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Abstract

In this paper, we propose and analyze a reaction-diffusion model for predator-prey interaction, featuring both prey and predator taxis mediated by nonlocal sensing. Both predator and prey densities are governed by parabolic equations. The prey and predator detect each other indirectly by means of odor or visibility fields, modeled by elliptic equations. We provide uniform estimates in Lebesgue spaces which lead to boundedness and the global well-posedness for the system. Numerical experiments are presented and discussed, allowing us to showcase the dynamical properties of the solutions.

1 Introduction

The mathematical modeling of predator-prey interactions has a long and rich history. The basic dynamics are given by the system of Lotka–Volterra (here in nondimensional form)

$$\begin{cases} u' = \alpha\, uw - u \\ w' = \beta w(1 - w - u), \end{cases} \tag{1.1}$$

where $u(t) \geq 0$ represents the predator and $w(t) \geq 0$ the prey at time $t \geq 0$. According to system (1.1), the predator population decreases exponentially in the absence of prey, while the prey follows a logistic growth law. Interactions between predators and prey are modeled by a mass-action law benefitting the predator and depleting the prey. The system's main feature is the global asymptotic stability of its only nontrivial steady state $(u, w) = (\frac{\alpha-1}{\alpha}, \frac{1}{\alpha})$, which, for $\alpha > 1$, expresses a balance between predation, prey reproduction, and predators' natural death rate.

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When spatial movement is expected to influence the dynamics, it is natural to consider predator and prey densities $u(t,x)$ and $w(t,x)$ depending on a spatial variable x in some physical domain $\Omega \subset \mathbb{R}^2$. Then, one can introduce spatial diffusion and advection to model foraging movement, the spreading of the population in a territory, and/or movement in a preferred direction.

In particular, besides spatial diffusion, a reasonable assumption is that the prey tries to evade the predator, while the predator tries to chase the prey. This can be modeled by introducing advection terms into the equations. Thus, the predators advect towards regions of higher prey density, while prey advects away from regions of higher predator density. Variants of this idea have been considered, for instance, in [1, 9, 11, 12, 14, 17, 22, 26, 27, 28, 30, 31].

Recently, indirect prey- and predator-taxis have been introduced as a mechanism allowing pursuit and evasion [11, 12, 26, 23]. This supposes that the advection velocities are mediated by some indirect signal, which may be an odor, a chemical, a field of visual detection, or seen as a potential. In this spirit, following [11, 12, 26, 23], we consider the predator-prey system with pursuit, evasion and non-local sensing (already written in a non-dimensional form)

$$\begin{cases} \partial_t u - \Delta u + \nabla\cdot \big(u\,\nabla p\big) = \alpha\, uw - u \\ \partial_t w - D_w \Delta w - \nabla\cdot \big(w\,\nabla q\big) = \beta w(1-w-u) \\ -D_p\Delta p = \delta_w w - \delta_p p \\ -D_q \Delta q = \delta_u u - \delta_q q, \end{cases} \tag{1.2}$$

for $t>0$, x in a bounded, open $\Omega \subset \mathbb{R}^2$, supplemented with boundary conditions

$$\nabla u \cdot \mathbf{n} = \nabla w \cdot \mathbf{n} = \nabla p \cdot \mathbf{n} = \nabla q \cdot \mathbf{n} = 0 \qquad \text{in } \partial\Omega \tag{1.3}$$

and initial data $u_0(x), w_0(x)$. Here, $u(t,x)$ is the predator density, $w(t,x)$ is the prey density, $p(t,x)$ is the odor produced by the prey, $q(t,x)$ is the odor produced by the predator, and $\alpha, \beta, D_w, D_p, D_q, \delta_p, \delta_q$ are non-negative non-dimensional constants — see the Appendix for the physical meaning of the constants and details on the non-dimensionalisation procedure. System (1.2) states that the predator is attracted to the odor p of the prey w, which solves a (steady-state) diffusion equation with source proportional to w, while the prey is repelled by the odor q produced by the predator.

Notice that the equations for the odors of the prey and predator are elliptic, rather than parabolic. This is justified in cases where the diffusion of the odor happens in a much faster time scale than the movement of individuals, which is reasonable on a variety of ecological settings.

Note also that we refer to p and q as “odors” but these quantities do not necessarily model chemical odors. They may be more generally interpreted as potentials representing the chance of an animal being detected at a distance by, e.g., visual means.

1.1 Main contributions and outline of the paper

In light of the recent mathematical results in [12, 23], our main contributions are the introduction of the predator-prey Lotka-Volterra dynamics and the numerical simulations. As we will see, the interaction terms on the right-hand side make the analysis more involved, in particular with respect to the derivation of L^∞ estimates. Indeed, while it is shown in the above-mentioned works that the system with no population or interaction dynamics does not originate finite time blow-up of the solutions, the predator equation dynamics introduces a quadratic term uw so it is not obvious that the boundedness property remains valid. We shall see that the attractive-repulsive nature of the advection terms continues to ensure boundedness of the solutions. Moreover, the property of instantaneous boundedness of the solution even for unbounded initial data, observed already in [12], remains valid in this setting.

An outline of the paper follows. In Section 2, we present our main well-posedness result. Next, in Section 3 we derive our main *a priori* estimates in L^γ spaces, for $\gamma \in [1, \infty]$. This will allow us, in Section 4, to construct strong and then weak solutions to the system (1.2), completing the proof of our well-posedness result. Finally, in Section 5 we detail an implicit-explicit two-step finite volume method for the approximation of system (1.2) and present some numerical experiments.

2 Main Results

The main results of this paper are concerned with the well-posedness and Lebesgue integrability of weak solutions of the system (1.2). We follow a strategy similar to [2, 12], making use of fine *a priori* estimates in Lebesgue spaces, and a De Giorgi level-set method [13] to obtain boundedness of the solutions. Still, the character of the present system introduces several changes in the analysis with respect to the results in [2, 12].

The system (1.2) is, mathematically, of chemotaxis type [18, 19]. As is well known, such systems may exhibit blow-up of solutions in finite time, see for example the review [15]. Therefore, it is not obvious at first glance whether solutions might also possess blow-up behavior. As we shall see in the following analysis, the indirect nature of the sensing, as well as the attraction-repulsion behavior, prevent the densities from becoming infinite in finite time.

We say that the quadruple (u, w, p, q) is a weak solution of the system (1.2) if it satisfies:

1. $(u, w) \in L^2(0, T; H^1(\Omega))$ and $(\partial_t u, \partial_t w) \in L^2(0, T, [H^1(\Omega)]^*)$, and
2. For any test function $\xi \in C^\infty([0, \infty) \times \Omega)$ compactly supported in $[0, T) \times \overline{\Omega}$, we have

$$\int_0^T \int_\Omega (-u\partial_t \xi + (\nabla u - u\nabla p)\cdot \nabla \xi)(t,x)\, dx\, dt$$
$$= \int_\Omega u_0(x)\xi(0,x)\, dx + \int_0^T \int_\Omega (uw-u)\xi(t,x)\, dx\, dt,$$
$$\int_0^T \int_\Omega (-w\partial_t \xi + (\nabla w - w\nabla q)\cdot \nabla \xi)(t,x)\, dx\, dt$$
$$= \int_\Omega w_0(x)\xi(0,x)\, dx + \int_0^T \int_\Omega w(1-w-u)\xi(t,x)\, dx\, dt,$$
$$\int_\Omega \nabla q\cdot\nabla\xi\, dx = \int_\Omega (u-q)\xi(t,x)\, dx$$
and
$$\int_\Omega \nabla p\cdot\nabla\xi\, dx = \int_\Omega (w-p)\xi(t,x)\, dx.$$
Note that while the biologically relevant regime is when $\alpha > 1$, the value of $\alpha > 0$ does not change the results, so we present most of our analysis with α, as well as all the remaining constants in system (1.2), set to 1.

We will suppose throughout the paper that the initial data (u_0, w_0) is non-negative and has finite mass
$$\int_\Omega u_0 + w_0\, dx = \mathcal{M} < \infty.$$
and that $\Omega \subset \mathbb{R}^2$ is a bounded domain of class C^2. In what follows, we will use the notation γ^+ to denote an arbitrary fixed number in $(\gamma, +\infty)$. The main results regarding the system (1.2) are collected here:

Theorem 2.1. *Let the initial data u_0, w_0 be in $L^1 \cap L^{2^+}(\Omega)$. Then, the system (1.2) has a unique non-negative weak solution. The following estimates are satisfied by the solutions for any $0 \le t \le T < \infty$, $\gamma \in (1,\infty)$:*

(i) *$L^\gamma-$integrability*
$$\|u(t)\|_\gamma + \|w(t)\|_\gamma \le C(\gamma, \mathcal{M})\Big(1 + \frac{1}{t^{(1/\gamma')^+}}\Big).$$

(ii) *If $u_0, w_0 \in L^\gamma(\Omega)$, then **(i)** becomes*
$$\|u(t)\|_\gamma + \|w(t)\|_\gamma \le C(\gamma, \mathcal{M}, \|u_0\|_\gamma, \|w_0\|_\gamma),$$

(iii) *For $(u_0, w_0) \in L^1(\Omega)$, we have the $L^\infty-$integrability*
$$\|u(t)\|_\infty + \|w(t)\|_\infty \le C\left(1 + \frac{1}{t^{1^+}}\right).$$

(iv) *If $(u_0, w_0) \in L^\infty(\Omega)$, then **(iii)** becomes*
$$\max\{\|u(t)\|_\infty, \|w(t)\|_\infty\} \le C(\mathcal{M}, \|u_0\|_\infty, \|w_0\|_\infty), \qquad t \ge 0,$$

*for some C independent of t, depending on $L^\gamma-$norms of data in **(ii)** and depending on $L^\infty-$norms of the data in (**iv**).*

Let us clarify the notation γ^+ used in the statement of the theorem. For instance, statement (i) means exactly that for every $\delta > 2$, if the initial data are in $L^1 \cap L^\delta$, then for every $\gamma \in [1, \infty)$ and every $\epsilon > 0$ it holds

$$\|u(t)\|_\gamma + \|w(t)\|_\gamma \leq C(\gamma, \mathcal{M})\Big(1 + \frac{1}{t^{(1/\gamma')+\epsilon}}\Big).$$

Also, it is clear that $L^\alpha \subset L^1$ for $\alpha > 1$, since Ω is bounded. Still, we often write $u_0 \in L^1 \cap L^\alpha$ to emphasize the fact that we are dealing with finite mass initial data.

3 Analysis of the system (1.2)

In this section we provide *a priori* estimates which will be used in the well-posedness results. To establish the existence of a weak solution, we must first prove that strong, or classical, solutions exist. We say that (u, w, p, q) is a classical solution of the system (1.2) if it satisfies:

1. $(u, w, p, q) \in C([0, T]; L^2(\Omega))$ and each of the terms in the system (1.2) are well defined functions in $L^2((0, T) \times \Omega)$,
2. The equations on (1.2) are satisfied almost everywhere, and
3. The initial data $(u, w)|_{t=0} = (u_0, w_0)$ and boundary conditions (1.3) are satisfied almost everywhere.

An essential feature of solutions of (1.2) is the following mass estimate:

Proposition 3.1 (Mass estimate)**.** *Let (u, w, p, q) be sufficiently smooth non-negative solutions of the system* (1.2) *with the boundary conditions* (1.3)*. Then there exists a constant $\mathcal{M}$ depending on $\|u_0\|_1$, $\|w_0\|_1$, α, β and $|\Omega|$, but not on t, such that for all $t > 0$,*

$$\int_\Omega w(t) + u(t)\, dx \leq \mathcal{M}. \tag{3.1}$$

Proof. Integrating the first and second equations of (1.2) and using the Neumann boundary conditions (1.3), we find

$$\frac{d}{dt}\int_\Omega w + \frac{\beta}{\alpha} u\, dx \leq \beta \int_\Omega w\, dx - \beta \int_\Omega w^2\, dx - \frac{\beta}{\alpha}\int_\Omega u\, dx.$$

From $\beta(w - w^2) \leq \frac{(\beta+1)^2}{4\beta} - w$ we get, with $\zeta(t) := \int_\Omega w + \frac{\beta}{\alpha} u\, dx$,

$$\begin{aligned}
&\frac{d}{dt}\zeta(t) + \zeta(t) \leq C|\Omega| \\
\Rightarrow\ &\frac{d}{dt}(e^t\zeta(t)) \leq e^t C|\Omega| \\
\Rightarrow\ &\zeta(t) \leq e^{-t}\zeta(0) + (1 - e^{-t})C|\Omega|.
\end{aligned}$$

The conclusion of the proposition readily follows. □

3.1 A priori estimates in L^γ

In this section we prove that data $(u_0, w_0) \in L^1(\Omega)$ generate instantaneous L^γ-integrability, with $\gamma > 1$, for classical non-negative solutions of (1.2).

Proposition 3.2 (A priori estimates in L^γ)**.** *Let (u, w, p, q) be sufficiently smooth nonnegative solutions of the system* (1.2) *with boundary condition* (1.3) *and integrable initial data, and let $t > 0$ be arbitrary. Then, for any $\gamma \in (1, \infty)$ we have the estimate*

$$\|u(t)\|_\gamma + \|w(t)\|_\gamma \le C(\gamma, \mathcal{M})\Big(1 + \frac{1}{t^{(1/\gamma')^+}}\Big). \tag{3.2}$$

Moreover, if $u_0, w_0 \in L^\gamma(\Omega)$, then actually

$$\|u(t)\|_\gamma + \|w(t)\|_\gamma \le C(\gamma, \mathcal{M}, \|u_0\|_\gamma, \|w_0\|_\gamma), \tag{3.3}$$

for some $C > 0$ depending on $\mathcal{M}$ and $L^\gamma-$norms of the data, but independent of t.

Proof. We will frequently use the Gagliardo–Nirenberg–Sobolev (GNS) inequality in two dimensions (see e. g. [20]), holding for any $\alpha \ge 1$,

$$\begin{aligned}\int_\Omega \xi^{\alpha+1}\,dx &\le C(\Omega,\alpha)\|\xi\|_1\ \|\xi^{\alpha/2}\|_{H^1}^2 \\ &\le C(\Omega,\alpha)\int_\Omega \xi\,dx\left(\int_\Omega \xi^\alpha\,dx + \int_\Omega |\nabla(\xi^{\alpha/2})|^2\,dx\right),\end{aligned} \tag{3.4}$$

as well as the interpolation inequality

$$\|u\|_\gamma \le \|u\|_1^{1-\theta}\|u\|_{\gamma+1}^\theta\,, \qquad \theta = \tfrac{\gamma^2-1}{\gamma^2} \in (0,1)\,. \tag{3.5}$$

We start by multiplying the second equation of (1.2) by $w^{\gamma-1}$ and integrating, to find

$$\begin{aligned}&\frac{d}{dt}\frac{1}{\gamma}\int_\Omega w^\gamma\,dx + \frac{\gamma-1}{\gamma}\int_\Omega \nabla q\cdot\nabla(w^\gamma)\,dx + (\gamma-1)\int_\Omega w^{\gamma-2}|\nabla w|^2\,dx \\ &\quad= \int_\Omega w^\gamma\,dx - \int_\Omega u w^\gamma\,dx.\end{aligned}$$

Now multiply the fourth equation of (1.2) by w^γ and integrate by parts to get

$$-\int_\Omega \nabla q\cdot\nabla(w^\gamma)\,dx \le \int_\Omega q w^\gamma\,dx. \tag{3.6}$$

Also using

$$\int_\Omega w^{\gamma-2}|\nabla w|^2\,dx = 4\frac{\gamma-1}{\gamma^2}\int_\Omega |\nabla w^{\gamma/2}|^2\,dx,$$

we find, discarding nonpositive terms,

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^\gamma\,dx &+ 4\frac{\gamma-1}{\gamma}\int_\Omega |\nabla w^{\gamma/2}|^2\,dx\\ &\le \gamma\int_\Omega w^\gamma\,dx + (\gamma-1)\int_\Omega q w^\gamma\,dx\end{aligned} \tag{3.7}$$

Let us consider the terms on the right-hand side of the previous inequality. Take a small $\epsilon > 0$ to be specified later. We use the following consequence of Young's inequality, $qw^\gamma \le \epsilon w^{\gamma+1} + \epsilon^{-\gamma} q^{\gamma+1}$, and also the inequality

$$\int_\Omega w^\gamma\,dx \le C\|w\|_{\gamma+1}^{\frac{\gamma^2-1}{\gamma}} \le C + \epsilon\|w\|_{\gamma+1}^{\gamma+1},$$

which is obtained from Young's inequality, the mass estimate (3.1), and the interpolation inequality (3.5). Therefore, for some constant C depending on γ, $\mathcal{M}$, and ϵ,

$$\begin{aligned}\gamma\int_\Omega w^\gamma\,dx &+ (\gamma-1)\int_\Omega q w^\gamma\,dx\\ &\le C + C\epsilon\int_\Omega w^{\gamma+1}\,dx + C\int_\Omega q^{\gamma+1}\,dx.\end{aligned}$$

This way, we find

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^\gamma\,dx &+ 4\frac{\gamma-1}{\gamma}\int_\Omega |\nabla w^{\gamma/2}|^2\,dx\\ &\le C + C\epsilon\int_\Omega w^{\gamma+1}\,dx + C\int_\Omega q^{\gamma+1}\,dx\end{aligned}$$

and from the GNS inequality (3.4) and ϵ sufficiently small,

$$\begin{aligned}\frac{d}{dt}\int_\Omega w^\gamma\,dx &+ C\int_\Omega w^{\gamma+1}\,dx\\ &\le C + C\int_\Omega w^\gamma\,dx + C\int_\Omega q^{\gamma+1}\,dx.\end{aligned}$$

Again, from $\int_\Omega w^\gamma\,dx \le C + \epsilon\int_\Omega w^{\gamma+1}\,dx$, we get

$$\frac{d}{dt}\int_\Omega w^\gamma\,dx + C\int_\Omega w^{\gamma+1}\,dx \le C + C\int_\Omega q^{\gamma+1}\,dx. \tag{3.8}$$

for some C depending on γ, $\mathcal{M}$, the GNS constant, and the parameters of the system.

To deal with the last term on the right-hand side of (3.8), multiply the fourth equation of (1.2) by $q^{\gamma-1}$ to get

$$\begin{aligned}\int_\Omega |\nabla q^{\gamma/2}|^2\,dx &= \int_\Omega u q^{\gamma-1}\,dx\\ &\le C\int_\Omega u^\gamma\,dx + C\int_\Omega q^\gamma\,dx.\end{aligned}$$

From the GNS inequality (3.4) we deduce that

$$\begin{aligned}\int_\Omega q^{\gamma+1}\,dx &\le C\int_\Omega u^\gamma\,dx + C\int_\Omega q^\gamma\,dx\\ &\le C\int_\Omega u^\gamma\,dx + C\epsilon\int_\Omega q^{\gamma+1}\,dx + C.\end{aligned}$$

Choosing ϵ small, we find

$$\int_\Omega q^{\gamma+1}\,dx \le C\int_\Omega u^\gamma\,dx + C. \tag{3.9}$$

In view of (3.9), the estimate (3.8) becomes

$$\frac{d}{dt}\int_\Omega w^\gamma\,dx + C\int_\Omega w^{\gamma+1}\,dx \le C + C\int_\Omega u^\gamma\,dx. \tag{3.10}$$

Note that the gain from w being the prey population, and thus having a repulsive behavior, is reflected in the lower power u^γ.

In contrast, performing very similar computations using the first and third equations of the system (1.2), so that instead of (3.6) only

$$\int_\Omega \nabla p\cdot\nabla(u^\gamma)\,dx \le \int_\Omega wu^\gamma\,dx.$$

is valid, we find that for $\alpha \ge 2$

$$\frac{d}{dt}\int_\Omega u^\alpha\,dx + C\int_\Omega u^{\alpha+1}\,dx \le C + C\int_\Omega w^{\alpha+1}\,dx. \tag{3.11}$$

Adding (3.10) and (3.11) gives

$$\frac{d}{dt}\int_\Omega w^\gamma + u^\alpha\,dx + C\int_\Omega w^{\gamma+1} + u^{\alpha+1}\,dx \le C|\Omega| + C\int_\Omega w^{\alpha+1} + \int_\Omega u^\gamma\,dx.$$

It is clear that to conveniently bound the terms on the right-hand side using the left-hand side, we should take $\alpha < \gamma < \alpha+1$.

Now, we make use of the interpolation inequalities

$$\begin{aligned}&\|u\|_\gamma \le \|u\|_1^{1-\theta_1}\|u\|_{\alpha+1}^{\theta_1}, \qquad &&\theta_1 = \tfrac{(\gamma-1)(\alpha+1)}{\gamma\alpha} \in (0,1),\\ &\|w\|_{\alpha+1} \le \|w\|_1^{1-\theta_2}\|w\|_{\gamma+1}^{\theta_2}, &&\theta_2 = \tfrac{\alpha(\gamma+1)}{\gamma(\alpha+1)} \in (0,1).\end{aligned}$$

Recalling the mass estimate (3.1), we get

$$\frac{d}{dt}\Big(\|w\|_\gamma^\gamma + \|u\|_\alpha^\alpha\Big) + C\Big(\|w\|_{\gamma+1}^{\gamma+1} + \|u\|_{\alpha+1}^{\alpha+1}\Big) \le C + C\Big(\|w\|_{\gamma+1}^{\theta_2(\alpha+1)} + \|u\|_{\alpha+1}^{\theta_1\gamma}\Big),$$

for C depending on γ, α, and $\mathcal{M}$. Now, $\theta_2(\alpha+1) < \gamma+1$ and $\theta_1\gamma < \alpha+1$, so using Young's inequality with a sufficiently small ϵ allows the terms on the right-hand side to be absorbed into the left-hand side. This gives

$$\frac{d}{dt}\Big(\|w\|_\gamma^\gamma + \|u\|_\alpha^\alpha\Big) + C\Big(\|w\|_{\gamma+1}^{\gamma+1} + \|u\|_{\alpha+1}^{\alpha+1}\Big) \le C$$

for C depending on γ, α, and $\mathcal{M}$. Next, use the inequality (3.5) to find

$$\frac{d}{dt}\Big(\|w\|_\gamma^\gamma + \|u\|_\alpha^\alpha\Big) + C\Big(\big(\|w\|_\gamma^\gamma\big)^{\frac{\gamma}{\gamma-1}} + \big(\|u\|_\alpha^\alpha\big)^{\frac{\alpha}{\alpha-1}}\Big) \le C$$

and so, from $\big(\|u\|_\alpha^\alpha\big)^{\frac{\alpha}{\alpha-1}} \le \big(\|u\|_\alpha^\alpha\big)^{\frac{\gamma}{\gamma-1}}$ and convexity of the power function, we find, setting

$$\eta(t) := \|w\|_\gamma^\gamma + \|u\|_\alpha^\alpha,$$

that

$$\eta'(t) + C\eta(t)^{\frac{\gamma}{\gamma-1}} \le C.$$

Now use the ODE comparison result from [2, Corollary A.2] to conclude

$$\eta(t) \le C\big(1 + t^{1-\gamma}\big).$$

In view of the definition of η, one finds

$$\|w\|_\gamma \le C\big(1 + t^{\frac{1-\gamma}{\gamma}}\big) = C\Big(1 + \frac{1}{t^{1/\gamma'}}\Big)$$

and, taking γ as close to α from above as desired,

$$\|u\|_\alpha \le C\big(1 + t^{\frac{1-\gamma}{\alpha}}\big) = C\Big(1 + \frac{1}{t^{(1/\alpha')^+}}\Big),$$

for C depending on γ, α, and $\mathcal{M}$. This proves the estimate (3.2). The uniform estimate (3.3) follows from the afore-mentioned ODE comparison results in [2]. This concludes the proof of Proposition 3.2. $\square$

3.2 L^∞ estimates

In this section we prove two boundedness results adopting De Giorgi's energy method (see [13] and [2, 12, 7, 8, 21] for related applications of the method). First, we consider initial data u_0 and w_0 only in $L^1(\Omega)$, and obtain an estimate of the type

$$\|u(t)\|_\infty + \|w(t)\|_\infty \le C(\mathcal{M})\left(1 + \frac{1}{t^{1^+}}\right),$$

for some constant $C > 0$ depending on $\mathcal{M}$, but not on $T > 0$. Then, we will suppose that the initial data u_0 and w_0 are in $L^\infty(\Omega)$. Then we can upgrade the estimates above to

$$\max\{\|u(t)\|_\infty, \|w(t)\|_\infty\} \le C(\mathcal{M}, \|u_0\|_\infty, \|w_0\|_\infty), \qquad t \ge 0,$$

where C now depends also on the L^∞ norms of initial data.

3.2.1 Initial data in L^1

The main result in this section is the following:

Proposition 3.3. *Let (u, w, p, q) be a sufficiently smooth non-negative solution of the system* (1.2) *with the boundary conditions (1.3) and let $T > 0$ be arbitrary. Then, for all $t \in (0, T]$, we have the estimate*

$$\|u(t)\|_\infty + \|w(t)\|_\infty \leq C(\mathcal{M}) \left(1 + \frac{1}{t^{1^+}}\right),$$

where the constant C is independent of $T > 0$.

Let us begin by recording here the following L^∞ estimates, which are a consequence of elliptic regularity for the last two equations of the system (1.2) and Proposition 3.2.

$$\|\nabla p(t)\|_\infty \leq C(\Omega)\|p(t)\|_{W^{2,2^+}} \leq C(\Omega)\|u(t)\|_{2^+} \leq C\left(1 + \frac{1}{t^{(1/2)^+}}\right),$$

and

$$\|\nabla q(t)\|_\infty \leq C(\Omega)\|q(t)\|_{W^{2,2^+}} \leq C(\Omega)\|w(t)\|_{2^+} \leq C\left(1 + \frac{1}{t^{(1/2)^+}}\right). \tag{3.12}$$

The first step in the proof of Proposition 3.3 is a boundedness result valid on each interval (t_*, T) with $t_* > 0$.

Lemma 3.4. *Let (u, w, p, q) be as in Proposition 3.3, and let $t_* > 0$. Then, there exist constants $M, N > 0$ depending on t_*, T, and $u(t_*), w(t_*)$ such that $0 \leq u(t, x) \leq M$ and $0 \leq w(t, x) \leq N$ almost everywhere on $(t_*, T) \times \Omega$.*

Proof. Although the structure of the proof is the similar to corresponding results in [2, 12], we show the details since the rather involved calculations depend heavily on the structure of the system. Consider (u, p) a non-negative, sufficiently smooth solution to the general problem

$$\begin{aligned} &\partial_t u - \Delta u + \nabla \cdot (u \nabla p) = f \\ &-\Delta p = w - p \end{aligned} \tag{3.13}$$

in $[0, T] \times \Omega$ with the boundary conditions

$$\nabla u \cdot \mathbf{n}|_{\partial\Omega} = \nabla p \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.14}$$

where $\mathbf{n}$ is the outward unit normal vector to $\partial\Omega$, f and w are known, and w satisfies

$$\|w(t)\|_\gamma \leq C\left(1 + \frac{1}{t^{(1/\gamma')^+}}\right), \qquad \gamma > 1.$$

Similarly, take a non-negative solution (w, q) to the problem

$$\begin{aligned} &\partial_t w - \Delta w - \nabla \cdot (w \nabla q) = g \\ &-\Delta q = w - q \end{aligned} \tag{3.15}$$

$[0,T] \times \Omega$ with boundary conditions

$$\nabla w \cdot \mathbf{n}|_{\partial\Omega} = \nabla q \cdot \mathbf{n}|_{\partial\Omega} = 0, \tag{3.16}$$

g and u given and u satisfying

$$\|u(t)\|_\gamma \le C\left(1 + \frac{1}{t^{(1/\gamma')^+}}\right), \qquad \gamma > 1.$$

We denote $V_\lambda = \{(t,x) \in \Omega \times [0,T],\ u(t,x) > \lambda\}$ and define

$$u_\lambda = (u - \lambda)\, \mathbf{1}_{V_\lambda}.$$

Multiplying the first equation of (3.13) by u_λ, integrating in space and using (3.14), we obtain

$$\frac{d}{dt}\int_\Omega u_\lambda^2\, dx + 2\int_\Omega |\nabla u_\lambda|^2\, dx \le -2\int_\Omega u\nabla p \cdot \nabla u_\lambda\, dx + 2\int_\Omega f_+ u_\lambda\, dx.$$

For the first term on the right-hand side note that using Young's inequality, we find

$$-2\int_\Omega u\nabla p \cdot \nabla u_\lambda\, dx \le \int_\Omega |u\nabla p|^2\, \mathbf{1}_{V_\lambda}\, dx + \int_\Omega |\nabla u_\lambda|^2\, dx.$$

Thus,

$$\frac{d}{dt}\int_\Omega u_\lambda^2\, dx + \int_\Omega |\nabla u_\lambda|^2\, dx \le \int_\Omega |u\nabla p|^2\, \mathbf{1}_{V_\lambda}\, dx + 2\int_\Omega f_+ u_\lambda\, dx. \tag{3.17}$$

Let $M > 0$ be a constant to be defined later, let $t_* > 0$, and set

$$\lambda_k = \left(1 - \frac{1}{2^k}\right) M, \qquad t_k = \left(1 - \frac{1}{2^{k+1}}\right)\, t_*$$

for $k = 0, 1, 2, \ldots$. Define the energy functional

$$U_k := \sup_{t\in[t_k,T]} \int_\Omega u_k^2\, dx + \int_{t_k}^T \int_\Omega |\nabla u_k|^2\, dx\, dt, \tag{3.18}$$

where we use the notation $u_k = u_{\lambda_k}$. Similarly, we define for w the energy functional

$$W_k := \sup_{t\in[t_k,T]} \int_\Omega w_k^2(t)\, dx + \int_{t_k}^T |\nabla w_k|^2\, dx\, dt \tag{3.19}$$

with the same definitions of λ_k (with N instead M) and t_k, for some $N > 0$ to be chosen later.

With $\lambda = \lambda_k$ on (3.17), integrating over $[s,t]$, we obtain

$$\begin{aligned}\int_\Omega u_k^2(t)\, dx + \int_s^t \int_\Omega |\nabla u_k|^2\, dx\, ds \quad \le \quad & \int_\Omega u_k^2(s)\, dx + \int_s^t \int_\Omega |u\nabla p|^2\, \mathbf{1}_{V_k}\, dx\, dt \\ & + 2\int_s^t \int_\Omega f_+ u_k\, dx\, ds.\end{aligned}$$

We use this relation with $t_{k-1} \leq s \leq t_k \leq t \leq T$ to check that

$$\frac{U_k}{2} \leq \int_\Omega u_k^2(s)\, dx + \int_{t_{k-1}}^T \int_\Omega |u\nabla p|^2 \, \mathbf{1}_{V_k}\, dx\, dt + 2\int_{t_{k-1}}^T \int_\Omega f_+ u_k \, dx\, dt.$$

Integrating with respect to s over $[t_{k-1}, t_k]$, bearing in mind that $t_k - t_{k-1} = t_*/2^k$ and only the first term on the right-hand side of this inequality depends on s, we obtain

$$\begin{aligned} U_k \quad &\leq \quad \frac{2^{k+1}}{t_*}\int_{t_{k-1}}^T \int_\Omega u_k^2(s)\, dx\, ds + 2\int_{t_{k-1}}^T \int_\Omega |u\nabla p|^2 \, \mathbf{1}_{V_k}\, dx\, dt \\ &\qquad +4\int_{t_{k-1}}^T \int_\Omega f_+ u_k \, dx\, dt \\ &=: \quad I_1 + I_2 + I_3. \end{aligned}$$

Now, we are going to introduce a series of estimates for I_1, I_2 and I_3. For this, we will use the Gagliardo-Nirenberg interpolation inequality in $\mathbb{R}^n$ (see e.g. [5]) and a key estimate for $\mathbf{1}_{V_k}$. Note that we are temporarily performing the analysis in n dimensions. We have

$$\|u\|_p^p \leq C\|u\|_{H^1}^{\alpha p}\|u\|_2^{(1-\alpha)p} \quad \text{with} \quad 1 = \left(\frac{1}{2} - \frac{1}{n}\right)\alpha p + \frac{1-\alpha}{2}\, p. \tag{3.20}$$

which holds for any $\alpha \in [0,1]$ and $1 \leq p \leq \infty$. Choosing the parameter $\alpha \in [0,1]$ such that $\alpha p = 2$, it follows that $p = 2\frac{n+2}{n}$ and

$$\|u\|_p^p \leq C\|u\|_{H^1}^2 \|u\|_2^{p-2}. \tag{3.21}$$

Observe also that $u_k > 0$ implies $u > \lambda_k$, therefore $u - \lambda_{k-1} > \lambda_k - \lambda_{k-1} = \frac{M}{2^k}$. We also have $u - \lambda_k < u - \lambda_{k-1}$, thus, which a simple computation,

$$\mathbf{1}_{V_k} \leq \left(\frac{2^k}{M}\, u_{k-1}\right)^a. \tag{3.22}$$

holds for all $a \geq 0$.

- Estimate for I_1

First, note that for $a \geq 0$ we have

$$I_1 = \frac{2^{k+1}}{t_*}\int_{t_{k-1}}^T \int_\Omega u_k^2 \, \mathbf{1}_{V_k}\, dx\, ds \leq \frac{2^{k+1}}{t_*}\frac{2^{ka}}{M^a}\int_{t_{k-1}}^T \int_\Omega u_{k-1}^{2+a}\, dx\, ds.$$

We choose a in (3.22) such that $2 + a = p = 2\frac{n+2}{n}$, so $a = \frac{4}{n}$. Thus,

$$\begin{aligned} I_1 \quad &\leq \quad 2\,\frac{2^{\frac{4+n}{n}k}}{M^{\frac{4}{n}} t_*}\int_{t_{k-1}}^T \int_\Omega u_{k-1}^{2\frac{n+2}{n}}\, dx\, ds \\ &\leq \quad 2C(\Omega, n)\,\frac{2^{\frac{4+n}{n}k}}{M^{\frac{4}{n}} t_*}\int_{t_{k-1}}^T (\|u_{k-1}\|_2^2 + \|\nabla u_{k-1}\|_2^2)\, \|u_{k-1}\|_2^{2\frac{n+2}{n}-2}\, ds \\ &\leq \quad 2C(\Omega, n)\,\frac{2^{\frac{4+n}{n}k}}{M^{\frac{4}{n}} t_*} U_{k-1}^{\frac{n+2}{n}-1} \int_{t_{k-1}}^T (\|u_{k-1}\|_2^2 + \|\nabla u_{k-1}\|_2^2)\, ds. \end{aligned}$$

Note that $\int_{t_{k-1}}^T (\|u_{k-1}\|_2^2 + \|\nabla u_{k-1}\|_2^2)\, ds \leq (T+1)U_{k-1}$, which leads to

$$I_1 \leq C(1+T)\, \frac{2^{\frac{4+n}{n}k}}{M^{\frac{4}{n}} t_*}\, U_{k-1}^{\frac{n+2}{n}}. \tag{3.23}$$

- Estimate for I_2

We have that

$$\begin{aligned} I_2 &= 2\int_{t_{k-1}}^T \int_\Omega |u\nabla p|^2\, \mathbf{1}_{V_k}\, dx\, dt \\ &\leq \left(\sup_{t\geq \frac{t_*}{2}} \|u\nabla p\|_{2q'}^2\right) \int_{t_{k-1}}^T \left[\int_\Omega \mathbf{1}_{V_k}\, dx\right]^{\frac{1}{q}}\, dt. \end{aligned}$$

Now, choosing $a = p$ in (3.22), we obtain

$$\begin{aligned} \int_{t_{k-1}}^T \left[\int_\Omega \mathbf{1}_{V_k}\, dx\right]^{\frac{1}{q}}\, dt &\leq \frac{2^{\frac{pk}{q}}}{M^{\frac{p}{q}}} \int_{t_{k-1}}^T \left[\int_\Omega u_{k-1}^p\, dx\right]^{\frac{1}{q}}\, dt \\ &\overset{(3.21)}{\leq} C\, \frac{2^{\frac{pk}{q}}}{M^{\frac{p}{q}}} \int_{t_{k-1}}^T \|u_{k-1}\|_{H^1}^{\frac{p}{q}\alpha}\, \|u_{k-1}\|_2^{(1-\alpha)\frac{p}{q}}\, dt, \end{aligned}$$

where we need $q > 1$. Thus,

$$I_2 \leq C\frac{2^{\frac{pk}{q}}}{M^{\frac{p}{q}}} \left(\sup_{t\geq \frac{t_*}{2}} \|u\nabla p\|_{2q'}^2\right) \int_{t_{k-1}}^T \|u_{k-1}\|_{H^1}^{\frac{p}{q}\alpha}\, \|u_{k-1}\|_2^{(1-\alpha)\frac{p}{q}}\, dt,$$

where we used (3.21) with the relation (3.20). We choose $\alpha \in (0,1)$ satisfying $\frac{\alpha p}{q} = 2$ to find

$$\frac{1}{q} = \frac{p}{2q} - \frac{2}{n} \quad \Rightarrow \quad p = 2\left(\frac{n+2q}{n}\right).$$

Observe that $\alpha = \frac{2q}{p}$ implies $\alpha = \dfrac{qn}{n+2q}$, then

$$0 < \alpha < 1 \quad \Rightarrow \quad 0 \leq 2 \leq \frac{p}{q} \quad \Rightarrow \quad 0 < 2q < 2\left(\frac{n+2q}{n}\right),$$

where q is such that $1 < q < \frac{n}{n-2}$. Thus,

$$\begin{aligned} I_2 &\leq C\frac{2^{2\frac{k}{\alpha}}}{M^{\frac{2}{\alpha}}} \left(\sup_{t\geq \frac{t_*}{2}} \|u\nabla p\|_{2q'}^2\right) \int_{t_{k-1}}^T \|u_{k-1}\|_{H^1}^2\, \|u_{k-1}\|_2^{\frac{2}{\alpha}-2}\, dt \\ &\leq C\frac{2^{2\frac{k}{\alpha}}}{M^{\frac{2}{\alpha}}} \left(\sup_{t\geq \frac{t_*}{2}} \|u\nabla p\|_{2q'}^2\right) (1+T)\, U_{k-1}^{\frac{1}{\alpha}}, \end{aligned}$$

where in this last step we proceeded as in the estimate for I_1. Therefore

$$I_2 \leq C(1+T)\, \frac{2^{2\frac{k}{\alpha}}}{M^{\frac{2}{\alpha}}} \left(\sup_{t\geq \frac{t_*}{2}} \|u\nabla p\|_{2q'}^2\right)\, U_{k-1}^{\frac{1}{\alpha}}. \tag{3.24}$$

- Estimates for I_3

Proceeding as we did for the estimate of I_2, using that $u_k \le u\mathbf{1}_{V_k}$, we have

$$\begin{aligned} I_3 &\overset{a=p}{\le} 4\left(\sup_{t\ge\frac{t_*}{2}} \|f_+u\|_{q'}\right) \frac{2^{\frac{kp}{q}}}{M^{\frac{p}{q}}} \int_{t_{k-1}}^T \|u_{k-1}\|_p^{\frac{p}{q}}\, ds \\ &\le C\left(\sup_{t\ge\frac{t_*}{2}} \|f_+u\|_{q'}\right) \frac{2^{\frac{kp}{q}}}{M^{\frac{p}{q}}} \int_{t_{k-1}}^T \|u_{k-1}\|_{H^1}^{\frac{p\alpha}{q}} \|u_{k-1}\|_2^{(1-\alpha)\frac{p}{q}}\, ds \\ &\le C\left(\sup_{t\ge\frac{t_*}{2}} \|f_+u\|_{q'}\right) \frac{2^{\frac{2k}{\alpha}}}{M^{\frac{2}{\alpha}}}\,(1+T)\, U_{k-1}^{\frac{1}{\alpha}}, \end{aligned}$$

where α, p and q are the same of the estimate for I_2. Since $\sup\limits_{t\ge\frac{t_*}{2}} \|f_+u\|_{q'} \le \sup\limits_{t\ge\frac{t_*}{2}} \{\|f_+\|_{2q'}\|u\|_{2q'}\}$, we find

$$I_3 \le C(1+T)\left(\sup_{t\ge\frac{t_*}{2}} \|f_+\|_{2q'}\|u\|_{2q'}\right) \frac{2^{\frac{2k}{\alpha}}}{M^{\frac{2}{\alpha}}}\, U_{k-1}^{\frac{1}{\alpha}}. \tag{3.25}$$

Combining (3.23), (3.24) e (3.25), we obtain

$$\begin{aligned} &U_k \le C(1+T)\times \\ &\times\left[\frac{2^{\frac{4+n}{n}k}}{M^{\frac{4}{n}}t_*}\, U_{k-1}^{\frac{n+2}{n}} + \frac{2^{2\frac{k}{\alpha}}}{M^{\frac{2}{\alpha}}}\left(\sup_{t\ge\frac{t_*}{2}} \|u\nabla p\|_{2q'}^2 + \sup_{t\ge\frac{t_*}{2}} \{\|f_+\|_{2q'}\|u\|_{2q'}\}\right) U_{k-1}^{\frac{1}{\alpha}}\right]. \end{aligned} \tag{3.26}$$

Now we are going to restrict our reasoning to the case $n=2$, where we have $\alpha = \frac{q}{q+1}$ and the advantage that we can take $1<q<\infty$, since $1<q<\frac{n}{n-2}=\infty$. Using Proposition 3.2, we have

$$\|u\nabla p\|_{2q'}^2 \le \|\nabla p\|_\infty^2 \|u\|_{2q'}^2 \le C\left(1+\frac{1}{t^{\left(\frac{2q+1}{q}\right)^+}}\right).$$

Likewise,

$$\|f_+\|_{2q'}\|u\|_{2q'} \le C\|f_+\|_{2q'}\left(1+\frac{1}{t^{\left(\frac{q+1}{2q}\right)^+}}\right) \tag{3.27}$$

and particularizing to $f_+ = uw$,

$$\|f_+\|_{2q'} = \left(\int_\Omega [uw]^{2q'}\, dx\right)^{\frac{1}{2q'}} \le \|u\|_{4q'}\|w\|_{4q'} \le C\left(1+\frac{1}{t^{\left(\frac{3q+1}{2q}\right)^+}}\right).$$

Therefore,

$$\sup_{t\ge\frac{t_*}{2}} \left\{\|\nabla p\|_\infty^2\|u\|_{2q'}^2 + \|f_+\|_{2q'}\|u\|_{2q'}\right\} \le C\left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right) \tag{3.28}$$

Now, substituting these results in (3.26), we obtain

$$U_k \le C(1+T)\left[\frac{2^{3k}}{M^2 t_*}\, U_{k-1}^2 + \left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right)\, \frac{2^{\frac{2(q+1)}{q}k}}{M^{\frac{2(q+1)}{q}}}\, U_{k-1}^{\frac{q+1}{q}}\right]. \tag{3.29}$$

We are going to prove that there exists a constant $a \in (0,1)$ depending only on q such that $U_k \le a^k U_0$ for all $k \in \mathbb{N}$. First, set $V_k = a^k U_0$. Then applying the recurrence relation defined by the right-hand side of (3.29) to V_k gives

$$\begin{aligned} & C(1+T)\left[\frac{2^{3k}}{M^2 t_*}\, V_{k-1}^2 + \left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right)\, \frac{2^{\frac{2(q+1)}{q}k}}{M^{\frac{2(q+1)}{q}}}\, V_{k-1}^{\frac{q+1}{q}}\right] \\ & = C(1+T)\left[\frac{(2^3 a)^k}{M^2 t_* a^2}\, U_0 + \left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right)\, \frac{(2^{\frac{2(q+1)}{q}} a^{\frac{1}{q}})^k}{M^{\frac{2(q+1)}{q}} a^{\frac{q+1}{q}}}\, U_0^{\frac{1}{q}}\right]\, V_k. \end{aligned} \tag{3.30}$$

Now we choose a such that $\max\{2^3 a,\ 2^{\frac{2(q+1)}{q}} a^{\frac{1}{q}}\} < 1$. So, the last line of (3.30) is bounded by

$$(3.30) \le C(1+T)\left[\frac{U_0}{M^2 t_* a^2} + \left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right)\frac{U_0^{\frac{1}{q}}}{M^{\frac{2(q+1)}{q}} a^{\frac{q+1}{q}}}\right] V_k.$$

Now, choosing M so that

$$\max\left\{\frac{CU_0}{a^2 M^2 t_*},\ \frac{CU_0^{\frac{1}{q}}}{a^{\frac{q+1}{q}} M^{\frac{2(q+1)}{q}}}\left(1+\frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}}\right)\right\} \le \frac{1}{2C(1+T)},$$

we find that

$$(3.30) \le V_k.$$

In other words, V_k is a supersolution of the recurrence relation defined by (3.29). By a comparison principle, we have $U_k \le a^k U_0 \underset{k\to+\infty}{\longrightarrow} 0$. Thus, we obtain

$$\int_{t_*/2}^{T}\int_\Omega u^2(t,x)\, \mathbf{1}_{\{u(t,x)\ge M(1-1/2^k)\}}\, dx\, dt \le U_k \overset{k\to+\infty}{\to} 0.$$

Fatou's lemma implies

$$\frac{1}{T-t_*/2}\int_{t_*/2}^{T}\int_\Omega u^2(t,x)\, \mathbf{1}_{\{u(t,x)\ge M\}}\, dx\, dt = 0,$$

which in turn implies $0 \le u(t,x) \le M$ almost everywhere on $(t_*/2, T)\times\Omega$. For $t_*/2 < 1$ we can determine M explicitly via

$$M = \max\left\{\sqrt{\frac{2C(1+T)U_0}{a^2 t_*}},\ \sqrt{\frac{(2C(1+T))^{\frac{q}{q+1}} U_0^{\frac{1}{q+1}}}{a}\ \frac{2}{t_*^{\left(\frac{2q+1}{2(q+1)}\right)^+}}}\right\}.$$

Taking $q \searrow 1$, we have

$$M = \max \left\{ \sqrt{\frac{2C(1+T)U_0}{a^2 t_*}}, \sqrt{\frac{(2C(1+T))^{\frac{1}{2}} U_0^{\frac{1}{2}}}{a}} \, \frac{2}{t_*^{(3/4)^+}} \right\}, \tag{3.31}$$

A very similar computation using (3.15),(3.19), particularizing to $g_+ = w$ gives

$$W_k \le C(1+T) \left[\frac{2^{3k}}{N^2 t_*} \, W_{k-1}^2 + \left(1 + \frac{1}{t_*^{\left(\frac{2q+1}{q}\right)^+}} \right) \frac{2^{\frac{2(q+1)}{q} k}}{N^{\frac{2(q+1)}{q}}} \, W_{k-1}^{\frac{q+1}{q}} \right].$$

Then, we obtain $N > 0$ as before so that $0 \le w(t,x) \le N$ almost everywhere on $(t_*/2, T) \times \Omega$ and when $0 < t_*/2 < 1$, N can be estimated by

$$N = \max \left\{ \sqrt{\frac{2C(1+T)W_0}{a^2 t_*}}, \sqrt{\frac{(2C(1+T))^{\frac{1}{2}} W_0^{\frac{q}{2}}}{a}} \, \frac{2}{t_*^{(3/4)^+}} \right\}. \tag{3.32}$$

Renaming $t_*/2$ as t_* concludes the proof of Lemma 3.4. □

Proof of Proposition 3.3. In Lemma 3.4 we found $M, N > 0$ such that $0 \le u(t,x) \le M$ and $0 \le w(t,x) \le N$ almost everywhere on $(t_*, T) \times \Omega$. From (3.31) and (3.32) we see that M and N depend directly on U_0 and W_0, respectively. First, we are going to obtain appropriate estimates for these terms, from which the $L^\infty-$bound for w and u in the time interval $(t_*, 1)$ follows. After this, we will extend the estimate for general large intervals. Proceeding as we did in (3.17) for the particular case $f_+ = uw$, we have

$$\begin{aligned} \frac{d}{dt} \int_\Omega u^2 \, dx + 2 \int_\Omega |\nabla u|^2 \, dx &\le 2 \int_\Omega u \nabla u \nabla p \, dx + 2 \int_\Omega f_+ u \, dx \\ &\le \int_\Omega \nabla u^2 \nabla p \, dx + 2 \int_\Omega u^2 w \, dx \\ &\le 3 \int_\Omega u^2 w \, dx, \end{aligned} \tag{3.33}$$

where, in the last step, we used

$$\int_\Omega \nabla u^2 \nabla p \, dx \le \int_\Omega u^2 w \, dx - \int_\Omega p u^2 \, dx \le \int_\Omega u^2 w \, dx,$$

which follows from the first equation of (3.13). Observe that

$$\int_\Omega u^2 w \, dx \le \|u\|_3^3 + \|w\|_3^3 \le C \left(1 + \frac{1}{t^2} \right).$$

Substituting this in (3.33), we obtain

$$\frac{d}{dt} \int_\Omega u^2 \, dx + \int_\Omega |\nabla u|^2 \, dx \le C \left(1 + \frac{1}{t^2} \right). \tag{3.34}$$

Integrating over $[s,t]$, we find

$$\begin{aligned}\int_\Omega u^2(t)\,dx + \int_s^t \int_\Omega |\nabla u(t')|^2\,dx\,dt' &\le \int_\Omega u^2(s)\,dx + C\int_s^t \left(1+\frac{1}{t'^{2^+}}\right)dt' \\ &\le C\left(1+\frac{1}{s^{1^+}}\right).\end{aligned}$$

Observe that from this inequality we conclude

$$U_0 \le C\left(1+\frac{1}{t_*^{1^+}}\right).$$

Now, via (3.31), using the estimates made previously, we get for $t \ge t_*$

$$\|u(t)\|_\infty \le\; M \le \frac{C}{t_*^{1^+}} \tag{3.35}$$

where we remember that $0 < t_* < 1$. For W_0, a similar computation with (3.15) and particularizing $g = w$ leads to

$$W_0 \le C(T+1)\left(1+\frac{1}{t_*^{1^+}}\right).$$

Now, via (3.32), we conclude for $t \ge t_*$

$$\|w(t)\|_\infty \le\; N \le \frac{C}{t_*^{1^+}}, \tag{3.36}$$

where we recall that $0 < t_* < 1$.

These estimates are valid whenever $0 < t_* \le t \le T = 1$, but the same reasoning can be applied for any $T > 0$ and it would provide a bound depending on T. In order to justify that the same L^∞-estimate can be made uniform with respect to T, we proceeding by extending this estimate in a similar way as was done in [2]: let $t_1 \in (t_*, T - t_*)$ and note that the shifted functions $w_{t_1}(t,x) = w(t+t_1,x)$ and $u_{t_1}(t,x) = u(t+t_1,x)$ are still solutions of the same problem with initial data $w_{t_1}(0,x) = w(t_1,x)$ and $u_{t_1}(0,x) = u(t_1,x)$, and the appropriate right-hand side. Since the constant C doesn't change due to the Proposition 3.1, we pick $t_1 \in (0,T)$ and repeat the same arguments to w_{t_1} and u_{t_1}, which leads the same L^∞-bounds (3.35) and (3.36) for w and u on the interval $[t_*, T]$. It means that (3.35) and (3.36) happen for $[t_* + t_1, T + t_1]$, that is, we extend (3.35) and (3.36) over $[t_*, T + t_1]$. We can repeat this procedure, completing the proof of Proposition 3.3. □

3.2.2 Initial data in L^∞

We suppose now we have initial data u_0 and w_0 in $L^\infty(\Omega)$. We will slightly modify the analysis made before in order to obtain a better estimate.

Proposition 3.5. *Let u_0 and w_0 be initial data in $L^\infty(\Omega)$. The estimate* (3.3) *can be upgraded by adding to the constant C the dependence of L^∞-norms on the initial data, getting*

$$\max\{\|u(t)\|_\infty, \|w(t)\|_\infty\} \leq C(\mathcal{M}, \|u_0\|_\infty, \|w_0\|_\infty), \qquad t \geq 0,$$

where the constant $C > 0$ is independent of $T > 0$.

Proof. First of all, by Proposition 3.2, there exist a constant $A > 0$ such that

$$A := \left\{ \sup_{s\geq 0} \|u(s)\|_{2^+}, \sup_{s\geq 0} \|w(s)\|_{2^+} \right\} < \infty.$$

We change the definition $t_k = (1 - 1/2^k)t_*$, and observe now that $t_0 = 0$. Note first that (3.28) becomes

$$\sup_{t\geq \frac{t_*}{2}} \left\{ \|\nabla p\|_\infty^2 \|u\|_{2q'}^2 + \|f_+\|_{2q'} \|u\|_{2q'} \right\} \leq C, \tag{3.37}$$

since $\|\nabla p(t)\|_\infty \leq \|u(t)\|_{2^+} \leq A < \infty$. In this way, (3.29) becomes

$$U_k \leq C \left[\frac{2^{3k}}{M^2 t_*} U_{k-1}^2 + \frac{2^{\frac{2(q+1)}{q}k}}{M^{\frac{2(q+1)}{q}}} U_{k-1}^{\frac{q+1}{q}} \right], \tag{3.38}$$

with a constant C depending on $\mathcal{M}$, A and T, but independent of t_* and k. We are going to proceed as we did after (3.29), but now with (3.38), relying on the fact that here we have

$$U_0 \leq C(1 + T) \leq \tilde{C},$$

where the constant $\tilde{C}$ depends on $\mathcal{M}$, A and T. Let us now see that there exist $a \in (0, 1)$, $M > 0$, so that $U_k \leq a^k U_0$, for all k. So, taking the right-hand side of (3.38) applied to $V_k := a^k U_0$, we find

$$\begin{aligned} &C \left[\frac{2^{3k}}{M^2 t_*} V_{k-1}^2 + \frac{2^{\frac{2(q+1)}{q}k}}{M^{\frac{2(q+1)}{q}}} V_{k-1}^{\frac{q+1}{q}} \right] \\ &\leq C \left[\frac{(2^3 a)^k}{M^2 t_* a^2} U_0 + \frac{(2^2 a)^{\frac{q+1}{q}k}}{M^{\frac{2(q+1)}{q}} a^{\frac{2(q+1)}{q}}} U_0^{\frac{1}{q}} \right] V_k. \end{aligned} \tag{3.39}$$

Then, taking a so that $2^3 a < 1$,

$$(3.39) \leq C \left[\frac{U_0}{M^2 t_* a^2} + \frac{U_0^{\frac{1}{q}}}{M^{\frac{2(q+1)}{q}} a^{\frac{2(q+1)}{q}}} \right] a^k U_0.$$

Choosing $M > 0$ so that

$$0 < \max \left\{ \frac{C U_0}{t_* a^2}, \left(\frac{C U_0^{\frac{1}{q}}}{a^{\frac{2(q+1)}{q}}} \right)^{\frac{q}{q+1}} \right\} \leq \frac{M^2}{2},$$

we get that V_k is a supersolution of the recurrence defined by (3.38), and so $U_k \le a^k U_0 \underset{k\to+\infty}{\longrightarrow} 0$. Observe also that for $t<1$

$$\max\left\{\left(\frac{CU_0}{t_* a^2}\right)^{1/2}, \left(\frac{CU_0^{\frac{1}{q}}}{a^{\frac{2(q+1)}{q}}}\right)^{\frac{q}{2(q+1)}}\right\} \le C\left(1+\frac{1}{\sqrt{t_*}}\right) =: M.$$

Thus, we have $0 \le u(t,x) \le M$, whence it follows that

$$0 \le u(t,x) \le C\left(1+\frac{1}{\sqrt{t}}\right). \tag{3.40}$$

We get the same estimate for w proceeding exactly in the same way for W_k, which leads to

$$0 \le w(t,x) \le C\left(1+\frac{1}{\sqrt{t}}\right). \tag{3.41}$$

From this point, the deduction of the uniform estimate of Proposition 3.5 using (3.40) and (3.41) is very similar to [2, Prop.3.2], so we refer the reader to that work for details. ☐

4 Construction of classical and weak solutions

The *a priori* estimates of the previous sections will now allow us to prove the global well-posedness of the system (1.2). The first step is to prove existence and uniqueness of classical solutions with smooth initial data. For this, we will use the Banach fixed-point theorem. A stability result for such solutions will be obtained and the existence of a weak solution follows as a consequence.

4.1 Construction of classical solutions

Let $w_0 \in C_c^\infty(\Omega)$ and define the set

$$\Upsilon = \{\xi \in C([0,T];L^2(\Omega)) \cap L^2(0,T;H^1(\Omega)),\ \ 0 \le \xi(t,x) \le 2\|w_0\|_\infty\},$$

equipped with the norm

$$\|u\|_\Upsilon = \sup_{0\le t\le T} \|u(t,\cdot)\|_{L^2}.$$

Note that Υ with the metric induced by $\|\cdot\|_\Upsilon$ is a complete metric space. In this section we prove the following theorem:

Theorem 4.1. *Let u_0 and $w_0 \in C_c^\infty(\Omega)$ be non-negative initial dada. Then, for all $T>0$ the system* (1.2) *supplemented with the boundary condition (1.3) admits a unique non-negative classical solution. This solution satisfies the estimates obtained in Propositions 3.2, 3.3 and 3.5.*

The main step to prove this theorem is the use of the following lemma, whose proof is standard and can be found in [2, Thm. 3.1].:

Lemma 4.2. *Let ψ be a smooth solution of*

$$\begin{array}{rcl} \partial_t \psi - \Delta\psi + \nabla\cdot(B\psi) + b\psi &=& 0 \\ \nabla\psi\cdot\mathbf{n} \;=\; B\cdot\mathbf{n} &=& 0, \\ \psi(0) &=& \psi_0 \geq 0 \end{array} \tag{4.1}$$

with $b, B, \nabla\cdot B \in L^\infty$. Then,

$$0 \leq \psi(t,x) \leq \|w_0\|_\infty \, e^{(\|b\|_\infty + \|\nabla\cdot B\|_\infty)t}. \tag{4.2}$$

Now let $\phi \in \Upsilon$ and let $p = p[\phi]$ be the solution of

$$\left\{ \begin{array}{rcl} p - \Delta p &=& \phi \\ \nabla p\cdot\mathbf{n} &=& 0. \end{array} \right.$$

Linear theory guarantees that there exists a unique solution $p \in H^1(\Omega)$, and, since $\phi(t) \in L^2(\Omega)\cap L^\infty(\Omega)$ and Ω is smooth, we have $p(t) \in H^2(\Omega)$ with $\|p(t)\|_{H^2} \leq C\|\phi(t)\|_{L^2}$ almost everywhere in time. Therefore we have $p(t), \nabla p(t)$ and $\Delta p(t) \in L^\infty(\Omega)$. Now we associate $u = u[\phi]$ the solution of

$$\left\{ \begin{array}{rcll} \partial_t u - \Delta u + \nabla\cdot(u\nabla p) + (1-\phi)u &=& 0 & \text{in } \Omega, \\ \nabla u\cdot\mathbf{n} &=& 0 & \text{in } \partial\Omega \\ u(0) &=& u_0. & \end{array} \right.$$

By linear theory, u is unique and such that $u \in L^2(0,T;H^1(\Omega))\cap C([0,T],L^2(\Omega))$. Linear theory still guarantees that there exists some constant $C > 0$ depending on T and Ω such that $\|u\|_{L^\infty(0,T;H^1(\Omega))} \leq C\|u_0\|_{H^1(\Omega)}$. Now, we associate $q = q[\phi]$ the solution of

$$\left\{ \begin{array}{rcl} q - \Delta q &=& u[\phi] \\ \nabla q\cdot\mathbf{n} &=& 0. \end{array} \right.$$

The same arguments lead to the existence of a unique solution $q \in H^1(\Omega)$, which satisfies $q(t) \in H^2(\Omega)$ with $\|q(t)\|_{H^2} \leq C\|u(t)\|_{L^2}$, then $q(t), \nabla q(t), \Delta q(t) \in L^\infty(\Omega)$. Finally, we associate $w = w[\phi]$ the solution of

$$\left\{ \begin{array}{rcll} \partial_t w - \Delta w - \nabla\cdot(w\nabla q) + \beta w(u+\phi-1) &=& 0 & \text{in } \Omega, \\ \nabla w\cdot\mathbf{n} &=& 0 & \text{in } \partial\Omega, \\ w(0) &=& w_0, & \end{array} \right.$$

where $w[\phi]$ is the only weak solution and such that $w \in L^2(0,T;H^1(\Omega))\cap C([0,T],L^2(\Omega))$.

Lemma 4.3. *For $T > 0$ small enough, $w[\phi] \in \Upsilon$.*

Proof. Applying Lemma 4.2 for $w[\phi]$,

$$0 \le w(t,x) \le \|w_0\|_\infty \, e^{(\|u+\phi-1\|_\infty + \|\Delta q\|_\infty)t}.$$

Note that all terms in the exponential can be bounded by $C\|\phi\|_\infty$, with $C > 0$ depending on the data of the problem. Thus, for $T > 0$ small enough we have $0 \le w(t,x) \le 2\|w_0\|_\infty$, which means $w[\phi] \in \Upsilon$. $\square$

Lemma 4.4. *$\Phi : \phi \in \Upsilon \mapsto w[\phi] \in \Upsilon$ is a contraction on $[0,T]$ for some $T > 0$.*

Proof. Let $\phi_1, \phi_2 \in \Upsilon$ and define $\overline{w} = w_1 - w_2$, where w_1 and w_2 are the respective associated solutions, and so forth. It's easy to check that

$$\partial_t \overline{w} - \Delta \overline{w} - \nabla \cdot (w_2 \nabla \overline{p}) - \nabla \cdot (\overline{w} \nabla p_2) = \overline{w} - w_1 \overline{\phi} - \overline{w}\phi_2 - w_1 \overline{u} - \overline{w} u_2.$$

Multiplying by $\overline{w}$ and integrating in space, we obtain

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_\Omega \overline{w}^2 \, dx + \int_\Omega |\nabla \overline{w}|^2 \, dx + \int_\Omega \nabla \overline{w} \cdot w_2 \nabla \overline{p} \, dx + \int_\Omega \nabla \overline{w} \cdot \overline{w} \nabla p_2 \, dx \\ \le C\left(\|\overline{w}(t)\|_2^2 + \|\overline{u}(t)\|_2^2 + \|\overline{\phi}(t)\|_2^2\right), \end{aligned} \tag{4.3}$$

where we used the estimates for w_i, u_i and ϕ_i guaranteed by Lemma 4.2 and by the definition of Υ, respectively. Substituting

$$\left| \int_\Omega w_2 \nabla \overline{w} \nabla \overline{p} \, dx \right| \le \frac{1}{2}\|\nabla \overline{w}\|_2^2 + C\|\overline{\phi}(t)\|_2^2$$

and

$$\left| \int_\Omega \overline{w} \nabla \overline{w} \nabla p_2 \, dx \right| \le \frac{1}{2}\|\nabla \overline{w}\|_2^2 + C\|\overline{w}(t)\|_2^2$$

in (4.3), we find

$$\frac{d}{dt}\int_\Omega \overline{w}^2 \, dx \le C\left(\|\overline{w}(t)\|_2^2 + \|\overline{u}(t)\|_2^2 + \|\overline{\phi}(t)\|_2^2\right).$$

By Gronwall's lemma, it follows that

$$\begin{aligned} \int_\Omega \overline{w}^2 \, dx &\le e^{Kt}\int_0^t \int_\Omega |\overline{u}(s)|^2 + |\overline{\phi}(s)|^2 \, dx \, ds \\ &\le e^{Kt}\int_0^t \int_\Omega |\overline{u}(s)|^2 \, dx \, ds + \left[\sup_{0\le s\le t} \int_\Omega |\overline{\phi}(s)|^2 \, dx\right] te^{Kt}. \end{aligned} \tag{4.4}$$

for some constant $K > 0$ which may change from line to line. Likewise, for $\overline{u} = u_1 - u_2$, we get

$$\int_\Omega |\overline{u}(t)|^2 \, dx \le Ce^{Kt}\int_0^t \int_\Omega |\overline{\phi}(s)|^2 \, dx \, ds. \tag{4.5}$$

Note that

$$e^{Kt}\int_0^t \int_\Omega |\overline{u}(s)|^2 \, dx \, ds \leq C e^{K't} \, \frac{t^2}{2} \, \left[\sup_{0\leq s\leq t} \int_\Omega |\overline{\phi}(s)|^2 \, dx \right]. \tag{4.6}$$

Combining (4.5) and (4.6) with (4.4), we conclude that

$$\int_\Omega |\overline{w}(t)|^2 \, dx \leq M e^{Kt} \, \left(\frac{t^2}{2} + t\right) \sup_{0\leq s\leq t} \int_\Omega |\overline{\phi}(s)|^2 \, dx,$$

for all $t \in [0,T]$. Thus, for $T > 0$ small enough, Φ is a contraction. $\square$

This is enough to prove Theorem 4.1. We take $\tilde{T} > 0$ being the smallest guaranteed by Lemma 4.3 and Lemma 4.4. By the fixed point theorem, the result follows for small time. Extension to $[0,T]$ is done in a standard way.

We say that the system of equations (1.2) is *stable* on $L^\infty(0,T;(L^1 \cap L^{2^+})(\Omega))$ when, given two pairs of initial data $u_{0,i}, w_{0,i} \in (L^1 \cap L^{2^+})(\Omega)$, the respective classical solutions u_i and w_i admit $C > 0$ depending only on $\mathcal{M}$ and on L^{2^+} −norms of the data such that

$$\begin{aligned} &\|u_1(t) - u_2(t)\|_2 + \|w_1(t) - w_2(t)\|_2 \\ &\quad \leq C \left(\|u_{1,0}(t) - u_{2,0}(t)\|_2 + \|w_{1,0}(t) - w_{2,0}(t)\|_2\right) \, e^{C(\mathcal{M})t}, \quad t \geq 0. \end{aligned} \tag{4.7}$$

Proposition 4.5. *The system* (1.2) *is stable (in the sense of* (4.7)*) on* $L^\infty(0,T;(L^1 \cap L^{2^+})(\Omega))$.

We omit the proof as it is an easy adaptation of [2, Prop.3.3].

4.2 Global well-posedness of weak solutions

Now we are ready to state and prove a well-posedness result for weak solutions:

Theorem 4.6. *Fix an arbitrary* $T > 0$ *and assume non-negative initial data* $u_0, w_0 \in (L^1 \cap L^{2^+})(\Omega)$. *Then, there exists a unique non-negative weak solution for the system* (1.2). *This solution satisfies the estimates of Proposition 3.2 and 3.3-3.5.*

Proof. Take a sequence of non-negative initial data $u_{0,k}, w_{0,k} \in C_c^\infty(\Omega)$ with $u_{0,k} \to u_0$ and $w_{0,k} \to w_0$ both strongly in $(L^1 \cap L^{2^+})(\Omega)$. Theorem 4.1 guarantees sequences u_k, w_k, p_k and q_k in $C([0,T], L^2(\Omega))$ solutions of the system (1.2) for each pair of data $u_{0,k}, w_{0,k} \in C_c^\infty(\Omega)$,

$$(1.2)_k \qquad \left\{ \begin{array}{rcl} \partial_t u_k - \Delta u_k + \nabla \cdot (u_k \nabla p_k) &=& u_k w_k - u_k \\ \partial_t w_k - \Delta w_k - \nabla \cdot (w_k \nabla q_k) &=& w_k(1 - w_k - u_k) \\ -\Delta p_k &=& w_k - p_k \\ -\Delta q_k &=& u_k - q_k \end{array} \right.$$

$k \in \mathbb{N}$. The following convergence properties hold:

i) $u_k \to u$ and $w_k \to w$ strongly in $L^\infty(0,T;L^2(\Omega))$.

ii) $\nabla p_k \to \nabla p$ in $L^\infty(0,T;L^{2^+}(\Omega))$ and $\nabla q_k \to \nabla q$ in $L^\infty(0,T;L^{2^+}(\Omega))$.

iii) $\nabla p_k \to \nabla p$ and $\nabla q_k \to \nabla q$ both strongly in $L^\infty(0,T;H^1(\Omega))$.

iv) $u_k \to u$ and $w_k \to w$ both strongly in $L^2(0,T;H^1(\Omega))$.

v) $\partial_t u_k \to \partial_t u$ and $\partial_t w_k \to \partial_t w$ weakly in $L^2(0,T;[H^1(\Omega)]^*)$.

vi) $u_k\nabla p_k \to u\nabla p$ and $w_k\nabla q_k \to w\nabla q$ in $L^1(0,T;[H^1(\Omega)]^*)$.

In fact, Proposition 3.3 guarantees that u_k and w_k are uniformly bounded in $L^\infty(0,T;L^{2^+}(\Omega))$ with respect to $k \geq 1$, and Proposition 4.5 guarantees that u_k and w_k are Cauchy sequences in $L^\infty(0,T;L^2(\Omega))$, getting **i)**. We can check **ii)** easily by elliptic regularity. Now, let $\overline{p} = p_k - p_l$ with $k,l \in \mathbb{N}$, and denote $\overline{w} = w_k - w_l$. Multiplying by $\overline{p}$ the difference of the third equations of (1.2) and integrating in space, we get

$$\int_\Omega \overline{p}^2\,dx + \int_\Omega |\nabla\overline{p}|^2\,dx \leq \int_\Omega \overline{w}^2\,dx$$

Similarly, with the fourth equation we compute

$$\int_\Omega \overline{q}^2\,dx + \int_\Omega |\nabla\overline{q}|^2\,dx \leq \int_\Omega \overline{u}^2\,dx.$$

Thus, using Proposition 4.5, we have

$$\|\overline{p}(t)\|_{H^1} + \|\overline{q}(t)\|_{H^1} \leq C\left(\|u_{0,k}(t) - u_{0,l}(t)\|_2 + \|w_{0,k}(t) - w_{0,l}(t)\|_2\right)\, e^{C(\mathcal{M})T}.$$

This means that p^k and q^k are Cauchy sequences in $L^\infty(0,T;H^1(\Omega))$, so we get **iii)**. Multiplying the first line of (1.2) by u_k and integrating in space, multiplying the second line of (1.2) by w_k and integrating in space, we get

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_\Omega u_k^2\,dx + \int_\Omega |\nabla u_k|^2\,dx &\leq \frac{1}{2}\int_\Omega |\nabla u_k|^2\,dx + C\int_\Omega u_k^2\,dx \\
\frac{1}{2}\frac{d}{dt}\int_\Omega w_k^2\,dx + \int_\Omega |\nabla w_k|^2\,dx &\leq C\int_\Omega w_k^2\,dx.
\end{aligned}$$

Thus, Gronwall lema implies that combining both inequalities above with **i)**, we obtain **iv)**. Since $H^1(\Omega) \hookrightarrow L^r(\Omega)$, $r \in [1,\infty)$ it's easy to check that $\partial_t u_k(t) = \nabla\cdot(\nabla u_k - u_k\nabla p_k) + u_k w_k - u_k \in H^{-1}$ and $\partial_t w_k(t) = \nabla\cdot(\nabla w_k + w_k\nabla q_k) + w_k - w_k^2 - u_k w_k \in H^{-1}$, getting that $\partial_t u_k$ and $\partial_t w_k$ are bounded in $L^2(0,T;[H^1(\Omega)]^*)$, so **v)**. Thus, (u,w,p,q) is a weak solution of (1.2). The condition $(u(0),w(0)) = (u_0,w_0)$ is satisfied by continuity at $t = 0$ which follows from the estimate on the time derivatives. Finally, using the approximating by classical solutions we can check that the stability result from Proposition 4.5 holds for weak solutions. This can be used to prove uniqueness. □

5 Numerical experiments

In order to show some of the relevant features of the system, we provide in this section the details of a numerical simulation of (1.2). Our goal is to present an implicit-explicit finite volume scheme and showcase some numerical results exhibiting the system's main features, namely, evasive behavior of the prey, and chasing by the predator.

We consider the system (1.2) in a rectangular domain $\Omega = [0, L_x] \times [0, L_y]$, where we introduce a cartesian mesh consisting of the cells $I_{i,j} := [x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}] \times [y_{j-\frac{1}{2}}, y_{j+\frac{1}{2}}]$, which for the sake of simplicity, are assumed square with uniform size, so $|I_{i,j}| := h^2$ for all i and j. Consider a step size $\Delta t > 0$ to discretize the time interval $(0, T)$. Let $N > 0$ the smallest integer such that $N\Delta t \leq T$ and set $t^n := n\Delta t$ for $n \in \{0, N\}$. The cell average of a quantity v at time t is defined by

$$\overline{v}_{i,j}(t) := \frac{1}{h^2} \int_{I_{i,j}} v(t, \mathbf{x}) d\mathbf{x},$$

and define $\overline{v}^n_{i,j} := \overline{v}_{i,j}(t^n)$. Note that in this section we use $\mathbf{x} = (x, y)$ to denote the spatial variable. Let $f_k(u, w)$, $k = 1, 2$, be the reactive terms in the right-hand side of the first two equations in (1.2). Then, the terms

$$\frac{1}{h^2} \int_{I_{i,j}} f_k(u(t, \mathbf{x}), w(t, \mathbf{x})) dx, \quad k = 1, 2$$

are approximated by $f_{k,i,j} := f_k(\overline{u}_{i,j}, \overline{w}_{i,j}), k = 1, 2$. The Laplacian on a Cartesian grid is discretized via

$$\begin{aligned}
\Delta_{i,j} u &:= \frac{1}{h}(F_{i+\frac{1}{2},j} - F_{i-\frac{1}{2},j}) + \frac{1}{h}(F_{i,j+\frac{1}{2}} - F_{i,j-\frac{1}{2}}),\\
F_{i+\frac{1}{2},j} &:= \frac{1}{h}(\overline{u}_{i+1,j} - \overline{u}_{i,j}), \quad F_{i,j+\frac{1}{2}} := \frac{1}{h}(\overline{u}_{i,j+1} - \overline{u}_{i,j}).
\end{aligned}$$

The numerical fluxes in the x- and y- directions are respectively

$$\mathcal{F}^1_{i+\frac{1}{2},j}(\mathbf{p}) = \begin{cases} \overline{u}_{i,j}\overline{p}_{i+\frac{1}{2},j} & \text{if } \overline{p}_{i+\frac{1}{2},j} > 0 \\ \overline{u}_{i+1,j}\overline{p}_{i+\frac{1}{2},j} & \text{if } \overline{p}_{i+\frac{1}{2},j} < 0, \end{cases} \qquad \overline{p}_{i+\frac{1}{2},j} = \frac{\overline{p}_{i+1,j} - \overline{p}_{i,j}}{h}, \tag{5.1}$$

and

$$\mathcal{F}^1_{i,j+\frac{1}{2}}(\mathbf{p}) = \begin{cases} \overline{u}_{i,j}\overline{p}_{i,j+\frac{1}{2}} & \text{if } \overline{p}_{i,j+\frac{1}{2}} > 0 \\ \overline{u}_{i,j+1}\overline{p}_{i,j+\frac{1}{2}} & \text{if } \overline{p}_{i,j+\frac{1}{2}} < 0, \end{cases} \qquad \overline{p}_{i,j+\frac{1}{2}} = \frac{\overline{p}_{i,j+1} - \overline{p}_{i,j}}{h}, \tag{5.2}$$

and in a similar way for $\mathcal{F}^2_{i+\frac{1}{2},j}(\mathbf{q})$ and $\mathcal{F}^2_{i,j+\frac{1}{2}}(\mathbf{q})$. Finally we incorporate a first-order Euler time integration for the u and w components. The diffusive terms are treated in an implicit form and an explicit form is used for the convective and reactive terms. The initial data are approximated by their cell averages,

$$\overline{u}^0_{i,j} := \frac{1}{h^2} \int_{I_{i,j}} u_0(\mathbf{x}) d\mathbf{x}, \quad \overline{w}^0_{i,j} := \frac{1}{h^2} \int_{I_{i,j}} w_0(\mathbf{x}) d\mathbf{x}.$$

To advance the numerical solution from t^n to $t^{n+1} = t^n + \Delta t$, we use the following finite volume scheme: given $\mathbf{u}^n = (\overline{u}^n_{i,j})$ and $\mathbf{w}^n = (\overline{w}^n_{i,j})$ for all cells $I_{i,j}$ at time $t = t^n$, the unknown values $\mathbf{u}^{n+1}$ and $\mathbf{w}^{n+1}$ are determined by the following two steps implicit-explicit scheme:

Step 1 solve for $\mathbf{p} = (\overline{p}_{i,j})$ and $\mathbf{q} = (\overline{q}_{i,j})$

$$-D_p \Delta_h \mathbf{p} + \delta_p \mathcal{I} \mathbf{p} = \delta_w \mathcal{I} \mathbf{w}^n \tag{5.3a}$$

$$-D_q \Delta_h \mathbf{q} + \delta_q \mathcal{I} \mathbf{q} = \delta_u \mathcal{I} \mathbf{u}^n. \tag{5.3b}$$

Step 2 solve for $\mathbf{u}^{n+1} = (\overline{u}^{n+1}_{i,j})$ and $\mathbf{w}^{n+1} = (\overline{w}^{n+1}_{i,j})$

$$\begin{aligned}\overline{u}^{n+1}_{i,j} - \Delta t \Delta_{i,j} u^{n+1} &= \overline{u}^n_{i,j} + \Delta t f^n_{1,i,j} \\ &+ \Delta t \left(\frac{\mathcal{F}^1_{i+\frac{1}{2},j}(\mathbf{p}) - \mathcal{F}^1_{i-\frac{1}{2},j}(\mathbf{p})}{h} + \frac{\mathcal{F}^1_{i,j+\frac{1}{2}}(\mathbf{p}) - \mathcal{F}^1_{i,j-\frac{1}{2}}(\mathbf{p})}{h} \right)\end{aligned} \tag{5.4a}$$

$$\begin{aligned}\overline{w}^{n+1}_{i,j} - \Delta t D_w \Delta_{i,j} w^{n+1} &= \overline{w}^n_{i,j} + \Delta t f^n_{2,i,j} \\ &+ \Delta t \left(\frac{\mathcal{F}^2_{i+\frac{1}{2},j}(\mathbf{q}) - \mathcal{F}^2_{i-\frac{1}{2},j}(\mathbf{q})}{h} + \frac{\mathcal{F}^2_{i,j+\frac{1}{2}}(\mathbf{q}) - \mathcal{F}^2_{i,j-\frac{1}{2}}(\mathbf{q})}{h} \right)\end{aligned} \tag{5.4b}$$

where we have used the notation $\Delta_h = (\Delta_{i,j})$ to indicate the matrix of the discrete Laplacian operator, and $\mathcal{I}$ is the identity matrix.

Theorem 5.1. *Suppose that $f_1(u, w) = u(\alpha w - 1)$ and $f_2(u, w) = \beta w(1 - u - w)$ Then the solutions $\overline{p}_{i,j}$, $\overline{q}_{i,j}$ and $\overline{u}^{n+1}_{i,j}$, $\overline{w}^{n+1}_{i,j}$ of the finite volume scheme* (5.3a)-(5.3b) *and* (5.4a)-(5.4b) *respectively, are nonnegatives for all i, j provided $\overline{u}^n_{i,j}$, $\overline{w}^n_{i,j}$ are nonnegative for all i, j, and the following CFL-like condition is satisfied:*

$$\frac{\Delta t}{h} \leq \min \left\{ \frac{1}{2a}, \frac{1}{2b}, \frac{1}{K} \right\}, \tag{5.5}$$

where

$$a = \max_{i,j} \{ |\overline{p}_{i+\frac{1}{2},j}|, |\overline{q}_{i+\frac{1}{2},j}| \}, \quad b = \max_{i,j} \{ |\overline{p}_{i,j+\frac{1}{2}}|, |\overline{q}_{i,j+\frac{1}{2}}| \},$$

$$K = \|f_{1,u}\|_\infty + \|f_{1,w}\|_\infty + \|f_{2,u}\|_\infty + \|f_{2,w}\|_\infty.$$

Proof. In (5.3a)-(5.3b), we have a linear system of algebraic equations for $\overline{p}_{i,j}$ and $\overline{q}_{i,j}$ which need to be solve in each time step t^n. However, observe that the matrix of these linear systems are diagonally dominant, which guarantee the existence of solution and the positivity of $p_{i,j}$ and $q_{i,j}$. Each system of equations (5.4a)-(5.4b) can be seen as a linear system for $\overline{u}^{n+1}_{i,j}$ and $\overline{w}^{n+1}_{i,j}$ respectively, where the right side is positive according with the CFL condition (5.5) (see [10, 16, 6]) which guarantees the positivity of $\overline{u}^{n+1}_{i,j}$ and $\overline{w}^{n+1}_{i,j}$. □

Each system of linear algebraic equations for $\overline{p}_{i,j}$, $\overline{q}_{i,j}$ and $\overline{u}^{n+1}_{i,j}$, $\overline{w}^{n+1}_{i,j}$ can be solved by using an accurate and efficient linear algebraic solver.

5.1 Test 1: chasing and evasion

In this numerical test, shown in Figure 5.1, we suppress the terms on the right-hand side of (1.2), to ignore the population dynamics and emphasize the effect of the pursuit and evasion. We can see that the predator starts to chase the prey even though at first any direct contact with it would be very small (due to diffusion only). The prey takes evasive action immediately. Note that by choosing large δ_u, δ_w in this example, we see from Tables 1, 2 that this may be interpreted as saying that the chemical sensitivity of the predator and prey are large compared to their diffusion rates. Therefore, we expect that the movement observed in Figure 5.1 is due to the attraction and repulsion terms and not so much to the diffusion. The numerical parameters are a 400 by 400 spatial cell grid, and a time step of 0.01.

5.2 Test 2: full dynamics

In this test, shown in Figure 5.2, we set some generic parameters in the system (1.2) in order to observe the full behavior. We can see now the predator-prey interaction taking place, as the densities of the two species fluctuate more widely in relation to the previous example, due to the predator's population growth from predation. After some time, the solution seems to exhibit wave-like interaction patterns with decreasing amplitudes, stabilizing around the values predicted by the equilibrium point $(\frac{\alpha-1}{\alpha}, \frac{1}{\alpha}) = (0.9, 0.1)$. Although here we show the solution only until $t = 10$, computation up to larger times, not shown here, confirm this behavior. The numerical parameters are a 400 by 400 spatial cell grid, and a time step of 0.01.

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Appendix

The dimensional version of system (1.2) reads as follows:

$$
\begin{cases}
\partial_t u - \alpha_u \Delta u + \nabla\cdot \big(u\,\beta_u \nabla p\big) = \overline{\alpha}\,w\,u - \overline{\beta}\,u \\
\partial_t w - \alpha_w \Delta w - \nabla\cdot \big(w\,\beta_w \nabla q\big) = \gamma\,w(1 - w/K_w) - \delta\,w\,u \\
-\,\alpha_p \Delta p = \overline{\delta}_w w - \overline{\delta}_p p \\
-\,\alpha_q \Delta q = \overline{\delta}_u u - \overline{\delta}_q q,
\end{cases}
\tag{5.6}
$$

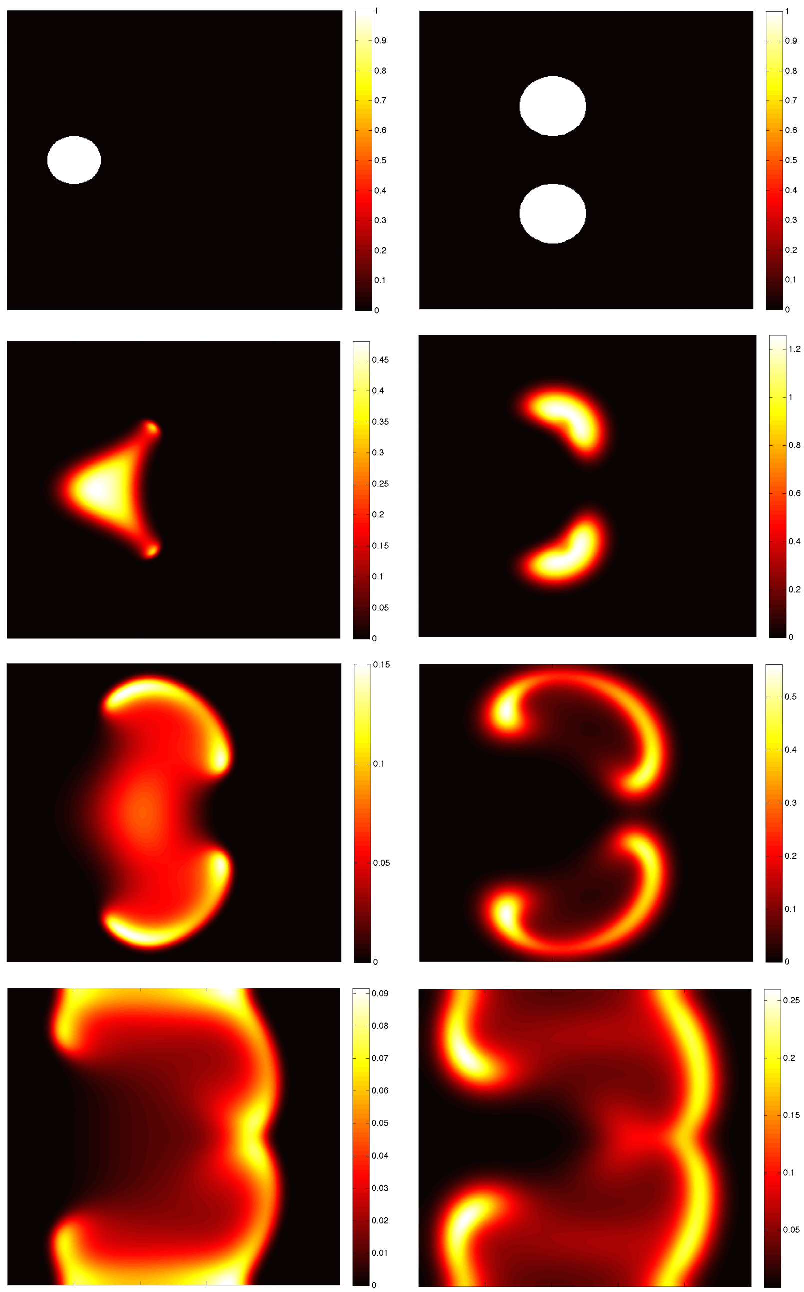

Figure 1: Numerical solution of system (1.2) with only pursuit and evasion. Left column is the predator and right column is the prey. Shown times are, from top to bottom, $t = 0, 1, 4, 8$. Parameters in this simulation: $f_1(u, w) = f_2(u, w) = 0$, $\delta_w = \delta_u = 100$, $\delta_p = \delta_q = 0.1$, $D_w = 1$, $D_p, D_q = 10$.

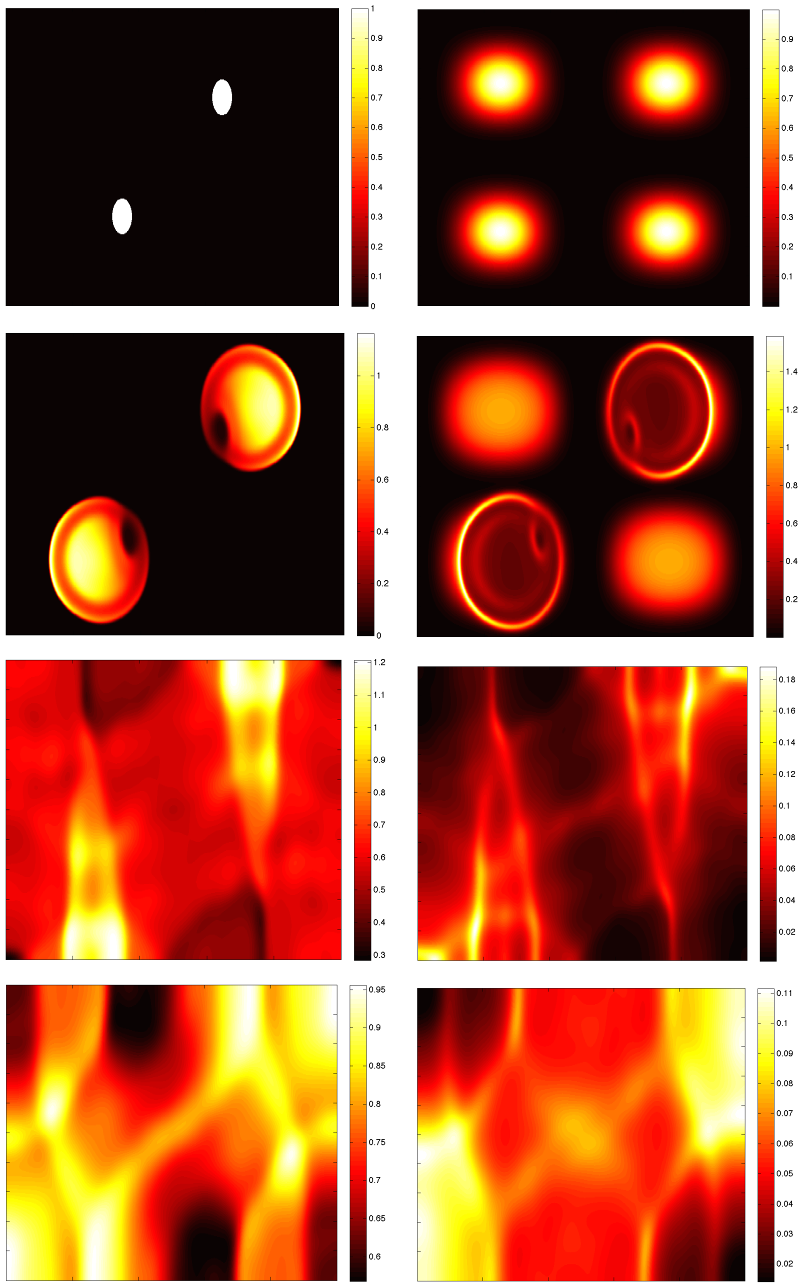

Figure 2: Numerical solution of system (1.2) with predator-prey dynamics, pursuit, and evasion. Left column is the predator and right column is the prey. Shown times are, from top to bottom, $t = 0, 1, 5, 10$. Parameters in this simulation: $\alpha = 10, \beta = 2, \delta_w = 100, \delta_u = 50, \delta_p = \delta_q = 1,\ D_w = 1,\ D_p = 3, D_q = 2$.

supplemented with appropriate no-flux boundary conditions similar to (1.3).

In Table 1 we present the physical meaning of the parameters appearing in (5.6), and in Table 2 we show the dimensionless parameters appearing in system (1.2), obtained after a standard non-dimensionalization procedure applied to the system (5.6).

Table 1: Physical parameters in system (5.6). Here, ℓ denotes length, t denotes time, *bio* denotes some measure of predator or prey quantity or mass, and *odor* denotes some measure of "odor".

Parameter	Units	Physical meaning
α_u, α_w	ℓ^2/t	Diffusion rate of predators and prey
α_p, α_q	ℓ^2/t	Diffusion rate of prey and predator odor
$\overline{\alpha}$	$(bio\, t)^{-1}$	Predator growth rate from predation
$\overline{\beta}$	t^{-1}	Predator death rate
γ	t^{-1}	Prey growth rate
K_w	$\dfrac{bio}{\ell^2}$	Prey carrying capacity
δ	$(bio\, t)^{-1}$	Prey death rate from predation
β_u, β_w	$\dfrac{\ell^4}{t \cdot odor}$	Predator (resp. prey) odor sensitivity
$\overline{\delta}_w, \overline{\delta}_u$	$\dfrac{odor}{t \cdot bio}$	Prey (resp. predator) odor production rate
$\overline{\delta}_p, \overline{\delta}_q$	t^{-1}	Prey (resp. predator) odor degradation rate

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Table 2: Dimensionless parameters in system (1.2). Here, ℓ denotes length, t denotes time, *bio* denotes some measure of predator or prey quantity or mass, and *odor* denotes some measure of "odor".

Dimensionless parameter	Physical meaning
$D_w = \alpha_w/\alpha_u$	Prey diffusion rate relative to predator diffusion rate
$\alpha = \overline{\alpha}K_w/\overline{\beta}$	Predator efficiency relative to death rate
$\beta = \gamma/\overline{\beta}$	Prey growth rate relative to predator death rate
$D_p = \alpha_p/\alpha_u$	Prey odor diffusion rate relative to predator diffusion rate
$D_q = \alpha_q/\alpha_u$	Predator odor diffusion rate relative to predator diffusion rate
$\delta_w = \beta_u K_w \overline{\delta}_w/(\alpha_u\overline{\beta})$	Normalized prey odor production rate
$\delta_u = \beta_w \gamma \overline{\delta}_u/(\alpha_u\overline{\beta\delta})$	Normalized prey odor production rate
$\delta_p = \overline{\delta}_p/\overline{\beta}$	Prey odor degradation rate relative to predator death rate
$\delta_q = \overline{\delta}_q/\overline{\beta}$	Predator odor degradation rate relative to predator death rate

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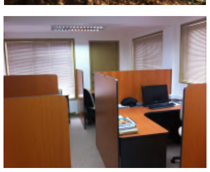

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