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A note on the existence and stability of an inverse problem for a SIS model

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Abstract

In this work, we discuss the existence and stability of an inverse problem arising from the determination of the reaction coefficients for a SIS model. The study is motivated by a remark regarding the final discussion of the recent paper *[H. Xiang and B. Liu, Computers and Mathematical with Applications, 70:805–819, 2015]*. The weak point of the work of H. Xiang and B. Liu is that the proofs of existence and stability results are valid only for the one-dimensional case. Here, we introduce an appropriate framework which is also valid in the multidimensional case and that generalizes the previous results.

Keywords: inverse problem, SIS epidemic reaction-diffusion model, optimal control methods
2010 MSC: 92D30, 49K20, 49N45, 49N90

1. Introduction

In this paper, we are interested in an inverse problem that originates in mathematical epidemiology theory. We consider that the dynamic processes of transmitted disease is governed by a SIS reaction-diffusion system for the susceptible and infected individuals population densities. We assume that the diffusion matrix is the identity and the reaction term consists of two terms modelling the infection process and the recovery process, which are governed by frequency-dependent transmission function and a proportional law, respectively. The inverse problem is motivated by the practical situation where state variables can be measured with relative ease, on the other hand, however, the rates of disease transmission and disease recovery, which are the coefficients of the model, are very costly or even infeasible. Thus, in the inverse problem, we want to perform an estimation of the reaction coefficients: disease transmission and disease recovery rates.

Let us precise the definition of the inverse problem. Indeed, we assume that the direct problem is given by the following reaction-diffusion system

$$\frac{\partial S}{\partial t} - \Delta S = -\beta(\mathbf{x})\frac{SI}{S+I} + \gamma(\mathbf{x})I, \qquad (\mathbf{x},t) \in Q_T := \Omega \times [0,T], \tag{1}$$

$$\frac{\partial I}{\partial t} - \Delta I = \beta(\mathbf{x})\frac{SI}{S+I} - \gamma(\mathbf{x})I, \qquad (\mathbf{x},t) \in Q_T, \tag{2}$$

$$\nabla S \cdot \mathbf{n} = \nabla I \cdot \mathbf{n} = 0, \qquad (\mathbf{x},t) \in \Gamma := \partial\Omega \times [0,T], \tag{3}$$

$$S(\mathbf{x},0) = S_0(\mathbf{x}), \quad I(\mathbf{x},0) = I_0(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{4}$$

where the open bounded set $\Omega \subset \mathbb{R}^d$ represents the physical domain where live the population; $S(\mathbf{x},t)$ and $I(\mathbf{x},t)$ are the population densities for susceptible and infected individuals at location $\mathbf{x}$ and time

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t, respectively; S_0 and I_0 are the initial densities of susceptible an infected individuals, respectively; the functions β and γ are the rates of disease transmission and disease recovery, respectively; and $\mathbf{n}$ is the unit external normal to $\partial\Omega$. If we consider that there are experimental observations on Ω at $t = T$ of the densities for susceptible and infected individuals given by S^{obs} and I^{obs}, respectively. Then, the inverse problem of estimating the unknown coefficients β and γ is defined as follows:

> **Inverse Problem.** *Given $T > 0$ and the set of functions $\{S_0, I_0, S^{obs}, I^{obs}\}$ defined on Ω, find the functions β and γ such that $(S, I)(\mathbf{x}, T) = (S^{obs}, I^{obs})(\mathbf{x})$ for $\mathbf{x} \in \Omega$ with (S, I) the solution of (1)-(4).*

This inverse problem falls into the class of model calibration or parameter identification problems.

The field of inverse problems for partial differential equations is an area with rapidly progress in the last years [1, 12, 13]. In particular, in the case of inverse problem arising in the coefficients identification in diffusion equations, including reaction terms or convection terms, have been addressed in the literature of the last decades, see for instance [2, 4, 5, 6, 7, 8, 10, 17, 18, 19, 20]. The list is not exhaustive and there is more contributions focused in different topics: applications, numerical analysis and theoretical analysis. It is well known that most of the existing inverse problems are not well posed in the sense of Hadamard and the study of uniqueness and stability of solutions are relevant topics, since imply that we have enough data to determine an object and convergence of numerical o regularized to the solutions sought, respectively.

In this paper, we investigate the existence and stability of solutions of the inverse problem. Indeed, to analyze the inverse problem, we consider a reformulation of the inverse problem like an optimal control problem of the following type:

> **Optimal control problem.** *The inverse problem may be recasting as the optimization problem*
>
> $$\inf\ J(\beta, \gamma) \quad \textit{subject to } (\beta, \gamma) \in U_{ad}(\Omega) \textit{ and } (S, I) \textit{ a solution of } (1)\text{-}(4). \tag{5}$$
>
> *where the cost functional J and the admissible set $U_{ad}(\Omega)$ are appropriately defined.*

Then, in order to get the existence of at least one solution of the optimization problem (5), we precise the definition of $U_{ad}(\Omega)$ and assume that Ω is an open bounded and convex set of $\mathbb{R}^d$. The hypothesis on Ω is considered in order to get the required compactness embedding which permits the pass to the limit in a minimizing sequence. Now, in the case of the stability result we study the continuous dependence of the inverse problem solution with respect to the observations.

The rest of this note is organized in two sections: in Section 2 we present the general notation and the main result and in Section 3 we give the proof of the main result.

2. Main result

We consider the standard notation of functions spaces used in the analysis of parabolic equations, see for instance [14, 15, 16]. In particular, we use the notations $C^{k,\alpha}(\overline{\Omega})$ with $k \in \mathbb{N}$ and $\alpha \in]0, 1]$, $L^p(\Omega)$ with $p \geq 1$, $W^{m,p}(\Omega)$ with $m \in \mathbb{N}$ and $p \geq 1$, for the Banach spaces of Hölder $k-$times continuously and whose k^{th}-partial derivatives are Hölder continuous with exponent α; the space of all functions from Ω to $\mathbb{R}$ which are p-integrable in the sense of Lebesgue; and the usual Sobolev spaces, respectively. In particular, we consider the notations $C^{\alpha}(\overline{\Omega})$ and $H^m(\overline{\Omega})$ instead of $C^{0,\alpha}(\overline{\Omega})$ and $W^{m,2}(\Omega)$, respectively.

We consider the admissible set $U_{ad}(\Omega)$ and the functional $J : U_{ad}(\Omega) \to \mathbb{R}$ defined as follows

$$U_{ad}(\Omega) = \mathscr{A}(\Omega) \cap \Big[H^{[\![d/2]\!]+1}(\Omega) \times H^{[\![d/2]\!]+1}(\Omega) \Big], \tag{6}$$

$$J(\beta, \gamma) := \frac{1}{2} \Big[\|S(\cdot, T) - S^{obs}\|^2_{L^2(\Omega)} + \|I(\cdot, T) - I^{obs}\|^2_{L^2(\Omega)} \Big] + \frac{\Gamma}{2} \Big[\|\nabla\beta\|^2_{L^2(\Omega)} + \|\nabla\gamma\|^2_{L^2(\Omega)} \Big], \tag{7}$$

with $[\![\cdot]\!]$ the integer part function, $\Gamma \in \mathbb{R}^+$ an appropriate regularization parameter and

$$\mathscr{A}(\Omega) = \Big\{ (\beta, \gamma) \in C^{\alpha}(\overline{\Omega}) \times C^{\alpha}(\overline{\Omega}) \ : \quad \mathrm{Ran}(\beta) \times \mathrm{Ran}(\gamma) \subseteq [\underline{b}, \overline{b}] \times [\underline{r}, \overline{r}] \subset]0, 1[^2, \quad \nabla\beta,\ \nabla\gamma \in L^2(\Omega) \Big\}, \tag{8}$$

where $\mathrm{Ran}(f)$ denotes the range of a function f. We note that $U_{ad}(\Omega) = \mathscr{A}(\Omega)$ when $d = 1$ and coincides with the admissible set considered by Xiang and Liu in [20].

We consider the following set of assumptions:

(H0) The open bounded and convex set Ω is such that $\partial\Omega$ is C^1.

(H1) The initial conditions S_0 and I_0 are belong to $C^{2,\alpha}(\overline{\Omega})$ and satisfy the inequalities

$$S_0(\mathbf{x}) \geq 0, \quad I_0(\mathbf{x}) \geq 0, \quad \int_\Omega I_0(\mathbf{x})d\mathbf{x} > 0, \quad S_0(\mathbf{x}) + I_0(\mathbf{x}) \geq \phi_0 > 0,$$

on Ω, for some positive constant ϕ_0;

(H2) The observation functions S^{obs} and I^{obs} are belong to $L^2(\Omega)$.

In a broad sense, the role of hypotheses are the following: (H0) is necessary to get the appropriate compactness used to prove the existence of solutions for the inverse problem, (H1) is necessary to get the well-posedness and strictly positive behavior of the solution for the direct problem, and (H2) is necessary to get that the stability result.

The existence and the uniqueness can be developed by the Shauder's theory for parabolic equations [14, 15, 16]. Meanwhile, the positive behavior of the solution is a consequence of the maximum principle. More precisely we have the following result.

Theorem 2.1. *Consider that the following hypotheses (H0)-(H1) are satisfied. If $(\beta,\gamma) \in C^\alpha(\overline{\Omega}) \times C^\alpha(\overline{\Omega})$, the initial boundary value problem* (1)-(4) *admits a unique positive classical solution (S, I), such that S and I are belong to $C^{2+\alpha,1+\alpha/2}(\overline{Q}_T)$ and also S and I are bounded on $\overline{Q}_T$, for any given $T \in \mathbb{R}^+$.*

On the other hand, we consider that the functions P and Q are solutions of the following backward boundary value problem

$$\frac{\partial P}{\partial t} + \Delta P = \overline{\beta}(\mathbf{x})\frac{\overline{I}^2}{(\overline{S}+\overline{I})^2}(P-Q), \qquad (\mathbf{x},t) \in Q_T := \Omega \times [0,T], \tag{9}$$

$$\frac{\partial Q}{\partial t} + \Delta Q = \left(\overline{\beta}(\mathbf{x})\frac{\overline{S}^2}{(\overline{S}+\overline{I})^2} - \overline{\gamma}(\mathbf{x})\right)(P-Q), \qquad (\mathbf{x},t) \in Q_T, \tag{10}$$

$$\nabla P \cdot \mathbf{n} = \nabla Q \cdot \mathbf{n} = 0, \qquad (\mathbf{x},t) \in \Gamma := \partial\Omega \times [0,T], \tag{11}$$

$$P(\mathbf{x},T) = \overline{S}(\mathbf{x},T) - S^{obs}(\mathbf{x}), \quad Q(\mathbf{x},T) = \overline{I}(\mathbf{x},T) - I^{obs}(\mathbf{x}), \qquad \mathbf{x} \in \Omega, \tag{12}$$

where $(\overline{\beta},\overline{\gamma}) \in U_{ad}$ and $(\overline{S},\overline{I})$ is the corresponding solution of (1)-(4) with $(\overline{\beta},\overline{\gamma})$ instead of (β,γ).

Theorem 2.2. *Assume that the hypotheses (H0)-(H3) are satisfied. Then, the assertions*

(i) There exists at least one solution of (5).

(ii) Let us consider $(\overline{\beta},\overline{\gamma})$ is the solution of (5) *and $(\overline{S},\overline{I})$ the corresponding solutions of* (1)-(4) *with $(\overline{\beta},\overline{\gamma})$ instead of (β,γ). Then, the adjoint system to* (1)-(4) *is given by the system* (9)-(12)*. Moreover, the solution* (9)-(12) *is bounded in $L^\infty(0,t;H^2(\Omega))$ for almost all time t in $]0,T]$. In particular the solution* (9)-(12) *is bounded in $L^\infty(0,t;L^\infty(\Omega))$ for almost all time t in $]0,T]$.*

(iii) Let us consider $\overline{S},\overline{I},\overline{\beta},\overline{\gamma},P$ and Q as is given in (ii). Then, the following inequality

$$\int_{Q_T}\left[(\hat{\beta}-\overline{\beta})\frac{\overline{S}\,\overline{I}}{\overline{S}+\overline{I}} - (\hat{\gamma}-\overline{\gamma})\overline{I}\right](P-Q)d\mathbf{x}dt + \Gamma\Big[\int_\Omega \nabla\overline{\beta}\cdot\nabla(\hat{\beta}-\overline{\beta})d\mathbf{x} + \int_\Omega \nabla\overline{\gamma}\cdot\nabla(\hat{\gamma}-\overline{\gamma})d\mathbf{x}\Big] \geq 0, \quad \forall(\hat{\beta},\hat{\gamma}) \in U_{ad}(\Omega), \tag{13}$$

is satisfied.

(iv) The mapping $(\beta,\gamma)\mapsto(S,I)$ is continuous from $U_{ad}(\Omega)\subset[L^2(\Omega)]^2$ to $L^\infty(0,t;L^2(\Omega))$ for almost all time t in $]0,T]$.

(v) The mapping $(\beta,\gamma,S^{obs},S^{obs})\mapsto(P,Q)$ is continuous from $U_{ad}(\Omega)\times L^2(\Omega)\times L^2(\Omega)\subset[L^2(\Omega)]^4$ to $L^\infty(0,t;L^2(\Omega))$ for almost all time t in $]0,T]$.

(vi) Given $\mathbf{c}=(c_1,c_2)\in\mathbb{R}^2_+$ (fix) define the set $\mathscr{U}_{\mathbf{c}}(\Omega)=\Big\{(\beta,\gamma)\in U_{ad}(\Omega)\,:\,\int_\Omega(\beta,\gamma)d\mathbf{x}=(c_1,c_2)\Big\}$. Then, there exist $\overline{\Gamma}\in\mathbb{R}^+$ such that the solution of (5) is uniquely defined, up an additive constant, on $\mathscr{U}_{\mathbf{c}}(\Omega)$ in the $L^2(\Omega)$ sense for any regularization parameter $\Gamma>\overline{\Gamma}$.

are satisfied.

Our results on generalize for $d\geq 1$ the recent results obtained by Xiang and Liu [20] in the case $d=1$.

3. Proof of main result: Theorem 2.2

3.1. Proof of (i)

We note that $U_{ad}(\Omega)\neq\emptyset$ and $J(\beta,\gamma)$ is bounded for any $(\beta,\gamma)\in U_{ad}(\Omega)$. The fact that $U_{ad}(\Omega)\neq\emptyset$ follows for instance by considering the pair of functions $(\beta,\gamma)(\mathbf{x})=(\underline{b}+\overline{b},\underline{r}+\overline{r})/2$, which is belong to $U_{ad}(\Omega)$. The boundedness of J is deduced by the following three facts: the bounded behavior of S and T on $\overline{Q}_T$ as consequence of Theorem 2.1, the hypothesis (H2) and the fact that $\nabla\beta,\ \nabla\gamma\in L^2(\Omega)$ by the definition of $U_{ad}(\Omega)$. Then, we can consider that $\{(\beta_n,\gamma_n)\}\subset\mathscr{U}$ is a minimizing sequence of J and also we can introduce the notation $C^\top\in\mathbb{R}^+$ such that $J(\beta_n,\gamma_n)<C^\top$.

On the other hand, we claim the compact embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^\alpha(\Omega)$ for $\alpha\in]0,1/2]$. Indeed, it can be deduced using two results. First, by Theorem 6[9, pp. 270], we have the Sobolev embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^\theta(\Omega)$ with $\theta=1/2$ for d odd and $\theta\in]0,1[$ for d even. Then, for all d we have the continuous embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^{1/2}(\Omega)$. Second, by Theorem 1.3.1[3, pp. 11], we have the compact embedding $C^{1/2}(\Omega)\subset C^\alpha(\Omega)$ for all $\alpha\in]0,1/2]$. Hence our claim follows from the chain of embeddings $H^{[\![d/2]\!]+1}(\Omega)\subset C^{1/2}(\Omega)\subset C^\alpha(\Omega)$ for all $\alpha\in]0,1/2]$.

The compact embedding $H^{[\![d/2]\!]+1}(\Omega)\subset C^\alpha(\Omega)$ for $\alpha\in]0,1/2]$, implies that the minimizing sequence $\{(\beta_n,\gamma_n)\}$ is bounded in the strong topology of $C^\alpha(\overline{\Omega})\times C^\alpha(\overline{\Omega})$ for all $\alpha\in]0,1/2]$, since there exists a positive constant C (independent of β,γ and n) such that

$$\|\beta_n\|_{C^\alpha(\overline{\Omega})}+\|\gamma_n\|_{C^\alpha(\overline{\Omega})}\leq C\Big(\|\beta_n\|_{H^{[\![d/2]\!]+1}(\Omega)}+\|\gamma_n\|_{H^{[\![d/2]\!]+1}(\Omega)}\Big),\quad\forall\alpha\in]0,1/2].$$

Now, we note that the right hand is bounded by the fact that $\beta_n,\gamma_n\in H^{[\![d/2]\!]+1}(\Omega)$, see the definition of $U_{ad}(\Omega)$ given on (6).

Let us denote by (S_n,I_n) the solution of the initial boundary value problem (1)-(4) corresponding to (β_n,γ_n). Then, by considering the fact that $\{(\beta_n,\gamma_n)\}$ is belong to $C^\alpha(\overline{\Omega})\times C^\alpha(\overline{\Omega})$ for all $\alpha\in]0,1/2]$, by Theorem 2.1, we have that S_n and I_n are belong to the Hölder space $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ and also $\{(S_n,I_n)\}$ is a bounded sequence in the strong topology of $C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)\times C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T)$ for all $\alpha\in]0,1/2]$.

The boundedness of the minimizing sequence and the corresponding sequence $\{(S_n,I_n)\}$, implies that there exist

$$(\overline{\beta},\overline{\gamma})\in\Big[C^{1/2}(\Omega)\times C^{1/2}(\Omega)\Big]\cap U_{ad}(\Omega),\qquad(\overline{S},\overline{T})\in C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T)\times C^{2+\frac{1}{2},1+\frac{1}{4}}(\overline{Q}_T),$$

and the subsequences again labeled by $\{(\beta_n,\gamma_n)\}$ and $\{(S_n,I_n)\}$ such that

$$\beta_n\to\overline{\beta},\quad\gamma_n\to\overline{\gamma}\quad\text{uniformly on }C^\alpha(\Omega),\tag{14}$$

$$S_n\to\overline{S},\quad I_n\to\overline{I}\quad\text{uniformly on }C^{\alpha,\frac{\alpha}{2}}(\overline{Q}_T)\cap C^{2+\alpha,1+\frac{\alpha}{2}}(\overline{Q}_T).\tag{15}$$

Moreover, we can deduce that $(\overline{S},\overline{I})$ is the solution of the initial boundary value problem (1)-(4) corresponding to the coefficients $(\overline{\beta},\overline{\gamma})$.

Hence, by Lebesgue's dominated convergence theorem, the weak lower-semicontinuity of L^2 norm, and the definition of the minimizing sequence, we have that

$$J(\overline{\beta},\overline{\gamma}) \leq \lim_{n\to\infty} J(\beta_n,\gamma_n) = \inf_{(\beta,\gamma)\in U_{ad}(\Omega)} J(\beta,\gamma). \tag{16}$$

Then, $(\overline{\beta},\overline{\gamma})$ is a solution of (5).

3.2. Proof of (ii)

The proof of that (9)-(12) is the adjoint system for (1)-(4) we can follow by the standard arguments in optimal control theory, see for instance [11]. Now, in order to get the $L^\infty(0,t;H^2(\Omega))$ estimates, let us consider an arbitrary $t\in]0,T]$ and we claim that

$$\|P(\cdot,t)\|^2_{L^2(\Omega)} + \|Q(\cdot,t)\|^2_{L^2(\Omega)} \leq C, \tag{17}$$
$$\|\nabla P(\cdot,t)\|_{L^2(\Omega)} + \|\nabla Q(\cdot,t)\|_{L^2(\Omega)} \leq C, \tag{18}$$
$$\|\Delta P(\cdot,t)\|_{L^2(\Omega)} + \|\Delta Q(\cdot,t)\|_{L^2(\Omega)} \leq C, \tag{19}$$
$$\|P(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \quad \|Q(\cdot,t)\|_{L^\infty(\Omega)} \leq C, \tag{20}$$

for a some positive generic constants C. We can prove the claims (17)-(20) by energy estimates for an initial value problem equivalent to (9)-(12). Indeed, in order to transform in an initial boundary problem we introduce the change of variable $\tau = T-t$ for $t\in[0,T]$. Moreover, consider the notation $w_1(\cdot,\tau) = P(\cdot,T-\tau)$, $w_2(\cdot,\tau) = Q(\cdot,T-\tau)$, $S^*(\cdot,\tau) = \bar{S}(\cdot,T-\tau)$, and $I^*(\cdot,\tau) = \bar{I}(\cdot,T-\tau)$. Then, the adjoint system (9)-(12) is equivalent to the system

$$(w_1)_\tau - \Delta w_1 = \overline{\beta}(\mathbf{x})\left(\frac{I^*}{S^*+I^*}\right)^2 (w_1-w_2), \qquad \text{in } Q_T, \tag{21}$$
$$(w_2)_\tau - \Delta w_2 = \overline{\beta}(\mathbf{x})\left(\frac{S^*}{S^*+I^*}\right)^2 (w_1-w_2) - \bar{\gamma}(\mathbf{x})(w_1-w_2), \qquad \text{in } Q_T, \tag{22}$$
$$\nabla w_1\cdot\mathbf{n} = \nabla w_2\cdot\mathbf{n} = 0, \qquad \text{on } \Gamma, \tag{23}$$
$$w_1(\mathbf{x},0) = \bar{S}(\mathbf{x},T) - S^{obs}(\mathbf{x}), \quad w_2(\mathbf{x},0) = \bar{I}(\mathbf{x},T) - I^{obs}(\mathbf{x}), \qquad \text{in } \Omega. \tag{24}$$

Now, we proceed to get the energy estimates for (21)-(24).

In order to prove (17) and (18), we test (21) by w_1 and (22) by w_2, and sum the results to get that

$$\begin{aligned}
\frac{1}{2}\frac{d}{d\tau}\Big(\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big) &+ \|\nabla w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|\nabla w_2(\cdot,\tau)\|^2_{L^2(\Omega)} \\
&\leq \int_\Omega |\overline{\beta}(\mathbf{x})|\left(\frac{I^*}{S^*+I^*}\right)^2 |w_1^2 - w_1w_2|\,d\mathbf{x} \\
&\quad + \int_\Omega \left(|\overline{\beta}(\mathbf{x})|\left(\frac{S^*}{S^*+I^*}\right)^2 + |\overline{\gamma}(\mathbf{x})|\right)|w_1w_2 - w_2^2|\,d\mathbf{x} \\
&\leq (\bar{b}+\bar{r})\left[\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\right].
\end{aligned} \tag{25}$$

Then, from the Gronwall inequality, we obtain

$$\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)} \leq \exp\Big(2(\bar{b}+\bar{r})T\Big)\Big(\|w_1(\cdot,0)\|^2_{L^2(\Omega)} + \|w_2(\cdot,0)\|^2_{L^2(\Omega)}\Big),$$

which implies (17). Now, from (25) and (17), we have that

$$\|\nabla w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|\nabla w_2(\cdot,\tau)\|^2_{L^2(\Omega)} \leq (\bar{b}+\bar{r})\exp\Big(2(\bar{b}+\bar{r})T\Big)\Big(\|w_1(\cdot,0)\|^2_{L^2(\Omega)} + \|w_2(\cdot,0)\|^2_{L^2(\Omega)}\Big),$$

and we can follow the estimate (18).

On the other hand, using the fact that

$$\int_\Omega (w_i)_\tau \Delta w_i \, d\mathbf{x} = -\int_\Omega \nabla[(w_i)_\tau] \cdot \nabla w_i \, d\mathbf{x} + \int_{\partial\Omega} (w_i)_\tau \nabla(w_i) \cdot \mathbf{n} \, dS = -\frac{1}{2}\frac{d}{d\tau}\|w_i(\cdot,\tau)\|^2_{L_2(\Omega)},$$

for $i = 1, 2$. We note that, multiplying (21) by Δw_1, multiplying (22) by Δw_2, integrating on Ω, and adding the results, we deduce that

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\Big(\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big) + \|\Delta w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|\Delta w_2(\cdot,\tau)\|^2_{L^2(\Omega)} \\ &\qquad \le (\bar{b}+\bar{r})\Big[2\epsilon\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + 2\epsilon\|w_2(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\Delta w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \frac{1}{2\epsilon}\|\Delta w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big], \end{aligned}$$

for any $\epsilon > 0$. Then, we have that

$$\begin{aligned} &\frac{1}{2}\frac{d}{d\tau}\Big(\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big) + \left(1 - \frac{(\bar{b}+\bar{r})}{2\epsilon}\right)\Big(\|\Delta w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|\Delta w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big) \\ &\qquad \le 2\epsilon(\bar{b}+\bar{r})\Big[\|w_1(\cdot,\tau)\|^2_{L^2(\Omega)} + \|w_2(\cdot,\tau)\|^2_{L^2(\Omega)}\Big]. \end{aligned}$$

Now, by selecting $\epsilon > (\bar{b}+\bar{r})/2$ and using the estimate (17) we get the inequality (19).

¿From (17), (18) and (19), we have that the norm of $P(\cdot,t)$ and $Q(\cdot,t)$ are bounded in the norm of $H^2(\Omega)$ for any $t \in]0,T]$. Thus, by the standard embedding theorem of $H^2(\Omega) \subset L^\infty(\Omega)$, we easily deduce (20) and conclude the proof of the item *(ii)*.

3.3. Proof of (iii)

We can prove the inequality (13) by straightforward generalization to the multidimensional case the one dimensional arguments given on the proof Theorem 3.3 in [20].

3.4. Proof of (iv)

Let us consider the set of functions $\{S, I\}$ and $\{\hat{S}, \hat{I}\}$ solutions to the direct problem (1)-(4) and with the coefficients $\{\beta, \gamma\}$ and $\{\hat{\beta}, \hat{\gamma}\}$, respectively. Then, we can prove that there exist the positive constant C such that the inequality

$$\|(\hat{S}-S)(\cdot,t)\|^2_{L^2(\Omega)} + \|(\hat{I}-I)(\cdot,t)\|^2_{L^2(\Omega)} \le C\Big(\|\hat{\beta}-\beta\|^2_{L^2(\Omega)} + \|\hat{\gamma}-\gamma\|^2_{L^2(\Omega)}\Big), \tag{26}$$

holds for any $t \in [0,T]$. Now, by notational convenience we consider $\delta S, \delta I, \delta\beta$ and $\delta\gamma$ defined as follows

$$\delta S = \hat{S} - S, \quad \delta I = \hat{I} - I, \quad \delta\beta = \hat{\beta} - \beta, \quad \delta\gamma = \hat{\gamma} - \gamma. \tag{27}$$

Then, from the systems (1)-(4) for (S, I) and $(\hat{S}, \hat{S})$ we have that $(\delta S, \delta I)$ satisfy the initial boundary value problem

$$(\delta S)_t - \Delta(\delta S) = -\hat{\beta}(\mathbf{x})\left(\frac{\hat{S}}{\hat{S}+\hat{I}} - \frac{S}{S+I}\right) - \delta\beta(\mathbf{x})\left(\frac{\hat{S}}{\hat{S}+\hat{I}}\right) + \hat{\gamma}(\mathbf{x})\delta I + \gamma(\mathbf{x})I, \qquad \text{in } \in Q_T, \tag{28}$$

$$(\delta I)_t - \Delta(\delta I) = \hat{\beta}(\mathbf{x})\left(\frac{\hat{S}}{\hat{S}+\hat{I}} - \frac{S}{S+I}\right) + \delta\beta(\mathbf{x})\left(\frac{\hat{S}}{\hat{S}+\hat{I}}\right) - \hat{\gamma}(\mathbf{x})\delta I - \gamma(\mathbf{x})I, \qquad \text{in } \in Q_T, \tag{29}$$

$$\nabla(\delta S)\cdot\mathbf{n} = \nabla(\delta I)\cdot\mathbf{n} = 0, \qquad \text{on } \in \Gamma, \tag{30}$$

$$(\delta S)(\mathbf{x},0) = (\delta I)(\mathbf{x},0) = 0, \qquad \text{in } \Omega. \tag{31}$$

Moreover, using the positivity of the solutions for the direct problem given by Theorem 2.1 and the lower bound of the total population by ϕ_0 (see proof of Lemma 4.1 in [20]), we observe that

$$\left|\frac{\hat{S}}{\hat{S}+\hat{I}}-\frac{S}{S+I}\right|=\left|\frac{\hat{S}(\hat{I}-I)}{(\hat{S}+\hat{I})(\hat{S}+I)}+\frac{I(\hat{S}-S)}{(\hat{S}+I)(S+I)}\right|\leq\frac{1}{\phi_0}\Big(|\delta S|+|\delta I|\Big). \tag{32}$$

Now, to prove (26), we test the equations (28) and (29) by δS and δI, respectively. Then, adding the results and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\delta S(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta I(\cdot,t)\|^2_{L^2(\Omega)}\right)+\|\nabla(\delta S)(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla(\delta I)(\cdot,t)\|^2_{L^2(\Omega)}\\
&\quad\leq\int_\Omega|\hat{\beta}(\mathbf{x})|\left|\frac{\hat{S}}{\hat{S}+\hat{I}}-\frac{S}{S+I}\right||\delta S|\,d\mathbf{x}+\int_\Omega|\delta\beta(\mathbf{x})|\left|\frac{\hat{S}}{\hat{S}+\hat{I}}\right||\delta S|\,d\mathbf{x}+\int_\Omega|\hat{\gamma}(\mathbf{x})||\delta I||\delta S|\,d\mathbf{x}\\
&\qquad+\int_\Omega|\delta\gamma(\mathbf{x})||I||\delta S|\,d\mathbf{x}+\int_\Omega|\hat{\beta}(\mathbf{x})|\left|\frac{\hat{S}}{\hat{S}+\hat{I}}-\frac{S}{S+I}\right||\delta I|\,d\mathbf{x}+\int_\Omega|\delta\beta(\mathbf{x})|\left|\frac{S}{S+I}\right||\delta I|\,d\mathbf{x}\\
&\qquad+\int_\Omega|\hat{\gamma}(\mathbf{x})||\delta I|^2\,d\mathbf{x}+\int_\Omega|\delta\gamma(\mathbf{x})||I||\delta I|\,d\mathbf{x}\\
&\quad\leq D_1\Big(\|\delta S(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta I(\cdot,t)\|^2_{L^2(\Omega)}\Big)+D_2\Big(\|\delta\beta\|^2_{L^2(\Omega)}+\|\delta\gamma\|^2_{L^2(\Omega)}\Big),
\end{aligned}\tag{33}$$

where $D_1=(4\bar{b}+\phi_0(1+3\overline{r}+\|I\|_{L^\infty(Q_T)}))/2\phi_0$ and $D_2=\max\Big\{1,\|I\|_{L^\infty(Q_T)}\Big\}$. Then, applying the Gronwall inequality, we deduce that

$$\|\delta S(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta I(\cdot,t)\|^2_{L^2(\Omega)}\leq e^{D_1T}\Big(\|\delta S_0\|^2_{L^2(\Omega)}+\|\delta I_0\|^2_{L^2(\Omega)}\Big)+D_2T\Big(\|\delta\beta\|^2_{L^2(\Omega)}+\|\delta\gamma\|^2_{L^2(\Omega)}\Big),$$

which implies (26) by using (31) and we conclude the proof of the item *(iv)*.

3.5. *Proof of (v)*

Let us consider the functions $S,I,\hat{S},\hat{I},\beta,\gamma,\hat{\beta}$ and $\hat{\gamma}$ as is given in the proof of the item *(iv)*. Additionally, we consider $\{\hat{P},\hat{Q}\}$ and $\{P,Q\}$ solution of the adjoint problem (9)-(12) with $\{\hat{S},\hat{I},\hat{\beta},\hat{\gamma},\hat{S}^{obs},\hat{I}^{obs}\}$ and $\{S,I,\beta,\gamma,S^{obs},I^{obs}\}$, respectively. Then, to prove the desired continuous dependence, we need to prove the existence of two positive constants $\tilde{C}_1$ and $\tilde{C}_2$ such that the inequality

$$\begin{aligned}
&\|(\hat{P}-P)(\cdot,t)\|^2_{L^2(\Omega)}+\|(\hat{Q}-Q)(\cdot,t)\|^2_{L^2(\Omega)}\\
&\qquad\leq\tilde{C}_1\Big(\|\hat{\beta}-\beta\|^2_{L^2(\Omega)}+\|\hat{\gamma}-\gamma\|^2_{L^2(\Omega)}\Big)+\tilde{C}_2\Big(\|\hat{S}^{obs}-S^{obs}\|^2_{L^2(\Omega)}+\|\hat{I}^{obs}-I^{obs}\|^2_{L^2(\Omega)}\Big)
\end{aligned}\tag{34}$$

holds for any $t\in[0,T]$. Indeed, additionally to the notation given in (27), we consider that $\delta P=\hat{P}-P$ and $\delta Q=\hat{Q}-Q$ which satisfy the system

$$(\delta P)_t+\Delta(\delta P)=\hat{\beta}(\mathbf{x})\left(\frac{\hat{I}}{\hat{S}+\hat{I}}\right)^2(\hat{P}-\hat{Q})-\beta(\mathbf{x})\left(\frac{I}{S+I}\right)^2(P-Q),\quad\text{in } Q_T, \tag{35}$$

$$(\delta Q)_t+\Delta(\delta Q)=\left(\hat{\beta}(\mathbf{x})\left(\frac{\hat{S}}{\hat{S}+\hat{I}}\right)^2-\hat{\gamma}(\mathbf{x})\right)(\hat{P}-\hat{Q})-\left(\beta(\mathbf{x})\left(\frac{S}{S+I}\right)^2-\gamma(\mathbf{x})\right)(P-Q),\quad\text{in } Q_T, \tag{36}$$

$$\nabla(\delta P)\cdot\mathbf{n}=\nabla(\delta Q)\cdot\mathbf{n}=0,\quad\text{on }\Gamma, \tag{37}$$

$$(\delta P)(\mathbf{x},T)=\delta S(\mathbf{x},T)-\Big(\hat{S}^{obs}(\mathbf{x})-S^{obs}(\mathbf{x})\Big),\quad\text{in }\Omega, \tag{38}$$

$$(\delta Q)(\mathbf{x},T)=\delta I(\mathbf{x},T)-\Big(\hat{I}^{obs}(\mathbf{x})-I^{obs}(\mathbf{x})\Big),\quad\text{in }\Omega. \tag{39}$$

Moreover, we can easily prove that the algebraic identity

$$\begin{aligned}&\hat{\zeta}\ \hat{\mathbb{A}}(\hat{P}-\hat{Q})-\zeta\ \mathbb{A}(P-Q)\\&\quad=\Big(\hat{\zeta}-\zeta\Big)\hat{\mathbb{A}}\hat{P}+\zeta\ \Big(\hat{\mathbb{A}}-\mathbb{A}\Big)\hat{P}+\zeta\ \mathbb{A}\delta P-\Big(\hat{\zeta}-\zeta\Big)\hat{\mathbb{A}}\hat{Q}-\zeta\ \Big(\hat{\mathbb{A}}-\mathbb{A}\Big)\hat{Q}-\zeta\ \mathbb{A}\delta Q\end{aligned}\tag{40}$$

is valid for any selection of $\hat{\zeta},\zeta,\hat{\mathbb{A}}$, and $\mathbb{A}$.

By selecting $(\hat{\zeta},\zeta,\hat{\mathbb{A}},\mathbb{A})=\Big(\hat{\beta},\beta,\hat{S}^2/(\hat{S}+\hat{I})^2,S^2/(S+I)^2\Big)$, we have that (40) implies that the right hand side (RHS) of equation (35) can be rewritten and bounded as follows

$$\begin{aligned}\text{RHS of (35)}&=\Big(\hat{\beta}-\beta\Big)\left(\frac{\hat{I}}{\hat{S}+\hat{I}}\right)^2\hat{P}+\beta\ \left[\left(\frac{\hat{I}}{\hat{S}+\hat{I}}\right)^2-\left(\frac{I}{S+I}\right)^2\right]\hat{P}+\beta\ \left(\frac{I}{S+I}\right)^2\delta P\\&\quad-\Big(\hat{\beta}-\beta\Big)\left(\frac{\hat{I}}{\hat{S}+\hat{I}}\right)^2\hat{Q}-\beta\ \left[\left(\frac{\hat{I}}{\hat{S}+\hat{I}}\right)^2-\left(\frac{I}{S+I}\right)^2\right]\hat{Q}-\beta\ \left(\frac{I}{S+I}\right)^2\delta Q\\&\le\Big(\|\hat{P}\|_{L^\infty(Q_T)}+\|\hat{Q}\|_{L^\infty(Q_T)}\Big)\left(|\delta\beta|+\frac{1}{\phi_0}\Big(|\delta S|+|\delta I|\Big)\right)+\bar{b}\Big(|\delta P|+|\delta Q|\Big).\end{aligned}\tag{41}$$

Similarly, we deduce that

$$\text{RHS of (36)}\le\Big(\|\hat{P}\|_{L^\infty(Q_T)}+\|\hat{Q}\|_{L^\infty(Q_T)}\Big)\left(|\delta\beta|+|\delta\gamma|+\frac{1}{\phi_0}\Big(|\delta S|+|\delta I|\Big)\right)+(\bar{b}+\bar{r})\Big(|\delta P|+|\delta Q|\Big).\tag{42}$$

Then, by testing (35) and (36) by δP and δQ, respectively; using the estimations (41)-(42); and applying the Cauchy-Schwarz inequality, we get

$$\begin{aligned}&-\frac{d}{dt}\Big(\|\delta P(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta Q(\cdot,t)\|^2_{L^2(\Omega)}\Big)+2\Big(\|\nabla\delta P(\cdot,t)\|^2_{L^2(\Omega)}+\|\nabla\delta Q(\cdot,t)\|^2_{L^2(\Omega)}\Big)\\&\quad\le\tilde{E}_1\Big[\|\delta P(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta Q(\cdot,t)\|^2_{L^2(\Omega)}\Big]\\&\quad\quad+\tilde{E}_2\Big(\|\delta S(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta I(\cdot,t)\|^2_{L^2(\Omega)}\Big)\\&\quad\quad+\tilde{E}_0\Big(\|\delta\beta\|^2_{L^2(\Omega)}+\|\delta\gamma\|^2_{L^2(\Omega)}\Big),\end{aligned}$$

with $\tilde{E}_0=2\Big(\|\hat{P}\|_{L^\infty(Q_T)}+\|\hat{Q}\|_{L^\infty(Q_T)}\Big)$, $\tilde{E}_1=\tilde{E}_0\left(2+(\phi_0)^{-1}\right)+4(\bar{b}+\bar{r})$ and $\tilde{E}_2=\tilde{E}_0/\phi_0$. ¿From the Theorem 2.2-*(ii)* we have that $\tilde{E}_i$, $i=0,1,2$ are constants. Then, applying the estimate (26) and rearranging some terms, we deduce that

$$-\frac{d}{dt}\left(e^{\tilde{E}_1t}\Big[\|\delta p_1(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta p_2(\cdot,t)\|^2_{L^2(\Omega)}\Big]\right)\le(\tilde{E}_2C+\tilde{E}_3)\Big(\|\delta\beta\|^2_{L^2(\Omega)}+\|\delta\gamma\|^2_{L^2(\Omega)}\Big),$$

and integrating on $[t,T]$, we have that

$$\begin{aligned}e^{\tilde{E}_1t}\Big[\|\delta p_1(\cdot,t)\|^2_{L^2(\Omega)}+\|\delta p_2(\cdot,t)\|^2_{L^2(\Omega)}\Big]&\le e^{\tilde{E}_1T}\Big[\|\delta p_1(\cdot,T)\|^2_{L^2(\Omega)}+\|\delta p_2(\cdot,T)\|^2_{L^2(\Omega)}\Big]\\&\quad+T(\tilde{E}_2C+\tilde{E}_3)e^{\tilde{C}_1T}\Big(\|\delta\beta\|^2_{L^2(\Omega)}+\|\delta\gamma\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

Hence, we can deduce (34) by application of the end condition (38)-(39).

3.6. Proof of (vi)

Using the fact that (β,γ) is a solution of (5), from (13) we have that

$$\int_{Q_T}\left[(\hat{\beta}-\beta)\frac{SI}{S+I}-(\hat{\gamma}-\gamma)I\right](P-Q)d\mathbf{x}dt$$

$$+\Gamma\Big[\int_\Omega \nabla\beta\cdot\nabla(\hat\beta-\beta)d\mathbf{x}+\int_\Omega \nabla\gamma\cdot\nabla(\hat\gamma-\gamma)d\mathbf{x}\Big]\geq 0,\quad \forall(\hat\beta,\hat\gamma)\in U_{ad}(\Omega), \tag{43}$$

with (P,Q) a solution of the adjoint problem (9)-(12) with (β,γ) instead of $(\overline{\beta},\overline{\gamma})$. Analogously for $(\tilde\beta,\tilde\gamma)$ solution of (5) with $(\tilde S^{obs},\tilde I^{obs})$ as observations, we have (13) implies that

$$\int_{Q_T}\left[(\hat\beta-\tilde\beta)\frac{\tilde S\ \tilde I}{\tilde S+\tilde I}-(\hat\gamma-\tilde\gamma)\tilde I\right](\tilde P-\tilde Q)d\mathbf{x}dt$$
$$+\Gamma\Big[\int_\Omega \nabla\tilde\beta\cdot\nabla(\hat\beta-\tilde\beta)d\mathbf{x}+\int_\Omega \nabla\tilde\gamma\cdot\nabla(\hat\gamma-\tilde\gamma)d\mathbf{x}\Big]\geq 0,\quad \forall(\hat\beta,\hat\gamma)\in U_{ad}(\Omega), \tag{44}$$

with $(\tilde P,\tilde Q)$ a solution of the adjoint problem (9)-(12) with $(\tilde\beta,\tilde\gamma)$ instead of $(\overline{\beta},\overline{\gamma})$, $(\tilde S\ \ \tilde I)$ instead of $(\overline{S}\ \ \overline{I})$ and $(\tilde S^{obs},\tilde I^{obs})$ instead of (S^{obs},I^{obs}). Then, selecting $(\hat\beta,\hat\gamma)=(\overline{\beta},\overline{\gamma})$ in (43) and $(\hat\beta,\hat\gamma)=(\beta,\gamma)$ in (44), rearranging some terms and applying the Cauchyy-Schwarz we deduce that

$$\begin{aligned}&\Gamma\Big[\|\nabla(\tilde\beta-\beta)\|^2_{L^2(\Omega)}+\|\nabla(\tilde\gamma-\gamma)\|^2_{L^2(\Omega)}\Big]\\&\leq\int_{Q_T}|\tilde\beta-\beta|\left|\frac{\tilde S\ \tilde I}{\tilde S+\tilde I}(\tilde P-\tilde Q)-\frac{SI}{S+I}(P-Q)\right|d\mathbf{x}dt+\int_{Q_T}|\tilde\gamma-\gamma|\left|\tilde I(\tilde P-\tilde Q)-I(P-Q)\right|\mathbf{x}dt.\end{aligned}\tag{45}$$

¿From the identity (40) with $(\hat\zeta,\zeta,\hat{\mathbb{A}},\mathbb{A})=\Big(1,1,\hat S^2/(\hat S+\hat I)^2,S^2/(S+I)^2\Big)$, the estimate (32), and with $(\hat\zeta,\zeta,\hat{\mathbb{A}},\mathbb{A})=\Big(1,1,\hat I,I\Big)$, we observe that

$$\begin{aligned}&\left|\frac{\tilde S\ \tilde I}{\tilde S+\tilde I}(\tilde P-\tilde Q)-\frac{SI}{S+I}(P-Q)\right|\\&\quad=\left|\left(\frac{\tilde S\ \tilde I}{\tilde S+\tilde I}-\frac{SI}{S+I}\right)\tilde P+\frac{SI}{S+I}(\tilde P-P)-\left(\frac{\tilde S\ \tilde I}{\tilde S+\tilde I}-\frac{SI}{S+I}\right)\tilde Q-\frac{SI}{S+I}(\tilde Q-Q)\right|\\&\quad\leq\frac{1}{\phi_0}\left[\|\tilde P\|_{L^\infty(Q_T)}+\|\tilde Q\|_{L^\infty(Q_T)}\right]\left[|\tilde S-S|+|\tilde I-I|\right]+\|S\|_{L^\infty(Q_T)}\Big[|\tilde P-P|+|\tilde Q-Q|\Big],\\&\left|\tilde I(\tilde P-\tilde Q)-I(P-Q)\right|\leq\left[\|\tilde P\|_{L^\infty(Q_T)}+\|\tilde Q\|_{L^\infty(Q_T)}\right]\left[|\tilde S-S|+|\tilde I-I|\right]\\&\quad+\|I\|_{L^\infty(Q_T)}\Big[|\tilde P-P|+|\tilde Q-Q|\Big].\end{aligned}$$

Then in (45) by applying the Cauchyy-Schwarz inequality, using the continuous dependence results of items *(iv)- (v)*, and rearranging some terms, we deduce that

$$\begin{aligned}&\Gamma\Big[\|\nabla(\tilde\beta-\beta)\|^2_{L^2(\Omega)}+\|\nabla(\tilde\gamma-\gamma)\|^2_{L^2(\Omega)}\Big]\\&\quad\leq\Theta_1\Big[\|\tilde\beta-\beta\|^2_{L^2(\Omega)}+\|\tilde\gamma-\gamma\|^2_{L^2(\Omega)}\Big]+\Theta_2\Big[\|\tilde S-S\|^2_{L^\infty(0,T;L^2(\Omega)}+\|\tilde I-I\|^2_{L^\infty(0,T;L^2(\Omega))}\Big]\\&\qquad+\Theta_3\Big[\|\tilde P-P\|^2_{L^\infty(0,T;L^2(\Omega))}+\|\tilde Q-Q\|^2_{L^\infty(0,T;L^2(\Omega))}\Big]\\&\quad\leq\Big[\Theta_1+\Theta_2+\Theta_3\Big]\Big[\|\tilde\beta-\beta\|^2_{L^2(\Omega)}+\|\tilde\gamma-\gamma\|^2_{L^2(\Omega)}\Big]+\Theta_3\Big[\|\tilde S^{obs}-S\|^2_{L^2(\Omega)}+\|\tilde I^{obs}-I\|^2_{L^2(\Omega)}\Big]\end{aligned}\tag{46}$$

with

$$\Theta_1=T\ \max\left\{\frac{1}{\phi_0}\left[\|\tilde P\|_{L^\infty(Q_T)}+\|\tilde Q\|_{L^\infty(Q_T)}\right]+\|\tilde S\|_{L^\infty(Q_T)},\|\tilde P\|_{L^\infty(Q_T)}+\|\tilde Q\|_{L^\infty(Q_T)}\right\},$$
$$\Theta_2=\frac{1}{2}\left(1+\frac{1}{\phi_0}\right)\left[\|\tilde P\|_{L^\infty(Q_T)}+\|\tilde Q\|_{L^\infty(Q_T)}\right],\quad \Theta_3=\frac{1}{2}\left[\|S\|_{L^\infty(Q_T)}+\|I\|_{L^\infty(Q_T)}\right].$$

Now, considering that $(\hat{\beta},\hat{\gamma}),(\beta,\gamma)\in\mathscr{U}_{\mathbf{c}}(\Omega)$, by the generalized Poincaré inequality, we have that there exist a positive constant C_{poi} such that

$$\begin{aligned}&\|\hat{\beta}-\beta\|^2_{L^2(\Omega)}+\|\hat{\gamma}-\gamma\|^2_{L^2(\Omega)}\\&\quad\le C_{poi}\Big(\|\nabla(\hat{\beta}-\beta)\|^2_{L^2(\Omega)}+\|\nabla(\hat{\gamma}-\gamma)\|^2_{L^2(\Omega)}+\|\hat{\beta}-\beta\|^2_{L^1(\Omega)}+\|\hat{\gamma}-\gamma\|^2_{L^1(\Omega)}\Big)\\&\quad=C_{poi}\Big(\|\nabla(\hat{\beta}-\beta)\|^2_{L^2(\Omega)}+\|\nabla(\hat{\gamma}-\gamma)\|^2_{L^2(\Omega)}\Big).\end{aligned}$$

¿From (46) and selecting $\overline{\Gamma}=(\Theta_1+\Theta_2+\Theta_3)C_{poi}$, we have that

$$\Big(\Gamma-\overline{\Gamma}\Big)\Big[\|\nabla(\hat{\beta}-\beta)\|^2_{L^2(\Omega)}+\|\nabla(\hat{\gamma}-\gamma)\|^2_{L^2(\Omega)}\Big]\le\Upsilon_2\Big[\|\hat{S}^{obs}-S^{obs}\|^2_{L^2(\Omega)}+\|\hat{I}^{obs}-I^{obs}\|^2_{L^2(\Omega)}\Big],$$

which implies the desired uniqueness result given in the item-*(vi)*.

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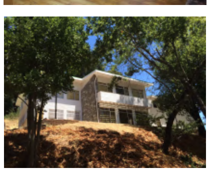

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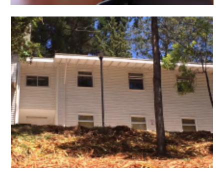

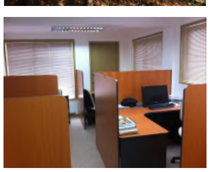

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