|\psi\|_{\mathbf{L}^\infty}} \right\}. \tag{3.1}$$

Proof. By induction, assume that $0 \leq \rho_j^n \leq \rho_{\max}$ for all $j \in \mathbb{Z}$. Let us consider $j \neq 0$ and set $v(\rho) := k_s \psi(\rho)$ for $s = l, r$. In this case, we can observe that

$$\begin{aligned}
\rho_j^{n+1} &= \rho_j^n - \lambda \left(\rho_j^n v(R_{j+1/2}^n) - \rho_{j-1}^n v(R_{j-1/2}^n) \right) \\
&\leq \rho_j^n + \lambda \left(\rho_{\max} v(R_{j-1/2}^n) - \rho_j^n v(R_{j+1/2}^n) \right) \\
&\leq \rho_j^n + \lambda \left((\rho_{\max} - \rho_j^n) v(R_{j-1/2}^n) + \rho_j^n (v(R_{j-1/2}^n) - v(R_{j+1/2}^n)) \right),
\end{aligned}$$

Using the hypothesis (1.2) on the kernel function ω_η and computing,

$$\begin{aligned}
R_{j-1/2}^n - R_{j+1/2}^n &= \Delta x \left(\sum_{k=0}^{N-1} \omega_k \rho_{j+k}^n - \sum_{k=0}^{N-1} \omega_k \rho_{j+k+1}^n \right) \\
&= \Delta x \left(\omega_0 \rho_j^n + \sum_{k=1}^{N-1} (\omega_k - \omega_{k-1}) \rho_{j+k}^n + \underbrace{\omega_N}_{:=0} \rho_{j+N} - \omega_{N-1} \rho_{j+N}^n \right) \\
&= \Delta x \left(\omega_0 \rho_j^n + \sum_{k=1}^{N} (\omega_k - \omega_{k-1}) \rho_{j+k}^n \right) \\
&\leq \Delta x \left(\omega_0 \rho_j^n + \sum_{k=1}^{N} (\omega_k - \omega_{k-1}) \rho_{\max} \right) \\
&= \Delta x \omega_0 \left(\rho_j^n - \rho_{\max} \right),
\end{aligned}$$

we write

$$\begin{aligned}
\rho_j^{n+1} &\leq \rho_j^n + \lambda \left((\rho_{\max} - \rho_j^n) v(R_{j-1/2}^n) - \rho_j^n v'(R_j^n) \Delta x \omega_0 \left(\rho_{\max} - \rho_j^n \right) \right), \\
&\leq \rho_j^n + \lambda \left(k_s \|\psi\|_{\mathbf{L}^\infty} + \Delta x \omega_0 \rho_{\max} k_s \|\psi'\|_{\mathbf{L}^\infty} \right) \left(\rho_{\max} - \rho_j^n \right).
\end{aligned}$$

Under the CFL condition (3.1), we conclude $\rho_j^{n+1} \leq \rho_{\max}$ for all $j \in \mathbb{Z}^*$.
For $j = 0$, recalling the hypothesis $v_l(\rho) \leq v_r(\rho)$, we obtain

$$\begin{aligned}
\rho_0^{n+1} &= \rho_0^n - \lambda \left(\rho_0^n v_r(R_{1/2}^n) - \rho_{-1}^n v_l(R_{-1/2}^n) \right) \\
&\leq \rho_0^n + \lambda \left(\rho_{\max} v_l(R_{-1/2}^n) - \rho_0^n v_r(R_{1/2}^n) \right) \\
&\leq \rho_0^n + \lambda \left(\rho_{\max} v_r(R_{-1/2}^n) - \rho_0^n v_r(R_{1/2}^n) \right),
\end{aligned}$$

and we proceed as before. To prove the positivity $\rho_j^{n+1} \geq 0$, we observe that

$$\rho_j^{n+1} = \rho_j^n - \lambda \left(\rho_j^n v(R_{j+1/2}^n) - \rho_{j-1}^n v(R_{j-1/2}^n) \right)$$

$$\begin{aligned}&\geq \rho_j^n \left(1-\lambda v(R^n_{j+1/2})\right)\\ &\geq 0.\end{aligned}$$

This concludes the proof. □

Remark 3.1. *If $v_l(\rho) > v_r(\rho)$ we can compute*

$$\begin{aligned}\rho_0^{n+1} &= \rho_0^n - \lambda\left(\rho_0^n v_r(R^n_{1/2}) - \rho^n_{-1} v_l(R^n_{-1/2})\right)\\ &\leq \rho_0^n + \lambda\left(\|\rho^n\|_{\mathbf{L}^\infty} v_l(R^n_{-1/2}) - \rho_0^n v_l(R_{1/2}) \pm \rho_0^n v_l(R^n_{-1/2}) \pm \rho_0^n v_l(R^n_{1/2})\right)\\ &\leq \rho_0^n + \lambda\left((\|\rho^n\|_{\mathbf{L}^\infty} - \rho_0^n) v_l(R^n_{-1/2}) + \rho_0^n(v_l(R^n_{-1/2}) - v_l(R^n_{1/2})) + \rho_0^n(v_l(R^n_{1/2}) - v_r(R^n_{1/2}))\right)\\ &\leq \rho_0^n + \lambda\left(\|v_l\|_{\mathbf{L}^\infty} + \Delta x\omega_0\|\rho^n\|_{\mathbf{L}^\infty}\|v_l'\|_{\mathbf{L}^\infty}\right)\left(\|\rho^n\|_{\mathbf{L}^\infty} - \rho_0^n\right) + \lambda\|\rho^n\|_{\mathbf{L}^\infty}\|v_l - v_r\|_{\mathbf{L}^\infty}\\ &\leq \|\rho^n\|_{\mathbf{L}^\infty}(1+\lambda\|v_l - v_r\|_{\mathbf{L}^\infty})\\ &\leq \rho_{\max}(1 + \frac{(k_l-k_r)\|\psi\|_{\mathbf{L}^\infty}}{\Delta x\omega_\eta(0)\rho_{\max}k_l\|\psi'\|_{\mathbf{L}^\infty} + k_l\|\psi\|_{\mathbf{L}^\infty}}).\end{aligned}$$

We notice that we are not able to recover an upper bound for ρ_j^n independent from the mesh in this case. See also [11, Remark 2.1].

Lemma 3.2 ($\mathbf{L}^1$ norm)**.** *Let hypotheses* (1.3) *hold. If $\rho_0 \in \mathbf{L}^1(\mathbb{R};\mathbb{R}^+)$ then under the CFL condition* (3.1)*, the approximate solution ρ_Δ constructed through the finite volume scheme* (2.1)-(2.2) *satisfies*

$$\|\rho_\Delta(t,\cdot)\|_{\mathbf{L}^1} = \|\rho_0\|_{\mathbf{L}^1}, \quad \text{for all } t>0. \tag{3.2}$$

Proof. By induction, suppose that (3.2) holds for $t^n = n\Delta t$. Thanks to the positivity and the conservative form of the numerical scheme (2.1) we have

$$\left\|\rho^{n+1}\right\|_{\mathbf{L}^1} = \Delta x\sum_{j\in\mathbb{Z}}\rho_j^{n+1} = \|\rho^n\|_{\mathbf{L}^1}.$$

□

We now prove the $\mathbf{L}^1$-continuity in time by following the idea introduced in [22].
For the sake of simplicity we use the following notation throughout the proof, let us define

$$v^n_{j+1/2} := \begin{cases} v_l(R^n_{j+1/2}), & \text{if } j \leq -1/2,\\ v_r(R^n_{j+1/2}), & \text{if } j \geq 1/2.\end{cases}$$

Lemma 3.3. *Set $N_T = \lfloor T/\Delta t\rfloor$. Let $\rho_0 \in \mathbf{BV}(\mathbb{R};[0,\rho_{\max}])$ with $\|\rho_0\|_{\mathbf{L}^1} < +\infty$. Assume that the CFL condition* (3.1) *holds. Then, for $n = 0, ..., N_T - 1$*

$$\Delta x\sum_{j\in\mathbb{Z}}|\rho_j^{n+1} - \rho_j^n| \leq \Delta t\, C(T), \tag{3.3}$$

where

$$C(T) = e^{(2T\rho_{\max}k_r\|\psi'\|_{\mathbf{L}^\infty})}\left(\|\psi\|_{\mathbf{L}^\infty} + \rho_{\max}\|\psi'\|_{\mathbf{L}^\infty}\|\omega\|_{\mathbf{L}^1}\right)k_r\, TV(\rho_0).$$

Proof. Fix $j \in \mathbb{Z}$, by (2.1) we have

$$\begin{aligned}
&\rho_j^{n+1}-\rho_j^n \\
&\quad=\rho_j^n - \rho_j^{n-1} - \lambda\left(\rho_j^n v_{j+1/2}^n \pm \rho_j^{n-1} v_{j+1/2}^n - \rho_j^{n-1} v_{j+1/2}^{n-1} - \rho_{j-1}^n v_{j-1/2}^n \pm \rho_{j-1}^{n-1} v_{j-1/2}^n + \rho_{j-1}^{n-1} v_{j-1/2}^{n-1}\right) \\
&\quad=\rho_j^n - \rho_j^{n-1} - \lambda\Big((\rho_j^n - \rho_j^{n-1}) v_{j+1/2}^n + \rho_j^{n-1}(v_{j+1/2}^n - v_{j+1/2}^{n-1}) \\
&\qquad - (\rho_{j-1}^n - \rho_{j-1}^{n-1}) v_{j-1/2}^n - \rho_{j-1}^{n-1}(v_{j-1/2}^n - v_{j-1/2}^{n-1})\Big).
\end{aligned}$$

Observe that

$$v_{j+1/2}^n - v_{j+1/2}^{n-1} = v'(R_{j+1/2}^{n-1/2})(R_{j+1/2}^n - R_{j+1/2}^{n-1}) = v'(R_{j+1/2}^{n-1/2})\Delta x \sum_{k=0}^{N-1} \omega_k(\rho_{j+k}^n - \rho_{j+k}^{n-1}).$$

We write

$$\begin{aligned}
\rho_j^{n+1} - \rho_j^n &= (1 - \lambda(v_{j+1/2}^n - \Delta x \omega_1 \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})))(\rho_j^n - \rho_j^{n-1}) \\
&\quad - \lambda v'(R_{j+1/2}^{n-1/2})\rho_j^{n-1}\Delta x \sum_{k=0}^{N-1} \omega_k(\rho_{j+k}^n - \rho_{j+k}^{n-1}) + \lambda v_{j-1/2}^n(\rho_{j-1}^n - \rho_{j-1}^{n-1}) \\
&\quad + \lambda \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})\Delta x \sum_{k=0,k\neq 1}^{N-1} \omega_k(\rho_{j-1+k}^n - \rho_{j-1+k}^{n-1}).
\end{aligned}$$

Thanks to the CFL condition (3.1)

$$1 - \lambda(v_{j+1/2}^n - \Delta x \omega_1 \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})) \geq 0.$$

Taking the absolute value, we obtain

$$\begin{aligned}
|\rho_j^{n+1} - \rho_j^n| &\leq (1 - \lambda(v_{j+1/2}^n - \Delta x \omega_1 \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})))|\rho_j^n - \rho_j^{n-1}| \\
&\quad - \lambda v'(R_{j+1/2}^{n-1/2})\rho_j^{n-1}\Delta x \sum_{k=0}^{N-1} \omega_k|\rho_{j+k}^n - \rho_{j+k}^{n-1}| + \lambda v_{j-1/2}^n|\rho_{j-1}^n - \rho_{j-1}^{n-1}| \\
&\quad - \lambda \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})\Delta x \sum_{k=0,k\neq 1}^{N-1} \omega_k|\rho_{j-1+k}^n - \rho_{j-1+k}^{n-1}|.
\end{aligned}$$

Multiplying by Δx and summing over j, we get

$$\begin{aligned}
\sum_{j\in\mathbb{Z}} \Delta x|\rho_j^{n+1} - \rho_j^n| &\leq \sum_{j\in\mathbb{Z}} \Delta x|\rho_j^n - \rho_j^{n-1}| \\
&\quad - 2\lambda \sum_{j\in\mathbb{Z}} \Delta x \rho_{j-1}^{n-1} v'(R_{j-1/2}^{n-1/2})\Delta x \sum_{k=0,k\neq 1}^{N-1} \omega_k|\rho_{j-1+k}^n - \rho_{j-1+k}^{n-1}| \\
&\leq \sum_{j\in\mathbb{Z}} \Delta x|\rho_j^n - \rho_j^{n-1}| \\
&\quad + 2\Delta t \rho_{\max}\|v'\|_{\mathbf{L}^\infty}\Delta x \sum_{k=0,k\neq 1}^{N-1} \omega_k \sum_{j\in\mathbb{Z}} \Delta x|\rho_{j-1+k}^n - \rho_{j-1+k}^{n-1}|
\end{aligned}$$

$$\leq \left(1 + 2\Delta t \rho_{\max} \|v'\|_{\mathbf{L}^\infty} \Delta x \sum_{k=0}^{N-1} \omega_k\right) \sum_{j\in\mathbb{Z}} \Delta x |\rho_j^n - \rho_j^{n-1}|.$$

Thus,

$$\sum_{j\in\mathbb{Z}} \Delta x |\rho_j^{n+1} - \rho_j^n| \leq e^{(2n\Delta t \rho_{\max}\|v'\|_{\mathbf{L}^\infty})} \sum_{j\in\mathbb{Z}} \Delta x |\rho_j^1 - \rho_j^0|.$$

On the other hand,

$$\begin{aligned}\sum_{j\in\mathbb{Z}} \Delta x |\rho_j^1 - \rho_j^0| \leq \Delta t \sum_{j<0} |\rho_j^0 v_l(R_{j+1/2}^n) - \rho_{j-1}^0 v_l(R_{j-1/2}^n)| + \Delta t |\rho_{-1}^0 v_l(R_{-1/2}^0) - \rho_0^0 v_r(R_{1/2}^0)| \\ + \Delta t \sum_{j>0} |\rho_j^0 v_r(R_{j+1/2}^0) - \rho_{j-1}^0 v_r(R_{j-1/2}^0)|.\end{aligned}$$

The first term of the right-hand side can be estimated as

$$\begin{aligned}&\sum_{j<0} |\rho_j^0 v_l(R_{j+1/2}^0) - \rho_{j-1}^0 v_l(R_{j-1/2}^0)| \\ &\leq \|v_l\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_j^0 - \rho_{j-1}^0| + \rho_{\max} \|v_l'\|_{\mathbf{L}^\infty} \sum_{j<0} \left|R_{j+1/2}^n - R_{j-1/2}^n\right| \\ &\leq \|v_l\|_{\mathbf{L}^\infty} \sum_{j<0} |\rho_j^0 - \rho_{j-1}^0| + \rho_{\max} \|v_l'\|_{\mathbf{L}^\infty} \sum_{j<0} \sum_{k=0}^{N-1} \Delta x \omega_k \left|\rho_{j+k+1}^0 - \rho_{j+k}^0\right| \\ &\leq \left(\|v_l\|_{\mathbf{L}^\infty} + \rho_{\max} \|v_l'\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1}\right) \sum_{j<0} |\rho_j^0 - \rho_{j-1}^0|.\end{aligned}$$

Analogously,

$$\sum_{j>0} |\rho_j^0 v_r(R_{j+1/2}^0) - \rho_{j-1}^0 v_r(R_{j-1/2}^0)| \leq \left(\|v_r\|_{\mathbf{L}^\infty} + \rho_{\max} \|v_r'\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1}\right) \sum_{j>0} |\rho_j^0 - \rho_{j-1}^0|.$$

and by hypothesis (1.3)

$$|\rho_{-1}^0 v_l(R_{-1/2}^0) - \rho_0^0 v_r(R_{1/2}^0)| \leq |\rho_{-1}^0 v_r(R_{-1/2}^0) - \rho_0^0 v_r(R_{1/2}^0)|.$$

Finally,

$$\sum_{j\in\mathbb{Z}} \Delta x |\rho_j^1 - \rho_j^0| \leq \Delta t \left(\|\psi\|_{\mathbf{L}^\infty} + \rho_{\max} \|\psi'\|_{\mathbf{L}^\infty} \|\omega\|_{\mathbf{L}^1}\right) k_r \mathrm{TV}(\rho_0).$$

This completes the proof. □

3.1. **Spatial BV estimates.**

Lemma 3.4. *Let $\rho_0 \in \mathbf{L}^\infty \cap \mathbf{BV}(\mathbb{R}; [0, \rho_{\max}])$. Assume that the CFL condition (3.1) holds. For any interval $[a, b] \subset \mathbb{R}$ such that $0 \notin [a, b]$, fix $q > 0$ such that $2q < \min\{|a|, |b|\}$ and $q > \Delta x$. Then, for any $n = 1, ..., N_T - 1$ the following estimate holds:*

$$\sum_{j\in\mathbf{J}_a^b} \left|\rho_{j+1}^n - \rho_j^n\right| \leq e^{2\mathcal{K}T} \left(TV(\rho_0) + 2\frac{C(T)}{q} + \mathcal{K}_2 T\right),$$

with $\mathbf{J}_a^b = \{j \in \mathbb{Z} : a \leq x_j \leq b\}$.

Proof. Let

$$\mathcal{M}_\Delta = \left\{ j \in \mathbb{Z} : x_{j-1/2} \in [a - q - \Delta x, a] \right\}, \quad \mathcal{N}_\Delta = \left\{ j \in \mathbb{Z} : x_{j+1/2} \in [b, b + q + \Delta x] \right\}.$$

By the assumptions on q, observe that there are at least 2 elements in each of the sets above, i.e. $|\mathcal{M}_\Delta|, |\mathcal{N}_\Delta| \geq 2$. Moreover, $|\mathcal{M}_\Delta| \Delta x \geq q$ and $|\mathcal{N}_\Delta| \Delta x \geq q$. By Lemma 3.3 there exists a constant $C(T)$ such that

$$\Delta x \sum_{n=0}^{N_T-1} \sum_{j \in \mathbb{Z}} |\rho_j^{n+1} - \rho_j^n| \leq C(T), \tag{3.4}$$

with $C(T)$ as in Lemma 3.3. When restricting the sum over j in the set $\mathcal{M}_\Delta$, respectively $\mathcal{N}_\Delta$, it follows that

$$\Delta x \sum_{n=0}^{N_T-1} \sum_{j \in \mathcal{M}_\Delta} |\rho_j^{n+1} - \rho_j^n| \leq C(T), \quad \Delta x \sum_{n=0}^{N_T-1} \sum_{j \in \mathcal{N}_\Delta} |\rho_j^{n+1} - \rho_j^n| \leq C(T). \tag{3.5}$$

Let us choose $j_a \in \mathcal{M}_\Delta$ and j_b with $j_b + 1 \in \mathcal{N}_\Delta$ such that

$$\sum_{n=0}^{N_T-1} |\rho_{j_a}^{n+1} - \rho_{j_a}^n| = \min_{j \in \mathcal{M}_\Delta} \sum_{n=0}^{N_T-1} |\rho_j^{n+1} - \rho_j^n|,$$
$$\sum_{n=0}^{N_T-1} |\rho_{j_b+1}^{n+1} - \rho_{j_b+1}^n| = \min_{j \in \mathcal{N}_\Delta} \sum_{n=0}^{N_T-1} |\rho_j^{n+1} - \rho_j^n|.$$

Thus,

$$\sum_{n=0}^{N_T-1} |\rho_{j_a}^{n+1} - \rho_{j_a}^n| \leq \frac{C}{|\mathcal{M}_\Delta| \Delta x} \leq \frac{C(T)}{q},$$
$$\sum_{n=0}^{N_T-1} |\rho_{j_b+1}^{n+1} - \rho_{j_b+1}^n| \leq \frac{C}{|\mathcal{N}_\Delta| \Delta x} \leq \frac{C(T)}{q}.$$

We observe that

$$\sum_{j=j_a}^{j_b} \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| = \left| \rho_{j_a+1}^{n+1} - \rho_{j_a}^{n+1} \right| + \sum_{j=j_a+1}^{j_b-1} \left| \rho_{j+1}^{n+1} - \rho_j^{n+1} \right| + \left| \rho_{j_b+1}^{n+1} - \rho_{j_b}^{n+1} \right|. \tag{3.6}$$

Let us focus on the central sum on the right-hand side of (3.6).

We write

$$\rho_{j+1}^{n+1} - \rho_j^{n+1} = \mathcal{A}_j^n - \lambda \mathcal{B}_j^n,$$

with

$$\mathcal{A}_j^n := (1 - \lambda v_{j+3/2}^n)(\rho_{j+1}^n - \rho_j^n) + \lambda v_{j+1/2}^n (\rho_j^n - \rho_{j-1}^n),$$
$$\mathcal{B}_j^n := \rho_j^n (v_{j+3/2}^n - v_{j+1/2}^n) - \rho_{j-1}^n (v_{j+1/2}^n - v_{j-1/2}^n).$$

Taking the absolute value and summing,

$$\sum_{j=j_a+1}^{j_b-1} |\mathcal{A}_j^n| \leq \sum_{j=j_a+1}^{j_b-1} |\rho_{j+1}^n - \rho_j^n| + \lambda v_{j_a+3/2}^n |\rho_{j_a+1}^n - \rho_{j_a}^n| - \lambda v_{j_b+1/2}^n |\rho_{j_b}^n - \rho_{j_b-1}^n|.$$

On the other hand,

$$\begin{aligned}
\mathcal{B}_j^n =&\rho_j(v_{j+3/2}^n - v_{j+1/2}^n) - \rho_{j-1}(v_{j+1/2}^n - v_{j-1/2}^n)\\
=&\rho_j v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n - R_{j+1/2}^n\right) - \rho_{j-1}v'(\tilde{R}_j^n)\left(R_{j+1/2}^n - R_{j-1/2}^n\right)\\
=&\rho_j v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n - R_{j+1/2}^n\right)\\
&\pm \rho_{j-1}v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n - R_{j+1/2}^n\right) - \rho_{j-1}v'(\tilde{R}_j^n)\left(R_{j+1/2}^n - R_{j-1/2}^n\right)\\
=&(\rho_j - \rho_{j-1})v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n - R_{j+1/2}^n\right)\\
&+ \rho_{j-1}v'(\tilde{R}_{j+1}^n)\left(R_{j+3/2}^n - 2R_{j+1/2}^n + R_{j-1/2}^n\right)\\
&+ \rho_{j-1}v''(\bar{R}_{j+1/2}^n)\left(R_{j+1/2}^n - R_{j-1/2}^n\right)\left(\tilde{R}_{j+1}^n - \tilde{R}_j^n\right),
\end{aligned}$$

where $\tilde{R}_j^n \in \mathcal{I}(R_{j-1/2}^n, R_{j+1/2}^n)$ and $\bar{R}_{j+1/2}^n \in \mathcal{I}(\tilde{R}_j^n, \tilde{R}_{j+1}^n)$.

By the assumptions (1.3) on the kernel function and defining $\omega_N := 0$, we get

$$\begin{aligned}
\left|R_{j+1/2}^n - R_{j-1/2}^n\right| &= \left|\Delta x \sum_{k=0}^{N-1}\omega_k(\rho_{j+k+1}^n - \rho_{j+k}^n)\right|\\
&= \Delta x\left|-\omega_0\rho_j^n + \sum_{k=0}^{N-1}(\omega_{k-1} - \omega_k)\rho_{j+k}^n + \omega_{N-1}\rho_{j+N}^n\right|\\
&\le \Delta x\left(\omega_\eta(0)\rho_{\max} + \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\right).
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
|R_{j+3/2}^n - 2R_{j+1/2}^n + R_{j-1/2}^n| &= \left|\Delta x\left(\sum_{k=0}^{N-1}\omega_k\rho_{j+k+2}^n - 2\sum_{k=0}^{N-1}\omega_k\rho_{j+k+1}^n + \sum_{k=0}^{N-1}\omega_k\rho_{j+k}^n\right)\right|\\
&= \Bigg|\Delta x\Bigg(\sum_{k=1}^{N-2}(\omega_{k-1} - 2\omega_k + \omega_{k+1})\rho_{j+k+1}^n\\
&\quad + \omega_{N-1}(\rho_{j+N+1}^n - \rho_{j+N}^n) + \rho_{j+N}^n(\omega_{N-2} - \omega_{N-1})\\
&\quad + \omega_0(\rho_j^n - \rho_{j+1}^n) + \rho_{j+1}^n(\omega_1 - \omega_0)\Bigg)\Bigg|\\
&= \Bigg|\Delta x\Bigg(\sum_{k=1}^{N-1}(\omega_{k-1} - 2\omega_k + \omega_{k+1})\rho_{j+k+1}^n\\
&\quad + \frac{1}{\Delta x}(\omega_{N-1}\underbrace{-\omega_N}_{:=0})\rho_{j+N+1}^n\Delta x\\
&\quad + \omega_0(\rho_j^n - \rho_{j+1}^n) + \Delta x\rho_{j+1}^n\frac{\omega_1 - \omega_0}{\Delta x}\Bigg)\Bigg|\\
&\le (\Delta x)^2\|\omega''\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + 2(\Delta x)^2\rho_{\max}\|\omega'\|_{\mathbf{L}^\infty} + \Delta x\omega_0|\rho_j - \rho_{j+1}|.
\end{aligned}$$

Now, we compute, $\left|\tilde{R}^n_{j+1} - \tilde{R}^n_j\right|$,

$$\left|\tilde{R}^n_{j+1} - \tilde{R}^n_j\right| = \left|\theta R^n_{j+3/2} + (1-\theta)R^n_{j+1/2} - \mu R^n_{j+1/2} - (1-\mu)R^n_{j-1/2}\right| \leq 3\Delta x \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1},$$

for some $\theta, \mu \in [0,1]$. We end up with

$$\begin{aligned}
\left|\mathcal{B}^n_j\right| \leq & \Delta x(\omega_\eta(0)\rho_{\max} + \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1})\,\|v'\|_{\mathbf{L}^\infty}\left|\rho^n_j - \rho^n_{j-1}\right| \\
& + (\Delta x)^2\left(\|\omega''\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\|v'\|_{\mathbf{L}^\infty} + 2\rho_{\max}\|\omega'\|_{\mathbf{L}^\infty}\|v'\|_{\mathbf{L}^\infty}\right)\left|\rho^n_{j-1}\right| \\
& + \Delta x\omega_\eta(0)\,\|v'\|_{\mathbf{L}^\infty}\rho_{\max}\left|\rho^n_{j+1} - \rho^n_j\right| \\
& + 3(\Delta x)^2\left(\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + \omega_\eta(0)\rho_{\max}\right)^2\|v''\|_{\mathbf{L}^\infty}\left|\rho^n_{j-1}\right|.
\end{aligned}$$

Summing,

$$\begin{aligned}
\lambda\sum_{j=j_a+1}^{j_b-1}|\mathcal{B}^n_j| \leq & \ \Delta t(\omega_\eta(0)\rho_{\max} + \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1})\,\|v'\|_{\mathbf{L}^\infty})\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j - \rho^n_{j-1}\right| \\
& + \Delta t\Delta x\left(\|\omega''\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\|v'\|_{\mathbf{L}^\infty} + 2\rho_{\max}\|\omega'\|_{\mathbf{L}^\infty}\|v'\|_{\mathbf{L}^\infty}\right)\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| \\
& + \Delta t\omega_\eta(0)\,\|v'\|_{\mathbf{L}^\infty}\rho_{\max}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1} - \rho^n_j\right| \\
& + 3\Delta t\Delta x\left(\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + \omega_\eta(0)\rho_{\max}\right)^2\|v''\|_{\mathbf{L}^\infty}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| \\
\leq & \ \Delta t\mathcal{K}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j - \rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| + \Delta t\mathcal{K}_2\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1} - \rho^n_j\right|,
\end{aligned}$$

where $\mathcal{K} = (\omega_\eta(0)\rho_{\max} + \|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1})\,\|v'\|_{\mathbf{L}^\infty}$,

$\mathcal{K}_1 = \left(\|\omega''\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1}\|v'\|_{\mathbf{L}^\infty} + 2\rho_{\max}\|\omega'\|_{\mathbf{L}^\infty}\|v'\|_{\mathbf{L}^\infty}\right) + 3\left(\|\omega'\|_{\mathbf{L}^\infty}\|\rho\|_{\mathbf{L}^1} + \omega_\eta(0)\rho_{\max}\right)^2\|v''\|_{\mathbf{L}^\infty}$,

and $\mathcal{K}_2 = \omega_\eta(0)\,\|v'\|_{\mathbf{L}^\infty}\rho_{\max}$.

We are left with the boundary terms in (3.6). For $j = j_a$, we have

$$\begin{aligned}
\rho^{n+1}_{j_a+1} - \rho^{n+1}_{j_a} =& \rho^n_{j_a+1} - \lambda v^n_{j+3/2}(\rho^n_{j_a+1} - \rho^n_{j_a}) + \lambda\rho^n_{j_a}(v^n_{j+3/2} - v^n_{j+1/2}) - \rho^{n+1}_{j_a} \pm \rho^n_{j_a} \\
=& (1 - \lambda v^n_{j+3/2})\rho^n_{j_a+1} + \lambda v^n_{j+3/2}\rho^n_{j_a} + \lambda\rho^n_{j_a}(v^n_{j+3/2} - v^n_{j+1/2}) - \rho^{n+1}_{j_a} \pm \rho^n_{j_a} \\
=& (1 - \lambda v^n_{j+3/2})(\rho^n_{j_a+1} - \rho^n_{j_a}) \\
& + \lambda\rho^n_{j_a}(v^n_{j+3/2} - v^n_{j+1/2}) \\
& + \rho^n_{j_a} - \rho^{n+1}_{j_a}.
\end{aligned}$$

Proceed similarly for $j = j_b$

$$\begin{aligned}
\rho^{n+1}_{j_b+1} - \rho^{n+1}_{j_b} =& \rho^{n+1}_{j_b+1} - \rho^n_{j_b} + \lambda v^n_{j_b+1/2}(\rho^n_{j_b} - \rho^n_{j_b-1}) - \lambda\rho^n_{j_b-1}(v^n_{j_b+1/2} - v^n_{j_b-1/2}) \pm \rho^n_{j_b+1} \\
=& \rho^{n+1}_{j_b+1} - \rho^n_{j_b+1} \\
& + \rho^n_{j_b+1} - \rho^n_{j_b}
\end{aligned}$$

$$+\lambda v^n_{j_b+1/2}(\rho^n_{j_b}-\rho^n_{j_b-1})$$
$$-\lambda \rho^n_{j_b-1}(v^n_{j_b+1/2}-v^n_{j_b-1/2}).$$

Collecting the terms, taking the absolute value and summing over j

$$\begin{aligned}
\sum_{j=j_a}^{j_b}\left|\rho^{n+1}_{j+1}-\rho^{n+1}_j\right| &= \left|\rho^{n+1}_{j_a+1}-\rho^{n+1}_{j_a}\right| + \sum_{j=j_a+1}^{j_b-1}\left|\rho^{n+1}_{j+1}-\rho^{n+1}_j\right| + \left|\rho^{n+1}_{j_b+1}-\rho^{n+1}_{j_b}\right| \\
&\le \left|\rho^{n+1}_{j_a+1}-\rho^{n+1}_{j_a}\right| + \sum_{j=j_a+1}^{j_b-1}\left(|\mathcal{A}^n_j| + \lambda\left|\mathcal{B}^n_j\right|\right) + \left|\rho^{n+1}_{j_b+1}-\rho^{n+1}_{j_b}\right| \\
&\le (1-\lambda v^n_{j_a+3/2})\left|\rho^n_{j_a+1}-\rho^n_{j_a}\right| + \lambda\rho^n_{j_a}\left|v^n_{j_a+3/2}-v^n_{j_a+1/2}\right| + \left|\rho^n_{j_a}-\rho^{n+1}_{j_a}\right| \\
&\quad + \sum_{j=j_a+1}^{j_b-1}|\rho^n_{j+1}-\rho^n_j| + \lambda v^n_{j_a+3/2}|\rho^n_{j_a+1}-\rho^n_{j_a}| - \lambda v^n_{j_b+1/2}|\rho^n_{j_b}-\rho^n_{j_b-1}| \\
&\quad + \Delta t\mathcal{K}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j-\rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| + \Delta t\mathcal{K}_2\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1}-\rho^n_j\right| \\
&\quad + \left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| + \left|\rho^n_{j_b+1}-\rho^n_{j_b}\right| \\
&\quad + \lambda v^n_{j_b+1/2}\left|\rho^n_{j_b}-\rho^n_{j_b-1}\right| + \lambda\rho^n_{j_b-1}\left|v^n_{j_b+1/2}-v^n_{j_b-1/2}\right| \\
&= \left|\rho^n_{j_a}-\rho^{n+1}_{j_a}\right| + \sum_{j=j_a}^{j_b}|\rho^n_{j+1}-\rho^n_j| \\
&\quad + \lambda\rho^n_{j_a}\left|v^n_{j_a+3/2}-v^n_{j_a+1/2}\right| + \left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| \\
&\quad + \Delta t\mathcal{K}\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_j-\rho^n_{j-1}\right| + \Delta t\Delta x\mathcal{K}_1\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j-1}\right| + \Delta t\mathcal{K}_2\sum_{j=j_a+1}^{j_b-1}\left|\rho^n_{j+1}-\rho^n_j\right| \\
&\quad + \lambda\rho^n_{j_b-1}\left|v^n_{j_b+1/2}-v^n_{j_b-1/2}\right| \\
&\le \left|\rho^n_{j_a}-\rho^{n+1}_{j_a}\right| + (1+2\Delta t\mathcal{K})\sum_{j=j_a}^{j_b}|\rho^n_{j+1}-\rho^n_j| + \left|\rho^{n+1}_{j_b+1}-\rho^n_{j_b+1}\right| + \Delta t\mathcal{K}_3,
\end{aligned}$$

where $\mathcal{K}_3 = 2\rho_{\max}\mathcal{K} + \|\rho\|_{\mathbf{L}^1}\mathcal{K}_1$. By a standard iterative procedure we can deduce, for $1 \le n < N_T - 1$,

$$\sum_{j=j_a}^{j_b}\left|\rho^{n+1}_{j+1}-\rho^{n+1}_j\right| \le e^{2\mathcal{K}T}\left(\sum_{j=j_a}^{j_b}|\rho^0_{j+1}-\rho^0_j| + 2\frac{C(T)}{q} + \mathcal{K}_3 T\right).$$

This concludes the proof because $[a,b] \subseteq [x_{j_a}, x_{j_b+1}]$. □

3.2. **Discrete Entropy Inequality.** Next we show that the approximate solution obtained by the scheme (2.1) fulfills a discrete entropy inequality. Let us define

$$G_{j+1/2}(u) = uv^n_{j+1/2}, \qquad \mathcal{F}^c_{j+1/2}(u) := G_{j+1/2}(u\vee c) - G_{j+1/2}(u\wedge c)$$

with $a\vee b = \max\{a,b\}$ and $a\wedge b = \min\{a,b\}$.

Lemma 3.5. *Let ρ_j^n for $j \in \mathbb{Z}$ and $n \in \mathbb{N}$ given by* (2.1)*, and let the CFL condition* (3.1) *and the hypothesis* (1.3) *hold. Then we have*

$$\begin{aligned} &\left|\rho_j^{n+1} - c\right| - \left|\rho_j^n - c\right| \\ &\quad + \lambda(\mathcal{F}^c_{j+1/2}(\rho_j^n) - \mathcal{F}^c_{j-1/2}(\rho_{j-1}^n)) + \lambda \operatorname{sgn}(\rho_j^{n+1} - c)c(v^n_{j+1/2} - v^n_{j-1/2}) \leq 0 \end{aligned} \tag{3.8}$$

for all $j \in \mathbb{Z}$, $n \in \mathbb{N}$ and $c \in [0, \rho_{\max}]$.

Proof. For a complete proof see [17, Section 3.4]. □

3.3. **Convergence to entropy solution.**

Theorem 3.1. *Let $\rho_0 \in \mathbf{BV} \cap \mathbf{L}^\infty(\mathbb{R}, [0, \rho_{\max}])$. Let $\Delta x \to 0$ with $\lambda = \frac{\Delta t}{\Delta x}$ constant and satisfying the CFL condition* (3.1)*. The sequence of approximate solution ρ_Δ constructed through finite volume scheme* (2.1)-(2.2) *converges in $\mathbf{L}^1_{\text{loc}}$ to a function in $\mathbf{L}^\infty([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ such that $\|\rho\|_{\mathbf{L}^1} = \|\rho_0\|_{\mathbf{L}^1}$.*

Proof. Lemma 3.1 ensures that the sequence of approximate solutions ρ_Δ is bounded in $\mathbf{L}^\infty$. Lemma 3.3 proves the $\mathbf{L}^1-$continuity in time of the sequence ρ_Δ, while Lemma 3.4 guarantees a bound on the spatial total variation in any interval $[a, b]$ not containing $x = 0$. Applying standard compactness results we have that for any interval $[a, b]$ not containing $x = 0$, there exists a subsequence, still denoted by ρ_Δ converging in $\mathbf{L}^1([0,T] \times [a, b]; [0, \rho_{\max}])$. Let us take a countable set of intervals $[a_i, b_i]$ such that $\cup_i [a_i, b_i] = \mathbb{R}^*$, using a standard diagonal process, we can extract a subsequence, still denoted by ρ_Δ, converging in $\mathbf{L}^1_{\text{loc}}([0,T] \times \mathbb{R}; [0, \rho_{\max}])$ and almost everywhere in $[0,T] \times \mathbb{R}$, to a function $\rho \in \mathbf{L}^\infty([0,T] \times \mathbb{R}; [0, \rho_{\max}])$. □

Lemma 3.6. *Let $\rho(t, x)$ be a weak solution constructed as the limit of approximations ρ_Δ generated by the Godunov scheme* (2.1) *and let $c \in [0, \rho_{\max}]$. Let $\varphi \in \mathcal{D}(\mathbb{R}^* \times [0, T))$. Then the following entropy inequality is satisfied:*

$$\begin{aligned} &\int_0^T \int_{\mathbb{R}} (|\rho - c|\varphi_t \mathrm{d}x\mathrm{d}t \\ &\qquad + \int_0^T \int_{\mathbb{R}} \operatorname{sgn}(\rho - c)(f(t, x, \rho) - f(t, x, c))\varphi_x \,\mathrm{d}x\mathrm{d}t \\ &\qquad - \int_0^T \int_{\mathbb{R}} \operatorname{sgn}(\rho - c)\partial_x f(t, x, c)\varphi \,\mathrm{d}x\mathrm{d}t \\ &\qquad + \int_{-\infty}^{\infty} |\rho_0(x) - c|\varphi(0, x)\mathrm{d}x \geq 0. \end{aligned} \tag{3.9}$$

Proof. Let φ be a test function of the type described in the statement of the lemma and set $\varphi_j^n = \varphi(t^n, x_j)$. Let us denote $\Delta_- p_j = p_j - p_{j-1}$. We multiply the cell entropy inequality (3.8) by $\varphi_j^n \Delta x$, and then sum by parts to get

$$\Delta x \Delta t \sum_{n \geq 0} \sum_{j \in \mathbb{Z}} \left|\rho_j^{n+1} - c\right| (\varphi_j^{n+1} - \varphi_j^n)/\Delta t + \Delta x \sum_j \left|\rho_j^0 - c\right| \varphi_j^0 \tag{3.10}$$

$$+\Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z}}\mathcal{F}^c_{j-1/2}\Delta_-\varphi^n_j/\Delta x \tag{3.11}$$

$$-\Delta x\Delta t\sum_{n\geq 0}\sum_{j\in\mathbb{Z}}\operatorname{sgn}(\rho^{n+1}_j-c)c\,\Delta_- v_{j+1/2}\,\varphi^n_j/\Delta x\geq 0. \tag{3.12}$$

By Lebesgue's dominated convergence theorem as $\Delta:=(\Delta x,\Delta t)\to 0$,

$$(3.10)\to\int_0^T\int_{\mathbb{R}}|\rho-c|\,\varphi_t\mathrm{d}x\mathrm{d}t+\int_{-\infty}^{\infty}|\rho_0(x)-c|\varphi(0,x)\mathrm{d}x,$$

and

$$(3.11)\to\int_0^T\int_{\mathbb{R}}\operatorname{sgn}(\rho-c)(f(t,x,\rho)-f(t,x,c))\varphi_x\,\mathrm{d}x\mathrm{d}t.$$

Now let us study the sum (3.12) and we have

$$(3.12)=$$

$$-\Delta x\Delta t\sum_{n\geq 0}\sum_{\substack{j\in\mathbb{Z}\\ j\leq -1}}\operatorname{sgn}(\rho^{n+1}_j-c)c\,\Delta_- v_{j+1/2}\,\varphi^n_j/\Delta x \tag{3.13}$$

$$-\Delta x\Delta t\sum_{n\geq 0}\sum_{\substack{j\in\mathbb{Z}\\ j\geq 1}}\operatorname{sgn}(\rho^{n+1}_j-c)c\,\Delta_- v_{j+1/2}\,\varphi^n_j/\Delta x \tag{3.14}$$

$$-\Delta x\Delta t\sum_{n\geq 0}\operatorname{sgn}(\rho^{n+1}_0-c)c\,\Delta_- v_{1/2}\,\varphi^n_0/\Delta x \tag{3.15}$$

The support of the test function φ does not include the discontinuity flux point 0, for this reason we consider $\varphi_0=0$ according to our discretization. The sum (3.15) is equal to zero because $\varphi_0=0$. Finally,

$$(3.13)+(3.14)\to-\int_0^T\int_{-\infty}^0\operatorname{sgn}(\rho-c)\partial_x f(t,x,c)\varphi\,\mathrm{d}x\mathrm{d}t-\int_0^T\int_0^{\infty}\operatorname{sgn}(\rho-c)\partial_x f(t,x,c)\varphi\,\mathrm{d}x\mathrm{d}t.$$

□

Lemma 3.7. *Let $\rho(t,x)$ be a weak solution constructed as the limit of approximations ρ_Δ generated by the scheme (2.1) and let $c\in[0,\rho_{\max}]$. Let $\varphi\in\mathbf{C^1_c}(\mathbb{R}\times[0,T))$. Then the following entropy inequality is satisfied:*

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}&|\rho-c|\,\varphi_t+\operatorname{sgn}(\rho-c)(f(t,x,\rho)-f(t,x,c))\,\partial_x\varphi\,\mathrm{d}x\,\mathrm{d}t\\&+\int_0^T\int_{\mathbb{R}_*}|\partial_x f(t,x,c)|\varphi\,\mathrm{d}x\,\mathrm{d}t+\int_{\mathbb{R}}|\rho_0(x)-c|\,\varphi(0,x)\mathrm{d}x\\&+\int_0^T|(k_r-k_l)c\,\psi(\rho*\omega_\eta)|\,\varphi(t,0)\mathrm{d}t\geq 0.\end{aligned}$$

Proof. Let φ be a test function of the type described in the statement of the lemma and set $\varphi^n_j=\varphi(t^n,x_j)$. There exist $T>0$ and $R>0$ such that $\varphi(t,x)=0$ for $t>T$ and $|x|>R$. Our starting point is the following cell entropy inequality which is a consequence of (3.8).

$$\left|\rho^{n+1}_j-c\right|\leq\left|\rho^n_j-c\right|-\lambda\Delta_-\mathcal{F}^c_{j+1/2}+\lambda\left|c\Delta_- v^n_{j+1/2}\right| \tag{3.16}$$

We multiply (3.16) by $\varphi_j^n \Delta x$, and then sum by parts to get

$$\Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left|\rho_j^{n+1} - c\right|(\varphi_j^{n+1} - \varphi_j^n)/\Delta t + \Delta x \sum_j \left|\rho_j^0 - c\right| \varphi_j^0 \tag{3.17}$$

$$+ \Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \mathcal{F}^c_{j-1/2}(\Delta_- \varphi_j^n/\Delta x) \tag{3.18}$$

$$+ \Delta x \Delta t \sum_{n\geq 0}\sum_{j\in\mathbb{Z}} \left|c\Delta_- v^n_{j+1/2}\right| \phi_j^n/\Delta x \geq 0. \tag{3.19}$$

By Lebesgue's dominated convergence theorem as $\Delta := (\Delta x, \Delta t) \to 0$,

$$(3.17) \to \int_0^T \int_{\mathbb{R}} |\rho - c|\, \varphi_t \mathrm{d}x\mathrm{d}t + \int_{-\infty}^{\infty} |\rho_0(x) - c|\varphi(0,x)\mathrm{d}x.$$

Following the same standard arguments as in Lemma 3.6, the sum (3.18) converges to

$$\int_0^T \int_{\mathbb{R}} \operatorname{sgn}(\rho - c)(f(t,x,\rho) - f(t,x,c))\, \partial_x \varphi \,\mathrm{d}x\,\mathrm{d}t.$$

Now we can rewrite the sum (3.19)

$$\Delta x \Delta t \sum_{n\geq 0}\sum_{\substack{j\in\mathbb{Z}\\ j\leq -1}} \left|c\Delta_- v^n_{j+1/2}\right| \varphi_j^n/\Delta x \tag{3.20}$$

$$+ \Delta x \Delta t \sum_{n\geq 0}\sum_{\substack{j\in\mathbb{Z}\\ j\geq 1}} \left|c\Delta_- v^n_{j+1/2}\right| \varphi_j^n/\Delta x \tag{3.21}$$

$$+ \Delta t \sum_{n\geq 0} \left|c\Delta_- v^n_{1/2}\right| \varphi_0^n. \tag{3.22}$$

At this point, we can observe that as $\Delta := (\Delta x, \Delta t) \to 0$

$$(3.20) + (3.21) \to \int_0^T \int_{\mathbb{R}\setminus\{0\}} |f(t,x,c)_x| \varphi \,\mathrm{d}x\,\mathrm{d}t$$

$$(3.22) \to \int_0^T |(k_r - k_l) c\, \psi(\rho * \omega_\eta)|\, \varphi(t,0)\mathrm{d}t.$$

□

Theorem 3.2. *Let $\rho(t,x)$ be a weak solution constructed as the limit of approximations ρ_Δ generated by the scheme* (2.1) *and let $c \in [0, \rho_{\max}]$. Then $\rho(t,x)$ is an entropy solution satisfying the Definition* 1.2.

Proof. Let $0 \leq \varphi \in \mathbf{C_c^1}([0,T)\times\mathbb{R})$. We set $\varphi_j^n = \varphi(t^n, x_j)$. For $\varepsilon > 0$, define the set

$$\sigma_0^\varepsilon = \{(t,x) \in [0,T)\times\mathbb{R} | x \in (-\varepsilon, \varepsilon),\, t \in [0,T)\}.$$

For each sufficiently small $\varepsilon > 0$ we can write the test function φ as a sum of two test functions, one having support away from the set $\Sigma := \sigma_0^\varepsilon$ and the other with support in Σ. We take test functions ψ^ε, $\alpha^\varepsilon \in \mathbf{C_c^1}([0,T)\times\mathbb{R})$ such that

$$\varphi(t,x) = \psi^\varepsilon(t,x) + \alpha^\varepsilon(t,x), \quad 0 \leq \psi^\varepsilon(t,x) \leq \varphi(t,x), \quad 0 \leq \alpha^\varepsilon(t,x) \leq \varphi(t,x),$$

where ψ^ε has support located around the jump in 0

$$\begin{aligned}\operatorname{supp}(\psi^\varepsilon) &\subseteq \sigma_0^\varepsilon,\\ \psi^\varepsilon(t,0) &= \varphi(t,0),\end{aligned}$$

and α^ε vanishes around the jump, i.e.

$$\operatorname{supp}(\alpha^\varepsilon) \subseteq [0,T)\times\mathbb{R}^*.$$

We can take this decomposition in such way that

$$\alpha^\varepsilon \to \varphi \quad \text{ in } \mathbf{L}^1([0,T)\times\mathbb{R}), \quad \psi^\varepsilon \to 0 \quad \text{ in } \mathbf{L}^1([0,T)\times\mathbb{R}) \tag{3.23}$$

as $\varepsilon \to 0$. By applying Lemma 3.6 with the test function α^ε and Lemma 3.7 with ψ^ε, and summing the two entropy inequalities, using $\varphi = \psi^\varepsilon + \alpha^\varepsilon$ along with $\psi^\varepsilon(0,t) = \varphi(0,t)$ to get

$$\begin{aligned}\int_0^T\int_{\mathbb{R}} &(|\rho-c|\varphi_t \mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(\rho-c)(f(t,x,\rho)-f(t,x,c))\varphi_x \mathrm{d}x\mathrm{d}t\\ &-\int_0^T\int_{\mathbb{R}} \operatorname{sgn}(\rho-c) f(t,x,c)_x \alpha^\varepsilon \mathrm{d}x\mathrm{d}t\\ &+\int_0^T\int_{\mathbb{R}^*} |f(t,x,c))_x|\,\psi^\varepsilon \mathrm{d}x\mathrm{d}t\\ &+\int_0^T |(k_r-k_l)c\,\psi(\rho*\omega_\eta)|\,\varphi(t,0)\mathrm{d}t\\ &+\int_{-\infty}^{\infty} |\rho_0(x)-c|\phi(0,x)\mathrm{d}x \geq 0.\end{aligned}$$

Thanks to (3.23), we can complete the proof by sending $\varepsilon \to 0$. □

3.4. $\mathbf{L}^1$-Stability and uniqueness.

Theorem 3.3. *Assume the hypothesis* (1.3)*. If ρ and $\tilde\rho$ are two entropy solutions of* (1.1) *in the sense of Definition* (1.2) *, the following inequality holds*

$$\|\rho(t,\cdot)-\tilde\rho(t,\cdot)\|_{\mathbf{L}^1(\mathbb{R})} \leq e^{K(T)t}\,\|\rho(0,\cdot)-\tilde\rho(0,\cdot)\|_{\mathbf{L}^1(\mathbb{R})}\,, \tag{3.24}$$

for almost every $0<t<T$ and some suitable constant $K(T)>0$.

Proof. For a complete proof see [11, Section 4]. □

4. Numerical tests

In the following numerical tests, we solve (1.1) numerically in the intervals $x \in [-2,2]$ and $t \in [0,T]$. We propose two tests in order to illustrate the dynamics of the model (1.1) and compare with the local cases, by using the Godunov-type scheme for different discretizations.

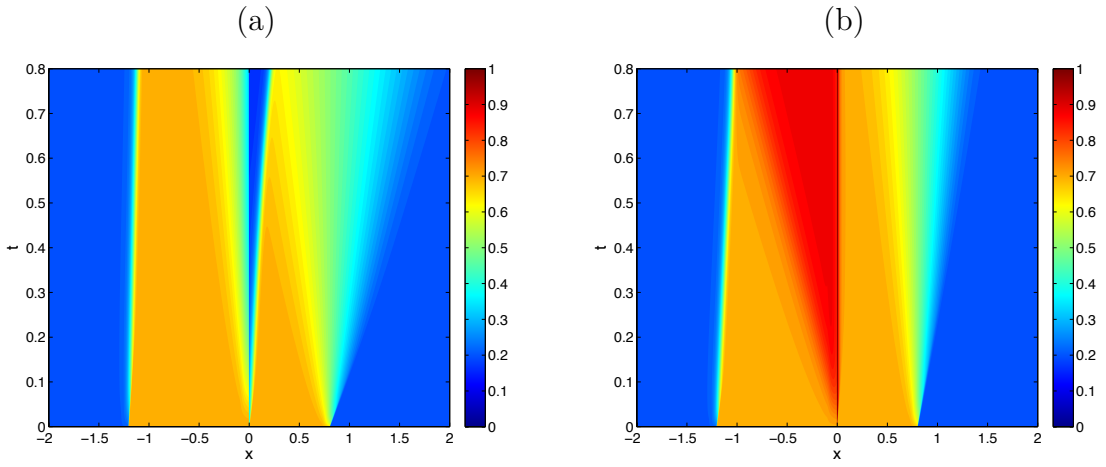

Figure 1. Numerical approximation at time $t \in [0, 0.8]$. (a) Example 1 Case $v_l(\rho) < v_r(\rho)$. (b) Example 2 Case $v_l(\rho) > v_r(\rho)$. Note that in the right figure the maximal density exceeds 1 but it is still visualized by the dark red color.

For each integration, we set Δt to satisfy the CFL condition (3.1).

For the tests we consider $\omega_\eta(x) = 3\frac{(\eta - x)^2}{\eta^3}$ for $0 \leq x \leq \eta$, and absorbing boundary conditions. The reference solution is computed according to the scheme (2.1)-(2.2) and $\Delta x = 1/3200$.

4.1. **Example 1 Case $v_l(\rho) < v_r(\rho)$.** We consider the initial condition

$$\rho_0(x) = \begin{cases} 0.7 & x \in [-1.2, 0.8] \\ 0.2 & \text{otherwise} \end{cases}$$

and velocity functions $v_l(\rho) = (1-\rho)$ and $v_r(\rho) = 2(1-\rho)$ which satisfy the hypothesis (1.3). In Fig 1 we show the evolution of $\rho_\Delta(\cdot, t)$ for $t \in [0, 0.8]$ with $\Delta x = 1/400$ and we can notice a shock emerging at position $x = 0$ due to the discontinuity in the flux function. In Fig 2 we display numerical approximation at simulation time $T = 0.7$ and compare numerical approximations for $\Delta x = 1/100$ and $\Delta x = 1/400$ with respect to the reference solution. The $\mathbf{L}^1$-error for different Δx is computed in Table 1.

	Example 1		Example 2	
Δx	$\mathbf{L}^1$-error	E.O.A.	$\mathbf{L}^1$-error	E.O.A.
$\frac{1}{100}$	$2.7e-2$	–	$1.9e-2$	–
$\frac{1}{200}$	$1.4e-2$	0.9	$1.1e-2$	0.8
$\frac{1}{400}$	$6.5e-3$	1.0	$5.8e-3$	0.9
$\frac{1}{800}$	$2.9e-3$	1.1	$2.9e-3$	1.0
$\frac{1}{1600}$	$1.0e-3$	1.5	$1.2e-3$	1.3

Table 1. Examples 1 and 2. $\mathbf{L}^1$-error and Experimental Order of Accuracy.

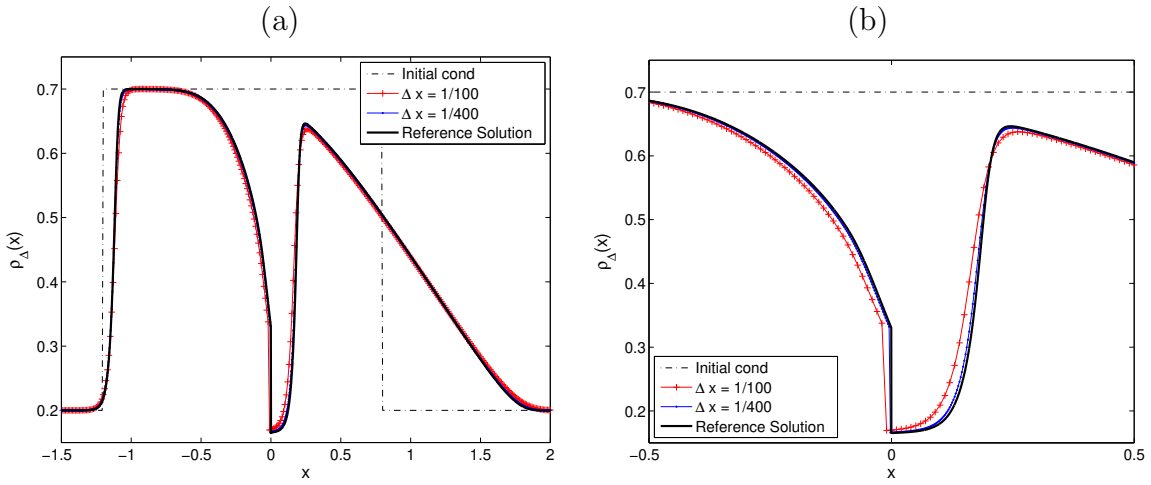

Figure 2. Example 1. Case $v_l(\rho) < v_r(\rho)$. (a) Numerical approximation at time $T = 0.7$ and comparison with the reference solution. (b) Zooming in a specific area.

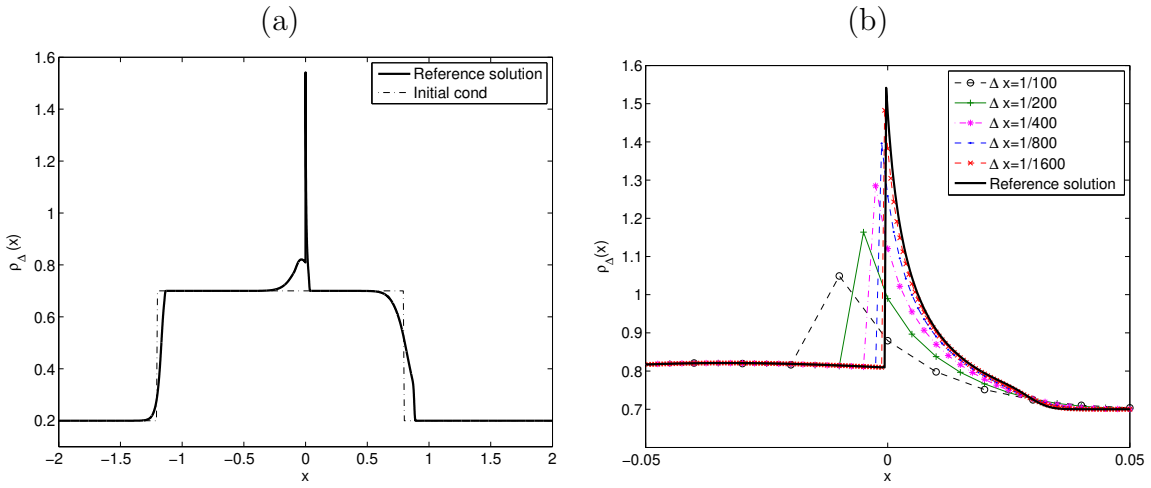

Figure 3. Example 2. Case $v_l(\rho) > v_r(\rho)$. (a) Numerical approximation at time $T = 0.1$ (b) Zooming in a specific area.

4.2. **Example 2: Case $v_l(\rho) > v_r(\rho)$.** In this example, we consider the same parameters as in Example 1 with $k_l = 2 > 1 = k_r$, this means that the velocity function doesn't satisfies the hypothesis (1.3). In this case, the result of Lemma 3.1 is not valid. In Figs 3 we can observe that solution at time $T = 0.1$ is greater than 1 even when initial condition satisfies $\rho_0 \in [0, 1]$. Furthermore, we can observe that the approximate solution seems to converge, at least numerically, to a function that does not satisfy Lemma 3.1, see Table 1.

4.3. **Example 3: Limit $\eta \to 0^+$.** In this example, we investigate the numerical convergence of the approximate solution computed with the numerical scheme (2.1)-(2.2) to the solution of the local conservation law with discontinuous flux under hypothesis (1.3), as the support of the kernel function ω_η tends to 0^+. In particular, we consider an approximation at $T = 0.7$ with fixed $\Delta x = 1/1600$ and $\eta = \{0.1, 0.05, 0.01, 0.005\}$. To evaluate the convergence, we compute the $\mathbf{L}^1$ distance between the approximate solution of the non-local problem with a

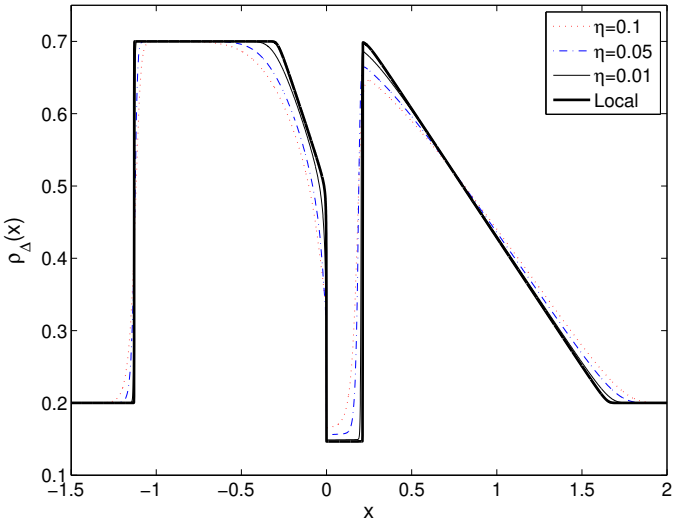

Figure 4. Example 3. Limit $\eta \to 0^+$, numerical approximations at final time $T = 0.7$ with $\Delta x = 1/3200$.

η	0.1	0.05	0.01	0.005
$\mathbf{L}^1$ distance	9.6e-2	6.1e-2	1.6e-2	7.8e-3

Table 2. Example 3. $\mathbf{L}^1$ distance between the approximate solutions to the nonlocal problem and the local problem for different values of η at $T = 0.7$ with $\Delta x = 1/1600$.

given η and the results of the classical Godunov scheme for the corresponding local problem. In Table 2, we can observe than the $\mathbf{L}^1$ distance goes to zero when $\eta \to 0^+$. The results are illustrated in Fig 4.

5. Conclusions and discussions

In this paper, we studied a nonlocal conservation law whose flux function is of the form $H(-x)\rho v_l(\omega_\eta * \rho) + H(x)\rho v_r(\omega_\eta * \rho)$, with a single spatial discontinuity at $x = 0$ and the velocity functions satisfies the hypothesis (1.3). We approximated the problem through a Godunov-type numerical scheme, which is a general version of the scheme proposed in [17], and provided $\mathbf{L}^\infty$ and $\mathbf{BV}$ estimates for the approximate solutions. Numerical simulations illustrate the dynamics of the studied model and corroborate the convergence of the numerical scheme. The limit model as the kernel support tends to zero is numerically investigated. In the case $k_l < k_r$ we have proved the well-possedness, i.e., existence and uniqueness of a weak entropy solution. On the contrary, in the case $k_r < k_l$, we are not able to prove $\mathbf{L}^\infty$ and $\mathbf{BV}$ bounds, and we can see that the solution exceeds 1 from the numerical tests. For this reason, this case is not suitable to describe traffic flow scenarios. In a future work, we would like to consider more general velocity functions allowing the fluxes to cross, like in the *local* discontinuous cases, see [22, 23, 24].

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