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NUMERICAL APPROXIMATION OF THE DISPLACEMENT FORMULATION OF THE AXISYMMETRIC ACOUSTIC VIBRATION PROBLEM.*

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Abstract. The aim of this paper is to study the numerical approximation of the displacement formulation of the acoustic eigenvalue problem in the axisymmetric case. We show that spurious eigenvalues appears when lowest order triangular Raviart-Thomas elements are used to discretize the problem. We propose an alternative weak formulation of the spectral problem which allows us to avoid this drawback. A discretization based on the same finite elements is proposed and analyzed. Quasi-optimal order spectral convergence is proved, as well as absence of spurious modes. Numerical experiments are reported which agree with the theoretical results.

Key word. eigenvalue problem, axisymmetric acoustic, finite element method

AMS subject classifications. 65N25, 65N15, 65N30

1. Introduction. The aim of this paper is to study the numerical approximation of a displacement formulation of the axisymmetric acoustic eigenvalue problem. More precisely, we focus on axisymmetric solutions of the following three-dimensional eigenvalue problem: find λ and $\breve{\boldsymbol{u}} \neq \mathbf{0}$ such that

$$\begin{aligned} -\nabla(\operatorname{div}\breve{\boldsymbol{u}}) &= \lambda\breve{\boldsymbol{u}} \quad \text{in } \Omega \subset \mathbb{R}^3, \\ \breve{\boldsymbol{u}}\cdot\breve{\boldsymbol{n}} &= 0 \quad \text{on } \partial\Omega, \end{aligned}$$

where $\breve{\boldsymbol{u}} = (u_x, u_y, u_z)^\top$ and $\breve{\boldsymbol{n}}$ is a unit normal vector. The classical weak formulation of the previous problem reads: find $(\lambda, \breve{\boldsymbol{u}}) \in \mathbb{R} \times \mathrm{H}_0(\operatorname{div};\Omega)$, such that

$$\int_\Omega \operatorname{div}\breve{\boldsymbol{u}} \operatorname{div}\breve{\boldsymbol{v}} = \lambda \int_\Omega \breve{\boldsymbol{u}}\cdot\breve{\boldsymbol{v}} \qquad \forall\ \breve{\boldsymbol{v}} \in \mathrm{H}_0(\operatorname{div};\Omega), \tag{1.1}$$

where $\mathrm{H}_0(\operatorname{div};\Omega) := \left\{\boldsymbol{v} \in \mathrm{L}^2(\Omega)^3 : \operatorname{div}\boldsymbol{v} \in \mathrm{L}^2(\Omega), \boldsymbol{v}\cdot\breve{\boldsymbol{n}} = 0 \text{ on } \partial\Omega\right\}$. It is known that this problem is well defined and it has a countable sequence of eigenpairs $(\lambda_n, \breve{\boldsymbol{u}}_n), n \in \mathbb{N}$, with $\lambda_n \to \infty$. $\lambda = 0$ is also an eigenvalue of the problem with associated eigenspace $\mathrm{H}_0(\operatorname{div}^0;\Omega) := \{\boldsymbol{v} \in \mathrm{H}_0(\operatorname{div};\Omega) : \operatorname{div}\boldsymbol{v} = 0 \text{ in } \Omega\}$. From the physical point of view these eigenfunctions are spurious solutions of the acoustic problem.

We restrict our attention to the case where the 3D domain Ω is a volume of revolution about the z-axis and look for solutions to (1.1) that are independent of the angular variable θ. In such a case, in order to reduce the dimension and thereby the computational effort, it is convenient to consider a cylindrical coordinate system (r, θ, z). The attached difficulty usually resides in the analysis and derivation of proper schemes to discretize axisymmetric formulations, due to the presence of singularities on the rotation axis $r = 0$ associated to the factor $1/r$ that appears in some integrals.

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We write $\breve{\boldsymbol{u}}$ in cylindrical coordinates $\breve{\boldsymbol{u}} = (\breve{u}_r, \breve{u}_\theta, \breve{u}_z)^T = \breve{u}_r \boldsymbol{e_z} + \breve{u}_\theta \boldsymbol{e_\theta} + \breve{u}_z \boldsymbol{e}_z$. Since we look for solutions of (1.1) that do not depend on θ, we define $D := \{(r, z) \in (0, \infty) \times \mathbb{R} : (r, 0, z) \in \Omega\}$ which corresponds to a half section of Ω. Moreover, we set (in cylindrical coordinates) $\widetilde{u}_\theta(r, z) := \breve{u}_\theta(r, \theta, z)$ and

$$\widetilde{\boldsymbol{u}}(r, z) := (\widetilde{u}_r(r, z), \widetilde{u}_z(r, z))^\top \text{ with } \widetilde{u}_r(r, z) := \breve{u}_r(r, \theta, z) \text{ and } \widetilde{u}_z(r, z) := \breve{u}_z(r, \theta, z),$$

for all $\theta \in \mathbb{R}$ and $(r, z) \in D$. A change of variable in (1.1) leads to the following formulation:

Problem 1. *Find* $(\lambda, \widetilde{\boldsymbol{u}}) \in \mathbb{R} \times \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$*, such that*

$$\int_D \mathrm{div}_{\mathrm{axi}}\, \widetilde{\boldsymbol{u}}\, \mathrm{div}_{\mathrm{axi}}\, \widetilde{\boldsymbol{v}}\, r\, dr\, dz = \lambda \int_D \widetilde{\boldsymbol{u}} \cdot \widetilde{\boldsymbol{v}}\, r\, dr\, dz \qquad \forall \widetilde{\boldsymbol{v}} \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D), \tag{1.2}$$

where $\mathrm{div}_{\mathrm{axi}}$ is the axisymmetric divergence operator defined for all $\boldsymbol{v} = (v_r, v_z)$ by

$$\mathrm{div}_{\mathrm{axi}}(\boldsymbol{v}) := \partial_r v_r + \frac{1}{r} v_r + \partial_z v_z = \frac{1}{r}\, \mathrm{div}(r\boldsymbol{v}), \tag{1.3}$$

and $\mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$ is an appropriate Sobolev space, whose definition will be recalled in Section 2.

Remark 1.1. *For any function $\widetilde{u}_\theta(r, z)$, we have that $\widetilde{u}_\theta \boldsymbol{e_\theta}$ is divergence-free and, hence, it is an eigenfunction of (1.1) with associated eigenvalue $\lambda = 0$. However, as claimed above, these eigenfunctions are spurious solutions of the acoustic problem.*

A similar procedure to solve an axisymmetric three-dimensional Darcy problem has been considered in [7, 14]. In this case, the authors were able to prove that discretizing by Raviart-Thomas (RT) elements leads to a convergent method. Moreover they proved error estimates which have been confirmed by numerical experiments.

Following this approach, Problem 1 could be in principle discretized by the same kind of finite elements (lowest-order RT). However, the tests we performed show that this discretization introduces spurious eigenvalues interspersed among the actual ones of the acoustic problem. To report numerical evidence of this behavior, we applied this approach to a rectangular box. Figure 1 shows the eigenvalues computed in the range $[0, 4]$ (which contains approximations of the 4 smallest positive eigenvalues of Problem 1) on different meshes. We also include in this figure the exact eigenvalues. It can be clearly seen from Figure 1 that discretizing this weak formulation leads to spurious modes.

To give a hint about why these spurious eigenvalues appear, we need to characterize the eigenspace of $\lambda = 0$. In Problem 1, the infinite-dimensional eigenspace associated with $\lambda = 0$, consists of pure rotational motions which are not physically relevant since they do not induce gradients of pressure. However, a suitable numerical approximation should take care of them. Otherwise, spurious modes may appear (see, for instance, [11] where spurious modes are reported in a similar fluid-structure interaction problem). These spurious modes are non-vanishing eigenvalues of the discrete problem that are approximations of $\lambda = 0$. They arise as a consequence of the fact that, in this discretization, the eigenspace associated with $\lambda = 0$ is very small. In fact, it is easy to check that lowest-order RT elements with a vanishing $\mathrm{div}_{\mathrm{axi}}$ are locally of the form $\widetilde{\boldsymbol{u}} = (0, b)^\top, b \in \mathbb{R}$, namely, vertical translations. However, Problem 1 has infinitely many other solutions with $u_r \neq 0$, which are approximated in this discretization by eigenpairs with $\lambda > 0$. Because of this, in the discretized

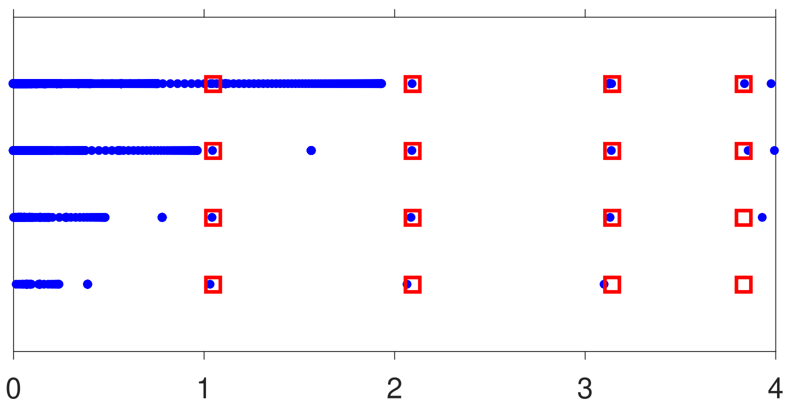

Fig. 1: Eigenvalues computed by solving Problem 1 with lowest-order RT elements (dots) and exact eigenvalues (squares) on four meshes with 192, 768, 3072 and 12288 triangles (from bottom to top).

problem, the eigenvalue $\lambda = 0$ is approximated by several spurious eigenvalues which are interspersed among the physical ones.

To avoid this drawback, in this paper we propose a new variational formulation equivalent to Problem 1 whose discretization will not introduce spurious eigenvalues. The discretization is also based on lowest-order RT finite elements but for a different variable $\boldsymbol{u}(r,z) := r\widetilde{\boldsymbol{u}}(r,z)$. By using the spectral theory for non-compact operators from [4, 5], we prove its spectral convergence and establish quasi-optimal-order error estimates.

The outline of the paper is as follows. In Section 2, we introduce some function spaces that will be used in the sequel. Then, in Section 3, we give an alternative weak formulation to Problem 1 and prove that it is equivalent to the spectral problem for a self-adjoint compact operator. This allows us to obtain a thorough characterization of the solutions of the eigenproblem. In Section 4, we introduce a finite element discretization. We prove quasi-optimal-order spectral convergence and absence of spurious modes. Finally, in Section 5, we report numerical tests that allow us to asses the convergence properties of the method and to check that it is not polluted with spurious modes, thus confirming the theoretical results.

2. Weighted Sobolev spaces. In this section we define appropriate weighted Sobolev spaces that will be used in the sequel and establish some of their properties. More general results can be found in [7, 3, 12, 10, 2].

Let $D \subset (0,\infty)\times\mathbb{R}$ be a convex polygonal domain. Let Ω be the solid of revolution generated by D. In order to prove optimal order of convergence for our numerical method we will also assume the following shift property:

H.1 for all $\breve{f} \in \mathrm{H}^1(\Omega)$ that does not depend on the angular coordinate θ, the solution $\breve{q} \in \mathrm{H}^1(\Omega)$ of

$$-\Delta\breve{q} = \breve{f} \quad \text{in} \quad \Omega, \qquad \frac{\partial\breve{q}}{\partial\breve{\boldsymbol{n}}} = 0 \quad \text{on} \quad \partial\Omega,$$

satisfies $\breve{q} \in \mathrm{H}^3(\Omega)$.

Let us emphasize that this assumption is only needed to prove quasi-optimal order of convergence. In fact, convergence holds true without the need of this assumption.

For $\omega \subset D$, $\alpha \in \mathbb{R}$ and $p \in [1,\infty)$, let $\mathrm{L}^p_\alpha(\omega)$ be the weighted Lebesgue space of measurable functions v on ω bounded in the norm

$$\|v\|_{\mathrm{L}^p_\alpha(\omega)} := \left(\int_\omega |v|^p r^\alpha \, dr\, dz\right)^{1/p}.$$

We denote by $\mathrm{H}^k_1(\omega)$ the weighted Sobolev spaces of functions in $\mathrm{L}^2_1(\omega)$, whose weak derivatives up to order k are in $\mathrm{L}^2_1(\omega)$. The following result will be used in the sequel (see [12, Remarque 4.1]).

Remark 2.1. *There holds* $\mathrm{H}^1_1(D) \hookrightarrow \mathrm{L}^2_{-1+\epsilon}(D)$ *continuously for all* $\epsilon > 0$.

Throughout the paper, we will use the following Hilbert spaces:

$$\begin{aligned}
\widetilde{\mathrm{H}}^1_1(D) &:= \mathrm{H}^1_1(D) \cap \mathrm{L}^2_{-1}(D),\\
\widehat{\mathrm{H}}^2_1(D) &:= \left\{v \in \mathrm{H}^2_1(D) : \partial_r v \in \mathrm{L}^2_{-1}(D)\right\},\\
\mathrm{H}_1(\mathrm{div}_{\mathrm{axi}}; D) &:= \left\{\boldsymbol{v} \in \mathrm{L}^2_1(D)^2 : \ \mathrm{div}_{\mathrm{axi}}\, \boldsymbol{v} \in \mathrm{L}^2_1(D)\right\},\\
\mathrm{H}_{-1}(\mathrm{div}; D) &:= \left\{\boldsymbol{v} \in \mathrm{L}^2_{-1}(D)^2 : \ \mathrm{div}\, \boldsymbol{v} \in \mathrm{L}^2_{-1}(D)\right\},
\end{aligned}$$

where $\mathrm{div}_{\mathrm{axi}}$ has been defined in (1.3), with their respective norms defined by

$$\begin{aligned}
\|v\|^2_{\widetilde{\mathrm{H}}^1_1(D)} &:= \|v\|^2_{\mathrm{H}^1_1(D)} + \|v\|^2_{\mathrm{L}^2_{-1}(D)},\\
\|v\|^2_{\widehat{\mathrm{H}}^2_1(D)} &:= \|v\|^2_{\mathrm{H}^2_1(D)} + \|\partial_r v\|^2_{\mathrm{L}^2_{-1}(D)},\\
\|\boldsymbol{v}\|^2_{\mathrm{H}_1(\mathrm{div}_{\mathrm{axi}}; D)} &:= \|\boldsymbol{v}\|^2_{\mathrm{L}^2_1(D)} + \|\mathrm{div}_{\mathrm{axi}}\, \boldsymbol{v}\|^2_{\mathrm{L}^2_1(D)},\\
\|\boldsymbol{v}\|^2_{\mathrm{H}_{-1}(\mathrm{div}; D)} &:= \|\boldsymbol{v}\|^2_{\mathrm{L}^2_{-1}(D)} + \|\mathrm{div}\, \boldsymbol{v}\|^2_{\mathrm{L}^2_{-1}(D)}.
\end{aligned}$$

On the boundary, let Γ_0 be the intersection of ∂D and the z -axis, namely, $\Gamma_0 := \{(r,z) \in \partial D : r = 0\}$ and $\Gamma := \partial D \setminus \Gamma_0$. Finally, we define the following spaces:

$$\begin{aligned}
\mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D) &:= \{\boldsymbol{v} \in \mathrm{H}_1(\mathrm{div}_{\mathrm{axi}}; D) : \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \partial D\},\\
\mathrm{H}^0_{-1}(\mathrm{div}; D) &:= \{\boldsymbol{v} \in \mathrm{H}_{-1}(\mathrm{div}; D) : \ \boldsymbol{v}\cdot\boldsymbol{n} = 0 \ \text{ on } \partial D\},\\
\widetilde{\boldsymbol{K}} &:= \left\{\boldsymbol{u} \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D) : \ \mathrm{div}_{\mathrm{axi}} \ \boldsymbol{u} = 0 \ \text{ in } D\right\},
\end{aligned}$$

where $\boldsymbol{n} := (n_r, n_z)$ denotes the outward unit normal.

The following lemma gives a Helmholtz-like decomposition of $\mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$, which will be used below.

Lemma 2.2. *Let* $\widetilde{\boldsymbol{G}} := \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D) \cap \nabla(\mathrm{H}^1_1(D))$. *Then,*

$$\mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D) := \mathrm{H}^1_1(D) \oplus \widetilde{\boldsymbol{G}},$$

is an orthogonal decomposition in both, $\mathrm{L}^2_1(D)^2$ *and* $\mathrm{H}_1(\mathrm{div}_{\mathrm{axi}}; D)$. *Moreover, for all* $\boldsymbol{v} \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$, *if* $\boldsymbol{v} = \hat{\boldsymbol{\chi}} + \hat{\boldsymbol{\eta}}$ *with* $\hat{\boldsymbol{\chi}} \in \widetilde{\boldsymbol{G}}$ *and* $\hat{\boldsymbol{\eta}} \in \widetilde{\boldsymbol{K}}$, *then*

$$\|\hat{\boldsymbol{\chi}}\|_{\widetilde{\mathrm{H}}^1_1(D)\times \mathrm{H}^1_1(D)} \leq C\, \|\mathrm{div}_{\mathrm{axi}}\, \hat{\boldsymbol{\chi}}\|_{\mathrm{L}^2_1(D)}. \tag{2.1}$$

Proof. The orthogonal decomposition is a consequence of [3, Lemma 2]. Thus, for any $\boldsymbol{v} \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$ we write $\boldsymbol{v} = \hat{\boldsymbol{\chi}} + \hat{\boldsymbol{\eta}}$ with $\hat{\boldsymbol{\chi}} = \nabla p$, $p \in \mathrm{H}^1_1(D)$ and $\hat{\boldsymbol{\eta}} = (\hat{\eta}_r, \hat{\eta}_z) \in \widetilde{\boldsymbol{K}}$. We define $\forall (r,\theta,z)$ polar coordinate of a point in Ω

$$\breve{p}(r,\theta,z) := p(r,z), \quad \breve{\boldsymbol{\chi}}(r,\theta,z) := \nabla \breve{p}(r,\theta,z) = \partial_r p(r,z)\boldsymbol{e}_r + \partial_z p(r,z)\boldsymbol{e}_z,$$

$$\breve{\boldsymbol{\eta}}(r,\theta,z) := \hat{\eta}_r(r,z)\boldsymbol{e}_r + \hat{\eta}_z(r,z)\boldsymbol{e}_z \quad \text{and} \quad \breve{\boldsymbol{v}} := \breve{\boldsymbol{\chi}} + \breve{\boldsymbol{\eta}}.$$

Since $\boldsymbol{v} \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$ and $\hat{\boldsymbol{\eta}} \in \widetilde{\boldsymbol{K}}$, then $\breve{\boldsymbol{v}} \in \mathrm{H}_0(\mathrm{div};\Omega)$ and $\mathrm{div}\,\breve{\boldsymbol{\eta}} = 0$. Moreover, $\breve{p} \in \mathrm{H}^1(\Omega)/\mathbb{R}$ is the solution to

$$\int_\Omega \nabla\breve{p}\cdot\nabla\breve{q} = -\int_\Omega \breve{q}\,\mathrm{div}\,\breve{\boldsymbol{v}} \qquad \forall \breve{q} \in \mathrm{H}^1(\Omega)/\mathbb{R}.$$

From the convexity assumption on Ω we know that $\breve{p} \in \mathrm{H}^2(\Omega)$ and it satisfies (see [9, Theorem 1.8])

$$\|\nabla\breve{p}\|_{\mathrm{H}^1(\Omega)} \le C\,\|\mathrm{div}\,\breve{\boldsymbol{v}}\|_{\mathrm{L}^2(\Omega)}.$$

Therefore, estimate (2.1) follows from the previous inequality and the fact that $\|\mathrm{div}\,\breve{\boldsymbol{v}}\|_{\mathrm{L}^2(\Omega)} = \|\mathrm{div}\,\breve{\boldsymbol{\chi}}\|_{\mathrm{L}^2(\Omega)} = \sqrt{2\pi}\,\|\mathrm{div}_{\mathrm{axi}}\,\hat{\boldsymbol{\chi}}\|_{\mathrm{L}^2_1(D)}$ and $\|\nabla\breve{p}\|_{\mathrm{H}^1(\Omega)} = \|\breve{\boldsymbol{\chi}}\|_{\mathrm{H}^1(\Omega)} = \sqrt{2\pi}\,\|\hat{\boldsymbol{\chi}}\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)}$, where the last equality is consequence of the isomorphism between the space of axisymmetric functions in $\mathrm{H}^1(\Omega)^3$ and $\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)$ (see [2]).

Throughout the paper, C with or without subscripts will be used for positive constants not necessarily the same at each occurrence, but always independent of r.

3. Weak formulation for the axisymmetric problem. As claimed in the introduction, a direct discretization of Problem 1 by RT elements leads to spurious eigenvalues. In what follows we introduce an alternative formulation which overcomes this drawback and will lead, after discretization, to a well posed generalized eigenvalue problem.

For the analysis of Problem 1, we consider the change of variable $\boldsymbol{u} := r\widetilde{\boldsymbol{u}}$ which leads to the following:

Problem 2. *Find* $(\lambda,\boldsymbol{u}) \in \mathbb{R}\times\mathrm{H}^0_{-1}(\mathrm{div}; D)$*, such that*

$$\int_D \frac{1}{r}\,\mathrm{div}\,\boldsymbol{u}\,\mathrm{div}\,\boldsymbol{v}\,dr\,dz = \lambda\int_D \frac{1}{r}\boldsymbol{u}\cdot\boldsymbol{v}\,dr\,dz \qquad \forall\boldsymbol{v}\in\mathrm{H}^0_{-1}(\mathrm{div}; D). \tag{3.1}$$

Problems 1 and 2 are equivalent in the sense that $(\lambda,\widetilde{\boldsymbol{u}})$ is a solution of Problem 1 if and only if $(\lambda, r\widetilde{\boldsymbol{u}})$ solves Problem 2. In fact, this is an immediate consequence of the following elementary relation:

Lemma 3.1. $\boldsymbol{v}\in\mathrm{H}^0_{-1}(\mathrm{div}; D)$ *if and only if* $\boldsymbol{v}/r \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$*. Moreover*

$$\left\|\frac{\boldsymbol{v}}{r}\right\|_{\mathrm{L}^2_1(D)} = \|\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)} \qquad \text{and} \qquad \left\|\mathrm{div}_{\mathrm{axi}}\left(\frac{\boldsymbol{v}}{r}\right)\right\|_{\mathrm{L}^2_1(D)} = \|\mathrm{div}\,\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)}. \tag{3.2}$$

In what follows we will show that the discretization of Problem 2 does not lead to spurious modes. Thus, we focus on its analysis. With this aim we introduce the solution operator:

$$\begin{aligned} T:\ \mathrm{H}^0_{-1}(\mathrm{div}; D) &\longrightarrow \mathrm{H}^0_{-1}(\mathrm{div}; D),\\ \boldsymbol{f} &\longmapsto T\boldsymbol{f} := \boldsymbol{w}, \end{aligned}$$

with $\boldsymbol{w} \in \mathrm{H}^0_{-1}(\mathrm{div}; D)$ such that

(3.3)
$$\int_D \frac{1}{r} \operatorname{div} \boldsymbol{w} \operatorname{div} \boldsymbol{v} \, dr\, dz + \int_D \frac{1}{r} \boldsymbol{w} \cdot \boldsymbol{v} \, dr\, dz = \int_D \frac{1}{r} \boldsymbol{f} \cdot \boldsymbol{v} \, dr\, dz \qquad \forall \boldsymbol{v} \in \mathrm{H}^0_{-1}(\mathrm{div}; D).$$

The well-posedness of problem (3.3) is a direct consequence of Lax-Milgram lemma, whence T is well-defined, self-adjoint and continuous. Note that $T\boldsymbol{u} = \mu \boldsymbol{u}$, with $\mu \neq 0$, if and only if $(\lambda, \boldsymbol{u})$ is a solution of Problem 2 with $\lambda + 1 = \frac{1}{\mu}$. Moreover, since the eigenvalues of Problem 2 are positive, then those of T satisfy $0 < \mu \leq 1$.

Clearly $\mu = 1$ is an eigenvalue of T (correspondingly, $\lambda = 0$ is an eigenvalue of Problem 2) with associated eigenspace

$$\boldsymbol{K} := \left\{ \boldsymbol{u} \in \mathrm{H}^0_{-1}(\mathrm{div}; D) : \ \operatorname{div} \boldsymbol{u} = 0 \ \in D \right\}, \tag{3.4}$$

which is a closed subspace of $\mathrm{H}^0_{-1}(\mathrm{div}; D)$.

The following lemma provides an orthogonal decomposition of $\mathrm{H}^0_{-1}(\mathrm{div}; D)$.

Lemma 3.2. *Let $\boldsymbol{G} := \{ r\boldsymbol{v} : \boldsymbol{v} \in \widetilde{\boldsymbol{G}} \}$. Then,*

$$\mathrm{H}^0_{-1}(\mathrm{div}; D) := \ \boldsymbol{K} \oplus \boldsymbol{G}, \tag{3.5}$$

is an orthogonal decomposition in $\mathrm{L}^2_{-1}(D)$ and $\mathrm{H}^0_{-1}(\mathrm{div}; D)$. Moreover, for all $\boldsymbol{v} \in \mathrm{H}^0_{-1}(\mathrm{div}; D)$, if $\boldsymbol{v} = \boldsymbol{\chi} + \boldsymbol{\eta}$ with $\boldsymbol{\chi} \in \boldsymbol{G}$ and $\boldsymbol{\eta} \in \boldsymbol{K}$,

$$\left\| \frac{\boldsymbol{\chi}}{r} \right\|_{\widetilde{\mathrm{H}}^1_1(D) \times \mathrm{H}^1_1(D)} \leq C \left\| \operatorname{div} \boldsymbol{\chi} \right\|_{\mathrm{L}^2_{-1}(D)}. \tag{3.6}$$

Proof. Let $\boldsymbol{u} \in \mathrm{H}^0_{-1}(\mathrm{div}; D)$. From Lemma 3.1 it follows that $\boldsymbol{u}/r \in \mathrm{H}^0_1(\mathrm{div}_{\mathrm{axi}}; D)$ and, from Lemma 2.2 we write

$$\frac{\boldsymbol{u}}{r} = \hat{\boldsymbol{\eta}} + \hat{\boldsymbol{\chi}},$$

where $\hat{\boldsymbol{\chi}} \in \widetilde{\boldsymbol{G}}$ and $\hat{\boldsymbol{\eta}} \in \widetilde{\boldsymbol{K}}$. Let $\boldsymbol{\eta} := r\hat{\boldsymbol{\eta}}$ and $\boldsymbol{\chi} := r\hat{\boldsymbol{\chi}}$. Clearly $\boldsymbol{\chi}$ belongs to $\boldsymbol{G}$ and $\boldsymbol{\eta} \in \boldsymbol{K}$, the latter because (cf. (1.3))

$$\operatorname{div}(\boldsymbol{\eta}) = \operatorname{div}(r\hat{\boldsymbol{\eta}}) = r \operatorname{div}_{\mathrm{axi}}(\hat{\boldsymbol{\eta}}) = 0.$$

Moreover, $\boldsymbol{\eta}$ and $\boldsymbol{\chi}$ are $\mathrm{L}^2_{-1}(D)$-orthogonal since $\hat{\boldsymbol{\eta}}$ and $\hat{\boldsymbol{\chi}}$ are $\mathrm{L}^2_1(D)$-orthogonal (see Lemma 2.2), namely,

$$\int_D \frac{1}{r} \boldsymbol{\eta} \cdot \boldsymbol{\chi} = \int_D \frac{1}{r} (r\hat{\boldsymbol{\eta}}) \cdot (r\hat{\boldsymbol{\chi}}) = 0.$$

Finally, (3.6) follows from (2.1), (3.2) and the fact that $\hat{\boldsymbol{\chi}} = \boldsymbol{\chi}/r$.

We notice that $T|_{\boldsymbol{K}}$ is not compact. In fact, $T|_{\boldsymbol{K}}$ is the identity on the infinite-dimensional subspace $\boldsymbol{K} \subset \mathrm{H}^0_{-1}(\mathrm{div}; D)$. However, we will show that $\boldsymbol{G}$ is an invariant subspace for T and

$$\widehat{T} := T|_{\boldsymbol{G}} : \boldsymbol{G} \to \boldsymbol{G}$$

is compact. Therefore, since $\sigma(T) = \sigma(\widehat{T}) \cup \{1\}$ (cf. [1]) to obtain a spectral characterization of T it is enough to know the spectrum of $\widehat{T}$. The following lemma shows an additional regularity result which will be used with this aim.

Lemma 3.3. *If $\boldsymbol{f} \in \boldsymbol{G}$, then $\boldsymbol{u} = \widehat{T}\boldsymbol{f} \in \boldsymbol{G}$ satisfies*

$$\left\| \frac{\operatorname{div} \boldsymbol{u}}{r} \right\|_{\mathrm{H}^1_1(D)} + \left\| \frac{\boldsymbol{u}}{r} \right\|_{\widetilde{\mathrm{H}}^1_1(D) \times \mathrm{H}^1_1(D)} \leq C \ \left\| \boldsymbol{f} \right\|_{\mathrm{L}^2_{-1}(D)}. \tag{3.7}$$

Consequently, $\widehat{T} : \boldsymbol{G} \to \boldsymbol{G}$ is compact.

Proof. Let $\boldsymbol{f} \in \boldsymbol{G}$ and $\boldsymbol{u} = \widehat{T}\boldsymbol{f} \in \boldsymbol{G}$. By taking $\boldsymbol{w} = \boldsymbol{u}$ in (3.3) we have that

$$\left\|\frac{\boldsymbol{u}}{r}\right\|_{\mathrm{L}^2_1(D)} + \left\|\frac{\operatorname{div}\boldsymbol{u}}{r}\right\|_{\mathrm{L}^2_1(D)} \leq \|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)}. \tag{3.8}$$

Thus, the bound in (3.7) for $\boldsymbol{u}/r$ follows from this equation and estimate (3.6). On the other hand, since $\mathcal{D}(D)^2 \subset \mathrm{H}^0_{-1}(\operatorname{div}; D)$, by integration by parts in (3.3) we obtain

$$\nabla\left(\frac{1}{r}\operatorname{div}\boldsymbol{u}\right) = \frac{\boldsymbol{u}}{r} - \frac{\boldsymbol{f}}{r} \ \in \widetilde{\boldsymbol{G}}. \tag{3.9}$$

and hence from (3.8)

$$\left\|\nabla\left(\frac{1}{r}\operatorname{div}\boldsymbol{u}\right)\right\|_{\mathrm{L}^2_1(D)} \leq \left\|\frac{\boldsymbol{u}}{r}\right\|_{\mathrm{L}^2_1(D)} + \|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)} \leq C\,\|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)}.$$

Then, the estimate in (3.7) for $\left\|\frac{\operatorname{div}\boldsymbol{u}}{r}\right\|_{\mathrm{H}^1_1(D)}$ follows from the above equation and (3.8).

Next, to prove that $\widehat{T}$ is compact, we show that for any bounded sequence $\{\boldsymbol{f}_n\}_{n\in\mathbb{N}} \in \boldsymbol{G}$, the sequence $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}} := \{\widehat{T}\boldsymbol{f}_n\}_{n\in\mathbb{N}} \in \boldsymbol{G}$ contains a converging subsequence. From the definition of $\boldsymbol{G}$, $\boldsymbol{u}_n = r\widetilde{\boldsymbol{u}}_n$ with $\widetilde{\boldsymbol{u}}_n \in \widetilde{\boldsymbol{G}}, n \in \mathbb{N}$. Moreover, from the definition of $\operatorname{div}_{\mathrm{axi}}$ (cf. (1.3)) and estimate (3.7) we notice that $\{\widetilde{\boldsymbol{u}}_n\}_{n\in\mathbb{N}}$ satisfies

$$\|\widetilde{\boldsymbol{u}}\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} + \|\operatorname{div}_{\mathrm{axi}}\widetilde{\boldsymbol{u}}\|_{\mathrm{H}^1_1(D)} \leq C\ \ \|\boldsymbol{f}_n\|_{\mathrm{L}^2_{-1}(D)}.$$

Then $\{\widetilde{\boldsymbol{u}}_n\}_{n\in\mathbb{N}}$ and $\{\operatorname{div}_{\mathrm{axi}}\widetilde{\boldsymbol{u}}_n\}_{n\in\mathbb{N}}$ are bounded in $\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)$ and $\mathrm{H}^1_1(D)$, respectively. On the other hand, we recall that the embedding $\mathrm{H}^1_1(D) \hookrightarrow \mathrm{L}^2_1(D)$ is compact (see, [12, Lemme 4.2]). Thus, since the embedding $\mathrm{H}^1(\Omega)^3 \hookrightarrow \mathrm{L}^2(\Omega)^3$ is also compact, then the embedding $\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D) \hookrightarrow \mathrm{L}^2_1(D)^2$ is compact, too. Therefore it follows that $\{\widetilde{\boldsymbol{u}}_n\}_{n\in\mathbb{N}}$ and $\{\operatorname{div}_{\mathrm{axi}}\widetilde{\boldsymbol{u}}_n\}_{n\in\mathbb{N}}$ contain a converging (not relabeled) subsequence such that $\widetilde{\boldsymbol{u}}_n \to \widetilde{\boldsymbol{u}} \in \mathrm{L}^2_1(D)^2$ and $\operatorname{div}_{\mathrm{axi}}\widetilde{\boldsymbol{u}}_n \to \operatorname{div}_{\mathrm{axi}}\widetilde{\boldsymbol{u}} \in \mathrm{L}^2_1(D)$. Hence, there exists a converging (not relabeled) subsequence of $\{\boldsymbol{u}_n\}_{n\in\mathbb{N}}$ such that $\boldsymbol{u}_n \to r\widetilde{\boldsymbol{u}}$ in $\mathrm{H}_{-1}(\operatorname{div}; D)$, which proves the result.

From the previous result we obtain the following spectral characterization of T.

THEOREM 3.4. *The spectrum of T decomposes as follows:*

$$\sigma(T) = \{1\} \cup \{\mu_n\}_{n\in\mathbb{N}} \cup \{0\}.$$

Moreover:

- $\mu = 1$ *is an eigenvalue of T with infinite-dimensional eigenspace $\boldsymbol{K}$;*
- $\{\mu_n\}_{n\in\mathbb{N}}$ *is a sequence of finite-multiplicity eigenvalues $\mu_n \in (0,1)$, $n \in \mathbb{N}$, which converge to 0;*
- $\mu = 0$ *is not an eigenvalue of T.*

Proof. As claimed above $\mu = 1$ is an eigenvalue of T with corresponding eigenspace $\boldsymbol{K}$ and $\sigma(T) = \sigma(\widehat{T}) \cup \{1\}$. Thus, the spectralcharacterization of T is a consequence of the compactness of $\widehat{T}$. Moreover, from the relation between the eigenvalues of T and the solution of Problem 2, it is easy to check that $0 < \mu_n < 1$. Finally, it is clear that $\mu = 0$ is not an eigenvalue of $\widehat{T}$.

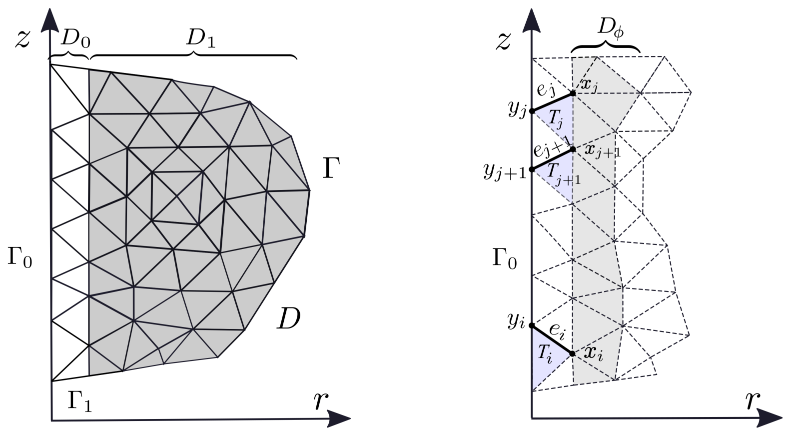

Fig. 2: Left: Decomposition of the finite element triangulation $\mathcal{T}_h$. Right: Notations used in the proof of Lemma 4.4.

4. Finite element approximation. In this section, we introduce a Galerkin approximation of Problem 2 and prove some convergence results. We assume that $\{\mathcal{T}_h\}_{h>0}$ is a regular family of partitions of $\overline{D}$ into triangles T; parameter h stands for the mesh-size and, from now on, we assume that any generic constant denoted by C is not only independent of r but also independent of h. We denote

$$\mathcal{T}_h^0 := \{T \in \mathcal{T}_h : T \cap \Gamma_0 \neq \emptyset\}, \qquad \mathcal{T}_h^1 = \mathcal{T}_h \setminus \mathcal{T}_h^0,$$

and define the open sets $D_0, D_1 \subset D$ such that $\overline{D}_0 := \bigcup_{T\in\mathcal{T}_h^0} T$, and $\overline{D}_1 := \bigcup_{T\in\mathcal{T}_h^1} T$. We also define $\Gamma_1 := \partial D_0 \cap \partial D_1$. We assume that the meshes are such that Γ_1 is parallel to the z -axis (see Figure 2 left). For any $T \in \mathcal{T}_h$, we define $r_{\max}(T) := \max\{r : (r,z) \in T\}$ and $r_{\min}(T) := \min\{r : (r,z) \in T\}$. The following inequalities hold: for all $T \in \mathcal{T}_h^1$,

$$r_{\min}(T) \geq Ch_T, \qquad r_{\max}(T) \leq Cr_{\min}(T). \tag{4.1}$$

Moreover, clearly, $r_{\max}(T) < Ch_T \ \forall T \in \mathcal{T}_h^0$.

For space discretization we use lowest-order RT elements:

(4.2)
$$\boldsymbol{R}_h := \left\{\boldsymbol{v}_h \in \mathrm{H}_{-1}(\mathrm{div}; D) : \ \boldsymbol{v}_h|_T \in \boldsymbol{RT}(T) \quad \forall T \in \mathcal{T}_h \quad \text{and} \quad \boldsymbol{v}_h|_T := \boldsymbol{0} \quad \forall T \in \mathcal{T}_h^0\right\},$$

where

$$\boldsymbol{RT}(T) := \left\{\boldsymbol{v}_h \in \mathbb{P}_1(T)^2 : \ \boldsymbol{v}_h(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x}, \ \boldsymbol{a} \in \mathbb{R}^2, \ b \in \mathbb{R}, \ \boldsymbol{x} \in T\right\}.$$

Whence, the natural approximation space for $\mathrm{H}^0_{-1}(\mathrm{div}; D)$ is

$$\boldsymbol{R}_h^0 := \boldsymbol{R}_h \cap \mathrm{H}^0_{-1}(\mathrm{div}; D),$$

and the Galerkin approximation of Problem 2 reads as follows:

Problem 3. *Find $(\lambda_h, \boldsymbol{u}_h) \in \mathbb{R} \times \boldsymbol{R}_h^0$, such that*

$$\int_D \frac{1}{r} \operatorname{div} \boldsymbol{u}_h \operatorname{div} \boldsymbol{v}_h drdz = \lambda \int_D \frac{1}{r} \boldsymbol{u}_h \cdot \boldsymbol{v}_h drdz \qquad \forall \boldsymbol{v} \in \boldsymbol{R}_h^0. \tag{4.3}$$

We introduce the corresponding discrete solution operator:

$$\begin{aligned} T_h :\ \mathrm{H}^0_{-1}(\mathrm{div}; D) &\longrightarrow \mathrm{H}^0_{-1}(\mathrm{div}; D), \\ \boldsymbol{f} &\longmapsto T_h \boldsymbol{f} := \boldsymbol{w}_h, \end{aligned}$$

with $\boldsymbol{w}_h \in \boldsymbol{R}_h^0$ such that

$$\int_D \frac{1}{r} \operatorname{div} \boldsymbol{w}_h \operatorname{div} \boldsymbol{v}_h \, dr\, dz + \int_D \frac{1}{r} \boldsymbol{w}_h \cdot \boldsymbol{v}_h \, dr\, dz = \int_D \frac{1}{r} \boldsymbol{f} \cdot \boldsymbol{v}_h \, dr\, dz \qquad \forall \boldsymbol{v}_h \in \boldsymbol{R}_h^0. \tag{4.4}$$

As a consequence of Lax-Milgram Lemma, T_h is well-defined, self-adjoint, continuous and with finite range. Clearly λ_h is an eigenvalue of Problem 3 if an only if $\frac{1}{1+\lambda_h} \in \sigma(T_h)$. To prove convergence and error estimates for the proposed Galerkin scheme we will use the results on spectral approximation for non-compact operators from [4, 5]. With this aim, we consider the operator T_h restricted to $\boldsymbol{R}_h^0$, which is an invariant subspace of this operator.

To use the theory from [4, 5] in our case, we need to prove the following two properties:

P1: $\lim_{h\to 0} \left\| (T - T_h)\,|_{\boldsymbol{R}_h^0} \right\|_{\mathrm{H}_{-1}(\mathrm{div};D)} = 0.$

P2: for each eigenfunction $\boldsymbol{u}$ of T associated with $\lambda \neq 1$,

$$\lim_{h\to 0} \inf_{\boldsymbol{v}_h \in \boldsymbol{R}_h^0} \|\boldsymbol{u} - \boldsymbol{v}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)} = 0.$$

To prove property P1, we will establish some preliminary results. Let I_h^R be the classical Raviart-Thomas interpolant in $\mathrm{H}(\mathrm{div}, D)$ (see [15]), we introduce $I_h : \mathrm{H}_{-1}(\mathrm{div}; D) \cap \mathrm{H}^1(D) \to \boldsymbol{R}_h^0$ such that $I_h \boldsymbol{u}$ is defined differently in the triangles in $\mathcal{T}_h^0$ and $\mathcal{T}_h^1$. On the former, we just define $I_h \boldsymbol{u} = \boldsymbol{0}$. On the latter, $I_h \boldsymbol{u}$ is the classical RT interpolant modified in such a way that $I_h \boldsymbol{u} \in \mathrm{H}(\mathrm{div}, D)$. With this aim, the degrees of freedom associated with the edges ℓ lying on Γ_1 are defined as $I_h \boldsymbol{u}|_\ell \cdot \boldsymbol{n} = 0$.

Next, we give an estimate for $I_h^R \boldsymbol{u} - I_h \boldsymbol{u}$ in the $\mathrm{H}_{-1}(\mathrm{div}; D)$-norm which will be used in the sequel. With this aim the consider the following lemma.

Lemma 4.1. *Let $\boldsymbol{f} \in \boldsymbol{G}$ and $\boldsymbol{u} = \widehat{T} \boldsymbol{f} \in \boldsymbol{G}$ then, there exists $q \in (2, 6)$ such that*

$$\left\| \nabla \left(\frac{u_r}{r} \right) \right\|_{\mathrm{L}^q_1(D)} + \left\| \nabla \left(\frac{u_z}{r} \right) \right\|_{\mathrm{L}^q_1(D)} \leq C \, \|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)} . \tag{4.5}$$

Additionally, if **H.1** *holds true, then*

$$\left\| \frac{\boldsymbol{u}}{r} \right\|_{\mathrm{H}^2_1(D) \times \widehat{\mathrm{H}}^2_1(D)} \leq C \ \|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)} . \tag{4.6}$$

Proof. Let $\boldsymbol{f} \in \boldsymbol{G}$ and $\boldsymbol{u} = \widehat{T} \boldsymbol{f} \in \boldsymbol{G}$ then, from Lemma 3.3 and the definition of $\boldsymbol{G}$ it follows that there exists $\widetilde{\boldsymbol{u}} = \boldsymbol{u}/r \in \widetilde{\boldsymbol{G}}$ such that $(\operatorname{div} \boldsymbol{u})/r = \operatorname{div}_{\mathrm{axi}} \widetilde{\boldsymbol{u}} \in \mathrm{H}^1_1(D)$ and

$$\|\operatorname{div}_{\mathrm{axi}} \widetilde{\boldsymbol{u}}\|_{\mathrm{H}^1_1(D)} \leq C \ \|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)} . \tag{4.7}$$

By proceeding as in Lemma 2.2 we write $\widetilde{\boldsymbol{u}} = \nabla p$, $p \in \mathrm{H}^1_1(D)$ and define for every point in Ω with polar coordinate (r, θ, z)

$$\breve{p}(r, \theta, z) := p(r, z), \qquad \breve{\boldsymbol{u}}(r, \theta, z) := \nabla \breve{p}(r, \theta, z) = \partial_r p(r, z) \boldsymbol{e}_r + \partial_z p(r, z) \boldsymbol{e}_z$$

where $\breve{p} \in \mathrm{H}^1(\Omega)/\mathbb{R}$ is the solution to

$$\int_\Omega \nabla\breve{p}\cdot\nabla\breve{q} = -\int_\Omega \breve{q}\,\mathrm{div}\,\breve{\boldsymbol{u}} \qquad \forall\breve{q}\in\ \mathrm{H}^1(\Omega)/\mathbb{R}.$$

Since $\mathrm{div}\,\breve{\boldsymbol{u}}(r,\theta,z) = \mathrm{div}_{\mathrm{axi}}\,\widetilde{\boldsymbol{u}}(r,z)$, from (4.7) it follows that $\mathrm{div}\,\breve{\boldsymbol{u}}\in\mathrm{H}^1(\Omega)$ and

$$\|\mathrm{div}\,\breve{\boldsymbol{u}}\|_{\mathrm{H}^1(\Omega)} = \sqrt{2\pi}\,\|\mathrm{div}_{\mathrm{axi}}\,\widetilde{\boldsymbol{u}}\|_{\mathrm{H}^1_1(D)} \le C\,\|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)}. \tag{4.8}$$

Therefore, from the convexity of Ω and additional regularity results for the Laplacian it follows that there exists $s\in(0,1)$ such that $\breve{p}\in\mathrm{H}^{2+s}(\Omega)$ and

$$\|\breve{p}\|_{\mathrm{H}^{2+s}(\Omega)} \le C\,\|\mathrm{div}\,\breve{\boldsymbol{u}}\|_{\mathrm{H}^1(\Omega)}.$$

We notice that $\breve{\boldsymbol{u}} = \nabla\breve{p}\in\mathrm{H}^{1+s}(\Omega)^3$ and thus $\nabla\breve{u}_r,\nabla\breve{u}_z\in\mathrm{H}^s(\Omega)^3$. On the other hand, since the embedding $\mathrm{H}^s(\Omega)\hookrightarrow\mathrm{L}^q(\Omega)$ is continuous for $q = 6/(3-2s)\in(2,6)$ (see [6, Theorem 6.7]), from the previous estimate and (4.8) we obtain that

$$\begin{aligned}\|\nabla\breve{u}_r\|_{\mathrm{L}^q(\Omega)} + \|\nabla\breve{u}_z\|_{\mathrm{L}^q(\Omega)} &\le C\left(\|\nabla\breve{u}_r\|_{\mathrm{H}^s(\Omega)} + \|\nabla\breve{u}_z\|_{\mathrm{H}^s(\Omega)}\right)\\ &\le C\,\|\breve{\boldsymbol{u}}\|_{\mathrm{H}^{1+s}(\Omega)} \le C\,\|\boldsymbol{f}\|_{\mathrm{L}^2_{-1}(D)}.\end{aligned}$$

Estimate (4.5) follows from the previous inequality and the relation between $\breve{\boldsymbol{u}}$, $\widetilde{\boldsymbol{u}}$ and $\boldsymbol{u}$. Finally, if **H.1** holds true, then $\breve{p}\in\mathrm{H}^3(\Omega)$ and

$$\|\nabla\breve{p}\|_{\mathrm{H}^2(\Omega)} \le C\ \ \|\mathrm{div}\,\breve{\boldsymbol{u}}\|_{\mathrm{H}^1(\Omega)}.$$

Therefore, estimate (4.6) follows from the previous bound, (4.8) and the fact that $\|\nabla\breve{p}\|_{2,\Omega} = \|\breve{\boldsymbol{u}}\|_{2,\Omega} \ge C\,\|\widetilde{\boldsymbol{u}}\|_{\mathrm{H}^2_1(D)\times\widehat{\mathrm{H}}^2_1(D)}$, where the last inequality is a consequence of the isomorphism between the space of axisymmetric functions in $\mathrm{H}^2(\Omega)^3$ and $\mathrm{H}^2_1(D)\times\widehat{\mathrm{H}}^2_1(D)$ (see [2]).

LEMMA 4.2. *There exists $C>0$ such that, for each $\boldsymbol{f}\in\boldsymbol{G}$ and $\boldsymbol{u} := T\boldsymbol{f}$*

$$\sum_{T\in\mathcal{T}_h^1} \left\|I_h^R\boldsymbol{u} - I_h\boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch^2\,\|\boldsymbol{f}\|^2_{\mathrm{L}^2_{-1}(D)}. \tag{4.9}$$

Moreover for some $q\in(2,6)$

$$\sum_{T\in\mathcal{T}_h^1} \left\|\mathrm{div}(I_h^R\boldsymbol{u} - I_h\boldsymbol{u})\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch^{2-4/q}\,\|\boldsymbol{f}\|^2_{\mathrm{L}^2_{-1}(D)}. \tag{4.10}$$

Additionally, if **H.1** *holds true then, for all $\epsilon>0$, there exist $C>0$ such that*

$$\sum_{T\in\mathcal{T}_h^1} \left\|\mathrm{div}(I_h^R\boldsymbol{u} - I_h\boldsymbol{u})\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch^{2-2\epsilon}\,\|\boldsymbol{f}\|^2_{\mathrm{L}^2_{-1}(D)}. \tag{4.11}$$

Proof. Let $\mathcal{T}_h^a$ the set of triangles in $\mathcal{T}_h^1$ with an edge ℓ lying on Γ_1. We notice that $I_h^R\boldsymbol{u} - I_h\boldsymbol{u} = \boldsymbol{0}$ for all $T\in\mathcal{T}_h^1\setminus\mathcal{T}_h^a$ and thus

$$\sum_{T\in\mathcal{T}_h^1} \left\|I_h^R\boldsymbol{u} - I_h\boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)} = \sum_{T\in\mathcal{T}_h^a} \left\|I_h^R\boldsymbol{u} - I_h\boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)}.$$

Let $\boldsymbol{\varphi}_\ell$ be the standard basis function of lowest-order RT elements associated with edge ℓ, and $T \in \mathcal{T}_h^a$ such that $\Gamma_1 \cap T = \ell$. Then

$$\left\|I_h^R \boldsymbol{u} - I_h \boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)} = \left(\frac{1}{|\ell|}\int_\ell |\boldsymbol{u}\cdot\boldsymbol{n}|\right)^2 \|\boldsymbol{\varphi}_\ell\|^2_{\mathrm{L}^2_{-1}(T)}, \tag{4.12}$$

$$\left\|\operatorname{div}(I_h^R \boldsymbol{u} - I_h \boldsymbol{u})\right\|^2_{\mathrm{L}^2_{-1}(T)} = \left(\frac{1}{|\ell|}\int_\ell |\boldsymbol{u}\cdot\boldsymbol{n}|\right)^2 \|\operatorname{div}\boldsymbol{\varphi}_\ell\|^2_{\mathrm{L}^2_{-1}(T)}. \tag{4.13}$$

It is straightforward to bound the norms of the basis functions as follows:

$$\|\boldsymbol{\varphi}_\ell\|^2_{\mathrm{L}^2_{-1}(T)} = \int_T \frac{1}{r}\boldsymbol{\varphi}_\ell^2 \le \frac{1}{r_{\min(T)}}\int_T \boldsymbol{\varphi}_\ell^2 \le \frac{2|T|}{3h_T} \le Ch_T, \tag{4.14}$$

$$\|\operatorname{div}\boldsymbol{\varphi}_\ell\|^2_{\mathrm{L}^2_{-1}(T)} = \int_T \frac{1}{r}(\operatorname{div}\boldsymbol{\varphi}_\ell)^2 \le \frac{1}{r_{\min(T)}}\int_T (\operatorname{div}\boldsymbol{\varphi}_\ell)^2 \le \frac{C}{h_T}. \tag{4.15}$$

On the other hand, from the assumption on Γ_1, Cauchy-Schwarz inequality and a suitable trace theorem (see [3, Lemma 4]) we have

$$\frac{1}{|\ell|}\int_\ell |\boldsymbol{u}\cdot\boldsymbol{n}| \le C\left(\int_\ell r\left|\frac{u_r}{r}\right|^2\right)^{1/2} \le C\left\{h_{T_\ell}^{-1/2}\left\|\frac{u_r}{r}\right\|_{\mathrm{L}^2_1(T_\ell)} + h_{T_\ell}^{1/2}\left|\frac{u_r}{r}\right|_{\mathrm{H}^1_1(T_\ell)}\right\}, \tag{4.16}$$

here $T_\ell \in \mathcal{T}_h^0$ is such that $\Gamma_1 \cap T_\ell = \ell$. From (4.12), (4.14) and the previous estimates we obtain

$$\sum_{T\in\mathcal{T}_h^1}\left\|I_h^R \boldsymbol{u} - I_h \boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch\sum_{\ell\subset\Gamma_1}\left(\int_\ell \frac{|\boldsymbol{u}\cdot\boldsymbol{n}|}{|\ell|}\right)^2 \le C\left(\left\|\frac{u_r}{r}\right\|^2_{\mathrm{L}^2_1(D_0)} + h^2\left|\frac{u_r}{r}\right|^2_{\mathrm{H}^1_1(D_0)}\right).$$

Since $\boldsymbol{f} \in \boldsymbol{G}$, $\boldsymbol{u} \in \boldsymbol{G}$, too, we have that $u_r/r \in \widetilde{\mathrm{H}}^1_1(D)$ and thus vanishes on Γ_0. Then (4.9) follows from a Poincaré-like inequality and Lemma 3.3:

$$\sum_{T\in\mathcal{T}_h^1}\left\|I_h^R \boldsymbol{u} - I_h \boldsymbol{u}\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch^2\left|\frac{u_r}{r}\right|^2_{\mathrm{H}^1_1(D_0)} \le Ch^2\left\|\frac{\boldsymbol{u}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} \le Ch^2\|\boldsymbol{f}\|^2_{\mathrm{L}^2_{-1}(D)}.$$

To prove (4.10) we proceed as above but we apply (4.13) and (4.15) instead of (4.12) and (4.14), respectively

$$\sum_{T\in\mathcal{T}_h^1}\left\|\operatorname{div}(I_h^R \boldsymbol{u} - I_h \boldsymbol{u})\right\|^2_{\mathrm{L}^2_{-1}(T)} \le Ch^{-1}\sum_{l\subset\Gamma_1}\left(\int_l \frac{1}{|l|}|\boldsymbol{u}\cdot\boldsymbol{n}|\right)^2 \le \left|\frac{u_r}{r}\right|^2_{\mathrm{H}^1(D_0)}. \tag{4.17}$$

To estimate the last term of the previous equation we recall that there exists $s \in (0,1)$ such that $|\nabla(u_r/r)| \in \mathrm{L}^q_1(D)$, for $q = 6/(3-2s) \in (2,6)$ (cf. Lemma 4.1). Moreover, we notice that $r|\nabla(u_r/r)|^2$ can be written as $r|\nabla(u_r/r)|^2 = r^{\frac{2}{q}}|\nabla(u_r/r)|^2 r^{1-\frac{2}{q}}$ where $r^{\frac{2}{q}}|\nabla(u_r/r)|^2 \in \mathrm{L}^{\frac{q}{2}}(D)$ and $r^{1-\frac{2}{q}} \in \mathrm{L}^{q^*}(D)$, here q^* is such that $1/q^* + 2/q = 1$. Thus, from Hölder inequality we obtain

$$\left|\frac{u_r}{r}\right|^2_{\mathrm{H}^1_1(D_0)} \le \left(\int_{D_0} r\left|\nabla\left(\frac{u_r}{r}\right)\right|^q\right)^{\frac{2}{q}}\left(\int_{D_0} r\right)^{\frac{1}{q^*}} \le Ch^{2-\frac{4}{q}}\left\|\nabla\left(\frac{u_r}{r}\right)\right\|^2_{\mathrm{L}^q_1(D)}.$$

Hence, (4.10) follows from the previous inequalities and Lemma 4.1 (cf. (4.5)).

On the other hand, when **H.1** holds true, from Lemma 4.1 (cf. (4.6)) it follows that $\nabla(u_r/r) \in \mathrm{H}^1_1(D)^2$. Then, the last term of (4.17) can be bounded by applying Remark 2.1 as follows

$$\left|\frac{u_r}{r}\right|^2_{\mathrm{H}^1_1(D_0)} \leq Ch^{2-2\epsilon}\left\|\nabla\left(\frac{u_r}{r}\right)\right\|^2_{\mathrm{L}^2_{-1+2\epsilon}(D)} \leq Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{u}}{r}\right\|^2_{\mathrm{H}^2_1(D)},$$

for all $\epsilon > 0$. Therefore (4.11) follows from (4.17) the previous inequality and Lemma 4.1 (cf. (4.6)).

Remark 4.3. *Constant C in (4.11) depends on ϵ. Therefore, from now on, whenever (4.11) is used, the generic constant C will denote a constant independent of r and h but depending on ϵ.*

Next we notice that $\mu_h = 1$ is an eigenvalue of T_h with associated eigenspace

$$\boldsymbol{K}_h := \left\{\boldsymbol{v}_h \in \boldsymbol{R}^0_h : \operatorname{div}\boldsymbol{v}_h = 0\right\} \subset \boldsymbol{K},$$

so that T_h restricted to $\boldsymbol{K}_h$ is the identity, too. Let $\boldsymbol{G}_h := \boldsymbol{K}_h^{\perp_{\boldsymbol{R}_h}}$. We also notice that $\boldsymbol{G}_h \not\subset \boldsymbol{G}$. However, the following lemma shows that the divergence-free terms in the Helmholtz decomposition of $\boldsymbol{G}_h$ are asymptotically negligible.

Lemma 4.4. *For $\boldsymbol{f}_h \in \boldsymbol{G}_h$, there exist $\boldsymbol{\chi} \in \boldsymbol{G}$ and $\boldsymbol{\eta} \in \boldsymbol{K}$ such that $\boldsymbol{f}_h = \boldsymbol{\chi} + \boldsymbol{\eta}$ and there hold:*

a) $\frac{\boldsymbol{\chi}}{r} \in \widetilde{\mathrm{H}}^1_1(D) \times \mathrm{H}^1_1(D)$ and $\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} \leq C\left\|\operatorname{div} \boldsymbol{f}_h\right\|_{\mathrm{L}^2_{-1}(D)}$,

b) $\|\boldsymbol{\eta}\|_{\mathrm{L}^2_{-1}(D)} \leq Ch^{1-\epsilon}\left\|\operatorname{div}\boldsymbol{f}_h\right\|_{\mathrm{L}^2_{-1}(D)}$ for all $\epsilon > 0$.

Proof. Since $\boldsymbol{f}_h \in \boldsymbol{G}_h \subset \mathrm{H}^0_{-1}(\mathrm{div}; D)$, the decomposition $\boldsymbol{f}_h = \boldsymbol{\chi}+\boldsymbol{\eta}$ follows from Lemma 3.2. Moreover,

$$\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} \leq C\left\|\operatorname{div}\boldsymbol{\chi}\right\|_{\mathrm{L}^2_{-1}(D)} = C\left\|\operatorname{div}\boldsymbol{f}_h\right\|_{\mathrm{L}^2_{-1}(D)},$$

where the first inequality follows from (3.6). Thus we conclude (a). To prove (b) we first notice that $\boldsymbol{f}_h = \boldsymbol{0}$ in D_0 (cf. (4.2)), then

$$\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)} = \int_D \frac{1}{r}\boldsymbol{\eta}\cdot(\boldsymbol{f}_h - \boldsymbol{\chi}) = \int_{D_0}\frac{1}{r}\boldsymbol{\eta}\cdot\boldsymbol{\chi} + \int_{D_1}\frac{1}{r}\boldsymbol{\eta}\cdot(\boldsymbol{f}_h - \boldsymbol{\chi}).$$

We add and subtract the classical RT interpolant I^R_h and write $\boldsymbol{\eta} = \boldsymbol{f}_h - \boldsymbol{\chi}$

$$\begin{aligned}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)} &= -\int_{D_0}\frac{1}{r}\boldsymbol{\eta}\cdot\boldsymbol{\chi} + \int_{D_1}\frac{1}{r}\boldsymbol{\eta}\cdot\left(\boldsymbol{f}_h - I^R_h\boldsymbol{\chi}\right) + \int_{D_1}\frac{1}{r}\boldsymbol{\eta}\cdot\left(I^R_h\boldsymbol{\chi} - \boldsymbol{\chi}\right)\\ &= -\int_{D_0}\frac{1}{r}\boldsymbol{\eta}\cdot\boldsymbol{\chi} - \int_{D_1}\frac{1}{r}\boldsymbol{\chi}\cdot\left(\boldsymbol{f}_h - I^R_h\boldsymbol{\chi}\right) + \int_{D_1}\frac{1}{r}\boldsymbol{f}_h\cdot\left(\boldsymbol{f}_h - I^R_h\boldsymbol{\chi}\right)\\ &\quad + \int_{D_1}\frac{1}{r}\boldsymbol{\eta}\cdot\left(I^R_h\boldsymbol{\chi} - \boldsymbol{\chi}\right) = I_1 + I_2 + I_3 + I_4.\end{aligned} \tag{4.18}$$

Now we estimate each term separately. We begin with the term I_1 by using the Cauchy-Schwarz and Young inequalities,

$$I_1 \leq \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D_0)}\|\boldsymbol{\eta}\|_{\mathrm{L}^2_{-1}(D_0)} \leq \frac{3}{2}\|\boldsymbol{\chi}\|^2_{\mathrm{L}^2_{-1}(D_0)} + \frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}.$$

The first term on the right-hand side can be estimated by using Remark 2.1 and recalling that χ_r/r belongs to $\mathrm{L}^2_{-1}(D)$ (cf. (3.6)):

$$\begin{aligned}\|\boldsymbol{\chi}\|^2_{1/r,D_0} &\leq \sum_{T\in\mathcal{T}_h^0}\left(r^2_{\max(T)}\int_T \frac{1}{r}\left|\frac{\chi_r}{r}\right|^2 + r^{2-2\epsilon}_{\max(T)}\int_T \frac{1}{r^{1-2\epsilon}}\left|\frac{\chi_z}{r}\right|^2\right)\\
&\leq Ch^2\left\|\frac{\chi_r}{r}\right\|^2_{\mathrm{L}^2_{-1}(D)} + Ch^{2-2\epsilon}\left\|\frac{\chi_z}{r}\right\|^2_{\mathrm{L}^2_{-1+2\epsilon}(D)}\\
&\leq Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)},\end{aligned}\tag{4.19}$$

for all $\epsilon > 0$. Then, from the two previous inequalities we have

$$I_1 \leq Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} + \frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}.\tag{4.20}$$

To deal with I_2 we notice that $\operatorname{div}\boldsymbol{\chi} = \operatorname{div}\boldsymbol{f}_h$ and recall that (see, [8, Lemma 3.7])

$$\operatorname{div}\left(I_h^R\boldsymbol{\chi}\right) = \mathcal{P}_h\left(\operatorname{div}\boldsymbol{\chi}\right) = \mathcal{P}_h\left(\operatorname{div}\boldsymbol{f}_h\right) = \operatorname{div}\boldsymbol{f}_h \quad \text{in } D,\tag{4.21}$$

where $\mathcal{P}_h$ is the L^2-projection onto the space of piecewise constant functions. We note that although $\operatorname{div}\left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right) = 0$ and $\boldsymbol{\chi}$ belongs to $\boldsymbol{G}$, the term I_2 does not vanishes because $I_h^R\boldsymbol{\chi} \notin \boldsymbol{R}_h^0$ and thus $(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}) \notin \boldsymbol{K}_h$.

Since $\boldsymbol{\chi}\cdot\boldsymbol{n} = 0$ on ∂D and $\boldsymbol{\chi} = r\,\nabla q$ with $q \in \mathrm{H}^1_1(D)$ then, by integration by parts we obtain

$$I_2 = -\int_{D_1}\nabla q\cdot\left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right) = \int_{D_1} q\operatorname{div}\left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right) - \int_{\partial D_1} I_h^R\boldsymbol{\chi}\cdot\boldsymbol{n}q = -\int_{\Gamma_1} I_h^R\boldsymbol{\chi}\cdot\boldsymbol{n}q.\tag{4.22}$$

To estimate the last term in the previous equation we recall that $\operatorname{div}\boldsymbol{f}_h = 0$ in D_0 thus, from (4.21), $\operatorname{div} I_h^R\boldsymbol{\chi} = 0$ in D_0, too. Therefore, by integration by parts again, we have

$$\int_{\Gamma_1} I_h^R\boldsymbol{\chi}\cdot\boldsymbol{n}q = \int_{D_0}\nabla q\cdot I_h^R\boldsymbol{\chi} + \int_{D_0} q\operatorname{div} I_h^R\boldsymbol{\chi} = \int_{D_0}\frac{\boldsymbol{\chi}}{r}\cdot(I_h^R\boldsymbol{\chi} - \boldsymbol{\chi}) + \|\boldsymbol{\chi}\|^2_{\mathrm{L}^2_{-1}(D_0)}.\tag{4.23}$$

By applying standard error estimates for the RT interpolant it follows that

$$\int_{D_0}\frac{\boldsymbol{\chi}}{r}\cdot(I_h^R\boldsymbol{\chi} - \boldsymbol{\chi}) \leq \left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\mathrm{L}^2(D_0)}\left\|\boldsymbol{\chi} - I_h^R\boldsymbol{\chi}\right\|_{\mathrm{L}^2(D_0)} \leq Ch\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\mathrm{L}^2(D_0)}\|\nabla\boldsymbol{\chi}\|_{\mathrm{L}^2(D_0)}.\tag{4.24}$$

Since $\nabla\boldsymbol{\chi} = r\nabla(\boldsymbol{\chi}/r) + (\chi_r, 0)^\top$ it is straightforward to write

$$\|\nabla\boldsymbol{\chi}\|_{\mathrm{L}^2(D_0)} \leq \left\|r\nabla\left(\frac{\boldsymbol{\chi}}{r}\right)\right\|_{\mathrm{L}^2(D_0)} + \|\chi_r\|_{\mathrm{L}^2(D_0)} \leq Ch^{1/2}\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)}.\tag{4.25}$$

From (4.22)-(4.25), (4.19), by estimating the term $\|\boldsymbol{\chi}/r\|_{\mathrm{L}^2(D_0)}$ as in (4.19) it follows that

$$I_2 \leq \|\boldsymbol{\chi}\|^2_{\mathrm{L}^2_{-1}(D_0)} + Ch^{3/2}\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\mathrm{L}^2(D_0)}\left\|\frac{\boldsymbol{\chi}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} \leq Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)},\tag{4.26}$$

for all $\epsilon > 0$. To estimate I_3 we first recall that

$$\begin{aligned}\operatorname{div}\left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right) &= 0 \quad \text{in } D\\
\left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right)\cdot\boldsymbol{n} &= 0 \quad \text{on } \partial D.\end{aligned}\tag{4.27}$$

From [13] it follows that $\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}$ belongs to $\mathbf{curl}(\mathcal{L}_h^0)$ where $\mathcal{L}_h^0$ is defined by:

$$\mathcal{L}_h^0 := \left\{\psi_h \in \mathrm{H}_0^1(D): \ \psi_h|_T \in \mathbb{P}_1(T) \ \ \forall T \in \mathcal{T}_h\right\}.$$

Let $\{\phi_j\}_{j=1}^N$ be the nodal basis of $\mathcal{L}_h^0$ and $\{\boldsymbol{x}_j\}_{j=1}^N$ the set of inner vertices of the triangulation. Recall that $\phi_j(\boldsymbol{x}_i) = \delta_{ij}$ for $i, j = 1, \cdots, N$. We order these basis functions so that the first N_1 of them correspond to vertices on the boundary Γ_1. Then, there exist α_i, $i = 1, \ldots, N$ such that

$$\boldsymbol{f}_h - I_h^R\boldsymbol{\chi} = \sum_{j=1}^{N} \alpha_i \,\mathbf{curl}\,\phi_i = \sum_{i=1}^{N_1} \alpha_i \,\mathbf{curl}\,\phi_i + \sum_{i=N_1+1}^{N} \alpha_i \,\mathbf{curl}\,\phi_i. \tag{4.28}$$

We notice that $\mathbf{curl}\,\phi_j \in \boldsymbol{K}_h$ for $j = N_1+1, \cdots, N$. Since $\boldsymbol{f}_h \in \boldsymbol{G}_h$, these $\mathbf{curl}\,\phi_j$ are orthogonal to $\boldsymbol{f}_h$ in $\mathrm{L}^2_{-1}(D)$. Therefore I_3 can be rewritten as follows

$$I_3 = \int_{D_1} \frac{1}{r}\boldsymbol{f}_h \cdot \sum_{i=1}^{N_1} \alpha_i \,\mathbf{curl}\,\phi_i + \sum_{i=N_1+1}^{N} \alpha_i \int_D \frac{1}{r}\boldsymbol{f}_h \cdot \mathbf{curl}\,\phi_i = \sum_{i=1}^{N_1} \int_{D_1} \frac{1}{r}\boldsymbol{f}_h \cdot \alpha_i \,\mathbf{curl}\,\phi_i.$$

To estimate the right-hand side of the previous equation we apply Cauchy-Schwarz and Young inequalities, the decomposition $\boldsymbol{f}_h = \boldsymbol{\chi} + \boldsymbol{\eta}$ and the fact that $\|\mathbf{curl}\,\phi_i\|_{\mathrm{L}^2(D_1)} \leq C$ for all $i = 1, \cdots, N_1$ with a constant C which only depends on the regularity of the meshes

$$\begin{aligned} I_3 &\leq \frac{1}{12} \sum_{i=1}^{N_1} \int_{D_1 \cap \mathrm{supp}\phi_i} \frac{1}{r}|\boldsymbol{f}_h|^2 + \frac{C}{h} \sum_{i=1}^{N_1} \int_{D_1 \cap \mathrm{supp}\phi_i} |\alpha_i \,\mathbf{curl}\,\phi_i|^2 \\ &\leq \frac{1}{6}\,\|\boldsymbol{\chi}\|^2_{\mathrm{L}^2_{-1}(D_0 \cup D_\phi)} + \frac{1}{6}\,\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)} + \frac{C}{h} \sum_{i=1}^{N_1} |\alpha_i|^2, \end{aligned} \tag{4.29}$$

where $D_\phi := D_1 \cap \{\cup_{i=1}^{N_1} \mathrm{supp}(\phi_i)\}$ (see Figure 2 right). Next, we write α_i, $i = 1, \cdots, N_1$ in terms of $\boldsymbol{\chi}$. With this aim we consider a set of edges $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{N_1}\}$ in $\mathcal{T}_h^0$ such that for $i = 1, \ldots, N_1$, $\boldsymbol{x}_i$ is an endpoint of $\boldsymbol{e}_i$ while the other endpoint $\boldsymbol{y}_i$ is on Γ_0 (see Figure 2 right). We also define $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_{N_1}\}$ a set of normal vectors associated with edges $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{N_1}\}$. We multiply (4.28) by $\boldsymbol{n}_i$ and integrate over $\boldsymbol{e}_i$, $i = 1, \ldots, N_1$. Then, since $\boldsymbol{f}_h$ vanishes in $\mathcal{T}_h^0$ and, from the properties of the RT interpolant we have

$$\sum_{j=1}^{N} \alpha_j \int_{\boldsymbol{e}_i} \mathbf{curl}\,\phi_j \cdot \boldsymbol{n}_i = \int_{\boldsymbol{e}_i} \left(\boldsymbol{f}_h - I_h^R\boldsymbol{\chi}\right) \cdot \boldsymbol{n}_i = \int_{\boldsymbol{e}_i} -I_h^R\boldsymbol{\chi} \cdot \boldsymbol{n}_i = \mp \int_{\boldsymbol{e}_i} \boldsymbol{\chi} \cdot \boldsymbol{n}_i.$$

For $j = 1, \ldots, N$ and $i = 1, \ldots, N_1$, $\int_{\boldsymbol{e}_i} \mathbf{curl}\,\phi_j \cdot \boldsymbol{n}_i = \pm(\phi_j(\boldsymbol{x}_i) - \phi_j(\boldsymbol{y}_i)) = \pm\delta_{ij}$. Thus, from the previous equality we obtain

$$|\alpha_i| = \left|\int_{\boldsymbol{e}_i} \boldsymbol{\chi} \cdot \boldsymbol{n}_i\right| \qquad i = 1, \cdots, N_1.$$

We use the previous equality to estimate the last term of (4.29). Let $T_i \in \mathcal{T}_h^0$ be such that $\boldsymbol{e}_i$ is an edge of $T_i, i = 1, \cdots, N_1$ (see Figure 2 right), by proceeding as in

Lemma 4.2 (cf. (4.16)) and (4.19) it follows that

$$\begin{aligned}
\frac{1}{h}\sum_{i=1}^{N_1}|\alpha_i|^2 &\le \frac{1}{h}\sum_{i=1}^{N_1}\left(\int_{\boldsymbol{e}_i}|\boldsymbol{\chi}\cdot\boldsymbol{n}_i|\right)^2\\
&\le \frac{C}{h}\sum_{i=1}^{N_1}h^2\int_{\boldsymbol{e}_i}\left|\frac{\boldsymbol{\chi}}{r}\right|^2 r\\
&\le Ch\sum_{i=1}^{N_1}\left\{h_{T_i}^{-1}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\mathrm{L}^2_1(T_i)}+h_{T_i}\left|\frac{\boldsymbol{\chi}}{r}\right|^2_{\mathrm{H}^1_1(T_i)}\right\}\\
&\le Ch\left(h^{-1}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\mathrm{L}^2_1(D_0)}+h\left|\frac{\boldsymbol{\chi}}{r}\right|^2_{\mathrm{H}^1_1(D_0)}\right)\\
&\le Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)},
\end{aligned}$$

for all $\epsilon>0$. We bound I_3 from (4.29) and the previous inequality

$$\begin{aligned}
I_3 &\le \frac{1}{6}\|\boldsymbol{\chi}\|^2_{\mathrm{L}^2_{-1}(D_0\cup D_\phi)}+\frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}+Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)}\\
&\le \frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}+Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)},
\end{aligned}\tag{4.30}$$

where we have estimated the first term on the right-hand side of (4.29) by proceeding as in (4.19). Finally we estimate the term I_4 of (4.18). By using Cauchy-Schwarz and Young inequalities we obtain

$$I_4 \le \|\boldsymbol{\chi}-I_h\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D_1)}\|\boldsymbol{\eta}\|_{\mathrm{L}^2_{-1}(D_1)} \le \frac{3}{2}\sum_{T\in\mathcal{T}_h^1}\int_T\frac{1}{r}|\boldsymbol{\chi}-I_h^R\boldsymbol{\chi}|^2+\frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}.$$

To bound the first term on the right-hand side of the previous equation, we use standard error estimates for the RT interpolant, the fact that χ_r/r belongs to $\mathrm{L}^2_{-1}(D)$ (cf. (3.6)) and property (4.1):

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h^1}\int_T\frac{1}{r}|\boldsymbol{\chi}-I_h^R\boldsymbol{\chi}|^2 &\le \sum_{T\in\mathcal{T}_h^1}\frac{C}{r_{\min(T)}}h_T^2\int_T|\nabla\boldsymbol{\chi}|^2\\
&\le \sum_{T\in\mathcal{T}_h^1}\frac{C}{r_{\min(T)}}h_T^2\left\{\int_T\left|r\nabla\left(\frac{\boldsymbol{\chi}}{r}\right)\right|^2+\int_T\left|\frac{\chi_r}{r}\right|^2\right\}\\
&\le \sum_{T\in\mathcal{T}_h^1}\frac{C\,r_{\max(T)}}{r_{\min(T)}}h_T^2\left\{\int_T r\left|\nabla\left(\frac{\boldsymbol{\chi}}{r}\right)\right|^2+\int_T\frac{1}{r}\left|\frac{\chi_r}{r}\right|^2\right\}.
\end{aligned}\tag{4.31}$$

Then, from the two previous estimates we obtain

$$I_4 \le Ch^2\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)}+\frac{1}{6}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)}.\tag{4.32}$$

Therefore, from (4.18), (4.20),(4.26), (4.30) and (4.32) it follows that

$$\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)} \le Ch^{2-2\epsilon}\left\|\frac{\boldsymbol{\chi}}{r}\right\|^2_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)}+\frac{1}{2}\|\boldsymbol{\eta}\|^2_{\mathrm{L}^2_{-1}(D)},$$

for all $\epsilon>0$. Estimate (b) is consequence of the previous inequality and (a).

Now we are ready to prove the following result, from which we will derive property P1.

LEMMA 4.5. *There exists $C > 0$ such that, for all $\boldsymbol{f}_h \in \boldsymbol{G}_h$,*

$$\|(T - T_h)\, \boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)} \leq C h^{1-2/q} \|\boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)},$$

for some $q \in (2, 6)$.

Proof. Given $\boldsymbol{f}_h \in \boldsymbol{G}_h$, let $\boldsymbol{\chi} \in \boldsymbol{G}$ and $\boldsymbol{\eta} \in \boldsymbol{K}$ be as in Lemma 4.4. Let $\boldsymbol{w} := T\boldsymbol{\chi}$ and $\boldsymbol{w}_h := T_h\boldsymbol{\chi}$. The following Cea estimate follows immediately from the definitions of T and T_h:

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)} \leq C \inf_{\boldsymbol{v}_h \in \boldsymbol{R}_h^0} \|\boldsymbol{w} - \boldsymbol{v}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)}.$$

Then, by setting $\boldsymbol{v}_h := I_h\boldsymbol{w}$ it follows that

$$\|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathrm{H}_{-1}(\mathrm{div};D)} \leq C \left(\|\boldsymbol{w} - I_h\boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D)} + \|\mathrm{div}(\boldsymbol{w} - I_h\boldsymbol{w})\|_{\mathrm{L}^2_{-1}(D)} \right). \tag{4.33}$$

Since $I_h\boldsymbol{w}$ vanishes in D_0, to estimate the first term on the right-hand side of the previous equation we decompose D into D_0 and D_1, and then we add and subtract the classical RT interpolant I_h^R:

$$\|\boldsymbol{w} - I_h\boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D)} \leq C \left(\|\boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D_0)} + \left\|\boldsymbol{w} - I_h^R\boldsymbol{w}\right\|_{\mathrm{L}^2_{-1}(D_1)} + \left\|I_h\boldsymbol{w} - I_h^R\boldsymbol{w}\right\|_{\mathrm{L}^2_{-1}(D_1)} \right). \tag{4.34}$$

We estimate the first and second term on the right-hand side of the previous equation by proceeding as in Lemma 4.4 (cf. (4.19) and (4.31), respectively)

$$\|\boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D_0)} \leq C h^{1-\epsilon} \left\|\frac{\boldsymbol{w}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times \mathrm{H}^1_1(D)}, \quad \left\|\boldsymbol{w} - I_h^R\boldsymbol{w}\right\|_{\mathrm{L}^2_{-1}(D_1)} \leq C h \left\|\frac{\boldsymbol{w}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times \mathrm{H}^1_1(D)}$$

for all $\epsilon > 0$. Thus, from the previous inequalities, by applying Lemma 4.2 (cf. (4.9)) to estimate the last term of (4.34), we obtain

$$\|\boldsymbol{w} - I_h\boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D)} \leq C \left(h^{1-\epsilon} \left\|\frac{\boldsymbol{w}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times \mathrm{H}^1_1(D)} + h\, \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D)} \right) \leq C h^{1-\epsilon} \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D)} \tag{4.35}$$

for all $\epsilon > 0$, where the last inequality is a consequence of Lemma 3.3. Similarly, we decompose the last term in (4.33) as follows:

$$\begin{aligned} &\|\mathrm{div}(\boldsymbol{w} - I_h\boldsymbol{w})\|_{\mathrm{L}^2_{-1}(D)} \\ &\quad \leq C \left(\|\mathrm{div}\, \boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D_0)} + \left\|\mathrm{div}(I_h\boldsymbol{w} - I_h^R\boldsymbol{w})\right\|_{\mathrm{L}^2_{-1}(D_1)} + \left\|\mathrm{div}(\boldsymbol{w} - I_h^R\boldsymbol{w})\right\|_{\mathrm{L}^2_{-1}(D_1)} \right). \end{aligned} \tag{4.36}$$

To estimate each term of the previous equation, we proceed as above. Since $(\mathrm{div}\, \boldsymbol{w})/r \in \mathrm{H}^1_1(D)$ (see Lemma 3.3), the first term can be estimated by proceeding as in Lemma 4.4 (cf. (4.19)), whereas for the second term we apply Lemma 4.2 (4.10):

$$\begin{aligned} \|\mathrm{div}\, \boldsymbol{w}\|_{\mathrm{L}^2_{-1}(D_0)} &\leq C h^{1-\epsilon} \left\|\frac{\mathrm{div}\, \boldsymbol{w}}{r}\right\|_{\mathrm{H}^1_1(D)}, \\ \left\|\mathrm{div}(\boldsymbol{w} - I_h^R\boldsymbol{w})\right\|_{\mathrm{L}^2_{-1}(D_1)} &\leq C h^{1-2/q} \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D)}, \end{aligned} \tag{4.37}$$

for all $\epsilon > 0$ and for some $q \in (2,6)$. Next, the last term in (4.36) is bounded by using standard error estimates of the RT interpolant as follows

(4.38)
$$\left\|\operatorname{div}(\boldsymbol{w} - I_h^R \boldsymbol{w})\right\|^2_{\mathrm{L}^2_{-1}(D_1)} = \sum_{T\in\mathcal{T}_h^1} \int_T \frac{1}{r} |\operatorname{div}(\boldsymbol{w} - I_h^R\boldsymbol{w})|^2 \leq \sum_{T\in\mathcal{T}_h^1} \frac{Ch_T^2}{r_{\min(T)}} \int_T |\nabla \operatorname{div} \boldsymbol{w}|^2.$$

To estimate the last term of the previous equation we proceed as in Lemma 4.4 (cf. (4.31)):

$$\begin{aligned}
\sum_{T\in\mathcal{T}_h^1} \frac{Ch_T^2}{r_{\min(T)}} \int_T |\nabla \operatorname{div} \boldsymbol{w}|^2 &\leq \sum_{T\in\mathcal{T}_h^1} \frac{Ch_T^2}{r_{\min(T)}} \left\{ \int_T \left| r\nabla \left(\frac{\operatorname{div}\boldsymbol{w}}{r}\right)\right|^2 + \int_T \left|\frac{\operatorname{div}\boldsymbol{w}}{r}\right|^2 \right\} \\
&\leq \sum_{T\in\mathcal{T}_h^1} \frac{Ch_T^2 r_{\max(T)}}{r_{\min(T)}} \int_T r \left|\nabla\left(\frac{\operatorname{div}\boldsymbol{w}}{r}\right)\right|^2 + \sum_{T\in\mathcal{T}_h^1} Ch_T^{2(1-\epsilon)} \int_T \frac{1}{r^{1-2\epsilon}} \left|\frac{\operatorname{div}\boldsymbol{w}}{r}\right|^2 \\
&\leq \sum_{T\in\mathcal{T}_h^1} Ch_T^2 \left\|\frac{\operatorname{div}\boldsymbol{w}}{r}\right\|^2_{\mathrm{H}^1_1(T)} + \sum_{T\in\mathcal{T}_h^1} Ch_T^{2(1-\epsilon)} \int_T \frac{1}{r^{1-2\epsilon}} \left|\frac{\operatorname{div}\boldsymbol{w}}{r}\right|^2, \qquad (4.39)
\end{aligned}$$

for all $\epsilon > 0$. We apply Remark 2.1 to estimate the last term of the previous inequality

$$\sum_{T\in\mathcal{T}_h^1} Ch_T^{2(1-\epsilon)} \int_T \frac{1}{r^{1-2\epsilon}} \left|\frac{\operatorname{div}\boldsymbol{w}}{r}\right|^2 \leq Ch^{2(1-\epsilon)} \left\|\frac{\operatorname{div}\boldsymbol{w}}{r}\right\|^2_{\mathrm{H}^1_1(D)}. \qquad (4.40)$$

Hence, from (4.38)-(4.40) we obtain that, for all $\epsilon > 0$,

$$\left\|\operatorname{div}(\boldsymbol{w} - I_h^R\boldsymbol{w})\right\|_{\mathrm{L}^2_{-1}(D_1)} \leq Ch^{1-\epsilon} \left\|\frac{\operatorname{div}\boldsymbol{w}}{r}\right\|_{\mathrm{H}^1_1(D)}.$$

Next we return to (4.36) and using (4.37), the previous inequality and Lemma 3.3 we write

$$\|\operatorname{div}(\boldsymbol{w} - I_h\boldsymbol{w})\|_{\mathrm{L}^2_{-1}(D)} \leq Ch^{1-2/q} \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D)} \quad \text{for some } q \in (2,6). \qquad (4.41)$$

Now we are in a position to end the proof. From (4.33), (4.35), (4.41), and the fact that $\boldsymbol{K} \perp \boldsymbol{G}$ in $\mathrm{L}^2_{-1}(D)$, we have

$$\begin{aligned}
\|(T - T_h)\,\boldsymbol{\chi}\|_{\mathrm{H}_{-1}(\operatorname{div};D)} &= \|\boldsymbol{w} - \boldsymbol{w}_h\|_{\mathrm{H}_{-1}(\operatorname{div};D)} \\
&\leq Ch^{1-2/q} \|\boldsymbol{\chi}\|_{\mathrm{L}^2_{-1}(D)} \leq Ch^{1-2/q} \|\boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\operatorname{div};D)},
\end{aligned}$$

for some $q \in (2,6)$. On the other hand, for $\boldsymbol{\eta} \in \boldsymbol{K}$, since $T\boldsymbol{\eta} = \boldsymbol{\eta}$ and $T_h\boldsymbol{\eta}$ is the Galerkin projection of $\boldsymbol{\eta}$ onto $\boldsymbol{R}_h^0$, by using Lemma 4.4 (b) we obtain

$$\|(T - T_h)\,\boldsymbol{\eta}\|_{\mathrm{H}_{-1}(\operatorname{div};D)} \leq \|\boldsymbol{\eta}\|_{\mathrm{H}_{-1}(\operatorname{div};D)} = \|\boldsymbol{\eta}\|_{\mathrm{L}^2_{-1}(D)} \leq Ch^{1-\epsilon} \|\boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\operatorname{div};D)},$$

for all $\epsilon > 0$. Therefore, from the two previous estimates it follows that

$$\begin{aligned}
\|(T - T_h)\,\boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\operatorname{div};D)} &\leq \|(T - T_h)\,\boldsymbol{\chi}\|_{\mathrm{H}_{-1}(\operatorname{div};D)} + \|(T - T_h)\,\boldsymbol{\eta}\|_{\mathrm{H}_{-1}(\operatorname{div};D)} \\
&\leq Ch^{1-2/q} \|\boldsymbol{f}_h\|_{\mathrm{H}_{-1}(\operatorname{div};D)},
\end{aligned}$$

for some $q \in (2,6)$ and we conclude the proof.

Property P1 clearly follows from the previous lemma and the fact that T and T_h coincide on $\boldsymbol{K}_h$. As a first consequence, we have the next result (see, [4, Theorem 1]).

THEOREM 4.6. *Let $J \subset \mathbb{R}$ be an open set containing $\sigma(T)$. Then, there exists $h_0 > 0$ such that $\sigma(T_h) \subset J \ \forall h < h_0$.*

As a consequence of the above theorem, we know that the proposed numerical method does not introduce spurious modes, as it happens instead when the same elements are used to approximate Problem 1 (see Section 1).

To prove P2 we notice that any eigenvector $\boldsymbol{v}$ of T satisfies; $\boldsymbol{v}/r \in \widetilde{\mathrm{H}}^1_1(D) \times \mathrm{H}^1_1(D)$ and $(\operatorname{div} \boldsymbol{v})/r \in \mathrm{H}^1_1(D)$, which implies that $I_h \boldsymbol{v}$ is well defined and belongs to $\boldsymbol{R}^0_h$. Thus, property P2 follows by proceeding as in Lemma 4.5 (cf. (4.33)).

Now, we are in a position to write the main result of this paper related to the convergence of the proposed scheme.

THEOREM 4.7. *Let $\mu \in \sigma(T)$ be an eigenvalue of finite-multiplicity m. Let $\boldsymbol{\mathcal{E}}$ be the corresponding eigenspace. There exists $h_0 > 0$ such that, for all $h < h_0$, $\sigma(T_h)$ contains exactly m eigenvalues $\mu_h^{(1)}, \ldots, \mu_h^{(m)}$ (repeated according to their respective multiplicities) such that*

$$\mu_h^{(i)} \underset{h \to 0}{\longrightarrow} \mu, \qquad i = 1, \ldots, m.$$

Let $\boldsymbol{\mathcal{E}}_h$ be the direct sum of the corresponding eigenspaces. Let

$$\gamma_h := \delta\left(\boldsymbol{\mathcal{E}}, \boldsymbol{R}^0_h\right) := \sup_{\substack{\boldsymbol{v} \in \boldsymbol{\mathcal{E}} \\ \|\boldsymbol{v}\|_{\mathrm{H}_{-1}(\operatorname{div}; D)} = 1}} \inf_{\boldsymbol{v}_h \in \boldsymbol{R}^0_h} \|\boldsymbol{v} - \boldsymbol{v}_h\|_{\mathrm{H}_{-1}(\operatorname{div}; D)}$$

and

$$\widehat{\delta}\left(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h\right) := \max\left\{\delta\left(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h\right), \delta\left(\boldsymbol{\mathcal{E}}_h, \boldsymbol{\mathcal{E}}\right)\right\}.$$

Then,

$$\widehat{\delta}\left(\boldsymbol{\mathcal{E}}, \boldsymbol{\mathcal{E}}_h\right) \leq C \gamma_h,$$

and

$$\max_{1 \leq i \leq m} \left|\mu - \mu_h^{(i)}\right| \leq C \gamma_h^2.$$

Proof. Since we have already proved that properties P1 and P2 hold true, the results are direct consequences of [4, Section 2] and Theorems 1 and 3 from [5].

To conclude spectral convergence we only need an appropriate estimate for the term γ_h. In the following theorem we will prove two of them. The first one is valid under more general conditions but it leads to a sub-optimal order of convergence. For the second one we need to assume **H.1** but it leads quasi-optimal error estimates.

THEOREM 4.8. *Let γ_h be as in Theorem 4.7. Then, there exist $q \in (2, 6)$ and $C > 0$ such that*

$$\gamma_h \leq C h^{1 - 2/q}. \tag{4.42}$$

Moreover, if **H.1** *holds true, then for all $\epsilon > 0$, there exists $C > 0$ such that*

$$\gamma_h \leq C h^{1 - \epsilon}. \tag{4.43}$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{E}}$ be such that $\|\boldsymbol{v}\|_{\mathrm{H}_{-1}(\mathrm{div};D)} = 1$. Since $T\boldsymbol{v} = \mu\boldsymbol{v}$, from Lemma 3.3 it follows that

$$\left\|\frac{\boldsymbol{v}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} + \left\|\frac{\operatorname{div}\boldsymbol{v}}{r}\right\|_{\mathrm{H}^1_1(D)} \le \frac{C}{\mu}\,\|\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)} \le \frac{C}{\mu}.$$

Since $\boldsymbol{v} \in \boldsymbol{\mathcal{E}} \subset \mathrm{H}^0_{-1}(\mathrm{div};D)$, we have that $\boldsymbol{v}/r \in \widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)$ and $(\operatorname{div}\boldsymbol{v})/r \in \mathrm{H}^1_1(D)$, then $I_h\boldsymbol{v}$ is well defined and belongs to $\boldsymbol{R}^0_h$. Therefore, by proceeding as in Lemma 4.5, we obtain

$$\delta\left(\boldsymbol{\mathcal{E}}, \boldsymbol{R}_h\right) \le \sup_{\substack{\boldsymbol{v}\in\boldsymbol{\mathcal{E}}\\ \|\boldsymbol{v}\|_{\mathrm{H}_{-1}(\mathrm{div};D)}=1}} \|\boldsymbol{v} - I_h\boldsymbol{v}\|_{\mathrm{H}_{-1}(\mathrm{div};D)}$$

$$\le Ch^{1-\epsilon}\left(\left\|\frac{\boldsymbol{v}}{r}\right\|_{\widetilde{\mathrm{H}}^1_1(D)\times\mathrm{H}^1_1(D)} + \left\|\frac{\operatorname{div}\boldsymbol{v}}{r}\right\|_{\mathrm{H}^1_1(D)}\right) + C\left\|I^R_h\boldsymbol{v} - I_h\boldsymbol{v}\right\|_{\mathrm{H}_{-1}(\mathrm{div},D_1)}, \tag{4.44}$$

for all $\epsilon > 0$. Estimate (4.42) follows from the previous equations and Lemma 4.2 (cf. (4.9) and (4.10)):

$$\delta\left(\boldsymbol{\mathcal{E}}, \boldsymbol{R}_h\right) \le \frac{C}{\mu}h^{1-\epsilon}\|\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)} + \frac{C}{\mu}h^{1-2/q}\|\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)} \le \frac{C}{\mu}h^{1-2/q},$$

for some $q \in (2,6)$. On the other hand, if **H.1** holds true, then by applying Lemma 4.2 (cf. (4.9) and (4.11)) to estimate the last term on the right-hand side of (4.44) we obtain

$$\delta\left(\boldsymbol{\mathcal{E}}, \boldsymbol{R}_h\right) \le \frac{C}{\mu}h^{1-\epsilon}\,\|\boldsymbol{v}\|_{\mathrm{L}^2_{-1}(D)} \le \frac{C}{\mu}h^{1-\epsilon}.$$

Thus, we end the proof.

5. Numerical experiments.

We have developed a MATLAB code based on lowest-order RT elements to solve Problem 3. We report in this section some numerical experiments which agree with the theoretical results proved in the previous sections.

5.1. Validation.

As a first numerical test, we have solved a particular problem with a known analytical solution, which allow us to validate the numerical implementation and to check the performance and convergence properties of the scheme. The domain Ω is a cylinder with radius 1 and height 3. On the other hand, we notice that the eigenvalues of (1.1) correspond to those of the Helmholtz equation with Neumann boundary conditions. By separation of variables it can be shown that the four lowest positive eigenvalues related with axisymmetric solutions are: $\lambda_1 = \pi/3, \lambda_2 = 2\pi/3, \lambda_3 = \pi$ and $\lambda_4 = 3.831705$. All these eigenvalues have multiplicity one. Then, from Theorem 4.7 it follows that, for h small enough, there exist eigenvalues $\lambda_{h,i}$, $i = 1,\dots,4$, solution to Problem 3 such that

$$\lambda_{h,i} \underset{h\to 0}{\longrightarrow} \lambda_i, \qquad i = 1,\dots,4.$$

The code has been used on several meshes $\mathcal{T}_h$ with different levels of refinement; we identify each mesh by its respective mesh size h. Table 1 shows the computed eigenvalues for different meshes, the exact eigenvalue and the computed orders of convergence.

Table 1: Convergence of the eigenvalues in the rectangle with regular meshes of size h.

h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$
0.125000	1.042919	2.086140	3.129950	3.937973
0.020833	1.047058	2.094126	3.141215	3.835302
0.011364	1.047154	2.094311	3.141474	3.832830
0.007813	1.047176	2.094354	3.141534	3.832252
0.005952	1.047185	2.094371	3.141558	3.832030
0.004808	1.047189	2.094379	3.141570	3.831920
0.004032	1.047192	2.094383	3.141576	3.831859
$\lambda_{\rm ex}$	1.047197	2.094395	3.141592	3.831705
Rates	1.912658	1.911716	1.910318	1.906234

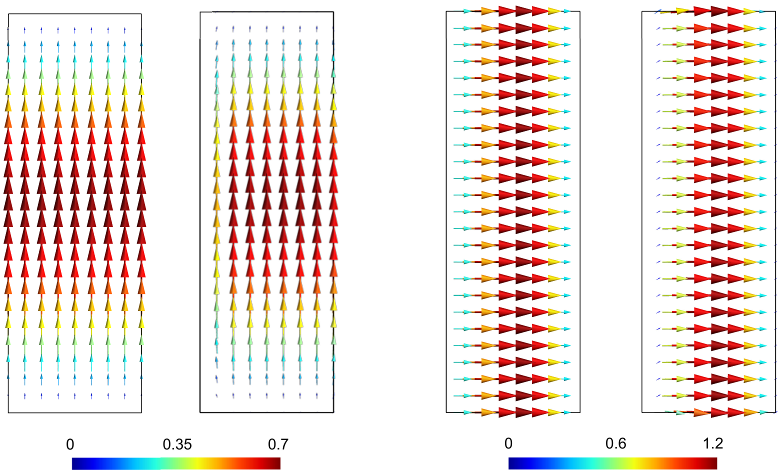

Fig. 3: From left to right: Analytical and computed displacement fields $\widetilde{\boldsymbol{u}}$ for the eigenfunction corresponding to λ_1 and λ_4, respectively.

For such a domain, assumption **H.1** holds true, so that the theoretical result predict an order of convergence $\mathcal{O}(h^{2(1-\epsilon)})$, for all $\epsilon > 0$. It can be seen from the last row of Table 1 that the obtained results confirm the theoretical one. We report in Figure 3 the analytical and the computed displacement fields for the eigenfunctions corresponding to the first and the fourth smallest eigenvalues. We chose the latter, because it is the first one in which the radial component does not vanish.

It can be seen from Figure 3 that the error concentrates in the vicinity of the symmetry axis. This seems quite reasonable, because of the constraint $\boldsymbol{u} = \mathbf{0}$ in D_0 imposed on the discrete space (cf. (4.2)). However, in spite of this fact, Table 1 clearly shows that this constraint does not pervert the convergence of the method.

5.2. Eigenvalues in a non-convex domain. For this test we have chosen an L-shaped domain $D := (0,1)\times(0,3)\setminus\{[0.5,1)\times[1,3)\}$. We have used uniform meshes as those shown in Figure 4. We report in Table 2 the four lowest eigenvalues computed by solving Problem 3. The table includes estimated orders of convergence, as well as more accurate values of the eigenvalues obtained by means of a least-square fitting of

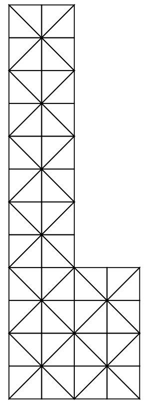

Fig. 4: Coarse initial mesh for the L-shape domain.

Table 2: Convergence of the eigenvalues in the L-shape domain with regular meshes of size h.

h	$\lambda_{h,1}$	$\lambda_{h,2}$	$\lambda_{h,3}$	$\lambda_{h,4}$
0.006667	0.834722	2.115595	3.141505	3.575267
0.005882	0.834746	2.115646	3.141524	3.575313
0.005263	0.834764	2.115684	3.141537	3.575348
0.004762	0.834778	2.115714	3.141547	3.575376
0.004348	0.834789	2.115737	3.141554	3.575399
0.004000	0.834797	2.115755	3.141560	3.575418
$\hat{\lambda}_{i,\mathrm{ex}}$	0.834858	2.115891	3.141593	3.575590
Rates	1.569000	1.520800	1.909800	1.230400

the model: $\lambda_{h,i} = \hat{\lambda}_{i,\mathrm{ex}} + Ch^t$.

In this case, the eigenfunctions corresponding to the first second and fourth lowest positive eigenvalues present a singularity at the reentrant corner. Instead, the eigenfunction corresponding to the third eigenvalue, which can be analytically computed, is smooth. All these agree with the results computed in Table 2. Moreover, the singular character of the eigenfunctions can be observed from Figures 5-8.

6. Conclusions. We have proposed a finite element method to solve a displacement formulation of the axisymmetric acoustic eigenvalue problem. We have shown that, in contrast to the Cartesian setting, spurious eigenvalues appear when lowest-order triangular RT elements are used to discretize the problem. Although this type of elements has been used in different axisymmetric problems, this behavior has not been documented. We have proposed an alternative weak formulation of the spectral problem which allowed us to avoid this drawback. The discretization is also based on lowest-order RT finite elements but for a different variable $\boldsymbol{u}(r,z) := r\widetilde{\boldsymbol{u}}(r,z)$. Spectral convergence and quasi-optimal-order error estimates have been proved by using the spectral theory for non-compact operators from [4, 5]. We have reported several illustrative numerical examples that allowed us to asses the convergence properties of the method and to check that it is not polluted with spurious modes. Notice that the techniques proposed in this article can be applied to other axisymmetric problems

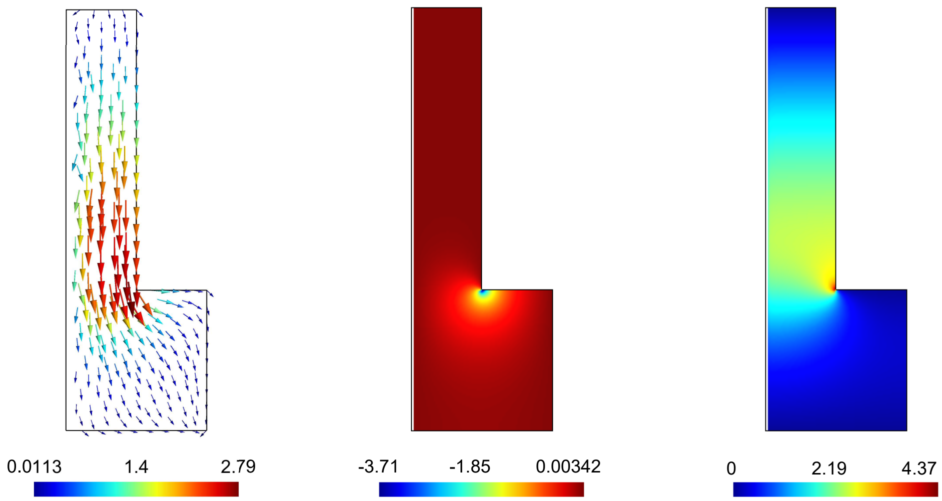

Fig. 5: Computed displacement field $\widetilde{\boldsymbol{u}} = (\widetilde{u}_r, \widetilde{u}_z)$ for the eigenfunction corresponding to $\lambda_{h,1}$. From left to right: $\widetilde{\boldsymbol{u}}$, $\widetilde{u}_r$ and $\widetilde{u}_z$.

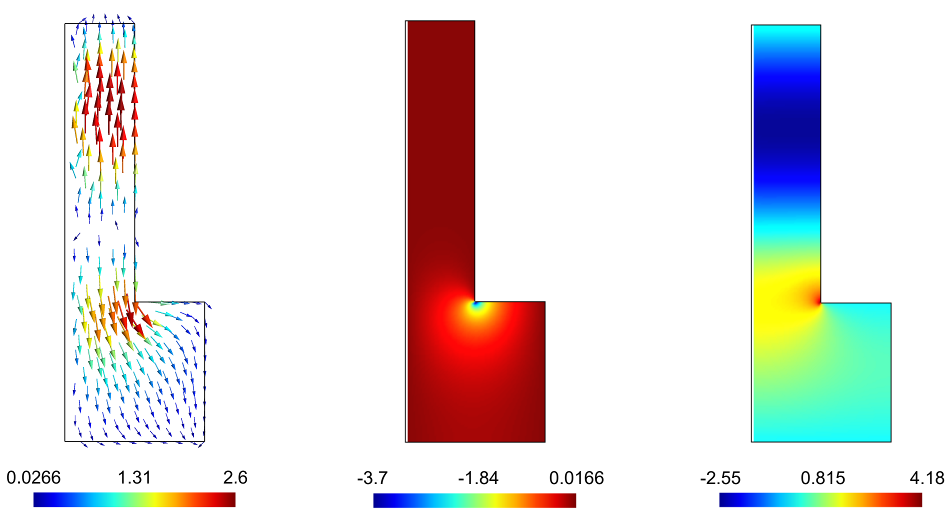

Fig. 6: Computed displacement field $\widetilde{\boldsymbol{u}} = (\widetilde{u}_r, \widetilde{u}_z)$ for the eigenfunction corresponding to $\lambda_{h,2}$. From left to right: $\widetilde{\boldsymbol{u}}$, $\widetilde{u}_r$ and $\widetilde{u}_z$.

such as elastoacoustic transient problems appearing in several settings, for instance, propagation of noise or seismic waves.

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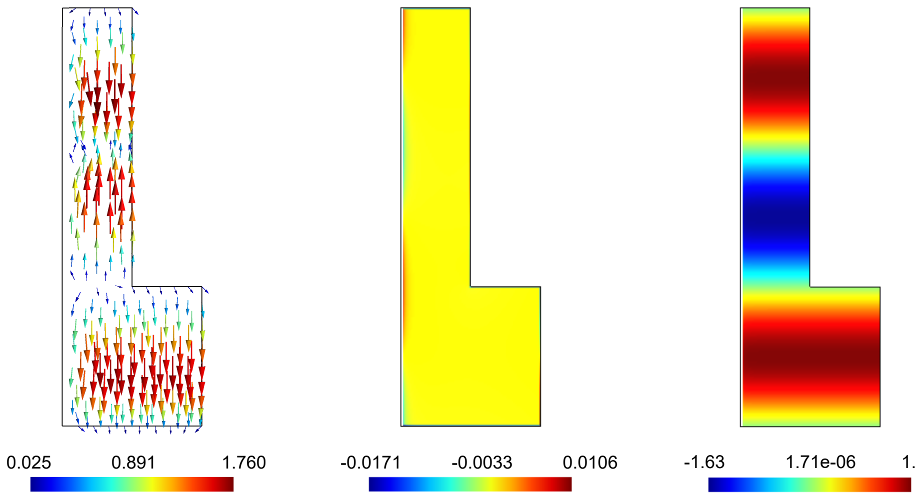

Fig. 7: Computed displacement field $\widetilde{\boldsymbol{u}} = (\widetilde{u}_r, \widetilde{u}_z)$ for the eigenfunction corresponding to $\lambda_{h,3}$. From left to right: $\widetilde{\boldsymbol{u}}$, $\widetilde{u}_r$ and $\widetilde{u}_z$. This eigenfunction can be analytically computed and it does not depend on the coordinate r and $\widetilde{u}_r$ vanishes.

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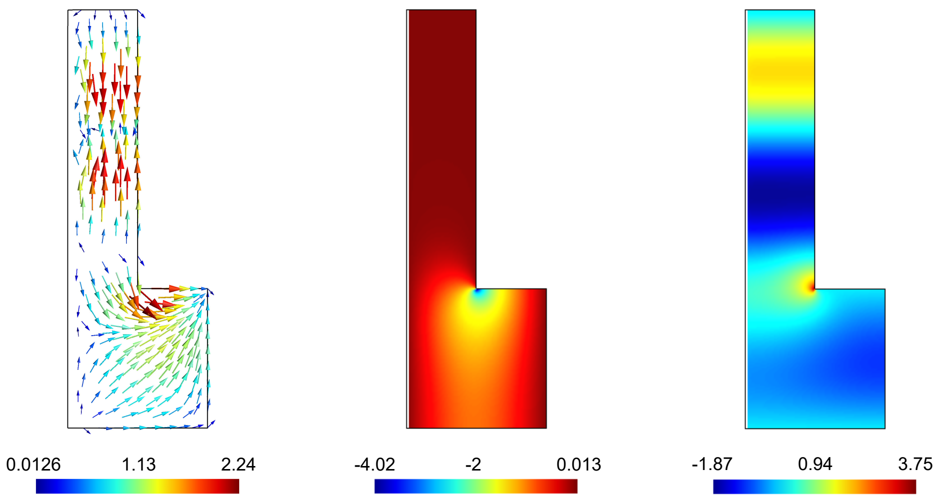

Fig. 8: Computed displacement field $\widetilde{\boldsymbol{u}} = (\widetilde{u}_r, \widetilde{u}_z)$ for the eigenfunction corresponding to $\lambda_{h,4}$. From left to right: $\widetilde{\boldsymbol{u}}$, $\widetilde{u}_r$ and $\widetilde{u}_z$.

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