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CENTRO DE INVESTIGACIÓN EN
INGENIERÍA MATEMÁTICA (CI²MA)



A posteriori error estimates for mixed virtual element methods

ANDREA CANGIANI, MAURICIO MUNAR

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A POSTERIORI ERROR ESTIMATES FOR MIXED VIRTUAL ELEMENT METHODS

ANDREA CANGIANI¹ AND MAURICIO MUNAR²

Abstract. We present an a posteriori error analysis for the mixed virtual element method (mixed VEM) applied to second order elliptic equations in divergence form with mixed boundary conditions. The resulting error estimator is of residual-type. It only depends on quantities directly available from the VEM solution and applies on very general polygonal meshes. The proof of the upper bound relies on a global inf-sup condition, a suitable Helmholtz decomposition, and the local approximation properties of a Clément-type interpolant. In turn, standard inverse inequalities and localization techniques based on bubble functions are the main tools yielding the lower bound. Via the inclusion of a fully local postprocessing of the mixed VEM solution, we also show that the estimator provides a reliable and efficient control on the broken $H(\text{div})$ -norm error between the exact and the postprocessed flux. Numerical examples confirm the theoretical properties of our estimator, and show that it can be effectively used to drive an adaptive mesh refinement algorithm.

Mathematics Subject Classification. 65N30, 65N12.

1. INTRODUCTION

The Virtual Element Method (VEM) was originally introduced in [5] for the solution of elliptic problems, followed by the mixed VEM proposed in [15]. Subsequently, new mixed VEMs have been analysed for the solution of the Stokes, Navier-Stokes, and Brinkman problem [3, 8, 9, 17, 18, 21, 30, 34].

One of the defining characteristics of the VEM is that it allows for the use of very general polygonal and polyhedral meshes. As such, it naturally lends itself as a flexible solution step within automatically adaptive algorithms. Indeed, mesh refinement and coarsening strategies can be implemented very easily and efficiently as, for instance, hanging nodes are simply treated as new nodes, with no detrimental affect on the quality of the approximation. Moreover, it has been shown that the VEM in primal form allows for extremely aggressive mesh adaptation producing strongly solution-adapted polygonal meshes [20].

In this respect, the design and analysis of adaptive mesh refinement strategies based on robust a posteriori error indicators for the VEM approach and, in particular, for the mixed-VEM is an attractive proposition.

Several error estimators have been proposed in the context of VEM for primal forms (see, e.g., [12, 13, 20, 24, 31, 32]). Firstly, the authors of [12] proposed a posteriori error bounds for the C^1 -conforming VEM for the two-dimensional Poisson problem. Next, a posteriori error bounds for the C^0 -conforming VEM for the discretization of second-order linear elliptic reaction-convection-diffusion problems with nonconstant coefficients in two and three dimension were proposed in [20], whereas a residual-based a posteriori error estimator for the VEM

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¹ Department of Mathematics, University of Nottingham, University Park, Nottingham NG7 2RD, UK,
email: Andrea.Cangiani@nottingham.ac.uk

² CI²MA and Departamento de Ingeniería Matemática, Universidad de Concepción, Casilla 160-C, Concepción, Chile,
email: mmunar@ci2ma.udec.cl

discretization of the Poisson problem with discontinuous diffusivity coefficient was introduced and analysed in [13]. Moreover, in [31] and [32], the authors developed a posteriori error analysis of a VEM approach for the Steklov eigenvalue problem and the spectral analysis for the elasticity equations, respectively. Finally, in [24] a general recovery-based a posteriori error estimation framework for the VEM of arbitrary order on general polygonal/polyhedral meshes has been developed. A posteriori analyses of other techniques of mixed-type on general meshes have been presented in [25] for the Mixed High-Order method, in [10] for the Mimetic Finite Difference method, and in [36] for lowest-order locally conservative methods. However, to the best of our knowledge, *no* a posteriori error analysis for mixed VEM is available from the literature.

The aim of this paper is to introduce the basic tools to develop the a posteriori error analysis for the mixed VEM. To this end, we consider a second order elliptic equation in divergence form with mixed boundary condition, discretised using the basic mixed VEM of [7]. As usual in the VEM approach, we introduce fully computable approximations for the virtual approximation of the flux variable and establish its corresponding a priori error estimates. In particular, in order to improve on the sub-optimal order provided by the computable component of the flux variable in the broken $H(\text{div})$ -norm, observed numerically in [29], we follow [18] and construct by postprocessing second computable approximation of the flux variable, which has an optimal rate of convergence in the aforementioned norm.

The a posteriori error analysis is based on a global inf-sup condition coming from the well-posedness of the continuous problem. Upper bounds are shown for the scalar variable in the L^2 -norm, the VEM flux variable in the $H(\text{div})$ -norm, its projection in the L^2 -norm, and postprocessing in the broken $H(\text{div})$ -norm. The proof uses properties of the interpolation operator associated to the virtual subspace of the flux variable and Clément-type interpolation operators, together with a suitable Helmholtz decomposition. Moreover, some inverse inequalities and localization techniques based on bubble functions will serve to show a lower bound for the error. In this way, we are able to establish the equivalence up to virtual inconsistency terms between the error and the error estimator for the postprocessing of the virtual element approximation, measured in the broken $H(\text{div})$ -norm.

1.1. Outline

The remainder of the paper has been structured as follows. In what is left of this section, we introduce some standard notations and the required functional spaces. In Section 2 we introduce the model problem and presents the associate variational formulation. In Section 3, we present the mixed virtual element scheme. The a posteriori error analysis is laid down in details in Section 5. In Section 6, we propose an adaptive algorithm and test its effectiveness with some numerical examples. Finally, in Section 7 we give some concluding remarks.

1.2. Preliminaries

Let us assume that $\Omega \subset \mathbb{R}^2$ be a bounded domain with polygonal boundary Γ . We denote by ν the outward unit normal vector to the boundary Γ . Moreover, we assume that Γ admits a disjoint partition $\Gamma = \bar{\Gamma}_D \cup \bar{\Gamma}_N$, where Γ_D and Γ_N are open subsets of Γ , with $|\Gamma_D|, |\Gamma_N| \neq 0$.

For $s \geq 0$, the symbol $|\cdot|_{s,\Omega}$ stands for the norm of the Hilbertian Sobolev spaces $H^s(\Omega)$, with the convention $H^0(\Omega) := L^2(\Omega)$. We also define the Hilbert space

$$H(\text{div}; \Omega) := \left\{ \boldsymbol{\tau} \in [L^2(\Omega)]^2 : \text{div } \boldsymbol{\tau} \in L^2(\Omega) \right\},$$

whose norm is given by $\|\boldsymbol{\tau}\|_{\text{div};\Omega}^2 := \|\boldsymbol{\tau}\|_{0,\Omega}^2 + \|\text{div } \boldsymbol{\tau}\|_{0,\Omega}^2$. Hereafter, we use the following notation for any vector field $\boldsymbol{\tau} = (\tau_i)_{i=1,2}$ and any scalar field v :

$$\text{div } \boldsymbol{\tau} := \partial_1 \tau_1 + \partial_2 \tau_2 \quad \text{rot } \boldsymbol{\tau} := \partial_1 \tau_2 - \partial_2 \tau_1 \quad \text{and} \quad \mathbf{rot } v := (\partial_2 v, -\partial_1 v)^t.$$

Additionally, we need to introduce the following spaces

$$H := \left\{ \boldsymbol{\tau} \in H(\text{div}; \Omega) : \boldsymbol{\tau} \cdot \nu = 0 \quad \text{on } \Gamma_N \right\} \quad \text{and} \quad Q := L^2(\Omega), \quad (1)$$

endorsed with the norms

$$\|\tau\|_H := \|\tau\|_{0,\Omega} + \|\operatorname{div} \tau\|_{0,\Omega} \quad \text{and} \quad \|v\|_Q := \|v\|_{0,\Omega}.$$

Furthermore, we make use of the product space $H \times Q$ with the norm

$$\|(\tau, v)\|_{H \times Q} := \|\tau\|_H + \|v\|_Q.$$

In addition, we will denote with c and C , with or without subscripts, tildes, or hats, a generic constant independent of the mesh parameter h , which may take different values in different occurrences.

2. THE MODEL PROBLEM

We consider the problem

$$-\operatorname{div}(\kappa \nabla u) = f \quad \text{in } \Omega, \quad u = g \quad \text{on } \Gamma_D \quad \text{and} \quad (\kappa \nabla u) \cdot \nu = 0 \quad \text{on } \Gamma_N, \quad (2)$$

where $f \in L^2(\Omega)$, $g \in H^{1/2}(\Gamma_D)$ and $\kappa \in [L^\infty(\Omega)]^{2 \times 2}$ is an uniformly positive definite tensor, which is assumed to be known. In particular, we denote by κ^* the positive constant satisfying

$$\kappa^{-1} \zeta \cdot \zeta \geq \kappa^* |\zeta|^2, \quad \forall \zeta \in [L^2(\Omega)]^2. \quad (3)$$

By introducing the flux variable $\sigma := \kappa \nabla u$ in Ω as additional unknown, a mixed variational formulation of (2) becomes:

Find $(\sigma, u) \in H \times Q$ such that

$$\begin{aligned} a(\sigma, \tau) + b(\tau, u) &= \langle \tau \cdot \nu, g \rangle_{\Gamma_D} \quad \forall \tau \in H, \\ b(\sigma, v) &= - \int_{\Omega} f v \quad \forall v \in Q, \end{aligned} \quad (4)$$

where $\langle \cdot, \cdot \rangle$ stands for the duality pairing between $H^{-1/2}(\Gamma_D) \rightarrow H^{1/2}(\Gamma_D)$. In turn, $a : H \times H \rightarrow \mathbb{R}$ and $b : H \times Q \rightarrow \mathbb{R}$ are the bounded bilinear forms defined by

$$a(\sigma, \tau) := \int_{\Omega} \kappa^{-1} \sigma \cdot \tau, \quad \text{and} \quad b(\tau, u) := \int_{\Omega} u \operatorname{div} \tau. \quad (5)$$

Under the assumptions on κ, f and g , the existence and uniqueness of the weak solution of (4) is consequence of the Babuška-Brezzi theory.

3. THE VIRTUAL ELEMENT METHOD

Let $\{\mathcal{T}_h\}_{h>0}$ be a family of decompositions of Ω into open non-overlapping polygonal elements. Then, for each $K \in \mathcal{T}_h$ we denote its diameter by h_K , and also, as usual, $h := \max \{h_K : K \in \mathcal{T}_h\}$. In what follows we make the following mesh regularity assumptions which are standard in this context (cf. [5, 15]).

Assumption 3.1. *The family of decompositions $\{\mathcal{T}_h\}_{h>0}$ satisfies:*

- a) *the ratio between the shortest edge and the diameter h_K of K is bigger than $C_{\mathcal{T}}$, and*
- b) *K is star-shaped with respect to a ball B of radius $C_{\mathcal{T}} h_K$ and center $\mathbf{x}_B \in K$.*

Remark 3.2. The above assumptions imply that each $K \in \mathcal{T}_h$ is simply connected and that there exists an integer $N_{\mathcal{T}}$ (depending only on $C_{\mathcal{T}}$), such that the numbers of edges of each $K \in \mathcal{T}_h$ is bounded above by $N_{\mathcal{T}}$. Moreover, as each element K is star-shaped, it admits a sub-triangulation \mathcal{T}_h^K obtained by joining each vertex of K with a point with respect to which K is starred. And the uniform bound on the diameter of the mesh edges ensures that the resulting global triangulation $\widehat{\mathcal{T}}_h := \bigcup_{K \in \mathcal{T}_h} \mathcal{T}_h^K$ is shape-regular. We finally note that the

above assumptions allow for very general possibly non-convex polygonal elements. In particular, they permit the natural incorporation of so-called hanging nodes, thus completely avoiding the need of removing hanging nodes typical of standard mesh adaptation algorithms.

Now, given an integer $\ell \geq 0$ and $\mathcal{O} \subseteq \mathbb{R}^d$, $d = 1, 2$, we denote by $\mathbb{P}_\ell(\mathcal{O})$ the space of polynomials on \mathcal{O} of degree up to ℓ . Then, given an edge $e \in \partial K$ with barycentric x_e and diameter h_e , we denote the following set of $(\ell + 1)$ normalized monomials on e

$$\mathcal{B}_\ell(e) := \left\{ \left(\frac{x - x_e}{h_e} \right)^j \right\}_{0 \leq j \leq \ell},$$

which certainly constitutes a basis on $\mathbb{P}_\ell(e)$. Similarly, on $K \in \mathcal{T}_h$ with barycenter \mathbf{x}_K , we define the following set of $\frac{1}{2}(\ell + 1)(\ell + 2)$ normalized monomials

$$\mathcal{B}_\ell(K) := \left\{ \left(\frac{\mathbf{x} - \mathbf{x}_K}{h_K} \right)^\alpha \right\}_{0 \leq |\alpha| \leq \ell},$$

which is a basis of $\mathbb{P}_\ell(K)$. Notice that in the definition of $\mathcal{B}_\ell(K)$ above, we made use of the multi-index notation, that is, given $\mathbf{x} := (x_1, x_2)^\mathbf{t} \in \mathbb{R}^2$ and $\alpha := (\alpha_1, \alpha_2)^\mathbf{t}$, with non-negative integers α_1, α_2 , we set $\mathbf{x}^\alpha := x_1^{\alpha_1} x_2^{\alpha_2}$ and $|\alpha| := \alpha_1 + \alpha_2$.

We further let $\mathcal{G}_\ell(K)$ be a basis of $(\nabla \mathbb{P}_{\ell+1}(K)) \cap [\mathbb{P}_\ell(K)]^2$, whereas with $\mathcal{G}_\ell^\perp(K)$ we denote a basis of the $[\mathbb{L}^2(K)]^2$ -orthogonal of $\mathcal{G}_\ell(K)$ in $[\mathbb{P}_\ell(K)]^2$.

Throughout the paper, we denote by $\Pi_k^0 : \mathbb{L}^2(K) \rightarrow \mathbb{P}_k(K)$ the $\mathbb{L}^2(K)$ -orthogonal projection onto the space $\mathbb{P}_k(K)$, for any $K \in \mathcal{T}_h$ and $k \geq 0$. In addition, we will make use of a vectorial version of the aforementioned projector, which is denoted by $\mathbf{\Pi}_k^0$. The following approximation properties of these projectors are well-known:

$$\|v - \Pi_k^0(v)\|_{0,K} \leq Ch_K^m |v|_{m,K} \quad \text{and} \quad \|\boldsymbol{\tau} - \mathbf{\Pi}_k^0(\boldsymbol{\tau})\|_{0,K} \leq Ch_K^m |\boldsymbol{\tau}|_{m,K} \quad (6)$$

for all $K \in \mathcal{T}_h$, and for all $v \in \mathbb{H}^m(K)$, $\boldsymbol{\tau} \in [\mathbb{H}^m(K)]^2$, with $m \in \{0, 1, \dots, k+1\}$.

4. VIRTUAL SUBSPACES AND ITS APPROXIMATION PROPERTIES

For any integer $k \geq 0$, we introduce the finite dimensional subspaces of H and Q , respectively, given by

$$H_h := \left\{ \boldsymbol{\tau} \in H : \boldsymbol{\tau}|_K \in H_h^K \quad \forall K \in \mathcal{T}_h \right\}, \quad (7)$$

and

$$Q_h := \left\{ v \in Q : v|_K \in Q_h^K \quad \forall K \in \mathcal{T}_h \right\}, \quad (8)$$

where $Q_h^K := \mathbb{P}_k(K)$, and H_h^K is the virtual element space introduced in [7, Section 3.1]. This is defined by

$$H_h^K := \left\{ \boldsymbol{\tau} \in \mathbb{H}(\text{div}; K) \cap \mathbb{H}(\text{rot}; K) : \boldsymbol{\tau} \cdot \boldsymbol{\nu}|_e \in \mathbb{P}_k(e) \quad \forall \text{edge } e \in \partial K, \right. \\ \left. \text{div } \boldsymbol{\tau} \in \mathbb{P}_k(K) \quad \text{and} \quad \text{rot } \boldsymbol{\tau} \in \mathbb{P}_{k-1}(K) \right\}. \quad (9)$$

and is characterised by the following degrees of freedom (cf. [6,7]):

$$\begin{aligned} \int_e q(\boldsymbol{\tau} \cdot \boldsymbol{\nu}) & \quad \forall q \in \mathcal{B}_k(e), \quad \forall \text{edge } e \text{ in } \mathcal{T}_h, \\ \int_K \boldsymbol{\tau} \cdot \nabla q & \quad \forall q \in \mathcal{B}_k(K) \setminus \{1\}, \quad \forall K \in \mathcal{T}_h, \\ \int_K \boldsymbol{\tau} \cdot \boldsymbol{\eta} & \quad \forall \boldsymbol{\eta} \in \mathcal{G}_k^\perp(K), \quad \forall K \in \mathcal{T}_h. \end{aligned} \quad (10)$$

As was remarked in [7, Section 3.2] (see also [6, Section 3.5]), the degrees of freedom (10) allow the explicit computation of the projection $\boldsymbol{\Pi}_k^0(\boldsymbol{\tau})$ using only the degrees of freedom of $\boldsymbol{\tau}$. Moreover, collected together, the local degrees of freedom (10) provide a set of degrees of freedom for the global virtual element space H_h .

For each $\boldsymbol{\tau} \in H$ such that $\boldsymbol{\tau}|_K \in [\mathbf{H}^1(K)]^2$ for all $K \in \mathcal{T}_h$, we may denote by $\boldsymbol{\tau}_I \in H_h$ the Lagrange interpolant of $\boldsymbol{\tau}$ with respect to the degrees of freedom (10). For each $q \in \mathcal{B}_k(K)$ we find that

$$\int_K q \operatorname{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_I) = - \int_K (\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \nabla q + \int_{\partial K} q(\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \boldsymbol{\nu} = 0,$$

which, thanks to the fact that $\operatorname{div} \boldsymbol{\tau}_I \in P_k(K)$, implies the commutative property

$$\operatorname{div} \boldsymbol{\tau}_I = \boldsymbol{\Pi}_k^0(\operatorname{div} \boldsymbol{\tau}) \quad \forall \boldsymbol{\tau} \in [\mathbf{H}^1(K)]^2. \quad (11)$$

Hence we have the following approximation error estimates [5,7].

Lemma 4.1. *Let r be an integer such that $1 \leq r \leq k+1$. Then, there exists a constant $C > 0$, independent of K , such that for each $\boldsymbol{\tau} \in [\mathbf{H}^r(K)]^2$ such that $\operatorname{div} \boldsymbol{\tau} \in \mathbf{H}^r(K)$ there holds*

$$\|\boldsymbol{\tau} - \boldsymbol{\tau}_I\|_{\operatorname{div};K} \leq C h_K^r \left\{ |\boldsymbol{\tau}|_{r,K} + |\operatorname{div} \boldsymbol{\tau}|_{r,K} \right\} \quad \forall K \in \mathcal{T}_h. \quad (12)$$

Proof. The bound on the divergence term follows from (11) and (6). The result then follows from classical arguments [14]. \square

4.1. Discrete formulation

We now aim to define a virtual scheme for our problem (4) based on the discrete spaces (7) and (8). To this end, we first notice that the bilinear form b (cf.(5)) is explicitly computable for all $(\boldsymbol{\tau}, v) \in H_h \times Q_h$, just by accessing the degrees of freedom (10). On the contrary, for each $K \in \mathcal{T}_h$, the local version $a^K : H_h^K \times H_h^K \rightarrow \mathbb{R}$ of the bilinear form a , which, is defined for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in H_h^K \times H_h^K$ by

$$a^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \int_K \boldsymbol{\kappa}^{-1} \boldsymbol{\zeta} \cdot \boldsymbol{\tau}, \quad (13)$$

is not explicitly computable for $\boldsymbol{\zeta}, \boldsymbol{\tau} \in H_h^K$ since in general $\boldsymbol{\zeta}$ and $\boldsymbol{\tau}$ are not known explicitly on the whole of K . In order to deal with this difficulty, we follow [7, Section 3.3] and introduce a local bilinear form $a_h^K : H_h^K \times H_h^K \rightarrow \mathbb{R}$ defined by

$$a_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) := a^K(\boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}), \boldsymbol{\Pi}_k^0(\boldsymbol{\tau})) + \mathcal{S}^K(\boldsymbol{\zeta} - \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}), \boldsymbol{\tau} - \boldsymbol{\Pi}_k^0(\boldsymbol{\tau})), \quad (14)$$

where $\mathcal{S}^K : H_h^K \times H_h^K \rightarrow \mathbb{R}$ is any symmetric and positive definite bilinear form such that

$$\widehat{c}_0 a^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \leq \mathcal{S}^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \leq \widehat{c}_1 a^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in H_h^K, \quad \text{with } \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}) = 0, \quad (15)$$

with constants $\widehat{c}_0, \widehat{c}_1 > 0$ which depend only on the shape regularity constant $C_{\mathcal{T}}$ and on $\boldsymbol{\kappa}$. In particular, to define \mathcal{S}^K we can consider the bilinear form associated to the identity matrix in $\mathbb{R}^{n_k^K}$ with respect to the local basis determined by the degrees of freedom (10), and where $n_k^K = \dim H_h^K$. (cf. [5, 15])

The following two lemmas establish the properties of the bilinear form a_h^K and the consistency error between a_h^K and a^K , respectively.

Lemma 4.2. *For all $K \in \mathcal{T}_h$, there holds*

$$\text{(Consistency)} \quad a_h^K(p, \boldsymbol{\zeta}) = \int_K \boldsymbol{\kappa}^{-1} p \cdot \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}) \quad \forall p \in [\mathbb{P}_k(K)]^2 \quad \text{and} \quad \forall \boldsymbol{\zeta} \in H_h^K,$$

and further, there exist constants $\alpha_*, \alpha^* > 0$, such that

$$\text{(Stability)} \quad \alpha_* a^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \leq a_h^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \leq \alpha^* a^K(\boldsymbol{\zeta}, \boldsymbol{\zeta}) \quad \forall \boldsymbol{\zeta} \in H_h^K, \forall K \in \mathcal{T}_h.$$

Proof. We refer to [7] and [11] for the details. \square

Lemma 4.3. *There exists a constant $C > 0$, depending only on $\boldsymbol{\kappa}, \widehat{c}_1$ and α^* , such that*

$$(a_h^K - a^K)(\boldsymbol{\zeta}, \boldsymbol{\tau}) \leq C \left\{ \|\boldsymbol{\zeta} - \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta})\|_{0,K} + \|\boldsymbol{\kappa}^{-1} \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}) - \boldsymbol{\Pi}_k^0(\boldsymbol{\kappa}^{-1} \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}))\|_{0,K} \right\} \|\boldsymbol{\tau}\|_{0,K}$$

for all $\boldsymbol{\zeta}, \boldsymbol{\tau} \in H_h^K$ and for all $K \in \mathcal{T}_h$.

Proof. We have that

$$\begin{aligned} (a_h^K - a^K)(\boldsymbol{\zeta}, \boldsymbol{\tau}) &= - \int_K \left\{ \boldsymbol{\kappa}^{-1} \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}) - \boldsymbol{\Pi}_k^0(\boldsymbol{\kappa}^{-1} \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta})) \right\} \cdot (\boldsymbol{\tau} - \boldsymbol{\Pi}_k^0(\boldsymbol{\tau})) - \int_K \boldsymbol{\kappa}^{-1} (\boldsymbol{\zeta} - \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta})) \cdot \boldsymbol{\tau} \\ &\quad + \mathcal{S}^K(\boldsymbol{\zeta} - \boldsymbol{\Pi}_k^0(\boldsymbol{\zeta}), \boldsymbol{\tau} - \boldsymbol{\Pi}_k^0(\boldsymbol{\tau})). \end{aligned}$$

The results now follows from Cauchy-Schwarz inequality and the properties of the bilinear \mathcal{S}^K . \square

According to the definition (14) the global discrete bilinear form $a_h : H_h \times H_h \rightarrow \mathbb{R}$ can now be defined summing together the local contribution (14), that is

$$a_h(\boldsymbol{\zeta}, \boldsymbol{\tau}) := \sum_{K \in \mathcal{T}_h} a_h^K(\boldsymbol{\zeta}, \boldsymbol{\tau}) \quad \forall \boldsymbol{\zeta}, \boldsymbol{\tau} \in H_h. \quad (16)$$

In this way, the virtual element method associated with the formulation (4) reads:

Find $(\boldsymbol{\sigma}_h, u_h) \in H_h \times Q_h$ such that

$$\begin{aligned} a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) + b(\boldsymbol{\tau}_h, u_h) &= \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} \quad \forall \boldsymbol{\tau}_h \in H_h, \\ b(\boldsymbol{\sigma}_h, v_h) &= - \int_{\Omega} f v_h \quad \forall v_h \in Q_h. \end{aligned} \quad (17)$$

The well-posedness of (17) follows from Lemma 4.2 and of the well-posedness of (4). In addition, we have the following result about the a priori error estimates for the schemes (4) and (17).

Theorem 4.4. *Let $(\boldsymbol{\sigma}, u) \in H \times Q$ and $(\boldsymbol{\sigma}_h, u_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete schemes (4) and (17), respectively. In addition, assume that for some $s \in [1, k+1]$ there hold $\boldsymbol{\sigma}|_K \in \mathbf{H}^s(K)$ and $\operatorname{div} \boldsymbol{\sigma}|_K, u|_K \in H^s(K)$ for each $K \in \mathcal{T}_h$. Then, there exist a positive constant $C > 0$, independent of h , such that*

$$\|(\boldsymbol{\sigma}, u) - (\boldsymbol{\sigma}_h, u_h)\|_{H \times Q} \leq Ch^s \left\{ \sum_{K \in \mathcal{T}_h} |\boldsymbol{\sigma}|_{s,K}^2 + |\operatorname{div} \boldsymbol{\sigma}|_{s,K}^2 + |u|_{s,K}^2 \right\}^{1/2}. \quad (18)$$

Proof. The result is consequence of [15, Theorem 6.1] and of a straightforward application of the approximation properties provided by (6) and Lemma 4.1. \square

4.2. Computable approximations

A first fully computable approximation $\widehat{\boldsymbol{\sigma}}_h \in Q$ of the VEM solution $\boldsymbol{\sigma}_h \in H$ is given by

$$\widehat{\boldsymbol{\sigma}}_h := \mathbf{\Pi}_k^0(\boldsymbol{\sigma}_h). \quad (19)$$

The corresponding a priori error estimates for the error $\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_Q$ immediately follows from the foregoing Theorem 4.4 and the triangle inequality.

Theorem 4.5. *Let $(\boldsymbol{\sigma}, u) \in H \times Q$ and $(\boldsymbol{\sigma}_h, u_h) \in H_h \times Q_h$ be the unique solutions of the continuous and discrete schemes (4) and (17), respectively. In addition, assume that for some $s \in [1, k+1]$ there hold $\boldsymbol{\sigma}|_K \in \mathbf{H}^s(K)$ and $u|_K \in H^s(K)$ for each $K \in \mathcal{T}_h$. Then, there exists a positive constant $C > 0$, independent of h , such that*

$$\|\boldsymbol{\sigma} - \widehat{\boldsymbol{\sigma}}_h\|_Q + \|u - u_h\|_Q \leq Ch^s \left\{ \sum_{K \in \mathcal{T}_h} |\boldsymbol{\sigma}|_{s,K}^2 + |u|_{s,K}^2 \right\}^{1/2}. \quad (20)$$

Next, motivated by the non-satisfactory order provided by $\widehat{\boldsymbol{\sigma}}_h$ in the broken $H(\text{div})$ -norm (see [29, Section 5] for numerical evidences of this fact), we proceed as in [18, Section 5.3] (see also [19]) and construct, by local postprocessing, a second approximation $\boldsymbol{\sigma}_h^*$ for the flux variable $\boldsymbol{\sigma}$ which has an optimal rate of convergence in such norm. To this end, for each $K \in \mathcal{T}_h$ we let $(\cdot, \cdot)_{\text{div};K}$ be the usual $H(\text{div}; K)$ -inner product with induced norm $\|\cdot\|_{\text{div};K}$ and let $\boldsymbol{\sigma}_h^*|_K := \boldsymbol{\sigma}_{h,K}^* \in [P_{k+1}(K)]^2$ be the unique solution of the local problem

$$(\boldsymbol{\sigma}_{h,K}^*, \boldsymbol{\tau}_h)_{\text{div};K} = \int_K \widehat{\boldsymbol{\sigma}}_h \cdot \boldsymbol{\tau}_h + \int_K \text{div } \boldsymbol{\sigma}_h \text{ div } \boldsymbol{\tau}_h \quad \forall \boldsymbol{\tau}_h \in [P_{k+1}(K)]^2. \quad (21)$$

We stress that $\boldsymbol{\sigma}_{h,K}^*$ can be explicitly computed for each $K \in \mathcal{T}_h$, independently. Then, the rate of convergence for the broken $H(\text{div}; \Omega)$ -norm of $\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*$ is established as follows.

Theorem 4.6. *Assume that the hypotheses of Theorem 4.4 are satisfied. Then, there exists a positive constant C , independent of h , such that*

$$\left\{ \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,K}^*\|_{\text{div};K}^2 \right\}^{1/2} \leq Ch^s \left\{ \sum_{K \in \mathcal{T}_h} |\boldsymbol{\sigma}|_{s,K}^2 + |\text{div } \boldsymbol{\sigma}|_{s,K}^2 \right\}^{1/2}. \quad (22)$$

Proof. See [18, Section 5.3, Theorem 5.5]. \square

5. A POSTERIORI ERROR ANALYSIS

In this section we develop a residual-based a posteriori error analysis for the mixed virtual element scheme (17). The proof of the a posteriori upper bound on the error is based on a global inf-sup condition, (cf. [2]), and a suitable Helmholtz decomposition; the lower bound is derived as usual via techniques based on bubble functions together with inverse inequalities.

5.1. Preliminaries

We let $\mathcal{E}_h = \mathcal{E}_h(\Omega) \cup \mathcal{E}_h(\Gamma_D) \cup \mathcal{E}_h(\Gamma_N)$ be the set of all edges of \mathcal{T}_h , where $\mathcal{E}_h(\Omega) := \{e \in \mathcal{E}_h : e \subseteq \Omega\}$, $\mathcal{E}_h(\Gamma_D) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_D\}$, and $\mathcal{E}_h(\Gamma_N) := \{e \in \mathcal{E}_h : e \subseteq \Gamma_N\}$. And, for a given $K \in \mathcal{T}_h$, we denote by $\mathcal{E}(K) \subset \mathcal{E}_h$ the set of edges of K . Given an edge $e \in \mathcal{E}_h$, we let h_e be its length and we fix a unit normal

vector $\nu_e := (\nu_1, \nu_2)^\mathbf{t}$ and let $s_e := (-\nu_2, \nu_1)^\mathbf{t}$ be the corresponding unit tangential vector along e . However, when no confusion arises, we simply write ν and s instead of ν_e and s_e , respectively. Now, given $\zeta \in [L^2(\Omega)]^2$, for each $K \in \mathcal{T}_h$ and $e \in \mathcal{E}_h(\Omega) \cap \mathcal{E}(K)$ we denote by $\llbracket \zeta \cdot s \rrbracket$ the tangential jump of ζ across e , that is $\llbracket \zeta \cdot s \rrbracket := (\zeta|_K - \zeta|_{K'})|_e \cdot s$, where K and K' are the elements of \mathcal{T}_h having e as a common edge.

We first recall the conforming VEM spaces from [5], which will be used as an auxiliary space in the a posteriori analysis below. Given $k \geq 0$, we consider the space defined by

$$V_h := \{v \in H^1(\Omega) : v|_{\partial K} \in \mathbb{B}_k(\partial K) \text{ and } \Delta v \in P_{k-1}(K) \quad \forall K \in \mathcal{T}_h\},$$

where

$$\mathbb{B}_k(\partial K) := \{v \in C(\partial K) : v|_e \in P_{k+1}(e) \quad \forall \text{ edge } e \subseteq \partial K\}.$$

It has been shown in [33, Section 4, Proposition 4.2] that there exists an interpolation operator $\mathcal{I}_h : H^1(\Omega) \rightarrow V_h$, such that there holds

$$\|v - \mathcal{I}_h(v)\|_{0,K} + h_K |v - \mathcal{I}_h(v)|_{1,K} \leq c_1 h_K \|v\|_{1,K} \quad \forall v \in H^1(K). \quad (23)$$

From this, using a scaled trace inequality, the Cauchy-Schwarz inequality, and Assumption 3.1 it follows that

$$\|v - \mathcal{I}_h(v)\|_{0,e} \leq c_2 h_e^{1/2} \|v\|_{1,K} \quad \forall e \in \mathcal{E}_h. \quad (24)$$

We now let $H_{\Gamma_N}^1(\Omega) := \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_N\}$ and consider the virtual element subspace given by

$$\tilde{V}_h := V_h \cap H_{\Gamma_N}^1(\Omega). \quad (25)$$

Also, we introduce, analogously as before, the interpolation operator $\tilde{\mathcal{I}}_h : H_{\Gamma_N}^1(\Omega) \rightarrow \tilde{V}_h$ such that $\tilde{\mathcal{I}}_h := \mathcal{I}_h|_{H_{\Gamma_N}^1(\Omega)}$. In addition, the following lemma establishes an important relation between the virtual spaces \tilde{V}_h and H_h (cf.(7)).

Lemma 5.1. *For $k \geq 0$, given $v \in \tilde{V}_h$ we have $\mathbf{rot} v \in H_h$.*

Proof. Given $v \in \tilde{V}_h$, it is easy to see that $\mathbf{rot} v \in H$. Moreover, given $K \in \mathcal{T}_h$, we observe that $\mathbf{rot}(\mathbf{rot} v) = -\Delta v \in P_{k-1}(K)$. Furthermore, following [6, Section 8, Theorem 3], we have that $\mathbf{rot} v \cdot \nu|_e = \nabla v \cdot s|_e \in P_k(e)$ for all edge $e \in \partial K$. Hence, we conclude that $\mathbf{rot} v|_K \in H_h^K$ for all $K \in \mathcal{T}_h$. \square

We now recall from [16, Section 3.3] some preliminary notations and technical results. For each element $K \in \mathcal{T}_h$ we first define $\tilde{K} := T_K(K)$, where $T_K : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the bijective affine mapping defined by

$$T_K(\mathbf{x}) := \frac{\mathbf{x} - \mathbf{x}_B}{h_K} \quad \forall \mathbf{x} \in \mathbb{R}^2.$$

Then, as it was remarked in [16, Section 3.3], it is easy to see that the diameter $h_{\tilde{K}}$ of \tilde{K} is 1, the shortest edge of \tilde{K} is bigger than $C_{\mathcal{T}}$ (which follows from Assumption 3.1), and \tilde{K} is star-shaped with respect to a ball \tilde{B} of radius $C_{\mathcal{T}}$ and centered at the origin. Then, by connecting each vertex of \tilde{K} to the center of \tilde{B} , that is to the origin, we generate a partition of \tilde{K} into $d_{\tilde{K}}$ triangles $\tilde{\Delta}_i$, $i \in \{1, 2, \dots, d_{\tilde{K}}\}$, where $d_{\tilde{K}} \leq N_{\mathcal{T}}$, and for which the minimum angle condition is satisfied. The later means that there exists a constant $c_{\mathcal{T}} > 0$, depending only on $C_{\mathcal{T}}$ and $N_{\mathcal{T}}$, such that $\tilde{h}_i(\tilde{\rho}_i)^{-1} \leq c_{\mathcal{T}} \quad \forall i \in \{1, 2, \dots, d_{\tilde{K}}\}$, where \tilde{h}_i is the diameter of $\tilde{\Delta}_i$ and $\tilde{\rho}_i$ is the diameter of the largest ball contained in $\tilde{\Delta}_i$. We also let $\tilde{\Delta}$ be the canonical triangle of \mathbb{R}^2 with corresponding parameters \tilde{h} and $\tilde{\rho}$. In what follows, given $K \in \mathcal{T}_h$ and $\zeta \in [H^1(K)]^2$, we let $\tilde{\zeta} := \zeta \circ T_K^{-1} \in [H^1(\tilde{K})]^2$. With this notation at hand, we prove the following interpolation error bound for normal components of H^1 functions on edges which generalises to the VEM setting on polygons the analogous result for mixed-FEM given by Lemma 3.18 in [27].

Lemma 5.2. *There exists a constant $c_3 > 0$, independent of h , such that for all $\boldsymbol{\tau} \in [\mathbf{H}^1(\Omega)]^2$, there holds*

$$\|(\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \boldsymbol{\nu}_e\|_{0,e} \leq c_3 h_e^{1/2} |\boldsymbol{\tau}|_{1,K} \quad \forall e \in \mathcal{E}_h, \quad (26)$$

where K is any element of \mathcal{T}_h such that $K \in \omega_e$.

Proof. The proof is based on the availability of the sub-triangulation of the scaled element \tilde{K} (cf. Remark 3.2), and follows along the lines of the proof of Lemma 3.18 in [27]. Let $e \in \mathcal{E}_h$ and $K \in \mathcal{T}_h$ such that $K \in \omega_e$, and let \tilde{e} be the edge of $\partial\tilde{K}$, such that $e = T_K^{-1}(\tilde{e})$. We further define $T_e := T_K|_e$. Now, given $\boldsymbol{\tau} \in [\mathbf{H}^1(K)]^2$, we know from (7) and the definition of $\boldsymbol{\tau}_I$, respectively, that $\boldsymbol{\tau}_I \cdot \boldsymbol{\nu}|_e \in \mathbf{P}_k(e)$ and

$$\int_e q(\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \boldsymbol{\nu} = 0 \quad \forall q \in \mathcal{B}_k(e).$$

In turns, this implies that

$$\boldsymbol{\tau}_I \cdot \boldsymbol{\nu}_e = \Pi_k^e(\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e),$$

where $\Pi_k^e : L^2(e) \rightarrow \mathbf{P}_k(e)$ is the orthogonal projector. Then, it is easy to see that $\widetilde{\Pi_k^e(v)} = \Pi_k^{\tilde{e}}(\tilde{v}) \quad \forall v \in L^2(e)$, where $\Pi_k^{\tilde{e}} : L^2(\tilde{e}) \rightarrow \mathbf{P}_k(\tilde{e})$ is the corresponding orthogonal projector. Hence, we obtain

$$\begin{aligned} \|(\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \boldsymbol{\nu}_e\|_{0,e} &= \|\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e - \Pi_k^e(\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e)\|_{0,e} = \frac{h_e^{1/2}}{h_{\tilde{e}}^{1/2}} \|\widetilde{\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e} - \Pi_k^e(\widetilde{\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e})\|_{0,\tilde{e}} \\ &= \frac{h_e^{1/2}}{h_{\tilde{e}}^{1/2}} \|\widetilde{\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e} - \Pi_k^{\tilde{e}}(\widetilde{\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e})\|_{0,\tilde{e}} \leq \frac{h_e^{1/2}}{h_{\tilde{e}}^{1/2}} \|\widetilde{\boldsymbol{\tau} \cdot \boldsymbol{\nu}_e}\|_{0,\tilde{e}} \leq \frac{h_e^{1/2}}{h_{\tilde{e}}^{1/2}} \|\tilde{\boldsymbol{\tau}}\|_{0,\tilde{e}}. \end{aligned} \quad (27)$$

Now, let $\tilde{\Delta}$ be the triangle formed connecting the end points of \tilde{e} to the center of \tilde{B} and consider $\hat{\boldsymbol{\tau}} := \tilde{\boldsymbol{\tau}}|_{\tilde{\Delta}} \circ F \in [\mathbf{H}^1(\hat{\Delta})]^2$, where $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is the bijective linear mapping defined by $F(\mathbf{x}) := B\mathbf{x} \quad \forall \mathbf{x} \in \mathbb{R}^2$, with $B \in \mathbb{R}^{2 \times 2}$ invertible, such that $F(\hat{\Delta}) = \tilde{\Delta}$. Let \hat{e} be the edge of $\partial\hat{\Delta}$ such that $\tilde{e} = F(\hat{e})$, then

$$\|\tilde{\boldsymbol{\tau}}\|_{0,\tilde{e}} = \frac{h_{\tilde{e}}^{1/2}}{h_{\hat{e}}^{1/2}} \|\hat{\boldsymbol{\tau}}\|_{0,\hat{e}} = \hat{C} h_{\tilde{e}}^{1/2} \|\hat{\boldsymbol{\tau}}\|_{0,\hat{e}}. \quad (28)$$

Now, considering $\varphi \in C^\infty(\hat{\Delta})$ such that $\varphi \equiv 1$ in a neighbourhood of \hat{e} , and $\varphi \equiv 0$ in a neighbourhood of the vertex opposite to \hat{e} , and applying the trace theorem in $\mathbf{H}^1(\hat{\Delta})$, the Friedrichs-Poincaré inequality, and the Leibniz rule, we get

$$\|\hat{\boldsymbol{\tau}}\|_{0,\hat{e}} = \|\hat{\boldsymbol{\tau}}\varphi\|_{0,\hat{e}} \leq \|\hat{\boldsymbol{\tau}}\varphi\|_{0,\partial\hat{\Delta}} \leq \gamma_{\text{tr}} \|\hat{\boldsymbol{\tau}}\varphi\|_{1,\hat{\Delta}} \leq \gamma_{\text{tr}} C_p |\hat{\boldsymbol{\tau}}\varphi|_{1,\hat{\Delta}} \leq C_\varphi \gamma_{\text{tr}} C_p |\hat{\boldsymbol{\tau}}|_{1,\hat{\Delta}}.$$

Using this to bound (28) and replacing the resulting bound in (27) we deduce that

$$\begin{aligned} \|(\boldsymbol{\tau} - \boldsymbol{\tau}_I) \cdot \boldsymbol{\nu}_e\|_{0,e} &\leq C_\varphi \gamma_{\text{tr}} C_p \hat{C} h_e^{1/2} |\hat{\boldsymbol{\tau}}|_{1,\hat{\Delta}} \leq \hat{C}_1 C_\varphi \gamma_{\text{tr}} C_p \hat{C} h_e^{1/2} |\tilde{\boldsymbol{\tau}}|_{1,\tilde{\Delta}} \\ &\leq \hat{C}_1 C_\varphi \gamma_{\text{tr}} C_p \hat{C} h_e^{1/2} |\tilde{\boldsymbol{\tau}}|_{1,\tilde{K}} \leq c_3 h_e^{1/2} |\boldsymbol{\tau}|_{1,K}, \end{aligned}$$

where $c_3 := C_1 \hat{C}_1 C_\varphi \gamma_{\text{tr}} C_p \hat{C}$, with \hat{C}_1 and C_1 , the \mathbf{H}^1 -seminorm scaly constants on $\tilde{\Delta}$ and K , respectively, thus concluding the proof. \square

5.2. A posteriori error estimator

Let $(\sigma_h, u_h) \in H_h \times Q_h$ be the unique solution of (17). In addition, let $\widehat{\sigma}_h, \sigma_h^*$ be the discrete approximations introduced in (19) and (21), respectively. For each $K \in \mathcal{T}_h$, we define the following local and computable error indicators:

$$\begin{aligned} \Phi_K^2 &:= \|f + \operatorname{div} \sigma_h\|_{0,K}^2, & \Lambda_{1,K}^2 &:= \|\widehat{\sigma}_h - \sigma_h^*\|_{0,K}^2, & \Upsilon_K^2 &:= \|(\kappa^{-1} - \kappa_h)\sigma_h^*\|_{0,K}^2, \\ \Psi_K^2 &:= \sum_{i=1}^2 \Psi_{i,K}^2, & \eta_K^2 &:= \sum_{i=1}^2 \eta_{i,K}^2, & \theta_K^2 &:= \sum_{i=1}^3 \theta_{i,K}^2, \end{aligned}$$

where

$$\begin{aligned} \Psi_{1,K}^2 &:= \|\sigma_h - \widehat{\sigma}_h\|_{0,K}^2, & \Psi_{2,K}^2 &:= \|\kappa^{-1}\widehat{\sigma}_h - \Pi_k^0(\kappa^{-1}\widehat{\sigma}_h)\|_{0,K}^2, \\ \eta_{1,K}^2 &:= h_K^2 \|\kappa_h \sigma_h^* - \nabla u_h\|_{0,K}^2, & \eta_{2,K}^2 &:= \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma_D)} h_e \|u_h - g\|_{0,e}^2, \\ \theta_{1,K}^2 &:= h_K^2 \|\operatorname{rot}(\kappa_h \sigma_h^*)\|_{0,K}^2, & \theta_{2,K}^2 &:= \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Omega)} h_e \|[\kappa_h \sigma_h^* \cdot s]\|_{0,e}^2, \\ \theta_{3,K}^2 &:= \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma_D)} h_e \left\| \kappa_h \sigma_h^* \cdot s - \frac{dg}{ds} \right\|_{0,e}^2, \end{aligned}$$

and κ_h is a piecewise-polynomial approximation of κ^{-1} .

Remark 5.3. Notice that from the residual character of the indicators, the computability of each local term becomes clear. This is the case for all terms apart from $\Psi_{1,K}$ which is not directly computable but is immediately bounded by a computable term using the stability property of Lemma 4.2. As such, this term represents, together with $\Psi_{2,K}$, a bound on the error related to the inconsistency between the continuous and discrete bilinear forms, a^K and a_h^K , (cf. Lemma 4.3). We further observe that the last term in $\theta_{3,K}$ requires the trace g to be more regular. This assumption will be stated and clarified below in Lemma 5.9.

Remark 5.4. If κ is piecewise-constant on each $K \in \mathcal{T}_h$, we have that Υ_K and $\Psi_{2,K}$ are null, whereas if we use homogeneous boundary conditions on Γ_D , we deduce that $\eta_{2,K}$ is null.

Remark 5.5. Through the a posteriori analysis below, it will be clear that the same terms but *without* the postprocessing, hence with $\widehat{\sigma}_h$ in place of σ_h^* everywhere, also constitute an a posteriori bound for the error $\|\sigma - \sigma_h\|_H$. However, as we shall see, the introduction of σ_h^* will permit us to include an optimal bound on the broken $H(\operatorname{div}; \Omega)$ -norm of computable quantities.

5.3. Upper bound

We proceed with the following preliminary estimate

Lemma 5.6. *Let $(\sigma, u) \in H \times Q$ and $(\sigma_h, u_h) \in H_h \times Q_h$ be the unique solutions of (4) and (17), respectively. In addition, let σ_h^* be the discrete approximation introduced in (21). Then, there exists a positive constant C , independent of h , such that*

$$C \|(\sigma, u) - (\sigma_h, u_h)\|_{H \times Q} \leq \left\{ \sum_{K \in \mathcal{T}_h} \Phi_K^2 + \Upsilon_K^2 + \Psi_K^2 + \Lambda_{1,K}^2 \right\}^{1/2} + \sup_{\substack{\tau \in H \\ \tau \neq 0}} \frac{E(\tau)}{\|\tau\|_H}, \quad (29)$$

where

$$E(\boldsymbol{\tau}) := - \int_{\Omega} \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* \cdot (\boldsymbol{\tau} - \boldsymbol{\tau}_h) - \int_{\Omega} u_h \operatorname{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) + \langle (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D}, \quad (30)$$

for all $\boldsymbol{\tau}_h \in H_h$ such that $\|\boldsymbol{\tau}_h\|_Q \leq C\|\boldsymbol{\tau}\|_H$ for some positive constant C independent of $\boldsymbol{\tau}$.

Proof. Consider the bounded linear operator $\mathcal{A} : H \times Q \rightarrow (H \times Q)'$ induced by the left hand-side of (4), that is, the linear operator defined by

$$[\mathcal{A}(\boldsymbol{\rho}, z), (\boldsymbol{\tau}, v)] := a(\boldsymbol{\rho}, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, z) + b(\boldsymbol{\rho}, v). \quad (31)$$

From the well-posedness of the variational formulation (4), we know that \mathcal{A} is an isomorphism. In particular, there exists a positive constant C , such that

$$C\|(\boldsymbol{\rho}, z)\|_{H \times Q} \leq \sup_{\substack{(\boldsymbol{\tau}, v) \in H \times Q \\ (\boldsymbol{\tau}, v) \neq \mathbf{0}}} \frac{[\mathcal{A}(\boldsymbol{\rho}, z), (\boldsymbol{\tau}, v)]}{\|(\boldsymbol{\tau}, v)\|_{H \times Q}}.$$

Now, applying the foregoing equation to $(\boldsymbol{\rho}, z) := (\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)$, from (31), we get

$$\begin{aligned} C\|(\boldsymbol{\sigma}, u) - (\boldsymbol{\sigma}_h, u_h)\|_{H \times Q} &\leq \sup_{\substack{(\boldsymbol{\tau}, v) \in H \times Q \\ (\boldsymbol{\tau}, v) \neq \mathbf{0}}} \frac{a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u - u_h) + b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v)}{\|(\boldsymbol{\tau}, v)\|_{H \times Q}} \\ &\leq \sup_{\substack{v \in Q \\ v \neq 0}} \frac{b(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v)}{\|v\|_Q} + \sup_{\substack{\boldsymbol{\tau} \in H \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u - u_h)}{\|\boldsymbol{\tau}\|_H} \\ &\leq \|f + \operatorname{div} \boldsymbol{\sigma}_h\|_{0, \Omega} + \sup_{\substack{\boldsymbol{\tau} \in H \\ \boldsymbol{\tau} \neq \mathbf{0}}} \frac{a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u - u_h)}{\|\boldsymbol{\tau}\|_H}, \end{aligned} \quad (32)$$

and it remains to bound the second term above. To this end, given $\boldsymbol{\tau} \in H$ and any $\boldsymbol{\tau}_h \in H_h$, from (4) and (17), we have that

$$\begin{aligned} a(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b(\boldsymbol{\tau}, u - u_h) &= -a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, u_h) + \langle \boldsymbol{\tau} \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} \\ &= \langle \boldsymbol{\tau}_h \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} - a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b(\boldsymbol{\tau}, u_h) + \langle (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} \\ &= a_h(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - a(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b(\boldsymbol{\tau} - \boldsymbol{\tau}_h, u_h) + \langle (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} \\ &= (a_h - a)(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - a(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \\ &\quad - a(\boldsymbol{\sigma}_h^* - \boldsymbol{\kappa} \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \\ &\quad - a(\boldsymbol{\kappa} \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) - b(\boldsymbol{\tau} - \boldsymbol{\tau}_h, u_h) + \langle (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} \\ &=: I + II + III. \end{aligned}$$

Now, in what follows we take in particular $\boldsymbol{\tau}_h \in H_h$ with $\|\boldsymbol{\tau}_h\|_Q \leq C\|\boldsymbol{\tau}\|_H$ for some positive constant C independent of $\boldsymbol{\tau}$. For I and II , we use the bound of Lemma 4.3 and the Cauchy-Schwarz inequality to deduce

$$I := (a_h - a)(\boldsymbol{\sigma}_h, \boldsymbol{\tau}_h) - a(\boldsymbol{\sigma}_h - \boldsymbol{\sigma}_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \leq C \left\{ \sum_{K \in \mathcal{T}_h} \Psi_K^2 + \Lambda_{1,K}^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_H, \quad (33)$$

and

$$II := -a(\boldsymbol{\sigma}_h^* - \boldsymbol{\kappa} \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) \leq C \left\{ \sum_{K \in \mathcal{T}_h} \Upsilon_K^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_H, \quad (34)$$

whereas bearing mind the functional E (cf.(30)), we use the definitions (5) to get

$$III := -a(\kappa\kappa_h\sigma_h^*, \boldsymbol{\tau} - \boldsymbol{\tau}_h) - b(\boldsymbol{\tau} - \boldsymbol{\tau}_h, u_h) + \langle (\boldsymbol{\tau} - \boldsymbol{\tau}_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D} = E(\boldsymbol{\tau}). \quad (35)$$

Finally, replacing (33)-(35) into (32), we conclude the proof. \square

We now aim to bound the supremum on the right hand-side of (29), for which we need a suitable choice of $\boldsymbol{\tau}_h \in H_h$ such that $\|\boldsymbol{\tau}_h\|_Q \leq \|\boldsymbol{\tau}\|_H$. To this end, in what follow we assume that the boundary Γ is such that Γ_N is contained in a convex part of Ω . More precisely, we make use of the following result.

Lemma 5.7. *Assume that Ω is a connected domain and that Γ_N is contained in the boundary of a convex part of Ω , that is there exists a convex domain B such that $\Omega \subset B$ and $\Gamma_N \subseteq \partial B$. Then, for each $\boldsymbol{\tau} \in H$ (cf.(1)), there exist $\boldsymbol{\zeta} \in \mathbf{H}^1(\Omega)$ with $\boldsymbol{\zeta} \cdot \boldsymbol{\nu} = 0$ on Γ_N and $\chi \in \mathbf{H}_{\Gamma_N}^1(\Omega)$ (cf. Section 5.1) such that*

$$\boldsymbol{\tau} = \boldsymbol{\zeta} + \mathbf{rot} \chi \quad \text{in } \Omega, \quad \operatorname{div} \boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\tau} \quad \text{in } \Omega, \quad (36)$$

and

$$\|\boldsymbol{\zeta}\|_{1,\Omega} + \|\chi\|_{1,\Omega} \leq C \|\boldsymbol{\tau}\|_{\operatorname{div};\Omega}, \quad (37)$$

with a positive constant C independent of $\boldsymbol{\tau}$.

Proof. See [2, Lemma 3.9] for more details. \square

Now, for $\boldsymbol{\tau} \in H$ from Lemmas 5.1 and 5.7, we define $\chi_h := \tilde{\mathcal{I}}_h(\chi) \in \tilde{V}_h$, and set

$$\boldsymbol{\tau}_h := \boldsymbol{\zeta}_I + \mathbf{rot} \chi_h \in H_h, \quad (38)$$

as its associated discrete Helmholtz decomposition. Now, it follows from (38), the triangle inequality, (6), (23) and (37) that

$$\|\boldsymbol{\tau}_h\|_Q \leq \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_I\|_Q + \|\boldsymbol{\zeta}\|_Q + |\chi - \chi_h|_{1,\Omega} + |\chi|_{1,\Omega} \leq C \|\boldsymbol{\tau}\|_H,$$

with a positive constant C independent of $\boldsymbol{\tau}$. Next, we can write

$$\boldsymbol{\tau} - \boldsymbol{\tau}_h = \boldsymbol{\zeta} - \boldsymbol{\zeta}_I + \mathbf{rot}(\chi - \chi_h), \quad (39)$$

from which, using (11), and the fact that $\operatorname{div} \boldsymbol{\zeta} = \operatorname{div} \boldsymbol{\tau}$ in Ω , we deduce

$$\int_{\Omega} u_h \operatorname{div}(\boldsymbol{\tau} - \boldsymbol{\tau}_h) = \int_{\Omega} u_h \operatorname{div}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_I) = 0. \quad (40)$$

Then, using the choice for $\boldsymbol{\tau}_h$ given by (38) to bound the supremum in (29), replacing (39) and (40) into (30), we find that $E(\boldsymbol{\tau}) = E_1(\boldsymbol{\zeta}) + E_2(\chi)$ where

$$E_1(\boldsymbol{\zeta}) := - \int_{\Omega} \kappa_h \sigma_h^* \cdot (\boldsymbol{\zeta} - \boldsymbol{\zeta}_I) + \langle (\boldsymbol{\zeta} - \boldsymbol{\zeta}_I) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D}, \quad (41)$$

and

$$E_2(\chi) := - \int_{\Omega} \kappa_h \sigma_h^* \cdot \mathbf{rot}(\chi - \chi_h) + \langle \mathbf{rot}(\chi - \chi_h) \cdot \boldsymbol{\nu}, g \rangle_{\Gamma_D}. \quad (42)$$

The following two lemmas provide the upper bounds for $|E_1(\boldsymbol{\zeta})|$ and $|E_2(\chi)|$.

Lemma 5.8. *There exists $C > 0$, independent of h , such that*

$$|E_1(\boldsymbol{\zeta})| \leq C \left\{ \sum_{K \in \mathcal{T}} \eta_K^2 \right\}^{1/2} \|\boldsymbol{\tau}\|_{\operatorname{div};\Omega}.$$

Proof. We rewrite the second term in $E_1(\zeta)$ as:

$$\langle (\zeta - \zeta_I) \cdot \nu, g \rangle_{\Gamma_D} = \sum_{e \in \mathcal{E}_h(\Gamma_D)} \int_e g(\zeta - \zeta_I) \cdot \nu. \quad (43)$$

Next, since $u_h|_K \in P_k(K)$, we have

$$\int_e u_h(\zeta - \zeta_I) \cdot \nu = 0 \quad \forall e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma_D),$$

and

$$\int_K (\zeta - \zeta_I) \cdot \nabla u_h = 0,$$

for all $K \in \mathcal{T}_h$. Hence, using the above expressions, we can write

$$E_1(\zeta) = - \sum_{K \in \mathcal{T}_h} \left\{ \int_K (\kappa_h \sigma_h^* - \nabla u_h) \cdot (\zeta - \zeta_I) + \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma_D)} \int_e (u_h - g)(\zeta - \zeta_I) \cdot \nu \right\},$$

from which, applying the Cauchy-Schwarz inequality, the approximation properties (6) and (26), and the fact $\|\zeta\|_{1,\Omega} \leq \|\tau\|_{\text{div};\Omega}$, we obtain the required estimate. \square

Lemma 5.9. *Assume that $g \in H^1(\Gamma_D)$. Then, there exists $C > 0$, independent of h , such that*

$$|E_2(\chi)| \leq C \left\{ \sum_{K \in \mathcal{T}} \theta_K^2 \right\}^{1/2} \|\tau\|_{\text{div};\Omega}.$$

Proof. We proceed as in the proof of the Lemma 3.11 in [2]. Integrating by parts on each $K \in \mathcal{T}_h$, using that $\text{rot}(\chi - \chi_h) \cdot \nu = \frac{d}{ds}(\chi - \chi_h)$, noting that $\frac{dg}{ds} \in L^2(\Gamma_D)$, and using the fact that $\chi|_{\Gamma_N} = \chi_h|_{\Gamma_N} = 0$, we get

$$\begin{aligned} E_2(\chi) &= - \sum_{K \in \mathcal{T}_h} \int_K \kappa_h \sigma_h^* \cdot \text{rot}(\chi - \chi_h) + \left\langle \frac{d}{ds}(\chi - \chi_h), g \right\rangle_{\Gamma_D} \\ &= - \sum_{K \in \mathcal{T}_h} \left\{ \int_K \text{rot}(\kappa_h \sigma_h^*)(\chi - \chi_h) - \int_{\partial K} (\kappa_h \sigma_h^* \cdot s_K)(\chi - \chi_h) \right\} - \int_{\Gamma_D} \frac{dg}{ds}(\chi - \chi_h) \\ &= - \sum_{K \in \mathcal{T}_h} \left\{ \int_K \text{rot}(\kappa_h \sigma_h^*)(\chi - \chi_h) - \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Omega)} \int_e [\kappa_h \sigma_h^* \cdot s](\chi - \chi_h) \right. \\ &\quad \left. - \sum_{e \in \mathcal{E}(K) \cap \mathcal{E}_h(\Gamma_D)} \int_e \left(\kappa_h \sigma_h^* \cdot s - \frac{dg}{ds} \right) (\chi - \chi_h) \right\}. \end{aligned}$$

In this way, since $\chi_h = \tilde{\mathcal{I}}_h(\chi)$, applying the Cauchy-Schwarz inequality to each term in the above expression and making use of the approximation properties (23) and (24) and the fact that the number of elements in ω_e is bounded, we conclude the proof. \square

Finally, from Lemmas 5.6, 5.8 and 5.9 we deduce an upper bound for the global error.

Theorem 5.10. *Let $(\sigma, u) \in H \times Q$ and $(\sigma_h, u_h) \in H_h \times Q_h$ be the unique solutions of the problem (4) and (17), respectively. Then, there exists a positive constant C , independent of h , such that*

$$\|(\sigma, u) - (\sigma_h, u_h)\|_{H \times Q} \leq C \left\{ \sum_{K \in \mathcal{T}_h} \Phi_K^2 + \Upsilon_K^2 + \Psi_K^2 + \Lambda_{1,K}^2 + \eta_K^2 + \theta_K^2 \right\}^{1/2}.$$

We recall from the discussion in Section 4.2 that the corresponding result for the computable quantity $\widehat{\sigma}_h$ is only to be expected in the L^2 -norm. Instead, for the error using the postprocessing flux we are able to obtain the following result in line with Theorem 5.10.

Theorem 5.11. *Let $(\sigma, u) \in H \times Q$ and $(\sigma_h, u_h) \in H_h \times Q_h$ be the unique solutions of the problem (4) and (17), respectively. In addition, let σ_h^* be the discrete postprocessing introduced in (21). Then, there exists a positive constant C , independent of h , such that*

$$\left\{ \sum_{K \in \overline{\mathcal{T}}_h} \|\sigma - \sigma_{h,K}^*\|_{\text{div};K}^2 \right\}^{1/2} + \|u - u_h\|_Q \leq C \left\{ \sum_{K \in \overline{\mathcal{T}}_h} \Phi_K^2 + \Upsilon_K^2 + \Psi_K^2 + \Lambda_K^2 + \eta_K^2 + \theta_K^2 \right\}^{1/2},$$

with

$$\Lambda_K^2 := \sum_{i=1}^2 \Lambda_{i,K}^2 \quad \text{where} \quad \Lambda_{2,K}^2 := \|\text{div } \sigma_h - \text{div } \sigma_h^*\|_{0,K}^2.$$

Proof. From the triangle inequality, we have

$$\begin{aligned} \|\sigma - \sigma_{h,K}^*\|_{\text{div};K} &\leq \|\sigma - \sigma_h\|_{\text{div};K} + \|\sigma_h - \sigma_{h,K}^*\|_{0,K} + \|\text{div } \sigma_h - \text{div } \sigma_{h,K}^*\|_{0,K} \\ &\leq \|\sigma - \sigma_h\|_{\text{div};K} + \|\sigma_h - \widehat{\sigma}_h\|_{0,K} + \|\widehat{\sigma}_h - \sigma_{h,K}^*\|_{0,K} + \|\text{div } \sigma_h - \text{div } \sigma_{h,K}^*\|_{0,K}. \end{aligned}$$

Then, since $H(\text{div}; \Omega) \subset H(\text{div}; \mathcal{T}_h)$ and using the definition of Ψ_K^2 and Λ_K^2 , we get

$$\left\{ \sum_{K \in \overline{\mathcal{T}}_h} \|\sigma - \sigma_{h,K}^*\|_{\text{div};K}^2 \right\}^{1/2} \leq C \left\{ \|\sigma - \sigma_h\|_H + \left\{ \sum_{K \in \overline{\mathcal{T}}_h} \Psi_K^2 + \Lambda_K^2 \right\}^{1/2} \right\}.$$

Therefore, the result is consequence of the foregoing equation and the Theorem 5.10. \square

5.4. Lower bound

In this section we derive suitable upper bounds for the terms defining the local error indicators. First, using that $f = -\text{div } \sigma$ in Ω we have that

$$\Phi_K^2 = \|\text{div}(\sigma - \sigma_h)\|_{0,K}^2 \leq 2 \left\{ \|\sigma - \sigma_h^*\|_{\text{div};K}^2 + \Lambda_{1,K}^2 \right\}. \quad (44)$$

Moreover, adding and subtracting σ , we easily have

$$\begin{aligned} \Lambda_K^2 &= \|\widehat{\sigma}_h - \sigma_h^*\|_{0,K}^2 + \|\text{div } \sigma_h - \text{div } \sigma_h^*\|_{0,K}^2 \\ &\leq 2 \left\{ \|\sigma - \widehat{\sigma}_h\|_{0,K}^2 + \|\sigma - \sigma_h^*\|_{\text{div};K}^2 + \Phi_K^2 \right\}. \end{aligned} \quad (45)$$

In addition, proceeding as in [20, Lemma 18], we deduce

$$\Psi_{2,K}^2 \leq C \left\{ \Lambda_{1,K}^2 + \|\sigma - \sigma_h^*\|_{0,K}^2 + \|\kappa^{-1} \sigma - \Pi_k^0(\kappa^{-1} \sigma)\|_{0,K}^2 \right\}, \quad (46)$$

with C depending only on κ and \widehat{c}_0 .

Remark 5.12. Again by adding and subtracting σ we have

$$\Psi_{1,K}^2 = \|\sigma_h - \widehat{\sigma}_h\|_{0,K}^2 \leq 2 \left\{ \|\sigma - \sigma_h\|_{0,K}^2 + \|\sigma - \widehat{\sigma}_h\|_{0,K}^2 \right\}. \quad (47)$$

This does provide a lower bound, although in terms of the error $\sigma - \sigma_h$. Here we have chosen, instead, to leave this term as is, interpreting it as a sort of oscillation term representing the virtual inconsistency of the method.

The upper bounds of the terms which depend on the mesh parameters h_K and h_e , will be derived next. To this end, we proceed similarly as in [22] and [23] and apply the technique based on bubble functions, together with inverse inequalities. Following [20, Section 4] and [31, Section 3], given $K \in \mathcal{T}_h$, a bubble function ψ_K can be constructed piecewise as the sum of the (polynomial) barycentric bubble functions (cf. [1, 35]) on each triangle of the shape-regular sub-triangulation of the mesh element K discussed in the Section 4. Further, an edge bubble function ψ_e , $e \in \partial K$, is a piecewise quadratic function attaining the value 1 at the mid-point of e and vanishing on the triangles that do not contain e on their boundary. Furthermore, given $k \geq 0$, there exists an extension operator $L : C(e) \rightarrow C(K)$ that satisfies $L(p) \in \mathbb{P}_k(K)$ and $L(p)|_e = p$ for all $p \in \mathbb{P}_k(e)$ (cf. [31, Remark 3.1]). Further properties of ψ_K , ψ_e , and L are stated in the following lemma. See [20, Section 4] and [31, Section 3] for more details.

Lemma 5.13. *Given $k \geq 0$ and $K \in \mathcal{T}_h$, there exists a positive constant C_{bub} , independent of h_K such that*

$$C_{\text{bub}}^{-1} \|q\|_{0,K}^2 \leq \|\psi_K^{1/2} q\|_{0,K}^2 \leq C_{\text{bub}} \|q\|_{0,K}^2 \quad \forall q \in \mathbb{P}_k(K), \quad (48)$$

and

$$C_{\text{bub}}^{-1} \|q\|_{0,K} \leq \|\psi_K q\|_{0,K} + h_K |\psi_K q|_{1,K} \leq C_{\text{bub}} \|q\|_{0,K} \quad \forall q \in \mathbb{P}_k(K). \quad (49)$$

In addition, given $e \in \partial K$, there hold

$$C_{\text{bub}}^{-1} \|q\|_{0,e}^2 \leq \|\psi_e^{1/2} q\|_{0,e}^2 \leq C_{\text{bub}} \|q\|_{0,e}^2 \quad \forall q \in \mathbb{P}_k(e), \quad (50)$$

and

$$h_K^{-1/2} \|\psi_e L(q)\|_{0,K} + h_K^{1/2} |\psi_e L(q)|_{1,K} \leq C_{\text{bub}} \|q\|_{0,e} \quad \forall q \in \mathbb{P}_k(e), \quad (51)$$

where $K \in \omega_e$.

We start the analysis bounding the terms defining $\eta_{1,K}^2$ and $\eta_{2,K}^2$.

Lemma 5.14. *There exists a constant $C > 0$, independent of h , such that*

$$h_K^2 \|\kappa_h \sigma_h^* - \nabla u_h\|_{0,K}^2 \leq C \left\{ h_K^2 \|\sigma - \sigma_h^*\|_{0,K}^2 + h_K^2 \|(\kappa^{-1} - \kappa_h) \sigma_h^*\|_{0,K}^2 + \|u - u_h\|_{0,K}^2 \right\} \quad \forall K \in \mathcal{T}_h.$$

Proof. It is a slight modification of the proof of the Lemma 6.3 in [22] (see also Lemma 5.5 in [26]). Given $K \in \mathcal{T}_h$ we denote $\gamma_K := \kappa_h \sigma_h^* - \nabla u_h \in [\mathbb{P}_\ell(K)]^2$ for some $\ell \geq 0$. Then, applying (48), using that $\kappa^{-1} \sigma = \nabla u$ in Ω , and integrating by parts, we find that

$$\begin{aligned} C_{\text{bub}}^{-1} \|\gamma_K\|_{0,K}^2 &\leq \|\psi_K^{1/2} \gamma_K\|_{0,K}^2 = \int_K \psi_K \gamma_K \cdot \{\kappa_h \sigma_h^* - \nabla u_h\} \\ &= \int_K \psi_K \gamma_K \cdot \{(\kappa_h - \kappa^{-1}) \sigma_h^* + \kappa^{-1} \sigma_h^* - \kappa^{-1} \sigma + \nabla(u - u_h)\} \\ &= - \int_K \psi_K \gamma_K \cdot (\kappa^{-1} - \kappa_h) \sigma_h^* - \int_K \psi_K \gamma_K \cdot \kappa^{-1} (\sigma - \sigma_h^*) \\ &\quad - \int_K \text{div}(\psi_K \gamma_K) (u - u_h). \end{aligned}$$

Then, applying the Cauchy-Schwarz inequality, the estimate (49), and setting $C_\kappa := \max\{1, \|\kappa^{-1}\|\}$, we get

$$\begin{aligned} C_{\text{bub}}^{-1} \|\gamma_K\|_{0,K}^2 &\leq C_\kappa \left\{ \|\psi_K \gamma_K\|_{0,K} \left\{ \|(\kappa^{-1} - \kappa_h) \sigma_h^*\|_{0,K} + \|\sigma - \sigma_h^*\|_{0,K} \right\} + |\psi_K \gamma_K|_{1,K} \|u - u_h\|_{0,K} \right\} \\ &\leq C_\kappa C_{\text{bub}} \left\{ \|(\kappa^{-1} - \kappa_h) \sigma_h^*\|_{0,K} + \|\sigma - \sigma_h^*\|_{0,K} + h_K^{-1} \|u - u_h\|_{0,K} \right\} \|\gamma_K\|_{0,K} \\ &\leq 2C_\kappa C_{\text{bub}} \left\{ \|(\kappa^{-1} - \kappa_h) \sigma_h^*\|_{0,K}^2 + \|\sigma - \sigma_h^*\|_{0,K}^2 + h_K^{-2} \|u - u_h\|_{0,K}^2 \right\}^{1/2} \|\gamma_K\|_{0,K}, \end{aligned}$$

whence, the proof is concluded. \square

Lemma 5.15. *There exists a constant $C > 0$, independent of h , such that*

$$h_e \|g - u_h\|_{0,e}^2 \leq C \left\{ h_K^2 \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K}^2 + h_K^2 \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K}^2 + \|u - u_h\|_{0,K}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Gamma_D),$$

where $K \in \mathcal{T}_h$ is such that $e \in \partial K$.

Proof. We proceed as in the proof of the Lemma 4.14 in [28]. We consider $e \in \mathcal{E}_h(\Gamma_D)$ and $K \in \mathcal{T}_h$ such that $e \in \partial K$. Then, applying a trace inequality, together with the fact that $u = g$ on Γ_D and $\boldsymbol{\sigma} = \boldsymbol{\kappa} \nabla u$ in Ω , we get

$$\begin{aligned} \|g - u_h\|_{0,e}^2 &= \|u - u_h\|_{0,e}^2 \leq C_{\text{tr}} \left\{ h_K^{-1} \|u - u_h\|_{0,K}^2 + h_K \|u - u_h\|_{1,K}^2 \right\} \\ &\leq 2C_{\boldsymbol{\kappa}} C_{\text{tr}} \left\{ h_K^{-1} \|u - u_h\|_{0,K}^2 + h_K \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K}^2 + \|\boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* - \nabla u_h\|_{0,K}^2 + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K}^2 \right\} \right\}, \end{aligned}$$

with $C_{\boldsymbol{\kappa}}$ as in the proof of Lemma 5.14. From this, using the bound $h_e \leq h_K$ and the estimate of Lemma 5.14 we obtain the result. \square

The following result is required in view of proving upper bounds for the terms defining θ_K^2 .

Lemma 5.16. *Let $\boldsymbol{\zeta}_h \in [\mathbf{L}^2(\Omega)]^2$ be a piecewise polynomial of degree $k \geq 0$ on each $K \in \mathcal{T}_h$. In addition let $\boldsymbol{\zeta} \in [\mathbf{L}^2(\Omega)]^2$ be such that $\text{rot}(\boldsymbol{\zeta}) = 0$ in Ω . Then, there exists $C > 0$, depending only on C_{bub} , such that*

$$\|\text{rot} \boldsymbol{\zeta}_h\|_{0,K} \leq C h_K^{-1} \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,K} \quad \forall K \in \mathcal{T}_h, \quad (52)$$

and

$$\|[\boldsymbol{\zeta}_h \cdot \mathbf{s}_e]\|_{0,e} \leq C h_e^{-1/2} \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,\omega_e} \quad \forall e \in \mathcal{E}_h(\Omega). \quad (53)$$

Proof. To show (52), we proceed as in the proof of Lemma 4.3 in [4]. Applying (48), observing that $\psi_K = 0$ on ∂K , and using the Cauchy-Schwarz inequality, we get

$$\begin{aligned} C_{\text{bub}}^{-1} \|\text{rot} \boldsymbol{\zeta}_h\|_{0,K}^2 &\leq \|\psi_K^{1/2} \text{rot} \boldsymbol{\zeta}_h\|_{0,K}^2 = - \int_K \psi_K \text{rot} \boldsymbol{\zeta}_h \text{rot}(\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) \\ &= \int_K (\boldsymbol{\zeta} - \boldsymbol{\zeta}_h) \cdot \mathbf{rot}(\psi_K \text{rot} \boldsymbol{\zeta}_h) \leq \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,K} |\psi_K \text{rot} \boldsymbol{\zeta}_h|_{1,K}. \end{aligned}$$

Then, from inverse inequality (49), we deduce (52).

The estimate (53) follows from a slight modification of the proof of [4, Lemma 4.4]. Indeed, given $e \in \mathcal{E}_h(\Omega)$, we let $J_h := [\boldsymbol{\zeta}_h \cdot \mathbf{s}_e] \in \mathbf{P}_k(e)$. Then, utilizing (50), the fact that $[\boldsymbol{\zeta} \cdot \mathbf{s}_e] = 0$ a.e on e , and integrating by parts on each $K \in \mathcal{T}_h$, we get

$$\begin{aligned} C_{\text{bub}}^{-1} \|J_h\|_{0,e}^2 &\leq \|\psi_e^{1/2} J_h\|_{0,e}^2 = \|\psi_e^{1/2} L(J_h)\|_{0,e}^2 = \int_e \psi_e L(J_h) [\boldsymbol{\zeta}_h \cdot \mathbf{s}] \\ &= \int_{\omega_e} (\boldsymbol{\zeta}_h - \boldsymbol{\zeta}) \cdot \mathbf{rot}(\psi_e L(J_h)) + \int_{\omega_e} \psi_e L(J_h) \text{rot} \boldsymbol{\zeta}_h, \end{aligned}$$

which, using the Cauchy-Schwarz inequality, the estimates (51) and (52), and the fact that $h_e \leq h_K$, yields

$$\begin{aligned} C_{\text{bub}}^{-1} \|J_h\|_{0,e}^2 &\leq |\psi_e L(J_h)|_{1,\omega_e} \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,\omega_e} + \|\psi_e L(J_h)\|_{0,\omega_e} \|\text{rot} \boldsymbol{\zeta}_h\|_{0,\omega_e} \\ &\leq 2N_{\mathcal{T}} C_{\text{bub}} h_e^{-1/2} \|\boldsymbol{\zeta} - \boldsymbol{\zeta}_h\|_{0,\omega_e} \|J_h\|_{0,e}, \end{aligned}$$

whence, we conclude the proof of (53). \square

Lemma 5.17. *There exists $C > 0$, independent of h , such that*

$$h_K^2 \|\text{rot}(\boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*)\|_{0,K}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K}^2 + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Omega),$$

and

$$h_e \|\llbracket \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* \cdot \mathbf{s}_e \rrbracket\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K}^2 + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Omega),$$

where $K \in \mathcal{T}_h$ is such that $K \in \omega_e$.

Proof. It suffices to apply Lemma 5.16 with $\boldsymbol{\zeta}_h := \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*$ and $\boldsymbol{\zeta} := \boldsymbol{\kappa}^{-1} \boldsymbol{\sigma} = \nabla u$, and the triangle inequality. \square

Lemma 5.18. *Assume that $\frac{dg}{ds}$ is piecewise polynomial on Γ_D . Then, there exists $C > 0$, independent of h , such that*

$$h_e \left\| \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* \cdot \mathbf{s} - \frac{dg}{ds} \right\|_{0,e}^2 \leq C \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K}^2 + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K}^2 \right\} \quad \forall e \in \mathcal{E}_h(\Gamma_D), \quad (54)$$

where $K \in \mathcal{T}_h$ is such that $K \in \omega_e$.

Proof. We proceed as in the proof of Lemma 4.15 in [28] (see also Lemma 5.7 in [26]). Given $e \in \mathcal{E}_h(\Gamma_D)$ and $K \in \omega_e$, we denote $\gamma_e := \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* \cdot \mathbf{s} - \frac{dg}{ds} \in \mathbb{P}_\ell(e)$ for some $\ell \geq 0$. Then, applying (50), the fact that $\nabla u \cdot \mathbf{s} = \frac{dg}{ds}$, integrating by parts and using that $\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma} = \nabla u$ in Ω , we obtain that

$$\begin{aligned} C_{\text{bub}}^{-1} \|\gamma_e\|_{0,e}^2 &\leq \|\psi_e^{1/2} \gamma_e\|_{0,e}^2 = \int_e \psi_e \gamma_e \left\{ \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^* \cdot \mathbf{s} - \nabla u \cdot \mathbf{s} \right\} \\ &= - \int_{\partial K} \psi_e L(\gamma_e) \left\{ (\boldsymbol{\kappa}^{-1} \boldsymbol{\sigma} - \boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*) \cdot \mathbf{s} \right\} \\ &= - \int_K \mathbf{rot}(\psi_e L(\gamma_e)) \cdot (\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^* - \int_K \mathbf{rot}(\psi_e L(\gamma_e)) \cdot \boldsymbol{\kappa}^{-1} \left\{ \boldsymbol{\sigma} - \boldsymbol{\sigma}_h^* \right\} \\ &\quad + \int_K \psi_e L(\gamma_e) \mathbf{rot}(\boldsymbol{\kappa}_h \boldsymbol{\sigma}_h^*). \end{aligned}$$

Next, applying the Cauchy-Schwarz inequality, Lemma 5.17, the estimate (51), and the fact that $h_e \leq h_K$ we get

$$\begin{aligned} C_{\text{bub}}^{-1} \|\gamma_e\|_{0,e}^2 &\leq C \left\{ |\psi_e L(\gamma_e)|_{1,K} + h_K^{-1} \|\psi_e L(\gamma_e)\|_{0,K} \right\} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K} + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K} \right\} \\ &\leq C h_e^{-1/2} \left\{ \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_h^*\|_{0,K} + \|(\boldsymbol{\kappa}^{-1} - \boldsymbol{\kappa}_h) \boldsymbol{\sigma}_h^*\|_{0,K} \right\} \|\gamma_e\|_{0,e}, \end{aligned}$$

and the proof is complete. \square

If $\frac{dg}{ds}$ is not piecewise polynomial but sufficiently smooth, Lemma 5.18 can still be proven with higher order terms given by the errors arising from suitable polynomial approximations appearing in (54).

Finally, a lower bound is obtained from estimates (44)-(46), together with Lemmata 5.14 throughout 5.18, after summing up over $K \in \mathcal{T}_h$ and using the fact that the number of elements on each domain ω_e is bounded.

6. NUMERICAL TESTS

In this section, we present three numerical tests confirming the upper and lower bounds, derived in Section 5, for the a posteriori error estimator of Theorem 5.11, and showing the behaviour of the associated adaptive

algorithm. We begin by introducing additional notations. In what follows, N stands for the total number of degrees of freedom of (17), that is,

$$N := (k + 1) \times \{\text{number of edges } e \in \mathcal{T}_h\} + \frac{(k + 2)(3k + 1)}{2} \times \{\text{number of elements } K \in \mathcal{T}_h\}.$$

Also, the individual errors are defined by

$$\mathbf{e}(\boldsymbol{\sigma}) := \left\{ \sum_{K \in \mathcal{T}_h} \|\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,K}^*\|_{\text{div};K}^2 \right\}^{1/2}, \quad \mathbf{e}(u) := \|u - u_h\|_{0,\Omega}, \quad \text{and} \quad \mathbf{e}(\boldsymbol{\sigma}, u) := \left\{ [\mathbf{e}(\boldsymbol{\sigma})]^2 + [\mathbf{e}(u)]^2 \right\}^{1/2},$$

whereas the associated experimental rates of convergence are given by

$$\mathbf{r}(\cdot) := -2 \frac{\log(\mathbf{e}(\cdot)/\mathbf{e}'(\cdot))}{\log(N/N')},$$

where \mathbf{e} and \mathbf{e}' denote the corresponding errors for two consecutive meshes with N and N' denote the corresponding degrees of freedom of each decomposition. Denote by Θ the a posteriori error estimator of Theorem 5.11. The effectivity of the estimator Θ is given by

$$\text{eff}(\Theta) := \frac{\mathbf{e}(\boldsymbol{\sigma}, u)}{\Theta}.$$

For the tests that include adaptivity, we use the strategy:

- (i) Start with a coarse mesh \mathcal{T}_h .
- (ii) Solve the discrete problem on the current mesh \mathcal{T}_h .
- (iii) Compute local indicators for each $K \in \mathcal{T}_h$.
- (iv) Mark each $K' \in \mathcal{T}_h$ such that

$$\Theta_{K'} \geq \beta \max_{K \in \mathcal{T}_h} \Theta_K,$$

with $\beta \in [0, 1]$ and we refine using the midpoint of each edge of each element and connecting this to its barycentre. Here, we use $\beta = 0.5$.

- (v) Update \mathcal{T}_h with the new mesh and go to step (ii).

Hereafter, in all numerical tests we have $\boldsymbol{\kappa} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and we consider domains Ω satisfying Lemma 5.7. In this case, we have that Υ_K^2 and $\Psi_{2,K}^2$ are null for each $K \in \mathcal{T}_h$ (cf. Remark 5.4 in Section 5.2).

6.1. Test 1. Smooth solution: behaviour of the estimator under uniform refinement

For this test case, we consider $\Omega := (0, 1)^2$ with $\Gamma_D := \left\{ (w, 0), (0, w) \in \Omega : 0 \leq w \leq 1 \right\}$ and $\Gamma_N := \Gamma \setminus \bar{\Gamma}_D$. The source term f and the boundary data g are chosen such that the exact solution is given by $u(x, y) = \cos(\pi x) \cos(\pi y)$.

Table 1 shows the convergence history of the error for each variable and the estimator on a sequence of uniformly refined hexagonal meshes, indicating that both converge at the optimal rate for polynomial degrees $k = 0, 1, 2$. Moreover, the effectivity remains bounded. In addition, we see from Table 2 that each term of the error estimator converge with optimal order $k + 1$.

k	N	$e(\sigma)$	$r(\sigma)$	$e(u)$	$r(u)$	$e(\sigma, u)$	$r(\sigma, u)$	Θ	$r(\Theta)$	$eff(\Theta)$
0	589	1.0959e+00	--	5.5406e-02	--	1.0973e+00	--	1.2946e+00	--	8.4760e-01
	3469	4.3834e-01	1.0336	2.1963e-02	1.0437	4.3889e-01	1.0336	5.1997e-01	1.0288	8.4405e-01
	8749	2.7388e-01	1.0167	1.3708e-02	1.0191	2.7423e-01	1.0167	3.2511e-01	1.0153	8.4348e-01
	19605	1.8247e-01	1.0067	9.1290e-03	1.0077	1.8270e-01	1.0067	2.1667e-01	1.0059	8.4323e-01
	43805	1.2165e-01	1.0087	6.0851e-03	1.0090	1.2180e-01	1.0087	1.4447e-01	1.0082	8.4307e-01
1	1766	5.9382e-02	--	3.0274e-03	--	5.9459e-02	--	7.2082e-02	--	8.2488e-01
	10406	9.5820e-03	2.0569	4.8269e-04	2.0704	9.5941e-03	2.0569	1.1588e-02	2.0611	8.2791e-01
	26246	3.7498e-03	2.0282	1.8859e-04	2.0317	3.7546e-03	2.0282	4.5310e-03	2.0301	8.2863e-01
	58814	1.6674e-03	2.0088	8.3810e-05	2.0103	1.6695e-03	2.0088	2.0135e-03	2.0105	8.2918e-01
	131414	7.4166e-04	2.0154	3.7268e-05	2.0160	7.4260e-04	2.0154	8.9537e-04	2.0159	8.2938e-01
2	3384	2.2263e-03	--	1.1132e-04	--	2.2290e-03	--	3.6826e-03	--	6.0529e-01
	19944	1.4599e-04	3.0718	7.2751e-06	3.0757	1.4618e-04	3.0718	2.3901e-04	3.0835	6.1158e-01
	50304	3.5848e-05	3.0358	1.7855e-06	3.0368	3.5892e-05	3.0358	5.8541e-05	3.0412	6.1312e-01
	112726	1.0645e-05	3.0097	5.3011e-07	3.0101	1.0658e-05	3.0097	1.7356e-05	3.0135	6.1405e-01
	251876	3.1615e-06	3.0200	1.5743e-07	3.0202	3.1654e-06	3.0200	5.1503e-06	3.0222	6.1460e-01

TABLE 1. Test 1. Convergence history for an uniformly generated sequence of hexagonal meshes.

k	N	Φ	$r(\Phi)$	η	$e(\eta)$	θ	$r(\theta)$	Ψ	$r(\Psi)$	Λ	$r(\Lambda)$
0	589	1.0813e+00	--	2.4480e-01	--	4.1938e-01	--	4.9543e-01	--	1.5941e-01	--
	3469	4.3269e-01	1.0331	9.9066e-02	1.0204	1.7027e-01	1.0167	1.9984e-01	1.0240	6.6419e-02	0.9874
	8749	2.7038e-01	1.0166	6.2010e-02	1.0129	1.0674e-01	1.0096	1.2490e-01	1.0162	4.1913e-02	0.9954
	19605	1.8014e-01	1.0066	4.1346e-02	1.0047	7.1146e-02	1.0056	8.3291e-02	1.0043	2.8040e-02	0.9963
	43805	1.2010e-01	1.0086	2.7577e-02	1.0075	4.7506e-02	1.0047	5.5492e-02	1.0103	1.8769e-02	0.9987
1	1766	5.8895e-02	--	2.6740e-02	--	2.4564e-02	--	1.7131e-02	--	1.0741e-02	--
	10406	9.5028e-03	2.0569	4.3181e-03	2.0560	4.0707e-03	2.0268	2.3959e-03	2.2182	1.7408e-03	2.0519
	26246	3.7187e-03	2.0283	1.6914e-03	2.0262	1.6028e-03	2.0150	8.9830e-04	2.1208	6.8185e-04	2.0263
	58814	1.6535e-03	2.0089	7.5239e-04	2.0079	7.1485e-04	2.0014	3.8855e-04	2.0774	3.0292e-04	2.0111
	131414	7.3548e-04	2.0154	3.3486e-04	2.0139	3.1868e-04	2.0097	1.6997e-04	2.0569	1.3484e-04	2.0135
2	3384	2.1867e-03	--	1.5283e-03	--	7.9424e-04	--	2.3731e-03	--	4.2607e-04	--
	19944	1.4347e-04	3.0713	9.9799e-05	3.0766	5.3189e-05	3.0482	1.5168e-04	3.1008	2.7317e-05	3.0973
	50304	3.5232e-05	3.0356	2.4480e-05	3.0379	1.3119e-05	3.0261	3.7011e-05	3.0494	6.6740e-06	3.0466
	112726	1.0462e-05	3.0096	7.2624e-06	3.0120	3.9066e-06	3.0027	1.0949e-05	3.0189	1.9758e-06	3.0172
	251876	3.1074e-06	3.0200	2.1565e-06	3.0206	1.1618e-06	3.0168	3.2445e-06	3.0257	5.8587e-07	3.0240

TABLE 2. Test 1. Convergence history of the terms composing the estimator using hexagonal meshes.

6.2. Test 2. Solution with a sharp layer: uniform vs adaptive refinement

We consider $\Omega := (0, 1)^2$ with $\Gamma_D := \{(w, 0), (0, w) \in \Omega : 0 \leq w \leq 1\}$ and $\Gamma_N := \Gamma \setminus \bar{\Gamma}_D$, and choose f and g such that the exact solution is given by

$$u(x, y) = (x - 1)^2(y - 1)^2 \left(\frac{1}{x + 0.1} + \frac{1}{1 + y} \right) \quad \text{in } \Omega.$$

Note that u and ∇u are singular along the lines $x = -0.1$ and $y = -1$. Both such lines are outside Ω , but we expect regions of high gradients in the vicinity of the left boundary. From Figure 1 we observe, as expected, that the adaptive methods outperforms uniform refinement. Indeed, initially the adaptive method superconverges until, once the steep layer is resolved, both methods converge at the theoretical rate, namely $k + 1$. This is clearly shown in Table 3, where the rates of convergence of the global error and the estimator at each step of

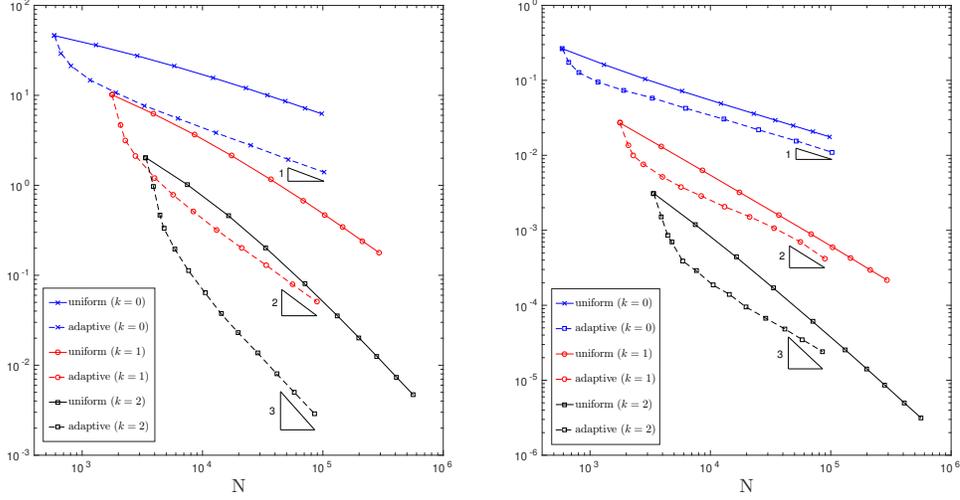


FIGURE 1. Test 2. Convergence history under uniform and the adaptive refinement of hexagonal meshes (cf. Figure 3). The error $e(\boldsymbol{\sigma})$ (left) and $e(u)$ (right).

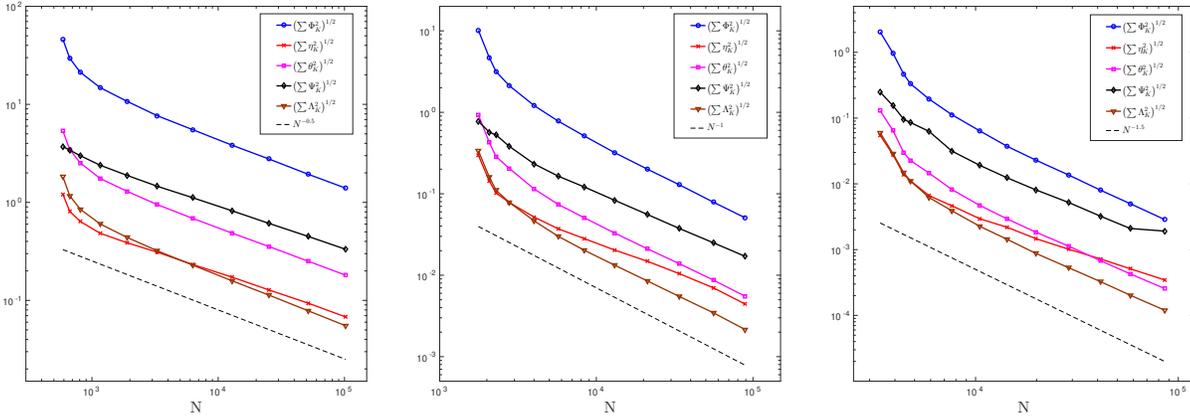


FIGURE 2. Test 2. Convergence history of the components of the estimator under adaptive refinement of hexagonal meshes (cf. Figure 3 below). For $k = 0$ (left), $k = 1$ (centre), and $k = 2$ (right).

the adaptive algorithm are reported together with the effectivity index. As shown in Figure 2, all terms in the error estimator follow precisely the same behaviour.

Some intermediate meshes obtained with adaptive strategy are displayed in Figure 3. Notice here that the adapted meshes concentrate the refinements in the proximity of the line $x = 0$, confirming that the adaptive algorithm is able to target the regions with high gradients of the solution.

k	N	$e(\boldsymbol{\sigma}, u)$	$r(\boldsymbol{\sigma}, u)$	Θ	$r(\Theta)$	$\text{eff}(\Theta)$
0	589	4.6114e+01	--	4.6592e+01	--	0.9896
	668	2.9471e+01	7.1146	2.9879e+01	7.0595	0.9863
	809	2.1279e+01	3.4013	2.1642e+01	3.3681	0.9832
	1163	1.4820e+01	1.9932	1.5118e+01	1.9769	0.9803
	1902	1.0735e+01	1.3112	1.0978e+01	1.3011	0.9779
	3290	7.6690e+00	1.2275	7.8661e+00	1.2165	0.9749
	6272	5.5309e+00	1.0131	5.6850e+00	1.0066	0.9729
	12928	3.8304e+00	1.0158	3.9474e+00	1.0086	0.9704
1	1766	1.0162e+01	--	1.0241e+01	--	0.9923
	2072	4.7026e+00	9.6441	4.7595e+00	9.5903	0.9880
	2288	3.1654e+00	7.9834	3.2240e+00	7.8561	0.9818
	2782	2.1237e+00	4.0834	2.1691e+00	4.0545	0.9791
	4014	1.2081e+00	3.0775	1.2367e+00	3.0652	0.9769
	5706	7.8010e-01	2.4868	8.0163e-01	2.4652	0.9732
	8368	5.1334e-01	2.1859	5.3063e-01	2.1550	0.9674
	13090	3.1982e-01	2.1151	3.3270e-01	2.0867	0.9613
	21158	2.0077e-01	1.9394	2.0991e-01	1.9183	0.9564
2	3384	2.0312e+00	--	2.0513e+00	--	0.9902
	3913	9.7026e-01	10.1734	9.8510e-01	10.0996	0.9849
	4422	4.6358e-01	12.0796	4.7458e-01	11.9440	0.9768
	4771	3.3337e-01	8.6810	3.4515e-01	8.3847	0.9659
	5899	1.9436e-01	5.0845	2.0487e-01	4.9154	0.9487
	7640	1.1258e-01	4.2231	1.1729e-01	4.3134	0.9598
	10494	6.3836e-02	3.5747	6.6868e-02	3.5407	0.9547
	14293	3.7310e-02	3.4766	3.9482e-02	3.4105	0.9450
	19800	2.3005e-02	2.9673	2.4509e-02	2.9259	0.9386

TABLE 3. Test 2. The behaviour of the global error and the estimator under adaptive refinement of hexagonal meshes (cf. Figure 3). The effectivity of the estimator is reported in the right-most column.

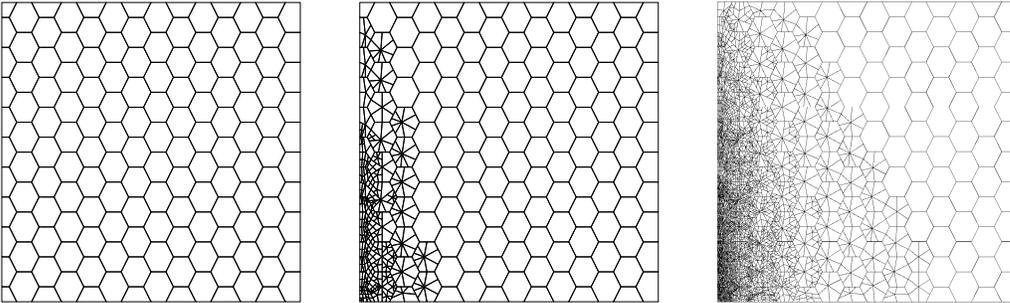


FIGURE 3. Test 2. Some meshes from the adaptive refinement sequence obtained with $k = 1$: initial (left), after 5 refinement steps (centre), and after 10 refinement steps (right).

6.3. Test 3. L-shaped domain solution: adaptive refinement

We consider $\Omega := (-1, -1)^2 \setminus (0, 1)^2$ with $\Gamma_N := \left\{ (-1, w), (w, -1) \in \Omega : -1 \leq w \leq 1 \right\}$ and $\Gamma_D := \Gamma \setminus \bar{\Gamma}_N$, and choose f and g such that the exact solution is given by

$$u(x, y) = \frac{(x+1)^2(y+1)^2}{\sqrt{(x-0.1)^2 + (y-0.1)^2}} \quad \text{in } \Omega.$$

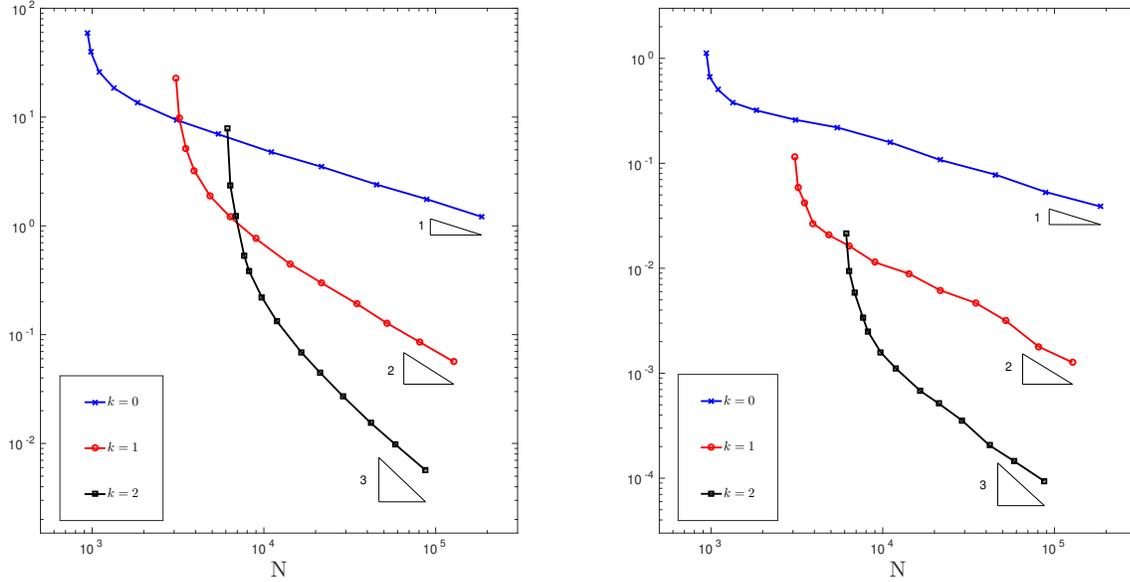


FIGURE 4. Test 3. Errors curves for the adaptive strategy using distorted quadrilateral meshes, (cf. Figure 5 below). The error $e(\boldsymbol{\sigma})$ (left) and the error $e(u)$ (right).

Note that Ω is an L-shaped domain and that u and ∇u are singular at the point $(0.1, 0.1)$, which is just outside of Ω . Hence, we should expect regions of high gradients around the origin, which is the middle corner of the L-shaped domain. In Figure 4 and Table 4 we display the convergence history of the adaptive method. Finally, Figure 5 shows how the adaptive strategy correctly refines in a neighbourhood of the origin. We also notice that increasing the order of the method allows for a less aggressive refinement.

7. CONCLUSIONS

We have derived a posteriori error estimates for a mixed-VEM approach for a second order elliptic equation in divergence form with mixed boundary conditions. We have proved upper and lower bounds for the error between the true solution and both the VEM approximation and a computable postprocessing of the VEM approximation. In particular, the postprocessing permitted us to obtain optimal error estimates in the broken $H(\text{div})$ -norm, whereas for the directly computable projection of the virtual element approximation, it is only possible to prove error estimates in the L^2 -norm. Arguments based in the inf-sup global condition, suitable Helmholtz decompositions and a type Clément-type interpolant were used to derive the upper bound. The lower bound was obtained, in classical fashion, by using localisation techniques of bubble functions. We have also proposed an adaptive algorithm based on the fully local and computable error estimator derived from the a posteriori error analysis. Its performance and effectiveness was illustrated through some numerical test. The extension of the present analysis to other relevant problems, such as the Stokes system, will be the subject of future works.

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k	N	$e(\sigma, u)$	$r(\sigma, u)$	Θ	$r(\Theta)$	$\text{eff}(\Theta)$
0	940	5.9057e+01	--	6.1812e+01	--	0.9554
	982	3.9730e+01	18.1365	4.1712e+01	17.9953	0.9525
	1096	2.6136e+01	7.6263	2.7724e+01	7.4387	0.9427
	1337	1.8521e+01	3.4653	2.0096e+01	3.2376	0.9216
	1838	1.3548e+01	1.9650	1.4905e+01	1.8783	0.9090
	3098	9.4108e+00	1.3959	1.0536e+01	1.3286	0.8932
	5420	6.9603e+00	1.0786	7.8716e+00	1.0426	0.8842
	11100	4.7501e+00	1.0659	5.4231e+00	1.0395	0.8759
1	3080	2.2588e+01	--	2.4078e+01	--	0.9381
	3212	9.7746e+00	39.9209	1.0379e+01	40.1065	0.9418
	3516	5.1580e+00	14.1379	5.6331e+00	13.5160	0.9157
	3936	3.2117e+00	8.3964	3.5285e+00	8.2911	0.9102
	4866	1.8770e+00	5.0650	2.1510e+00	4.6667	0.8726
	6342	1.2110e+00	3.3083	1.4262e+00	3.1021	0.8491
	8966	7.6293e-01	2.6687	9.0810e-01	2.6076	0.8401
	14202	4.4611e-01	2.3334	5.5115e-01	2.1714	0.8094
21684	2.9875e-01	1.8948	3.6841e-01	1.9037	0.8109	
2	6120	7.8275e+00	--	8.6432e+00	--	0.9056
	6378	2.3483e+00	58.3131	2.5850e+00	58.4627	0.9084
	6851	1.2364e+00	17.9329	1.4322e+00	16.5094	0.8633
	7676	5.3466e-01	14.7464	6.6401e-01	13.5204	0.8052
	8200	3.8435e-01	9.9969	4.6526e-01	10.7732	0.8261
	9738	2.2031e-01	6.4748	2.7509e-01	6.1138	0.8009
	11895	1.3259e-01	5.0756	1.7407e-01	4.5744	0.7617
	16581	6.8309e-02	3.9936	9.0940e-02	3.9096	0.7512
21183	4.4949e-02	3.4173	6.1021e-02	3.2578	0.7366	

TABLE 4. Test 3. The behaviour of the global error and the estimator using the adaptive strategy. The effectivity of the estimator is reported in the right-most column.

REFERENCES

- [1] M. AINSWORTH AND J. T. ODEN, *A posteriori error estimation in finite element analysis*, Pure and Applied Mathematics (New York), Wiley-Interscience [John Wiley & Sons], New York, 2000.
- [2] M. ALVAREZ, G. N. GATICA, AND R. RUIZ-BAIER, *A posteriori error analysis for a viscous flow-transport problem*, ESAIM Math. Model. Numer. Anal., 50 (2016), pp. 1789–1816.
- [3] P. F. ANTONIETTI, L. BEIRÃO DA VEIGA, D. MORA, AND M. VERANI, *A stream virtual element formulation of the Stokes problem on polygonal meshes*, SIAM J. Numer. Anal., 52 (2014), pp. 386–404.
- [4] T. P. BARRIOS, G. N. GATICA, M. A. GONZÁLEZ, AND N. HEUER, *A residual based a posteriori error estimator for an augmented mixed finite element method in linear elasticity*, M2AN Math. Model. Numer. Anal., 40 (2006), pp. 843–869 (2007).
- [5] L. BEIRÃO DA VEIGA, F. BREZZI, A. CANGIANI, G. MANZINI, L. MARINI, AND A. RUSSO, *Basic principles of virtual element methods*, Math. Models Methods Appl. Sci., 23 (2013), pp. 199–214.
- [6] L. BEIRÃO DA VEIGA, F. BREZZI, L. MARINI, AND A. RUSSO, *$H(\text{div})$ and $H(\text{curl})$ -conforming virtual element method*, Numer. Math., 133 (2016), pp. 303–332.
- [7] ———, *Mixed virtual element methods for general second order elliptic problems on polygonal meshes*, ESAIM Math. Model. Numer. Anal., 50 (2016), pp. 727–747.
- [8] L. BEIRÃO DA VEIGA, C. LOVADINA, AND G. VACCA, *Divergence free virtual elements for the Stokes problem on polygonal meshes*, ESAIM Math. Model. Numer. Anal., 51 (2017), pp. 509–535.
- [9] L. BEIRÃO DA VEIGA, C. LOVADINA, AND G. VACCA, *Virtual elements for the Navier-Stokes problem on polygonal meshes*, SIAM J. Numer. Anal., 56 (2018), pp. 1210–1242.
- [10] L. BEIRÃO DA VEIGA AND G. MANZINI, *An a posteriori error estimator for the mimetic finite difference approximation of elliptic problems*, Internat. J. Numer. Methods Engrg., 76 (2008), pp. 1696–1723.

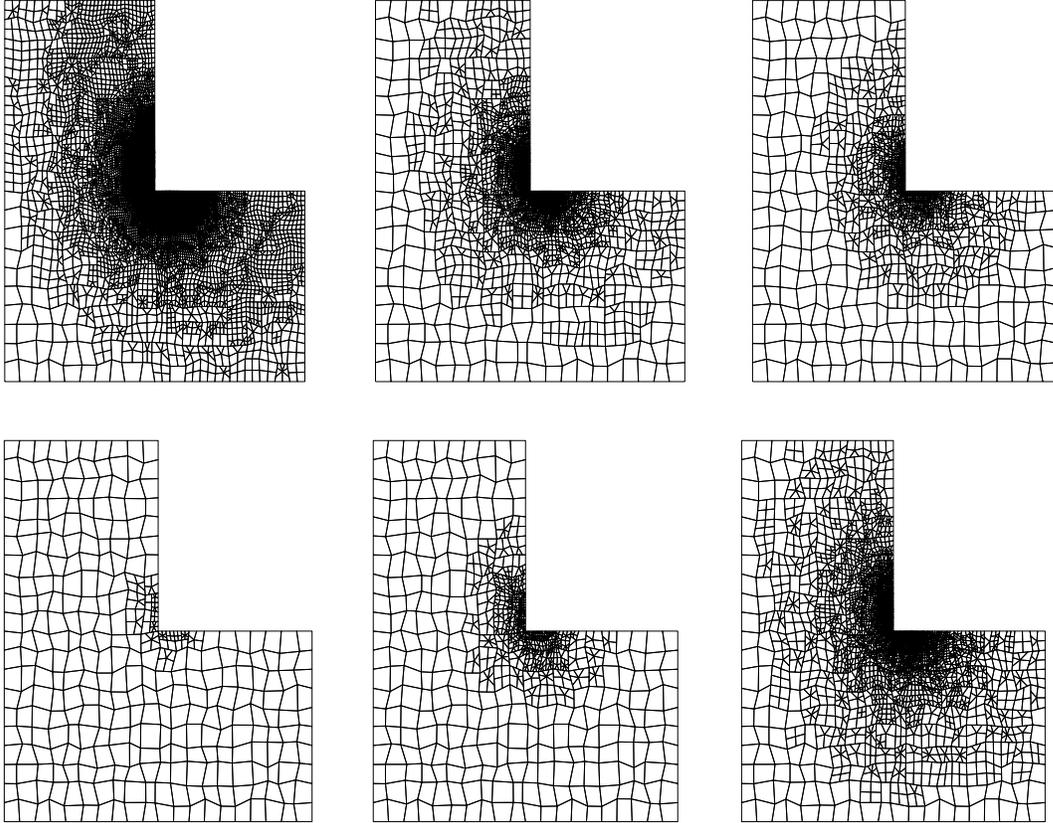


FIGURE 5. Test 3. (Top) The mesh after ten adaptive refinements with $k = 0$ (left), $k = 1$ (centre) and $k = 2$ (right). (Below) Some meshes from the adaptive refinement sequence for $k = 2$: after 3 refinement steps (left), 8 refinement steps (centre), and 15 refinement steps (right).

- [11] L. BEIRÃO DA VEIGA AND G. MANZINI, *A virtual element method with arbitrary regularity*, IMA J. Numer. Anal., 34 (2014), pp. 759–781.
- [12] L. BEIRÃO DA VEIGA AND G. MANZINI, *Residual a posteriori error estimation for the virtual element method for elliptic problems*, ESAIM Math. Model. Numer. Anal., 49 (2015), pp. 577–599.
- [13] S. BERRONE AND A. BORIO, *A residual a posteriori error estimate for the Virtual Element Method*, Math. Models Methods Appl. Sci., 27 (2017), pp. 1423–1458.
- [14] S. C. BRENNER AND L. R. SCOTT, *The mathematical theory of finite element methods*, vol. 15 of Texts in Applied Mathematics, Springer, New York, third ed., 2008.
- [15] F. BREZZI, R. S. FALK, AND L. MARINI, *Basic principles of mixed virtual element methods*, ESAIM Math. Model. Numer. Anal., 48 (2014), pp. 1227–1240.
- [16] E. CÁCERES AND G. GATICA, *A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem*, IMA J. Numer. Anal., 37 (2017), pp. 296–331.
- [17] E. CÁCERES AND G. N. GATICA, *A mixed virtual element method for the pseudostress-velocity formulation of the Stokes problem*, IMA J. Numer. Anal., 37 (2017), pp. 296–331.
- [18] E. CÁCERES, G. N. GATICA, AND F. A. SEQUEIRA, *A mixed virtual element method for the Brinkman problem*, Math. Models Methods Appl. Sci., 27 (2017), pp. 707–743.
- [19] E. CÁCERES, G. N. GATICA, AND F. A. SEQUEIRA, *A mixed virtual element method for quasi-Newtonian Stokes flows*, SIAM J. Numer. Anal., 56 (2018), pp. 317–343.

- [20] A. CANGIANI, E. H. GEORGOULIS, T. PRYER, AND O. J. SUTTON, *A posteriori error estimates for the virtual element method*, Numer. Math., 137 (2017), pp. 857–893.
- [21] A. CANGIANI, V. GYRYA, AND G. MANZINI, *The nonconforming virtual element method for the Stokes equations*, SIAM J. Numer. Anal., 54 (2016), pp. 3411–3435.
- [22] C. CARSTENSEN, *A posteriori error estimate for the mixed finite element method*, Math. Comp., 66 (1997), pp. 465–476.
- [23] C. CARSTENSEN AND G. DOLZMANN, *A posteriori error estimates for mixed FEM in elasticity*, Numer. Math., 81 (1998), pp. 187–209.
- [24] H. CHI, L. BEIRÃO DA VEIGA, AND G. H. PAULINO, *A simple and effective gradient recovery scheme and a posteriori error estimator for the virtual element method (VEM)*, Comput. Methods Appl. Mech. Engrg., 347 (2019), pp. 21–58.
- [25] D. A. DI PIETRO AND R. SPECOGNA, *An a posteriori-driven adaptive mixed high-order method with application to electrostatics*, J. Comput. Phys., 326 (2016), pp. 35–55.
- [26] G. N. GATICA, *A note on the efficiency of residual-based a-posteriori error estimators for some mixed finite element methods*, Electron. Trans. Numer. Anal., 17 (2004), pp. 218–233.
- [27] ———, *A simple introduction to the mixed finite element method*, SpringerBriefs in Mathematics, Springer, Cham, 2014. Theory and applications.
- [28] G. N. GATICA, A. MÁRQUEZ, AND M. A. SÁNCHEZ, *Analysis of a velocity-pressure-pseudostress formulation for the stationary Stokes equations*, Comput. Methods Appl. Mech. Engrg., 199 (2010), pp. 1064–1079.
- [29] G. N. GATICA, M. MUNAR, AND F. A. SEQUEIRA, *A mixed virtual element method for a nonlinear Brinkman model of porous media flow*, Calcolo, 55 (2018), pp. Art. 21, 36.
- [30] G. N. GATICA, M. MUNAR, AND F. A. SEQUEIRA, *A mixed virtual element method for the Navier-Stokes equations*, Math. Models Methods Appl. Sci., 28 (2018), pp. 2719–2762.
- [31] D. MORA, RIVERA, AND R. RODRÍGUEZ, *A posteriori error estimates for a Virtual Element Method for the Steklov eigenvalue problem*, Comput. Math. Appl., 74 (2017), pp. 2172–2190.
- [32] D. MORA AND G. RIVERA, *A priori and a posteriori error estimates for a virtual element spectral analysis for the elasticity equations*, IMA Journal of Numerical Analysis. DOI: <https://doi.org/10.1093/imanum/dry063>.
- [33] D. MORA, G. RIVERA, AND R. RODRÍGUEZ, *A virtual element method for the Steklov eigenvalue problem*, Math. Models Methods Appl. Sci., 25 (2015), pp. 1421–1445.
- [34] G. VACCA, *An H^1 -conforming virtual element for Darcy and Brinkman equations*, Math. Models Methods Appl. Sci., 28 (2018), pp. 159–194.
- [35] R. VERFURTH, *A review of a posteriori error estimation*, in and Adaptive Mesh-Refinement Techniques, Wiley - Teubner, 1996.
- [36] M. VOHRALÍK AND S. YOUSEF, *A simple a posteriori estimate on general polytopal meshes with applications to complex porous media flows*, Comput. Methods Appl. Mech. Engrg., 331 (2018), pp. 728–760.

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INGENIERÍA MATEMÁTICA (CI²MA)
Universidad de Concepción**



Casilla 160-C, Concepción, Chile
Tel.: 56-41-2661324/2661554/2661316
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