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WELL-POSEDNESS, EXPONENTIAL DECAY ESTIMATE AND NUMERICAL RESULTS FOR THE HIGH ORDER NONLINEAR SCHRÖDINGER EQUATION WITH LOCALIZED DISSIPATION

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ABSTRACT. In this work, we study the existence at the L^2 – level as well as the stability for the high order nonlinear Schrödinger equation in a bounded interval with a localized damping term. To prove the existence, we employ the method devised by Bisognin et al., [8]. To prove the exponential stabilization, with these approximations, we use multipliers techniques found in Bisognin et al., [8] and Linares and Pazoto, [23]. In addition, we implement a precise and efficient code to study the energy decay of the high order nonlinear Schrödinger equation.

1. INTRODUCTION

1.1. **Description of the Problem.** In this work we will study the Non-Linear Schrödinger (HNLS) equation with extra high-order terms:

(1.1)

$$\begin{cases} i\,u_t + a_1\,u_{xx} + a_2\,|u|^2\,u + i\left[a_3\,u_{xxx} + a_4\,\left(|u|^2\,u\right)_x + a_5\,u\,\left(|u|^2\right)_x + a(x)\,u\right] = 0 & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,0) = u_0 & \text{in } (0,L) \end{cases}$$

so that the real constants $a_1, a_3 > 0$ and $a_i \neq 0$, $i = 2,4,5$. Let's assume that $a(x)$ is a non-negative real valued function belonging to $L^\infty(0,L)$ and moreover we will assume that

$$a(x) \geqslant a_0 > 0 \text{ a.e. in an open, non-empty subset } \omega \text{ of } (0,L), \tag{1.2}$$

where the damping is acting effectively.

1.2. **Main Goal, Methodology and Previous Results.** The main objective of the present manuscript is to prove the existence and uniqueness for mild solutions to problem (1.1) and, in addition, that those solutions decay exponentially and uniformly to zero in L^2- norm, that is, there exist positive constants C, γ, such that

$$E(T) \leq Ce^{-\gamma t}E(0),\ \forall\, t \geq T_0, \tag{1.3}$$

where $E(t)$ is given in (2.1) for all mild (L^2-level) solutions to problem (1.1) provided that the initial data u_0 are taken in bounded sets of $L^2(\Omega)$.

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To this end, the following tools are considered:

- In order to prove the well-posedness for a mild solution to problem (1.1) we borrowed ideas due to Bisognin et al., [8] using a contraction mapping argument.
- To prove the exponential stabilization, with these approximations, we use multipliers techniques found in Bisognin et al., [8] and Linares and Pazoto, [23]. Under these circumstances, the condition (1.2) imposed on $a(x)$ is crucial to handle the energy in L^2-level. Indeed, when $a(x) \geq a_0 > 0$ almost everywhere in $\mathbb{R}^+$, it is very simple to prove that the energy $E(t)$ decays exponentially as t tends to infinity. The problem of stabilization when the damping is effective only on a subset of $\mathbb{R}^+$ is harder. In this work, we treat with this situation. More precisely, our purpose is to prove the exponential decay given in (1.3). This can be stated in the following equivalent form: Find $T > 0$ and $C > 0$ such that

$$E(0) \leq C \left\{ \frac{a_3}{2} \int_0^T |u_x(0,t)|^2 \, dt + 2 \int_0^T \int_0^L a(x) |u(x,t)|^2 \, dx \, dt \right\} \tag{1.4}$$

holds for every finite energy solution of (1.1). In fact, taking into account (1.4) and (3.30), we shall show that $E(T) \leq \gamma \, E(0)$, which combined with the semigroup property allow us to derive the exponential decay for $E(t)$.

However, the desired estimate (1.4) will not hold directly since lower order additional terms will appear. So, to absorb them we shall use the so-called compactness-uniqueness argument that reduces the question to a unique continuation problem that will be solved by applying the result due to Carvajal and Panthee, [17]. The authors proved the following unique continuation result (Theorem 1.1, page 189): We consider the following problem:

$$u_t + i\,\alpha\, u_{xx} + i\,\gamma\, |u|^2\, u + \beta\, u_{xxx} + \delta\, |u|^2\, u_x + \epsilon\, u^2\, \bar{u}_x = 0, \quad x, t \in \mathbb{R}, \tag{1.5}$$

where $\alpha, \beta \in \mathbb{R}, \beta \neq 0, \gamma, \delta, \epsilon \in \mathbb{C}$ and $u = u(x,t)$ is a complex valued function.

We have the following unique continuation result:

Theorem 1.1. *Let $u \in C([t_1, t_2]; H^s) \cap C^1([t_1, t_2]; H^1)$, $s \geq 4$, be a solution of the equation* (1.5) *with $\alpha, \beta, \gamma, \delta, \epsilon \in \mathbb{R}$ and $\beta \neq 0$. If there exists $t_1 < t_2$ such that*

$$\operatorname{supp} u(\cdot, t_j) \subset (-\infty, a),\ j = 1, 2 \tag{1.6}$$

$$\textit{or} \ \operatorname{supp} u(\cdot, t_j) \subset (b, \infty),\ j = 1, 2\,. \tag{1.7}$$

Then $u(t) = 0$ for all $t \in [t_1, t_2]$.

Remark 1.2. *The problem* (1.1) *can be rewritten as the following problem*

$$\begin{cases} i\, u_t + a_1\, u_{xx} + f(u) + i \left[a_3\, u_{xxx} + a(x)\, u \right] = 0 & \textit{in } (0,L) \times (0,\infty) \\ u(0,t) = u(L,t) = 0 & \textit{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \textit{for all } t \geqslant 0 \\ u(x,0) = u_0 & \textit{in } (0,L) \end{cases} \tag{1.8}$$

where

$$f(u) = a_2\,|u|^2\,u + i\left[a_4\,\left(|u|^2\,u\right)_x + a_5\,u\,\left(|u|^2\right)_x\right] \tag{1.9}$$
$$= a_2\,|u|^2\,u + i\left[\,(2\,a_4 + a_5)\,|u|^2\,u_x + (a_4 + a_5)\,u^2\,\bar{u}_x\right].$$

Then, in the proof of the exponential decay, we can use Theorem 1.1 working with the problem (1.8) *instead* (1.1).

But due to the lack of regularity of the solutions we are dealing with, i.e., finite energy solutions, the unique continuation result presented in Theorem (1.1) may not directly be applied. To overcome this problem, we proceed as in Bisognin et al., [8] and Linares and Pazoto, [23] and we first guarantee that solutions are smooth enough.

In what follows we would like to mention some important papers in connection with the subject of the present article. Initially, let us consider the following initial boundary value problem of the higher - order nonlinear Schrödinger equation with localized damping:

$$\begin{cases} i\,u_t + a_1\,u_{xx} + |u|^2\,u + i\,a_3\,u_{xxx} + i\,a(x)\,u = 0 & \text{in } (0,L)\,\times\,(0,\infty) \\ u(0,t) = u(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,0) = u_0 & \text{in } (0,L), \end{cases} \tag{1.10}$$

where $a_1, a_3 \in \mathbb{R}, a_3 \neq 0$ and the damping $a \in C^\infty(0,L)$ satisfies (1.2). We observe that the problem (1.10) is a particular case of the problem (1.1) considering $a_2 = 1$ and $a_4 = a_5 = 0$. Bisognin et al., [8] proved the exponential decay in L^2– level. Using compactness arguments, the smoothing effect of the KdV equation on the line and the unique continuation results, the authors deduced the exponential decay in time of the solutions of the linear equation and a local uniform stabilization result of the solutions of the nonlinear equation when the localized damping is active simultaneously only in a neighborhood of both extremes $x = 0, x = L$. In order to prove the result, the authors used multipliers together with compactness arguments and smoothing properties proved by Sepúlveda and Vera, [42] and the Unique Continuation Principle valid for this problem given in Bisognin and Vera, [9].

Later, Alves, Sepúlveda and Vera, [1] studied local and global existence and smoothing properties of the problem (1.10) with $a \equiv 0$. In this situation, the authors verified gain in regularity for this equation. Specifically, they were proved conditions on this problem for which initial data u_0 possessing sufficient decay at infinity and minimal amount of regularity will lead to a unique solution $u(t) \in C^\infty(\mathbb{R})$ for $0 < t < T$, where T is the existence time of the solution.

Ceballos et al.,[16], analyzed directly the exact boundary controllability problem for the higher order Schrödinger equation with $a \equiv 0$ by adapting a method which combines the Hilbert Uniqueness Method (HUM) and multiplier techniques.

The equation (1.1) plays an important rule in soliton theory. It has applications in the propagation of femtosecond optical pulses in a monomode optical fiber, accounting for additional effects such as third order dispersion, self-steeping of the pulse, and self-frequency shift (see [20]). But we can also consider it as a generalization of the classical NLS equation using $a_3 = a_4 = a_5 = 0$ which can describe the electric field envelope of a laser beam in a medium with Kerr nonlinearity as described by Kodama, [25]. If we also take $a_1 = a_2 = 0, a_3 = a_4 = -a_5 = 6$, we can obtain the modified Korteweg-de Vries (KdV) equation:

$$u_t + u_{xxx} + 6\,u^2\,u_x = 0$$

which studies, for example, surface waves on conducting nonviscous incompressible liquid under the presence of a transverse electric field as introduced by Perel'man et al, [39]. The KdV equation has also great importance in the study of surface water waves (see Kortweg and De Vries [26]).

A damping of the type $a(x)u$ was introduced in Menzala et al., [31] to stabilize the KdV system inspired in the work of Rosier, [41]. More precisely, considering the damping localized at a subset $\omega \subset (0, L)$ containing nonempty neighborhoods of the end-points of an interval, it was shown that solutions of both linear and nonlinear problems for the KdV equation decay, independently on $L > 0$. In Pazoto, [37] it was proved that the same holds without cumbersome restrictions on $\omega \subset (0, L)$. Linares and Pazoto, [23] proved the exponential stabilization of the Korteweg–de Vries equation in the right half-line under the effect of the same localized damping term $a(x)\,u$. Araruna et al., [2] proved the exponential decay in L^2 - level for the modified Kawahara equation posed in a bounded bounded interval under the presence of a localized damping term $a(x)\,u$ satisfying where the function $a(\cdot)$ satisfies (1.2). Cavalcanti et al., [13] studied the well-posedness and the asymptotic behavior of solutions of a KdV- Burgers equation subject to a localized dissipation mechanism with indefinite sign:

$$u_t - u_{xx} + u_{xxx} + u\,u_x + \lambda(x)\,u = 0\, x \in \mathbb{R},\, t > 0,\, \lambda \in L^\infty(\mathbb{R})$$

such that a sufficient condition criteria for the exponential decay has been established.

The study of decay rate estimates for weakly full damped semilinear focusing and defocusing Schrödinger equations ($a_4 = a_5 = 0$)

$$iy_t + \Delta y \pm |y|^2 y + iay = 0 \ \text{ in } \Omega \times (0, \infty),\ a > 0, \tag{1.11}$$

where Ω is a bounded domain of $\mathbb{R}^n$, with zero Dirichlet boundary condition, has been considered by Tsutsumi [44] where exponential stabilization of H^k-solutions ($k = 1, 2$) is established. For this purpose, smallness on the initial data is assumed. Later on, Özsarı, Kalantarov and Lasiecka [34] generalized the previous result mentioned above (at least for the defocusing case) by considering inhomogeneous Dirichlet boundary conditions. Smallness on the initial data is also assumed for proving decay rates estimates in H^2-norm. In H^1-norm, no smallness is required. Indeed, the result for H^1-solutions obtained in [34] is strong in the sense that it is independent of the dimension of the domain and the smallness of the initial data.

On the other hand, regarding the exponential stability for the semilinear defocusing Schrödinger equation, subject to a linear damping locally distributed and posed in unbounded domains, namely,

$$iy_t + \Delta y - |y|^2 y + ia(x)y = 0 \ \text{ in } \mathbb{R}^n \times (0, \infty),\ n = 1, 2, \tag{1.12}$$

(here $a(x) \geq a_0 > 0$ for $||x|| > R > 0$), we would like to mention the works of the authors Cavalcanti et. al. [14], [15]. In order to achieve the desired goal, the authors make use of two main ingredients in the proof: (i) To establish an unique continuation property associated with regular and mild solutions of the non-damped problem $iy_t + \Delta y - |y|^2 y = 0$ restricted to a fixed ball of radius $r > R$; (ii) To employ a smoothing effect as established, for instance, in Constantin and

Saut [18]. In the same spirit of Cavalcanti et al., [14] we can also mention the following works by Natali, [32] regarding the one-dimensional case and Natali, [33] in the two-dimensional case.

Cavalcanti et al., [10] proved the existence in H^1-norm as well as the stability for the damped defocusing Schrödinger equation using the following model:

$$\begin{cases} i\,\partial_t\, y + \Delta\, y - |y|^p\, y + i\,\lambda(x,t)\, y = 0 & \text{in } \mathbb{R}^n \times (0,\infty) \\ y(0) = y_0 & \text{in } \mathbb{R}^n, \end{cases} \tag{1.13}$$

where $n \geq 1, p > 0$. The damping coefficient $\lambda(x,t)$ may vanish at infinity and satisfies the following conditions:

$$\lambda \in C_b([0,\infty);\, W^{1,\infty}(\mathbb{R}^n)), \quad \lambda(x,t) > 0,\, \forall\, x \in \mathbb{R}^n,\, \forall t \geq 0. \tag{1.14}$$

To prove the existence, the authors employed the method devised by Özsarı et al. [34]. In particular, when $n = 1$ or $n = 2$, the uniqueness is obtained. Decay estimates for the L^2-norm and $(H^1 \cap L^{p+2})-$norm are established with the help of direct multipliers method, coupled with refined energy estimates and a lower semi-continuity argument.
Finally, we would like to mention some relevant works about Schrödinger equation in connection with the subject of the present paper, namely, [3], [4], [5], [11], [13], [15], [22],[28], [32], [33], [35], [44].

Concerning the numerical results, and regarding the finite difference method, we'd like to mention one of the first proposals from Delfour et al [19] to solve the problem (1.11) for $a(x) \equiv 0$. Their main contribution was the way the term $|u|^2u$ was discretized in order to preserve the numerical charge. To this end, the way they've proceeded was as follows for $t^n := \Delta t n,\, n \in \mathbb{N}$:

$$|u(t_n)|^2 u(t_n) \approx \frac{|u^{n+1}|^2 + |u^n|^2}{2} u^{n+\frac{1}{2}}, \quad u^{n+\frac{1}{2}} := \frac{u^{n+1} + u^n}{2}$$

It is worh noting that this ways of discretize the nonlinear term doesn't affect the energy preservation. Meanwhile, Pazoto et al. [38] proposed a finite differences scheme to solve the following KdV problem for $u = u(x,t)$:

$$u_t + u_{xxx} + u^4 u_x + u_x + a(x)u = 0, \qquad (x,t) \in (0,L) \times (0,+\infty) \tag{1.15}$$

$$u(0,t) = u(L,t) = 0, \qquad t \in (0,+\infty) \tag{1.16}$$

$$u_x(L,t) = 0, \qquad t \in (0,+\infty) \tag{1.17}$$

$$u(x,0) = u_0(x),\; x \in (0,L) \tag{1.18}$$

for $a \in L^\infty(0,L) : a(x) \geq a_0 > 0,$ a. e. in Ω, and Ω a nonempty open subset of $(0,L)$. The power nonlinearity $u^4 u_x =: F(u)$ was rewritten using algebraic identities widely used in finite differences analysis, and aiming to achieve two conservation properties:

$$(u, F(u)) = 0$$

$$(u, F(u))_x = -\frac{1}{6}|u|_6^6$$

This was done in order to obtain H_0^1-estimates for the numerical solution of the problem. Finally, following the spirit of both works, Cavalcanti et al. [12] proposed a new Finite Difference Scheme which solves problem (1.1) for $a(x) \equiv 0$. The nonlinear terms multiplied by a_4 and a_5 were rewritten as a convex combination in order to achieve complete conservation of energy. The scheme also achieves an almost-conservation of the numerical charge when $3a_2a_3 = a_1(3a_4+a_5)$; this is, the numerical charge has an order of conservation of $\mathcal{O}(\Delta t+\Delta x^2)$.

This paper uses that numerical scheme, while adding the damping term $a(x)$ to our calculations, in order to prove and achieve exponential decay for the numerical energy.

Our paper is organized as follows: Section 2 is devoted to notations and statement of main results. In section 3 we present the proofs of the well-posedness to problem (1.1) and in section 4, we present the proofs of the exponential stability. Finally, the section 5 is devoted to the study of the numerical results of the problem (1.1).

2. Notations and statement of the Main results

We consider the space $L^2(0,L)$ of complex valued functions on $\mathbb{R}^d$ endowed with the inner product

$$(y,z)_{L^2(0,L)} = \mathrm{Re}\int_0^L y(x)\overline{z}(x)\,dx$$

with the corresponding norm

$$||y||^2_{L^2(0,L)} = (y,y)_{L^2(0,L)}.$$

We also consider the Sobolev space $H^1(0,L)$ endowed with scalar product

$$(y,z)_{H^1(0,L)} = (y_x,z_x)_{L^2(0,L)}.$$

The energy is defined by:

$$E(t) := \frac{1}{2}\left\|u(x,t)\right\|^2_{L^2(0,L)}. \tag{2.1}$$

Employing the boundary conditions given in (1.1), we infer that

$$\frac{d}{dt}E(t) = -\frac{a_3}{2}\left|u_x(0,t)\right|^2 - \int_0^L a(x)|u(x,t)|^2\,dx,\ \forall\, t>0. \tag{2.2}$$

Since $a_3 > 0$ and by assumption on damping a, we observe that according to the above energy dissipation law, the energy $E(t)$ is a nonincreasing function of the time.

Now, we can state our main results:

Theorem 2.1 (Existence)**.** *Assume that $u_0 \in\ L^2(0,L)$ and the function $a(x)$ satisfies* (1.2)*. Then, the problem* (1.1) *possesses a unique mild solution*

$$u \in\ C([0,\infty);L^2(0,L))\ \cap\ L^2(0,\infty;H^1_0(0,L))\,. \tag{2.3}$$

Regarding to the exponential decay, we have the following result:

Theorem 2.2 (Exponential Decay)**.** *Let y be a mild solution to problem (1.1) given by Theorem 2.1. Assume that $a\in L^\infty(0,L)$ such that $a(x)\geq a_0>0$ a.e. in ω. Then, for any $L'>0$, there are $C=C(L')>0$ and $\gamma=\gamma(L')$ such that the following exponential decay holds*

$$E_0(t)\leq Ce^{-\gamma t}E_0(0)$$

where $E_0(t) := \frac{1}{2}\,||y(t)||^2_{L^2(0,L)}$, provided that $||y_0||_{L^2(0,L)}\leq L'$.

3. Existence of Solutions

As stated in the introduction, we used the ideas found in the works of Bisognin et al., [8].

3.1. **Linear System:** First of all, we consider the linear system, that is, assuming that $a_2 = a_4 = a_5 = 0$:

$$
\begin{cases} i\,u_t + a_1\,u_{xx} + i\left[a_3\,u_{xxx} + a(x)\,u\right] = 0 & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,0) = u_0 & \text{in } (0,L) \end{cases} \tag{3.1}
$$

Let assume that $a \equiv 0$. We consider the following operator:

$$
\begin{aligned} A : D(A) &\to L^2(0,L) \\ u \mapsto A(u) &:= -a_3\,u_{xxx} + i\,a_1\,u_{xx} \end{aligned} \tag{3.2}
$$

with the domain:

$$D(A) = \left\{v \in\, H^3(0,L);\, v(0) = v(L) = 0,\, v_x(L) = 0\right\}\,.$$

We have the following result:

Theorem 3.1. *Let $a \equiv 0$. Then, the operator A given in* (3.2) *generates a semigroup contraction $\left\{e^{tA}\right\}_{t=0}^{t=\infty}$ in $L^2(0,L)$.*

Proof. It is easy to prove that the operator A is closed. We claim that A is dissipative.

Indeed, performing integration by parts give us

$$
\begin{aligned} (A\,v, v)_{L^2(0,L)} &= \int_0^L (-a_3\,v_{xxx} + i\,a_{11}\,v_{xx})\,\bar{v}\,dx \\ &= -\frac{a_3}{2}\,|v_x(0,t)|^2 - i\,a_{11}\int_0^L |v_x|^2\,dx\,. \end{aligned}
$$

Hence,

$$\operatorname{Re}\,(A\,v, v)_{L^2(0,L)} = -\frac{a_3}{2}\,|u_x(0,t)|^2 \leq 0,$$

which proves that A is dissipative.

On the other hand, the adjoint of the operator A is given by

$$
\begin{aligned} A^* : D(H^*) &\to L^2(0,L) \\ u \mapsto A(u) &:= -a_3\,u_{xxx} - i\,a_{11}\,u_{xx} \end{aligned}
$$

with its domain

$$D(A^*) = \left\{v \in\, H^3(0,L);\, v(0) = v(L) = 0,\, v_x(0) = 0\right\} \subseteq\, L^2(0,L)\,.$$

A similar calculation shows that

$$\operatorname{Re}\,(A^*\,v, v)_{L^2(0,L)} = -\frac{a_3}{2}\,|v_x(0,t)|^2 \leq 0,$$

Because A^* is dissipative, then we have that A is maximal. Then, because A is maximal, and A and its adjoint are dissipative, we conclude the desired from the Lumer-Phillips theorem. □

The well-posedness of the linear system (3.1) can be handled in a similar way by considering the term $i\,a(x)\,u$ as a linear perturbation with the case $a \equiv 0$. Summary, we have:

Theorem 3.2. *Let $u_0 \in L^2(0,L)$. Then, the problem* (3.1) *possesses a unique solution in the class*

$$u \in C([0,\infty); H^3(0,L)) \cap C^1([0,\infty); L^2(0,L))\,. \tag{3.3}$$

In the sequel, regarding the linear problem (3.1), we have the following results:

Lemma 3.3. *The map*

$$u_0 \in L^2(0,L) \mapsto \mathrm{e}^{t\,A}\,u_0 \in X_T \equiv C([0,T];\, L^2(0,L)) \cap L^2(0,T;\, H_0^1(0,L))$$

is continuous.

Proof. In fact, for $u_0 \in L^2(0,L)$, let $u = \mathrm{e}^{t\,A}\,u_0$ be a solution of the problem (3.1) (with $a \equiv 0$). By Theorem (3.2), we have that $u \in C([0,T];\, L^2(0,L))$ and since the Schrödinger semigroup $\left\{\mathrm{e}^{t\,A}\right\}_{t=-\infty}^{t=\infty}$ is a contraction semigroup, we have the isometry:

$$\|u(t)\|_{L^2(0,L)} = \left\|\mathrm{e}^{t\,A}\,u_0\right\|_{L^2(0,L)} \leq \|u_0\|_{L^2(0,L)}\,. \tag{3.4}$$

To show that $u \in L^2(0,T;\, H_0^1(0,L))$, we consider $u_0 \in D(A)$. By density of $D(A)$ in $L^2(0,L)$, the result will be extended to an arbitrary initial data $u_0 \in L^2(0,L)$.

Multiplying the first equation of the problem (3.1) by $x\,\bar{u}$, we have

$$i\,x\,\bar{u}\,u_t + a_1\,x\,\bar{u}\,u_{xx} + i\left[a_3\,x\,\bar{u}\,u_{xxx} + a(x)\,|u|^2\right] = 0\,. \tag{3.5}$$

Applying the conjugate in (3.5), we have:

$$-\,i\,x\,u\,\bar{u}_t + a_1\,x\,u\,\bar{u}_{xx} + i\left[a_3\,x\,u\,\bar{u}_{xxx} + a(x)\,|u|^2\right] = 0\,. \tag{3.6}$$

Subtracting (3.5) and (3.6) and integrating over $x \in [0,L]$, we obtain

$$\frac{d}{dt}\int_0^L x\,|u|^2\,dx + 3\,a_3\int_0^L |u_x|^2\,dx - 2\,a_1\,\mathrm{Im}\int_0^L u_x\,\bar{u}\,dx + 2\int_0^L x\,a(x)\,|u|^2\,dx = 0\,. \tag{3.7}$$

Integrating (3.7) over $t \in [0,T]$, we arrive

$$\begin{aligned}&\underbrace{\int_0^L x\,|u(x,T)|^2\,dx}_{\geq 0} + 3\,a_3\int_0^T\!\!\int_0^L |u_x(x,t)|^2\,dx\,dt + \underbrace{\int_0^T\!\!\int_0^L x\,a(x)\,|u(x,t)|^2\,dx\,dt}_{\geq 0}\\ &= 2\,a_1\,\mathrm{Im}\int_0^T\int_0^L u_x\,\bar{u}\,dx\,dt\,.\end{aligned} \tag{3.8}$$

On the other hand, making use of Young's inequality, having in mind (3.4), we get

$$\begin{aligned}2\,a_1\ \mathrm{Im}\int_0^T\int_0^L u_x\,\bar u\,dx\,dt &\le \frac{a_1^2}{2\,a_3}\int_0^T\int_0^L |u|^2\,dx\,dt + 2\,a_3\int_0^T\int_0^L |u_x|^2\,dx\,dt\\ &\le \frac{a_1^2\,T}{2\,a_3}\,\|u_0\|^2_{L^2(0,T)} + 2\,a_3\int_0^T\int_0^L |u_x|^2\,dx\,dt\,.\end{aligned}\tag{3.9}$$

Combining (3.8) and (3.9), we obtain

$$\int_0^T\int_0^L |u_x(x,t)|^2\,dx\,dt \le \frac{a_1^2\,T}{2\,a_3^2}\,\|u_0\|^2_{L^2(0,T)},$$

that is,

$$\|u\|_{L^2(0,T;\,H_0^1(0,L))} \le \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|_{L^2(0,L)}\,.\tag{3.10}$$

Finally, combining (3.4) and (3.10), we concluded that

$$\|u\|_{X_T} \le \left(1 + \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\right)\,\|u_0\|_{L^2(0,L)},\tag{3.11}$$

which proves the desired.

□

Remark 3.4. *We observe that if* $u_0 \in L^2(0,L)$, *then* $u_x(0,\cdot)$ *makes sense. Indeed, as in the previous Lemma, we also assume that* $u_0 \in D(A)$ *and the result follows by density. Multiplying the first equation of* (3.1) *by* $\bar u$, *integrating in* $x \in (0,L)$, *one gets*

$$\frac{1}{2}\,\frac{d}{dt}\|u(t)\|^2_{L^2(0,L)} = -\frac{a_3}{2}\,|u_x(0,t)|^2 - \int_0^L a(x)|u(x,t)|^2\,dx\,.\tag{3.12}$$

Integrating over $t \in [0,T]$, *we have*

$$\frac{1}{2}\,\|u(t)\|^2_{L^2(0,L)} = \frac{1}{2}\,\|u_0\|^2_{L^2(0,L)} - \frac{a_3}{2}\int_0^t |u_x(0,t)|^2\,dt - \int_0^t\int_0^L a(x)|u(x,t)|^2\,dx\,dt\,.\tag{3.13}$$

Hence,

$$\begin{aligned}\frac{a_3}{2}\int_0^T |u_x(0,t)|^2\,dt &= -\frac{1}{2}\,\|u(t)\|^2_{L^2(0,L)} + \frac{1}{2}\,\|u_0\|^2_{L^2(0,L)} - \int_0^t\int_0^L a(x)|u(x,t)|^2\,dx\,dt\\ &\le \frac{1}{2}\,\|u_0\|^2_{L^2(0,L)},\end{aligned}$$

which proves the desired.

3.2. **Nonlinear System.** In this subsection, let's prove Theorem 3.2. As stated in the introduction, we used the ideas found in the works of Bisognin et al., [8]. Initially, let's analyze the local solutions of the problem (1.1).

3.2.1. *Local Solutions.* Let $T > 0$ and consider the following functional:

$$\begin{aligned} \Psi : X_T &\rightarrow X_T \\ u &\mapsto \Psi(u)(t) := \mathrm{e}^{t\,A}\, u_0 - \int_0^t \mathrm{e}^{i\,(t-\tau)\,\Delta}\, f(u)(\tau)\, d\tau \end{aligned} \tag{3.14}$$

where $e^{t\,A}$ is the Schrödinger group given above and $f(u)$ is given in (1.9). Here, the space X_T is endowed with the norm

$$\|u\|_{X_T} := \sup_{t\in[0,T]} \|u(t)\|_{L^2(0,L)} + \left[\int_0^T \|u(s)\|^2_{H^1_0(0,L)}\, ds \right]^{1/2}.$$

According to the isometry given in (3.4) and (3.10), it follows that the Schrödinger semigroup $\left\{\mathrm{e}^{t\,A}\right\}_{t=-\infty}^{t=\infty}$ corresponding to the linear system (3.1) satisfies

$$\|\mathrm{e}^{t\,A}\, u_0\|_{C([0,T],L^2(0,L))} = \sup_{t\in[0,T]} \|\mathrm{e}^{t\,A}\, u_0\|_{L^2(0,L))} = \sup_{t\in[0,T]} \|u_0\|_{L^2(0,L))} = \|u_0\|_{L^2(0,L)}\,, \tag{3.15}$$

$$\|\mathrm{e}^{t\,A}\, u_0\|_{L^2(0,T,H^1_0(0,L))} \leqslant \frac{a_1\, T^{1/2}}{2^{1/2}\, a_3}\, \|u_0\|_{L^2(0,L)}\,. \tag{3.16}$$

Moreover, we also have the called conservation of the Schrödinger flow:

$$\|\mathrm{e}^{t\,A}\, u_0\|_{H^1_0(0,L)} = \|u_0\|_{H^1_0(0,L)}\,. \tag{3.17}$$

We shall prove that the contraction mapping principle can be applied to $\Psi : B_R \rightarrow B_R$ where

$$B_R = \{v \in X_T;\ \|v\|_{X_T} \leq R\}$$

provided that R is suitably large and T is suitably small, so that Ψ possesses a fixed point in B_R.

In order to achieve this, we will need of the Lemmas 3.5 and 3.6 below:

Lemma 3.5. *Let $u \in X_T$ and $f(u)$ given in* (1.9). *Then*

$$\int_0^T \|f(u)\|_{L^2(0,L)}\, dt \leq \widetilde{C}\, \|u\|^2_{X_T}\,.$$

Lemma 3.6. *Let $u, v \in X_T$. Then,*

$$\int_0^T \|f(u) - f(v)\|_{L^2(0,L)}\, dt \leq C\, \left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\, \|u - v\|_{X_T}\,.$$

These Lemmas are technical issues and they will proved in the Section 4.

In the sequel, we will show that Ψ maps B_R into itself for R sufficiently large and T small enough, or in other words, we shall prove that for R large enough and T sufficiently small, we have

$$\|\Psi(u)\|_{X_T} \leq R, \tag{3.18}$$

provided that $\|u\|_{X_T} \leq R$.

In fact, let $u \in B_R$. By Lemma (3.5), we have that $f(u) \in L^1(0,T;\, L^2(0,L))$. From this, having in mind (3.14), (3.16) and the Hölder's inequality, we have

$$
\begin{aligned}
\|\Psi(u)\|_{H_0^1(0,L)} &\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + \left\| \int_0^t e^{i\,(t-\tau)\,\Delta}\, f(u)(\tau)\, d\tau \right\|_{H_0^1(0,L)} \\
&\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + \int_0^t \left\| e^{i\,(t-\tau)\,\Delta}\, f(u)(\tau) \right\|_{H_0^1(0,L)} d\tau \\
&\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + \left(\int_0^T d\tau \right)^{1/2} \left(\int_0^T \left\| e^{i\,(t-\tau)\,\Delta}\, f(u)(\tau) \right\|^2_{H_0^1(0,L)} d\tau \right)^{1/2} \\
&\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + T^{1/2} \int_0^T \|e^{i\,(t-\tau)\,\Delta}\, f(u)\|_{L^2(0,T;\, H_0^1(0,L))}\, ds \\
&\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + \frac{a_1\,T}{2^{1/2}\,a_3} \int_0^T \|f(u)\|_{L^2(0,L)}\, dt \\
&\leq \|e^{t\,A}\,u_0\|_{H_0^1(0,L)} + \frac{a_1\,\widetilde{C}\,T}{2^{1/2}\,a_3}\, \|u\|^2_{X_T},
\end{aligned}
\tag{3.19}
$$

where in the last inequality, we used again the Lemma 3.5.

Hence, employing the smoothing effect given in (3.16), we have

$$
\begin{aligned}
\|\Psi(u)\|_{L^2(0,T;\, H_0^1(0,L))} &\leq \|e^{t\,A}\,u_0\|_{L^2(0,T;\, H_0^1(0,L))} + \frac{a_1\,\widetilde{C}\,T}{2^{1/2}\,a_3}\, \|u\|^2_{X_T} \\
&\leq \|u_0\|_{L^2(0,L)} + \frac{a_1\,\widetilde{C}\,T}{2^{1/2}\,a_3}\, \|u\|^2_{X_T}
\end{aligned}
\tag{3.20}
$$

On the other hand, from (3.4), the Poincaré's inequality and the conservation of the Schrödinger flow given in (3.17), we get

$$
\begin{aligned}
\|\Psi(u)\|_{L^2(0,L)} &\leq \|e^{t\,A}\,u_0\|_{L^2(0,L)} + \left\| \int_0^t e^{i\,(t-\tau)\,\Delta}\, f(u)(\tau)\, d\tau \right\|_{L^2(0,L)} \\
&\leq \|u_0\|_{L^2(0,L)} + C \int_0^t \left\| f(u) \right\|_{H_0^1(0,L)} d\tau \\
&= \|u_0\|_{L^2(0,L)} + C \int_0^t \left\| e^{i\,(t-\tau)\,\Delta}\, f(u) \right\|_{H_0^1(0,L)} d\tau\,.
\end{aligned}
\tag{3.21}
$$

Employing the Hölder's inequality and (3.16), from (3.21), we have

$$
\begin{aligned}
\|\Psi(u)\|_{L^2(0,L)} &\leq \|u_0\|_{L^2(0,L)} + C \left(\int_0^T d\tau \right)^{1/2} \left(\int_0^T \left\| e^{i\,(t-\tau)\,\Delta}\, f(u)(\tau) \right\|^2_{H_0^1(0,L)} d\tau \right)^{1/2} \\
&\leq \|u_0\|_{L^2(0,L)} + \int_0^{T^{1/2}} \|e^{i\,(t-\tau)\,\Delta}\, f(u)\|_{L^2(0,T;\, H_0^1(0,L))}\, ds \\
&\leq \|u_0\|_{L^2(0,L)} + \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3} \int_0^T \|f(u)\|_{L^2(0,L)}\, dt \\
&\leq \|u_0\|_{L^2(0,L)} + \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\, \widetilde{C} \|u\|^2_{X_T}\,.
\end{aligned}
\tag{3.22}
$$

Thus,

$$\|\Psi(u)\|_{C([0,T];\,L^2(0,L))} \leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\widetilde{C}\|u\|^2_{X_T}\,. \tag{3.23}$$

From (3.19) and (3.23), we infer

$$\begin{aligned} \|\psi(u)\|_{X_T} &\leq \|u_0\|_{L^2(0,L)} + \frac{2\,\widetilde{C}\,a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u\|^2_{X_T} \\ &\leq \|u_0\|_{L^2(0,L)} + \frac{2\,\widetilde{C}\,a_1\,T^{1/2}\,R^2}{2^{1/2}\,a_3}\,. \end{aligned} \tag{3.24}$$

Our intention is to choose T small enough so that $\|u_0\|_{L^2(0,L)} + \frac{2\,\widetilde{C}\,a_1\,T^{1/2}\,R^2}{2^{1/2}\,a_3} < R$. For example, choose $R = 2\,\|u_0\|_{L^2(0,L)}$ and T small enough. Choosing T even smaller, that is,

$$\frac{2\,\widetilde{C}\,a_1\,T^{1/2}\,\|u_0\|_{L^2(0,L)}}{2^{1/2}\,a_3} < 1$$

so that we concluded that Ψ maps X_T into X_T.

Now, we shall prove that Ψ is a contraction in B_R, that is, there exists $\alpha \in\ (0,1)$ such that

$$\|\Psi(u) - \Psi(v)\|_{X_T} \leq\ \alpha\,\|u-v\|_{X_T};\ \forall\,u,v\ \in\ B_R. \tag{3.25}$$

Indeed, let $u, v \in\ B_R$. From (3.14), (3.16), we have

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{H^1_0(0,L)} &\leq \left\| \int_0^t \mathrm{e}^{i\,(t-\tau)\,\Delta}\,[f(u)-f(v)](\tau)\,d\tau \right\|_{H^1_0(0,L)} \\ &\leq \int_0^t \left\|\mathrm{e}^{i\,(t-\tau)\,\Delta}\,[f(u)-f(v)](\tau)\right\|_{H^1_0(0,L)} d\tau \\ &= \int_0^T \|\mathrm{e}^{i\,(t-\tau)\,\Delta}\,[f(u)-f(v)]\|_{L^2(0,T;\,H^1_0(0,L))}\,ds \\ &\leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\int_0^T \|f(u)-f(v)\|_{L^2(0,L)}\,d\tau \\ &\leq T^{1/2}\,C\,\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\,\|u-v\|_{X_T}\,. \end{aligned} \tag{3.26}$$

Hence,

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{L^2(0,T;\,H^1_0(0,L))} &\leq C\,T\,\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\,\|u-v\|_{X_T} \\ &\leq 2\,R^2\,C\,T\,\|u-v\|_{X_T}\,. \end{aligned} \tag{3.27}$$

Now, having in mind the same computations used in obtaining (3.22), we infer

$$\|\Psi(u) - \Psi(v)\|_{L^2(0,L)} \leq \frac{a_1\,C\,T^{1/2}}{2^{1/2}\,a_3}\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\,\|u-v\|_{X_T}\,. \tag{3.28}$$

Therefore,

$$\begin{aligned} \|\Psi(u) - \Psi(v)\|_{C([0,T];\,L^2(0,L))} &\leq \frac{a_1\,C\,T^{1/2}}{2^{1/2}\,a_3}\left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\,\|u-v\|_{X_T} \\ &\leq 2\,R^2\,C\,T^{1/2}\,\|u-v\|_{X_T}\,. \end{aligned} \tag{3.29}$$

Combining (3.27) and (3.29), we have

$$\|\Psi(u)-\Psi(v)\|_{X_T} \leq 2\,R^2\,C\,T^{1/2}\,\|u-v\|_{X_T}$$

The above estimate implies that if $2\,R^2\,C\,T^{1/2} < 1$, we obtain the desired as stated in (3.25). Therefore, we proved the local existence of mild solutions in X_T to (1.1). The uniqueness can be shown in the standard way using using Gronwall's inequality.

Remark 3.7. *Since* $u \in X_T$, *from* (3.5)*, we have that* $f(u) \in L^1(0,T;\, L^2(0,L))$. *Then, due to Pazy [*[36]*, Theorem 1.7, page 108], for every* $T' > T$, *the mild solution* u *is the uniform limit of regular solutions of* (1.1) *on* $[0,T']$. *From this result, the identity of energy* (2.1) *is guaranteed for mild solution* $u \in X_T$ *on* $[0,T']$ *and we can repeat the procedure used in Lemma 3.3 by density arguments.*

Due to Remark 3.7, multiplying the first equation of (1.1) by $\bar{u}$, integrating in $x \in (0,L)$, taking into account the boundary conditions and looking the real parts, we infer

$$\frac{1}{2}\,\frac{d}{dt}E(t) = -\frac{a_3}{2}\,|u_x(0,t)|^2 - \int_0^L a(x)|u(x,t)|^2\,dx \tag{3.30}$$

and, therefore, the energy $E(t)$ is non increasing function of the time variable t.

Integrating over $t \in [0,T]$, it follows that

$$\|u\|_{C([0,T];\,L^2(0,L))} \leq \|u_0\|_{L^2(0,L)}\,. \tag{3.31}$$

3.2.2. *Global Solutions.* Before to prove the existence of the global solutions of the problem (1.1), let's prove a useful estimate:

Lemma 3.8 (Smoothing effect)**.** *Let* $u \in X_T$ *a mild solution to the problem* (1.1). *Then,*

$$\|u\|_{L^2(0,T;\,H_0^1(\Omega))} \leq \frac{2^{1/2}\,T^{1/2}\,a_1}{a_3}\,\|u_0\|^2_{L^2(0,L)} + \frac{(3\,|a_4|+2\,|a_5|)^2}{4}\,\|u_0\|^6_{L^2(0,L)}\,. \tag{3.32}$$

Proof. First of all, having in mind Remark 1.2, let's work with the equivalent problem (1.5). So, combining the techniques used in (3.5) – (3.10), we get

$$\|u\|_{L^2(0,T;\,H_0^1(0,L))} \leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|_{L^2(0,L)} + |W|, \tag{3.33}$$

where

$$\begin{aligned} W &:= (2\,a_4+a_5)\ \mathrm{Re}\int_0^T\!\!\int_0^L x\,|u|^2\,u_x\,\bar{u}\,dx\,dt + (a_4+a_5)\ \mathrm{Re}\int_0^T\!\!\int_0^L x\,u^2\,\bar{u}_x\,\bar{u}\,dx\,dt \\ &:= W_1 + W_2\,. \end{aligned}$$

Integrating by parts, taking into account that $\mathrm{Re}(a\,\bar{b}) = \mathrm{Re}(\bar{a}\,b)$, for all $a, b \in \mathbb{C}$, the identity $\frac{1}{4}\,\left(|u|^4\right)_x = |u|^2\,\mathrm{Re}(u\,\bar{u}_x)$ and the boundary conditions, we have

$$
\begin{aligned}
W_1 &= (2\,a_4 + a_5)\ \mathrm{Re}\int_0^T\int_0^L x\,|u|^2\,u\,\bar{u}_x\,dx\,dt \\
&= (2\,a_4 + a_5)\int_0^T\int_0^L x\,\frac{1}{4}\,\left(|u|^4\right)_x\,dx\,dt \\
&= \frac{1}{4}\,(2\,a_4 + a_5)\left[\cancelto{0}{x\,|u|^4\Big|_0^L} - \int_0^T\int_0^L |u|^4\,dx\,dt\right] \\
&= -\frac{1}{4}\,(2\,a_4 + a_5)\int_0^T \|u(t)\|^4_{L^4(0,L)}\,dt\,.
\end{aligned}
\tag{3.34}
$$

Analogous computations give us

$$
\begin{aligned}
W_2 &\leq (|a_4| + |a_5|)\ \mathrm{Re}\int_0^T\int_0^L x\,|u|^2\,u\,\bar{u}_x\,dx\,dt \\
&= -\frac{1}{4}\,(|a_4| + |a_5|)\int_0^T \|u(t)\|^4_{L^4(0,L)}\,dt\,.
\end{aligned}
\tag{3.35}
$$

From (3.33) – (3.35), it results that

$$
\|u\|_{L^2(0,T;\,H_0^1(0,L))} \leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|_{L^2(0,L)} + \frac{1}{2}\,(3\,|a_4| + 2\,|a_5|)\int_0^T \|u(t)\|^4_{L^4(0,L)}\,dt\,.
\tag{3.36}
$$

In this moment, we appeal to the Gagliardo - Nirenberg inequality in one dimensional domains:

Lemma 3.9 (Gagliardo - Nirenberg inequality)**.** *Let q, r be any real numbers satisfying $1 \leqslant q \leqslant p \leqslant \infty$ and let j and m non-negative integers such that $j \leqslant m$. Then,*

$$
\|\partial^j\,u\|_{L^p(0,L)} \leqslant C\,\|\partial^m\,u\|^a_{L^r(0,L)}\,\|u\|^{1-a}_{L^q(0,L)}
$$

where $\frac{1}{p} = j + a\,\left(\frac{1}{r} - m\right) + \frac{1-a}{q}$ for all a in the interval $\frac{j}{m} \leqslant a \leqslant 1$ and M is a positive constant depending only on m, j, q, r and a.

Employing the Lemma 3.9 with $p = 4, j = 0, m = 1$ and $r = q = 2$, jointly with (3.31), we obtain

$$
\|u\|_{L^4(0,L)} \leq C\,\|u\|^{1/4}_{H_0^1(0,L)}\,\|u\|^{3/4}_{L^2(0,L)}\,.
\tag{3.37}
$$

Hence, from (3.36) and (3.37) the inequality $ab \leq \frac{1}{4\varepsilon}a^2 + \varepsilon\, b^2$, for all $\varepsilon > 0$, it follows that

$$\begin{aligned}\|u\|_{L^2(0,T;\,H_0^1(0,L))} &\leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|_{L^2(0,L)} + \frac{1}{2}\,(3\,|a_4| + 2\,|a_5|)\int_0^T \|u(t)\|_{H_0^1(0,L)}\,\|u(t)\|^3_{L^2(0,L)}\,dt \\ &\leq \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|_{L^2(0,L)} + \frac{(3\,|a_4| + 2\,a_5|)\;\varepsilon}{2}\int_0^T \|u(t)\|^2_{H_0^1(0,L)}\,dt \\ &+ \frac{3\,|a_4| + 2\,|a_5|}{8\,\varepsilon}\int_0^T \|u(t)\|^6_{L^2(0,L)}\,dt\,. \end{aligned} \tag{3.38}$$

Taking $\varepsilon = (3\,a_4 + 2\,a_5)^{-1}$, from (3.31) and (3.38), one gets

$$\begin{aligned}\|u\|_{L^2(0,T;\,H_0^1(0,L))} &\leq \frac{2^{1/2}\,T^{1/2}\,a_1}{a_3}\,\|u_0\|_{L^2(0,L)} + \frac{(3\,|a_4| + 2\,|a_5|)^2}{4}\int_0^T \|u(t)\|^6_{L^2(0,L)}\,dt \\ &\leq \frac{2^{1/2}\,T^{1/2}\,a_1}{a_3}\,\|u_0\|^2_{L^2(0,L)} + \frac{(3\,|a_4| + 2\,|a_5|)^2}{4}\,\|u_0\|^6_{L^2(0,L)}\,,\end{aligned}$$

which proves the desired. □

Now, let's prove the existence of global solutions. From previous subsection, we can extend the solution u to the maximal interval of existence $0 < t < T_{\max}$. Suppose that $T_{\max} < \infty$. Then, Combining (3.32) and (3.31), we obtain

$$\|u\|_{X_T} \leq \|u_0\|_{L^2(0,L)} + \frac{a_1\,T^{1/2}}{2^{1/2}\,a_3}\,\|u_0\|^2_{L^2(0,L)} + \frac{(3\,a_4 + 2\,a_5)^2\;T}{8}\,\|u_0\|^6_{L^2(0,L)}, \tag{3.39}$$

and we concluded that $T_{\max} = \infty$. Therefore, Theorem 2.1 is proved.

4. STABILIZATION

In the present section, we shall obtain the exponential decay in L^2- level of the problem (1.1). Our intention is to obtain an estimate of the energy in terms of the damping term plus a LOT (where LOT means a lower order term). From now on, we shall work with regular solutions. From density arguments the exponential stability remains true for mild solutions by density arguments (see Remark 3.7).

In the sequel, we proceed as the proof of Lemma 3.8. From (3.30), we have

$$\|u(t)\|^2_{L^2(0,L)} = \|u_0\|^2_{L^2(0,L)} - \frac{a_3}{2}\int_0^t |u_x(0,t)|^2\,dt - 2\int_0^t\int_0^L a(x)|u(x,t)|^2\,dx\,dt\,. \tag{4.1}$$

In the sequel, we proceed as the proof of Lemma 3.8. Multiplying the equation (1.1) by $(t-T)\,\bar{u}$, integrating over $x \in (0,L)$ and repeating the same procedure used in (3.5) – (3.10), we infer

$$\frac{d}{dt}\int_0^L (T-t)|u|^2\,dx + \int_0^L |u|^2\,dx + a_3\,(T-t)\,|u_x(0,t)|^2 + 2\int_0^L (T-t)\,a(x)\,|u|^2\,dx = 0\,. \tag{4.2}$$

Integrating over $t \in [0,T]$, we have

$$\int_0^L |u_0|^2\,dx = \frac{1}{T}\int_0^T\!\!\int_0^L |u|^2\,dx\,dt + \frac{a_3}{T}\int_0^T (T-t)\,|u_x(0,t)|^2\,dt + \frac{2}{T}\int_0^T\!\!\int_0^L (T-t)\,a(x)\,|u|^2\,dx\,dt. \tag{4.3}$$

Consequently,

$$\int_0^L |u_0|^2\,dx \le \frac{1}{T}\int_0^T\int_0^L |u|^2\,dx\,dt + a_3\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)\,|u|^2\,dx\,dt\,. \tag{4.4}$$

The next task is to absorb the LOT $:= \int_0^T\int_0^L |u|^2\,dx\,dt$. To this end, we have the following result:

Lemma 4.1. *Given $T > 0$ there exists $C = C(T) > 0$ such that every regular solution to problem (1.1) satisfies the inequality*

$$\int_0^T\int_0^L |u|^2\,dx\,dt \le C\left\{a_3\int_0^T |u_x(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)\,|u|^2\,dx\,dt\right\}, \tag{4.5}$$

provided the initial data are taken in bounded sets of $L^2(0,L)$.

Proof. We argue by contradiction. Assume that (4.5) does not hold. Then, there exists a sequence of initial data $\{u_{0,\mu}\}_{\mu\in\mathbb{N}} \in L^2(0,L)$, assumed to be taken in bounded sets of $L^2(0,L)$ and a sequence of regular solutions $\{u_\mu\}_{\mu\in\mathbb{N}}$ to problem (1.1), for all $\mu \in I\!N$, verifying

$$\lim_{\mu\to\infty}\frac{\int_0^T \|u_\mu(t)\|^2_{L^2(0,L)}\,dt}{a_3\,\int_0^T |u_{x,\mu}(0,t)|^2\,dt + 2\,\int_0^T\int_0^L a(x)\,|u_\mu|^2\,dx\,dt} = +\infty, \tag{4.6}$$

that is,

$$\lim_{\mu\to\infty}\frac{a_3\,\int_0^T |u_{x,\mu}(0,t)|^2\,dt + 2\,\int_0^T\int_0^L a(x)\,|u_\mu|^2\,dx\,dt}{\int_0^T \|u_\mu(t)\|^2_{L^2(0,L)}\,dt} = 0\,. \tag{4.7}$$

From (3.30), we know that the energy is a non increasing function on the parameter t, thus,

$$\frac{1}{2}\,\|u_\mu(t)\|^2_{L^2(0,L)} = E_\mu(t) \le E_\mu(0) = \frac{1}{2}\,\|u_{0,\mu}\|^2_{L^2(0,L)} \le M,\ \forall t \ge\ 0\,. \tag{4.8}$$

Then, from (4.7) and taking into account (4.8), we infer

$$a_3\int_0^T |u_{x,\mu}(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)\,|u_\mu|^2\,dx\,dt \longrightarrow 0 \quad \text{in} \quad L^2(0,T;L^2(0,L)), \tag{4.9}$$

consequently, from the assumption on $a(x)$, namely, $a(x) \ge a_0$ a.e. in ω , we arrive

$$u_\mu \ \to\ 0 \quad \text{strongly in} \quad L^2(0,T;L^2(\omega))\,. \tag{4.10}$$

On the other hand, from (4.8), we infer,

$$u_\mu \ \to\ u \quad \text{weakly in} \quad L^2(0,T;L^2(0,L))\,. \tag{4.11}$$

Hence, from (4.10) and (4.11) and the uniqueness of weak limit in $L^2(0,T;L^2(0,L))$, we conclude

$$u = 0 \quad \text{in} \quad \omega\,\times\,(0,T). \tag{4.12}$$

Now, combining (3.32) and (4.8) , there exists a subsequence of $\{u_\mu\}_{\mu\in\mathbb{N}}$, still denoted by the same form and $u \in L^\infty(0,T;H_0^1(0,L))$ such that

$$\{u_\mu\}\ \text{ is bounded in } \ L^2(0,T;H_0^1(0,L)) \tag{4.13}$$

and

$$u_\mu \ \overset{\star}{\rightharpoonup}\ u \text{ weak star in } L^2(0,T;H_0^1(0,L))\,. \tag{4.14}$$

On the other hand, again employing Lemma (3.9) for $p=6, j=0, m=1, r=q=2$, we have

$$\|u\|_{L^6(0,L)} \leq C\,\|u\|^{1/3}_{H^1_0(0,L)}\,\|u\|^{2/3}_{L^2(0,L)}\,. \tag{4.15}$$

Hence, employing Hölder inequality, (3.32) and (4.8), we have

$$\begin{aligned}
\|\,|u_\mu|^2\,u_\mu\,\|_{L^1(0,T;\,L^2(0,L))} &= \int_0^T \|u_\mu\|^3_{L^6(0,L)}\,dt \\
&\leq C\int_0^T \|u_\mu\|_{H^1_0(0,L)}\,\|u_\mu\|^2_{L^2(0,L)}\,dt \\
&\leq C\,T^{1/2}\,\|u_{\mu,0}\|^2_{L^2(0,L)}\int_0^T \|u_\mu\|^2_{H^1_0(0,L)}\;dt \\
&\leq C\,T^{1/2}\,M\,\|u_\mu\|_{L^2(0,T;\,H^1_0(0,L))} \\
&\leq C\,T^{1/2}\,M^2,
\end{aligned} \tag{4.16}$$

that is,

$$\{|u_\mu|^2\,u_\mu\} \quad \text{is bounded in } L^1(0,T;L^2(0,L)). \tag{4.17}$$

Consequently, there exists $\chi \in L^1(0,T;L^2(\Omega))$ such that

$$|u_\mu|^2\,u_\mu \rightharpoonup \chi \quad \text{weakly in } L^1(0,T;L^2(0,L)). \tag{4.18}$$

Employing Lemma (3.9) for $p=\infty, j=0, m=1, r=q=2$, we have

$$\|u\|_{L^\infty(0,L)} \leq C\,\|u\|^{1/2}_{H^1_0(0,L)}\,\|u\|^{1/2}_{L^2(0,L)}\,. \tag{4.19}$$

So, combining (4.19), (4.8) and (4.13), we have

$$\begin{aligned}
\|\,|u_\mu|^2\,u_{x,\mu}\|_{L^1(0,T;\,L^2(0,L))} &\leq \int_0^T \|u_\mu\|^2_{L^\infty(0,L)}\,\|u_\mu\|_{H^1_0(0,L)}\,dt \\
&\leq C\int_0^T \|u_\mu\|^2_{H^1_0(0,L)}\,\|u_\mu\|_{L^2(0,L)}\,dt \\
&\leq 2\sqrt{M}\,C\,\|u_\mu\|^2_{L^2(0,T;\,H^1_0(0,L))} \\
&\leq M_1,
\end{aligned} \tag{4.20}$$

hence,

$$\{|u_\mu|^2\,u_{x,\mu}\} \quad \text{is bounded in } L^1(0,T;L^2(0,L)). \tag{4.21}$$

Similar calculations give us

$$\{u^2_\mu\,\bar{u}_{x,\mu}\} \quad \text{is bounded in } L^1(0,T;L^2(0,L)). \tag{4.22}$$

On the other hand,

$$\begin{aligned}
|\langle u_{\mu,t},\varphi\rangle| \leq{}& a_1\|u_{\mu,xx}\|_{H^{-1}(0,L)}\,\|\varphi\|_{H^1_0(0,L)} + a_2\|\,|u_\mu|^2\,u_\mu\,\|_{L^2(0,L)}\,\|\varphi\|_{L^2(0,L)} \\
&+ a_3\,\|u_\mu\|_{H^1_0(0,L)}\,\|\varphi_{xx}\|_{L^2(0,L)} + a_4\|\,|u_\mu|^2\,u_{\mu,x}\,\|_{L^2(0,L)}\,\|\varphi\|_{L^2(0,L)} \\
&+ a_5\|\,u^2_\mu\,\bar{u}_{\mu,x}\,\|_{L^2(0,L)}\,\|\varphi\|_{L^2(0,L)},\ \forall\,\varphi\in\ H^2_0(0,L)\,.
\end{aligned} \tag{4.23}$$

From (4.13), (4.17), (4.21), (4.22) and (4.23), we infer

$$\{u_{t,\mu}\} \quad \text{is bounded in } L^2(0,T;H^{-2}(0,L)). \tag{4.24}$$

Now, making use of the embedded chain

$$H_0^1(0,L) \overset{c}{\hookrightarrow} L^2(0,L) \hookrightarrow H^{-2}(0,L), \tag{4.25}$$

it follows from the boundness (4.13) and (4.24) and employing Aubin-Lions Theorem (see [29], lemma 5.2 on page 57), that there exists a subsequence of $\{u_\mu\}$, still denote by the same form such that,

$$u_\mu \to u \quad \text{strongly in} \quad L^2(0,T;L^2(0,L))\,. \tag{4.26}$$

From (4.26), we have

$$|u_\mu|^2\, u_\mu \to |u|^2\, u \quad \text{a. e. in } (0,L)\,\times\,(0,T)\,. \tag{4.27}$$

So, combining (4.17) and (4.27), recalling Lions' lemma (see [29] lemma 1.3 on page 12), we have

$$|u_\mu|^2\, u_\mu \rightharpoonup |u|^2\, u \quad \text{in } L^1(0,T;L^2(0,L))\,. \tag{4.28}$$

As a consequence $\chi = |u|^2 u$ a.e. in $(0,L)$. Moreover, combining (4.14) and (4.26), we obtain

$$|u_\mu|^2\, u_{x,\mu} \rightharpoonup |u|^2\, u_x \quad \text{weakly in } L^2(0,T;L^2(0,L)), \tag{4.29}$$

$$u_\mu^2\, \bar{u}_{x,\mu} \rightharpoonup u^2\, \bar{u}_x \quad \text{weakly in } L^2(0,T;L^2(0,L))\,. \tag{4.30}$$

At this point we shall divide our proof into two cases: $u \neq 0$ and $u = 0$.

Case 1: $u \neq 0$

Initially, let us consider the sequence of problems:

$$\begin{cases} i\, u_{\mu,t} + a_1\, u_{\mu,xx} + i\left[a_3\, u_{\mu,xxx} + a(x)\, u_\mu\right] + f(u_\mu) = 0 & \text{in } (0,L)\,\times\,(0,\infty) \\ u_\mu(0,t) = u_\mu(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_{x,\mu}(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_\mu(x,0) = u_{0,\mu} & \text{in } (0,L), \end{cases} \tag{4.31}$$

where $f(u_\mu)$ is given in (1.9) .

From (4.14), (4.24), (4.28), (4.29) and (4.30), passing to the limit in (4.31), one gets

$$\begin{cases} i\, u_t + a_1\, u_{xx} + a_2\, |u|^2\, u + i\left[a_3\, u_{xxx} + a_4\, \left(|u|^2\, u\right)_x + a_5\, u\, \left(|u|^2\right)_x\right] = 0 & \text{in } \mathcal{D}'((0,L)\,\times\,(0,T)) \\ u(0,t) = u(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,t) = 0 & \text{in } \omega\,\times\,(0,L) \end{cases} \tag{4.32}$$

Due to Remark 1.2, the problem above can be rewritten as

$$\begin{cases} i\, u_t + a_1\, u_{xx} + f(u) + i\, a_3\, u_{xxx} = 0 & \text{in } \mathcal{D}'((0,L)\,\times\,(0,T)) \\ u(0,t) = u(L,t) = 0 & \text{for all } t \geqslant 0 \\ u_x(L,t) = 0 & \text{for all } t \geqslant 0 \\ u(x,t) = 0 & \text{in } \omega\,\times\,(0,L) \end{cases} \tag{4.33}$$

where $f(u)$ is given in (1.9).

As indicated in the introduction, the main difficulty to check the unique continuation property is the weak regularity of the solution under consideration, since to our best knowledge (see Carvajal, stated in the introduction), the existing results on unique continuation require that $u \in C([0,T];\, H^s(0,L)) \cap C^1([0,T];\, H^1(0,L))$ for $s \geq 4$. Therefore, the main task when checking unique continuation property is to show that the mild solution under consideration has, in fact, this property.

For the reader's convenience, we shall repeat (verbatim) the same arguments introduced by Bisognin et al., [8] and Linares and Pozoto, [23]. According to the structure of ω, we have that $u \equiv 0$ in $\{(0,\delta) \cup (L-\delta, L)\} \times (0,T)$. Now, let us introduce the extended function

(4.34) $$v(x,t) = \begin{cases} u(x,t), & \text{if } (x,t) \in (\delta, L-\delta) \times (0,T) \\ 0, & \text{if } (x,t) \in \{\mathbb{R} - (\delta, L-\delta)\} \times (0,T) \,. \end{cases}$$

Hence, $v = v(x,t)$ satisfies

(4.35)
$$\begin{cases} i\, v_t + a_1\, v_{xx} + a_2\, |v|^2\, v + i \left[a_3\, v_{xxx} + (2\, a_4 + a_5)\, |v|^2\, v_x + (a_4 + a_5)\, v^2\, \bar{v}_x\right] = 0 & \text{in } \mathbb{R} \times (0,T)) \\ v(x,t) = v_0(x) & \text{in } \mathbb{R}, \end{cases}$$

where,

(4.36) $$v_0(x) = \begin{cases} u_0(x), & \text{if } x \in (\delta, L-\delta) \\ 0, & \text{if } x \in \{\mathbb{R} - (\delta, L-\delta)\} \,. \end{cases}$$

If we consider $w(x,t) = v(x+t,t)$, then, w solves:

(4.37)
$$\begin{cases} i\, w_t + a_1\, w_{xx} + a_2\, |w|^2\, w + i \left[a_3\, w_{xxx} + (2\, a_4 + a_5)\, |w|^2\, w_x + (a_4 + a_5)\, w^2\, \bar{w}_x\right] = 0 & \text{in } \mathbb{R} \times (0,T)) \\ w(x,t) = v_0(x) & \text{in } \mathbb{R}, \end{cases}$$

Since v_0 has compact support and belongs to $L^2(\mathbb{R})$, we infer

$$\int_{\mathbb{R}} v_0^2(x)\, \mathrm{e}^{2\,\lambda\, x}\, dx < \infty, \quad \forall\, \lambda > 0.$$

Thus, by regularizing properties proved by Kato [[24], Theorem 2.1], $w \in C^\infty((0,L) \times (0,T))$ and, therefore, v is smooth as well. So, v possesses the required regularity to apply the unique continuation property given in Theorem 1.1. Moreover, the assumption (1.6) is verified observing

$$\operatorname{supp} v(\cdot,t) \subset (-\infty, L-\delta),\ t = 0, T.$$

Therefore, employing Theorem 1.1, we have that $v \equiv 0$, consequently, $u \equiv 0,\ x \in (0,L),\ t \in (0,T)$, which is a contradiction.

Case 2: $u = 0$

Now, we denote:

(4.38) $$c_\mu = \|u_\mu\|_{L^2(0,T;L^2(0,L))},\ \widetilde{u}_\mu = \frac{u_\mu}{c_\mu},$$

Dividing (4.31) by c_μ we obtain

$$\begin{cases} i\,\widetilde{u}_{\mu,t} + a_1\,\widetilde{u}_{\mu,xx} + g(\widetilde{u}_\mu) + i\left[a_3\,\widetilde{u}_{\mu,xxx} + a(x)\,\widetilde{u}_\mu\right] = 0 & \text{in } (0,L)\,\times\,(0,\infty) \\ \widetilde{u}_\mu(0,t) = \widetilde{u}_\mu(L,t) = 0 & \text{for all } t \geqslant 0 \\ \widetilde{u}_{x,\mu}(L,t) = 0 & \text{for all } t \geqslant 0 \\ \widetilde{u}_\mu(x,0) = \widetilde{u}_{0,\mu} & \text{in } (0,L) \end{cases} \tag{4.39}$$

where $g(\widetilde{u}_\mu) = a_2|u_\mu|^2\,\widetilde{u}_\mu + i\,a_4\,|u_\mu|^2\,\widetilde{u}_{\mu,x} + i\,a_5\,u_\mu^2\,\overline{\widetilde{u}}_{\mu,x}$.

Thus, taking (4.6) into account, we have

$$\lim_{\mu\to\infty} \frac{\int_0^T \|\widetilde{u}_\mu(t)\|^2_{L^2(0,L)}\,dt}{a_3\,\int_0^T |\widetilde{u}_{x,\mu}(0,t)|^2\,dt + 2\,\int_0^T\int_0^L a(x)\,|\widetilde{u}_\mu|^2\,dx\,dt} = +\infty \tag{4.40}$$

and from (4.38), we know that

$$\|\widetilde{u}_\mu\|^2_{L^2(0,T;L^2(0,L))} = 1\,. \tag{4.41}$$

Now, recalling (4.4), we observe

$$\int_0^L |\widetilde{u}_0|^2\,dx \le \frac{1}{T}\int_0^T\int_0^L |\widetilde{u}|^2\,dx\,dt + a_3\int_0^T |\widetilde{u}_{x,\mu}(0,t)|^2\,dt + 2\int_0^T\int_0^L a(x)\,|\widetilde{u}_\mu|^2\,dx\,dt\,. \tag{4.42}$$

Hence, from (4.40), (4.41) jointly with (4.42), we guarantee the existence of a constant $M_3 > 0$ such that

$$\|\widetilde{u}_{0,\mu}\|^2_{L^2(0,L)} \le \frac{1}{T} + M_2 := M_3\,. \tag{4.43}$$

From (4.8) and (4.43), it results that

$$\|\widetilde{u}_\mu(t)\|_{L^2(0,L)} \le \|\widetilde{u}_{0,\mu}\|_{L^2(0,L)} \le \sqrt{M_3}. \tag{4.44}$$

Moreover, combining (3.32) and (4.43), we obtain

$$\{\widetilde{u}_\mu\} \text{ is bounded in } \; L^2(0,T;H_0^1(0,L))\,. \tag{4.45}$$

Repeating the same arguments used in done in (4.8) – (4.23), we infer

$$\widetilde{u} = 0 \quad \text{a.e. in} \quad \omega\,\times\,(0,T) \tag{4.46}$$

$$\{\widetilde{u}_{t,\mu}\} \quad \text{is bounded in } L^2(0,T;H^{-2}(0,L)). \tag{4.47}$$

So, from (4.25), (4.45) and (4.47), employing Aubin – Lions theorem, there exists a subsequence of $\{\widetilde{u}_\mu\}$, still from now on will be denoted by the same notation, such that,

$$\widetilde{u}_\mu \longrightarrow \widetilde{u} \;\text{ strongly in }\; L^2(0,T,L^2(0,L))\,. \tag{4.48}$$

Now, recalling the fact that $u = 0$, from convergence given in (4.26), we obtain

$$u_\mu \to 0 \text{ strongly in } L^2(0,T;L^2(0,L)) \text{ and } c_\mu \to 0 \quad \text{when} \quad \mu \to +\infty. \tag{4.49}$$

So, from (4.38), (4.44), (4.45), (4.49) and (4.15), we get

$$\begin{aligned}
\| \, |u_\mu|^2 \, \widetilde{u}_\mu \, \|_{L^1(0,T;\, L^2(0,L))} &= c_\mu^2 \, \| \, |\widetilde{u}_\mu|^2 \, \widetilde{u}_\mu \, \|_{L^1(0,T;\, L^2(0,L))} \\
&= c_\mu^2 \int_0^T \|\widetilde{u}_\mu\|^3_{L^6(0,L)} \, dt \\
&\leq C \, c_\mu^2 \int_0^T \|\widetilde{u}_\mu\|_{H_0^1(0,L)} \, \|\widetilde{u}_\mu\|^2_{L^2(0,L)} \, dt \\
&\leq C \, T^{1/2} \, c_\mu^2 \, \|\widetilde{u}_{\mu,0}\|^2_{L^2(0,L)} \int_0^T \|\widetilde{u}_\mu\|^2_{H_0^1(0,L)} \, dt \\
&\leq C \, T^{1/2} \, c_\mu^2 \, M_3 \, \|\widetilde{u}\|_{L^2(0,T;\, H_0^1(0,L))} \\
&\leq C \, T^{1/2} \, C(M_3) \, c_\mu^2 \longrightarrow 0, \text{ when } \mu \to +\infty.
\end{aligned} \tag{4.50}$$

Performing similar computations done in (4.50), now, having in mind (4.19), we get

$$\| \, |u_\mu|^2 \, \widetilde{u}_{x,\mu} \, \|_{L^1(0,T;\, L^2(0,L))} \longrightarrow 0 \quad \text{in } L^1(0,T;\, L^2(0,L)), \tag{4.51}$$

$$\| \, u_\mu^2 \, \widetilde{\overline{u}}_{x,\mu} \, \|_{L^1(0,T;\, L^2(0,L))} \longrightarrow 0 \quad \text{in } L^1(0,T;\, L^2(0,L))\,. \tag{4.52}$$

Passing to the limit in (4.39) when $\mu \to +\infty$, taking (4.45), (4.46), (4.50), (4.51) and (4.52) into account, we arrive at

$$\begin{cases} i\,\widetilde{u}_t + a_1\,\widetilde{u}_{xx} + i\,a_3\,\widetilde{u}_{xxx} = 0 & \text{in } (0,L) \times (0,\infty) \\ \widetilde{u} = 0 & \text{a. e. in } \omega \times (0,T)\,. \end{cases} \tag{4.53}$$

Applying the Holmgrem's uniqueness theorem we conclude that

$$\widetilde{u} = 0 \quad \text{a. e. in } (0,L) \times (0,T) \tag{4.54}$$

and this contradicts (4.41) and (4.48). □

We observe that taking (4.4), the fact that the energy $E(t)$ is non increasing function of the time variable t into account and considering Lemma 4.1, we obtain the desired inequality, namely,

$$E(T) \leq E(0) \leq C \left\{ \frac{a_3}{2} \int_0^T |u_x(0,t)|^2 \, dt + 2 \int_0^T \int_0^L a(x) |u(x,t)|^2 \, dx \, dt \right\}, \tag{4.55}$$

where C is a positive constant. We observe that taking (4.1) into account, we have

$$E(T) = E(0) - \frac{a_3}{2} \int_0^T |u_x(0,t)|^2 \, dt - 2 \int_0^T \int_0^L a(x) |u(x,t)|^2 \, dx \, dt \tag{4.56}$$

Now, combining (4.55) and (4.56), we obtain,

$$\begin{aligned}
E(T) &\leq C \left\{ \frac{a_3}{2} \int_0^T |u_x(0,t)|^2 \, dt + 2 \int_0^T \int_0^L a(x) |u(x,t)|^2 \, dx \, dt \right\} \\
&= C \, \{E(0) - E(T)\}\,.
\end{aligned} \tag{4.57}$$

Therefore,

$$E(T) \;\leq\; \left(\frac{C}{1+C} \right) E(0). \tag{4.58}$$

Repeating the procedure for nT, $n \in I\!N$, we deduce

$$E(nT) \leq \frac{1}{(1+\hat{C})^n} E(0),$$

for all $T \geq T_0$.

Let us consider, now, $t \geq T_0$, then $t = nT_0 + r$, $0 \leq r < T_0$. Thus,

$$E(t) \leq E(t-r) = E(nT_0) \leq \frac{1}{(1+\hat{C})^n} E(0) = \frac{1}{(1+\hat{C})^{\frac{t-r}{T_0}}} E(0).$$

Setting $C_0 = \mathrm{e}^{\frac{r}{T_0}\ln(1+\hat{C})}$ and $\lambda_0 = \frac{\ln(1+\hat{C})}{T_0} > 0$, we obtain

$$E(t) \leq C_0 \, \mathrm{e}^{-\lambda_0 t} E(0); \;\; \forall t \geq T_0, \tag{4.59}$$

which proves the exponential decay for regular solutions to problem (1.1). Therefore, the exponential decay (4.59) remains true for mild solutions by density arguments and Theorem 2.2 is proved.

5. Finite Difference Method Approximation and Main Result

5.1. **Description of the Numerical Scheme.** For the sake of the following analysis, and for a given $M \in \mathbb{N}$, we will introduce the vector space

$$X_M := \left\{ u = [u_0 \, u_1 \, \dots \, u_M]^T \in \mathbb{C}^{M+1} : u_0 = u_{M-1} = u_M = 0 \right\}$$

Let us introduce the classical finite differences operators for complex-valued arrays:

$$\begin{aligned}
\left[\boldsymbol{D}^+ u\right]_j &:= \frac{u_{j+1} - u_j}{\Delta x} \\
\left[\boldsymbol{D}^- u\right]_j &:= \frac{u_j - u_{j-1}}{\Delta x} \\
\boldsymbol{D} u &:= \frac{1}{2}\Big(\boldsymbol{D}^+ u + \boldsymbol{D}^- u\Big) \\
\boldsymbol{D}^2 u &:= \boldsymbol{D}^+ \boldsymbol{D}^- u \\
\boldsymbol{D}^3 u &:= \boldsymbol{D}^+ \boldsymbol{D}^+ \boldsymbol{D}^- u
\end{aligned}$$

For $u, v \in X_M$, and $\Delta x < 1$, let us consider the discrete space $L^2(0, L)_\Delta$ of complex-valued vectors endowed with the inner product

$$(u, v)_2 := \sum_{j=1}^{M-1} u_j \overline{v}_j \Delta x \tag{5.60}$$

this induces a discrete version of the L^2 norm:

$$||u||_2^2 := (u, u)_2.$$

For $p \in [1, \infty)$, we can define the $L^p(0, L)_\Delta$ spaces in a similar fashion: we say that $u \in X_m$ is also in $L^p(0, L)_\Delta$ if

$$||u||_p := \left(\sum_{j=1}^{M-1} |u_j|^p \Delta x \right)^{\frac{1}{p}} < \infty.$$

We also say that $u \in L^\infty(0,L)_\Delta$ if

$$||u||_\infty := \max_{j\in[0,M-1]} |u_j| < \infty.$$

For $u, v \in X_M$, we will introduce the following inner product and their respective norm:

$$(u,v)_x := \sum_{j=1}^{M-1} j\Delta x^2 u_j \overline{v}_j, \qquad ||u||_x^2 := (u,u)_x \tag{5.61}$$

Finally, let us recall problem (1.1), and re-write the equation to solve as

$$i\, u_t + a_1\, u_{xx} + a_2\, |u|^2\, u + i\left[a_3\, u_{xxx} + a_4\, |u|^2\, u_x + (a_4+a_5)\, u\, \left(|u|^2\right)_x + a(x)\, u\right] = 0 \tag{5.62}$$

The following numerical scheme is a slight modification of the one proposed in [12]: for a given $u^0 \in X_M \cap L^2_{\Delta x}(0,L)$, and for $u^{n+\frac{1}{2}} := \frac{1}{2}(u^{n+1}+u^n)$, then $u^{n+1} \in X_M, n \in \mathbb{N}$, approximated solution of (1.1) at the time $t_{n+1} = (n+1)\Delta t,\ \Delta t < 1$, can be calculated using the following scheme:

$$\begin{aligned} i\boldsymbol{D}_t u^n + a_1 \boldsymbol{D}^2 u^{n+\frac{1}{2}} + a_2 |u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}} + ia_3 \boldsymbol{D}^3 u^{n+\frac{1}{2}} + ia_4 F_{a_4}(u^{n+1}) \\ + i(a_4+a_5) F_{a_4+a_5}(u^{n+1}) + iau^{n+\frac{1}{2}} = 0 \\ u^0 \in X_M \cap L^2_{\Delta x}(0,L) \text{ given.} \end{aligned} \tag{5.63}$$

where

$$F_{a_4}(u^p) := \frac{1}{2}\Big|\frac{u^p+u^n}{2}\Big|^2 \boldsymbol{D}\Big(\frac{u^p+u^n}{2}\Big) + \frac{1}{4}\boldsymbol{D}\left(\Big|\frac{u^p+u^n}{2}\Big|^2 \frac{u^p+u^n}{2}\right) - \frac{1}{4}\Big(\frac{u^p+u^n}{2}\Big)^2 \overline{\boldsymbol{D}\Big(\frac{u^p+u^n}{2}\Big)} \tag{5.64}$$

$$F_{a_4+a_5}(u^p) := \frac{u^p+u^n}{2}\boldsymbol{D}\left(\Big|\frac{u^p+u^n}{2}\Big|^2\right) \tag{5.65}$$

$$a \in \mathbb{R}^{M+1} : a_j = a(x_j), \quad x_j = x_0 + j\Delta x, \quad j = 0,1,\ldots,M.$$

The reason behind the definitions (5.64) and (5.65) lies in the scheme proposed in [12], which aimed to the preservation of the $L^2(0,L)_\Delta$ norm when $a(x) = 0, \forall x \in [0,L]$ and a centered finite difference approximation is considered for the third derivative. In the present scheme (5.63), however, a decentered approximation is used in order to obtain an L^2 estimate for $\boldsymbol{D}^2 u$. This fact will help us to describe the decay of the numerical energy, defined as

$$E^n := \frac{1}{2}||u^n||_2^2. \tag{5.66}$$

To this end, we will need some lemmas.

Lemma 5.1. *For* $a, b \in \mathbb{C}$*, for* $u, v \in X_M$*, and for* $j \in [0, M] \subset \mathbb{N}$ *we have*

$$Re\Big(b(\bar{b} - \bar{a})\Big) = \frac{1}{2}\Big(|b|^2 - |a|^2\Big) + \frac{1}{2}|b - a|^2 \tag{5.67}$$

$$Re\Big(a(\bar{b} - \bar{a})\Big) = \frac{1}{2}\Big(|b|^2 - |a|^2\Big) - \frac{1}{2}|b - a|^2 \tag{5.68}$$

$$D(u_j v_j) = u_{j+1}\frac{D^+ v_j}{2} + u_{j-1}\frac{D^- v_j}{2} + v_j D u_j \tag{5.69}$$

$$D^-(u_j v_j) = u_j D^- v_j + v_{j-1} D^- u_j \tag{5.70}$$

Proof. We will prove (5.67), (5.69) and (5.70). For (5.67), we have

$$\begin{aligned} b\Big(\bar{b} - \bar{a}\Big) &= |b|^2 - \bar{a}b \\ &= |b|^2 - \bar{a}\Big(b - a + a\Big) \\ &= |b|^2 - \bar{a}(b - a) - |a|^2 \\ &= |b|^2 - |a|^2 + (\bar{b} - \bar{a} - \bar{b})(b - a) \\ &= |b|^2 - |a|^2 - \bar{b}(b - a) + |b - a|^2 \end{aligned}$$

This, then, let us conclude (5.67). (5.68) can be proved using similar arguments. For (5.69), we have

$$\begin{aligned} D(u_j v_j) &= \frac{u_{j+1}v_{j+1} - u_{j-1}v_{j-1}}{2\Delta x} \\ &= \frac{1}{2\Delta x}\Big(u_{j+1} + u_{j+1}v_j - u_{j+1}v_j + u_{j-1}v_j - u_{j-1}v_j - u_{j-1}v_{j-1}\Big) \\ &= \frac{1}{2\Delta x}\Big(u_{j+1}(v_{j+1} - v_j) + u_{j-1}(v_j - v_{j-1}) + v_j(u_{j+1} - u_{j-1})\Big) \\ &= u_{j+1}\frac{D^+ v_j}{2} + u_{j-1}\frac{D^- v_j}{2} + v_j D u_j \end{aligned}$$

hence, (5.69) is proved. For (5.70),

$$\begin{aligned} D^+(u_j v_j) &= \frac{u_j v_j - u_{j-1}v_{j-1}}{\Delta x} \\ &= \frac{1}{\Delta x}\Big(u_j v_j + u_j v_{j-1} - u_j v_{j-1} - u_{j-1}v_{j-1}\Big) \\ &= \frac{1}{\Delta x}\Big(u_j(v_j - v_{j-1}) + v_{j-1}(u_j - u_{j-1})\Big) \\ &= u_j D^- v_j + v_{j-1} D^- u_j. \end{aligned}$$

Thus, (5.70) is proved, as well as the present lemma. □

A final lemma needs to be presented:

Lemma 5.2. *For $z, w \in X_M$, we have*

$$\left(\boldsymbol{D}^+ z, w\right)_x = -\left(z, \boldsymbol{D}^- w\right)_x + \Delta x\left(z, \boldsymbol{D}^- w\right)_2 - \left(z, w\right)_2 \tag{5.71}$$

$$\left(\boldsymbol{D}^- z, w\right)_x = -\left(z, \boldsymbol{D}^+ w\right)_x - \Delta x\left(z, \boldsymbol{D}^+ w\right)_2 - \left(z, w\right)_2 \tag{5.72}$$

$$\left(\boldsymbol{D} z, w\right)_x = -\left(z, \boldsymbol{D} w)_x + \frac{\Delta x}{2}(z, \boldsymbol{D}^- w\right)_2 - \frac{\Delta x}{2}\left(z, \boldsymbol{D}^+ w\right) - \left(z, w\right)_2 \tag{5.73}$$

$$Re\left(\boldsymbol{D}^+ z, z\right)_x = -\frac{1}{2}||z||_2^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+ z||_x^2 \tag{5.74}$$

$$Re\left(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- z, z\right)_x = -\frac{\Delta x}{2}|D^- z_1|^2 + \frac{3}{2}||\boldsymbol{D}^+ z||_2^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- z||_x^2 - \frac{\Delta x^2}{2}||\boldsymbol{D}^+\boldsymbol{D}^- z||_2^2 \tag{5.75}$$

Proof. Starting with (5.71), and using the definition (5.61), we have

$$\begin{aligned}
\left(\boldsymbol{D}^+ z, w\right)_x &= \sum_{j=1}^{M-1} j\Delta x^2 D^+ z_j \overline{w}_j \\
&= \sum_{j=1}^{M-1} \frac{1}{\Delta x}\Delta x^2 (j z_{j+1}\overline{w}_j - j z_j \overline{w}_j) \\
&= \sum_{j=2}^{M} \frac{1}{\Delta x}\Delta x^2 (j-1) z_j \overline{w}_{j-1} - \sum_{j=1}^{M-1} \frac{1}{\Delta x}\Delta x^2 j z_j \overline{w}_j \\
&= \frac{1}{\Delta x}\Delta x^2 (M-1) z_M \overline{w}_{M-1} + \sum_{j=1}^{M-1} \frac{1}{\Delta x}\Delta x^2 (j-1) z_j \overline{w}_{j-1} \\
&\quad - \frac{1}{\Delta x}\Delta x^2 (0) z_1 \overline{w}_0 - \sum_{j=1}^{M-1} \frac{1}{\Delta x}\Delta x^2 j z_j \overline{w}_j \\
&= \sum_{j=1}^{M-1} j\Delta x^2 z_j \frac{\overline{w}_{j-1} - \overline{w}_j}{\Delta x} - \sum_{j=1}^{M-1} \frac{1}{\Delta x}\Delta x^2 z_j \overline{w}_{j-1} \\
&= -\sum_{j=1}^{M-1} j\Delta x^2 z_j D^- \overline{w}_j - \sum_{j=1}^{M-1} \Delta x^2 z_j \frac{\overline{w}_{j-1} - \overline{w}_j}{\Delta x} - \sum_{j=1}^{M-1} z_j \overline{w}_j \Delta x \\
&= -\left(z, \boldsymbol{D}^- w\right)_x + \Delta x\left(z, \boldsymbol{D}^- w\right)_2 - \left(z, w\right)_2.
\end{aligned}$$

Thus, (5.71) is proved. (5.72) and (5.73) can both be demonstrated using similar arguments. For (5.74), and using (5.68), we have

$$
\begin{aligned}
Re\Big(\boldsymbol{D}^+z,z\Big)_x &= Re\left(\sum_{j=1}^{M-1} j\Delta x^2 \frac{z_{j+1}-z_j}{\Delta x}\overline{z}_j\right)\\
&= \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2\Big(|z_{j+1}|^2-|z_j|^2-|z_{j+1}-z_j|^2\Big)\\
&= \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2|z_{j+1}|^2 - \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2|z_j|^2 - \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2|z_{j+1}-z_j|^2\\
&= \sum_{j=2}^{M} \frac{1}{2\Delta x}(j-1)\Delta x^2|z_j|^2 - \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2|z_j|^2 - \sum_{j=1}^{M-1} \frac{1}{2\Delta x} j\Delta x^2|\Delta x D^+ z_j|^2\\
&= -\frac{1}{2}\sum_{j=2}^{M}|z_j|^2\Delta x - \frac{\Delta x}{2}|z_1|^2 - \frac{\Delta x}{2}\sum_{j=1}^{M-1} j\Delta x^2|D^+z_j|^2\\
&= -\frac{1}{2}||z||_2^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+z||_x^2.
\end{aligned}
$$

Hence, (5.74) is proved. Finally, in order to obtain (5.75), using the same reasoning as in the proof of (5.71) using $a=\boldsymbol{D}^+\boldsymbol{D}^-z$, and recalling that $z\in X_M$, we can write

$$
\begin{aligned}
\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^-z,z\Big)_x &= \Big(\boldsymbol{D}^+a,z\Big)_x\\
&= -\Big(a,\boldsymbol{D}^-z\Big)_x + \Delta x\Big(a,\boldsymbol{D}^-z\Big)_2 - \Big(a,z\Big)_2\\
&= -\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_x + \Delta x\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_2 - \Big(\boldsymbol{D}^+\boldsymbol{D}^-z,z\Big)_2\\
&= -\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_x + \Delta x\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_2 + ||\boldsymbol{D}^+z||_2^2 \qquad (5.76)
\end{aligned}
$$

Now we will extract the real part. Denoting $b:=\boldsymbol{D}^-z$, and using (5.74), we can re-write the first term in the right hand side of (5.76) as

$$
\begin{aligned}
Re\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_x &= Re\Big(\boldsymbol{D}^+b,b\Big)_x\\
&= -\frac{1}{2}||b||_2^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+b||_x^2\\
&= -\frac{1}{2}||\boldsymbol{D}^-z||_2^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^-z||_x^2
\end{aligned}
$$

Hence,

$$
Re\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^-z,z\Big)_x = \frac{1}{2}||\boldsymbol{D}^-z||_2^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^-z||_x^2 + \Delta x Re\Big(\boldsymbol{D}^+\boldsymbol{D}^-z,\boldsymbol{D}^-z\Big)_2 + ||\boldsymbol{D}^+z||_2^2.
$$

and because $||\boldsymbol{D}^+z||_2^2 = ||\boldsymbol{D}^-z||_2^2$, we have

$$Re\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- z, z\Big)_x = \frac{3}{2}||\boldsymbol{D}^+ z||_2^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- z||_x^2 + \Delta x Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- z, \boldsymbol{D}^- z\Big)_2 \tag{5.77}$$

Using again $b = \boldsymbol{D}^- z$, we can work the third term on the right hand side of (5.77):

$$\begin{aligned}
Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- z, \boldsymbol{D}^- z\Big)_2 &= Re\Big(\boldsymbol{D}^+ b, b\Big)_2 \\
&= Re\left(\sum_{j=1}^{M-1} \frac{b_{j+1} - b_j}{\Delta x} b_j \Delta x\right) \\
&= \sum_{j=1}^{M-1} \frac{1}{2\Delta x}\Big(|b_{j+1}|^2 - |b_j|^2 - |b_{j+1} - b_j|^2\Big)\Delta x \\
&= \frac{1}{2}\Big(|b_{M+1}|^2 - |b_1|^2\Big) - \sum_{j=1}^{M-1} \frac{1}{2\Delta x}|\Delta x D^+ b_j|^2 \Delta x \\
&= -\frac{|b_1|^2}{2} - \frac{\Delta x}{2}||\boldsymbol{D}^+ b||_2^2 \\
&= -\frac{|D^- z_1|^2}{2} - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- z||_2^2.
\end{aligned}$$

(5.75) can be then obtained combining this last result with (5.77).

□

Before presenting the next results, we will introduce some extension operators, presented already in [43], [38]; and originally, [29]. For $v \in X_M$ with $v = (v_j)_{j=0}^M$, for the space variable we define:

$$p_\Delta v_\Delta(x) = \begin{cases} \text{the continuous function, linear in each interval } [j\Delta x, (j+1)\Delta x] \\ \text{such that } p_\Delta v_\Delta(j\Delta x) = v_j, \quad j = 0, \dots, M \end{cases}$$

$$q_\Delta v_\Delta(x) = \begin{cases} \text{the step function, defined in each interval } \Big(\big(j - \frac{1}{2}\big)\Delta x, \big(j + \frac{1}{2}\big)\Delta x\Big) \cap (0, L) \\ \text{such that } q_\Delta v_\Delta(j\Delta x) = v_j, \quad j = 0, \dots, M \end{cases}$$

and for the time variable, we have

$$P_\Delta v_\Delta(x,t) = \begin{cases} \text{the continuous function, linear in each interval } [n\Delta t,(n+1)\Delta t] \\ \text{such that } P_\Delta v_\Delta(x,t_n) = p_\Delta v_\Delta^n(x), \quad n\in\mathbb{N}, x\in(0,L) \end{cases}$$

$$P_\Delta^{\frac{1}{2}} v_\Delta(x,t) = \begin{cases} \text{the continuous function, linear in each interval } [n\Delta t,(n+1)\Delta t] \\ \text{such that } P_\Delta^{\frac{1}{2}} v_\Delta(x,t_n) = \frac{1}{2}\Big(p_\Delta v_\Delta^n(x) + p_\Delta v_\Delta^{n+1}(x)\Big), \quad n\in\mathbb{N}, x\in(0,L) \end{cases}$$

$$Q_\Delta u_\Delta(x,t) = \begin{cases} \text{the step function, linear in each interval } [n\Delta t,(n+1)\Delta t] \\ \text{such that } Q_\Delta v_\Delta(x,t_n) = q_\Delta v_\Delta^n(x), \qquad t_n\le t\le t_{n+1}, n\in\mathbb{N}, x\in(0,L) \end{cases}$$

$$Q_\Delta^{\frac{1}{2}} u_\Delta(x,t) = \begin{cases} \text{the step function, linear in each interval } [n\Delta t,(n+1)\Delta t] \\ \text{such that } Q_\Delta^{\frac{1}{2}} v_\Delta(x,t_n) = \frac{1}{2}\Big(q_\Delta v_\Delta^n(x) + q_\Delta v_\Delta^{n+1}(x)\Big), \\ t_n\le t\le t_{n+1}, n\in\mathbb{N}, x\in(0,L) \end{cases}$$

With this, it is easy to see that

$$||Q_\Delta u_\Delta||^2_{L^2(0,T;L^2(0,L))} = \int_0^T\int_0^L |Q_\Delta u_\Delta(x,t)|^2 dxdt = \sum_{n=0}^N\sum_{j=0}^{M-1}|u_j^n|^2\Delta x\Delta t = \sum_{n=0}^N ||u^n||_2^2\Delta t$$

$$||p_\Delta u_\Delta||^2_{H_0^1(0,L)} = \int_0^L |(p_\Delta u_\Delta)_x|^2dx = \sum_{j=0}^{M-1}\left|\frac{u_{j+1}-u_j}{\Delta x}\right|^2\Delta x$$

In order to prove the main results of this section, we will present and prove two lemmas:

Lemma 5.3. *For $u\in X_M\cap L^\infty(0,L)_\Delta$, we have*

$$||q_\Delta u_\Delta||^2_{L^\infty(0,L)} \le 2||q_\Delta u_\Delta||_{L^2(0,L)}||p_\Delta u_\Delta||_{H_0^1(0,L)} \tag{5.78}$$

$$||q_\Delta u_\Delta||^4_{L^4(0,L)} \le 2||q_\Delta u_\Delta||^3_{L^2(0,T)}||p_\Delta u_\Delta||_{H_0^1(0,L)} \tag{5.79}$$

$$||q_\Delta u_\Delta||_{L^6(0,L))} \le 2^{\frac{1}{3}}||q_\Delta u_\Delta||^{\frac{2}{3}}_{L^2(0,L)}||p_\Delta u_\Delta||^{\frac{1}{3}}_{L^2(0,L)} \tag{5.80}$$

Proof. To prove (5.78), we will need the algebraic identity $(a^2-b^2)+(a-b)^2 = 2a(a-b)$ for any constants $a,b\in\mathbb{C}$. For $u_i\in u=[u_0\, u_1\, \ldots\, u_{M-1}\, u_M]^T\in\mathbb{C}^{M+1}$, we have:

$$\begin{aligned}
u_i^2 &= \frac{1}{2}\left[\sum_{j=1}^i (u_j^2-u_{j-1}^2) + \sum_{j=0}^{i-1}(u_{j+1}^2-u_j^2)\right] \\
&= \frac{1}{2}\left[\sum_{j=1}^i 2u_j(u_j-u_{j-1}) - (u_j-u_{j-1})^2\right] - \frac{1}{2}\left[\sum_{j=0}^{i-1} 2u_j(u_j-u_{j+1}) - (u_{j+1}-u_j)^2\right] \\
&= \sum_{j=1}^i\left[\Delta x\, u_jD^-u_j - \frac{\Delta x^2}{2}(D^-u_j)^2\right] + \sum_{j=0}^{i-1}\left[\Delta x\, u_jD^+u_j + \frac{\Delta x^2}{2}(D^+u_j)^2\right] \\
&= \sum_{j=1}^i \Delta x\, u_jD^-u_j + \sum_{j=0}^{i-1}\Delta x\, u_jD^+u_j + \sum_{j=0}^{i-1}\frac{\Delta x^2}{2}(D^+u_j)^2 - \sum_{j=0}^{i-1}\frac{\Delta x^2}{2}(D^+u_j)^2.
\end{aligned}$$

Taking the modulus at both sides, using Hölder Inequality, and recalling that $u_0 = 0$,

$$\begin{aligned}
|u_i|^2 &\leq \sum_{j=1}^{i} \Delta x \, |u_j D^+ u_j| + \sum_{j=1}^{i-1} \Delta x \, |u_j D^+ u_j| \\
&\leq \sqrt{\sum_{j=1}^{i} \Delta x |u_j|^2} \sqrt{\sum_{j=1}^{i} \Delta x |D^- u_j|^2} + \sqrt{\sum_{j=1}^{i-1} \Delta x \, |u_j|^2} \sqrt{\sum_{j=1}^{i-1} \Delta x \, |D^+ u_j|^2} \\
&\leq 2 \sqrt{\sum_{j=1}^{i} |u_j|^2 \Delta x} \sqrt{\sum_{j=1}^{i} |D^+ u_j|^2 \Delta x} \\
&\leq 2||u||_2 ||\boldsymbol{D}^+ u||_2.
\end{aligned}$$

Inequality (5.78) is then proved since this is valid to any $i = 0,\, 1,\, \ldots,\, M$. To get (5.79), and combining Hölder Inequality with (5.78),

$$\begin{aligned}
||q_\Delta u_\Delta||^4_{L^4(0,L)} &= \sum_{j=1}^{M-1} |u_j|^4 \Delta x \\
&\leq ||u||^2_\infty \sum_{j=1}^{M-1} |u_j|^2 \Delta x \\
&\leq 2||u||^2_2 ||\boldsymbol{D}^+ u||_2 ||u||^2_2 \\
&= 2||q_\Delta u_\Delta||^3_{L^2(0,T)} ||p_\Delta u_\Delta||_{H^1_0(0,L)}.
\end{aligned}$$

In order to conclude (5.80), we will again use Hölder inequality with (5.78) and (5.79):

$$\begin{aligned}
||q_\Delta u_\Delta||_{L^6(0,L))} &= \sum_{j=1}^{M-1} |u_j|^6 \Delta x \\
&\leq ||u||^2_\infty \sum_{j=1}^{M-1} |u_j|^4 \Delta x \\
&\leq 2||u||_2 ||\boldsymbol{D}^+ u||_2 \; 2||u||^3_2 ||\boldsymbol{D}^+ u||_2 \\
&= 4||u||^4_2 ||\boldsymbol{D}^+ u||^2_2
\end{aligned}$$

which can then lead us to conclude (5.80), and hence, the lemma is proved. □

Lemma 5.4. *Let* $\{u^n\}_{n\in\mathbb{N}}$ *be a sequence in* X_M *induced by the numerical scheme* (5.63)*, and let* $u^0 \in X_M \cap L^2_{\Delta x}(0, L)$ *such that* $||u^0||^2_2 \leq \frac{2a_3}{|a_4+a_5|}$*. Then, there exist some constant* $K = K(T, L) > 0$ *such that*

$$||Q_\Delta u_\Delta||^2_{L^\infty(0,T;L^2(0,L))} \leq ||u^0||_2^2, \quad \forall n \in \mathbb{N} \tag{5.81}$$

$$a_3||Q_\Delta^{\frac{1}{2}}\boldsymbol{D}^2 u_\Delta||^2_{L^2(0,T;L^2(0,L))} \leq \frac{1}{2\Delta x}||u^0||_2^2 \tag{5.82}$$

$$||P_\Delta^{\frac{1}{2}} u_\Delta||^2_{L^2((0,T);H_0^1(0,L))} \leq K||u^0||_2^2 \tag{5.83}$$

$$||Q_\Delta^{\frac{1}{2}}(|u|^2 u_x)_\Delta||^2_{L^2(0,T;L^2(0,L))} \leq K||q_\Delta u_\Delta^0||^6_{L^2(0,L)} \tag{5.84}$$

$$||q_\Delta(|u|_x^2)_\Delta||^2_{L^2(0,L)} \leq 32(1+\Delta x^2)||q_\Delta u_\Delta||^2_{L^\infty(0,L)}||p_\Delta u_\Delta||^2_{H_0^1(0,L)} \tag{5.85}$$

Proof. We start by multiplying (5.63) component-wise by $\Delta x \overline{u}_j^{n+\frac{1}{2}}$, sum over j and extract the imaginary part. This will lead us to

$$\frac{1}{2\Delta t}\Big(||u^{n+1}||_2^2 - ||u^n||_2^2\Big) - a_3 Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\Big)_2 + \Big(a u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 = 0 \tag{5.86}$$

Using (5.68), and using $z := \boldsymbol{D}^- u^{n+\frac{1}{2}}$, we can re-write the second term in (5.86) as

$$\begin{aligned}
Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\Big)_2 &= Re\Big(\boldsymbol{D}^+ z, z\Big)_2 \\
&= Re\left(\sum_{j=1}^{M-1} \frac{(z_{j+1}-z_j)\overline{z_j}}{\Delta x}\Delta x\right) \\
&= \sum_{j=1}^{M-1}\frac{1}{2\Delta x}\left(|z_{j+1}|^2 - |z_j|^2 - |z_{j+1}-z_j|^2\right)\Delta x \\
&= \frac{1}{2\Delta x}\Big(|z_M|^2 - |z_1|^2\Big)\Delta x - \sum_{j=1}^{M-1}\frac{1}{2\Delta x}|z_{j+1}-z_j|^2\Delta x \\
&= -\frac{1}{2}|z_1|^2 - \frac{\Delta x}{2}\sum_{j=1}^{M-1}|D^+ z_j|^2\Delta x \\
&= -\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 - \frac{\Delta x}{2}\sum_{j=1}^{M-1}|D^+D^- u_j^{n+\frac{1}{2}}|^2\Delta x
\end{aligned}$$

Therefore,

$$Re\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, \boldsymbol{D}^- u^{n+\frac{1}{2}}\Big)_2 = -\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 - \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2$$

and thus, (5.86) can be written as

$$\frac{||u^{n+1}||_2^2 - ||u^n||_2^2}{2\Delta t} + a_3\Big(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \frac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Big) + \sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x = 0. \tag{5.87}$$

Because $a_j > 0, \forall j = 0, 1, \ldots, M$, we can conclude (5.81) with ease from (5.87). Now, multiplying (5.87) by $2\Delta t$, dropping some terms, and summing for $n = 0, 1, \ldots, N$, we get

$$a_3 \Delta x \sum_{n=0}^{N} ||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 \Delta t \leq \sum_{n=0}^{N} ||u^n||_2^2 - ||u^{n+1}||_2^2 = ||u^0||_2^2 - ||u^{N+1}||_2^2 \tag{5.88}$$

and thus, (5.82) can be concluded. In order to prove (5.83), we need to multiply (5.63) component-wise by $j\Delta x u_j^{n+\frac{1}{2}}$, sum over $j = 0, 1, \ldots, M-1$, and extract the imaginary part. We have

$$\begin{aligned} i\Big(\boldsymbol{D}_t u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + a_1\Big(\boldsymbol{D}^+\boldsymbol{D} u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_2\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x \\ +ia_3\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D} u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x + ia_4\Big(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x \\ +i(a_4+a_5)\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x + i\Big(a u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = 0 \end{aligned} \tag{5.89}$$

We will study each term in (5.89). First, and using the definition (5.61), it is easy to see that

$$Im\bigg(i\Big(\boldsymbol{D}_t u^n, u^{n+\frac{1}{2}}\Big)_x\bigg) = \frac{1}{2\Delta t}(||u^{n+1}||_x^2 - ||u^n||_x^2) \tag{5.90}$$

Using (5.71), we can write

$$\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_x^2 + \Delta x||\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 - \Delta x\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2.$$

Hence,

$$Im\Big(\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = -\Delta x Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2. \tag{5.91}$$

We can also write

$$Im\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x = 0. \tag{5.92}$$

Using now the identity (5.75) for $z = u^{n+\frac{1}{2}}$, we have

$$\begin{aligned} Im\bigg(i\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\bigg) = -\tfrac{\Delta x}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \tfrac{3}{2}||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2 \\ +\tfrac{\Delta x}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_x^2 - \tfrac{\Delta x^2}{2}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2 \end{aligned} \tag{5.93}$$

Now we have to study the nonlinear terms defined in (5.64) and (5.65); this is, we have to work with

$$\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x := \left(\frac{1}{2}|u^{n+\frac{1}{2}}|^2\boldsymbol{D}\left(u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x \tag{5.94}$$

$$+\left(\frac{1}{4}\boldsymbol{D}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x \tag{5.95}$$

$$-\left(\frac{1}{4}\left(u^{n+\frac{1}{2}}\right)^2\boldsymbol{D}\overline{\left(u^{n+\frac{1}{2}}\right)}, u^{n+\frac{1}{2}}\right)_x$$

$$\left(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x := \left(u^{n+\frac{1}{2}}\boldsymbol{D}\left(|u^{n+\frac{1}{2}}|^2\right), u^{n+\frac{1}{2}}\right)_x \tag{5.96}$$

Using (5.73), we have

$$\begin{aligned}\left(\boldsymbol{D}(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}), u^{n+\frac{1}{2}}\right)_x &= -\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x + \frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^-u^{n+\frac{1}{2}}\right)_2\\ &\quad -\frac{\Delta x}{2}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^+u^{n+\frac{1}{2}}\right)_2 - \left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2\end{aligned} \tag{5.97}$$

and, at the same time,

$$\left(u^{n+\frac{1}{2}2}\boldsymbol{D}\overline{u}^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_x = \left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x \tag{5.98}$$

Combining (5.97) and (5.98) in (5.95), we get

$$\begin{aligned}\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x &= \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2\boldsymbol{D}\left(u^{n+\frac{1}{2}}\right), u^{n+\frac{1}{2}}\right)_x - \frac{1}{2}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}u^{n+\frac{1}{2}}\right)_x\\ &\quad + \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^-u^{n+\frac{1}{2}}\right)_2\\ &\quad - \frac{\Delta x}{8}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^+u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\right)_2\end{aligned}$$

and extracting the real part, we get

$$\begin{aligned}Re\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x &= \frac{\Delta x}{8}Re\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^-u^{n+\frac{1}{2}}\right)_2\\ &\quad - \frac{\Delta x}{8}Re\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^+u^{n+\frac{1}{2}}\right)_2 - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4\end{aligned} \tag{5.99}$$

and recalling that $\boldsymbol{D}^2u = \frac{\boldsymbol{D}^+u-\boldsymbol{D}^-u}{\Delta x}$

$$Re\left(F_{a_4}(u^{n+1}), u^{n+\frac{1}{2}}\right)_x = -\frac{\Delta x^2}{8}Re\left(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\right)_x - \frac{1}{4}||u^{n+\frac{1}{2}}||_4^4 \tag{5.100}$$

Finally, for the last nonlinear term, we get

$$\begin{aligned}\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x &= \Big(u^{n+\frac{1}{2}}\boldsymbol{D}(|u^{n+\frac{1}{2}}|^2), u^{n+\frac{1}{2}}\Big)_x \\ &= \Big(\boldsymbol{D}|u^{n+\frac{1}{2}}|^2, |u^{n+\frac{1}{2}}|^2\Big)_x \\ &= -\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}|u^{n+\frac{1}{2}}|^2\Big)_x + \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big)_2 \\ &\quad - \frac{\Delta x}{2}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 - ||u^{n+\frac{1}{2}}||_4^4\end{aligned}$$

and thus,

$$Re\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = \frac{\Delta x}{4}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^-|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{\Delta x}{4}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^+|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4$$

which, in turn, can be rewritten as

$$Re\Big(F_{a_4+a_5}(u^{n+1}), u^{n+\frac{1}{2}}\Big)_x = -\frac{\Delta x^2}{8}\Big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\Big)_2 - \frac{1}{2}||u^{n+\frac{1}{2}}||_4^4. \tag{5.101}$$

Combining together (5.90), (5.91), (5.92), (5.93), (5.100) and (5.101); multiplying by Δt, and summing from $n = 0$ to $n = N$, we obtain

$$\begin{aligned}0 = \frac{1}{2}(||u^{N+1}||_x^2 - ||u^0|_x^2) &- a_1\Delta x\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^-u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t \\ &+ a_3\Bigg(\frac{\Delta x}{2}\sum_{n=0}^{N}|D^-u_1|^2\Delta t + \frac{3}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t \\ &+ \frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_x^2\Delta t - \frac{\Delta x^2}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Delta t\Bigg) \\ &- a_4\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2u^{n+\frac{1}{2}}, \boldsymbol{D}^2u^{n+\frac{1}{2}}\big)_2\Delta t \\ &- (a_4 + a_5)\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N}\big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\big)_2\Delta t\Bigg) - \Big(\frac{3}{2}a_4 + a_5\Big)\Bigg(\frac{1}{2}\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_4^4\Delta t\Bigg) \\ &+ \sum_{n=0}^{N}\Big(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\Delta t\end{aligned}$$

Let us recall the fact that $a_1, a_3 > 0$. This can let us drop some terms in the above equality to get

$$
\begin{aligned}
\frac{1}{2}||u^0||_x^2 \geq & -a_1\Delta x\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t \\
& + \frac{3}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2\Delta t - a_3\frac{\Delta x^2}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Delta t \\
& - a_4\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\big)_2\Delta t \\
& - (a_4+a_5)\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N}\big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\big)_2\Delta t\Bigg) - \Big(\frac{3}{2}a_4 + a_5\Big)\Bigg(\frac{1}{2}\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_4^4\Delta t\Bigg) \\
& + \sum_{n=0}^{N}\Big(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_x\Delta t
\end{aligned}
$$

Meanwhile, let us recall equality (5.87). Multiplying it by Δt, and summing from $n = 0$ to N, we get

$$
\begin{aligned}
-a_3\frac{\Delta x}{2}\sum_{n=0}^{N}||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Delta t &= \frac{||u^{N+1}||_2^2 - ||u^0||_2^2}{2} + \frac{a_3}{2\Delta x}\sum_{n=0}^{N}|\boldsymbol{D}^- u_1^{n+\frac{1}{2}}|^2\Delta t + \sum_{n=0}^{N}\Big(au^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t \\
&\geq -\frac{1}{2}||u^0||_2^2
\end{aligned}
$$

thus, and after re-ordening terms,

$$
\begin{aligned}
\frac{1}{2}||u^0||_x^2 + \frac{\Delta x}{4}||u^0||_2^2 + a_1\Delta x\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^- u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2\Delta t & \\
+ |a_4|\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N} Re\big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\big)_2\Delta t\Bigg) & \\
+ |a_4+a_5|\Bigg(\frac{\Delta x^2}{8}\sum_{n=0}^{N}\big(|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2|u^{n+\frac{1}{2}}|^2\big)_2\Delta t\Bigg) & \\
+ \Big|\frac{3}{2}a_4 + a_5\Big|\Bigg(\frac{1}{2}\sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_4^4\Delta t\Bigg) & \\
\geq \frac{3}{2}a_3\sum_{n=0}^{N}||\boldsymbol{D}^+ u^{n+\frac{1}{2}}||_2^2\Delta t & \qquad (5.102)
\end{aligned}
$$

Using Young and Hölder inequalities, and for $T = N\Delta t$, we can demonstrate with ease the following inequalities:

$$\sum_{n=0}^{N} Im\Big(\boldsymbol{D}^{-}u^{n+\frac{1}{2}}, u^{n+\frac{1}{2}}\Big)_2 \Delta t \leq \frac{1}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t + \frac{T}{2}||u^0||_2^2 \tag{5.103}$$

$$\Delta x^2 \sum_{n=0}^{N} Re\Big(|u^{n+\frac{1}{2}}|^2 u^{n+\frac{1}{2}}, \boldsymbol{D}^2 u^{n+\frac{1}{2}}\Big)_2 \Delta t \leq \frac{||u^0||_2^2}{2}\left(\frac{\Delta x}{4a_3}\sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t + T\right) \tag{5.104}$$

$$(a_4 + a_5)\sum_{n=0}^{N} (|u^{n+\frac{1}{2}}|^2, \boldsymbol{D}^2 |u^{n+\frac{1}{2}}|^2)\Delta t \leq |a_4 + a_5| ||u^0||_2^2 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t \tag{5.105}$$

Combining those in (5.102),

$$\begin{aligned}
&\frac{1}{2}||u^0||_x^2 + \frac{\Delta x}{4}||u^0||_2^2 + a_1 \Delta x\left(\frac{1}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t + \frac{T}{2}||u^0||_2^2\right) \\
&\quad + \frac{|a_4|}{8}\frac{||u^0||_2^2}{2}\left(\frac{\Delta x}{4a_3}\sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t + T\right) \\
&\quad + \frac{\Delta x^2}{8}|a_4 + a_5| ||u^0||_2^2 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t \\
&\quad + \frac{1}{2}\Big|\frac{3}{4}a_4 + a_5\Big| \sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t \\
&\quad \geq \frac{3}{2}a_3 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t
\end{aligned} \tag{5.106}$$

and using the fact that

$$\sum_{n=0}^{N} ||u^{n+\frac{1}{2}}||_4^4 \Delta t \leq T||u^0||^4 + ||u^0||^2 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t$$

we can write

$$\begin{aligned}
&\frac{1}{2}||u^0||_x^2 + \frac{\Delta x}{4}||u^0||_2^2 + a_1 \Delta x\left(\frac{1}{2}\sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t + \frac{T}{2}||u^0||_2^2\right) \\
&+\left(\Big|\frac{a_4}{a_3}\Big|\frac{\Delta x ||u^0||_2^2}{64} + \frac{1}{2}\Big|\frac{3}{2}a_4 + a_5\Big|\right)\Big(T||u^0||_2^4 + ||u^0||_2^2 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t\Big) \\
&+a_4 T\frac{||u^0||_2^2}{16} + \frac{\Delta x^2}{8}|a_4 + a_5| ||u^0||_2^2 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t \geq \frac{3}{2}a_3 \sum_{n=0}^{N} ||\boldsymbol{D}^{+}u^{n+\frac{1}{2}}||_2^2 \Delta t
\end{aligned} \tag{5.107}$$

and reordening

$$\left(\frac{3}{2}a_3 - \frac{\Delta x}{2}a_1 - \left(\left|\frac{a_4}{a_3}\right|\frac{\Delta x}{64}||u^0||_2^2 + \frac{1}{2}\left|\frac{3}{2}a_4 + a_5\right| + \frac{\Delta x^2}{8}|a_4 + a_5|\right)||u^0||_2^2\right)\sum_{n=0}^{N}||\boldsymbol{D}^+u^{n+\frac{1}{2}}||_2^2\Delta t$$
$$\leq \left(\frac{L}{2} + \frac{\Delta x}{4} + a_1\Delta x\frac{T}{2} + T\left(\left|\frac{a_4}{a_3}\right|\frac{\Delta x||u^0||_2^2}{64} + \frac{1}{2}\left|\frac{3}{2}a_4 + a_5\right|\right)||u^0||_2^2\right)||u^0||_2^2 \tag{5.108}$$

because $||u^0||_2^2 < \dfrac{2a_3}{|a_4 + a_5|}$, and considering $\Delta x << 1$ we can infere the existance of the needed constant $K = K(T, L)$. Hence, (5.83) is proved. To prove (5.84), let us first note that:

$$\begin{aligned}\left|\left||u^{n+\frac{1}{2}}|^2\boldsymbol{D}u^{n+\frac{1}{2}}\right|\right|_2^2 &= \sum_{j=0}^{M-1}|u_j^{n+\frac{1}{2}}|^4|Du_j^{n+\frac{1}{2}}|^2\Delta x \\ &\leq ||u^{n+\frac{1}{2}}||_\infty^4\sum_{j=0}^{M-1}|Du_j^{n+\frac{1}{2}}|^2\Delta x.\end{aligned}$$

Hence, using (5.81) and (5.83),

$$\begin{aligned}\sum_{n=0}^{N}\left|\left||u^{n+\frac{1}{2}}|^2\boldsymbol{D}u^{n+\frac{1}{2}}\right|\right|_2^2\Delta t &\leq \sum_{n=0}^{N}||u^{n+\frac{1}{2}}||_\infty^4||\boldsymbol{D}u^{n+\frac{1}{2}}||_2^2\Delta t \\ &\leq \max_{n\in[0,N]}||u^{n+\frac{1}{2}}||_\infty^4\sum_{n=0}^{N}||\boldsymbol{D}u^{n+\frac{1}{2}}||_2^2\Delta t \\ &\leq ||u^0||_2^4\sum_{n=0}^{N}||\boldsymbol{D}u^{n+\frac{1}{2}}||_2^2\Delta t \\ &\leq K||u^0||_2^6\end{aligned}$$

proving then (5.84). To prove (5.85), we will again the identity $(a^2-b^2)+(a-b)^2 = 2a(a-b)$. For a $u_j \in u,\ i = 0,\ 1,\ \ldots,\ M$, we have:

$$\begin{aligned}D|u_j|^2 &= \frac{|u_{j+1}|^2 - |u_{j-1}|^2}{2\Delta x} \\ &= \frac{1}{2\Delta x}\Big(|u_{j+1}|^2 - |u_j|^2 + |u_j|^2 - |u_{j-1}|^2\Big) \\ &= \frac{1}{2\Delta x}\Big[2|u_j|(|u_j| - |u_{j-1}|) - (|u_j| - |u_{j-1}|)^2 \\ &\qquad + 2|u_j|(|u_{j+1}| - |u_j|) + (|u_{j+1}| - |u_j|)^2\Big] \\ &= |u_j|D^-|u_j| - \frac{\Delta x^2}{2}(D^-|u_j|)^2 + |u_j|D^+|u_j| + \frac{\Delta x^2}{2}(D^+|u_j|)^2.\end{aligned}$$

Taking the square at both sides, using inverse triangle inequality, and $D^2|u_j| \leq 4\frac{||u||_\infty}{\Delta x^2}$,

$$\begin{aligned}
(D|u_j|^2)^2 &= \left(|u_j|D^-|u_j| - \frac{\Delta x^2}{2}(D^-|u_j|)^2 + |u_j|D^+|u_j| + \frac{\Delta x^2}{2}(D^+|u_j|)^2\right)^2 \\
&= \left(2|u_j|D|u_j| + \frac{\Delta x^2}{2}\Big[(D^+|u_j|)^2 - (D^-|u_j|)^2\Big]\right)^2 \\
&= \left(2|u_j|D|u_j| + \frac{\Delta x^2}{2}\Big[(D^+|u_j| + D^-|u_j|)(D^+|u_j| - D^-|u_j|)\Big]\right)^2 \\
&= \left(2|u_j|D|u_j| + \Delta x^3 D|u_j|D^2|u_j|\right)^2 \\
&\leq 4\left(4|u_j|^2|Du_j|^2 + \Delta x^6|Du_j|^2(D^2|u_j|)^2\right) \\
&\leq 16||u||_\infty^2|Du_j|^2 + 16\Delta x^2||u||_\infty^2|Du_j|^2.
\end{aligned}$$

Summing over j will lead us to

$$\sum_{j=0}^{M-1}(D|u_j|^2)^2\Delta x \leq 32||u||_\infty^2||\boldsymbol{D}^+u||_2^2(1+\Delta x^2)$$

and hence, (5.85) is proved, and thus concluding the demonstration of the Lemma.

□

Now we are in conditions to state and prove the followig theorem:

Theorem 5.5. *Let $u_\Delta = \{u_m^n\}_{m\in\mathbb{N}}$ a sequence in X_M of solutions induced by the numerical scheme* (5.63)*, at a time $t_n = n\Delta t$, computed from a sequence of initial conditions $\{u_m^0\}_{m\in\mathbb{N}} \subset X_M$ using a timestep Δt and a spacestep Δx. If $u^0 \in L^2(0,L)_\Delta : ||u^0||_2^2 \leq \frac{2a_3}{|a_4+a_5|}$, then there is a subsequence, still denoted by $\{u_m^n\}_{m\in\mathbb{N}}$, such that*

$$Q_\Delta u_\Delta \to u \ \text{ strongly in } \ L^2(0,T;L^2(0,L)) \tag{5.109}$$

when $\Delta t, \Delta x \to 0$, and for u the weak solution of (1.1)

Proof. We will proceed as in the proof of Lemma 4.1. From (5.81), we infer the existance of a u such that

$$Q_\Delta u_\Delta \to u \ \text{ weakly in } \ L^2(0,T;L^2(0,L)) \tag{5.110}$$

From (5.81) and (5.83), we can also say that there exists a $u \in L^2(0,T;H_0^1(0,L))$ such that

$$\{Q_\Delta u_\Delta\} \ \text{ is bounded in } \ L^2(0,T;H_0^1(0,L)) \tag{5.111}$$

and thus

$$Q_\Delta u_\Delta \overset{\star}{\rightharpoonup} u \text{ weak star in } \ L^2(0,T;H_0^1(0,L)) \tag{5.112}$$

From (5.80) and (5.83), we have

$$\{Q_\Delta^{\frac{1}{2}}(|u|^2u)_\Delta\} \ \text{ is bounded in } \ L^2(0,T;L^2(0,L)) \tag{5.113}$$

And from (5.84) and (5.85),

$$\{Q_\Delta F_{a_4}(u)_\Delta\} \text{ is bounded in } L^2(0,T;L^2(0,L)) \tag{5.114}$$

$$\{Q_\Delta F_{a_4+a_5}(u)_\Delta\} \text{ is bounded in } L^2(0,T;L^2(0,L)) \tag{5.115}$$

Let us now consider a $\varphi \in H_0^2(0,L)$, with $\varphi_j^n = \varphi(x_j,t_n),\ 0 \le n \le N,\ 0 \le j \le M,$. Multiplying (5.63) by $\Delta t\Delta x\overline{\varphi}_j$, sum over j and then sum over n. We then get

$$\begin{aligned}\sum_{n=0}^{N}\Big(\boldsymbol{D}_t u_m^n,\varphi\Big)_2\Delta t &= ia_1\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac12},\varphi\Big)_2\Delta t - a_3\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac12},\varphi\Big)_2\Delta t\\ &\quad + a_2\sum_{n=0}^{N}\Big(|u_m^{n+\frac12}|^2u^{n+\frac12},\varphi\Big)_2\Delta t - a_4\sum_{n=0}^{N}\Big(F_{a_4}(u_m^{n+1}),\varphi\Big)_2\Delta t\\ &\quad - (a_4+a_5)\sum_{n=0}^{N}\Big(F_{a_4+a_5}(u_m^{n+1}),\varphi\Big)_2\Delta t - \sum_{n=0}^{N}\Big(au_m^{n+\frac12},\varphi\Big)_2\Delta t\end{aligned} \tag{5.116}$$

Our aim is to prove that the left hand side of (5.116) is bounded. From (5.81) and (5.83), and summing by parts, we get

$$\begin{aligned}\sum_{n=0}^{N}&\Big(\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac12},\varphi\Big)_2\Delta t+\sum_{n=0}^{N}\Big(\boldsymbol{D}^+\boldsymbol{D}^+\boldsymbol{D}^- u_m^{n+\frac12},\varphi\Big)_2\Delta t\\ &= \sum_{n=0}^{N} -\Big(\boldsymbol{D}^+u_m^{n+\frac12},\boldsymbol{D}^+\varphi\Big)_2\Delta t + \sum_{n=0}^{N}\Big(\boldsymbol{D}^+u_m^{n+\frac12},\boldsymbol{D}^+\boldsymbol{D}^-\varphi\Big)_2\Delta t\\ &\le \sum_{n=0}^{N}||\boldsymbol{D}^+u_m^{n+\frac12}||_2||\boldsymbol{D}^+\varphi||_2\Delta t + \sum_{n=0}^{N}||\boldsymbol{D}^+u_m^{n+\frac12}||_2||\boldsymbol{D}^+\boldsymbol{D}^-\varphi||_2\Delta t\\ &\le C_\varphi\Big(\sum_{n=0}^{N}||\boldsymbol{D}^+u_m^{n+\frac12}||_2\Delta t + \sum_{n=0}^{N}||\boldsymbol{D}^+u_m^{n+\frac12}||_2\Delta t\Big)\\ &\le 2C_\varphi K||u_m^0||_2\\ &< \infty\end{aligned}$$

since we are considering any $\varphi \in H_0^2(0,L)$, and combining (5.111), (5.113), (5.114) and (5.115) after using Cauchy-Schwarz Inequality in (5.116) , we get

$$\Big\{\frac{\partial}{\partial t}P_\Delta u_\Delta\Big\} \text{ is bounded in } L^2(0,T;H^{-2}(0,L)) \tag{5.117}$$

and as in the continuous case, because

$$H_0^1(0,L) \overset{c}{\hookrightarrow} L^2(0,L) \hookrightarrow H^{-2}(0,L),$$

and employing Aubin-Lions Theorem, there exists a subsequence of $\{u_m^n\}_{m\in\mathbb{N}}$, still denoted by the same form, such that,

$$Q_\Delta u_\Delta \longrightarrow u \quad \text{strongly in} \quad L^2(0,T;L^2(0,L))\,. \tag{5.118}$$

Now we will prove that u is the weak solution of (1.1). Thanks to (5.118), we have

$$|u_m^{n+\frac12}|u_m^{n+\frac12} \longrightarrow |u|^2u, \qquad \text{a.e. in } (0,L)\times(0,T) \tag{5.119}$$

using (5.119), and recalling again Lion's lemma [29], we will get

$$Q_{\Delta}^{\frac{1}{2}}(|u|^2u)_{\Delta} \rightharpoonup |u|^2u \text{ weakly in } L^2(0,T;L^2(0,L)). \tag{5.120}$$

furthermore, combining (5.118) and (5.112),

$$Q_{\Delta}(F_{a_4}(u))_{\Delta} \rightharpoonup \frac{1}{2}|u|^2u_x + \frac{1}{4}(|u|^2u)_x - \frac{1}{4}u^2\overline{u}_x \text{ weakly in } L^2(0,T;L^2(0,L)) \tag{5.121}$$

$$Q_{\Delta}(F_{a_4+a_5}(u))_{\Delta} \rightharpoonup u|u|_x^2 \text{ weakly in } L^2(0,T;L^2(0,L)) \tag{5.122}$$

Multiplying componentwise the numerical scheme (5.63) by $\Delta x \Delta t \phi_k^n$, sum by parts, and passing to the limit, is easy to see that $u = u(t_n)$ is, indeed, the weak solution of problem (1.1), and hence the Theorem is proved.

□

We will now proceed with the discret analog of the main result.

Theorem 5.6. *Consider a sequence $\{u^n\}_{n\in\mathbb{N}} \subset X_M$ induced by the numerical scheme* (5.63)*, and consider the function $a(x)$ and the set ω as defined in* (1.2)*. If $||u^0||_2^2 \leq \frac{2a_3}{|a_4+a_5|}$, and for $T_0 = n\Delta t > 0$, there exist a positive constant $C = C(T_0)$ and $\mu = \mu(T_0)$, both independent of Δx and Δt, such that the inequality*

$$E^n \leq C||u^0||_2^2 e^{-\mu n \Delta t}$$

holds for all $n > 0$.

As in Section 4, before proving the theorem we will state and prove a last lemma.

Lemma 5.7. *For $T_0 = n\Delta t$, with $n \in \mathbb{N}$, and for a sequence $\{u^n\}_{n\in\mathbb{N}}$ induced by the numerical scheme* (5.63) *such that $||u^0||_2^2 \leq \frac{2a_3}{|a_4+a_5|}$, there exist a constant $C = C(T_0)$, independent of Δt and Δx, such that*

$$||u^0||_2^2 \leq C\left(a_3\sum_{n=0}^{N}\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t\right) \tag{5.123}$$

Proof. Let $N \in \mathbb{N}$, and consider the numerical scheme (5.63). Multiplying it by $(N+1-n)\overline{u}^{n+\frac{1}{2}}\Delta t$ componentwise, extracting the imaginary part, summing over $n = 0, 1, \ldots, N$, and recalling the computations made in order to get (5.87), we will have

$$\begin{aligned} 0 = {} & \frac{1}{2}\Big(\big(\sum_{n=0}^{N}||u^{n+1}||_2^2\big) - (N+1)||u^0||_2^2\Big) \\ & + a_3\sum_{n=0}^{N}(N+1-n)\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t \\ & + \sum_{n=0}^{N}\sum_{j=1}^{M-1}(N+1-n)a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t \end{aligned} \tag{5.124}$$

rearranging terms and bounding,

$$
\begin{aligned}
||u^0||_2^2 \leq \frac{1}{2T_0}\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t + a_3\sum_{n=0}^{N}\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t \\
+\sum_{n=0}^{N}\sum_{j=1}^{M-1}a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t. \qquad (5.125)
\end{aligned}
$$

In order to prove (5.123) then, we must demonstrate the existance of a constant $C_1 = C_1(T_0)$ such that

$$
\begin{aligned}
\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t \leq C_1\Bigg(a_3\sum_{n=0}^{N}\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t \\
+\sum_{n=0}^{N}\sum_{j=1}^{M-1}a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t\Bigg). \qquad (5.126)
\end{aligned}
$$

We will proceed by contradiction. Hence, we must assume as true the opposite of (5.126); this is,

$$
\begin{aligned}
\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t > C_1\Bigg(a_3\sum_{n=0}^{N}\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t \\
+\sum_{n=0}^{N}\sum_{j=1}^{M-1}a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t\Bigg). \qquad (5.127)
\end{aligned}
$$

Since $||Q_\Delta u_\Delta||_{L^\infty(0,T;L^2(0,L))} < \infty$, we can extract a subsequence $\{u^{n_m}\}_{m\in\mathbb{R}}$, still denoting it by $\{u^n\}_{n\in\mathbb{N}}$, such that

$$
\lim_{\Delta x,\Delta t\to 0}\frac{\sum_{n=0}^{N}||u^{n+1}||_2^2\Delta t}{a_3\sum_{n=0}^{N}(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1}a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t} = +\infty \qquad (5.128)
$$

Let $\lambda^n \geq 0, \forall n \in \mathbb{N}$ such that $(\lambda^n)^2 = \sum_{k=0}^{n+1}||u^k||_2^2\Delta t$, and let us define $v^n := \frac{u^n}{\lambda^n}$. This induces the following sequence of numerical problems: find $v^{n+1} \in X_M$ such that

$$
\begin{aligned}
0 &= i\boldsymbol{D}_t v^n + a_1\boldsymbol{D}^2v^{n+\frac{1}{2}} + a_2(\lambda^n)^2|v^{n+\frac{1}{2}}|^2v^{n+\frac{1}{2}} + ia_3\boldsymbol{D}^3v^{n+\frac{1}{2}} \\
&\quad + (\lambda^n)^2F_{a_4}(v^{n+1}) + (\lambda^n)^2F_{a_4+a_5}(v^{n+1}) + iav^{n+\frac{1}{2}} \qquad (5.129)\\
v^0 &= u^0,\ u^0 \in X_M \cap L^2_{\Delta x}(0,L)
\end{aligned}
$$

where

$$
\sum_{n=0}^{N}||v^n||_2^2\Delta t = 1 \qquad (5.130)
$$

On the other hand, and due to (5.128), when $\Delta x, \Delta t \to 0$,

$$
a_3\sum_{n=0}^{N}\Big(\frac{1}{2}|D^-u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^-u^{n+\frac{1}{2}}||_2^2\Big)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1}a_j|u_j^{n+\frac{1}{2}}|^2\Delta x\Delta t \to 0 \qquad (5.131)
$$

and recalling (5.125), we conclude that v^0 is bounded in $L^2(0,L)_\Delta$. And by Theorem 5.5, we can extract a subsequence of $\{v^n\}_{n\in\mathbb{N}}$, still denoted by the same way, that

$$v^n \longrightarrow v(t_n) \text{ strongly on } L^2(0,T_0;L^2(0,L))$$

and by (5.130),

$$||v(t)||_{L^2(0,T_0,L^2(0,L))} = 1 \tag{5.132}$$

When passing to the limit in (5.131), we have

$$0 = \lim_{\Delta x,\Delta t\to 0} a_3 \sum_{n=0}^{N}\Big(\frac{1}{2}|D^- u_1^{n+\frac{1}{2}}|^2 + \Delta x||\boldsymbol{D}^+\boldsymbol{D}^- u^{n+\frac{1}{2}}||_2^2\Big)\Delta t + \sum_{n=0}^{N}\sum_{j=1}^{M-1} a_j |u_j^{n+\frac{1}{2}}|^2 \Delta x\Delta t$$
$$= \int_0^{T_0} |v(0,t)|^2 dt + 2\int_0^{T_0}\int_0^L a(x)|v|^2 dxdt$$

and thus, $v(x,t)\equiv 0$ for $(x,t)\in(\omega\times(0,T_0))$. From here, we must distinguish two scenarios:

Case 1: Let us extract a subsequence from $\{\lambda^n\}_{n\in\mathbb{N}}$, denoted by the same way, such that $\lambda^n\to 0$ when $n\to\infty$. This induces the following linear problem:

$$\begin{aligned} iv_t + a_1 v_{xx} + ia_3 v_{xxx} + iav = 0, &\quad (x,t)\in(0,L)\times(0,T_0)\\ v(0,t) = v(L,t) = 0, &\quad t\in[0,T_0]\\ v_t(L,t) = 0, &\quad t\in[0,T_0]\\ v(t=0) = u^0, &\quad u^0\in L^2(0,L)\end{aligned}$$

And again, by Holgrem's Theorem, we conclude that $v^n\equiv 0$, for $(x,t)\in(0,L)\times(0,T)$, which contradicts (5.132).

Case 2: There is a subsequence from $\{\lambda^n\}_{n\in\mathbb{N}}$, still denoted by λ^n; and there is a $\lambda>0$ such that $\lambda^n\to\lambda$. This converges to the IVP (1.1), and using Theorem 1.1 and the same arguments as in Section 4, we conclude that $v\equiv 0$ for $(x,t)\in(0,L)\times(0,T)$, and this is again a contradiction to (5.132).

This allows to conclude that (5.127) is false, and hence, the lemma is proved. □

In order to conclude the proof of Theorem 5.6, we can just follow again the arguments employed to prove equation (4.59) in Section 4.

5.2. **Numerical Results.** We will finally present some computational results using the numerical scheme proposed in this section.

5.2.1. *First case.* For a first numerical result, we will work with the following HNLS equation for $u=u(x,t),\ (x,t)\in(-60,60)\times(0,3]$:

$$iu_t + 0.001u_{xx} + |u|^2 u + i\Big(0.01u_{xxx} + 0.1|u|^2u_x + 0.05|u|_x^2 u + a(x)u\Big) = 0$$
$$u(x,0) = u^0(x) = A\,e^{irx}\,sech(x) \tag{5.133}$$

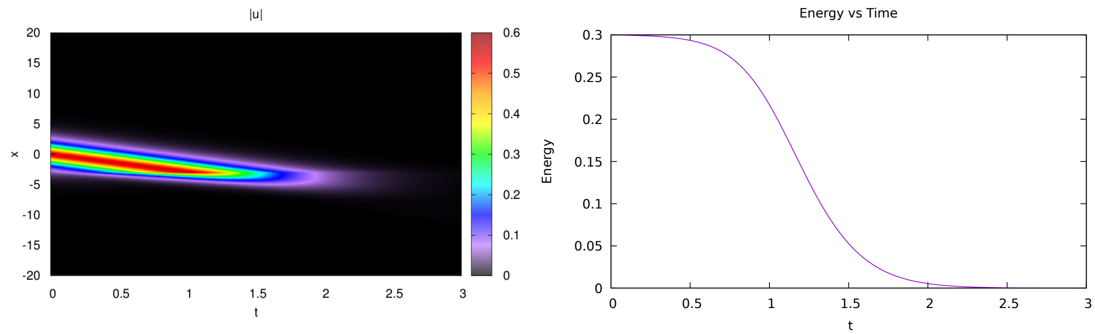

FIGURE 1. First case results. Left: time evolution of the absolute value of the solution. Right: evolution of the energy.

this is; $a_1 = 0.001$, $a_2 = 1$, $a_3 = 0.01$, $a_4 = 0.1$, and $a_5 = 0.04$. The initial condition is given by the analytical solution of the IVP (5.133) when $a(x) \equiv 0$; this is,

$$u(x,t) = A\, exp(i(nt + rx))\, sech(x + lt)$$

where,

$$A^2 = \frac{6a_3}{3a_4 + 2a_5} \quad r = \frac{-2a_1(3a_4 + 2a_5) + 6a_3}{-12a_3(a_4 + a_5)} \quad l = -2a_1 r - a_3(1 - 3r^2), \quad n = a_1(1 - r^2) - a_3 r(3 - r^2).$$

This solution was proposed first by Potasek in [40].

On the other hand, $a(x) = (1 + \sqrt{|x|}), \quad x \in (-8, -3)$, and in our calculations, $\Delta t = 0.001$ and $\Delta x = \frac{120}{2^{13}} \approx 0.015$. As shown in Figure 1 right, the energy decays at an exponential rate to zero, which is what we expected. While Figure 1 left shows how the soliton is getting *dissipated* after entering the damping zone, starting at $x = -3$.

5.2.2. *Second case.* A last case will be presented, regarding the following equation for $u = u(x,t),\ (x,t) \in (-60, 80) \times (0, 500]$

$$iu_t + 0.1u_{xx} + 2|u|^2 u + i\Big(0.001u_{xxx} + 0.01|u|^2 u_x + 0.1|u|_x^2 u + a(x)u\Big) = 0$$

(5.134) $$u(x,0) = u^0(x) = A\, e^{ix}\, sech(-Bx)$$

where we used an initial condition based on a solution propsed by Kumar et al. [27] when $a(x) \equiv 0$:

$$u(x,t) = A\, e^{i(-kt + \omega x)}\, sech(B(t - x))$$

where $v = 10$, $k = 0.001$, and

$$B = \pm\sqrt{\frac{k - a_1\omega^2 + a_3\omega^3}{3a_3\omega - a_1}} \quad A = \pm\sqrt{\frac{2(k - a_1\omega^2 + a_3\omega^3)}{a_4\omega - a_2}} \quad \omega = \frac{a_1 v \pm \sqrt{a_1^2 v^2 + 3a_3^2 v^3 B - 3a_3 v}}{3a_3 v}$$

Meanwhile, for the damping function we've considered $a(x) = 0.01,\ x > 10$; while for our computations we've used $\Delta t = 0.05$, $\Delta x = \frac{140}{2^{13}} \approx 0.017$.

As seen in Figure 2 right, the energy also decays following an exponential trend, while as seen in Figure 2 left, the soliton manages to enter the zone with the active damping, dissipating in the process.

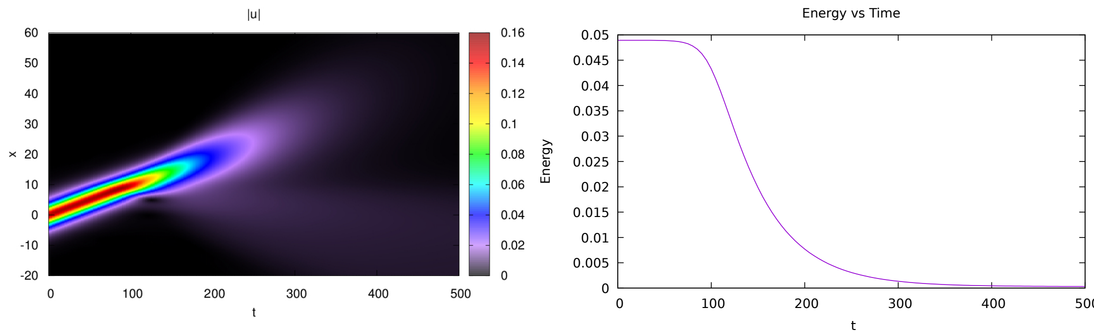

FIGURE 2. Second case results. Left: time evolution of the absolute value of the solution. Right: evolution of the energy.

APPENDIX

In this appendix, we shall prove the Lemmas 3.5 and 3.6.

Proof of the Lemma 3.5: From (1.9), we have

$$\begin{aligned} \int_0^T \|f(u)\|_{L^2(0,T)}\,dt &\le a_2 \int_0^T \|\,|u|^2\,u\,\|_{L^2(0,T)}\,dt + a_4 \int_0^T \|\,\left(|u|^2\,u\right)_x\,\|_{L^2(0,T)}\,dt \\ &\quad + a_5 \int_0^T \|\,u\,\left(|u|^2\right)_x\,\|_{L^2(0,T)}\,dt \\ &= I_1 + I_2 + I_3\,. \end{aligned} \tag{A.135}$$

Having in mind the embedding $H^1(0,L) \hookrightarrow L^\infty(0,L)$, we obtain

$$\begin{aligned} I_1 &\le a_2 \int_0^T \|u\|^2_{L^\infty(0,L)}\,\|u\|_{L^2(0,T)}\,dt \\ &\le \|u\|_{C([0,T];\,L^2(0,L))} \int_0^T \|u\|^2_{L^\infty(0,L)}\,dt \\ &\le C\,\|u\|_{X_T} \int_0^T \|u\|^2_{H_0^1(0,L)}\,dt \\ &= C\,\|u\|_{X_T}\,\|u\|_{L^2(0,T;\,H_0^1(0,L))} \\ &\le C\,\|u\|^2_{X_T}\,. \end{aligned} \tag{A.136}$$

On the other hand, since $|\,(|u|^2\,u)_x\,| \le 3\,|u|^2\,|u_x|$, we have

$$\begin{aligned} I_2 &\le 3\,a_4 \int_0^T \left[\int_0^L |u|^4\,|u_x|^2\,dx\right]^{1/2} dt \\ &\le 3\,a_4 \int_0^T \|u\|^2_{L^\infty(0,L)}\,\|u\|_{H_0^1(0,L)}\,dt\,. \end{aligned} \tag{A.137}$$

From (4.19) and (A.137) , it follows that

$$
\begin{aligned}
I_2 &\leq 3\, a_4 \int_0^T \|u\|^2_{L^\infty(0,L)}\, \|u\|_{H_0^1(0,L)}\, dt \\
&\leq C \int_0^T \|u\|_{L^2(0,L)}\, \|u\|^2_{H_0^1(0,L)}\, dt \\
&\leq \|u\|_{C([0,T];\, L^2(0,L))}\, \|u\|_{L^2(0,T;\, H_0^1(0,L))} \\
&\leq C\, \|u\|^2_{X_T}\,.
\end{aligned} \tag{A.138}
$$

Again, from (4.19) and the fact that $|\, u\ (|u|^2)_x\,| \leq 2\,|u|^2\,|u_x|$, we conclude

$$
\begin{aligned}
I_3 &\leq 2\, a_5 \int_0^T \left[\int_0^L |u|^4\, |u_x|^2\, dx\right]^{1/2} dt \\
&\leq 3\, a_4 \int_0^T \|u\|^2_{L^\infty(0,L)}\, \|u\|_{H_0^1(0,L)}\, dt \\
&\leq C \int_0^T \|u\|_{L^2(0,L)}\, \|u\|^2_{H_0^1(0,L)}\, dt \\
&\leq \|u\|_{C([0,T];\, L^2(0,L))}\, \|u\|_{L^2(0,T;\, H_0^1(0,L))} \\
&\leq C\, \|u\|^2_{X_T}\,.
\end{aligned} \tag{A.139}
$$

Therefore, the lemma is proved combining the estimates (A.135) – (A.139). □

Proof of the Lemma 3.6: From (1.9), we have

$$
\begin{aligned}
\int_0^T \|f(u)-f(v)\|_{L^2(0,L)}\, dt &\leq a_2 \int_0^T \|\, |u|^2\, u - |v|^2\, v\, \|_{L^2(0,L)}\, dt + a_4 \int_0^T \left\| \left(|u|^2\, u - |v|^2\, v\right)_x \right\|_{L^2(0,L)} dt \\
&+ a_5 \int_0^T \left\| u\ \left(|u|^2\right)_x - v\ \left(|v|^2\right)_x \right\|_{L^2(0,L)} dt \\
&:= J_1 + J_2 + J_3\,.
\end{aligned} \tag{A.140}
$$

Using the fact that $|\,|u|-|v|\,| \leq |u-v|$, the embedding $H_0^1(0,L) \hookrightarrow L^\infty(0,L)$ and adding and subtracting terms suitably, it yields

$$
\begin{aligned}
J_1 &= a_2 \int_0^T \|\, |u|^2\, (u-v)\|_{L^2(0,L)} + v\, (|u|^2 - |v|^2)\|_{L^2(0,L)}\, dt \\
&= a_2 \int_0^T \|\, |u|^2\, (u-v)\|_{L^2(0,L)} + \|\, |v|\ (|u|-|v|)\ \cdot\ (|u|+|v|)\, \|_{L^2(0,L)}\, dt \\
&\leq a_2 \int_0^T \|u\|^2_{L^\infty(0,L)}\, \|u-v\|_{L^2(0,L)}\, dt + \int_0^T \|v\|_{L^\infty(0,L)} \left(\|u\|_{L^\infty(0,L)} + \|v\|_{L^\infty(0,L)}\right) \|u-v\|_{L^2(0,L)}\, dt \\
&\leq a_2\, \|u-v\|_{C([0,T];\, L^2(0,L))} \left\{\int_0^T \|u\|^2_{L^\infty(0,L)}\, dt + \int_0^T \|v\|^2_{L^\infty(0,L)}\, dt\right\} \\
&\leq C \left\{\int_0^T \|u\|^2_{H_0^1(0,L)}\, dt + \int_0^T \|v\|^2_{H_0^1(0,L)}\, dt\right\} \|u-v\|_{X_T} \\
&\leq C \left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right)\, \|u-v\|_{X_T}\,.
\end{aligned} \tag{A.141}
$$

and

$$
\begin{aligned}
J_2 &\le a_4 \int_0^T \left\| u \, \left(|u|^2 - |v|^2\right)_x \right\|_{L^2(0,L)} \, dt + a_4 \int_0^T \left\| \left(|v|^2\right)_x \, (u-v) \right\|_{L^2(0,L)} \, dt \\
&\quad + a_4 \int_0^T \left\| \, |u|^2 \, (u-v)_x \right\|_{L^2(0,L)} \, dt + a_4 \, \left\| v_x \left(|u|^2 - |v|^2\right) \right\|_{L^2(0,L)} \\
&\quad =: J_{2,1} + J_{2,2} + J_{2,3} + J_{2,4} \, .
\end{aligned} \tag{A.142}
$$

We observe that

$$
\begin{aligned}
J_{2,1} &\le 2\, a_4 \int_0^T \|u\|_{L^\infty(0,L)} \, \|(u-v)\, u_x\|_{L^2(0,L)} \, dt + 2\, a_4 \int_0^T \|u\|_{L^\infty(0,L)} \, \|v\,(u-v)_x\|_{L^2(0,L)} \, dt \\
&\le C \, \|u-v\|_{C([0,T];\, L^2(0,L))} \int_0^T \|u\|^2_{H_0^1(0,L)} \, dt + 2\, a_4 \int_0^T \|u\|_{L^\infty(0,L)} \, \|v\|_{L^2(0,L)} \, \|u-v\|_{H_0^1(0,L)} \, dt \\
&\le C \, \|u-v\|_{C([0,T];\, L^2(0,L))} \int_0^T \|u\|^2_{H_0^1(0,L)} \, dt + 2\, a_4 \, \|v\|_{C([0,T];\, L^2(0,L))} \int_0^T \|u\|_{L^\infty(0,L)} \|u-v\|_{H_0^1(0,L)} \, dt \\
&\le C \, \|u\|^2_{X_T} \, \|u-v\|_{X_T} + C \, \|v\|_{X_T} \int_0^T \|u\|_{H_0^1(0,L)} \|u-v\|_{H_0^1(0,L)} \, dt \\
&\le C \, \|u\|_{X_T} \, \|u-v\|_{X_T} + C \, \|v\|_{X_T} \, \|u\|_{L^2(0,T);\, H_0^1(0,L)} \, \|u-v\|_{L^2(0,T;\, H_0^1(0,L))} \\
&\le C \, \left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right) \, \|u-v\|_{X_T} \, ;
\end{aligned} \tag{A.143}
$$

$$
\begin{aligned}
J_{2,2} &\le 2\, a_4 \int_0^T \|v\|_{L^\infty(0,L)} \, \|v\|_{H_0^1(0,L)} \, \|u-v\|_{L^2(0,L)} \, dt \\
&\le C \, \|u-v\|_{C([0,T];\, L^2(0,L))} \int_0^T \|u\|^2_{H_0^1(0,L)} \, dt \\
&\le C \, \|u\|^2_{X_T} \, \|u-v\|_{X_T} \, ;
\end{aligned} \tag{A.144}
$$

On the other hand, from (4.19) and Hölder's inequality, we get

$$
\begin{aligned}
J_{2,3} &\le 2\, a_4 \int_0^T \|u\|^2_{L^\infty(0,L)} \, \|u-v\|_{H_0^1(0,L)} \, dt \\
&\le C \int_0^T \|u\|_{H_0^1(0,L)} \, \|u\|_{L^2(0,L)} \, \|u-v\|_{H_0^1(0,L)} \, dt \\
&\le C \, \|u\|_{X_T} \, \|u\|_{L^2(0,T;\, H_0^1(0,L))} \, \|u-v\|_{L^2(0,T;\, H_0^1(0,L))} \\
&\le C \, \|u\|^2_{X_T} \, \|u-v\|_{X_T} \, ;
\end{aligned} \tag{A.145}
$$

$$
\begin{aligned}
J_{2,4} &\le 2\, a_4 \int_0^T \left(\|u\|_{L^\infty(0,L)} + \|v\|_{L^\infty(0,L)}\right) \, \|v\|_{H_0^1(0,L)} \, \|u-v\|_{L^2(0,L)} \, dt \\
&\le C \, \left[\int_0^T \|u\|^2_{H_0^1(0,L)} \, dt + \int_0^T \|v\|^2_{H_0^1(0,L)} \, dt\right] \|u-v\|_{X_T} \\
&\le C \, \left(\|u\|^2_{X_T} + \|v\|^2_{X_T}\right) \, \|u-v\|_{X_T} \, .
\end{aligned} \tag{A.146}
$$

Finally, by (A.143) and (A.144), we have

$$\begin{aligned} J_3 &\leq a_5 \int_0^T \left\| u \, \left(|u|^2 - |v|^2\right)_x \right\|_{L^2(0,L)} \, dt + a_5 \int_0^T \left\| \left(|v|^2\right)_x \, (u-v) \right\|_{L^2(0,L)} \, dt \\ &= J_{2,1} + J_{2,2} \\ &\leq C \, \left(\|u\|_{X_T}^2 + \|v\|_{X_T}^2 \right) \, \|u-v\|_{X_T} \, . \end{aligned} \tag{A.147}$$

Collecting the estimates (A.140) – (A.147), we obtain the desired. □

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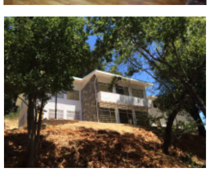

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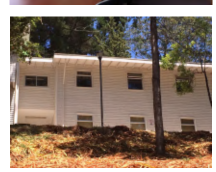
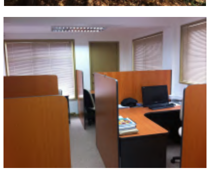

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