UNIVERSIDAD DE CONCEPCIÓN

Centro de Investigación en Ingeniería Matemática (CI^{2}MA)

Analysis of a new mixed-FEM for stationary incompressible magneto-hydrodynamics

Jessika Camaño, Carlos Garcia,
Ricardo Oyarzúa

PREPRINT 2020-13

SERIE DE PRE-PUBLICACIONES

Analysis of a new mixed-FEM for stationary incompressible magneto-hydrodynamics *

Jessika Camaño[†], Carlos García[‡] and Ricardo Oyarzúa[§]

Abstract

In this paper we propose and analyze a new mixed finite element method for a stationary magnetohydrodynamic (MHD) model. The method is based on the utilization of a new dual-mixed formulation recently introduced in [3] for the Navier-Stokes problem, which is coupled with a classical primal formulation for the Maxwell equations. The latter implies that the velocity and a pseudostress tensor relating the velocity gradient with the convective term for the hydrodynamic equations, together with the magnetic field and a Lagrange multiplier related with the divergence-free property of the magnetic field, become the main unknowns of the system. Then the associated Galerkin scheme can be defined by employing Raviart–Thomas elements of degree k for the aforementioned pseudostress tensor, discontinuous piecewise polynomial elements of degree k for the velocity, Nédélec elements of degree k for the magnetic field and Lagrange elements of degree k for the associated Lagrange multiplier. The analysis of the continuous and discrete problems are carried out by means of the Lax–Milgram lemma, the Banach–Nečas–Babuška and Banach fixed-point theorems, under a sufficiently small data assumption. In particular, the analysis for the discrete scheme can be carried out by means of a quiasi-uniformity assumption of the mesh. We also develop an a priori error analysis and show that the proposed finite element method is optimal convergent. Finally, some numerical results illustrating the good performance of the method are provided.

Key words: Incompressible magnetohydrodynamics, mixed finite element method, Banach spaces, Raviart–Thomas elements, Nédélec elements.

Mathematics subject classifications (1991): 65N15, 65N30, 76M10

*This research was partially supported by CONICYT-Chile through project Fondecyt 1180859 and project AFB170001 of the PIA Program: Concurso Apoyo a Centros Cientificos y Tecnologicos de Excelencia con Financiamiento Basal.

[†]Departamento de Matemática y Física Aplicadas, Universidad Católica de la Santísima Concepción, Casilla 297, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile,email: jecamano@ucsc.cl

[‡]Facultad de Matemáticas, Pontificia Universidad Católica de Chile, Santiago, Chile, email: cgarciv@mat.uc.cl

[§]GIMNAP-Departamento de Matemática, Universidad del Bío-Bío, Casilla 5-C, Concepción, Chile, and CI^{2}MA, Universidad de Concepción, Casilla 160-C, Concepción, Chile, email: royarzua@ubiobio.cl

1 Introduction

The numerical study of the dynamics of electrically conducting fluids in electromagnetic fields, best known as Magnetohydrodynamics (MHD), is a very active area of research, and the interest and number of scientific contributions in this discipline increase in time due to its applicability in different scientific disciplines, such as astrophysics, engineering related to liquid metals and controlled thermonuclear fusion. Since the mathematical model of MHD is a coupled system where the incompressible Navier–Stokes equations are coupled with the Maxwell's equations through the Lorentz force and Ohm's law, the numerical analysis community has been trying to bring together the best of both, electromagnetism and fluid dynamics communities, to develop better numerical methods to approximate the solution of the MHD model.

To begin the bibliographical discussion we start by mentioning one of the first works devoted to the analysis of finite element methods (FEM) for MHD, namely [18]. There, the authors develop the well-posedness and convergence analysis for a conforming FEM for MHD considering inf-sup stable velocity-pressure elements for the hydrodynamic variables and standard nodal finite elements, i.e, H^1-conforming elements for the magnetic field. An extention to [18] can be found in [15] where the authors propose a stabilized method for the three-field formulation considered in [18]. We emphasize that in both contributions, [18] and [15], the magnetic field is in $H^1(\Omega)^3$, which is only feasible if the domain is convex. To circunvent the latter, in [19] the authors introduce a mixed finite element method based on weighted regularization for the incompressible MHD system which can be used even in non-convex domains (see also [7]). Another way to circumvent this problem can be found [27] where the author imposes weakly the divergence-free condition of the magnetic field through the introduction of a Lagrange multiplier. By doing that, the magnetic field can be approximated by curl-conforming Nédélec elements and the convex domain assumption is no longer required.

MHD equations admit many different variational formulations which lead to different mathematical properties. In [22] the authors introduce a fully-DG method for a linearized incompressible MHD model problem based on the mixed method introduced in [27]. All the variables are aproximated by discontinuous finite element spaces and, as a consequence, the approach requires a large number of degrees of freedom. This drawback is overcome in [17] where the authors introduce a finite element discretization and instead of using discontinuous elements for all the unknowns, they use divergence conforming Brezzi-Douglas-Marini (BDM) elements for the approximation the velocity and curl-conforming Nédélec elements for the magnetic field. A different approach is presented in [23] where the authors proposed a mixed finite element discretization for the MHD problem where the Gauss's law for the magnetic field at discrete level is preserved (see also [24]).

According to the above discussion, and in order to contribute toward the development and analysis of new numerical methods to approximate the solution of MHD, in this paper we propose and analyze a new mixed finite element method for the stationary incompressible MHD system considering constant parameters. Our method combines the mixed finite element discretization that has been developed recently for the stationary Navier–Stokes problem in [3] and the mixed finite element method presented in [27] for the Maxwell's equations. More precisely, we adopt the methodology developed in [3] for the fluid equations, where the main unknowns are the velocity and a nonlinear pseudostress tensor depending nonlinearly on the velocity through the corresponding convective term, and in [27] for the electromagnetic unknowns. The pressure is eliminated by using the incompressibility condition, and can be recovered as a simple postprocess of the nonlinear pseudostress tensor. Thus, the resulting main unknowns of the system are the aforementioned pseudostress tensor, the velocity, the magnetic field and a Lagrange multiplier related with the divergence-free property of the magnetic field. As in [3], we introduce non-standard Banach spaces for the velocity and the nonlinear pseudostress tensor, and $H(\mathbf{curl})$ and H^1 for the magnetic field and Lagrange multiplier, respectively. A fixed-point setting resembling the approach applied in [6] is then utilized to study the well-posedness of the continuous and discrete schemes of our problem. The analysis is based on the Lax–Milgram lemma, the Banach–Nečas–Babuška theorem and the Banach's fixed–point theorem under a small data assumption. For the associated Galerkin scheme we employ Raviart-Thomas elements of degree k to approximate the nonlinear pseudostress tensor, discontinuous piecewise polynomials of degree k for the velocity, Nédélec elements of degree k for the magnetic field and continuous piecewise polynomials of degree k for the Lagrange multiplier. We also derive optimal a priori error estimates of our mixed scheme. Notice that, an important advantage of our formulation is that further variables of interest, such as the fluid pressure, the fluid vorticity and the fluid velocity gradient, can be computed as a simple postprocess of the finite element solutions with the same rate of convergence. In addition, similarly to [27], our analysis is carried out without assuming that the computational domain is convex as in [18] and [15].

We have organized the contents of this paper as follows. In Section 2 we introduce some standard notation and functional spaces. We reformulate the incompressible MHD problem as an equivalent set of equations and derive our mixed variational formulation. Next, in Section 3, we apply the classical Banach fixed point theorem, the Lax–Milgram lemma and the Banach–Nečas–Babuška theorem to prove unique solvability and stability of the continuous formulation. The corresponding Galerkin scheme is introduced and analysed by mimicking the theory developed for the continuous problem in Section 4. In addition, we establish the corresponding Céa's estimate and prove optimal convergence of the method. Finally, in Section 5, we provide some numerical results illustrating the performance of our mixed finite element method and confirming the theoretical rates of convergence.

2 The model problem and its mixed formulation

2.1 Preliminaries

Let us denote by $\Omega \subseteq \mathbb{R}^3$ a given bounded domain with polyhedral boundary Γ, and denote by $\mathbf{n}$ the outward unit normal vector on Γ. Standard notations will be adopted for Lebesgue spaces $\mathrm{L}^p(\Omega)$, with $p \in [1,\infty]$ and Sobolev spaces $\mathrm{W}^{r,p}(\Omega)$ with $r \geq 0$, endowed with the norms $\|\cdot\|_{\mathrm{L}^p(\Omega)}$ and $\|\cdot\|_{\mathrm{W}^{r,p}(\Omega)}$, respectively. Note that $\mathrm{W}^{0,p}(\Omega) = \mathrm{L}^p(\Omega)$ and if $p = 2$, we write $\mathrm{H}^r(\Omega)$ in place of $\mathrm{W}^{r,2}(\Omega)$, with the corresponding Lebesgue and Sobolev norms denoted by $\|\cdot\|_{0,\Omega}$ and $\|\cdot\|_{r,\Omega}$, respectively. We also write $|\cdot|_{r,\Omega}$ for the H^r-seminorm. In addition, $\mathrm{H}^{1/2}(\Gamma)$ is the spaces of traces of functions of $\mathrm{H}^1(\Omega)$ and $\mathrm{H}^{-1/2}(\Gamma)$ denotes its dual. With $\langle\cdot,\cdot\rangle$ we denote the corresponding product of duality between $\mathrm{H}^{1/2}(\Gamma)$ and $\mathrm{H}^{-1/2}(\Gamma)$. By $\mathbf{S}$ and $\mathbb{S}$ we will denote the corresponding vectorial and tensorial counterparts of the generic scalar functional space S, and, in the case of product spaces, we preserve the same type of symbol as the space of objects of higher dimensions, for example $\mathbb{U} = \mathbb{V} \times \mathbf{X} \times D$ or $\mathbf{U} = \mathbf{V} \times X$. In addition, we will denote by $\|(u,v)\| := \|(u,v)\|_{U\times V} := \|u\|_U + \|v\|_V$ the norm on the product space $U \times V$. In turn, for any vector fields $\mathbf{v} = (v_i)_{i=1,3}$ and $\mathbf{w} = (w_i)_{i=1,3}$ we set the gradient, divergence, tensor product and curl operators, respectively, as

$$\nabla \mathbf{v} := \left(\frac{\partial v_i}{\partial x_j}\right)_{i,j=1,3}, \qquad \operatorname{div} \mathbf{v} := \sum_{j=1}^{3} \frac{\partial v_j}{\partial x_j}, \qquad \mathbf{v} \otimes \mathbf{w} := (v_i w_j)_{i,j=1,3}$$

and

$$\operatorname{curl} \mathbf{v} := \left(\frac{\partial v_3}{\partial x_2} - \frac{\partial v_2}{\partial x_3}, \frac{\partial v_1}{\partial x_3} - \frac{\partial v_3}{\partial x_1}, \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}\right).$$

In addition, for any tensor fields $\boldsymbol{\tau} = (\tau_{ij})_{i,j=1,3}$ and $\boldsymbol{\zeta} = (\zeta_{ij})_{i,j=1,3}$, we let $\mathbf{div}\, \boldsymbol{\tau}$ be the divergence operator div acting along the rows of $\boldsymbol{\tau}$, and define the transpose, the trace, the tensor inner product, and the deviatoric tensor, respectively, as

$$\boldsymbol{\tau}^{\mathrm{t}} := (\boldsymbol{\tau}_{ji})_{i,j=1,3}, \qquad \operatorname{tr}(\boldsymbol{\tau}) := \sum_{i=1}^{3} \tau_{ii}, \qquad \boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{3} \tau_{ij}\zeta_{ij}, \qquad \boldsymbol{\tau}^{d} := \boldsymbol{\tau} - \frac{1}{3}\operatorname{tr}(\boldsymbol{\tau})\mathbb{I},$$

where $\mathbb{I}$ is the identity tensor in $\mathbb{R}^{3\times 3}$. The cross product of two vectors $\mathbf{u} = (u_1, u_2, u_3)$ and $\mathbf{v} = (v_1, v_2, v_3)$ in $\mathbb{R}^3$ is given by

$$\mathbf{u} \times \mathbf{v} := (u_2 v_3 - u_3 v_2, u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1).$$

For simplicity, in what follows we denote

$$(v, w)_\Omega := \int_\Omega v w, \quad (\mathbf{v}, \mathbf{w})_\Omega := \int_\Omega \mathbf{v} \cdot \mathbf{w}, \quad (\mathbf{v}, \mathbf{w})_\Gamma := \int_\Gamma \mathbf{u} \cdot \mathbf{v} \quad \text{and} \quad (\boldsymbol{\tau}, \boldsymbol{\zeta})_\Omega := \int_\Omega \boldsymbol{\tau} : \boldsymbol{\zeta}.$$

Furthermore, we recall that the Hilbert spaces

$$\mathbf{H}(\operatorname{div}; \Omega) := \left\{\boldsymbol{\tau} \in \mathbf{L}^2(\Omega) : \operatorname{div} \boldsymbol{\tau} \in \mathrm{L}^2(\Omega)\right\}$$

and

$$\mathbf{H}(\operatorname{curl}; \Omega) := \left\{\mathbf{v} \in \mathbf{L}^2(\Omega) : \operatorname{curl} \mathbf{v} \in \mathbf{L}^2(\Omega)\right\},$$

with norms $\|\boldsymbol{\tau}\|^2_{\operatorname{div};\Omega} := \|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\operatorname{div} \boldsymbol{\tau}\|^2_{0,\Omega}$ and $\|\mathbf{v}\|^2_{\operatorname{curl};\Omega} := \|\mathbf{v}\|^2_{0,\Omega} + \|\operatorname{curl} \mathbf{v}\|^2_{0,\Omega}$, respectively. Both spaces are standard in mixed problems and electromagnetism problems, respectively. We denote by $\mathbf{H}(\operatorname{div}^0; \Omega)$ the subspace of $\mathbf{H}(\operatorname{div}; \Omega)$ with divergence zero and in the sequel we will make use of the tensor version of $\mathbf{H}(\operatorname{div}; \Omega)$, namely

$$\mathbb{H}(\mathbf{div}; \Omega) := \left\{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) : \quad \mathbf{div}\, \boldsymbol{\tau} \in \mathbf{L}^2(\Omega)\right\},$$

whose norm will be denoted $\|\cdot\|_{\mathbf{div};\Omega}$. In turn, given $p \geq \frac{6}{5}$, in what follows we will also employ the non-standard Banach space $\mathbb{H}(\mathbf{div}_p, \Omega)$ defined by

$$\mathbb{H}(\mathbf{div}_p; \Omega) := \{\boldsymbol{\tau} \in \mathbb{L}^2(\Omega) \ : \ \mathbf{div}\, \boldsymbol{\tau} \in \mathbf{L}^p(\Omega)\},$$

endowed with the norm

$$\|\boldsymbol{\tau}\|_{\mathbf{div}_p;\Omega} := \left(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\mathbf{div}\, \boldsymbol{\tau}\|^2_{\mathbf{L}^p(\Omega)}\right)^{1/2}.$$

Also, we consider the following subspace of $\mathbf{H}(\operatorname{curl}; \Omega)$

$$\mathbf{H}_0(\operatorname{curl}; \Omega) := \left\{\mathbf{c} \in \mathbf{H}(\operatorname{curl}; \Omega) : \quad \mathbf{n} \times \mathbf{c} = \mathbf{0} \text{ on } \Gamma\right\}.$$

2.2 The stationary incompressible magneto-hydrodynamics problem

Let $\Omega \subseteq \mathbb{R}^3$ be a bounded Lipschitz polyhedron domain. For simplicity, we assume that Ω is simply-connected, and that its boundary is connected. Let ν, ν_m, $\mathbf{u}$ and p be the hydrodynamic viscosity, electromagnetic viscosity, velocity and pressure, respectively, of a viscous incompressible fluid occupying the region Ω, exposed to a magnetic field $\mathbf{b}$. Then, the movement of the fluid and the behavior of the magnetic field are described by the stationary incompressible magneto-hydrodynamic equations:

$$\begin{aligned}
-\nu\Delta\mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} + \nabla p - k(\mathrm{curl}\ \mathbf{b})\times\mathbf{b} &= \mathbf{f} \quad \text{in} \quad \Omega, \\
k\nu_m \mathrm{curl}\ (\mathrm{curl}\ \mathbf{b}) + \nabla r - k\mathrm{curl}\ (\mathbf{u}\times\mathbf{b}) &= \mathbf{g} \quad \text{in} \quad \Omega, \\
\mathrm{div}\ \mathbf{u} &= 0 \quad \text{in} \quad \Omega, \\
\mathrm{div}\ \mathbf{b} &= 0 \quad \text{in} \quad \Omega, \\
(p,1)_\Omega &= 0,
\end{aligned} \tag{2.1}$$

where $\mathbf{f}$ and $\mathbf{g}$ are source terms, k is the coupling number and the unknown r is the corresponding Lagrange multiplier associated with the divergence constraint on the magnetic field $\mathbf{b}$. In addition, we consider the following Dirichlet boundary conditions:

$$\mathbf{u} = \mathbf{u}_D \quad \text{on} \quad \Gamma, \qquad \mathbf{n}\times\mathbf{b} = \mathbf{0} \quad \text{on} \quad \Gamma \quad \text{and} \quad r = 0 \quad \text{on} \quad \Gamma, \tag{2.2}$$

where $\mathbf{u}_D$ is the prescribed velocity on Γ satisfying the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \mathbf{n} = 0. \tag{2.3}$$

As we already mentioned before, we are interested in deriving a mixed finite element method to approximate the solution of problem (2.1). To that end, we proceed analogously as in [3] and write (2.1) as an equivalent set of equations by introducing the pseudostress tensor

$$\boldsymbol{\sigma} := \nu\nabla\mathbf{u} - p\mathbb{I} - \mathbf{u}\otimes\mathbf{u} \quad \text{in} \quad \Omega. \tag{2.4}$$

Notice that from the incompressibility condition div $\mathbf{u} = \mathrm{tr}\,(\nabla\mathbf{u}) = 0$ in Ω, there hold

$$\mathbf{div}\,(\mathbf{u}\otimes\mathbf{u}) = (\mathbf{u}\cdot\nabla)\mathbf{u} \quad \text{in} \quad \Omega \quad \text{and} \quad \mathrm{tr}\,(\boldsymbol{\sigma}) = -3p - \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) \quad \text{in} \quad \Omega. \tag{2.5}$$

In particular, the second equation in (2.5) allows us to write the pressure p in terms of the tensor $\boldsymbol{\sigma}$ and the velocity $\mathbf{u}$ as

$$p = -\frac{1}{3}(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u})) \quad \text{in} \quad \Omega, \tag{2.6}$$

which in turn, together to (2.4), leads us to the equation

$$\boldsymbol{\sigma}^d = \nu\nabla\mathbf{u} - (\mathbf{u}\otimes\mathbf{u})^d \quad \text{in} \quad \Omega.$$

On the other hand, from (2.4), the first equation of (2.1) and (2.5), we easily get the equilibrium equation

$$-\,\mathbf{div}\,\boldsymbol{\sigma} - k(\mathrm{curl}\ \mathbf{b})\times\mathbf{b} = \mathbf{f} \quad \text{in} \quad \Omega\,.$$

Finally, from (2.6) we observe that the condition $(p,1)_\Omega = 0$, ensuring the uniqueness of solution of problem (2.1), is equivalent to

$$(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega \,=\, 0.$$

According to the above, we rewrite equations (2.1) equivalently as follows:

$$\begin{aligned} \boldsymbol{\sigma}^d &= \nu\nabla\mathbf{u} - \,(\mathbf{u}\otimes\mathbf{u})^d && \text{in} \quad \Omega, \\ -\,\mathbf{div}\,\boldsymbol{\sigma} - k(\mathrm{curl}\,\mathbf{b})\times\mathbf{b} &= \mathbf{f} && \text{in} \quad \Omega, \\ k\nu_m \mathrm{curl}\,(\mathrm{curl}\,\mathbf{b}) + \nabla r - k\mathrm{curl}\,(\mathbf{u}\times\mathbf{b}) &= \mathbf{g} && \text{in} \quad \Omega, \\ \mathrm{div}\,\mathbf{b} &= \mathbf{0} && \text{in} \quad \Omega, \\ (\mathrm{tr}\,(\boldsymbol{\sigma}),1)_\Omega + (\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1)_\Omega &= 0, \end{aligned} \tag{2.7}$$

along with the boundary conditions (2.2). Here the unknowns of the system are the tensor $\boldsymbol{\sigma}$, the velocity $\mathbf{u}$, the magnetic field $\mathbf{b}$ and the Lagrange multiplier r. The pressure p can be easily computed as a postprocess of the solution by using (2.6). In the sequel we employ the set of equations (2.7) to derive our mixed formulation.

2.3 Derivation of the mixed variational formulation

We begin by proceeding similarly to [3] for the first and second equations of (2.7), that is, we multiply the first equation of (2.7) by $\boldsymbol{\tau} \in \mathbb{M}$, integrate by parts, employ the Dirichlet boundary condition $\mathbf{u} = \mathbf{u}_D$ on Γ, and test the second equation of (2.7) by $\mathbf{v} \in \mathbf{Q}$ ($\mathbb{M}$ and $\mathbf{Q}$ will be specified later on), to obtain

$$\frac{1}{\nu}(\boldsymbol{\sigma}^d,\boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\,\boldsymbol{\tau},\mathbf{u})_\Omega \,+\, \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u},\boldsymbol{\tau}^d)_\Omega \,=\, \langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle, \quad \forall\,\boldsymbol{\tau}\in\mathbb{M}, \tag{2.8}$$

and

$$(\mathbf{div}\,\boldsymbol{\sigma},\mathbf{v})_\Omega + k((\mathrm{curl}\,\mathbf{b})\times\mathbf{b},\mathbf{v})_\Omega \,=\, -(\mathbf{f},\mathbf{v})_\Omega, \quad \forall\,\mathbf{v}\in\mathbf{Q}. \tag{2.9}$$

Next, for the third and fourth equations of (2.7) we proceed analogously to [27], that is, we multiply the third equation of (2.7) by $\mathbf{d} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega)$, integrate by parts, test the fourth equation of (2.7) by $z \in H_0^1(\Omega)$ and also integrate by parts, to get

$$k\nu_m(\mathrm{curl}\,\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega + (\nabla r,\mathbf{d})_\Omega - k(\mathbf{u}\times\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega \,=\, (\mathbf{g},\mathbf{d})_\Omega, \quad \forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega) \tag{2.10}$$

and

$$(\mathbf{b},\nabla z)_\Omega \,=\, 0, \quad \forall\, z\in H_0^1(\Omega). \tag{2.11}$$

In this way, at first we are interested in finding $\boldsymbol{\sigma}\in\mathbb{M}$, $\mathbf{u}\in\mathbf{Q}$, $\mathbf{b}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega)$ and $r\in H_0^1(\Omega)$ satisfying (2.8), (2.9), (2.10) and (2.11) and the condition $(\mathrm{tr}\,(\boldsymbol{\sigma}),1)_\Omega \,=\, -(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1)_\Omega$.

Now we turn to specify the spaces $\mathbb{M}$ and $\mathbf{Q}$. To that end, we first let

$$\mathbf{C} := \{\mathbf{d}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega) \,:\, (\mathbf{d},\nabla z)_\Omega \,=\, 0 \quad \forall\, z\in H_0^1(\Omega)\} = \mathbf{H}_0(\mathrm{curl}\,;\Omega)\,\cap\,\mathbf{H}(\mathrm{div}^{\,0};\Omega), \tag{2.12}$$

and observe that, since $\mathbf{b}$ satisfies (2.11), then $\mathbf{b}\in\mathbf{C}$ (see [16, Section I.2.2]). Then, since $\mathbf{C}$ is continuously embedded into $\mathbf{H}^s(\Omega)$ for some $s > 1/2$ (see [1, Proposition 3.7]), which in turn is continuously embedded into $\mathbf{L}^{3+\delta}(\Omega)$, for some $\delta > 0$ (see [26, Theorem 1.3.4]), we obtain

$$\|\mathbf{b}\|_{\mathbf{L}^{3+\delta}(\Omega)} \leq c_1\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}.$$

Therefore, using the embedding inequality

$$\|\mathbf{v}\|_{\mathbf{L}^q(\Omega)} \leq c_2 \|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \quad \forall\, q \in [1, 6), \tag{2.13}$$

and defininng $\delta^* := \frac{4\delta}{1+\delta} > 0$, it follows that

$$|((\mathrm{curl}\ \mathbf{d})\times\mathbf{b}, \mathbf{v})_\Omega| \leq \|\mathrm{curl}\ \mathbf{d}\|_{0,\Omega}\|\mathbf{b}\|_{\mathbf{L}^{3+\delta}(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^{6-\delta^*}(\Omega)} \leq C_s\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}, \tag{2.14}$$

for all $\mathbf{d} \in \mathbf{H}(\mathrm{curl}\,;\Omega)$ and $\mathbf{v} \in \mathbf{L}^6(\Omega)$, with C_s the resulting constant from the aforementioned embedding inequalities. In addition, if $\mathbf{u} \in \mathbf{L}^6(\Omega)$ we obtain

$$|(\mathbf{u} \times \mathbf{b}, \mathrm{curl}\ \mathbf{d})_\Omega| \leq \|\mathrm{curl}\ \mathbf{d}\|_{0,\Omega}\|\mathbf{b}\|_{\mathbf{L}^{3+\delta}(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^{6-\delta^*}(\Omega)} \leq C_s\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}, \tag{2.15}$$

for all $\mathbf{d} \in \mathbf{H}(\mathrm{curl}\,;\Omega)$.

According to the above, the terms $((\mathrm{curl}\ \mathbf{b})\times\mathbf{b}, \mathbf{v})_\Omega$ and $(\mathbf{u}\times\mathbf{b}, \mathrm{curl}\ \mathbf{d})_\Omega$ in (2.9) and (2.10), respectively, are well defined if we set $\mathbf{Q} := \mathbf{L}^6(\Omega)$ and consequently, $(\mathbf{div}\,\boldsymbol{\tau}, \mathbf{u})_\Omega$ and $(\mathbf{div}\,\boldsymbol{\sigma}, \mathbf{v})_\Omega$ in (2.8) and (2.9), respectively, are well defined if $\mathbf{div}\,\boldsymbol{\tau}, \mathbf{div}\,\boldsymbol{\sigma} \in \mathbb{M} := \mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)$.

Let us now define the space

$$\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) := \{\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\ :\ (\mathrm{tr}\,(\boldsymbol{\tau}), 1)_\Omega = 0\},$$

and recall that there holds

$$\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega) = \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\ \oplus\ P_0(\Omega)\mathbb{I}, \tag{2.16}$$

where $P_0(\Omega)$ is the space of constant polynomials on Ω. More precisely, each $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)$ can be decomposed uniquely as:

$$\boldsymbol{\tau} \,=\, \boldsymbol{\tau}_0 \,+\, c\,\mathbb{I}\,, \quad \text{with} \quad \boldsymbol{\tau}_0 \,\in\, \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \quad \text{and} \quad c \,:=\, \frac{1}{3\,|\Omega|}(\mathrm{tr}\ \boldsymbol{\tau}, 1)_\Omega \in \mathbb{R}\,.$$

Then, if we define the tensor

$$\boldsymbol{\sigma}_0 := \boldsymbol{\sigma} + \frac{1}{3|\Omega|}(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega\mathbb{I}, \tag{2.17}$$

it can be readily seen that

$$(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}), 1)_\Omega = 0 \quad \text{if and only if} \quad \boldsymbol{\sigma}_0 \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega).$$

Moreover, owing to (2.16) and the compatibility condition (2.3), after simple computations we realize that equation (2.8) can be rewritten in terms of $\boldsymbol{\sigma}_0$ equivalently as

$$\frac{1}{\nu}(\boldsymbol{\sigma}_0^d, \boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\ \boldsymbol{\tau}, \mathbf{u})_\Omega \,+\, \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u}, \boldsymbol{\tau}^d)_\Omega \,=\, \langle \boldsymbol{\tau}\mathbf{n}, \mathbf{u}_D\rangle, \quad \forall\,\boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega), \tag{2.18}$$

whereas (2.9) becomes:

$$(\mathbf{div}\ \boldsymbol{\sigma}_0, \mathbf{v})_\Omega + k((\mathrm{curl}\ \mathbf{b}) \times \mathbf{b}, \mathbf{v})_\Omega \,=\, -(\mathbf{f}, \mathbf{v})_\Omega, \quad \forall\,\mathbf{v} \in \mathbf{L}^6(\Omega). \tag{2.19}$$

Summarizing, instead of considering equations (2.8) and (2.9) in the system of equations, from now on we consider the new equations (2.18) and (2.19). In this way, for the sake of simplicity we drop the subscript 0 from the new unknown $\boldsymbol{\sigma}_0$, and at last arrive to our variational problem: Find $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b},r)\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\times\mathbf{L}^6(\Omega)\times\mathbf{H}_0(\mathrm{curl};\Omega)\times H_0^1(\Omega)$, such that:

$$\begin{aligned}
\frac{1}{\nu}(\boldsymbol{\sigma}^d,\boldsymbol{\tau}^d)_\Omega + (\mathbf{div}\,\boldsymbol{\tau},\mathbf{u})_\Omega + \frac{1}{\nu}(\mathbf{u}\otimes\mathbf{u},\boldsymbol{\tau}^d)_\Omega &= \langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega),\\
(\mathbf{div}\,\boldsymbol{\sigma},\mathbf{v})_\Omega + k((\mathrm{curl}\,\mathbf{b})\times\mathbf{b},\mathbf{v})_\Omega &= -(\mathbf{f},\mathbf{v})_\Omega && \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega),\\
k\nu_m(\mathrm{curl}\,\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega + (\nabla r,\mathbf{d})_\Omega - k(\mathbf{u}\times\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega &= (\mathbf{g},\mathbf{d})_\Omega && \forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega),\\
(\mathbf{b},\nabla z)_\Omega &= 0 && \forall\,z\in H_0^1(\Omega)\,.
\end{aligned}\tag{2.20}$$

Remark 2.1 *Notice that, if $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b},r)$ is the solution of (2.20), then according to (2.6) and (2.17), the post processing formula for the pressure p now reduces to*

$$p = -\frac{1}{3}\left(\mathrm{tr}\,(\boldsymbol{\sigma}) + \mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}) - \frac{1}{|\Omega|}(\mathrm{tr}\,(\mathbf{u}\otimes\mathbf{u}),1)_\Omega\right). \tag{2.21}$$

3 Analysis of the continuous problem

3.1 Preliminaries

The well-posedness of problem (2.20) will be addressed by means of a fixed-point strategy and a sufficiently small data assumption. To that end, and for easiness of presentation, we define the forms $a_f:\mathbb{H}(\mathbf{div}_{6/5};\Omega)\times\mathbb{H}(\mathbf{div}_{6/5};\Omega)\to\mathbb{R}$, $b_f:\mathbb{H}(\mathbf{div}_{6/5};\Omega)\times\mathbf{L}^6(\Omega)\to\mathbb{R}$, $c_f:\mathbf{L}^6(\Omega)\times\mathbf{L}^6(\Omega)\times\mathbb{H}(\mathbf{div}_{6/5};\Omega)\to\mathbb{R}$, $c_d:\mathbf{H}(\mathrm{curl};\Omega)\times\mathbf{C}\times\mathbf{L}^6(\Omega)\to\mathbb{R}$, $a_m:\mathbf{H}(\mathrm{curl};\Omega)\times\mathbf{H}(\mathrm{curl};\Omega)\to\mathbb{R}$, $b_m:\mathbf{H}(\mathrm{curl};\Omega)\times H_0^1(\Omega)\to\mathbb{R}$ and $c_m:\mathbf{L}^6(\Omega)\times\mathbf{C}\times\mathbf{H}(\mathrm{curl};\Omega)\to\mathbb{R}$, as

$$\begin{aligned}
a_f(\boldsymbol{\sigma},\boldsymbol{\tau}) &:= \tfrac{1}{\nu}(\boldsymbol{\sigma}^d,\boldsymbol{\tau}^d)_\Omega, & b_f(\boldsymbol{\tau},\mathbf{v}) &:= (\mathbf{div}\,\boldsymbol{\tau},\mathbf{u})_\Omega,\\
c_f(\mathbf{u};\mathbf{v},\boldsymbol{\tau}) &:= \tfrac{1}{\nu}(\mathbf{u}\otimes\mathbf{v},\boldsymbol{\tau}^d)_\Omega, & c_d(\mathbf{b};\mathbf{c},\mathbf{v}) &:= k((\mathrm{curl}\,\mathbf{b})\times\mathbf{c},\mathbf{v})_\Omega,\\
a_m(\mathbf{b},\mathbf{d}) &:= k\nu_m(\mathrm{curl}\,\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega, & b_m(\mathbf{d},z) &:= (\mathbf{d},\nabla z)_\Omega,\\
c_m(\mathbf{u};\mathbf{b},\mathbf{d}) &:= -k(\mathbf{u}\times\mathbf{b},\mathrm{curl}\,\mathbf{d})_\Omega,
\end{aligned}\tag{3.1}$$

where $\mathbf{C}$ is the subset of $\mathbf{H}_0(\mathrm{curl};\Omega)$ defined in (2.12), the functionals $F_1:\mathbb{H}(\mathbf{div}_{6/5};\Omega)\to\mathbb{R}$, $F_2:\mathbf{L}^6(\Omega)\to\mathbb{R}$ and $F_3:\mathbf{H}(\mathrm{curl};\Omega)\to\mathbb{R}$, as

$$F_1(\boldsymbol{\tau}) := \langle\boldsymbol{\tau}\mathbf{n},\mathbf{u}_D\rangle,\quad F_2(\mathbf{v}) := -(\mathbf{f},\mathbf{v})_\Omega,\quad F_3(\mathbf{d}) := (\mathbf{g},\mathbf{d})_\Omega,$$

and rewrite problem (2.20) in terms of these forms and functionals as follows: Find $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b},r)\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\times\mathbf{L}^6(\Omega)\times\mathbf{H}_0(\mathrm{curl};\Omega)\times H_0^1(\Omega)$, such that:

$$\begin{aligned}
a_f(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_f(\boldsymbol{\tau},\mathbf{u}) + c_f(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) && \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega),\\
b_f(\boldsymbol{\sigma},\mathbf{v}) + c_d(\mathbf{b};\mathbf{b},\mathbf{v}) &= F_2(\mathbf{v}) && \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega),\\
a_m(\mathbf{b},\mathbf{d}) + b_m(\mathbf{d},r) + c_m(\mathbf{u};\mathbf{b},\mathbf{d}) &= F_3(\mathbf{d}) && \forall\,\mathbf{d}\in\mathbf{H}_0(\mathrm{curl};\Omega),\\
b_m(\mathbf{b},z) &= 0 && \forall\,z\in H_0^1(\Omega).
\end{aligned}\tag{3.2}$$

In turn, recalling that the bilinear form b_m satisfies the inf-sup condition (see [27, Section 2.4] or [21, Section 5.4]):

$$\sup_{\mathbf{0}\neq\mathbf{d}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega)} \frac{b_m(\mathbf{d},z)}{\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}} \geq \beta_m \|z\|_{1,\Omega} \qquad \forall\, z \,\in\, H_0^1(\Omega), \tag{3.3}$$

with $\beta_m > 0$, analogously to [27], it is not difficult to see that (3.2) can be rewritten equivalently (to be proved next in Corollary 3.7) as the following reduced problem: Find $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b}) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)\times\mathbf{C}$, such that:

$$\begin{aligned} a_f(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, b_f(\boldsymbol{\tau},\mathbf{u}) \,+\, c_f(\mathbf{u};\mathbf{u},\boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega),\\ b_f(\boldsymbol{\sigma},\mathbf{v}) \,+\, c_d(\mathbf{b};\mathbf{b},\mathbf{v}) &= F_2(\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega),\\ a_m(\mathbf{b},\mathbf{d}) \,+\, c_m(\mathbf{u};\mathbf{b},\mathbf{d}) &= F_3(\mathbf{d}) \quad \forall\,\mathbf{d}\in\mathbf{C}. \end{aligned} \tag{3.4}$$

Therefore, to prove the well-posedness of (3.2) it suffices to apply our aforementioned fixed-point strategy to the reduced problem (3.4).

3.2 The fixed-point strategy

Here we describe the fixed-point strategy to be employed next to prove the well-posedness of (3.4). This strategy consists firstly in rewriting (3.4) as an equivalent fixed-point equation in terms of an operator $\mathcal{J}$, and secondly in proving existence of a unique fixed point of $\mathcal{J}$ by means of the classical Banach fixed-point theorem. We begin by introducing the associated fixed-point operator. To that end we first introduce the operator $S_m : \mathbf{L}^6(\Omega)\to\mathbf{C}$, as

$$S_m(\mathbf{w}) := \mathbf{b}, \quad \forall\,\mathbf{w}\in\mathbf{L}^6(\Omega),$$

where $\mathbf{b}$ is the element in $\mathbf{C}$ satisfying

$$a_m(\mathbf{b},\mathbf{d}) \,+\, c_m(\mathbf{w};\mathbf{b},\mathbf{d}) \,=\, F_3(\mathbf{d}) \qquad \forall\,\mathbf{d}\in\mathbf{C}. \tag{3.5}$$

In turn, we let $S_f : \mathbf{L}^6(\Omega)\times\mathbf{C}\to\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)$ be the operator defined by

$$S_f(\mathbf{w},\hat{\mathbf{b}}) := (S_{f,1}(\mathbf{w},\hat{\mathbf{b}}), S_{f,2}(\mathbf{w},\hat{\mathbf{b}})) = (\boldsymbol{\sigma},\mathbf{u}) \quad \forall\,(\mathbf{w},\hat{\mathbf{b}})\in\mathbf{L}^6(\Omega)\times\mathbf{C},$$

where $(\boldsymbol{\sigma},\mathbf{u})$ is the pair in $\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)$ satisfying

$$\begin{aligned} a_f(\boldsymbol{\sigma},\boldsymbol{\tau}) \,+\, b_f(\boldsymbol{\tau},\mathbf{u}) \,+\, c_f(\mathbf{w};\mathbf{u},\boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega),\\ b_f(\boldsymbol{\sigma},\mathbf{v}) &= F_{\hat{\mathbf{b}}}(\mathbf{v}) \quad \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega), \end{aligned} \tag{3.6}$$

with

$$F_{\hat{\mathbf{b}}}(\mathbf{v}) := F_2(\mathbf{v}) \,-\, c_d(\hat{\mathbf{b}};\hat{\mathbf{b}},\mathbf{v}), \quad \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega).$$

Then, we define

$$\mathcal{J} : \mathbf{L}^6(\Omega)\to\mathbf{L}^6(\Omega), \quad \mathbf{w}\to\mathcal{J}(\mathbf{w}) := S_{f,2}(\mathbf{w},S_m(\mathbf{w}))$$

and realize that solving (3.4) is equivalent to seeking a fixed-point of $\mathcal{J}$, that is: Find $\mathbf{u} \in \mathbf{L}^6(\Omega)$, such that

$$\mathcal{J}(\mathbf{u}) = \mathbf{u}. \tag{3.7}$$

Therefore, in what follows we focus on proving the existence of a unique fixed-point of $\mathcal{J}$. Before doing that, we need to prove that $\mathcal{J}$ is well defined. To that end, and since $\mathcal{J}$ is defined in terms of S_m and S_f, it suffices to prove that both operators S_m and S_f are well-defined separately.

3.3 Well-definiteness of S_m

It is clear that to prove the well-definiteness of operator S_m it suffices to prove the well-posedness of problem (3.5). In order to do that, we first need to establish the stability properties of the forms and functionals involved. We begin by recalling that the form a_m is continuous:

$$|a_m(\mathbf{b}, \mathbf{d})| \leq \kappa \nu_m \|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} \|\mathbf{d}\|_{\mathrm{curl}\,;\Omega} \quad \forall\, \mathbf{b}, \mathbf{d} \in \mathbf{H}(\mathrm{curl}\,;\Omega). \tag{3.8}$$

In addition, from (2.15) we have

$$|c_m(\mathbf{u}; \mathbf{b}, \mathbf{d})| \;\leq\; k C_s \|\mathbf{u}\|_{\mathbf{L}^6(\Omega)} \|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} \|\mathbf{d}\|_{\mathrm{curl}\,;\Omega} \quad \forall\, \mathbf{u} \in \mathbf{L}^6(\Omega),\, \mathbf{b} \in \mathbf{C},\, \mathbf{d} \in \mathbf{H}(\mathrm{curl}\,;\Omega). \tag{3.9}$$

In turn, recalling that there holds

$$\|\mathrm{curl}\; \mathbf{d}\|_{0,\Omega} \;\geq\; \alpha_m \|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}, \quad \forall\, \mathbf{d} \in \mathbf{C}, \tag{3.10}$$

with $\alpha_m > 0$ only depending of Ω (see [25, Corollary 3.51]), it readily follows that a_m is elliptic on $\mathbf{C}$, that is

$$a_m(\mathbf{d}, \mathbf{d}) \;\geq\; k \nu_m \alpha_m^2 \|\mathbf{d}\|^2_{\mathrm{curl}\,;\Omega} \quad \forall\, \mathbf{d} \in \mathbf{C}. \tag{3.11}$$

Finally, it is clear that the functional F_3 is bounded, that is

$$|F_3(\mathbf{d})| \leq \|\mathbf{g}\|_{0,\Omega} \|\mathbf{d}\|_{\mathrm{curl}\,;\Omega} \quad \forall\, \mathbf{d} \in \mathbf{H}(\mathrm{curl}\,;\Omega) \tag{3.12}$$

Now, we are in position of establishing the well-definiteness of operator S_m or equivalently, the well-posedness of (3.5).

Lemma 3.1 *Let $\mathbf{w} \in \mathbf{L}^6(\Omega)$ be such that*

$$\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \;\leq\; \frac{1}{2} \nu_m C_m, \tag{3.13}$$

with $C_m := \frac{\alpha_m^2}{C_s}$ a positive constant independent of the physical parameters. Then, there exists a unique $\mathbf{b} \in \mathbf{C}$ solution to (3.5). In addition, there holds

$$\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} \;\leq\; \frac{2}{k \nu_m \alpha_m^2} \|\mathbf{g}\|_{0,\Omega}. \tag{3.14}$$

Proof. First, from (3.11) and (3.9) it is clear that for all $\mathbf{d} \in \mathbf{C}$, there holds

$$\begin{aligned} a_m(\mathbf{d}, \mathbf{d}) \,+\, c_m(\mathbf{w}; \mathbf{d}, \mathbf{d}) \;&\geq\; k \nu_m \alpha_m^2 \|\mathbf{d}\|^2_{\mathrm{curl}\,;\Omega} - k C_s \|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \|\mathbf{d}\|^2_{\mathrm{curl}\,;\Omega} \\ &\geq\; k \left(\nu_m \alpha_m^2 - C_s \|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \right) \|\mathbf{d}\|^2_{\mathrm{curl}\,;\Omega}, \end{aligned}$$

which together to hypothesis (3.13), implies

$$a_m(\mathbf{d},\mathbf{d}) + c_m(\mathbf{w};\mathbf{d},\mathbf{d}) \geq \frac{1}{2}k\nu_m\alpha_m^2\|\mathbf{d}\|^2_{\mathrm{curl}\,;\Omega}, \quad \forall\,\mathbf{d}\in\mathbf{C}, \tag{3.15}$$

that is the bilinear form $a_m(\cdot,\cdot) + c_m(\mathbf{w};\cdot,\cdot)$ is elliptic on $\mathbf{C}$. In this way, the well-posedness of (3.5) follows straightforwardly from the Lax-Milgram lemma. In addition, from (3.15) and (3.12) it readily follows that the solution $\mathbf{b}$ satisfies (3.14), which concludes the proof. □

3.4 Well-definiteness of S_f

Similarly as in Section 3.3, to prove the well-definiteness of S_f in what follows we focus on proving the well-posedness of (3.6). In this regard, we point out that most of the results here are omitted and recall only the key properties since the desired result follows straightforwardly from [3, Section 3.2].

We start by noticing that, analogously to [3, Lemma 3.2], that is, proceeding as in [14, Lemma 2.3] and utilizing the Sobolev inequality

$$\|w\|_{L^q(\Omega)} \leq C\|w\|_{1,\Omega}, \quad \forall\, w\in H^1(\Omega), \quad \forall\, q\in[1,6], \tag{3.16}$$

with $q = 6$, it can be proved that

$$C_{f,1}\|\boldsymbol{\tau}\|^2_{0,\Omega} \leq \|\boldsymbol{\tau}^d\|^2_{0,\Omega} + \|\mathbf{div}\,\boldsymbol{\tau}\|_{\mathbf{L}^{6/5}(\Omega)} \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega), \tag{3.17}$$

with $C_{f,1} > 0$ depending only on Ω, which clearly implies

$$a_f(\boldsymbol{\tau},\boldsymbol{\tau}) = \frac{1}{\nu}\|\boldsymbol{\tau}^d\|^2_{0,\Omega} \geq \frac{C_{f,1}}{\nu}\|\boldsymbol{\tau}\|^2_{\mathbf{div}_{6/5}\,;\Omega}, \tag{3.18}$$

for all $\boldsymbol{\tau}\in Ker(b_f) := \{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) : b_f(\boldsymbol{\tau},\mathbf{v}) = 0 \quad \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega)\}$. In addition, it is easy to see that a_f and b_f are continuous forms, that is

$$|a_f(\boldsymbol{\sigma},\boldsymbol{\tau})| \leq \frac{1}{\nu}\|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5}\,;\Omega}\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5}\,;\Omega}, \quad \forall\,\boldsymbol{\sigma},\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega), \tag{3.19}$$

and

$$|b_f(\boldsymbol{\tau},\mathbf{v})| \leq \|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5}\,;\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}, \quad \forall\,\boldsymbol{\sigma}\in\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega), \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega). \tag{3.20}$$

Now we establish the inf-sup condition of b_f.

Lemma 3.2 *There exists $\beta_f > 0$ only depending on Ω, such that*

$$\sup_{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\setminus\{\mathbf{0}\}} \frac{b_f(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5}\,;\Omega}} \geq \beta_f\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \tag{3.21}$$

for all $\mathbf{v}\in\mathbf{L}^6(\Omega)$.

Proof. We proceed similarly to [3, Lemma 3.4] and [4, Lemma 2.1]. In fact, given $\mathbf{v}\in\mathbf{L}^6(\Omega)$, we let $\mathbf{h}(\mathbf{v}) := |\mathbf{v}|^4\mathbf{v}$ and observe that

$$(|\mathbf{h}(\mathbf{v})|^{6/5}, 1)_\Omega = (|\mathbf{v}|^6, 1)_\Omega < +\infty, \tag{3.22}$$

which implies that $\mathbf{h}(\mathbf{v}) \in \mathbf{L}^{6/5}(\Omega)$ and $\|\mathbf{h}(\mathbf{v})\|_{\mathbf{L}^{6/5}} = \|\mathbf{v}\|^5_{\mathbf{L}^6(\Omega)}$. Then, proceeding analogously to [3, Lemma 3.4] we can find $\tilde{\boldsymbol{\tau}} \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)$, such that

$$\mathbf{div}\,\tilde{\boldsymbol{\tau}} = \mathbf{h}(\mathbf{v}) \in \mathbf{L}^{6/5}(\Omega), \qquad (\mathrm{tr}\,(\tilde{\boldsymbol{\tau}}), 1)_\Omega = 0, \tag{3.23}$$

which owing to (3.16), satisfies

$$\|\tilde{\boldsymbol{\tau}}\|_{\mathbf{div}_{6/5}\,;\Omega} \le \beta_f^{-1}\|\mathbf{v}\|^5_{\mathbf{L}^6(\Omega)}, \tag{3.24}$$

with $\beta_f > 0$. In this way, from (3.23), and (3.24), it follows that

$$\sup_{\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\setminus\{\mathbf{0}\}} \frac{b_f(\boldsymbol{\tau},\mathbf{v})}{\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5}\,;\Omega}} \ge \beta_f\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)},$$

which concludes the proof. □

Now, let us define $\mathbf{A} : \big(\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)\big)\times\big(\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)\big)\to\mathbb{R}$, as

$$\mathbf{A}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := a_f(\boldsymbol{\sigma},\boldsymbol{\tau}) + b_f(\boldsymbol{\tau},\mathbf{u}) + b_f(\boldsymbol{\sigma},\mathbf{v}), \tag{3.25}$$

for all $(\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}) \in \mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)$, and recall that thanks to (3.18)-(3.21) and [10, Proposition 2.36], the bilinear form $\mathbf{A}$ satisfies:

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)} \frac{\mathbf{A}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \ge \gamma_f\|(\boldsymbol{\sigma},\mathbf{u})\|\,, \tag{3.26}$$

for all $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)$, with

$$\gamma_f := C_{f,2}\frac{min\{1,\nu\beta_f\}}{\nu\beta_f+1} \tag{3.27}$$

with constants $C_{f,2} > 0$ and $\beta_f > 0$ depending only on Ω. On the other hand, using (2.13) with $q = 4$, we obtain that the form c_f (cf. (3.1)) satisfies

$$|c_f(\mathbf{u};\mathbf{v},\boldsymbol{\tau})| \le \frac{C_{f,3}}{\nu}\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5}\,;\Omega} \quad \forall\,\mathbf{u},\mathbf{v}\in\mathbf{L}^6(\Omega), \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega), \tag{3.28}$$

wit $C_{f,3} > 0$ depending only on Ω. In turn, from (2.14), we have that

$$|c_d(\mathbf{d};\mathbf{b},\mathbf{v})| \le kC_s\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \quad \forall\,\mathbf{d}\in\mathbf{H}(\mathrm{curl}\,;\Omega),\ \mathbf{b}\in\mathbf{C},\ \mathbf{v}\in\mathbf{L}^6(\Omega), \tag{3.29}$$

and in particular, for a fixed $\hat{\mathbf{b}} \in \mathbf{C}$, the latter implies that

$$|F_{\hat{\mathbf{b}}}(\mathbf{v})| \le (\|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + kC_s\|\hat{\mathbf{b}}\|^2_{\mathrm{curl}\,;\Omega})\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \quad \forall\,\mathbf{v}\in\mathbf{L}^6(\Omega). \tag{3.30}$$

In addition, analogously to the proof of [3, Lemma 3.5], considering here the Sobolev inequality (3.16) with $q = 6$, for F_1 we have

$$|F_1(\boldsymbol{\tau})| \le C_\Gamma\|\boldsymbol{\tau}\|_{\mathbf{div}_{6/5}\,;\Omega}\|\mathbf{u}_D\|_{1/2,\Gamma} \quad \forall\,\boldsymbol{\tau}\in\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega). \tag{3.31}$$

Now, in order to establish the well-posedness of (3.6), for a fix $\mathbf{w} \in \mathbf{L}^6(\Omega)$ we define the bilinear form $\mathbf{A}_{\mathbf{w}} : (\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega))\times(\mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega))\to\mathbb{R}$ given by

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) := \mathbf{A}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) + c_f(\mathbf{w};\mathbf{u},\boldsymbol{\tau}). \tag{3.32}$$

For this bilinear form, from (3.26) and (3.28) we obtain the following result. For its proof we refer to [3, Theorem 3.6].

Lemma 3.3 *Let* $\mathbf{w} \in \mathbf{L}^6(\Omega)$ *be such that*

$$\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \leq \frac{\nu\gamma_f}{2C_{f,3}},$$

with γ_f *being the constant defined in* (3.27). *Then, the following inf-sup conditions holds:*

$$\sup_{\mathbf{0}\neq(\boldsymbol{\tau},\mathbf{v})\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\times\mathbf{L}^6(\Omega)} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v}))}{\|(\boldsymbol{\tau},\mathbf{v})\|} \geq \frac{\gamma_f}{2}\|(\boldsymbol{\sigma},\mathbf{u})\|\,, \tag{3.33}$$

for all $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \times \mathbf{L}^6(\Omega)$*, and*

$$\sup_{(\boldsymbol{\sigma},\mathbf{u})\in\mathbb{H}_0(\mathbf{div}_{6/5};\Omega)\times\mathbf{L}^6(\Omega)} \mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) > 0\,,$$

for all $(\boldsymbol{\tau},\mathbf{v}) \in \big(\mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \times \mathbf{L}^6(\Omega)\big)\backslash\{\mathbf{0}\}$.

Now we are in position of establishing the well-posedness of (3.6), or equivalently, the well-definiteness of S_f.

Lemma 3.4 *Let* $\mathbf{w} \in \mathbf{L}^6(\Omega)$ *be such that*

$$\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \leq \frac{\nu\gamma_f}{2C_{f,3}}, \tag{3.34}$$

with γ_f *being the constant defined in* (3.27) *and let* $\hat{\mathbf{b}} \in \mathbf{C}$. *Then, there exists a unique* $(\boldsymbol{\sigma},\mathbf{u}) \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \times \mathbf{L}^6(\Omega)$*, solution to* (3.6). *In addition, there holds*

$$\|(\boldsymbol{\sigma},\mathbf{u})\| \leq \frac{2}{\gamma_f}\left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + kC_s\|\hat{\mathbf{b}}\|^2_{\mathrm{curl};\Omega}\right). \tag{3.35}$$

Proof. Here, we proceed as in the proof of [3, Theorem 3.6] and rewrite problem (3.6) as

$$\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma},\mathbf{u}),(\boldsymbol{\tau},\mathbf{v})) = F_1(\boldsymbol{\tau}) + F_{\hat{b}}(\mathbf{v}).$$

Then, the well-posedness of (3.6) follows straightforwardly from Lemma 3.3 and the well-known Banach-Nečas-Babŭska theorem (cf. [10, Theorem 2.6]). Moreover, from (3.30) and (3.31) we obtain

$$|F_1(\boldsymbol{\tau}) + F_{\hat{b}}(\mathbf{v})| \leq \left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + kC_s\|\hat{\mathbf{b}}\|^2_{\mathrm{curl};\Omega}\right)\|(\boldsymbol{\tau},\mathbf{v})\|,$$

which clearly implies (3.35). □

3.5 Well-posedness of the continuous formulation

Recalling that problem (3.4) is equivalent to the fixed-point problem (3.7), in what follows we apply the classical Banach fixed-point theorem to obtain the desired well-posedness of (3.4). Before doing that, we notice that from Lemma 3.1 we have that for all $\mathbf{w} \in \mathbf{L}^6(\Omega)$, such that $\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \leq \frac{1}{2}\nu_m C_m$, there holds

$$\|S_m(\mathbf{w})\|_{\mathrm{curl};\Omega} \leq \frac{2}{k\nu_m\alpha_m^2}\|\mathbf{g}\|_{0,\Omega}$$

and from Lemma 3.4

$$\|S_{f,2}(\mathbf{w},\hat{\mathbf{b}})\|_{\mathbf{L}^6(\Omega)} \le \|S_f(\mathbf{w},\hat{\mathbf{b}})\| \le \frac{2}{\gamma_f}\left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + kC_s\|\hat{\mathbf{b}}\|^2_{\mathrm{curl}\,;\Omega}\right)$$

for all $(\mathbf{w},\hat{\mathbf{b}}) \in \mathbf{L}^6(\Omega)\times\mathbf{H}_0(\mathrm{curl}\,;\Omega)$, with $\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \le \frac{\nu\gamma_f}{2C_{f,3}}$. Hence, recalling that $\mathcal{J}(\mathbf{w}) = S_{f,2}(\mathbf{w}, S_m(\mathbf{w}))$, from the inequalities above, we deduce

$$\begin{aligned}\|\mathcal{J}(\mathbf{w})\|_{\mathbf{L}^6(\Omega)} &= \|S_{f,2}(\mathbf{w},S_m(\mathbf{w}))\|_{\mathbf{L}^6(\Omega)}\\ &\le \frac{2}{\gamma_f}\left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + kC_s\|S_m(\mathbf{w})\|^2_{\mathrm{curl}\,;\Omega}\right)\\ &\le \frac{2}{\gamma_f}\left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \frac{M_1}{\kappa\nu_m^2}\|\mathbf{g}\|^2_{0,\Omega}\right),\end{aligned}$$

with $M_1 = \frac{4C_s}{\alpha_m^4}$ independent of the physical parameters. Therefore, if we define the ball

$$\mathbf{K} := \left\{\mathbf{v}\in\mathbf{L}^6(\Omega) \,:\, \|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \,\le\, \mu(\mathbf{u}_D,\mathbf{f},\mathbf{g})\right\}, \tag{3.36}$$

where

$$\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g}) := \frac{2}{\gamma_f}\left(C_\Gamma\|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \frac{M_1}{\kappa\nu_m^2}\|\mathbf{g}\|^2_{0,\Omega}\right), \tag{3.37}$$

and redefine $\mathcal{J}$ on $\mathbf{K}$, that is $\mathcal{J}:\mathbf{K}\to\mathbf{K}$, we obtain that $\mathcal{J}$ is well-defined provided

$$\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g}) \,\le\, \frac{1}{2}\min\left\{\frac{\nu\gamma_f}{C_{f,3}}, \nu_m C_m\right\}. \tag{3.38}$$

This result is established next.

Lemma 3.5 *Assume that the data* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_D$ *satisfy* (3.38)*. Then, given* $\mathbf{w}\in\mathbf{K}$*, there exists a unique* $\mathbf{u}\in\mathbf{K}$ *such that* $\mathcal{J}(\mathbf{w}) = \mathbf{u}$*.*

Proof. If $\mathbf{w}$ belongs to $\mathbf{K}$ and the data satisfies (3.38), it is clear that the hypotheses of lemmas 3.1 and 3.4 hold. Then, the result follows straightforwardly from the well-definiteness of S_m and S_f. □

Now we turn to prove the main result if this section, namely, existence and uniqueness of solution of problem (3.4).

Theorem 3.6 *Assume that the data* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_D$ *satisfy* (3.38)*. Assume further that*

$$\frac{2}{\gamma_f}\left(\frac{M_2}{\kappa\nu_m^3}\|\mathbf{g}\|^2_{0,\Omega} + \frac{C_{f,3}}{\nu}\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g})\right) \,<\, 1, \tag{3.39}$$

with $M_2 := \alpha_m^2 M_1^2 > 0$*, independent of the physical parameters. Then, there exists a unique* $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b})\in\mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega)\times\mathbf{L}^6(\Omega)\times\mathbf{C}$ *solution to* (3.4) *. In addition, there hold*

$$\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)} + \|\boldsymbol{\sigma}\|_{\mathbf{div}_{6/5}\,;\Omega} \,\le\, \mu(\mathbf{u}_D,\mathbf{f},\mathbf{g}), \tag{3.40}$$

and

$$\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} \,\le\, \frac{2}{k\nu_m\alpha_m^2}\|\mathbf{g}\|_{0,\Omega}. \tag{3.41}$$

Proof. Let $\mathbf{w}_1$, $\mathbf{w}_2$, $\mathbf{u}_1$, $\mathbf{u}_2 \in \mathbf{K}$, be such that $\mathbf{u}_1 = \mathcal{J}(\mathbf{w}_1)$ and $\mathbf{u}_2 = \mathcal{J}(\mathbf{w}_2)$. From definition of $\mathcal{J}$ it follows that there exist unique $(\boldsymbol{\sigma}_1, \mathbf{u}_1, \mathbf{b}_1) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{C}$ and $(\boldsymbol{\sigma}_2, \mathbf{u}_2, \mathbf{b}_2) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{C}$, satisfying

$$\begin{aligned} a_f(\boldsymbol{\sigma}_1, \boldsymbol{\tau}) + b_f(\boldsymbol{\tau}, \mathbf{u}_1) + c_f(\mathbf{w}_1; \mathbf{u}_1, \boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega), \\ b_f(\boldsymbol{\sigma}_1, \mathbf{v}) &= F_{\mathbf{b}_1}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{L}^6(\Omega), \\ a_m(\mathbf{b}_1, \mathbf{d}) + c_m(\mathbf{w}_1; \mathbf{b}_1, \mathbf{d}) &= F_3(\mathbf{d}) \quad \forall\, \mathbf{d} \in \mathbf{C}, \end{aligned}$$

and

$$\begin{aligned} a_f(\boldsymbol{\sigma}_2, \boldsymbol{\tau}) + b_f(\boldsymbol{\tau}, \mathbf{u}_2) + c_f(\mathbf{w}_2; \mathbf{u}_2, \boldsymbol{\tau}) &= F_1(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega), \\ b_f(\boldsymbol{\sigma}_2, \mathbf{v}) &= F_{\mathbf{b}_2}(\mathbf{v}) \quad \forall\, \mathbf{v} \in \mathbf{L}^6(\Omega), \\ a_m(\mathbf{b}_2, \mathbf{d}) + c_m(\mathbf{w}_2; \mathbf{b}_2, \mathbf{d}) &= F_3(\mathbf{d}) \quad \forall\, \mathbf{d} \in \mathbf{C}. \end{aligned}$$

Then, subtracting both systems, and adding and subtracting suitable terms, we easily arrive at

$$\begin{aligned} a_f(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \boldsymbol{\tau}) + b_f(\boldsymbol{\tau}, \mathbf{u}_1 - \mathbf{u}_2) + c_f(\mathbf{w}_1; \mathbf{u}_1 - \mathbf{u}_2, \boldsymbol{\tau}) &= -c_f(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{u}_2, \boldsymbol{\tau}), \\ b_f(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{v}) + c_d(\mathbf{b}_1; \mathbf{b}_1 - \mathbf{b}_2, \mathbf{v}) &= -c_d(\mathbf{b}_1 - \mathbf{b}_2; \mathbf{b}_2, \mathbf{v}), \\ a_m(\mathbf{b}_1 - \mathbf{b}_2, \mathbf{d}) + c_m(\mathbf{w}_1; \mathbf{b}_1 - \mathbf{b}_2, \mathbf{d}) &= -c_m(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{b}_2, \mathbf{d}), \end{aligned} \tag{3.42}$$

In turn, since $\mathbf{w}_1$ belongs to $\mathbf{K}$, we have that the hypothesis of Lemma 3.1 holds and then the bilinear form $a_m(\cdot,\cdot) + c_m(\mathbf{w}_1;\cdot,\cdot)$ satisfies (3.15). Then, from the third equation of (3.42), and using the continuity of c_m (cf. (3.9)), we obtain

$$\begin{aligned} \tfrac{1}{2} k\nu_m \alpha_m^2 \|\mathbf{b}_1 - \mathbf{b}_2\|^2_{\mathrm{curl}\,;\Omega} &\le a_m(\mathbf{b}_1 - \mathbf{b}_2, \mathbf{b}_1 - \mathbf{b}_2) + c_m(\mathbf{w}_1; \mathbf{b}_1 - \mathbf{b}_2, \mathbf{b}_1 - \mathbf{b}_2) \\ &\le k C_s \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^6(\Omega)} \|\mathbf{b}_2\|_{\mathrm{curl}\,;\Omega} \|\mathbf{b}_1 - \mathbf{b}_2\|_{\mathrm{curl}\,;\Omega} \end{aligned}$$

which together to the fact that $\mathbf{b}_2 = S_m(\mathbf{w}_2)$ satisfies (3.14), implies

$$\|\mathbf{b}_1 - \mathbf{b}_2\|_{\mathrm{curl}\,;\Omega} \le \frac{M_1}{k\nu_m^2} \|\mathbf{g}\|_{0,\Omega} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^6(\Omega)}. \tag{3.43}$$

On the other hand, from the first and second equations of (3.42) and from the definition of the bilinear form $\mathbf{A}_{\mathbf{w}}$ (cf. (3.32)) we have

$$\begin{aligned} \mathbf{A}_{\mathbf{w}_1}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v})) = \; & -c_d(\mathbf{b}_1; \mathbf{b}_1 - \mathbf{b}_2, \mathbf{v}) - c_d(\mathbf{b}_1 - \mathbf{b}_2; \mathbf{b}_2, \mathbf{v}) \\ & -c_f(\mathbf{w}_1 - \mathbf{w}_2; \mathbf{u}_2, \boldsymbol{\tau}). \end{aligned}$$

Therefore, since $\mathbf{w}_1$ satisfies (3.34), from the latter identity, and from estimates (3.28), (3.29) and (3.33), it follows that

$$\begin{aligned} \frac{\gamma_f}{2} \|(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2)\| &\le \sup_{\mathbf{0} \neq (\boldsymbol{\tau}, \mathbf{v}) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega)} \frac{\mathbf{A}_{\mathbf{w}}((\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2, \mathbf{u}_1 - \mathbf{u}_2), (\boldsymbol{\tau}, \mathbf{v}))}{\|(\boldsymbol{\tau}, \mathbf{v})\|} \\ &\le \kappa C_s \|\mathbf{b}_1 - \mathbf{b}_2\|_{\mathrm{curl}\,;\Omega} \left(\|\mathbf{b}_1\|_{\mathrm{curl}\,;\Omega} + \|\mathbf{b}_2\|_{\mathrm{curl}\,;\Omega} \right) \\ &\quad + \frac{C_{f,3}}{\nu} \|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^6(\Omega)} \|\mathbf{u}_2\|_{\mathbf{L}^6(\Omega)}. \end{aligned}$$

Then, recalling that $\mathbf{b}_1$ and $\mathbf{b}_2$ satisfy (3.14) and $\mathbf{u}_2 \in \mathbf{K}$ (cf. (3.36)), using (3.43) from the latter it follows that

$$\|\mathbf{u}_1 - \mathbf{u}_2\|_{\mathbf{L}^6(\Omega)} \,\leq\, \frac{2}{\gamma_f}\left(\frac{M_2}{\kappa\nu_m^3}\|\mathbf{g}\|_{0,\Omega}^2 + \frac{C_{f,3}}{\nu}\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g})\right)\|\mathbf{w}_1 - \mathbf{w}_2\|_{\mathbf{L}^6(\Omega)}, \tag{3.44}$$

with $M_2 := \alpha_m^2 M_1^2 > 0$ independent of the physical parameters. In this way, from assumption (3.39) we clearly obtain that $\mathcal{J}$ is a contraction mapping.

Finally, if $\mathbf{u} \in \mathbf{K}$ is the unique fixed-point of $\mathcal{J}$, it is clear that $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{C}$ is the unique solution of (3.4) with $\mathbf{b} = S_m(\mathbf{u})$ and $(\boldsymbol{\sigma}, \mathbf{u}) = S_f(\mathbf{u}, S_m(\mathbf{u}))$. Then, estimates (3.40) and (3.41) follow from (3.35) and (3.14), respectively, which concludes the proof. □

We end this section by establishing the well-posedness of (3.2).

Corollary 3.7 *Let $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ and $\mathbf{g} \in \mathbf{L}^2(\Omega)$ such that* (3.38) *and* (3.39) *hold. Then, there exists a unique $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}, r) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{H}_0(\mathrm{curl}\,;\Omega) \times H_0^1(\Omega)$ solution to* (3.2)*. In addition, $(\boldsymbol{\sigma}, \mathbf{u})$ and $\mathbf{b}$ satisfy* (3.40) *and* (3.41)*, respectively, and for r there holds*

$$\|r\|_{1,\Omega} \leq \frac{1}{\beta_m}\left(\frac{2}{\alpha_m^2} + 1\right)\|\mathbf{g}\|_{0,\Omega} + \frac{\kappa C_s}{\beta_m}\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g}), \tag{3.45}$$

with $\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g})$ being the constant defined in (3.37)

Proof. We begin by proving the equivalence between (3.2) and (3.4). In fact, we first notice that if $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}, r) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{H}_0(\mathrm{curl}\,;\Omega) \times H_0^1(\Omega)$ is the unique solution of (3.2), then clearly $\mathbf{b} \in \mathbf{C}$ and $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ satisfies (3.4). On the other hand, let $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b})$ be the unique solution of (3.4) and let $\mathcal{F} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega)$ be the unique element in $\mathbf{H}_0(\mathrm{curl}\,;\Omega)$ (guaranteed by the Riesz representation theorem), such that

$$\langle \mathcal{F}, \mathbf{d}\rangle = F_3(\mathbf{d}) - a_m(\mathbf{b},\mathbf{d}) - c_m(\mathbf{u};\mathbf{b},\mathbf{d}), \quad \forall\,\mathbf{d} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega),$$

with $\langle\cdot,\cdot\rangle$ being the inner product of $\mathbf{H}_0(\mathrm{curl}\,;\Omega)$. From the third equation of (3.4) it is clear that $\langle \mathcal{F}, \mathbf{d}\rangle = 0$ for all $\mathbf{d} \in \mathbf{C}$, that is $\mathcal{F} \in \mathbf{C}^{\perp}$. Then, owing to the inf-sup condition (3.3), and according to [14, Lemma 2.1-ii], we deduce that there exists a unique $r \in H_0^1(\Omega)$, such that

$$b_m(\mathbf{d}, r) = \langle \mathcal{F}, \mathbf{d}\rangle = F_3(\mathbf{d}) - a_m(\mathbf{b},\mathbf{d}) - c_m(\mathbf{u};\mathbf{b},\mathbf{d}), \quad \forall\,\mathbf{d} \in \mathbf{H}_0(\mathrm{curl}\,;\Omega),$$

which implies that $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}, r)$ is the unique solution of (3.2). Therefore, since both problems (3.2) and (3.4) are equivalent, the well-posedness of (3.4) follows from Theorem 3.6.

Finally, for the estimate (3.45), using again the inf-sup condition (3.3), and employing estimates (3.8), (3.9) and (3.12), we obtain that

$$\begin{aligned}\beta_m\|r\|_{1,\Omega} \leq& \sup_{\mathbf{0}\neq\mathbf{d}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega)} \frac{|b_m(\mathbf{d}, r)|}{\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}} = \sup_{\mathbf{0}\neq\mathbf{d}\in\mathbf{H}_0(\mathrm{curl}\,;\Omega)} \frac{|F_3(\mathbf{d}) - a_m(\mathbf{b},\mathbf{d}) - c_m(\mathbf{u};\mathbf{b},\mathbf{d})|}{\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}} \\ \leq& \;\kappa\nu_m\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} + \kappa C_s\|\mathbf{u}\|_{\mathbf{L}^6(\Omega)} + \|\mathbf{g}\|_{0,\Omega},\end{aligned}$$

which together to (3.40) and (3.41) implies (3.45) and concludes the proof. □

4 Galerkin scheme

In this section we introduce the Galerkin scheme associated to problem (2.20) and study its solvability and convergence. We mention in advance that, as we shall see in the forthcoming subsections, the well-posedness analysis follows straightforwardly by adapting the results derived for the continuous problem to the discrete case, reason why most of the details are omitted.

4.1 The discrete problem

Let $\mathcal{T}_h$ be a regular family of triangulations of the polyhedral region $\overline{\Omega}$ made up of tetrahedra T in $\mathbb{R}^3$ of diameter h_T such that $\overline{\Omega} = \cup\{T : T \in \mathcal{T}_h\}$ and define $h := \max\{h_T : T \in \mathcal{T}_h\}$. Given an integer $l \geq 0$ and a subset S of $\mathbb{R}^3$, we denote by $P_l(S)$ the space of polynomials of total degree at most l defined on S, $\tilde{P}_l(S)$ the space of homogeneous polynomials of degree exactly k on S and $\mathbf{M}_l(S)$ the space of polynomials $\mathbf{p}$ in $\tilde{\mathbf{P}}_l(S)$ satisfying $\mathbf{p}(\mathbf{x}) \cdot \mathbf{x} = 0$ on S, where $\mathbf{x} := (x_1, x_2, x_3)^t$ is a generic vector of $\mathbb{R}^3$. Hence, for each integer $k \geq 0$ and for each $T \in \mathcal{T}_h$, we define the local Raviart–Thomas and Nédélec elements of order k (see for instance [2] and [25]), respectively by

$$\mathbf{RT}_k(T) := \mathbf{P}_k(T) \,\oplus\, \tilde{P}_k(T)\mathbf{x}\,, \qquad \text{and} \qquad \mathbf{N}_k(T) := \mathbf{P}_k(T) \oplus \mathbf{M}_{k+1}(T).$$

Then, for $k \geq 0$ we define the discrete spaces:

$$\begin{aligned}
\mathbb{H}_h &:= \left\{\boldsymbol{\tau}_h \in \mathbb{H}(\mathbf{div}\,;\Omega) \,:\, \mathbf{c}^t\boldsymbol{\tau}_h|_T \in \mathbf{RT}_k(T), \quad \forall\,\mathbf{c} \in \mathbb{R}^n \quad \forall\,T \in \mathcal{T}_h\right\}\,,\\
\mathbf{V}_h &:= \left\{\mathbf{v}_h \in \mathbf{L}^6(\Omega) \,:\, \mathbf{v}_h|_T \in \mathbf{P}_k(T), \quad \forall\,T \in \mathcal{T}_h\right\}\,,\\
\mathbf{D}_h &:= \{\mathbf{d}_h \in \mathbf{H}_0(\mathrm{curl}\,;\Omega) \,:\, \mathbf{d}_h|_T \in \mathbf{N}_k(T), \quad \forall\,T \in \mathcal{T}_h\}\,,\\
\mathrm{S}_h &:= \left\{z_h \in H^1_0(\Omega) \,:\, z_h|_T \in P_{k+1}(T), \quad \forall\,T \in \mathcal{T}_h\right\}\,.
\end{aligned}$$

Notice that $\mathbb{H}_h \subseteq \mathbb{H}(\mathbf{div}_{6/5}\,;\Omega)$. In this way, defining

$$\mathbb{H}_{h,0} := \left\{\boldsymbol{\tau}_h \in \mathbb{H}_h : \int_\Omega \mathrm{tr}\,(\boldsymbol{\tau}_h) = 0\right\} \subseteq \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega),$$

the Galerkin scheme associated to (3.2) reads: Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h, r_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{D}_h \times \mathrm{S}_h$, such that:

$$\begin{aligned}
a_f(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b_f(\boldsymbol{\tau}_h, \mathbf{u}_h) + c_f(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= F_1(\boldsymbol{\tau}_h) \quad \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_{h,0},\\
b_f(\boldsymbol{\sigma}_h, \mathbf{v}_h) + c_d(\mathbf{b}_h; \mathbf{b}_h, \mathbf{v}_h) &= F_2(\mathbf{v}_h) \quad \forall\,\mathbf{v}_h \in \mathbf{V}_h,\\
a_m(\mathbf{b}_h, \mathbf{d}_h) + b_m(\mathbf{d}_h, r_h) + c_m(\mathbf{u}_h; \mathbf{b}_h, \mathbf{d}_h) &= F_3(\mathbf{d}_h) \quad \forall\,\mathbf{d}_h \in \mathbf{D}_h,\\
b_m(\mathbf{b}_h, z_h) &= 0 \qquad\quad\; \forall\,z_h \in \mathrm{S}_h.
\end{aligned} \tag{4.1}$$

In turn, similarly to the continuous case, from [21, Section 5.4] we recall that the bilinear form b_m satisfies the discrete inf-sup condition:

$$\sup_{\mathbf{0} \neq \mathbf{d}_h \in \mathbf{D}_h} \frac{b_m(\mathbf{d}_h, z_h)}{\|\mathbf{d}_h\|_{\mathrm{curl}\,;\Omega}} \geq \beta_m \|z_h\|_{1,\Omega} \qquad \forall\,z_h \,\in\, \mathrm{S}_h, \tag{4.2}$$

with $\beta_m > 0$ being the same constant satisfying (3.3), which certainly is independent of h. Then we define the discrete version of $\mathbf{C}$, namely

$$\mathbf{C}_h := \{\mathbf{d}_h \in \mathbf{D}_h \,:\, (\mathbf{d}_h, \nabla z_h)_\Omega \,=\, 0, \quad \forall z_h \in \mathrm{S}_h\}\,, \tag{4.3}$$

and introduce the discrete version of (3.4): Find $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{C}_h$, such that:

$$\begin{aligned} a_f(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, b_f(\boldsymbol{\tau}_h, \mathbf{u}_h) \,+\, c_f(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= F_1(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}, \\ b_f(\boldsymbol{\sigma}_h, \mathbf{v}_h) \,+\, c_d(\mathbf{b}_h; \mathbf{b}_h, \mathbf{v}_h) &= F_2(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\ a_m(\mathbf{b}_h, \mathbf{d}_h) \,+\, c_m(\mathbf{u}_h; \mathbf{b}_h, \mathbf{d}_h) &= F_3(\mathbf{d}_h) \quad \forall\, \mathbf{d}_h \in \mathbf{C}_h. \end{aligned} \tag{4.4}$$

It is not difficult to see that, owing to the discrete inf-sup condition (4.2), problems (4.1) and (4.4) are equivalent. Then analogously to the continuous case, in what follows we focus on analyzing (4.4) through a fixed-point strategy. To that end we introduce the operator $S_{m,h} : \mathbf{V}_h \to \mathbf{C}_h$, as

$$S_{m,h}(\mathbf{w}_h) := \mathbf{b}_h, \quad \forall\, \mathbf{w}_h \in \mathbf{V}_h,$$

where $\mathbf{b}_h$ is the unique element in $\mathbf{C}_h$ satisfying

$$a_m(\mathbf{b}_h, \mathbf{d}_h) \,+\, c_m(\mathbf{w}_h; \mathbf{b}_h, \mathbf{d}_h) \,=\, F_3(\mathbf{d}_h) \qquad \forall\, \mathbf{d}_h \in \mathbf{C}_h. \tag{4.5}$$

In addition, we let $S_{f,h} : \mathbf{V}_h \times \mathbf{C}_h \to \mathbb{H}_{h,0} \times \mathbf{V}_h$ be the operator defined by

$$S_{f,h}(\mathbf{w}_h, \hat{\mathbf{b}}_h) := (S_{f,h,1}(\mathbf{w}_h, \hat{\mathbf{b}}_h), S_{f,h,2}(\mathbf{w}_h, \hat{\mathbf{b}}_h)) = (\boldsymbol{\sigma}_h, \mathbf{u}_h) \quad \forall\, (\mathbf{w}_h, \hat{\mathbf{b}}_h) \in \mathbf{V}_h \times \mathbf{C}_h,$$

where $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ is the pair in $\mathbb{H}_{h,0} \times \mathbf{V}_h$ satisfying

$$\begin{aligned} a_f(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) \,+\, b_f(\boldsymbol{\tau}_h, \mathbf{u}_h) \,+\, c_f(\mathbf{w}_h; \mathbf{u}_h, \boldsymbol{\tau}_h) &= F_1(\boldsymbol{\tau}_h) \qquad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}, \\ b_f(\boldsymbol{\sigma}_h, \mathbf{v}_h) &= F_{\hat{\mathbf{b}}_h}(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h\,, \end{aligned} \tag{4.6}$$

with

$$F_{\hat{\mathbf{b}}_h}(\mathbf{v}_h) := F_2(\mathbf{v}_h) \,-\, c_d(\hat{\mathbf{b}}_h; \hat{\mathbf{b}}_h, \mathbf{v}_h) \qquad \forall\, \mathbf{v}_h \in \mathbf{V}_h. \tag{4.7}$$

Then, the discrete fixed-point operator associated to problem (4.4) is given by

$$\mathcal{J}_h : \mathbf{V}_h \to \mathbf{V}_h, \quad \mathbf{w}_h \to \mathcal{J}_h(\mathbf{w}_h) := S_{f,h,2}(\mathbf{w}_h, S_{m,h}(\mathbf{w}_h)).$$

According to the above, to prove the well-posedness of (4.4), in what follows we prove the existence of a unique $\mathbf{u}_h \in \mathbf{V}_h$, such that

$$\mathcal{J}_h(\mathbf{u}_h) = \mathbf{u}_h. \tag{4.8}$$

In the next section we address the solvability analysis of problems (4.5) and (4.6), thus confirming that $S_{m,h}$ and $S_{f,h}$, and hence $\mathcal{J}_h$, are well defined, and prove the existence of a unique $\mathbf{u}_h \in \mathbf{V}_h$ satisfying (4.8) by means of the Banach fixed-point theorem and a suitable smallness assumption on the data.

4.2 Solvability analysis

We begin by establishing the stability properties of the forms involved on the discrete spaces. First, we observe that since our Galerkin scheme is based on the utilization of conforming discrete spaces, it is clear that (3.8), (3.12), (3.19), (3.20), (3.28) and (3.31) hold on the corresponding discrete spaces with the exact same constants. On the other hand, on $\mathbf{C}_h$ (cf. (4.3)) it is known that the following estimate holds

$$\|\mathrm{curl}\ \mathbf{d}_h\|_{0,\Omega} \ \geq\ \hat{\alpha}_m \|\mathbf{d}_h\|_{\mathrm{curl}\,;\Omega}, \quad \forall\, \mathbf{d}_h \in \mathbf{C}_h,$$

with $\hat{\alpha}_m > 0$, independent of the mesh-size h (cf. [21, Theorem 4.7]). This estimate implies the ellipticity of a_m on $\mathbf{C}_h$, that is

$$a_m(\mathbf{d}_h, \mathbf{d}_h) \ \geq\ k\nu_m \hat{\alpha}_m^2 \|\mathbf{d}_h\|^2_{\mathrm{curl}\,;\Omega}, \quad \forall\, \mathbf{d}_h \in \mathbf{C}_h.$$

Next, to provide the discrete versions of (3.9) and (3.29) we notice that the discrete kernel of b_m, namely $\mathbf{C}_h$ (cf. (4.3)), is not included in its continuous counterpart $\mathbf{C}$, and consequently, we cannot employ the embedding $\mathbf{C} \subseteq \mathbf{H}^s(\Omega)$ for some $s > 1/2$. In order to overcome this drawback, as we shall see in the following lemma, from now on we need to assume that the mesh is quasi-uniform.

Lemma 4.1 *Assume that $\mathcal{T}_h$ is a family of quasi-uniform triangulations. Then, there exist positive constants $\widehat{C}_s^m$ and $\widehat{C}_s^d$, independent of h and the physical parameters, such that*

$$|c_m(\mathbf{w}; \mathbf{b}, \mathbf{d})| \ \leq\ k\widehat{C}_s^m \|\mathbf{w}\|_{\mathbf{L}^6(\Omega)} \|\mathbf{b}\|_{\mathrm{curl}\,;\Omega} \|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}\,, \tag{4.9}$$

for all $(\mathbf{w}; \mathbf{b}, \mathbf{d}) \in \mathbf{V}_h \times (\mathbf{C} + \mathbf{C}_h) \times \mathbf{H}_0(\mathrm{curl}\,;\Omega)$ *and*

$$|c_d(\mathbf{h}; \mathbf{c}, \mathbf{v})| \ \leq\ k\widehat{C}_s^d \|\mathbf{h}\|_{\mathrm{curl}\,;\Omega} \|\mathbf{c}\|_{\mathrm{curl}\,;\Omega} \|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}\,, \tag{4.10}$$

for all $(\mathbf{h}; \mathbf{c}, \mathbf{v}) \in \mathbf{D}_h \times (\mathbf{C} + \mathbf{C}_h) \times \mathbf{L}^6(\Omega)$.

Proof. Let $(\mathbf{w}; \mathbf{b}, \mathbf{d}) \in \mathbf{V}_h \times \mathbf{C}_h \times \mathbf{H}_0(\mathrm{curl}\,;\Omega)$ and $(\mathbf{h}; \mathbf{c}, \mathbf{v}) \in \mathbf{D}_h \times \mathbf{C}_h \times \mathbf{L}^6(\Omega)$ (notice that (4.9) and (4.10) are direct consequences of (3.9) and (3.29), respectively, when $\mathbf{b}$, $\mathbf{c} \in \mathbf{C}$, since $\mathbf{V}_h \subseteq \mathbf{L}^6(\Omega)$ and $\mathbf{D}_h \subseteq \mathbf{H}_0(\mathrm{curl}\,;\Omega)$). In the sequel we proceed similarly to the proof of [27, Proposition 3.2]. To that end, we let $\mathbf{T} : \mathbf{C}_h \to \mathbf{C}$ be a linear operator such that (see [21, Section 4])

$$\mathrm{curl}\,(\mathbf{d}) = \mathrm{curl}\,(\mathbf{T}(\mathbf{d})) \quad \forall\, \mathbf{d} \in \mathbf{C}_h, \tag{4.11}$$

satisfying

$$\|\mathbf{d} - \mathbf{T}(\mathbf{d})\|_{0,\Omega} \leq C h^s \|\mathrm{curl}\,\mathbf{d}\|_{0,\Omega} \quad \forall\, \mathbf{d} \in \mathbf{C}_h, \tag{4.12}$$

where $s > 1/2$ is the parameter satisfying $\mathbf{C} \subseteq \mathbf{H}^s(\Omega)$ (see [21, Lemma 4.5]). On the other hand, let us recall that, owing to the quasi-uniformity of the mesh, the following inverse inequality holds (see [8, Theorem 3.2.6])

$$\|\xi\|_{L^q(\Omega)} \leq C h^{3(\frac{1}{q} - \frac{1}{p})} \|\xi\|_{L^p(\Omega)}, \quad 1 \leq p \leq q \leq \infty, \tag{4.13}$$

for all piecewise polynomial functions ξ and $C > 0$ independent of h.

For (4.9) we add and subtract $\mathbf{T}(\mathbf{b})$ in the second component of c_m, utilize the triangle inequality, the identity (4.11) and estimates (3.9) and (3.10), to obtain

$$|c_m(\mathbf{w};\mathbf{b},\mathbf{d})| \leq |c_m(\mathbf{w};\mathbf{b}-\mathbf{T}(\mathbf{b}),\mathbf{d})| + C_1\kappa\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)}\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}, \tag{4.14}$$

with $C_1 > 0$ independent of h and the physical parameters. To bound the remaining term $|c_m(\mathbf{w};\mathbf{b}-\mathbf{T}(\mathbf{b}),\mathbf{d})|$ we apply Hölder's inequality, estimate (4.12) and the inverse inequality (4.13) with $q=\infty$ and $p=6$, to obtain

$$\begin{aligned} |c_m(\mathbf{w};\mathbf{b}-\mathbf{T}(\mathbf{b}),\mathbf{d})| \leq\;& \kappa\|\mathbf{w}\|_{\mathbf{L}^\infty(\Omega)}\|\mathbf{b}-\mathbf{T}(\mathbf{b})\|_{0,\Omega}\|\mathrm{curl}\,\mathbf{d}\|_{0,\Omega} \\ \leq\;& \kappa C_2 h^{s-1/2}\|\mathbf{w}\|_{\mathbf{L}^6(\Omega)}\|\mathrm{curl}\,\mathbf{b}\|_{0,\Omega}\|\mathbf{d}\|_{\mathrm{curl}\,;\Omega}, \end{aligned} \tag{4.15}$$

with $C_2 > 0$ independent of h and the physical parameters. In this way, from (4.14) and (4.15) and since $s > 1/2$ we clearly obtain (4.9).

Now, for c_d we proceed similarly as for c_m, that is we add and subtract $\mathbf{T}(\mathbf{c})$ in the second component of c_d and apply the triangle inequality, the identity (4.11) and estimates (3.29) and (3.10) to get

$$\begin{aligned} |c_d(\mathbf{h};\mathbf{c},\mathbf{v})| \leq\;& |c_d(\mathbf{h};\mathbf{c}-\mathbf{T}(\mathbf{c}),\mathbf{v})| + |c_d(\mathbf{h};\mathbf{T}(\mathbf{c}),\mathbf{v})| \\ \leq\;& |c_d(\mathbf{h};\mathbf{c}-\mathbf{T}(\mathbf{c}),\mathbf{v})| + C_3\kappa\|\mathbf{h}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{c}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}. \end{aligned} \tag{4.16}$$

Then, by applying again Hölder's inequality, estimate (4.12) and the inverse inequality (4.13) with $q=3$ and $p=2$ to the term $|c_d(\mathbf{h};\mathbf{c}-\mathbf{T}(\mathbf{c}),\mathbf{v})|$ we have

$$\begin{aligned} |c_d(\mathbf{h};\mathbf{c}-\mathbf{T}(\mathbf{c}),\mathbf{v})| \leq\;& \kappa\|\mathrm{curl}\,\mathbf{h}\|_{\mathbf{L}^3(\Omega)}\|\mathbf{c}-\mathbf{T}(\mathbf{c})\|_{0,\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)} \\ \leq\;& \kappa C_4 h^{s-1/2}\|\mathbf{h}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{c}\|_{\mathrm{curl}\,;\Omega}\|\mathbf{v}\|_{\mathbf{L}^6(\Omega)}, \end{aligned}$$

which together with (4.16) and the fact that $s > 1/2$, imply (4.10) which concludes the proof. □

The following result establishes the well-definiteness of $S_{m,h}$.

Lemma 4.2 *Let $\mathbf{w}_h \in \mathbf{V}_h$ be such that*

$$\|\mathbf{w}_h\|_{\mathbf{L}^6(\Omega)} \leq \frac{1}{2}\nu_m\widehat{C}_m, \tag{4.17}$$

with $\widehat{C}_m := \frac{\hat{\alpha}_m^2}{\widehat{C}_s^m}$ a positive constant independent of mesh-size h and the physical parameters. Then, there exists a unique $\mathbf{b}_h \in \mathbf{C}_h$ solution to (4.5), thus operator $S_{m,h}$ is well-defined. In addition, there holds

$$\|\mathbf{b}_h\|_{\mathrm{curl}\,;\Omega} \leq \frac{2}{k\nu_m\hat{\alpha}_m^2}\|\mathbf{g}\|_{0,\Omega}.$$

Proof. Analogously to Lemma 3.1 we observe that estimate (4.17) implies that $a_m(\cdot,\cdot) + c_m(\mathbf{w}_h;\cdot,\cdot)$ is elliptic on $\mathbf{C}_h$, that is

$$a_m(\mathbf{d}_h,\mathbf{d}_h) + c_m(\mathbf{w}_h;\mathbf{d}_h,\mathbf{d}_h) \geq \frac{1}{2}k\nu_m\hat{\alpha}_m^2\|\mathbf{d}_h\|^2_{\mathrm{curl}\,;\Omega} \quad \forall\,\mathbf{d}_h \in \mathbf{C}_h. \tag{4.18}$$

Then, the result follows from the Lax-Milgram lemma. □

Now we address the well-definiteness of $S_{f,h}$ by adapting the results from Section 3.4 to the discrete case. As for the continuous case, these results follow straightforwardly from [3, Section 4.2].

We start by noticing that, since $\mathbf{div}\, \mathbb{H}_h \subseteq \mathbf{V}_h$, the discrete kernel of b_f can be characterized as follows

$$Ker_h(b_f) \,=\, \{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} : \mathbf{div}\, \boldsymbol{\tau}_h = 0 \quad \text{in} \quad \Omega\},$$

and then, owing to (3.17), there holds

$$a_f(\boldsymbol{\tau}_h, \boldsymbol{\tau}_h) \geq \frac{C_{f,1}}{\nu} \|\boldsymbol{\tau}_h\|^2_{\mathbf{div}_{6/5}\,;\Omega} \qquad \forall\, \boldsymbol{\tau}_h \in Ker_h(b_f). \tag{4.19}$$

Now, we adapt the proof of [3, Lemma 4.4] to derive the discrete version of (3.21). To that end, we need to introduce some preliminaries results. We begin by defining the space

$$\mathbf{Z} := \{\tau \in \mathbf{H}(\mathrm{div}_{6/5}\,; \Omega) \;:\; \tau|_T \in \mathbf{W}^{1,6/5}(T), \quad \forall\, T \in \mathcal{T}_h\}.$$

Then, we let

$$\Pi_h^k : \mathbf{Z} \to \mathbf{X}_h := \{\tau \in \mathbf{H}(\mathrm{div}\,; \Omega) \;:\; \tau|_T \in \mathbf{RT}_k(T), \quad \forall\, T \in \mathcal{T}_h\},$$

be the Raviart–Thomas interpolator operator, which is well defined on $\mathbf{Z}$ (see e.g. [10, Section 1.2.7]) and is characterized by the identities

$$\int_e (\Pi_h^k(\tau) \cdot \nu)\xi = \int_e (\tau \cdot \nu)\xi \qquad \forall\, \xi \in P_k(e),\ \forall \text{ edge or face } e \text{ of } \mathcal{T}_h,$$

and

$$\int_T \Pi_h^k(\tau) \cdot \psi \;=\; \int_T \tau \cdot \psi \qquad \forall\, \psi \in [P_{k-1}(T)]^n,\ \forall\, T \in \mathcal{T}_h \text{ (if } k \geq 1) \;.$$

In addition, it is well known (see e.g. [10, Lemma 1.41]) that the following identity holds

$$\mathrm{div}\,(\Pi_h^k(\tau)) = \mathcal{P}_h^k(\mathrm{div}\,\tau) \quad \forall\, \tau \in \mathbf{Z}, \tag{4.20}$$

where, for $1 \leq q \leq \infty$, $\mathcal{P}_h^k : L^q(\Omega) \to M_h := \{v \in \mathrm{L}^q(\Omega) \;:\; v|_K \in P_k(K) \quad \forall\, K \in \mathcal{T}_h\}$ is the operator satisfying

$$\int_\Omega (\mathcal{P}_h^k(v) - v) z_h = 0 \quad \forall\, z_h \in M_h,$$

and the following error estimate (see [10, Proposition 1.135, Section 1.6.3]): For each $0 \leq t \leq k+1$ and for each $w \in \mathrm{W}^{t,q}(\Omega)$, with $1 \leq q \leq \infty$, there holds

$$\|w - \mathcal{P}_h^k(w)\|_{\mathrm{L}^q(\Omega)} \leq C h^t |w|_{\mathrm{W}^{t,q}(\Omega)}. \tag{4.21}$$

Notice that for $q = 2$, $\mathcal{P}_h^k$ coincides with the usual orthogonal projection.

The following lemma establishes the local approximation properties of Π_h^k. See, for instance, [3, Lemma 4.2].

Lemma 4.3 *There exists $C_1 > 0$, independent of h, such that for each $\tau \in \mathbf{W}^{l+1,6/5}(T)$ with $0 \le l \le k$, and for each $0 \le m \le l+1$, there holds*

$$|\tau - \Pi_h^k(\tau)|_{\mathbf{W}^{m,6/5}(T)} \le C_1 \frac{h_T^{l+2}}{\rho_T^{m+1}} |\tau|_{\mathbf{W}^{l+1,6/5}(T)}, \tag{4.22}$$

where h_T is the diameter of T, that is $h_T = \max_{x,y\in T} \|x-y\|$, and ρ_T is the diameter of the largest sphere contained in T. Moreover, there exists $C_2 > 0$, independent of h, such that for each $\tau \in \mathbf{W}^{1,6/5}(T)$, with $\mathrm{div}\, \tau \in \mathrm{W}^{l+1,6/5}(T)$ and $0 \le l \le k$, and for each $0 \le m \le l+1$, there holds

$$|\mathrm{div}\, \tau - \mathrm{div}\, (\Pi_h^k(\tau))|_{\mathrm{W}^{m,6/5}(T)} \le C_2 \frac{h_T^{l+1}}{\rho_T^m} |\mathrm{div}\, \tau|_{\mathrm{W}^{l+1,6/5}(T)}. \tag{4.23}$$

Owing to the regularity of the mesh and from estimates (4.22) and (4.23), it is not difficult see that the following global estimate holds

$$\|\tau - \Pi_h^k(\tau)\|_{0,\Omega} + \|\mathrm{div}\, \tau - \mathrm{div}\, (\Pi_h(\tau))\|_{\mathrm{L}^{6/5}(\Omega)} \le c h^{l+1} \Big\{ |\tau|_{\mathbf{H}^{l+1}(\Omega)} + |\mathrm{div}\, \tau|_{\mathrm{W}^{l+1,6/5}(\Omega)} \Big\}, \tag{4.24}$$

for all $0 \le l \le k+1$, and for all $\tau \in \mathbf{H}^{l+1}(\Omega)$ with $\mathrm{div}\, \tau \in \mathrm{W}^{l+1,6/5}(\Omega)$.

In the sequel, it will be employed a tensor version of Π_h^k, say $\mathbf{\Pi}_h^k : \mathbb{Z}_p \to \mathbb{X}_h$ which is defined row–wise by Π_h^k, and a vector version of $\mathcal{P}_h^k$, say $\mathbf{P}_h^k$, defined element–wise by $\mathcal{P}_h^k$. Obviously, both $\mathbf{\Pi}_h^k$ and $\mathbf{P}_h^k$ also satisfy the properties described above

Remark 4.4 *Notice that from the regularity of the mesh and from (4.22) with $m = 0$ and $m = 1$, one can easily obtain, respectively, that*

$$\|\tau - \Pi_h^k(\tau)\|_{\mathbf{L}^{6/5}(T)} \le C_1 \frac{h_T^{l+2}}{\rho_T} |\tau|_{\mathbf{W}^{l+1,6/5}(T)} \le \hat{C}_1 h_T^{l+1} |\tau|_{\mathbf{W}^{l+1,6/5}(T)}$$

and

$$|\tau - \Pi_h^k(\tau)|_{\mathbf{W}^{1,6/5}(T)} \le C_2 \frac{h_T^{l+2}}{\rho_T^2} |\boldsymbol{\tau}|_{\mathbf{W}^{l+1,6/5}(T)} \le \hat{C}_2 h_T^{l} |\tau|_{\mathbf{W}^{l+1,6/5}(T)},$$

which combined with [10, Lemma 1.101] and the continuity of the embedding from $\mathrm{W}^{1,6/5}$ *into* L^2 *on the reference triangle (see the proof of [5, Lemma 5.4]), yield*

$$\|\tau - \Pi_h^k(\tau)\|_{0,T} \le C |\tau|_{\mathbf{W}^{1,6/5}(T)} \quad \forall\, \tau \in \mathbf{W}^{1,6/5}(T). \tag{4.25}$$

Now we are in position of establishing the discrete inf-sup condition of b_f.

Lemma 4.5 *There exists $\hat{\beta}_f > 0$, independent of h, such that*

$$\sup_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0} \backslash \{\mathbf{0}\}} \frac{b_f(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}} \ge \hat{\beta}_f \|\mathbf{v}_h\|_{\mathbf{L}^6(\Omega)} \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \tag{4.26}$$

Proof. In what follows we proceed similarly to the proof of [4, Lemma 3.3] and [3, Lemma 4.4]. In fact, given $\mathbf{v}_h \in \mathbf{V}_h$, we set

$$\mathbf{g}(\mathbf{v}_h) := \begin{cases} |\mathbf{v}_h|^4 \mathbf{v}_h & \text{in} \quad \Omega, \\ \mathbf{0} & \text{in} \quad B \backslash \overline{\Omega}, \end{cases}$$

where $B \subseteq \mathbb{R}^3$ is a open ball containing $\overline{\Omega}$. Then, since $\mathbf{g}(\mathbf{v}_h) \in \mathbf{L}^{6/5}(B)$, it follows that there exists a unique weak solution $\mathbf{z} \in \mathbf{W}^{2,6/5}(B) \cap \mathbf{W}_0^{1,6/5}(B)$ of the boundary value problem

$$-\Delta \mathbf{z} = \mathbf{g}(\mathbf{v}_h) \quad \text{in} \quad B \quad \text{and} \quad \mathbf{z} = \mathbf{0} \quad \text{on} \quad \partial B,$$

which satisfies

$$\|\mathbf{z}\|_{\mathbf{W}^{2,6/5}(\Omega)} \leq C \|\mathbf{g}(\mathbf{v}_h)\|_{\mathbf{L}^{6/5}(B)} = C \||\mathbf{v}_h|^4 \mathbf{v}_h\|_{\mathbf{L}^{6/5}(\Omega)} = C \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}, \tag{4.27}$$

with $C > 0$ (see e.g. [13]).

Hence, we set $\hat{\boldsymbol{\tau}} = -\nabla \mathbf{z}|_\Omega \in \mathbb{W}^{1,6/5}(\Omega)$, and observe from (4.27) that

$$\|\hat{\boldsymbol{\tau}}\|_{\mathbb{W}^{1,6/5}(\Omega)} \leq C \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}, \tag{4.28}$$

which together with the continuity of the embedding from $\mathrm{W}^{1,6/5}(\Omega)$ into $\mathrm{L}^2(\Omega)$, implies

$$\|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \leq C \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}. \tag{4.29}$$

Then, we define $\hat{\boldsymbol{\tau}}_h = \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) - \dfrac{1}{3|\Omega|} \Big(\mathrm{tr}\,(\boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}})), 1 \Big)_\Omega \mathbb{I} \in \mathbb{H}_{h,0}$ and observe from (4.20), that

$$\mathbf{div}\,\hat{\boldsymbol{\tau}}_h = \mathbf{P}_h^k(\mathbf{div}\,\hat{\boldsymbol{\tau}}) = \mathbf{P}_h^k(|\mathbf{v}_h|^4 \mathbf{v}_h). \tag{4.30}$$

In turn, utilizing the triangle inequality and estimates (4.25) and (4.29), we obtain

$$\begin{aligned}
\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} &\leq \left\| \hat{\boldsymbol{\tau}} - \frac{1}{3|\Omega|} (\mathrm{tr}\,(\hat{\boldsymbol{\tau}}), 1)_\Omega \mathbb{I} - \hat{\boldsymbol{\tau}}_h \right\|_{0,\Omega} + \left\| \hat{\boldsymbol{\tau}} - \frac{1}{3|\Omega|} (\mathrm{tr}\,(\hat{\boldsymbol{\tau}}), 1)_\Omega \mathbb{I} \right\|_{0,\Omega} \\
&= \left\| \hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) - \frac{1}{3|\Omega|} \Big(\mathrm{tr}\, \Big(\hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) \Big), 1 \Big)_\Omega \mathbb{I} \right\|_{0,\Omega} + \left\| \hat{\boldsymbol{\tau}} - \frac{1}{3|\Omega|} (\mathrm{tr}\,(\hat{\boldsymbol{\tau}}), 1)_\Omega \mathbb{I} \right\|_{0,\Omega} \\
&\leq \left\| \hat{\boldsymbol{\tau}} - \boldsymbol{\Pi}_h^k(\hat{\boldsymbol{\tau}}) \right\|_{0,\Omega} + \|\hat{\boldsymbol{\tau}}\|_{0,\Omega} \\
&\leq c_1 \left\{ \sum_{T \in \mathcal{T}_h} |\hat{\boldsymbol{\tau}}|^2_{\mathbb{W}^{1,6/5}(T)} \right\}^{1/2} + c_2 \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}, \\
&\leq c_1 |\hat{\boldsymbol{\tau}}|_{\mathbb{W}^{1,6/5}(\Omega)} + c_2 \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)},
\end{aligned}$$

which together with (4.28), imply

$$\|\hat{\boldsymbol{\tau}}_h\|_{0,\Omega} \leq C \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}. \tag{4.31}$$

Hence, using the fact that $\mathbf{P}_h^k$ is a continuous operator, from (4.30) and (4.31), we easily obtain

$$\|\hat{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{6/5};\Omega} = \left\{ \|\hat{\boldsymbol{\tau}}_h\|^2_{0,\Omega} + \|\mathbf{div}\,(\hat{\boldsymbol{\tau}}_h)\|^2_{\mathbf{L}^{6/5}(\Omega)} \right\}^{1/2} \leq \hat{c} \|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}, \tag{4.32}$$

with $\hat{c} > 0$ independent of h.

Therefore, from (4.30) and (4.32), we obtain

$$\sup_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}\backslash\{\mathbf{0}\}} \frac{b_f(\boldsymbol{\tau}_h, \mathbf{v}_h)}{\|\boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega}} \geq \frac{b_f(\hat{\boldsymbol{\tau}}_h, \mathbf{v}_h)}{\|\hat{\boldsymbol{\tau}}_h\|_{\mathbf{div}_{6/5};\Omega}} \geq \frac{1}{\hat{c}} \frac{(|\mathbf{v}_h|^4 \mathbf{v}_h, \mathbf{v}_h)_\Omega}{\|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}} \geq \frac{1}{\hat{c}} \frac{\|\mathbf{v}_h\|^6_{\mathbf{L}^6(\Omega)}}{\|\mathbf{v}_h\|^5_{\mathbf{L}^6(\Omega)}} = \frac{1}{\hat{c}} \|\mathbf{v}_h\|_{\mathbf{L}^6(\Omega)},$$

which concludes the proof with $\hat{\beta}_f = \dfrac{1}{\hat{c}}$. □

Owing to estimates (4.19) and (4.26) we obtain ([3, Theorem 3.6]) that the bilinear form $\mathbf{A}$ defined in (3.25) also satisfies the discrete inf-sup condition

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h} \frac{\mathbf{A}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h))}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|} \geq \hat{\gamma}_f \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|\,, \tag{4.33}$$

for all $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h$, with

$$\hat{\gamma}_f := \hat{C}_{f,2} \frac{min\{1, \nu\hat{\beta}_f\}}{\nu\hat{\beta}_f + 1} \tag{4.34}$$

with constants $\widehat{C}_{f,2} > 0$ and $\widehat{\beta}_f > 0$ independent of h and the physical parameters.

Making use of estimate (4.33) it can be easily proved the following result whose proof can be obtained by applying the same steps given in the proof of [3, Theorem 3.6].

Lemma 4.6 *Let $\mathbf{w}_h \in \mathbf{V}_h$ be such that*

$$\|\mathbf{w}_h\|_{\mathbf{L}^6(\Omega)} \leq \frac{\nu\hat{\gamma}_f}{2C_{f,3}}, \tag{4.35}$$

with $\hat{\gamma}_f$ being the constant defined in (4.34). Then, the bilinear form $\mathbf{A}_{\mathbf{w}_h}$ defined in (3.32) satisfies the following inf-sup condition:

$$\sup_{\mathbf{0} \neq (\boldsymbol{\tau}_h, \mathbf{v}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h} \frac{\mathbf{A}_{\mathbf{w}_h}((\boldsymbol{\sigma}_h, \mathbf{u}_h), (\boldsymbol{\tau}_h, \mathbf{v}_h))}{\|(\boldsymbol{\tau}_h, \mathbf{v}_h)\|} \geq \frac{\hat{\gamma}_f}{2} \|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\|, \tag{4.36}$$

for all $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h$.

Finally, for a fixed $\hat{\mathbf{b}}_h \in \mathbf{C}_h$ from (4.10) we have that $F_{\hat{\mathbf{b}}_h}$ (cf. (4.7)) is continuous on $\mathbf{V}_h$, that is

$$|F_{\hat{\mathbf{b}}_h}(\mathbf{v}_h)| \leq (\|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + k\widehat{C}^d_s \|\hat{\mathbf{b}}_h\|^2_{\mathrm{curl};\Omega}) \|\mathbf{v}_h\|_{\mathbf{L}^6(\Omega)} \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h.$$

Now we are in position of establishing the well-posedness of (4.6), or equivalently, the well-definiteness of $S_{f,h}$.

Lemma 4.7 *Let $\mathbf{w}_h \in \mathbf{V}_h$ be such that (4.35) holds and let $\hat{\mathbf{b}}_h \in \mathbf{C}_h$. Then, there exists a unique $(\boldsymbol{\sigma}_h, \mathbf{u}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h$, solution to (4.6). In addition, there holds*

$$\|(\boldsymbol{\sigma}_h, \mathbf{u}_h)\| \leq \frac{2}{\hat{\gamma}_f} \left(C_\Gamma \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + k\widehat{C}^d_s \|\hat{\mathbf{b}}_h\|^2_{\mathrm{curl};\Omega} \right).$$

Proof. Employing Lemma 4.6, the fact that for finite dimensional linear problems, surjectivity and injectivity are equivalent, and the well-known Banach–Nečas–Babŭska theorem, (cf. [10, Theorem 2.6]) we obtain the desired result. The rest of the arguments are omitted since they are analogous to those given in the proof of Lemma 3.4. □

Now, analogously to the continuous case we define the set (see Section 3.5)

$$\mathbf{K}_h := \left\{ \mathbf{v}_h \in \mathbf{V}_h \,:\, \|\mathbf{v}_h\|_{\mathbf{L}^6(\Omega)} \,\leq\, \widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) \right\},$$

where

$$\widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) := \frac{2}{\hat{\gamma}_f} \left(C_\Gamma \|\mathbf{u}_D\|_{1/2,\Gamma} + \|\mathbf{f}\|_{\mathbf{L}^{6/5}(\Omega)} + \frac{\widehat{M}_1}{\kappa \nu_m^2} \|\mathbf{g}\|_{0,\Omega}^2 \right), \tag{4.37}$$

with $\widehat{M}_1 = \frac{4\hat{C}_s^m}{\hat{\alpha}_m^4}$ and $\hat{\gamma}_f > 0$ given in (4.34), both independent of h and the physical parameters, and redefine $\mathcal{J}_h$ on $\mathbf{K}_h$, that is $\mathcal{J}_h : \mathbf{K}_h \to \mathbf{K}_h$.

Now we establish discrete counterpart of Lemma 3.5 providing the well-definiteness of $\mathcal{J}_h$ on $\mathbf{K}_h$.

Lemma 4.8 *Assume that the data* $\mathbf{f}$, $\mathbf{g}$ *and* $\mathbf{u}_D$ *satisfy*

$$\widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) \,\leq\, \frac{1}{2} \min \left\{ \frac{\nu \hat{\gamma}_f}{C_{f,3}}, \nu_m \widehat{C}_m \right\}. \tag{4.38}$$

Then, given $\mathbf{w}_h \in \mathbf{K}_h$, *there exists a unique* $\mathbf{u}_h \in \mathbf{K}_h$ *such that* $\mathcal{J}_h(\mathbf{w}_h) = \mathbf{u}_h$.

Now we state the main result of this section, namely, the well-posedness of (4.4). Its proof is omitted since it can be obtained repeating the same steps in the proof of Theorem 3.6.

Theorem 4.9 *Let* $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$ *such that* (4.38) *holds. Assume further that*

$$\frac{2}{\hat{\gamma}_f} \left(\frac{\widehat{M}_2}{k \nu_m^3} \|\mathbf{g}\|_{0,\Omega}^2 \,+\, \frac{C_{f,3}}{\nu} \widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) \right) \,<\, 1, \tag{4.39}$$

with $\widehat{M}_2 > 0$ *independent of* h *and the physical parameters. Then, there exists a unique* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{C}_h$ *solution to* (4.4). *In addition, there hold*

$$\|\mathbf{u}_h\|_{\mathbf{L}^6(\Omega)} + \|\boldsymbol{\sigma}_h\|_{\mathbf{div}_{6/5};\Omega} \,\leq \widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}), \tag{4.40}$$

and

$$\|\mathbf{b}_h\|_{\mathrm{curl};\Omega} \,\leq\, \frac{2}{k \nu_m \hat{\alpha}_m^2} \|\mathbf{g}\|_{0,\Omega}. \tag{4.41}$$

We end this section by establishing the well-posedness of (4.1), whose proof is omitted since it follows analogously to the proof of Corollary 3.7.

Corollary 4.10 *Let* $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$ *such that* (4.38) *and* (4.39) *hold. Then, there exists a unique* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h, r_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{D}_h \times \mathrm{S}_h$ *solution to* (4.1) *. In addition,* $(\boldsymbol{\sigma}_h, \mathbf{u}_h)$ *and* $\mathbf{b}_h$ *satisfy* (4.40) *and* (4.41), *respectively, and for* r_h *there holds*

$$\|r_h\|_{1,\Omega} \leq \frac{1}{\beta_m} \left(\frac{2}{\hat{\alpha}_m^2} + 1 \right) \|\mathbf{g}\|_{0,\Omega} + \frac{k \widehat{C}_s^m}{\beta_m} \widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g}),$$

with $\widehat{\mu}(\mathbf{u}_D, \mathbf{f}, \mathbf{g})$ *being the constant defined in* (4.37).

4.3 Cea's estimates

In this section we aim to provide the convergence analysis of the Galerkin scheme (4.1). To do that we introduce first some notations and useful results to be used next.

In order to simplify the subsequent analysis, we define

$$\mathbf{e}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \quad \mathbf{e}_{\mathbf{u}} := \mathbf{u} - \mathbf{u}_h, \quad \mathbf{e}_{\mathbf{b}} := \mathbf{b} - \mathbf{b}_h \quad e_r := r - r_h, \tag{4.42}$$

and for any $(\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{d}}_h, \hat{s}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{C}_h \times \mathrm{S}_h$, we write

$$\mathbf{e}_{\boldsymbol{\sigma}} = \boldsymbol{\xi}_{\boldsymbol{\sigma}} + \boldsymbol{\chi}_{\boldsymbol{\sigma}}, \quad \mathbf{e}_{\mathbf{u}} = \boldsymbol{\xi}_{\mathbf{u}} + \boldsymbol{\chi}_{\mathbf{u}}, \quad \mathbf{e}_{\mathbf{b}} = \boldsymbol{\xi}_{\mathbf{b}} + \boldsymbol{\chi}_{\mathbf{b}}, \quad e_r = \xi_r + \chi_r, \tag{4.43}$$

where

$$\boldsymbol{\xi}_{\boldsymbol{\sigma}} := \boldsymbol{\sigma} - \hat{\boldsymbol{\tau}}_h, \quad \boldsymbol{\chi}_{\boldsymbol{\sigma}} := \hat{\boldsymbol{\tau}}_h - \boldsymbol{\sigma}_h, \quad \boldsymbol{\xi}_{\mathbf{u}} := \mathbf{u} - \hat{\mathbf{v}}_h, \quad \boldsymbol{\chi}_{\mathbf{u}} := \hat{\mathbf{v}}_h - \mathbf{u}_h,$$

$$\boldsymbol{\xi}_{\mathbf{b}} := \mathbf{b} - \hat{\mathbf{d}}_h, \quad \boldsymbol{\chi}_{\mathbf{b}} := \hat{\mathbf{d}}_h - \mathbf{b}_h, \quad \xi_r := r - \hat{z}_h, \quad \chi_r := \hat{z}_h - r_h.$$

In turn we recall that owing to the inf-sup condition (4.2), the following inequality holds (see for instance [14, Theorem 2.6])

$$\inf_{\mathbf{d}_h \in \mathbf{C}_h} \|\mathbf{b} - \mathbf{d}_h\|_{\mathrm{curl}\,;\Omega} \,\le\, C \inf_{\mathbf{d}_h \in \mathbf{D}_h} \|\mathbf{b} - \mathbf{d}_h\|_{\mathrm{curl}\,;\Omega}, \tag{4.44}$$

with $C > 0$ independent of h and the physical parameters.

The corresponding Cea's estimate is established in the following theorem.

Theorem 4.11 *Let* $\mathbf{f} \in \mathbf{L}^{6/5}(\Omega)$, $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$ *and* $\mathbf{g} \in \mathbf{L}^2(\Omega)$ *be such that* (3.38), (3.39), (4.38) *and* (4.39) *holds. Assume further that*

$$\frac{2C_{f,3}}{\nu\hat{\gamma}_f}\mu(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) \,+\, \frac{\widehat{M}_3}{\hat{\gamma}_f k \nu_m^3}\|\mathbf{g}\|_{0,\Omega}^2 \,<\, 1 \tag{4.45}$$

with $\mu(\mathbf{u}_D, \mathbf{f}, \mathbf{g})$ *being the positive constant defined in* (3.37) *and* $\widehat{M}_3 > 0$ *independent of* h *and the physical parameters. Let* $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}, r) \in \mathbb{H}_0(\mathbf{div}_{6/5}\,;\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{H}_0(\mathrm{curl}\,;\Omega) \times H_0^1(\Omega)$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h, r_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{D}_h \times S_h$ *be the unique solutions of problems* (3.2) *and* (4.1), *respectively. Then, there exist positive constants* C_1, C_2, *independent of* h, *such that*

$$\begin{aligned}\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5}\,;\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\mathbf{e}_{\mathbf{b}}\|_{\mathrm{curl}\,;\Omega} + \|e_r\|_{1,\Omega} \le C_1\Big\{ \inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5}\,;\Omega} \\ + \inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{L}^6(\Omega)} + \inf_{\mathbf{d}_h \in \times \mathbf{D}_h} \|\mathbf{b} - \mathbf{d}_h\|_{\mathrm{curl}\,;\Omega} + \inf_{z_h \in \mathrm{S}_h} \|r - z_h\|_{1,\Omega} \Big\}.\end{aligned} \tag{4.46}$$

Proof. We begin observing by that assumptions (3.38), (3.39), (4.38) and (4.39) allow us to conclude that continuous and discrete problems (3.2) and (4.1) are well-posed. Now, we subtract (3.2) and (4.1) and obtain the Galerkin orthogonality property

$$\begin{aligned}
a_f(\mathbf{e}_{\boldsymbol{\sigma}}, \boldsymbol{\tau}_h) \,+\, b_f(\boldsymbol{\tau}_h, \mathbf{e}_{\mathbf{u}}) \,+\, [c_f(\mathbf{u}; \mathbf{u}, \boldsymbol{\tau}_h) - c_f(\mathbf{u}_h; \mathbf{u}_h, \boldsymbol{\tau}_h)] &= 0 \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}, \\
b_f(\mathbf{e}_{\boldsymbol{\sigma}}, \mathbf{v}_h) \,+\, [c_d(\mathbf{b}; \mathbf{b}, \mathbf{v}_h) - c_d(\mathbf{b}_h; \mathbf{b}_h, \mathbf{v}_h)] &= 0 \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h, \\
a_m(\mathbf{e}_{\mathbf{b}}, \mathbf{d}_h) \,+\, b_m(\mathbf{d}_h, e_r) \,+\, [c_m(\mathbf{u}; \mathbf{b}, \mathbf{d}_h) - c_m(\mathbf{u}_h; \mathbf{b}_h, \mathbf{d}_h)] &= 0 \quad \forall\, \mathbf{d}_h \in \mathbf{D}_h, \\
b_m(\mathbf{e}_{\mathbf{b}}, z_h) &= 0 \quad \forall\, z_h \in \mathrm{S}_h.
\end{aligned} \tag{4.47}$$

For the first two equations of (4.47) we add and subtract suitable terms and make use of (4.43) and the definition of $\mathbf{A_w}$ (cf. (3.32)) to arrive at

$$\mathbf{A}_{\mathbf{u}_h}((\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}}),(\boldsymbol{\tau}_h,\mathbf{v}_h)) \,=\, G_1(\boldsymbol{\tau}_h) + G_2(\mathbf{v}_h) \quad \forall\,\boldsymbol{\tau}_h \in \mathbb{H}_{h,0},\; \mathbf{v}_h \in \mathbf{V}_h, \tag{4.48}$$

where

$$\begin{aligned} G_1(\boldsymbol{\tau}_h) &= -a_f(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\boldsymbol{\tau}_h) - b_f(\boldsymbol{\tau}_h,\boldsymbol{\xi}_{\mathbf{u}}) - c_f(\mathbf{u}_h;\boldsymbol{\xi}_{\mathbf{u}},\boldsymbol{\tau}_h) - c_f(\boldsymbol{\xi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h) - c_f(\boldsymbol{\chi}_{\mathbf{u}};\mathbf{u},\boldsymbol{\tau}_h)\,, \\ G_2(\mathbf{v}_h) &= -b_f(\boldsymbol{\xi}_{\boldsymbol{\sigma}},\mathbf{v}_h) - [c_d(\mathbf{b};\mathbf{b},\mathbf{v}_h) - c_d(\mathbf{b}_h;\mathbf{b}_h,\mathbf{v}_h)]\,. \end{aligned}$$

Similarly, for the third and fourth equations of (4.47) we add and subtract suitable terms and make use of (4.43) to easily obtain

$$\begin{aligned} a_m(\boldsymbol{\chi}_{\mathbf{b}},\mathbf{d}_h) \,+\, b_m(\mathbf{d}_h,\chi_r) \,+\, c_m(\mathbf{u}_h;\boldsymbol{\chi}_{\mathbf{b}},\mathbf{d}_h) &= G_3(\mathbf{d}_h) & \forall\,\mathbf{d}_h \in \mathbf{D}_h, \\ b_m(\boldsymbol{\chi}_{\mathbf{b}},z_h) &= -b_m(\boldsymbol{\xi}_{\mathbf{b}},z_h) & \forall\,z_h \in \mathrm{S}_h, \end{aligned} \tag{4.49}$$

with

$$G_3(\mathbf{d}_h) \,=\, -a_m(\boldsymbol{\xi}_{\mathbf{b}},\mathbf{d}_h) - b_m(\mathbf{d}_h,\xi_r) - c_m(\mathbf{u}_h;\boldsymbol{\xi}_{\mathbf{b}},\mathbf{d}_h) - c_m(\boldsymbol{\xi}_{\mathbf{u}};\mathbf{b},\mathbf{d}_h) - c_m(\boldsymbol{\chi}_{\mathbf{u}};\mathbf{b},\mathbf{d}_h).$$

In particular, since $\boldsymbol{\chi}_{\mathbf{b}} \in \mathbf{C}_h$, from the first equation of (4.49) we have

$$a_m(\boldsymbol{\chi}_{\mathbf{b}},\mathbf{d}_h) \,+\, c_m(\mathbf{u}_h;\boldsymbol{\chi}_{\mathbf{b}},\mathbf{d}_h) \,=\, G_3(\mathbf{d}_h) \quad \forall\,\mathbf{d}_h \in \mathbf{C}_h. \tag{4.50}$$

Then, using that $\mathbf{u}_h \in \mathbf{K}_h$ and employing assumption (4.38) we observe that $\mathbf{u}_h$ satisfies (4.17), thus the bilinear form $a_m(\cdot,\cdot) + c_m(\mathbf{u}_h;\cdot,\cdot)$ satisfies (4.18), which together with (4.50) imply

$$\|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl}\,;\Omega} \,\leq\, \frac{2}{k\nu_m\hat{\alpha}_m^2}\|G_3\|_{\mathbf{C}_h'}. \tag{4.51}$$

In turn, noticing now that the fact that $\mathbf{u}_h \in \mathbf{K}_h$ together with (4.38) also imply that $\mathbf{u}_h$ satisfies (4.35), we make use of (4.36) and combine it with (4.48) to obtain

$$\|(\boldsymbol{\chi}_{\boldsymbol{\sigma}},\boldsymbol{\chi}_{\mathbf{u}})\| \,\leq\, \frac{2}{\hat{\gamma}_f}\left(\|G_1\|_{\mathbb{H}_{h,0}'} + \|G_2\|_{\mathbf{V}_h'}\right).$$

Now we turn to estimate the norm of the functionals G_1, G_2 and G_3. First, for G_3 we utilize the continuity of a_m and b_m and estimates (3.9) and (4.9) to obtain

$$\begin{aligned} \|G_3\|_{\mathbf{C}_h'} \leq \kappa\left(\nu_m + \widehat{C}_s^m\|\mathbf{u}_h\|_{\mathbf{L}^6(\Omega)}\right)\|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl}\,;\Omega} + \|\xi_r\|_{1,\Omega} \\ +\kappa C_s\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \kappa C_s\|\mathbf{b}\|_{\mathrm{curl}\,;\Omega}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}, \end{aligned}$$

which together to (3.41) and (4.40), implies

$$\|G_3\|_{\mathbf{C}_h'} \leq C\left(\|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl}\,;\Omega} + \|\xi_r\|_{1,\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}\right) + \frac{2C_s}{\nu_m\alpha_m^2}\|\mathbf{g}\|_{0,\Omega}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}, \tag{4.52}$$

with $C > 0$ independent of h. Similarly, for G_1 we make use of the continuity of a_f, b_f and c_f to obtain

$$\|G_1\|_{\mathbb{H}_{h,0}'} \leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5}\,;\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}\right) + \frac{C_{f,3}}{\nu}\mu(\mathbf{u}_D,\mathbf{f},\mathbf{g})\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}, \tag{4.53}$$

with $C > 0$ independent of h. Finally, for G_2 we first notice that after adding and subtracting suitable terms there holds

$$G_2(\mathbf{v}_h) = -b_f(\boldsymbol{\xi}_{\boldsymbol{\sigma}}, \mathbf{v}_h) - c_d(\boldsymbol{\xi}_{\mathbf{b}}; \mathbf{b}, \mathbf{v}_h) - c_d(\mathbf{b}_h; \boldsymbol{\xi}_{\mathbf{b}}, \mathbf{v}_h) - c_d(\boldsymbol{\chi}_{\mathbf{b}}; \mathbf{b}, \mathbf{v}_h) - c_d(\mathbf{b}_h; \boldsymbol{\chi}_{\mathbf{b}}, \mathbf{v}_h),$$

and then, we apply the continuity of b_f and estimates (3.29) and (4.10) to obtain

$$\|G_2\|_{\mathbf{V}_h'} \leq \|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \widetilde{C}\left(\|\mathbf{b}\|_{\mathrm{curl};\Omega} + \|\mathbf{b}_h\|_{\mathrm{curl};\Omega}\right)\left(\|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega}\right),$$

with $\widetilde{C} > 0$ independent of h, which together to (3.41) and (4.41) imply

$$\|G_2\|_{\mathbf{V}_h'} \leq C(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega}) + \frac{2\widetilde{C}}{\kappa\nu_m\alpha_m^2\widehat{\alpha}_m^2}(\alpha_m^2 + \widehat{\alpha}_m^2)\|\mathbf{g}\|_{0,\Omega}\|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega}. \tag{4.54}$$

In this way, from estimates (4.51)–(4.54) it follows that

$$\|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} \leq C\left(\|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\xi_r\|_{1,\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}\right) + \frac{c_1}{\kappa\nu_m^2}\|\mathbf{g}\|_{0,\Omega}\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} \tag{4.55}$$

with $c_1 = \frac{4C_s}{\widehat{\alpha}_m\alpha_m^2}$ and

$$\begin{aligned}\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} &\leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}\right)\\ &\quad + \frac{2C_{f,3}}{\hat{\gamma}_f\nu}\mu(\mathbf{u}_D, \mathbf{f}, \mathbf{g})\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \frac{c_2}{\hat{\gamma}_f\kappa\nu_m}\|\mathbf{g}\|_{0,\Omega}\|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega},\end{aligned}$$

with $c_2 = \frac{4\widetilde{C}}{\widehat{\alpha}_m\alpha_m^2}$, which combined yield

$$\begin{aligned}\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} &\leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\xi_r\|_{1,\Omega}\right)\\ &\quad + \left(\frac{2C_{f,3}}{\hat{\gamma}_f\nu}\mu(\mathbf{u}_D, \mathbf{f}, \mathbf{g}) + \frac{\widehat{M}_3}{\hat{\gamma}_f\kappa^2\nu_m^3}\|\mathbf{g}\|_{0,\Omega}^2\right)\|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)}\end{aligned} \tag{4.56}$$

with $\widehat{M}_3 = c_1c_2$. Therefore, from (4.55), (4.56) and assumption (4.45) it readily follows that

$$\|\boldsymbol{\chi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\chi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\boldsymbol{\chi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} \leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\xi_r\|_{1,\Omega}\right), \tag{4.57}$$

which together with (4.43) implies

$$\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\mathbf{e}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} \leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\xi_r\|_{1,\Omega}\right). \tag{4.58}$$

On the other hand, to estimate $\|e_r\|_{1,\Omega}$ we observe that from first equation of (4.49) we have that

$$b_m(\mathbf{d}_h, \chi_r) = -a_m(\boldsymbol{\chi}_{\mathbf{b}}, \mathbf{d}_h) - c_m(\mathbf{u}_h; \boldsymbol{\chi}_{\mathbf{b}}, \mathbf{d}_h) + G_3(\mathbf{d}_h) \quad \forall\, \mathbf{d}_h \in \mathbf{D}_h,$$

which combined with the inf-sup condition (4.2), the continuity of a_m and c_m, and estimates (4.40), (4.52) and (4.57), yields

$$\begin{aligned}\beta_m\|\chi_r\|_{1,\Omega} &\leq \sup_{\mathbf{0}\neq\mathbf{d}_h\in\mathbf{D}_h}\frac{b_m(\mathbf{d}_h, \chi_r)}{\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}}\\ &= \sup_{\mathbf{0}\neq\mathbf{d}_h\in\mathbf{D}_h}\frac{-a_m(\boldsymbol{\chi}_{\mathbf{b}}, \mathbf{d}_h) - c_m(\mathbf{u}_h; \boldsymbol{\xi}_{\mathbf{b}}, \mathbf{d}_h) + G_3(\mathbf{d}_h)}{\|\mathbf{d}_h\|_{\mathrm{curl};\Omega}}\\ &\leq C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\xi_r\|_{1,\Omega}\right),\end{aligned}$$

thus

$$\|e_r\|_{1,\Omega} \le \|\chi_r\|_{1,\Omega} + \|\xi_r\|_{1,\Omega} \le C\left(\|\boldsymbol{\xi}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\boldsymbol{\xi}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|\boldsymbol{\xi}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\xi_r\|_{1,\Omega}\right). \tag{4.59}$$

We end the proof by noticing that estimate (4.46) follows from (4.58), (4.59), (4.44) and the fact that $(\hat{\boldsymbol{\tau}}_h, \hat{\mathbf{v}}_h, \hat{\mathbf{d}}_h, \hat{z}_h) \in \mathbb{H}_{h,0} \times \mathbf{V}_h \times \mathbf{C}_h \times \mathrm{S}_h$ is arbitrary. □

4.4 Rates of convergence

In order to establish the rate of convergence of our Galerkin scheme (4.1) we first recall the approximation properties of the discrete spaces involved:

$$\inf_{\boldsymbol{\tau}_h \in \mathbb{H}_{h,0}} \|\boldsymbol{\sigma} - \boldsymbol{\tau}_h\|_{\mathbf{div}_{6/5};\Omega} \le Ch^{k+1}\left(\|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{\mathbf{W}^{k+1,6/5}(\Omega)}\right), \tag{4.60}$$

for all $\boldsymbol{\sigma} \in \mathbb{H}^{k+1}(\Omega)$, such that $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{k+1,6/5}(\Omega)$,

$$\inf_{\mathbf{v}_h \in \mathbf{V}_h} \|\mathbf{u} - \mathbf{v}_h\|_{\mathbf{L}^6(\Omega)} \le Ch^{k+1}\|\mathbf{u}\|_{\mathbf{W}^{k+1,6}(\Omega)}, \tag{4.61}$$

for all $\mathbf{u} \in \mathbf{W}^{k+1,6}(\Omega)$,

$$\inf_{\mathbf{d}_h \in \times \mathbf{D}_h} \|\mathbf{b} - \mathbf{d}_h\|_{\mathrm{curl};\Omega} \le Ch^{k+1}(\|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}\,\mathbf{b}\|_{k+1,\Omega}), \tag{4.62}$$

for all $\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$, such that $\mathrm{curl}\,\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$, and

$$\inf_{z_h \in \mathrm{S}_h} \|r - z_h\|_{1,\Omega} \le Ch^{k+1}\|r\|_{k+2,\Omega}, \tag{4.63}$$

for all $r \in H^{k+2}(\Omega)$. For (4.60) we refer to [3, eq. (4.8)], which is consequence of [10, Lemma B.67, Lemma 1.101] and [14, Section 3.4.4], for (4.61) we refer to [10, Proposition 1.134, Section 1.6.3], whereas for (4.62) and (4.63) we refer the reader to [25, Theorem 5.41] and [10, Proposition 1.134, Section 1.6.3] respectively.

Owing to the approximation properties listed above we can easily obtain the aforementioned theoretical rate of convergence associated to the Galerkin scheme (4.1)

Theorem 4.12 *Assume that the hypotheses of Theorem 4.11 hold. Let* $(\boldsymbol{\sigma}, \mathbf{u}, \mathbf{b}, r) \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \times \mathbf{L}^6(\Omega) \times \mathbf{H}_0(\mathrm{curl};\Omega) \times H_0^1(\Omega)$ *and* $(\boldsymbol{\sigma}_h, \mathbf{u}_h, \mathbf{b}_h, r_h) \in \mathbb{H}_{h,0} \times \mathbf{M}_h \times \mathbf{D}_h \times \mathrm{S}_h$ *be the unique solutions of* (3.2) *and* (4.1)*, respectively and assume further that* $\boldsymbol{\sigma} \in \mathbb{H}^{k+1}(\Omega)$ *with* $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{k+1,6/5}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$ *with* $\mathrm{curl}\,\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$ *and* $r \in H^{k+2}(\Omega)$*. Then there exists* $C > 0$*, independent of* h*, such that*

$$\begin{aligned}\|\mathbf{e}_{\boldsymbol{\sigma}}\|_{\mathbf{div}_{6/5};\Omega} + \|\mathbf{e}_{\mathbf{u}}\|_{\mathbf{L}^6(\Omega)} + \|\mathbf{e}_{\mathbf{b}}\|_{\mathrm{curl};\Omega} + \|e_r\|_{1,\Omega} \;\le\; Ch^{l+1}\Big(&\|\boldsymbol{\sigma}\|_{k+1,\Omega} + \|\mathbf{div}\,\boldsymbol{\sigma}\|_{\mathbf{W}^{k+1,6/5}(\Omega)} \\ &+ \|\mathbf{u}\|_{\mathbf{W}^{k+1,6}(\Omega)} + \|\mathbf{b}\|_{k+1,\Omega} + \|\mathrm{curl}\,\mathbf{b}\|_{k+1,\Omega} + \|r\|_{k+2,\Omega}\Big).\end{aligned}$$

Proof. The result is a straightforward application of Theorem 4.11, and estimates (4.60)-(4.63). □

4.5 Computing further variables of interest

Besides the variables approximated by the Galerkin scheme (4.1), with our approach we can also approximate further variables of interest. In fact, according to Remark 2.1 we first recall that at the continuous level the pressure can be recovered through the post-processed formula (2.21) and consequently we propose to approximate the pressure in terms if $\boldsymbol{\sigma}_h$ and $\mathbf{u}_h$ by:

$$p_h \;=\; -\frac{1}{3}\left(\operatorname{tr}(\boldsymbol{\sigma}_h) + \operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h) - \frac{1}{|\Omega|}(\operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h),1)_\Omega\right). \tag{4.64}$$

In addition, noticing that the vorticity $\boldsymbol{\omega} := \frac{1}{2}(\nabla\mathbf{u} - \nabla\mathbf{u}^t)$, the stress $\tilde{\boldsymbol{\sigma}} := \nu(\nabla\mathbf{u} + (\nabla\mathbf{u})^t) - p\mathbb{I}$ and the velocity gradient $\mathbf{G} = \nabla\mathbf{u}$ can be rewritten in terms of $\boldsymbol{\sigma}$ and $\mathbf{u}$ as follows

$$\begin{aligned}\tilde{\boldsymbol{\sigma}} \;&=\; \boldsymbol{\sigma}^d \,+\, (\mathbf{u}\otimes\mathbf{u})^d + \boldsymbol{\sigma}^t + \mathbf{u}\otimes\mathbf{u} - \frac{1}{3|\Omega|}(\operatorname{tr}(\mathbf{u}\otimes\mathbf{u}),1)_\Omega\mathbb{I},\\ \mathbf{G} \;&=\; \frac{1}{\nu}(\boldsymbol{\sigma}^d \,+\, (\mathbf{u}\otimes\mathbf{u})^d) \quad\text{and}\quad \boldsymbol{\omega} \;=\; \frac{1}{2\nu}(\boldsymbol{\sigma} - \boldsymbol{\sigma}^t),\end{aligned}$$

we propose the following discrete formulas for these quantities:

$$\begin{aligned}\tilde{\boldsymbol{\sigma}}_h \;&=\; \boldsymbol{\sigma}_h^d \,+\, (\mathbf{u}_h\otimes\mathbf{u}_h)^d + \boldsymbol{\sigma}_h^t + \mathbf{u}_h\otimes\mathbf{u}_h - \frac{1}{3|\Omega|}(\operatorname{tr}(\mathbf{u}_h\otimes\mathbf{u}_h),1)_\Omega\mathbb{I},\\ \mathbf{G}_h \;&=\; \frac{1}{\nu}(\boldsymbol{\sigma}_h^d \,+\, (\mathbf{u}_h\otimes\mathbf{u}_h)^d) \quad\text{and}\quad \boldsymbol{\omega}_h \;=\; \frac{1}{2\nu}(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^t).\end{aligned} \tag{4.65}$$

The following result establishes the theoretical rates of convergence for the aforementioned variables.

Corollary 4.13 *Assume that hypotheses of Theorem 4.11 hold. Let* $(\boldsymbol{\sigma},\mathbf{u},\mathbf{b},r) \in \mathbb{H}_0(\mathbf{div}_{6/5};\Omega) \times\mathbf{L}^6(\Omega)\times\mathbf{H}_0(\operatorname{curl};\Omega)\times H_0^1(\Omega)$ *and* $(\boldsymbol{\sigma}_h,\mathbf{u}_h,\mathbf{b}_h,r_h) \in \mathbb{H}_{h,0}\times\mathbf{M}_h\times\mathbf{D}_h\times\mathrm{S}_h$ *be the unique solutions of* (3.2) *and* (4.1)*, respectively and assume further that* $\boldsymbol{\sigma} \in \mathbb{H}^{k+1}(\Omega)$ *with* $\mathbf{div}\,\boldsymbol{\sigma} \in \mathbf{W}^{k+1,6/5}(\Omega)$, $\mathbf{u} \in \mathbf{W}^{k+1,6}(\Omega)$, $\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$ *with* $\operatorname{curl}\mathbf{b} \in \mathbf{H}^{k+1}(\Omega)$ *and* $r \in H^{k+2}(\Omega)$*. Finally, let* p_h, $\tilde{\boldsymbol{\sigma}}_h$, $\mathbf{G}_h$ *and* $\boldsymbol{\omega}_h$ *given by* (4.64) *and* (4.65)*. Then there exists* $\tilde{C} > 0$*, independent of* h*, such that*

$$\begin{aligned}&\|p - p_h\|_{0,\Omega} \,+\, \|\tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h\|_{0,\Omega} \,+\, \|\mathbf{G} - \mathbf{G}_h\|_{0,\Omega} \,+\, \|\boldsymbol{\omega} - \boldsymbol{\omega}_h\|_{0,\Omega} \;\le\; \tilde{C}h^{k+1}\Big(\|\boldsymbol{\sigma}\|_{k+1,\Omega}\\ &+\|\mathbf{div}\,\boldsymbol{\sigma}\|_{\mathbf{W}^{k+1,6/5}(\Omega)} \,+\, \|\mathbf{b}\|_{k+1,\Omega} \,+\, \|\operatorname{curl}\mathbf{b}\|_{k+1,\Omega} \,+\, \|\mathbf{u}\|_{\mathbf{W}^{k+1,6}(\Omega)} \,+\, \|r\|_{k+2,\Omega}\Big).\end{aligned}$$

Proof. Recalling that $\mathbf{u} \in \mathbf{K}$ and $\mathbf{u}_h \in \mathbf{K}_h$, and employing (2.13) with $q = 4$, it is not difficult to see that

$$\|\mathbf{u}\otimes\mathbf{u} - \mathbf{u}_h\otimes\mathbf{u}_h\|_{0,\Omega} \le C(\|\mathbf{u}\|_{\mathbf{L}^4(\Omega)} + \|\mathbf{u}_h\|_{\mathbf{L}^4(\Omega)})\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^4(\Omega)} \le C\|\mathbf{u} - \mathbf{u}_h\|_{\mathbf{L}^6(\Omega)}, \tag{4.66}$$

with $C > 0$, independent of h. Then, using (4.66), the result follows from Theorem 4.12. We omit further details. □

5 Numerical results

In this section we report two numerical examples that will show the performance of our finite element scheme. Our implementation is based on a *FreeFem++* code (see [20]), in conjunction with the direct linear solver UMFPACK (see [9]). Regarding the resolution of the non-linear problem, we utilize the algorithm utilized to define the fixed-point operator $\mathcal{J}_h$. More precisely, starting with $\mathbf{u}_h^0 \in \mathbf{V}_h$ (to be specified on each example), we propose the following iterative process: for each $i = 1, 2, \ldots,$ solve

$$\begin{aligned} a_m(\mathbf{b}_h^i, \mathbf{d}_h) + b_m(\mathbf{d}_h, r_h^i) + c_m(\mathbf{u}_h^{(i-1)}; \mathbf{b}_h^i, \mathbf{d}_h) &= F_3(\mathbf{d}_h) \quad \forall\, \mathbf{d}_h \in \mathbf{D}_h, \\ b_m(\mathbf{b}_h^i, z_h) &= 0 \qquad\quad \forall\, z_h \in \mathrm{S}_h, \end{aligned} \tag{5.1}$$

and

$$\begin{aligned} a_f(\boldsymbol{\sigma}_h^i, \boldsymbol{\tau}_h) + b_f(\boldsymbol{\tau}_h, \mathbf{u}_h^i) + c_f(\mathbf{u}_h^{(i-1)}; \mathbf{u}_h^i, \boldsymbol{\tau}_h) &= F_1(\boldsymbol{\tau}_h) \quad \forall\, \boldsymbol{\tau}_h \in \mathbb{H}_{h,0}, \\ b_f(\boldsymbol{\sigma}_h^i, \mathbf{v}_h) &= F_{\mathbf{b}_h^i}(\mathbf{v}_h) \quad \forall\, \mathbf{v}_h \in \mathbf{V}_h\,. \end{aligned} \tag{5.2}$$

The iterations are terminated once the relative error of the entire coefficient vectors between two consecutive iterates is sufficiently small, that is,

$$\frac{|\mathbf{coeff}^{m+1} - \mathbf{coeff}^{m}|}{|\mathbf{coeff}^{m+1}|} \le tol,$$

where $|\cdot|$ is the standard euclidean norm $\mathbb{R}^N$, with N denoting the total number of degrees of freedom defining the finite element subspaces $\mathbb{H}_h$, $\mathbf{V}_h$, $\mathbf{D}_h$ and S_h.

We now introduce some additional notations. Let us denote the experimental rates of convergence as

$$R(\boldsymbol{\sigma}) := \frac{\log(\mathbf{e}_{\boldsymbol{\sigma}}/\mathbf{e}'_{\boldsymbol{\sigma}})}{\log(h/h')}, \quad R(\mathbf{u}) := \frac{\log(\mathbf{e}_{\mathbf{u}}/\mathbf{e}'_{\mathbf{u}})}{\log(h/h')}, \quad R(\mathbf{b}) := \frac{\log(\mathbf{e}_{\mathbf{b}}/\mathbf{e}'_{\mathbf{b}})}{\log(h/h')}, \quad R(r) := \frac{\log(e_r/e'_r)}{\log(h/h')},$$

$$R(p) := \frac{\log(e_p/e_p)}{\log(h/h')}, \quad R(\boldsymbol{\omega}) := \frac{\log(\mathbf{e}_{\boldsymbol{\omega}}/\mathbf{e}'_{\boldsymbol{\omega}})}{\log(h/h')}, \quad R(\mathbf{G}) := \frac{\log(\mathbf{e}_{\mathbf{G}}/\mathbf{e}'_{\mathbf{G}})}{\log(h/h')}, \quad R(\tilde{\boldsymbol{\sigma}}) := \frac{\log(\mathbf{e}_{\tilde{\boldsymbol{\sigma}}}/\mathbf{e}'_{\tilde{\boldsymbol{\sigma}}})}{\log(h/h')},$$

where $\mathbf{e}_{\boldsymbol{\sigma}}$, $\mathbf{e}_{\mathbf{u}}$, $\mathbf{e}_{\mathbf{b}}$ and e_r are defined in (4.42),

$$e_p := p - p_h, \quad \mathbf{e}_{\boldsymbol{\omega}} := \boldsymbol{\omega} - \boldsymbol{\omega}_h, \quad \mathbf{e}_{\mathbf{G}} := \mathbf{G} - \mathbf{G}_h \quad \mathbf{e}_{\tilde{\boldsymbol{\sigma}}} := \tilde{\boldsymbol{\sigma}} - \tilde{\boldsymbol{\sigma}}_h, \tag{5.3}$$

and h and h' denote two consecutive meshsizes with errors $\mathbf{e}$ and $\mathbf{e}'$ (or e and e').

In our first example we illustrate the performance of our Galerkin scheme (4.1). Here, we choose the domain $\Omega := (0,1) \times (0,0.5) \times (0,0.5)$, te parameters $\nu = k = 1$ and $\nu_m = 1000$, the initial guess $\mathbf{u}_h^0 = \mathbf{0}$, and take $\mathbf{f}$ and $\mathbf{g}$ and $\mathbf{u}_D$ so that the exact solution is given by

$$\begin{aligned} \mathbf{u}(\mathbf{x}) &:= \begin{pmatrix} -x_1(x_2 - x_3)(x_2 + x_3) \\ x_2(x_1 - x_3)(x_1 + x_3) \\ -x_3(x_1 - x_2)(x_1 + x_2) \end{pmatrix}, \\ p(\mathbf{x}) &:= x_2 x_3 (x_1 - 0.5)\,, \\ \mathbf{b}(\mathbf{x}) &:= \operatorname{curl}\left(x_1^2 (x_2 - 0.5)^2 x_3^2 \cos(\pi x_3)^2 (1,1,1)^t\right), \\ r(\mathbf{x}) &:= x_1 x_2 x_3 (x_2 - 0.5)(x_3 - 0.5)(x_1 - 1.0). \end{aligned}$$

In Table 5.1, we summarize the convergence history for Example 3 considering a sequence of regular triangulations. We observe there that the rates of convergence $O(h)$ predicted by

Iter.	h	$\mathbf{e}_{\boldsymbol{\sigma}}$	$R(\boldsymbol{\sigma})$	$\mathbf{e}_{\mathbf{u}}$	$R(\mathbf{u})$	$\mathbf{e}_{\mathbf{b}}$	$R(\mathbf{b})$	e_r	$R(r)$
3	0.1768	0.0616	–	0.0210	–	0.0290	–	7.6573e-04	–
3	0.0884	0.0312	0.9799	0.0105	0.9919	0.0148	0.9692	3.9522e-04	0.9542
3	0.0589	0.0208	1.0000	0.0070	0.9976	0.0099	0.9915	2.6508e-04	0.9851
3	0.0442	0.0156	1.0037	0.0053	0.9988	0.0074	0.9963	1.9924e-04	0.9926
3	0.0354	0.0125	1.0043	0.0042	0.9993	0.0060	0.9981	1.5955e-04	0.9956

e_p	$R(p)$	$\mathbf{e}_{\boldsymbol{\omega}}$	$R(\boldsymbol{\omega})$	$\mathbf{e}_{\mathbf{G}}$	$R(\mathbf{G})$	$\mathbf{e}_{\tilde{\boldsymbol{\sigma}}}$	$R(\tilde{\boldsymbol{\sigma}})$
0.0088	–	0.0550	–	0.0730	–	0.0967	–
0.0041	1.0992	0.0263	1.0677	0.0366	0.9948	0.0515	0.9093
0.0025	1.2391	0.0173	1.0332	0.0245	0.9915	0.0350	0.9508
0.0017	1.2618	0.0129	1.0197	0.0184	0.9917	0.0265	0.9662
0.0013	1.2525	0.0103	1.0130	0.0148	0.9924	0.0213	0.9741

Table 5.1: EXAMPLE 1: Number of iterations, meshsizes, errors, rates of convergence, for the $\mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{N}_0 - \mathrm{P}_1$ approximation of the MHD problem, with $\nu = k = 1$ and $\nu_m = 1000$.

Theorem 4.12 and Corollary 4.13 are attained for the unknowns and for all the post-processed variables. Next, In Figure 5.1 we display some streamlines of $\mathbf{u}_h$, isosurfaces of p_h and the magnitud of the magnetic field $\mathbf{b}_h$ (from top to the bottom to the right) and we compare them with their exact counterparts (to the left). There we observe that the mixed finite element method provides accurate approximations to the unknowns.

Finally, in our second example we test our method in a non-convex domain. In fact, we consider the Fichera's corner domain $\Omega := (-1,1)^3 \backslash [0,1)^3$, where, due to the regularity of the Neumann problem (see [11, 12]), there holds $\mathbf{H}_0(\mathrm{curl}\,;\Omega) \cap \mathbf{H}(\mathrm{div}^0;\Omega) \subseteq \mathbf{H}^s(\Omega)$ for $s \in (\frac{1}{2}, \frac{2}{3})$. Here we set $\nu = \kappa = \nu_m = 1$, consider the initial guess as $\mathbf{u}_h^0 = \mathbf{0}$, and take $\mathbf{f}$ and $\mathbf{g}$ and $\mathbf{u}_D$ so that the exact solution is given by

$$\begin{aligned}
\mathbf{u}(\mathbf{x}) &:= \begin{pmatrix} -x_1(x_2 - x_3)(x_2 + x_3) \\ x_2(x_1 - x_3)(x_1 + x_3) \\ -x_3(x_1 - x_2)(x_1 + x_2) \end{pmatrix}, \\
p(\mathbf{x}) &:= x_1 x_2 x_3 - C_p\,, \\
\mathbf{b}(\mathbf{x}) &:= \mathrm{curl}\,\left(\sin^2(\pi x)\sin^2(\pi y)\sin^2(\pi z)(1,1,1)^t\right), \\
r(\mathbf{x}) &:= \sin(\pi x_1)\sin(\pi x_2)\sin(\pi x_3),
\end{aligned}$$

where C_p is chosen in such a way $\int_\Omega p = 0$. Next, in Table 5.2 we observe that the rates of convergence predicted by Theorem 4.12 and Corollary 4.13 are attained for all the variables.

References

[1] C. AMROUCHE, C. BERNARDI, M. DAUGE AND V. GIRAULT *Vector potentials in three-dimensional non-smooth domains*. Math. Models Appl. Sci. vol. 21,pp. 823–864, (1998).

[2] F. BREZZI AND M. FORTIN, *Mixed and Hybrid Finite Element Methods*. Springer Series in Computational Mathematics, 15. Springer-Verlag, New York, 1991.

Iter.	h	$\mathbf{e}_{\boldsymbol{\sigma}}$	$R(\boldsymbol{\sigma})$	$\mathbf{e}_{\mathbf{u}}$	$R(\mathbf{u})$	$\mathbf{e}_{\mathbf{b}}$	$R(\mathbf{b})$	e_r	$R(r)$
5	0.7071	60.0066	–	0.5044	–	39.3466	–	4.0629	–
5	0.3536	36.5120	0.7167	0.1930	1.3863	18.3152	1.1032	2.4151	0.7504
5	0.2357	23.7066	1.0652	0.1254	1.0620	12.5472	0.9328	1.6694	0.9108
5	0.1768	18.0103	0.9553	0.0938	1.0095	9.5006	0.9668	1.2684	0.9547
5	0.1414	14.4967	0.9726	0.0750	1.0029	7.6339	0.9804	1.0210	0.9727
5	0.1479	12.1062	0.9883	0.0625	1.0011	6.3766	0.9870	0.8536	0.9817

e_p	$R(p)$	$\mathbf{e}_{\boldsymbol{\omega}}$	$R(\boldsymbol{\omega})$	$\mathbf{e}_{\mathbf{G}}$	$R(\mathbf{G})$	$\mathbf{e}_{\tilde{\boldsymbol{\sigma}}}$	$R(\tilde{\boldsymbol{\sigma}})$
4.1262	–	3.2673	–	5.3569	–	9.6858	–
1.2108	1.7689	1.1712	1.4802	1.7248	1.6350	3.0799	1.6530
0.6664	1.4726	0.7146	1.2183	1.0343	1.2613	1.8242	1.2918
0.4156	1.6415	0.5101	1.1717	0.7287	1.2174	1.2395	1.3433
0.2819	1.7390	0.3932	1.1663	0.5575	1.1997	0.9169	1.3511
0.2034	1.7898	0.3189	1.1487	0.4502	1.1722	0.7201	1.3247

Table 5.2: Example 2: Number of iterations, meshsizes, errors, rates of convergence, for the $\mathbf{RT}_0 - \mathrm{P}_0 - \mathbf{N}_0 - \mathrm{P}_1$ approximation of the MHD problem, with $\nu = k = \nu_m = 1$.

[3] J. Camaño, C. García and R. Oyarzúa, *Analysis of a conservative mixed-FEM for the stationary Navier-Stokes problem.* Preprint 2018-25, Centro de Investigación en Ingeniería Matemática (CI²MA), UDEC, (2018).

[4] J. Camaño, C. Muñoz and R. Oyarzúa, *Numerical analysis of a dual-mixed problem in non-standard Banach spaces.* Electronic Transactions on Numerical Analysis, vol. 48, pp. 114-130, (2018).

[5] E. Colmenares, G.N. Gatica, S. Moraga, *A Banach spaces-based analysis of a new fully-mixed finite element method for the Boussinesq problem.* ESAIM: Mathematical Modelling and Numerical Analysis, DOI: https://doi.org/10.1051/m2an/2020007, (2020).

[6] E. Colmenares, G.N. Gatica and R. Oyarzúa, *Fixed point strategies for mixed variational formulations of the stationary Boussinesq problem.* Comptes Rendus Mathematique, vol. 354, 1, pp. 57–62, (2016).

[7] M. Costabel and M. Dauge, *Weighted regularization of Maxwell equations in polyhedral domains.* Numer. Math.94 (2002) 239–277.

[8] P.G. Ciarlet, *The finite Element Method for Elliptic Problems.* North-Holland, Amsterdam, New York, Oxford, (1978).

[9] T. Davis, *Algorithm 832: UMFPACK V4.3 - an unsymmetric-pattern multifrontal method.* ACM Transactions on Mathematical Software, vol. 30, pp. 196–199, (2004).

[10] A. Ern and J.-L.Guermond, *Theory and Practice of Finite Elements.* Applied Mathematical Sciences, Springer-Verlag, New-York, vol. 159,(2004).

[11] M. Dauge, *Elliptic boundary value problems on corner domain*, vol. 1341 of Lecture Notes in Mathematics, Springer-Verlag, Berlin, (1988).

[12] M. Dauge, *Regularity and singularities in polyhedral domains*, https://perso.univ-rennes1.fr/monique.dauge/publis/Talk_Karlsruhe08.pdf, (2008).

[13] S. J. Fromm, *Potential space estimates for Green potentials in convex domains.* Proceedings of the American Mathematical Society, vol. 119, no. 1, pp. 225–233, (1993).

[14] G.N. Gatica, *A Simple Introduction to the Mixed Finite Element Method. Theory and Applications* Springer Briefs in Mathematics, Springer, Cham Heidelberg New York Dordrecht London, (2014).

[15] J.-F. Gerbeau, *A stabilized finite element method for the incompressible magnetohydrodynamic equations* Numer. Math. 87 (2000) 83–111.

[16] V. Girault and P.-A. Raviart, Finite Element Approximation of the Navier–Stokes Equations. Lecture Notes in Mathematics, 749. Springer-Verlag, Berlin-New York, (1979).

[17] Ch. Greif, D. Li, D. Schötzau and X. Wei, *A mixed finite element method with exactly divergence-free velocities for incompressible magnetohydrodynamics.* Comput. Methods Appl. Mech. Engrg. , vol. 199, pp. 2840–2855, (2010).

[18] M.D. Gunzburger, A.J. Meir and J.S. Peterson, *On the existence and uniqueness and finite element approximation of solutions of the equations of stationary incompressible magnetohydrodynamics.* Math. Comp. vol. 56, 523–563 (1991).

[19] U. Hasler, A. Schneebeli and D. Schötzau, *Mixed finite element approximation of incompressible MHD problems based on weighted regularization.* Appl. Numer. Math. 51 (2004) 19–45.

[20] F. Hecht, *New development in FreeFem++.* J. Numer. Math, 20,(2012), 251–265.

[21] R. Hiptmair, *Finite elements in computational electromagnetism.* Acta Numerica, vol. 11, pp. 237–339, (2002).

[22] P. Houston, D. Schötzau and X. Wei, *A mixed DG method for linearized incompressible magnetohydrodynamics.* J. Sci. Comp. vol. 40, pp. 281–314, (2009).

[23] K. Hu and J. Xu *Structure-preserving finite element methods for statonary MHD models.* Math. Comp. vol. 88, 316, pp. 553–581, (2019).

[24] K. Hu, Y. Ma and J. Xu *Stable finite element methods preserving* $\nabla \cdot \boldsymbol{B} = 0$ *exactly for MHD models.* Math. Comp. vol. 88, 316, pp. 553–581, (2019).

[25] P. Monk, *Finite Element Methods for Maxwell's Equations* Numerical Mathematics and Scientific Computation, Springer, Oxford University Press, New York, (2003).

[26] A. Quarteroni and A. Valli *Numerical approximation of partial differential equations.* Springer Series in Computational Mathematics, 23. Springer-Verlag, Berlin, 1994.

[27] D. Schötzau, *Mixed finite element methods for stationary incompressible magnetohydrodynamics.* Numerische Mathematik , vol. 96, pp. 771–800, (2004).

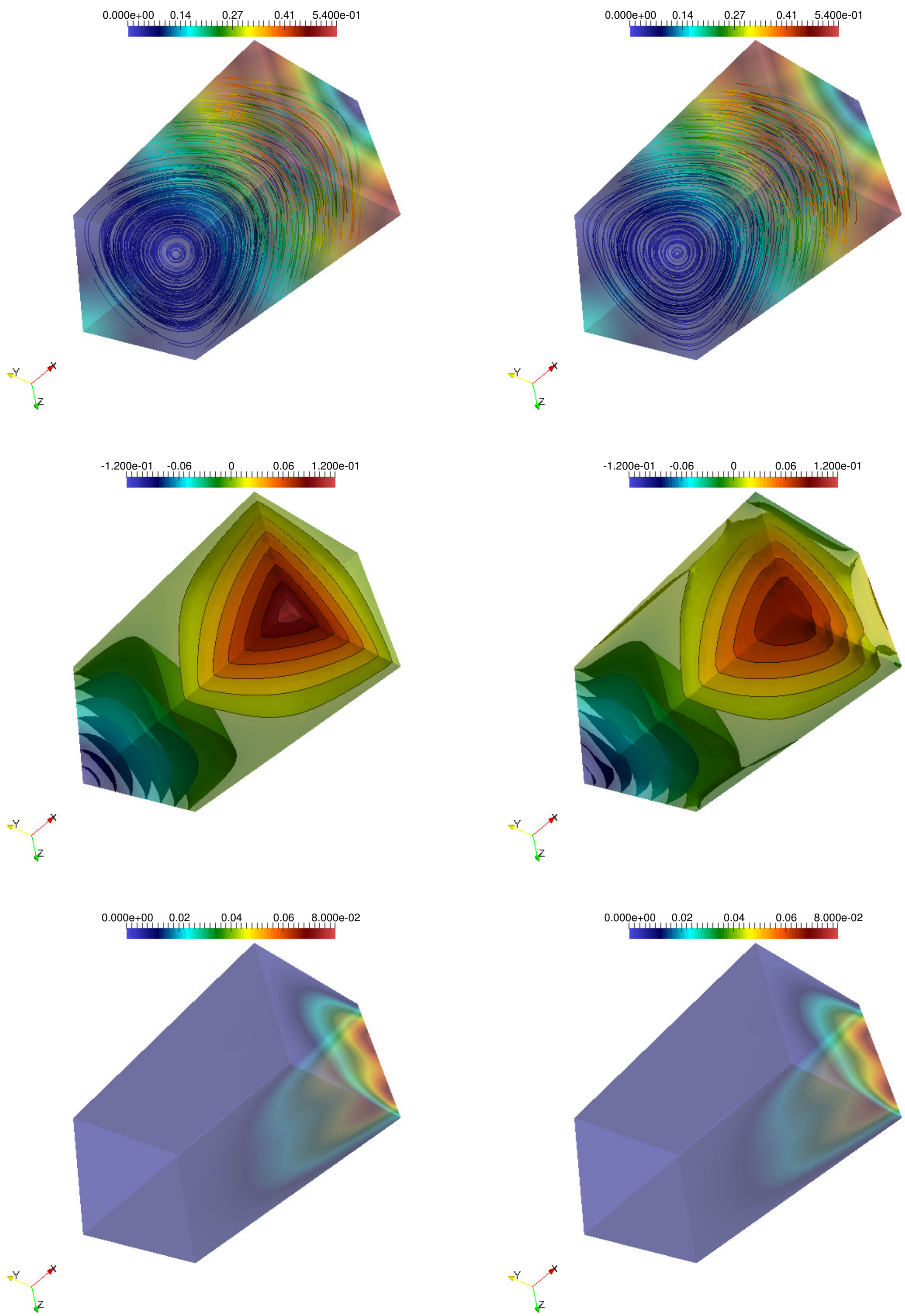


Figure 5.1: Example 1: Some streamlines of $\mathbf{u}_h$, isosurfaces of p_h and the magnitud of the magnetic field $\mathbf{b}_h$ (from top to the bottom to the right) and their exact counterparts (to the left).

Centro de Investigación en Ingeniería Matemática (CI²MA)

PRE-PUBLICACIONES 2020

2020-02 Gonzalo A. Benavides, Leonardo E. Figueroa: *Orthogonal polynomial projection error in Dunkl-Sobolev norms in the ball*

2020-03 Rodolfo Araya, Abner Poza, Frederic Valentin: *An adaptative multiscale hybrid-mixed method for the Oseen equations*

2020-04 Carlos Parés, David Zorío: *Lax Wendroff approximate Taylor methods with fast and optimized weighted essentially non-oscillatory reconstructions*

2020-05 Julio Aracena, Lilian Salinas: *Finding the fixed points of a Boolean network from a positive feedback vertex set*

2020-06 Tomás Barrios, Rommel Bustinza, Camila Campos: *A note on a posteriori error estimates for dual mixed methods*

2020-07 Raimund Bürger, Elvis Gavilán, Daniel Inzunza, Pep Mulet, Luis M. Villada: *Numerical simulation of forest fires by IMEX methods*

2020-08 Rommel Bustinza, Jonathan Munguia: *An a priori error analysis for a class of nonlinear elliptic problems applying the hybrid high-order method*

2020-09 Ngoc-Cuong Nguyen, Jaime Peraire, Manuel Solano, Sébastien Terrana: *An HDG method for non-matching meshes*

2020-10 Rommel Bustinza, Jonathan Munguia: *A mixed Hybrid High-Order formulation for linear interior transmission elliptic problems*

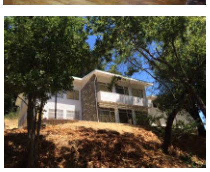
2020-11 Gabriel N. Gatica, Antonio Marquez, Salim Meddahi: *A mixed finite element method with reduced symmetry for the standard model in linear viscoelasticity*

2020-12 Raimund Bürger, Gerardo Chowell, Leidy Y. Lara-Diaz: *Measuring the distance between epidemic growth models*

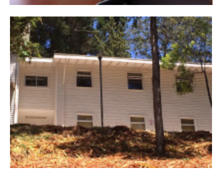
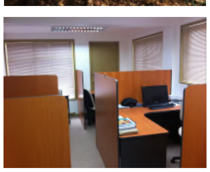
2020-13 Jessika Camaño, Carlos Garcia, Ricardo Oyarzúa: *Analysis of a new mixed-FEM for stationary incompressible magneto-hydrodynamics*

Para obtener copias de las Pre-Publicaciones, escribir o llamar a: Director, Centro de Investigación en Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile, Tel.: 41-2661324, o bien, visitar la página web del centro: http://www.ci2ma.udec.cl



Centro de Investigación en
Ingeniería Matemática (CI²MA)
Universidad de Concepción

Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
http://www.ci2ma.udec.cl