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This paper deals with the numerical analysis of a system of second-order in time partial differential equations modeling the vibrations of a coupled system that consists of an elastic solid in contact with an inviscid compressible fluid. We analyze a weak formulation with the unknowns in both media being the respective displacement fields. For its numerical approximation, we propose first a semi-discrete in space discretization based on standard Lagrangian elements in the solid and Raviart-Thomas elements in the fluid. We establish its wellposedness and derive error estimates in appropriate norms for the proposed scheme. In particular, we obtain an $\mathrm{L}^\infty(\mathrm{L}^2)$ optimal rate of convergence under minimal regularity assumptions of the solution, which are proved to hold for appropriate data of the problem. Then, we consider a fully discrete approximation based on a family of implicit finite difference schemes in time, from which we obtain optimal error estimates for sufficiently smooth solutions. Finally, we report some numerical results, which allow us to assess the performance of the method. These results also show that the numerical solution is not polluted by spurious modes as is the case with other alternative approaches.

Keywords: fluid-structure interaction; linear hyperbolic equations; non-conforming finite element discretization; error estimates.

1. Introduction

The aim of this paper is to analyze a numerical scheme to solve the elastoacoustic transient problem, namely, the evolution in time of a coupled system that consists of an elastic structure in contact with an acoustic fluid.

Different formulations have been tried to solve elastoacoustic problems, mainly in the frequency domain. While displacements are typically used for the solid, different variables have been used for the fluid: pressure (Zienkiewicz & Taylor (1991)), displacement potential (Morand & Ohayon (1979)), displacements (Kiefling & Feng (1976)), velocity potential (Everstine (1981)) or combinations of some of them (Morand & Ohayon (1995)). See also García *et al.* (2017a,b) for formulations in which pressure-stress variables are used instead of displacements for the solid.

We consider in this paper a pure displacements formulation, so that the same variable is used in both media, what makes easy to handle different interface conditions (see, for instance, Bermúdez & Rodríguez (1999)). A drawback of this formulation is the fact that the fluid displacements do not lie in

H^1 but in $\mathrm{H}(\mathrm{div})$. In spite of this, the first attempts to numerically solve the corresponding free-vibrations eigenproblem were based on using H^1 finite elements. However, such a procedure was readily seen to introduce spurious modes, which pollute the physical spectrum (Hamdi *et al.* (1978)). An alternative based on using $\mathrm{H}(\mathrm{div})$ elements in the fluid was proposed and analyzed in Bermúdez & Rodríguez (1994), Bermúdez *et al.* (1995) and Rodríguez & Solomin (1996), where in particular it was shown that the resulting method was free of spurious modes. In this paper, we consider a similar space discretization for the time domain elastoacoustic problem.

We consider first a continuous-time discrete Galerkin method and study its convergence. As in Bermúdez *et al.* (1995), we use a non-conforming space approximation based on lowest-order Lagrangian and Raviart-Thomas elements in the solid and the fluid domains, respectively. We prove optimal $\mathrm{L}^\infty(\mathrm{L}^2)$ error estimate of order $\mathscr{O}(h^r)$, where h is the mesh size and $r \in (0,1]$ depends on the regularity of the solution. The result is achieved under minimal regularity assumptions, which are proved to hold for appropriate data of the problem. The techniques used are based on classical results; see Baker (1976) for the wave equation and the more recent paper Basson & van Rensburg (2013) for an abstract setting. However, the results from the latter can not be directly applied to our setting due to the non-conforming character of the approximation.

Next, we study a fully discrete approximation resulting from applying a second-order Newmark-like scheme for the time discretization of the semidiscrete problem (see Bermúdez *et al.* (2003), where a similar approach was proved to be stable). Following the work of Karaa (2011) for the wave equation, we prove that the error exhibits a combined space-time asymptotic behavior of order $\mathscr{O}(h^r + \Delta t^2)$, where Δt is the time step.

The outline of the paper is as follows. In Section 2, we introduce the model and some functional spaces and obtain a well posed weak formulation. In Section 3, we introduce space discretizations for the solid and fluid displacements based on standard lowest-order Lagrangian and Raviart-Thomas elements, respectively. Then, we introduce a projector and use it to prove some properties that will be used for the error analysis. Section 4 is devoted to obtain an error estimate of the semi-discrete in space approximation under minimal regularity assumptions of the solution, which are proved to hold for appropriate data of the problem. In Section 5, we combine it with a family of implicit finite difference schemes in time and prove error estimates for the resulting full discretization. Finally, in Section 6, we report numerical results obtained for a test example, which show the convergence of the proposed numerical method. We also compare these results with those arising from an alternative H^1-approach usual in the engineering practice.

2. Problem statement

We consider a vessel completely filled with a fluid. Let Ω_{F} and $\Omega_{\mathrm{S}} \subset \mathbb{R}^d$, $d = 2,3$, be the polyhedral (polygonal for $d = 2$) domains occupied by the fluid and the solid, respectively, as shown in Fig. 1. Although all the forthcoming analysis holds true for $d = 2$ as well as $d = 3$, for the sake of definiteness we will use three-dimensional (3D) terminology throughout the paper. We assume that Ω_{F} is simply connected. Let Γ_{I} denote the interface between the solid and the fluid and $\boldsymbol{n}$ its unit normal vector pointing towards Ω_{S}. The exterior boundary of the solid is the union of Γ_{D} and Γ_{N}, the structure being fixed along Γ_{D}. Finally, let $\boldsymbol{\nu}$ be the unit outward normal vector along Γ_{N}.

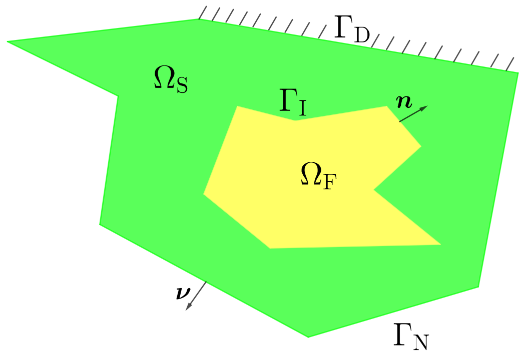


FIG. 1: Physical domain.

We consider the following notation for the physical quantities in the fluid:

- $\boldsymbol{u}^{\mathrm{F}}$ is the displacement vector,
- c is the sound velocity,
- ρ_{F} is the density

and in the solid:

- $\boldsymbol{u}^{\mathrm{S}}$ is the displacement vector,
- ρ_{S} is the density,
- λ_{S} and μ_{S} are the Lamé coefficients,
- $\boldsymbol{\varepsilon}\left(\boldsymbol{u}^{\mathrm{S}}\right)$ is the strain tensor defined by $\varepsilon_{ij} := \frac{1}{2}\left(\partial u_i^{\mathrm{S}}/\partial x_j + \partial u_j^{\mathrm{S}}/\partial x_i\right), \quad i,j = 1,\ldots,d,$
- $\boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right)$ is the stress tensor, which we assume related with the strains by Hooke's law, namely, $\sigma_{ij} = \lambda_{\mathrm{S}}\left(\sum_{k=1}^{d}\varepsilon_{kk}\right)\delta_{ij} + 2\mu_{\mathrm{S}}\varepsilon_{ij}, \quad i,j = 1,\ldots,d.$

When surface and volumetric loads $\boldsymbol{g}$ and $\boldsymbol{f}$ are applied on Γ_{N} and Ω_{S}, respectively, the equations governing the motion of the coupled system are

$$\rho_{\mathrm{S}}\partial_{tt}\boldsymbol{u}^{\mathrm{S}} - \mathbf{div}\,\boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right) = \boldsymbol{f} \quad \text{in } \Omega_{\mathrm{S}}, \tag{2.1}$$

$$\rho_{\mathrm{F}}\partial_{tt}\boldsymbol{u}^{\mathrm{F}} - \nabla\left(\rho_{\mathrm{F}}c^2\,\mathrm{div}\,\boldsymbol{u}^{\mathrm{F}}\right) = \mathbf{0} \quad \text{in } \Omega_{\mathrm{F}}, \tag{2.2}$$

$$\boldsymbol{u}^{\mathrm{S}}\cdot\boldsymbol{n} = \boldsymbol{u}^{\mathrm{F}}\cdot\boldsymbol{n} \quad \text{on } \Gamma_{\mathrm{I}}, \tag{2.3}$$

$$\boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right)\cdot\boldsymbol{n} = \rho_{\mathrm{F}}c^2\,\mathrm{div}\,\boldsymbol{u}^{\mathrm{F}}\boldsymbol{n} \quad \text{on } \Gamma_{\mathrm{I}}, \tag{2.4}$$

$$\boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right)\cdot\boldsymbol{\nu} = \boldsymbol{g} \quad \text{on } \Gamma_{\mathrm{N}}, \tag{2.5}$$

$$\boldsymbol{u}^{\mathrm{S}} = \mathbf{0} \quad \text{on } \Gamma_{\mathrm{D}}, \tag{2.6}$$

which must be completed with initial conditions

$$\boldsymbol{u}^{\mathrm{S}}(0) = \boldsymbol{u}_0^{\mathrm{S}}, \qquad \boldsymbol{u}^{\mathrm{F}}(0) = \boldsymbol{u}_0^{\mathrm{F}}, \qquad \partial_t\boldsymbol{u}^{\mathrm{S}}(0) = \boldsymbol{u}_1^{\mathrm{S}} \qquad \text{and} \qquad \partial_t\boldsymbol{u}^{\mathrm{F}}(0) = \boldsymbol{u}_1^{\mathrm{F}}. \tag{2.7}$$

Throughout this paper, we use standard notation for Sobolev spaces, norms and seminorms. Moreover, we introduce the spaces $\mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^d := \left\{\boldsymbol{v}^{\mathrm{S}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^d : \ \boldsymbol{v}^{\mathrm{S}} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{D}}\right\}$, which is a closed subspace of $\mathrm{H}^1(\Omega_{\mathrm{S}})^d$, $\mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}}) := \left\{\boldsymbol{v}^{\mathrm{F}} \in \mathrm{L}^2(\Omega_{\mathrm{F}})^d : \ \mathrm{div}\, \boldsymbol{v}^{\mathrm{F}} \in \mathrm{L}^2(\Omega_{\mathrm{F}})\right\}$, endowed with the norm defined by $\left\|\boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{H}(\mathrm{div},\Omega_{\mathrm{F}})} := \left\|\boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d} + \left\|\mathrm{div}\, \boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{L}^2(\Omega_{\mathrm{F}})}$, and

$$\mathrm{H}^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}) := \left\{\boldsymbol{v}^{\mathrm{F}} \in \mathrm{H}^{\alpha}(\Omega_{\mathrm{F}})^d : \ \mathrm{div}\, \boldsymbol{v}^{\mathrm{F}} \in \mathrm{H}^1(\Omega_{\mathrm{F}})\right\}, \qquad \alpha \geqslant 0,$$

with norm defined by $\left\|\boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{H}^{\alpha,1}(\mathrm{div},\Omega_{\mathrm{F}})} := \left\|\boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{H}^{\alpha}(\Omega_{\mathrm{F}})} + \left\|\mathrm{div}\, \boldsymbol{v}^{\mathrm{F}}\right\|^2_{\mathrm{H}^1(\Omega_{\mathrm{F}})}$. We define the product spaces

$$\boldsymbol{H} := \mathrm{L}^2(\Omega_{\mathrm{S}})^d \times \mathrm{L}^2(\Omega_{\mathrm{F}})^d \qquad \text{and} \qquad \boldsymbol{X} := \mathrm{H}^1_{\Gamma_{\mathrm{D}}}(\Omega_{\mathrm{S}})^d \times \mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}})$$

endowed with the corresponding product norms $\|\cdot\|_{\boldsymbol{H}}$ and $\|\cdot\|_{\boldsymbol{X}}$, respectively. We will use the notation $\boldsymbol{v} := \left(\boldsymbol{v}^{\mathrm{S}}, \boldsymbol{v}^{\mathrm{F}}\right)$ for functions in $\boldsymbol{H}$. $(\cdot,\cdot)$ for the classical inner product in $\boldsymbol{H}$. We will also denote $(\cdot,\cdot)_S$, where $S = \Omega_{\mathrm{S}}$, Ω_{F}, Γ_{N} or Γ_{I}, the respective $\mathrm{L}^2(S)^d$ inner product and $(\cdot,\cdot)_\rho$ the weighted inner product in $\boldsymbol{H}$ defined by

$$(\boldsymbol{u}, \boldsymbol{v})_\rho := \int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \boldsymbol{u}^{\mathrm{S}} \cdot \boldsymbol{v}^{\mathrm{S}} + \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \boldsymbol{u}^{\mathrm{F}} \cdot \boldsymbol{v}^{\mathrm{F}}.$$

Finally, we define the following spaces:

$$\boldsymbol{X}^{\alpha,\beta} := \mathrm{H}^{1+\beta}(\Omega_{\mathrm{S}})^d \times \mathrm{H}^{\alpha,1}(\mathrm{div}, \Omega_{\mathrm{F}}), \quad \alpha, \beta \geqslant 0, \qquad \boldsymbol{V} := \left\{\boldsymbol{v} \in \boldsymbol{X} : \ \boldsymbol{v}^{\mathrm{S}} \cdot \boldsymbol{n} = \boldsymbol{v}^{\mathrm{F}} \cdot \boldsymbol{n} \text{ on } \Gamma_{\mathrm{I}}\right\},$$

$$\boldsymbol{G} := \left\{(\boldsymbol{v}, \nabla q) : \ \boldsymbol{v} \in \mathrm{L}^2(\Omega_{\mathrm{S}})^d, \ q \in \mathrm{H}^1(\Omega_{\mathrm{F}})\right\} \qquad \text{and} \qquad \boldsymbol{K} := \{\boldsymbol{0}\} \times \mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{F}}),$$

where $\mathrm{H}_0(\mathrm{div}^0, \Omega_{\mathrm{F}}) := \left\{\boldsymbol{v}^{\mathrm{F}} \in \mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}}) : \ \mathrm{div}\, \boldsymbol{v}^{\mathrm{F}} = 0 \text{ in } \Omega_{\mathrm{F}} \text{ and } \boldsymbol{v}^{\mathrm{F}} \cdot \boldsymbol{n} = 0 \text{ on } \Gamma_{\mathrm{I}}\right\}$.

The following lemma gives a simple decomposition result of $\boldsymbol{H}$ and $\boldsymbol{V}$ which will be used below.

LEMMA 2.1 Let $\boldsymbol{G}_V := \boldsymbol{G} \cap \boldsymbol{V}$. Then,

(a) $\boldsymbol{H} = \boldsymbol{K} \oplus \boldsymbol{G}$ is an orthogonal decomposition in $\boldsymbol{H}$.

(b) $\boldsymbol{V} = \boldsymbol{K} \oplus \boldsymbol{G}_V$ is an orthogonal decomposition in both, $\boldsymbol{H}$ and $\boldsymbol{X}$ inner products.

(c) There exists $\widetilde{\alpha} > 1/2$ and $C > 0$ such that, for all $\left(\boldsymbol{\varphi}^{\mathrm{S}}, \boldsymbol{\varphi}^{\mathrm{F}}\right) \in \boldsymbol{G}_V$, there exists $q \in \mathrm{H}^{1+\widetilde{\alpha}}(\Omega_{\mathrm{F}})$, such that $\boldsymbol{\varphi}^{\mathrm{F}} = \nabla q$ and

$$\|\nabla q\|_{\mathrm{H}^{\widetilde{\alpha}}(\Omega_{\mathrm{F}})^d} \leqslant C\left[\left\|\mathrm{div}\, \boldsymbol{\varphi}^{\mathrm{F}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})} + \left\|\boldsymbol{\varphi}^{\mathrm{S}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right].$$

Proof. Notice that (a) follows immediately from the Helmholtz decomposition, whereas (b) follows from (a) and the fact that $\boldsymbol{K} \subset \boldsymbol{V}$. Note also that orthogonality in $\boldsymbol{X}$ and $\boldsymbol{H}$ coincide for functions in $\boldsymbol{K}$. Finally, let $\left(\boldsymbol{\varphi}^{\mathrm{S}}, \boldsymbol{\varphi}^{\mathrm{F}}\right) \in \boldsymbol{G}_V$. To prove (c), notice that $\boldsymbol{\varphi}^{\mathrm{F}} = \nabla q$, where q is a solution of the following Neumann problem:

$$\begin{cases} \mathrm{div}\,(\nabla q) = \mathrm{div}\, \boldsymbol{\varphi}^{\mathrm{F}} & \text{in } \Omega_{\mathrm{F}}, \\ \nabla q \cdot \boldsymbol{n} = \boldsymbol{\varphi}^{\mathrm{S}} \cdot \boldsymbol{n} & \text{on } \Gamma_{\mathrm{I}}. \end{cases} \tag{2.8}$$

The compatibility condition of this problem follows from the definition of the space $\boldsymbol{V}$. Moreover, from standard additional regularity results (Grisvard (2011); Dauge (1988)) we have that $\nabla q \in \mathrm{H}^{\widetilde{\alpha}}(\Omega_{\mathrm{F}})^d$ for some $\widetilde{\alpha} > 1/2$. ◻

To obtain a variational formulation of (2.1)–(2.7), we test equations (2.1) and (2.2) with $\boldsymbol{v}^{\mathrm{S}}$ and $\boldsymbol{v}^{\mathrm{F}}$ such that $\boldsymbol{v} = \left(\boldsymbol{v}^{\mathrm{S}}, \boldsymbol{v}^{\mathrm{F}}\right) \in \boldsymbol{V}$, respectively, to write

$$\int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \partial_{tt} \boldsymbol{u}^{\mathrm{S}} \cdot \boldsymbol{v}^{\mathrm{S}} + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right) : \boldsymbol{\varepsilon}\left(\boldsymbol{v}^{\mathrm{S}}\right) + \int_{\Gamma_{\mathrm{I}}} \boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right) \boldsymbol{n} \cdot \boldsymbol{v}^{\mathrm{S}} - \int_{\Gamma_{\mathrm{N}}} \boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right) \boldsymbol{\nu} \cdot \boldsymbol{v}^{\mathrm{S}} = \int_{\Omega_{\mathrm{S}}} \boldsymbol{f} \cdot \boldsymbol{v}^{\mathrm{S}}$$

and

$$\int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \partial_{tt} \boldsymbol{u}^{\mathrm{F}} \cdot \boldsymbol{v}^{\mathrm{F}} + \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \boldsymbol{u}^{\mathrm{F}} \operatorname{div} \boldsymbol{v}^{\mathrm{F}} - \int_{\Gamma_{\mathrm{I}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \boldsymbol{u}^{\mathrm{F}} \boldsymbol{v}^{\mathrm{F}} \cdot \boldsymbol{n} = 0.$$

By taking into account the interface and boundary conditions (2.4)–(2.5) and the fact that, for $\boldsymbol{v} \in \boldsymbol{V}$, $\boldsymbol{v}^{\mathrm{F}} \cdot \boldsymbol{n} = \boldsymbol{v}^{\mathrm{S}} \cdot \boldsymbol{n}$ on Γ_{I}, we arrive at the following problem.

Problem 2.1 Given $\left(\boldsymbol{u}_0^{\mathrm{S}}, \boldsymbol{u}_0^{\mathrm{F}}\right) \in \boldsymbol{V}$, $\left(\boldsymbol{u}_1^{\mathrm{S}}, \boldsymbol{u}_1^{\mathrm{F}}\right) \in \boldsymbol{H}$, $\boldsymbol{f} \in \mathrm{L}^2\left(0, T; \mathrm{L}(\Omega_{\mathrm{S}})^d\right)$ and $\boldsymbol{g} \in \mathrm{L}^2\left(0, T; \mathrm{L}^2(\Gamma_{\mathrm{N}})^d\right)$, find $\boldsymbol{u} \in \mathrm{L}^2(0, T; \boldsymbol{V})$ with $\partial_t \boldsymbol{u} \in \mathrm{L}^2(0, T; \boldsymbol{H})$ and $\partial_{tt} \boldsymbol{u} \in \mathrm{L}^2(0, T; \boldsymbol{X}')$ such that

$$\int_{\Omega_{\mathrm{S}}} \rho_{\mathrm{S}} \partial_{tt} \boldsymbol{u}^{\mathrm{S}} \cdot \boldsymbol{v}^{\mathrm{S}} + \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} \partial_{tt} \boldsymbol{u}^{\mathrm{F}} \cdot \boldsymbol{v}^{\mathrm{F}} + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}\left(\boldsymbol{u}^{\mathrm{S}}\right) : \boldsymbol{\varepsilon}\left(\boldsymbol{v}^{\mathrm{S}}\right) + \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \boldsymbol{u}^{\mathrm{F}} \operatorname{div} \boldsymbol{v}^{\mathrm{F}}$$
$$= \int_{\Omega_{\mathrm{S}}} \boldsymbol{f} \cdot \boldsymbol{v}^{\mathrm{S}} + \int_{\Gamma_{\mathrm{N}}} \boldsymbol{g} \cdot \boldsymbol{v}^{\mathrm{S}} \qquad \forall \boldsymbol{v} \in \boldsymbol{V} \quad (2.9)$$

and

$$\boldsymbol{u}^{\mathrm{S}}(0) = \boldsymbol{u}_0^{\mathrm{S}}, \qquad \boldsymbol{u}^{\mathrm{F}}(0) = \boldsymbol{u}_0^{\mathrm{F}}, \qquad \partial_t \boldsymbol{u}^{\mathrm{S}}(0) = \boldsymbol{u}_1^{\mathrm{S}} \qquad \text{and} \qquad \partial_t \boldsymbol{u}^{\mathrm{F}}(0) = \boldsymbol{u}_1^{\mathrm{F}}. \qquad (2.10)$$

Let us remark that the first two integrals in the equation above actually represent the duality pairing $\langle \partial_{tt} \boldsymbol{u}, \rho \boldsymbol{v} \rangle_{\boldsymbol{X}' \times \boldsymbol{X}}$, which is well defined. On the other hand, in view of (Dautray & Lions, 1992, Chapter XVIII, §1 Theorem 1), we known that $\boldsymbol{u} \in C(0, T; \boldsymbol{H})$ and $\partial_t \boldsymbol{u} \in C(0, T; \boldsymbol{X}')$. Consequently the equalities (2.10) above make sense.

Next, we define the bilinear symmetric form $a : \boldsymbol{X} \times \boldsymbol{X} \to \mathbb{R}$ by

$$a(\boldsymbol{v}, \boldsymbol{w}) := \int_{\Omega_{\mathrm{F}}} \rho_{\mathrm{F}} c^2 \operatorname{div} \boldsymbol{v}^{\mathrm{F}} \operatorname{div} \boldsymbol{w}^{\mathrm{F}} + \int_{\Omega_{\mathrm{S}}} \boldsymbol{\sigma}\left(\boldsymbol{v}^{\mathrm{S}}\right) : \boldsymbol{\varepsilon}\left(\boldsymbol{w}^{\mathrm{S}}\right), \qquad \boldsymbol{v} = (\boldsymbol{v}^{\mathrm{F}}, \boldsymbol{v}^{\mathrm{S}}),\ \boldsymbol{w} = (\boldsymbol{w}^{\mathrm{F}}, \boldsymbol{w}^{\mathrm{S}}) \in \boldsymbol{X}.$$

It is clear that, for all $\gamma > 0$, there exists $c > 0$ such that

$$a(\boldsymbol{v}, \boldsymbol{v}) + \gamma \|\boldsymbol{v}\|_{\boldsymbol{H}}^2 \geqslant c \|\boldsymbol{v}\|_{\boldsymbol{X}}^2 \qquad \forall \boldsymbol{v} \in \boldsymbol{X}. \qquad (2.11)$$

The existence and uniqueness of the solution $\boldsymbol{u}$ to Problem 2.1 is a consequence of the following result (see, (Bermúdez *et al.*, 2003, Theorem 1) and Santamarina (2002)).

THEOREM 2.2 (Existence) If $\boldsymbol{f} \in \mathrm{L}^2\left(0, T; \mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)$ and $\boldsymbol{g} \in \mathrm{H}^1\left(0, T; \mathrm{L}^2(\Gamma_{\mathrm{N}})^d\right)$, then there exists a unique solution to Problem 2.1, which satisfies

$$\boldsymbol{u} \in \mathrm{L}^\infty(0, T; \boldsymbol{V}) \qquad \text{and} \qquad \partial_t \boldsymbol{u} \in \mathrm{L}^\infty(0, T; \boldsymbol{H}).$$

3. Finite element space discretization

In this section, we introduce finite element spaces to approximate $\boldsymbol{V}$ in Problem 2.1. We notice that the equations for solid and fluid displacements involve different differential operators and functional spaces. Then, it makes sense to use different type of finite elements for each of them to discretize the variational problem. In fact, when those spaces are not chosen properly, non-physical oscillations may appear as will be shown in Section 6. In Bermúdez *et al.* (1995), a discretization avoiding spurious modes and leading to optimal order computation of eigenvalues and eigenfunctions was introduced and analyzed for the corresponding free-vibrations spectral problem. In this section we will use the same space discretization for the time-domain problem. Let us remark that a similar approach was proposed in Bermúdez *et al.* (2003), which was proved to be stable, although no convergence analysis was performed.

We consider a regular family of triangulations $\mathcal{T}_h^{\mathrm{F}}$ and $\mathcal{T}_h^{\mathrm{S}}$ of Ω_{F} and Ω_{S}, respectively, such that Γ_{D} and Γ_{N} are union of faces of tetrahedra in $\mathcal{T}_h^{\mathrm{S}}$. We also assume that the meshes are compatible on Γ_{I} in the sense that all faces of tetrahedra in $\mathcal{T}_h^{\mathrm{S}}$ lying on Γ_{I} are faces of tetrahedra in $\mathcal{T}_h^{\mathrm{F}}$, too.

For space discretization we use standard continuous piecewise linear elements for the solid displacement:

$$\boldsymbol{L}_h := \left\{ \boldsymbol{v}_h^{\mathrm{S}} \in \mathrm{H}^1(\Omega_{\mathrm{S}})^d : \ \boldsymbol{v}_h^{\mathrm{S}}|_T \in \mathbb{P}_1(T)^d \quad \forall T \in \mathcal{T}_h^{\mathrm{S}} \right\},$$

whereas for the fluid displacement we use lowest-order Raviart-Thomas elements:

$$\boldsymbol{R}_h := \left\{ \boldsymbol{v}_h^{\mathrm{F}} \in \mathrm{H}(\mathrm{div};\Omega_{\mathrm{F}}) : \ \boldsymbol{v}_h^{\mathrm{F}}|_T \in \boldsymbol{RT}_0(T) \quad \forall T \in \mathcal{T}_h^{\mathrm{F}} \right\},$$

where

$$\boldsymbol{RT}_0(T) := \left\{ \boldsymbol{v}_h^{\mathrm{F}} \in \mathbb{P}_1(T)^d : \ \boldsymbol{v}_h^{\mathrm{F}}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x}, \ \boldsymbol{a} \in \mathbb{R}^d, \ b \in \mathbb{R}, \ \boldsymbol{x} \in T \right\}.$$

Since imposing the kinematic constrain (2.4) in the discrete space would be too stringent, following Bermúdez *et al.* (1995) we do it in the following weak sense:

$$\int_E \boldsymbol{u}_h^{\mathrm{F}} \cdot \boldsymbol{n} = \int_E \boldsymbol{u}_h^{\mathrm{S}} \cdot \boldsymbol{n} \qquad \text{for all faces } E \subset \Gamma_{\mathrm{I}}. \tag{3.1}$$

Thus, the discrete analogue of $\boldsymbol{V}$ is

$$\boldsymbol{V}_h := \left\{ \left(\boldsymbol{u}_h^{\mathrm{S}}, \boldsymbol{u}_h^{\mathrm{F}}\right) \in \boldsymbol{L}_h \times \boldsymbol{R}_h \text{ satisfying (3.1) and } \boldsymbol{u}^{\mathrm{S}} = \boldsymbol{0} \text{ on } \Gamma_{\mathrm{D}} \right\}.$$

From the definition of the continuous space $\boldsymbol{V}$ and its discrete counterpart $\boldsymbol{V}_h$, it can be seen that we are dealing with a non-conforming approximation of Problem 2.1. Indeed, in general $\boldsymbol{u}_h^{\mathrm{F}} \cdot \boldsymbol{n} \neq \boldsymbol{u}_h^{\mathrm{S}} \cdot \boldsymbol{n}$ on Γ_{I} and, then, $\boldsymbol{V}_h \not\subset \boldsymbol{V}$.

To deal with this non-conforming approximation, the following estimate will be used in the sequel. The same result can be found in Bermúdez *et al.* (1995) in a two-dimensional (2D) setting. For the sake of completeness, we include an elementary proof. Here and thereafter, we will denote by C a generic positive constant, not necessarily the same at each occurrence, but always independent of the mesh-size h and, in the following sections, of the time-step Δt, too.

LEMMA 3.1 Let $\boldsymbol{z}_h = \left(\boldsymbol{z}_h^{\mathrm{S}}, \boldsymbol{z}_h^{\mathrm{F}}\right) \in \boldsymbol{V}_h$ and $\boldsymbol{w}^{\mathrm{F}} \in \mathrm{H}^{0,1}(\mathrm{div}, \Omega_{\mathrm{F}})$. Then,

$$\int_{\Gamma_{\mathrm{I}}} \mathrm{div}\, \boldsymbol{w}^{\mathrm{F}} \left(\boldsymbol{z}_h^{\mathrm{F}} - \boldsymbol{z}_h^{\mathrm{S}}\right) \cdot \boldsymbol{n} \leqslant Ch \left|\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{F}})} \left|\boldsymbol{z}_h^{\mathrm{S}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}.$$

Proof. Let $\boldsymbol{z}_h \in \boldsymbol{V}_h$ and $\boldsymbol{w}^{\mathrm{F}} \in \mathrm{H}(\mathrm{div}, \Omega_{\mathrm{F}})$ be such that $\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}} \in \mathrm{H}^1(\Omega_{\mathrm{F}})$. For any face $E \subset \Gamma_{\mathrm{I}}$, let $T_F \in \mathcal{T}_h^{\mathrm{F}}$ and $T_S \in \mathcal{T}_h^{\mathrm{S}}$ be the tetrahedra such that $\partial T_F \cap \partial T_F = E$. Let P_E denote the $\mathrm{L}^2(E)$-projection of $\mathrm{H}^{1/2}(E)$ onto the constants. Since $\boldsymbol{z}_h^{\mathrm{F}} \cdot \boldsymbol{n} = P_E\left(\boldsymbol{z}_h^{\mathrm{S}} \cdot \boldsymbol{n}\right)$, we have that

$$\begin{aligned}\int_E \mathrm{div}\, \boldsymbol{w}^{\mathrm{F}} \left(\boldsymbol{z}_h^{\mathrm{F}} - \boldsymbol{z}_h^{\mathrm{S}}\right) \cdot \boldsymbol{n} &= \int_E \left[\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}} - P_E\left(\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}}\right)\right] \left[P_E\left(\boldsymbol{z}_h^{\mathrm{S}} \cdot \boldsymbol{n}\right) - \boldsymbol{z}_h^{\mathrm{S}} \cdot \boldsymbol{n}\right] \\ &\leqslant \left\|\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}} - P_E\left(\mathrm{div}\, \boldsymbol{w}^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(E)} \left\|P_E\left(\boldsymbol{z}_h^{\mathrm{S}} \cdot \boldsymbol{n}\right) - \boldsymbol{z}_h^{\mathrm{S}} \cdot \boldsymbol{n}\right\|_{\mathrm{L}^2(E)}.\end{aligned}$$

If P_{T_F} denotes the $\mathrm{L}^2(T_F)$-projection of $\mathrm{H}^1(T_F)$ onto the constants, from a local trace theorem and standard error estimates we have

$$\begin{aligned}\left\|\operatorname{div}\boldsymbol{w}^{\mathrm{F}} - P_E\left(\operatorname{div}\boldsymbol{w}^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(E)} &\leqslant \left\|\operatorname{div}\boldsymbol{w}^{\mathrm{F}} - P_{T_F}\left(\operatorname{div}\boldsymbol{w}^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(E)} \\ &\leqslant C\left[h^{-1/2}\left\|\operatorname{div}\boldsymbol{w}^{\mathrm{F}} - P_{T_F}\left(\operatorname{div}\boldsymbol{w}^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(T_F)} + h^{1/2}\left|\operatorname{div}\boldsymbol{w}^{\mathrm{F}} - P_{T_F}\left(\operatorname{div}\boldsymbol{w}^{\mathrm{F}}\right)\right|_{\mathrm{H}^1(T_F)}\right] \\ &\leqslant Ch^{1/2}\left|\operatorname{div}\boldsymbol{w}^{\mathrm{F}}\right|_{\mathrm{H}^1(T_F)}.\end{aligned}$$

Similarly, $\left\|P_E\left(\boldsymbol{z}_h^{\mathrm{S}}\cdot\boldsymbol{n}\right) - \boldsymbol{z}_h^{\mathrm{S}}\cdot\boldsymbol{n}\right\|_{\mathrm{L}^2(E)} \leqslant Ch^{1/2}\left|\boldsymbol{z}_h^{\mathrm{S}}\right|_{\mathrm{H}^1(T_S)^d}$. Thus, the result follows from the two previous estimates. □

For the numerical analysis that will be performed in the following sections, we will use the elliptic projector $\mathbf{P}_h : \boldsymbol{V} \to \boldsymbol{V}_h$, defined for any $\boldsymbol{v} \in \boldsymbol{V}$ by

$$\mathbf{P}_h\boldsymbol{v} \in \boldsymbol{V}_h : \qquad a\left(\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{w}_h\right) + \left(\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}, \boldsymbol{w}_h\right)_\rho = 0 \qquad \forall \boldsymbol{w}_h \in \boldsymbol{V}_h. \tag{3.2}$$

From (2.11), it is clear that $\mathbf{P}_h : \boldsymbol{V} \to \boldsymbol{V}_h$ is a well posed continuous operator. In 2D, the following error estimate follows from (Bermúdez *et al.*, 1995, Theorem 5.2):

$$\left\|\boldsymbol{v} - \mathbf{P}_h\boldsymbol{v}\right\|_{\boldsymbol{X}} \leqslant C \inf_{\boldsymbol{w}_h\in\boldsymbol{V}_h}\left\|\boldsymbol{v} - \boldsymbol{w}_h\right\|_{\boldsymbol{X}} \leqslant Ch^{\min\{\alpha,\beta\}}\left\|\boldsymbol{v}\right\|_{\boldsymbol{X}^{\alpha,\beta}} \qquad \forall \boldsymbol{v} \in \boldsymbol{X}^{\alpha,\beta}\cap\boldsymbol{V} \tag{3.3}$$

with $\alpha \in \left(\frac{1}{2}, 1\right]$ and $\beta \in (0,1]$. Its extension to 3D is straightforward.

We will also need an approximation result for the projector $\mathbf{P}_h$ in the $\boldsymbol{H}$-norm for functions $\boldsymbol{v}$ not necessarily in $\boldsymbol{X}^{\alpha,\beta}$. With this end, we will use the following lemma, which can be proved by proceeding as in (Bermúdez *et al.*, 1995, Lemma 5.5).

LEMMA 3.2 Let $\boldsymbol{v}_h = \left(\boldsymbol{v}_h^{\mathrm{S}}, \boldsymbol{v}_h^{\mathrm{F}}\right) \in \boldsymbol{V}_h$ be such that

$$\left(\boldsymbol{v}_h, \boldsymbol{\varphi}_h\right)_\rho = 0 \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{K}\cap\boldsymbol{V}_h.$$

Then, there exist $\boldsymbol{v}_{\boldsymbol{K}} \in \boldsymbol{K}$ and $\boldsymbol{v}_{\boldsymbol{G}} \in \boldsymbol{G}$ such that $\boldsymbol{v}_h = \boldsymbol{v}_{\boldsymbol{K}} + \boldsymbol{v}_{\boldsymbol{G}}$ and

$$\left\|\boldsymbol{v}_{\boldsymbol{K}}\right\|_{\boldsymbol{H}} \leqslant Ch^\alpha\left[\left\|\operatorname{div}\boldsymbol{v}_h^{\mathrm{F}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})} + \left\|\boldsymbol{v}_h^{\mathrm{S}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right],$$

where $\alpha := \min\{\widetilde{\alpha}, 1\}$, with $\widetilde{\alpha}$ as in Lemma 2.1(c).

We will also use the $\boldsymbol{V}_h$ interpolant defined in (Bermúdez *et al.*, 1995, Section 5): $\mathbf{I}_h^V : \boldsymbol{X}^{\alpha,\beta}\cap\boldsymbol{V} \to \boldsymbol{V}_h$, with $\alpha \in \left(\frac{1}{2}, 1\right]$ and $\beta \in (0,1]$. It is proved in Theorem 5.2 from this reference that, in 2D,

$$\left\|\boldsymbol{v} - \mathbf{I}_h^V\boldsymbol{v}\right\|_{\boldsymbol{X}} \leqslant Ch^{\min\{\alpha,\beta\}}\left\|\boldsymbol{v}\right\|_{\boldsymbol{X}^{\alpha,\beta}}. \tag{3.4}$$

Once again, its extension to 3D is straightforward.

Now, we are in a position to prove an estimate for the projector $\mathbf{P}_h$ in the $\boldsymbol{H}$-norm by means of a duality argument.

LEMMA 3.3 There exist constants $C > 0$, $\alpha \in \left(\frac{1}{2}, 1\right]$ and $\beta \in (0,1]$ such that, for all $\boldsymbol{v} \in \boldsymbol{G}_V$,

$$\left\|\boldsymbol{v} - \mathbf{P}_h\boldsymbol{v}\right\|_{\boldsymbol{H}} \leqslant Ch^{\min\{\alpha,\beta\}}\left\|\boldsymbol{v}\right\|_{\boldsymbol{X}}.$$

Proof. Let $\boldsymbol{v} \in \boldsymbol{G}_V$. We consider the following auxiliary problem: Find $\boldsymbol{\varphi} = \left(\boldsymbol{\varphi}^{\mathrm{S}}, \boldsymbol{\varphi}^{\mathrm{F}}\right) \in \boldsymbol{V}$ such that

$$\widehat{a}(\boldsymbol{w},\boldsymbol{\varphi}) := a(\boldsymbol{w},\boldsymbol{\varphi}) + (\boldsymbol{w},\boldsymbol{\varphi})_\rho = (\boldsymbol{w}, \mathbf{P}_h\boldsymbol{v} - \boldsymbol{v})_\rho \qquad \forall \boldsymbol{w} \in \boldsymbol{V}. \tag{3.5}$$

From (2.11), it follows that there exists a unique $\boldsymbol{\varphi} \in \boldsymbol{V}$ solution of the above problem and that it satisfies

$$\|\boldsymbol{\varphi}\|_{\boldsymbol{X}} \leqslant C \|\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}}. \tag{3.6}$$

Moreover, it is easy to check that $\boldsymbol{\varphi} = \left(\boldsymbol{\varphi}^{\mathrm{S}}, \boldsymbol{\varphi}^{\mathrm{F}}\right)$ is the solution to the following problem:

$$\begin{aligned}
\mathbf{div}\,\boldsymbol{\sigma}\left(\boldsymbol{\varphi}^{\mathrm{S}}\right) + \rho_S \boldsymbol{\varphi}^{\mathrm{S}} &= \rho_S \left(\mathbf{P}_h\boldsymbol{v}\right)^{\mathrm{S}} - \rho_S \boldsymbol{v}^{\mathrm{S}} && \text{in } \Omega_{\mathrm{S}}, && (3.7)\\
\boldsymbol{\sigma}\left(\boldsymbol{\varphi}^{\mathrm{S}}\right)\cdot\boldsymbol{n} &= \rho_{\mathrm{F}} c^2 \operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}}\boldsymbol{n} && \text{on } \Gamma_{\mathrm{I}}, && (3.8)\\
\boldsymbol{\sigma}\left(\boldsymbol{\varphi}^{\mathrm{S}}\right)\cdot\boldsymbol{\nu} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{N}}, && (3.9)\\
\boldsymbol{\varphi}^{\mathrm{S}} &= \mathbf{0} && \text{on } \Gamma_{\mathrm{D}}, && (3.10)\\
\nabla\left(\rho_{\mathrm{F}} c^2 \operatorname{div}\left(\boldsymbol{\varphi}^{\mathrm{F}}\right)\right) + \rho_F \boldsymbol{\varphi}^{\mathrm{F}} &= \rho_F \left(\mathbf{P}_h\boldsymbol{v}\right)^{\mathrm{F}} - \rho_F \boldsymbol{v}^{\mathrm{F}} && \text{in } \Omega_{\mathrm{F}}, && (3.11)\\
\boldsymbol{\varphi}^{\mathrm{S}}\cdot\boldsymbol{n} &= \boldsymbol{\varphi}^{\mathrm{F}}\cdot\boldsymbol{n} && \text{on} \Gamma_{\mathrm{I}}. && (3.12)
\end{aligned}$$

We readily see from (3.11) that $\operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}} \in \mathrm{H}^1(\Omega_{\mathrm{F}})$ and

$$\left\|\nabla\left(\operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d} \leqslant C\left[\left\|\left(\mathbf{P}_h\boldsymbol{v}\right)^{\mathrm{F}} - \boldsymbol{v}^{\mathrm{F}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d} + \left\|\boldsymbol{\varphi}^{\mathrm{F}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d}\right].$$

Thus, from (3.6) we obtain

$$\left\|\operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{F}})} \leqslant C \|\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}}.$$

On the other hand, (3.7)–(3.10) is a standard problem of linear elasticity. Hence, by using classical additional regularity results for this problem (see Grisvard (2011); Dauge (1988)) it turns out that $\boldsymbol{\varphi}^{\mathrm{S}}$ belongs to $\mathrm{H}^{1+\widetilde{\beta}}(\Omega_{\mathrm{S}})^d$ for some $\widetilde{\beta} > 0$. Moreover, the following estimate holds true:

$$\left\|\boldsymbol{\varphi}^{\mathrm{S}}\right\|_{\mathrm{H}^{1+\widetilde{\beta}}(\Omega_{\mathrm{S}})} \leqslant C\left[\left\|\left(\mathbf{P}_h\boldsymbol{v}\right)^{\mathrm{S}} - \boldsymbol{v}^{\mathrm{S}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^d} + \left\|\operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{F}})}\right] \leqslant C \|\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}}.$$

Next, by using Lemma 2.1(b) we decompose $\boldsymbol{\varphi} = \boldsymbol{\varphi}_{\boldsymbol{K}} + \boldsymbol{\varphi}_{\boldsymbol{G}}$ with $\boldsymbol{\varphi}_{\boldsymbol{K}} \in \boldsymbol{K}$ and $\boldsymbol{\varphi}_{\boldsymbol{G}} \in \boldsymbol{G}_V$. From Lemma 2.1(c) it follows that $\boldsymbol{\varphi}_{\boldsymbol{G}} = \left(\boldsymbol{\varphi}^{\mathrm{S}}, \nabla q\right)$ with $\nabla q \in \mathrm{H}^{\widetilde{\alpha}}(\Omega_{\mathrm{F}})^d$ and $\widetilde{\alpha} > 1/2$. Moreover, since $\operatorname{div}(\nabla q) = \operatorname{div}\boldsymbol{\varphi}^{\mathrm{F}}$, from the above equations we obtain that

$$\boldsymbol{\varphi}_{\boldsymbol{G}} \in \mathrm{H}^{1+\widetilde{\beta}}(\Omega_{\mathrm{S}})^d \times \mathrm{H}^{\widetilde{\alpha},1}(\operatorname{div},\Omega_{\mathrm{F}}) \qquad \text{and} \qquad \|\boldsymbol{\varphi}_{\boldsymbol{G}}\|_{\boldsymbol{X}^{\widetilde{\alpha},\widetilde{\beta}}} \leqslant \|\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v}\|_{\boldsymbol{H}}. \tag{3.13}$$

Now, from Lemma 2.1(a) we write $\mathbf{P}_h\boldsymbol{v} = \boldsymbol{\eta} + \boldsymbol{\chi}$ with $\boldsymbol{\eta} \in \boldsymbol{G}$ and $\boldsymbol{\chi} \in \boldsymbol{K}$. By taking $\boldsymbol{w} \in \boldsymbol{K}$ in (3.5), it follows that $\boldsymbol{\varphi}_{\boldsymbol{K}} = \boldsymbol{\chi}$; in fact,

$$(\boldsymbol{w},\boldsymbol{\varphi}_{\boldsymbol{K}})_\rho = (\boldsymbol{w},\boldsymbol{\varphi})_\rho = (\boldsymbol{w},\mathbf{P}_h\boldsymbol{v} - \boldsymbol{v})_\rho = (\boldsymbol{w},\boldsymbol{\chi})_\rho \qquad \forall \boldsymbol{w} \in \boldsymbol{K}.$$

Moreover, from (3.2) it follows that

$$(\mathbf{P}_h\boldsymbol{v}, \boldsymbol{w}_h)_\rho = 0 \qquad \forall \boldsymbol{w}_h \in \boldsymbol{K} \cap \boldsymbol{V}_h$$

and, thus, from Lemma 3.2 we arrive at

$$\|\boldsymbol{\varphi}_{\boldsymbol{K}}\|_{\boldsymbol{H}} \leqslant Ch^{\alpha}\left[\left\|\operatorname{div}(\mathbf{P}_h\boldsymbol{v})^{\mathrm{F}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d} + \left\|(\mathbf{P}_h\boldsymbol{v})^{\mathrm{S}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right] \tag{3.14}$$

with $\alpha := \min\{\widetilde{\alpha},1\}$. To obtain the estimate of $\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}$ in the $\boldsymbol{H}$-norm, we take $\boldsymbol{w} = \mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}$ in (3.5) and, then, from (3.2) we write

$$(\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v},\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v})_\rho = \widehat{a}(\boldsymbol{\varphi},\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}) = \widehat{a}\left(\boldsymbol{\varphi}_{\boldsymbol{G}} - \mathbf{I}_h^V\boldsymbol{\varphi}_{\boldsymbol{G}},\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}\right) + (\boldsymbol{\varphi}_{\boldsymbol{K}},\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v})_\rho .$$

For the first term on the right-hand side above we use the following estimate (cf. (3.4) and (3.13)):

$$\left\|\boldsymbol{\varphi}_{\boldsymbol{G}} - \mathbf{I}_h^V\boldsymbol{\varphi}_{\boldsymbol{G}}\right\|_{\boldsymbol{X}} \leqslant Ch^{\min\{\alpha,\beta\}}\|\boldsymbol{\varphi}_{\boldsymbol{G}}\|_{\boldsymbol{X}^{\alpha,\beta}} \leqslant Ch^{\min\{\alpha,\beta\}}\|\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{H}},$$

where $\beta := \min\{\widetilde{\beta},1\}$. For the second one we use Cauchy-Schwarz inequality and estimate (3.14). Thus, we obtain

$$\|\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{H}} \leqslant Ch^{\min\{\alpha,\beta\}}\left(\|\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{X}} + \|\mathbf{P}_h\boldsymbol{v}\|_{\boldsymbol{X}}\right).$$

Therefore, the result follows from the previous inequality by estimating the right-hand side by using that $\|\mathbf{P}_h\boldsymbol{v}-\boldsymbol{v}\|_{\boldsymbol{X}} + \|\mathbf{P}_h\boldsymbol{v}\|_{\boldsymbol{X}} \leqslant C\|\boldsymbol{v}\|_{\boldsymbol{X}}$. □

REMARK 3.1 The numerical methods that will be introduced below will be proved to converge with order $\mathscr{O}\left(h^{\min\{\alpha,\beta\}}\right)$, where $\alpha := \min\{\widetilde{\alpha},1\}$ and $\beta := \min\{\widetilde{\beta},1\}$ with $\widetilde{\alpha}$ and $\widetilde{\beta}$ being the Sobolev exponents of the Neumann problem (2.8) and the linear elasticity problem (3.7)–(3.10), respectively.

4. The semidiscrete in space Galerkin approximation

The aim of this section is to introduce a semidiscrete in space Galerkin approximation of Problem 2.1 and to obtain error estimates under regularity assumptions of the solution that will be shown to hold for appropriate data of the problem. For the forthcoming analysis we will only need to assume that

$$\partial_t\boldsymbol{u} \in \mathrm{L}^1(0,T;\boldsymbol{V}) \qquad \text{and} \qquad \partial_{tt}\boldsymbol{u} \in \mathrm{L}^2(0,T;\boldsymbol{H}), \tag{4.1}$$

in order to prove an L^2-like error estimate for this semidiscrete approximation. Although this assumption could seem restrictive, in the following lemma we will show that it holds true under appropriate assumptions that include the following compatibility condition on $\boldsymbol{u}_0 := \left(\boldsymbol{u}_0^{\mathrm{S}},\boldsymbol{u}_0^{\mathrm{F}}\right)$:

$$\begin{aligned} &\mathbf{div}\left(\boldsymbol{\sigma}\left(\boldsymbol{u}_0^{\mathrm{S}}\right)\right) \in \mathrm{L}^2(\Omega_{\mathrm{S}})^d, && \nabla\left(\operatorname{div}\boldsymbol{u}_0^{\mathrm{F}}\right) \in \mathrm{L}^2(\Omega_{\mathrm{F}})^d,\\ &\boldsymbol{\sigma}\left(\boldsymbol{u}_0^{\mathrm{S}}\right)\cdot\boldsymbol{n} = \rho_{\mathrm{F}}c^2\operatorname{div}\boldsymbol{u}_0^{\mathrm{F}}\boldsymbol{n} \text{ on } \Gamma_{\mathrm{I}} \quad \text{and} && \boldsymbol{\sigma}\left(\boldsymbol{u}_0^{\mathrm{S}}\right)\cdot\boldsymbol{\nu} = \boldsymbol{g} \text{ on } \Gamma_{\mathrm{N}}. \end{aligned} \tag{4.2}$$

LEMMA 4.1 (Regularity) Let $\boldsymbol{u}$ be the solution to Problem 2.1. If $\boldsymbol{f} \in \mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)$, $\boldsymbol{g} = \mathbf{0}$, $\left(\boldsymbol{u}_0^{\mathrm{S}},\boldsymbol{u}_0^{\mathrm{F}}\right)$ satisfies (4.2) and $\left(\boldsymbol{u}_1^{\mathrm{S}},\boldsymbol{u}_1^{\mathrm{F}}\right) \in \boldsymbol{V}$, then

$$\boldsymbol{u} \in C^1(0,T;\boldsymbol{V}) \cap C^2(0,T;\boldsymbol{H}) \tag{4.3}$$

and, consequently, (4.1) holds true. Moreover, the following estimate holds:

$$\begin{aligned} &\|\partial_{tt}\boldsymbol{u}\|_{\mathrm{L}^\infty(0,T;\boldsymbol{H})} + \|\partial_t\boldsymbol{u}\|_{\mathrm{L}^\infty(0,T;\boldsymbol{X})}\\ &\qquad \leqslant C\left[\|\boldsymbol{u}_1\|_{\boldsymbol{V}} + \left\|\mathbf{div}\,\boldsymbol{\sigma}\left(\boldsymbol{u}_0^{\mathrm{S}}\right)\right\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^d} + \left\|\nabla\left(\operatorname{div}\boldsymbol{u}_0^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d} + \|\boldsymbol{f}\|_{\mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)}\right]. \end{aligned} \tag{4.4}$$

Proof. Since $C^1\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)$ is dense in $\mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)$, it is enough to prove the lemma for $\boldsymbol{f}$ in the former. Let $\boldsymbol{u}$ be the unique solution to Problem 2.1. We will derive (4.3) as a consequence of Theorem 2.1 from Basson & van Rensburg (2013) (see also (Showalter, 1994, Chapter IV)). With this aim we consider the following auxiliary problem: find $\boldsymbol{\phi} \in C(0,T;\boldsymbol{V})$ such that $\partial_t\boldsymbol{\phi}$ is continuous at 0 and for each $t\in[0,T]$, $\boldsymbol{\phi}(t)\in\boldsymbol{V}$, $\partial_t\boldsymbol{\phi}(t)\in\boldsymbol{V}$, $\partial_{tt}\boldsymbol{\phi}(t)\in\boldsymbol{H}$ and

$$(\partial_{tt}\boldsymbol{\phi},\boldsymbol{v})_\rho + 2\,(\partial_t\boldsymbol{\phi},\boldsymbol{v})_\rho + a\,(\boldsymbol{\phi},\boldsymbol{v}) + (\boldsymbol{\phi},\boldsymbol{v})_\rho = (\widehat{\boldsymbol{f}},\boldsymbol{v})_\mathrm{S} \qquad \forall\boldsymbol{v}\in\boldsymbol{V} \tag{4.5}$$

with $\widehat{\boldsymbol{f}} := e^t\boldsymbol{f} \in C^1\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)$ and initial conditions

$$\boldsymbol{\phi}(0) = \boldsymbol{u}_0 \quad\text{ and }\quad \partial_t\boldsymbol{\phi}(0) = \boldsymbol{u}_1 + \boldsymbol{u}_0. \tag{4.6}$$

In order to apply (Basson & van Rensburg, 2013, Theorem 2.1), we notice that assumptions (E.1)–(E.4) from this reference can be easily checked. Moreover, it is also necessary to check that there exits $\boldsymbol{u}_2\in\boldsymbol{H}$ such that

$$(\boldsymbol{u}_2,\boldsymbol{v})_\rho = a\,(\boldsymbol{u}_0,\boldsymbol{v}) \qquad \forall\boldsymbol{v}\in\boldsymbol{V}, \tag{4.7}$$

which according to (4.2) follows in our case for $\boldsymbol{u}_2 := \left(\mathbf{div}\,\boldsymbol{\sigma}\left(\boldsymbol{u}_0^\mathrm{S}\right), \nabla\left(\operatorname{div}\boldsymbol{u}_0^\mathrm{F}\right)\right)$. Thus we are in a position to apply (Basson & van Rensburg, 2013, Theorem 2.1) and conclude that problem (4.5)–(4.6) has a unique solution $\boldsymbol{\phi}$ that belongs to $C^1(0,T;\boldsymbol{V})\cap C^2(0,T;\boldsymbol{H})$. Moreover, it is straightforward to check that $e^{-t}\boldsymbol{\phi}$ is the solution to Problem 2.1, which allows us to conclude the first part of the lemma.

Estimate (4.4) follows from classical arguments which include the Faedo-Galerkin method (see, for instance, (Lions & Magenes, 1972, Chapter 5)). For the sake of completeness we include the main arguments. We (formally) differentiate (2.9) with respect to time and take $\boldsymbol{v} = \partial_{tt}\boldsymbol{u}(t)$ as a test function. Then, by integration by parts it follows that

$$\begin{aligned}\frac{1}{2}\,(\partial_{tt}\boldsymbol{u}(t),\partial_{tt}\boldsymbol{u}(t))_\rho + \frac{1}{2}a\,(\partial_t\boldsymbol{u}(t),\partial_t\boldsymbol{u}(t))&\\ = \frac{1}{2}\,(\partial_{tt}\boldsymbol{u}(0),\partial_{tt}\boldsymbol{u}(0))_\rho + \frac{1}{2}a\,(\partial_t\boldsymbol{u}(0),\partial_t\boldsymbol{u}(0)) &+ \int_0^t (\partial_t\boldsymbol{f}(s),\partial_{tt}\boldsymbol{u}(s))_\mathrm{S}\,ds.\end{aligned}$$

To estimate the first term on the right-hand side we notice that $(\partial_{tt}\boldsymbol{u}(0),\boldsymbol{v})_\rho = (\boldsymbol{f}(0),\boldsymbol{v}) - a\,(\boldsymbol{u}_0,\boldsymbol{v})$ for all $\boldsymbol{v}\in\boldsymbol{V}$. Then, from (4.7) it follows that

$$\begin{aligned}\|\partial_{tt}\boldsymbol{u}(t)\|_{\boldsymbol{H}}^2 + a\,(\partial_t\boldsymbol{u}(t),\partial_t\boldsymbol{u}(t))&\\ \leqslant C\left[\|\boldsymbol{f}(0)\|_{\mathrm{L}^2(\Omega_\mathrm{S})^d}^2 + \|\boldsymbol{u}_2\|_{\boldsymbol{H}}^2 + \|\boldsymbol{u}_1\|_{\boldsymbol{V}}^2 + \int_0^t \|\partial_t\boldsymbol{f}(s)\|_{\mathrm{L}^2(\Omega_\mathrm{S})^d}^2\,ds\right] &+ \int_0^t \|\partial_{tt}\boldsymbol{u}(s)\|_{\boldsymbol{H}}^2\,ds.\end{aligned}$$

Finally, (4.4) follows from this, Gronwall's inequality (see, Evans (2010)) and (2.11). □

From the regularity of $\boldsymbol{u}$ we can also obtain the following property of the solution.

LEMMA 4.2 Let $\boldsymbol{u}$ be the solution to Problem 2.1. If $\boldsymbol{u}_0,\boldsymbol{u}_1\in\boldsymbol{G}_V$, then $\boldsymbol{u}(t)$ and $\partial_t\boldsymbol{u}(t)$ belong to $\boldsymbol{G}_V$ a.e. $t\in(0,T)$.

Proof. We integrate equation (2.9) with respect to time once and twice, respectively, to obtain

$$(\partial_t\boldsymbol{u}(t),\boldsymbol{v})_\rho = (\boldsymbol{u}_1,\boldsymbol{v})_\rho + \int_0^t\left[-a(\boldsymbol{u},\boldsymbol{v}) + \int_{\Omega_\mathrm{S}}\boldsymbol{f}\cdot\boldsymbol{v}^\mathrm{S} + \int_{\Gamma_\mathrm{N}}\boldsymbol{g}\cdot\boldsymbol{v}^\mathrm{S}\right] \qquad \forall\boldsymbol{v}\in\boldsymbol{V}$$

and

$$(\boldsymbol{u}(t),\boldsymbol{v})_\rho = (\boldsymbol{u}_0,\boldsymbol{v})_\rho + t\,(\boldsymbol{u}_1,\boldsymbol{v})_\rho + \int_0^t\int_0^s\left[-a(\boldsymbol{u},\boldsymbol{v}) + \int_{\Omega_\mathrm{S}}\boldsymbol{f}\cdot\boldsymbol{v}^\mathrm{S} + \int_{\Gamma_\mathrm{N}}\boldsymbol{g}\cdot\boldsymbol{v}^\mathrm{S}\right] \qquad \forall \boldsymbol{v}\in\boldsymbol{V}.$$

Hence, the result follows from Lemma 2.1(b). □

Next, we consider the discrete space $\boldsymbol{V}_h$ defined in Section 3 and introduce the following semidiscrete Galerkin approximation of Problem 2.1.

Problem 4.1 Given $\boldsymbol{u}_{0h}$ and $\boldsymbol{u}_{1h}$ approximations in $\boldsymbol{V}_h$ of $\boldsymbol{u}_0$ and $\boldsymbol{u}_1$, respectively, $\boldsymbol{g}\in\mathrm{H}^1\left(0,T;\mathrm{L}^2(\Gamma_\mathrm{N})^d\right)$ and $\boldsymbol{f}\in\mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)$, find $\left(\boldsymbol{u}_h^\mathrm{S},\boldsymbol{u}_h^\mathrm{F}\right)\in C^2(0,T;\boldsymbol{V}_h)$ such that

$$\int_{\Omega_\mathrm{S}}\rho_\mathrm{S}\partial_{tt}\boldsymbol{u}_h^\mathrm{S}\cdot\boldsymbol{v}_h^\mathrm{S} + \int_{\Omega_\mathrm{F}}\rho_\mathrm{F}\partial_{tt}\boldsymbol{u}_h^\mathrm{F}\cdot\boldsymbol{v}_h^\mathrm{F} + \int_{\Omega_\mathrm{S}}\boldsymbol{\sigma}\left(\boldsymbol{u}_h^\mathrm{S}\right):\boldsymbol{\varepsilon}\left(\boldsymbol{v}_h^\mathrm{S}\right) + \int_{\Omega_\mathrm{F}}\rho_\mathrm{F}c^2\operatorname{div}\boldsymbol{u}_h^\mathrm{F}\operatorname{div}\boldsymbol{v}_h^\mathrm{F}$$
$$= \int_{\Gamma_\mathrm{N}}\boldsymbol{g}(t)\cdot\boldsymbol{v}_h^\mathrm{S} + \int_{\Omega_\mathrm{S}}\boldsymbol{f}(t)\cdot\boldsymbol{v}_h^\mathrm{S} \qquad \forall\left(\boldsymbol{v}_h^\mathrm{S},\boldsymbol{v}_h^\mathrm{F}\right)\in\boldsymbol{V}_h.$$

By choosing a basis of $\boldsymbol{V}_h$, the above problem can be written as a linear system of ordinary differential equations. Hence, it is well known that there exists a unique solution $\boldsymbol{u}_h$ to Problem 4.1 (see, for instance, Coddington (1961)).

To study the convergence of the semidiscrete scheme, we consider the projector $\mathbf{P}_h:\boldsymbol{V}\to\boldsymbol{V}_h$ defined in (3.2), for which we have the following property that follows by a density argument from (Basson & van Rensburg, 2013, Lemma 3.1) (see also Baker (1976)).

LEMMA 4.3 If $\boldsymbol{v}\in H^1(0,T;\boldsymbol{V})$, then $\mathbf{P}_h\boldsymbol{v}\in H^1(0,T;\boldsymbol{V})$ and $\partial_t\left(\mathbf{P}_h\boldsymbol{v}\right)(t) = \mathbf{P}_h\left(\partial_t\boldsymbol{v}\right)(t)$ a.e. $t\in(0,T)$.

From the definition of the continuous space $\boldsymbol{V}$ and its discrete approximation $\boldsymbol{V}_h$, it can be seen that we are dealing with a non-conforming approximation of Problem 2.1. Moreover, the Galerkin orthogonality of the error does not hold. In fact, under the regularity assumption (4.1), by taking appropriate test functions in (2.9), it follows that the solution $\boldsymbol{u}$ to Problem 2.1 satisfies (2.1)–(2.6) a.e $t\in(0,T)$. In particular, from (2.2) we have that $\nabla\left(\operatorname{div}\boldsymbol{u}^\mathrm{F}\right)\in\mathrm{L}^2\left(0,T;\mathrm{L}^2(\Omega_\mathrm{F})^d\right)$. Therefore, by testing (2.1)–(2.6) with $\boldsymbol{v}_h\in\boldsymbol{V}_h$ we obtain

$$\left(\partial_{tt}\boldsymbol{u},\boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{u},\boldsymbol{v}_h\right) = \int_{\Omega_\mathrm{S}}\boldsymbol{f}\cdot\boldsymbol{v}_h^\mathrm{S} + \int_{\Gamma_\mathrm{N}}\boldsymbol{g}\cdot\boldsymbol{v}_h^\mathrm{S} + \int_{\Gamma_\mathrm{I}}\rho_\mathrm{F}c^2\operatorname{div}\boldsymbol{u}^\mathrm{F}\left(\boldsymbol{v}_h^\mathrm{F}-\boldsymbol{v}_h^\mathrm{S}\right)\cdot\boldsymbol{n} \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \tag{4.8}$$

for all $t\in(0,T)$. Therefore, equation (2.9) holds for test functions in $\boldsymbol{V}$ but not in $\boldsymbol{V}_h$. Moreover, subtracting the above equation from that of Problem 4.1, we arrive at

$$\left(\partial_{tt}\boldsymbol{u}-\partial_{tt}\boldsymbol{u}_h,\boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{u}-\boldsymbol{u}_h,\boldsymbol{v}_h\right) = \int_{\Gamma_\mathrm{I}}\rho_\mathrm{F}c^2\operatorname{div}\boldsymbol{u}^\mathrm{F}\left(\boldsymbol{v}_h^\mathrm{F}-\boldsymbol{v}_h^\mathrm{S}\right)\cdot\boldsymbol{n} \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \tag{4.9}$$

Thus, from this (among others reasons mentioned in the sequel) we cannot apply known results for the semidiscrete approximation as those from Baker (1976) or from Basson & van Rensburg (2013) in a more abstract framework.

As usual, in order to estimate the error between the continuous and the semidiscrete in space solutions, we decompose

$$\boldsymbol{u}-\boldsymbol{u}_h = \boldsymbol{\eta}+\boldsymbol{\phi}, \qquad \text{where} \qquad \boldsymbol{\eta} := \boldsymbol{u}-\mathbf{P}_h\boldsymbol{u} \qquad \text{and} \qquad \boldsymbol{\phi} := \mathbf{P}_h\boldsymbol{u}-\boldsymbol{u}_h. \tag{4.10}$$

The term $\boldsymbol{\eta}$ will be bounded by using (3.3) and Lemma 3.3, whereas for $\boldsymbol{\phi}$ we have the following lemma.

LEMMA 4.4 Let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions to Problems 2.1 and 4.1, respectively. Let $\boldsymbol{\eta}$ and $\boldsymbol{\phi}$ be as in (4.10). Then,

$$\|\boldsymbol{\phi}\|_{\mathrm{L}^\infty(0,T;\boldsymbol{H})} + \left\|\int_0^t \boldsymbol{\phi}\right\|_{\mathrm{L}^\infty(0,T;\boldsymbol{X})} \leqslant C\left[\|\mathbf{P}_h\boldsymbol{u}_0 - \boldsymbol{u}_{0h}\|_{\boldsymbol{H}} + \|\boldsymbol{u}_1 - \boldsymbol{u}_{1h}\|_{\boldsymbol{H}} + \|\partial_t\boldsymbol{\eta}\|_{\mathrm{L}^1(0,T;\boldsymbol{H})} + \|\boldsymbol{\eta}\|_{\mathrm{L}^2(0,T;\boldsymbol{H})} + h\left\|\partial_{tt}\boldsymbol{u}^{\mathrm{F}}\right\|_{\mathrm{L}^2\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{F}})^d\right)}\right].$$

Proof. We proceed as in (Basson & van Rensburg, 2013, Section 4) and follow the approach from Baker (1976). With this end, we define $\boldsymbol{v}_h(t) := \int_t^\xi \boldsymbol{\phi} \in \boldsymbol{V}_h$, which implies that $\boldsymbol{v}_h(\xi) = 0$ and $\partial_t\boldsymbol{v}_h = -\boldsymbol{\phi}$. We consider the following equality, which is easy to check:

$$\frac{d}{dt}\left[\frac{1}{2}(\boldsymbol{\phi},\boldsymbol{\phi})_\rho - \frac{1}{2}a(\boldsymbol{v}_h,\boldsymbol{v}_h) + (\partial_t\boldsymbol{u} - \partial_t\boldsymbol{u}_h,\boldsymbol{v}_h)_\rho\right] = (\partial_t\boldsymbol{\phi},\boldsymbol{\phi})_\rho + a(\boldsymbol{\phi},\boldsymbol{v}_h) + (\partial_{tt}\boldsymbol{u} - \partial_{tt}\boldsymbol{u}_h,\boldsymbol{v}_h)_\rho - (\partial_t\boldsymbol{u} - \partial_t\boldsymbol{u}_h,\boldsymbol{\phi})_\rho.$$

Moreover, using (4.9), it is also easy to check that

$$(\partial_{tt}\boldsymbol{u} - \partial_{tt}\boldsymbol{u}_h,\boldsymbol{v}_h)_\rho + a(\boldsymbol{\phi},\boldsymbol{v}_h) = \int_{\Gamma_{\mathrm{I}}} \rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}}\left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n} + (\boldsymbol{\eta},\boldsymbol{v}_h)_\rho \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

Hence, by noticing that $\partial_t\boldsymbol{u} - \partial_t\boldsymbol{u}_h = \partial_t\boldsymbol{\eta} + \partial_t\boldsymbol{\phi}$, we arrive at the following equality:

$$\frac{d}{dt}\left[\frac{1}{2}(\boldsymbol{\phi},\boldsymbol{\phi})_\rho - \frac{1}{2}a(\boldsymbol{v}_h,\boldsymbol{v}_h) + (\partial_t\boldsymbol{u} - \partial_t\boldsymbol{u}_h,\boldsymbol{v}_h)_\rho\right] = \int_{\Gamma_{\mathrm{I}}} \rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}}\left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n} + (\boldsymbol{\eta},\boldsymbol{v}_h)_\rho - (\partial_t\boldsymbol{\eta},\boldsymbol{\phi})_\rho.$$

Now, we integrate the previous equation from 0 to ξ:

$$\frac{1}{2}(\boldsymbol{\phi}(\xi),\boldsymbol{\phi}(\xi))_\rho - \frac{1}{2}(\boldsymbol{\phi}(0),\boldsymbol{\phi}(0))_\rho + \frac{1}{2}a(\boldsymbol{v}_h(0),\boldsymbol{v}_h(0)) - (\partial_t\boldsymbol{u}(0) - \partial_t\boldsymbol{u}_h(0),\boldsymbol{v}_h(0))_\rho = -\int_0^\xi (\partial_t\boldsymbol{\eta},\boldsymbol{\phi})_\rho + \int_0^\xi (\boldsymbol{\eta},\boldsymbol{v}_h)_\rho + \int_0^\xi\left[\int_{\Gamma_{\mathrm{I}}} \rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}}\left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right].$$

Then, by integrating by parts in time, we can rewrite the last two terms on the right-hand side above as follows:

$$\int_0^\xi (\boldsymbol{\eta},\boldsymbol{v}_h)_\rho = \int_0^\xi\left(\partial_t\int_0^t \boldsymbol{\eta},\boldsymbol{v}_h\right)_\rho = \int_0^\xi\left(\int_0^t \boldsymbol{\eta},\boldsymbol{\phi}\right)_\rho.$$

Similarly,

$$\int_0^\xi\left(\rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} = \int_0^\xi\left(\int_0^t \rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\left(\boldsymbol{\phi}^{\mathrm{F}} - \boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} = \left(\int_0^\xi \rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^\xi\left(\boldsymbol{\phi}^{\mathrm{F}} - \boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} - \int_0^\xi\left(\rho_{\mathrm{F}}c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^t\left(\boldsymbol{\phi}^{\mathrm{F}} - \boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}.$$

Thus, from the last three equations we arrive at

$$\begin{aligned}(\boldsymbol{\phi}(\xi),\boldsymbol{\phi}(\xi))_\rho + a\left(\int_0^\xi \boldsymbol{\phi},\int_0^\xi \boldsymbol{\phi}\right)\\ = (\boldsymbol{\phi}(0),\boldsymbol{\phi}(0))_\rho + 2\left(\partial_t \boldsymbol{u}(0)-\partial_t \boldsymbol{u}_h(0),\int_0^\xi \boldsymbol{\phi}\right)_\rho - 2\int_0^\xi (\partial_t \boldsymbol{\eta},\boldsymbol{\phi})_\rho + 2\int_0^\xi\left(\int_0^t \boldsymbol{\eta},\boldsymbol{\phi}\right)_\rho\\ + 2\left(\int_0^\xi \rho_{\mathrm{F}} c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^\xi\left(\boldsymbol{\phi}^{\mathrm{F}}-\boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} - 2\int_0^\xi\left(\rho_{\mathrm{F}} c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^t\left(\boldsymbol{\phi}^{\mathrm{F}}-\boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}. \quad (4.11)\end{aligned}$$

Next, we estimate the terms on the right-hand side above. From Cauchy-Schwarz and Young's inequalities, it follows that for all $\gamma > 0$

$$\begin{aligned}\left(\partial_t \boldsymbol{u}(0)-\partial_t \boldsymbol{u}_h(0),\int_0^\xi \boldsymbol{\phi}\right)_\rho - \int_0^\xi (\partial_t \boldsymbol{\eta},\boldsymbol{\phi})_\rho + \int_0^\xi\left(\int_0^t \boldsymbol{\eta},\boldsymbol{\phi}\right)_\rho\\ \leqslant \frac{C}{\gamma}\left[T\left\|\partial_t \boldsymbol{u}(0)-\partial_t \boldsymbol{u}_h(0)\right\|_{\boldsymbol{H}} + \left\|\partial_t \boldsymbol{\eta}\right\|_{\mathrm{L}^1(0,T;\boldsymbol{H})} + T\left\|\boldsymbol{\eta}\right\|_{\mathrm{L}^2(0,T;\boldsymbol{H})}\right]^2 + \gamma\|\boldsymbol{\phi}\|^2_{\mathrm{L}^\infty(0,T;\boldsymbol{H})}.\end{aligned}$$

To estimate the last two terms of (4.11), we recall that $\operatorname{div}\boldsymbol{u}^{\mathrm{F}} \in \mathrm{L}^2\left(0,T;\mathrm{H}^1(\Omega_{\mathrm{F}})\right)$. Thus, these terms can be bounded by using Lemma 3.1 with $\boldsymbol{z}_h = \int_0^t \boldsymbol{\phi}$, (2.2) and the fact that $\left\|\boldsymbol{v}^{\mathrm{S}}\right\|^2_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d} \leqslant Ca(\boldsymbol{v},\boldsymbol{v})$ $\forall \boldsymbol{v} = \left(\boldsymbol{v}^{\mathrm{S}},\boldsymbol{v}^{\mathrm{F}}\right) \in \boldsymbol{X}$ as follows:

$$\begin{aligned}\left(\int_0^\xi \rho_{\mathrm{F}} c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^\xi\left(\boldsymbol{\phi}^{\mathrm{F}}-\boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} - \int_0^\xi\left(\rho_{\mathrm{F}} c^2 \operatorname{div}\boldsymbol{u}^{\mathrm{F}},\int_0^t\left(\boldsymbol{\phi}^{\mathrm{F}}-\boldsymbol{\phi}^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}\\ \leqslant C\left\{h\int_0^\xi \left|\operatorname{div}\boldsymbol{u}^{\mathrm{F}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{F}})}\left|\int_0^\xi \boldsymbol{\phi}^{\mathrm{S}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d} + h\int_0^\xi\left[\left|\operatorname{div}\boldsymbol{u}^{\mathrm{F}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{F}})}\left|\int_0^t \boldsymbol{\phi}^{\mathrm{S}}\right|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right]\right\}\\ \leqslant \frac{1}{4}\sup_{t\in[0,T]} a\left(\int_0^t \boldsymbol{\phi},\int_0^t \boldsymbol{\phi}\right) + Ch^2\left\|\partial_{tt}\boldsymbol{u}^{\mathrm{F}}\right\|^2_{\mathrm{L}^2\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{F}})^d\right)}. \quad (4.12)\end{aligned}$$

Finally the result follows from (4.11)–(4.12) by straightforward computations, which include taking the supremum over $0 \leqslant \xi \leqslant T$, choosing $\gamma = \frac{1}{4}\min\{\rho_{\mathrm{F}},\rho_{\mathrm{S}}\}$ and using the fact that $a(\cdot,\cdot) + (\cdot,\cdot)_\rho$ is a norm equivalent to the $\boldsymbol{X}$-norm. □

From Lemmas 3.3, 4.2 and 4.4, we obtain the following convergence result for the semidiscrete scheme.

THEOREM 4.2 For $\boldsymbol{u}_0,\boldsymbol{u}_1 \in \boldsymbol{G}_V$, let $\boldsymbol{u}$ and $\boldsymbol{u}_h$ be the solutions to Problems 2.1 and 4.1, respectively. If $\boldsymbol{u} \in \mathrm{L}^\infty(0,T;\boldsymbol{V})$, $\partial_t \boldsymbol{u} \in \mathrm{L}^1(0,T;\boldsymbol{V})$ and $\partial_{tt}\boldsymbol{u} \in \mathrm{L}^2(0,T;\boldsymbol{H})$, then

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{L}^\infty(0,T;\boldsymbol{H})} \leqslant Ch^{\min\{\alpha,\beta\}}\left[\|\partial_t \boldsymbol{u}\|_{\mathrm{L}^1(0,T;\boldsymbol{X})} + \|\boldsymbol{u}\|_{\mathrm{L}^\infty(0,T;\boldsymbol{X})} + \left\|\partial_{tt}\boldsymbol{u}^{\mathrm{F}}\right\|_{\mathrm{L}^2\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{F}})^d\right)} + \|\boldsymbol{u}_0\|_{\boldsymbol{X}}\right]\\ + C\left(\|\boldsymbol{u}_0-\boldsymbol{u}_{0h}\|_{\boldsymbol{H}} + \|\boldsymbol{u}_1-\boldsymbol{u}_{1h}\|_{\boldsymbol{H}}\right), \quad (4.13)\end{aligned}$$

with $\alpha \in (1/2,1]$ and $\beta \in (0,1]$ as in Remark 3.1.

REMARK 4.1 According to Lemma 4.1 and the previous theorem, if $\boldsymbol{g} = \boldsymbol{0}$, $\boldsymbol{f} \in \mathrm{H}^1(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d)$, $\boldsymbol{u}_0,\boldsymbol{u}_1 \in \boldsymbol{G}_V$ and (4.7) is satisfied, then $\boldsymbol{u}$ satisfies the regularity assumed in this theorem, so that, in such a case, (4.13) holds true without any further assumption. Moreover, if the discrete initial conditions are taken as $\boldsymbol{u}_{0h} := \mathbf{I}_h^V \boldsymbol{u}_0$ and $\boldsymbol{u}_{1h} := \mathbf{I}_h^V \boldsymbol{u}_1$, then, from (4.13) and Lemma 3.3, it follows that

$$\begin{aligned}&\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{L}^\infty(0,T;\boldsymbol{H})}\\ &\leqslant Ch^{\min\{\alpha,\beta\}}\left[\left\|\mathbf{div}\,\boldsymbol{\sigma}\left(\boldsymbol{u}_0^{\mathrm{S}}\right)\right\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^d}+\left\|\nabla\left(\operatorname{div}\boldsymbol{u}_0^{\mathrm{F}}\right)\right\|_{\mathrm{L}^2(\Omega_{\mathrm{F}})^d}+\|\boldsymbol{f}\|_{\mathrm{H}^1\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)}+\|\boldsymbol{u}_0\|_{\boldsymbol{X}}+\|\boldsymbol{u}_1\|_{\boldsymbol{X}}\right].\end{aligned}$$

Finally, if some further regularity of the solution is assumed, then we obtain the following $\boldsymbol{X}$-like error estimate for the semidiscrete approximation from Lemmas 3.3, 4.2 and 4.4 and estimate (3.3).

THEOREM 4.3 Under the assumptions of Theorem 4.2, if moreover $\boldsymbol{u} \in \mathrm{L}^\infty\left(0,T;\boldsymbol{X}^{\alpha,\beta}\right)$, then

$$\begin{aligned}\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\mathrm{L}^\infty(0,T;\boldsymbol{H})}+\left\|\int_0^t(\boldsymbol{u}-\boldsymbol{u}_h)\right\|_{\mathrm{L}^\infty(0,T;\boldsymbol{X})}&\leqslant C\left(\|\boldsymbol{u}_0-\boldsymbol{u}_{0h}\|_{\boldsymbol{H}}+\|\boldsymbol{u}_1-\boldsymbol{u}_{1h}\|_{\boldsymbol{H}}\right)\\ &+Ch^{\min\{\alpha,\beta\}}\left[\|\partial_t\boldsymbol{u}\|_{\mathrm{L}^1(0,T;\boldsymbol{X})}+\|\boldsymbol{u}\|_{\mathrm{L}^\infty\left(0,T;\boldsymbol{X}^{\alpha,\beta}\right)}+\left\|\partial_{tt}\boldsymbol{u}^{\mathrm{F}}\right\|_{\mathrm{L}^2\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{F}})^d\right)}+\|\boldsymbol{u}_0\|_{\boldsymbol{X}}\right],\end{aligned}$$

where $\alpha \in (1/2,1]$ and $\beta \in (0,1]$ are as in Remark 3.1.

5. Fully discrete Galerkin approximation

In this section, we introduce a fully discrete Galerkin approximation of Problem 2.1 based on finite elements and a Newmark's method for the space and time discretization, respectively. For the former we consider the finite element spaces introduced in Section 3. For the latter we introduce a partition of the time interval $[0,T]$ with step size $\Delta t = T/N$, $N \in \mathbb{N}$, and define $t_n := n\Delta t$, $n = 0,\ldots,N$. If $\boldsymbol{v}$ is regular enough with respect to t, we denote $\boldsymbol{v}^n := \boldsymbol{v}(t_n)$, $n = 0,\ldots,N$.

For the numerical scheme, we consider that $\boldsymbol{f}$ and $\boldsymbol{g}$ are continuous in time and that we dispose of an approximation $\boldsymbol{u}_{0h} \in \boldsymbol{V}_h$ of the initial data $\boldsymbol{u}_0$. We propose the following Newmark scheme (Newmark (1959)) with the initial step as in Karaa (2011) and a given value of the parameter $\theta \in [0,1]$:

Problem 5.1 Let $\boldsymbol{u}_{0h} \in \boldsymbol{V}_h$, $\boldsymbol{u}^1 \in \boldsymbol{H}$, $\boldsymbol{f} \in C\left(0,T;\in \mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)$ and $\boldsymbol{g} \in C\left(0,T;\mathrm{L}^2(\Gamma_{\mathrm{N}})^d\right)$ be given data.

- Let $\boldsymbol{u}_h^0 := \boldsymbol{u}_{0h}$.
- Let $\boldsymbol{u}_h^1 \in \boldsymbol{V}_h$ be the solution to

$$\begin{aligned}&\left(\boldsymbol{u}_h^1-\boldsymbol{u}_h^0,\boldsymbol{v}_h\right)_\rho+\Delta t^2\theta a\left(\boldsymbol{u}_h^1-\boldsymbol{u}_h^0,\boldsymbol{v}_h\right)\\ &\qquad=\Delta t\left(\boldsymbol{u}_1,\boldsymbol{v}_h\right)_\rho+\frac{\Delta t^2}{2}\left[\left(\boldsymbol{f}^0,\boldsymbol{v}_h^{\mathrm{S}}\right)_S+\left(\boldsymbol{g}^0,\boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}}-a\left(\boldsymbol{u}_{0h},\boldsymbol{v}_h\right)\right]\qquad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\end{aligned}\tag{5.1}$$

- For $n = 1,\ldots,N-1$, let $\boldsymbol{u}_h^{n+1} \in \boldsymbol{V}_h$ be the solution to

$$\left(\overline{\partial}_{tt}\boldsymbol{u}_h^n,\boldsymbol{v}_h\right)_\rho+a\left(\boldsymbol{u}_h^{n,\theta},\boldsymbol{v}_h\right)=\left(\boldsymbol{f}^{n,\theta},\boldsymbol{v}_h^{\mathrm{S}}\right)_S+\left(\boldsymbol{g}^{n,\theta},\boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}}\qquad\forall\boldsymbol{v}_h\in\boldsymbol{V}_h.\tag{5.2}$$

In the previous problem we have used the notation

$$\overline{\partial}_{tt}\boldsymbol{u}_h^n:=\frac{\boldsymbol{u}_h^{n+1}-2\boldsymbol{u}_h^n+\boldsymbol{u}_h^{n-1}}{\Delta t^2}\qquad\text{and}\qquad\boldsymbol{u}_h^{n,\theta}:=\theta\boldsymbol{u}_h^{n+1}+(1-2\theta)\boldsymbol{u}_h^n+\theta\boldsymbol{u}_h^{n-1}.$$

We recall that we are dealing with a non-conforming approximation of Problem 2.1. Notice that this fact does not allow us to apply classical results for the fully discrete approximation like those in Baker (1976), or more recent results like those in Karaa (2011).

In what follows, we will obtain error estimates for the fully discrete approximation proposed above. From now on we assume that $\theta \geqslant 1/4$, so that the scheme is unconditionally stable (see, for instance, Raviart & Thomas (1983)). For the forthcoming analysis we will assume that the solution $\boldsymbol{u}$ to Problem 2.1 satisfies

$$\partial_{tt}\boldsymbol{u} \in C(0,T;\boldsymbol{X}), \qquad \partial_{ttt}\boldsymbol{u} \in C(0,T;\boldsymbol{H}) \qquad \text{and} \qquad \partial_{tttt}\boldsymbol{u} \in \mathrm{L}^1(0,T;\boldsymbol{H}). \tag{5.3}$$

We notice that under the above regularity assumption, by taking appropriate test functions in (2.9), it follows that $\boldsymbol{u}$ satisfies (2.1)–(2.6) for all $t \in (0,T)$. In particular, from (2.2), we have that $\nabla\left(\operatorname{div}\boldsymbol{u}^{\mathrm{F}}\right) \in C(0,T;\mathrm{L}^2(\Omega_{\mathrm{F}})^d)$. Moreover, as was shown in the previous section, the solution $\boldsymbol{u}$ to Problem 2.1 satisfies (4.8).

To study the convergence of the fully discrete scheme at time t_n, $n = 1,\dots,N$, the error is decomposed as usual:

$$\boldsymbol{u}^n - \boldsymbol{u}_h^n = \boldsymbol{\eta}^n + \boldsymbol{\phi}^n, \qquad \text{where} \qquad \boldsymbol{\eta}^n := \boldsymbol{u}^n - \mathbf{P}_h\boldsymbol{u}^n \qquad \text{and} \qquad \boldsymbol{\phi}^n := \mathbf{P}_h\boldsymbol{u}^n - \boldsymbol{u}_h^n. \tag{5.4}$$

The term $\boldsymbol{\eta}^n$ will be bounded by using Lemma 3.3. Therefore, to bound the error, we only need to estimate $\boldsymbol{\phi}^n$. This is the aim of the forthcoming analysis. With this end, let us define the following functions of the space variable:

$$\boldsymbol{r}^m := \overline{\partial}_{tt}\mathbf{P}_h\boldsymbol{u}^m - (\partial_{tt}\boldsymbol{u})^{m,\theta}, \qquad \boldsymbol{R}^m := \Delta t\sum_{n=1}^{m}\boldsymbol{r}^n, \qquad \boldsymbol{P}^m := \Delta t\sum_{n=1}^{m}(\mathbf{I}-\mathbf{P}_h)\,\boldsymbol{u}^{n,\theta}, \quad m = 1,\dots,N-1.$$

Moreover, let $\mathscr{L}$, $\mathscr{J}^m$ and Θ^m in $\boldsymbol{V}_h'$, $m = 1,\dots,N-1$, be respectively defined for all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ as follows:

$$\begin{aligned}
\mathscr{L}(\boldsymbol{v}_h) &:= \Delta t^{-2}\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right)_\rho + \theta a\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right), \\
\mathscr{J}^m(\boldsymbol{v}_h) &:= \left(\rho_{\mathrm{F}}c^2\operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}, \left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}, \\
\Theta^m(\boldsymbol{v}_h) &:= \Delta t\sum_{n=1}^{m}\mathscr{J}^n(\boldsymbol{v}_h).
\end{aligned}$$

Finally, we define

$$\Phi^0 := -\frac{1}{2}\boldsymbol{\phi}^0 \qquad \text{and} \qquad \Phi^m := -\frac{1}{2}\boldsymbol{\phi}^0 + \sum_{n=0}^{m-1}\boldsymbol{\phi}^{n+1/2}, \quad m = 1,\dots,N,$$

where

$$\boldsymbol{\phi}^{n+1/2} := \frac{\boldsymbol{\phi}^{n+1} + \boldsymbol{\phi}^n}{2}, \qquad n = 0,\dots,N-1.$$

To prove an estimate for $\boldsymbol{\phi}^n$, $n = 1,\dots,N$, we follow several steps. First, we show the following estimate of $\boldsymbol{\phi}^n$ in terms of the above defined quantities.

LEMMA 5.1 For all $n = 1,\dots,N$, it follows that

$$\begin{aligned}
&(\boldsymbol{\phi}^n,\boldsymbol{\phi}^n)_\rho + \Delta t^2\left(\theta-\frac{1}{4}\right)a(\boldsymbol{\phi}^n,\boldsymbol{\phi}^n) + \Delta t^2 a(\Phi^n,\Phi^n)\\
&\qquad = \left(\boldsymbol{\phi}^0,\boldsymbol{\phi}^0\right)_\rho + \Delta t^2\theta a\left(\boldsymbol{\phi}^0,\boldsymbol{\phi}^0\right) + \Delta t\sum_{m=1}^{n-1}\left(\boldsymbol{R}^m+\boldsymbol{P}^m,\boldsymbol{\phi}^{m+1}+\boldsymbol{\phi}^m\right)_\rho\\
&\qquad\quad + 2\Delta t\sum_{m=1}^{n-1}\Theta^m\left(\Phi^{m+1}-\Phi^m\right) + 2\Delta t^2\mathscr{L}\left(\Phi^n-\Phi^0\right) \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h. \quad (5.5)
\end{aligned}$$

Proof. From the first assumption in (5.3), $\operatorname{div}\boldsymbol{u}^{\mathrm{F}} \in C\left(0,T;\mathrm{H}^1(\Omega_{\mathrm{F}})\right)$. Then, since $\boldsymbol{u}$ satisfies (4.8), we have

$$\left((\partial_{tt}\boldsymbol{u})^{n,\theta},\boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{u}^{n,\theta},\boldsymbol{v}_h\right) = \left(\boldsymbol{f}^{n,\theta},\boldsymbol{v}_h^{\mathrm{S}}\right)_S + \left(\boldsymbol{g}^{n,\theta},\boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}} + \int_{\Gamma_{\mathrm{I}}}\rho_{\mathrm{F}}c^2\operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{n,\theta}\left(\boldsymbol{v}_h^{\mathrm{F}}-\boldsymbol{v}_h^{\mathrm{S}}\right)\cdot\boldsymbol{n}, \quad (5.6)$$

for all $\boldsymbol{v}_h\in\boldsymbol{V}_h$ and $n = 1,\dots,N-1$. By subtracting (5.2) and (5.6) we arrive at

$$\left((\partial_{tt}\boldsymbol{u})^{n,\theta}-\overline{\partial}_{tt}\boldsymbol{u}_h^n,\boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{u}^{n,\theta}-\boldsymbol{u}_h^{n,\theta},\boldsymbol{v}_h\right) = \mathscr{J}^n(\boldsymbol{v}_h) \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h, \qquad n = 1,\dots,N-1.$$

Hence, using the definition of $\mathbf{P}_h$ (cf. (3.2)), we have that, for $n = 1,\dots,N-1$,

$$\left(\overline{\partial}_{tt}\boldsymbol{\phi}^n,\boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{\phi}^{n,\theta},\boldsymbol{v}_h\right) = (\boldsymbol{r}^n,\boldsymbol{v}_h)_\rho + \mathscr{J}^n(\boldsymbol{v}_h) + \left(\boldsymbol{u}^{n,\theta}-\mathbf{P}_h\boldsymbol{u}^{n,\theta},\boldsymbol{v}_h\right)_\rho \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \quad (5.7)$$

On the other hand, we notice that

$$\Delta t^2\left(\theta-\frac{1}{4}\right)\overline{\partial}_{tt}\boldsymbol{\phi}^n + \left(\frac{\boldsymbol{\phi}^{n+1/2}+\boldsymbol{\phi}^{n-1/2}}{2}\right) = \boldsymbol{\phi}^{n,\theta}.$$

Then, we can rewrite (5.7) as follows:

$$\begin{aligned}
&\left(\overline{\partial}_{tt}\boldsymbol{\phi}^n,\boldsymbol{v}_h\right)_\rho + \Delta t^2\left(\theta-\frac{1}{4}\right)a\left(\overline{\partial}_{tt}\boldsymbol{\phi}^n,\boldsymbol{v}_h\right) + \frac{1}{2}a\left(\boldsymbol{\phi}^{n+1/2}+\boldsymbol{\phi}^{n-1/2},\boldsymbol{v}_h\right)\\
&\qquad = (\boldsymbol{r}^n,\boldsymbol{v}_h)_\rho + \mathscr{J}^n(\boldsymbol{v}_h) + \left((\mathbf{I}-\mathbf{P}_h)\boldsymbol{u}^{n,\theta},\boldsymbol{v}_h\right)_\rho \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h.
\end{aligned}$$

Summing over n the above equation from $n = 1$ to $n = m$ and multiplying by Δt, we arrive at

$$\begin{aligned}
&\frac{1}{\Delta t}\left(\boldsymbol{\phi}^{m+1}-\boldsymbol{\phi}^m,\boldsymbol{v}_h\right)_\rho - \frac{1}{\Delta t}\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right)_\rho + \Delta t\left(\theta-\frac{1}{4}\right)a\left(\boldsymbol{\phi}^{m+1}-\boldsymbol{\phi}^m,\boldsymbol{v}_h\right)\\
&-\Delta t\left(\theta-\frac{1}{4}\right)a\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right) + \frac{\Delta t}{2}\sum_{n=1}^m a\left(\boldsymbol{\phi}^{n+1/2}+\boldsymbol{\phi}^{n-1/2},\boldsymbol{v}_h\right)\\
&\qquad = \Delta t\sum_{n=1}^m(\boldsymbol{r}^n,\boldsymbol{v}_h)_\rho + \Delta t\sum_{n=1}^m\mathscr{J}^n(\boldsymbol{v}_h) + \Delta t\sum_{n=1}^m\left((\mathbf{I}-\mathbf{P}_h)\boldsymbol{u}^{n,\theta},\boldsymbol{v}_h\right)_\rho \qquad \forall\boldsymbol{v}_h\in\boldsymbol{V}_h. \quad (5.8)
\end{aligned}$$

On the other hand, from the definition of Φ^m,

$$\sum_{n=1}^m\boldsymbol{\phi}_h^{n+1/2} + \sum_{n=1}^m\boldsymbol{\phi}_h^{n-1/2} + \frac{1}{2}\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0\right) = \Phi^{m+1}+\Phi^m, \qquad m = 0,\dots,N-1.$$

Hence,

$$-\frac{1}{\Delta t}\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right)_\rho-\Delta t\left(\theta-\frac{1}{4}\right)a\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right)+\frac{\Delta t}{2}\sum_{n=1}^{m}a\left(\boldsymbol{\phi}^{n+1/2}+\boldsymbol{\phi}^{n-1/2},\boldsymbol{v}_h\right)$$
$$=-\frac{1}{\Delta t}\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right)_\rho-\Delta t\theta a\left(\boldsymbol{\phi}^1-\boldsymbol{\phi}^0,\boldsymbol{v}_h\right)+\frac{\Delta t}{2}a\left(\Phi^{m+1}+\Phi^m,\boldsymbol{v}_h\right)_\rho \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h.$$

From the above equality and the definitions of $\boldsymbol{R}^m,\boldsymbol{P}^m,\Theta^m$ and $\mathscr{L}$, equation (5.8) can be rewritten as

$$\frac{1}{\Delta t}\left(\boldsymbol{\phi}^{m+1}-\boldsymbol{\phi}^m,\boldsymbol{v}_h\right)_\rho+\Delta t\left(\theta-\frac{1}{4}\right)a\left(\boldsymbol{\phi}^{m+1}-\boldsymbol{\phi}^m,\boldsymbol{v}_h\right)+\frac{\Delta t}{2}a\left(\Phi^{m+1}+\Phi^m,\boldsymbol{v}_h\right)_\rho$$
$$=\left(\boldsymbol{R}^m+\boldsymbol{P}^m,\boldsymbol{v}_h\right)_\rho+\Theta^m\left(\boldsymbol{v}_h\right)+\mathscr{L}\left(\boldsymbol{v}_h\right)\Delta t \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h, \qquad m=0,\dots,N-1.$$

We choose $\boldsymbol{v}_h=\boldsymbol{\phi}^{m+1}+\boldsymbol{\phi}^m=2\left(\Phi^{m+1}-\Phi^m\right)$ in the above equality and multiply by Δt to write

$$\left(\boldsymbol{\phi}^{m+1},\boldsymbol{\phi}^{m+1}\right)_\rho-\left(\boldsymbol{\phi}^m,\boldsymbol{\phi}^m\right)_\rho+\Delta t^2\left(\theta-\frac{1}{4}\right)\left[a\left(\boldsymbol{\phi}^{m+1},\boldsymbol{\phi}^{m+1}\right)-a\left(\boldsymbol{\phi}^m,\boldsymbol{\phi}^m\right)\right]$$
$$+\Delta t^2\left[a\left(\Phi^{m+1},\Phi^{m+1}\right)-a\left(\Phi^m,\Phi^m\right)\right]$$
$$=\Delta t\left(\boldsymbol{R}^m+\boldsymbol{P}^m,\boldsymbol{\phi}^{m+1}+\boldsymbol{\phi}^m\right)_\rho+2\Theta^m\left(\Phi^{m+1}-\Phi^m\right)+2\Delta t^2\mathscr{L}\left(\Phi^{m+1}-\Phi^m\right) \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h.$$

Summing over m from $m=0$ to $n-1$, for $1\leqslant n\leqslant N$, yields

$$\left(\boldsymbol{\phi}^n,\boldsymbol{\phi}^n\right)_\rho-\left(\boldsymbol{\phi}^0,\boldsymbol{\phi}^0\right)_\rho+\Delta t^2\left(\theta-\frac{1}{4}\right)\left[a\left(\boldsymbol{\phi}^n,\boldsymbol{\phi}^n\right)-a\left(\boldsymbol{\phi}^0,\boldsymbol{\phi}^0\right)\right]+\Delta t^2\left[a\left(\Phi^n,\Phi^n\right)-a\left(\Phi^0,\Phi^0\right)\right]$$
$$=\Delta t\sum_{m=1}^{n-1}\left(\boldsymbol{R}^m+\boldsymbol{P}^m,\boldsymbol{\phi}^{m+1}+\boldsymbol{\phi}^m\right)_\rho+2\Delta t\sum_{m=1}^{n-1}\Theta^m\left(\Phi^{m+1}-\Phi^m\right)+2\Delta t^2\mathscr{L}\left(\Phi^n-\Phi^0\right) \qquad \forall \boldsymbol{v}_h\in\boldsymbol{V}_h.$$

The result follows from the above equation and the fact that that $\Phi^0=-\frac{1}{2}\boldsymbol{\phi}^0$. □

Next, we need to deal with the terms on the right-hand side of the above lemma. Notice that $\boldsymbol{P}^m$ can be easily estimated by using Lemma 3.3:

$$\Delta t\sum_{m=0}^{N-1}\left\|\boldsymbol{P}^m\right\|_{\boldsymbol{H}}\leqslant Ch^{\min\{\alpha,\beta\}}\left\|\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})}. \tag{5.9}$$

For $\boldsymbol{R}^m$ we have the following estimate, where, once more, α and β are as in Remark 3.1:

LEMMA 5.2 There holds

$$\Delta t\sum_{m=1}^{N-1}\left\|\boldsymbol{R}^m\right\|_{\boldsymbol{H}}\leqslant C\left[h^{\min\{\alpha,\beta\}}\left\|\partial_{tt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})}+\Delta t^2\left\|\partial_{tttt}\boldsymbol{u}\right\|_{\mathrm{L}^1(0,T;\boldsymbol{H})}\right].$$

Proof. By triangle inequality we have

$$\left\|\boldsymbol{r}^n\right\|_{\boldsymbol{H}}=\left\|\overline{\partial}_{tt}\mathbf{P}_h\boldsymbol{u}^n-\left(\partial_{tt}\boldsymbol{u}\right)^{n,\theta}\right\|_{\boldsymbol{H}}\leqslant\left\|\overline{\partial}_{tt}\left(\mathbf{P}_h-\mathbf{I}\right)\boldsymbol{u}^n\right\|_{\boldsymbol{H}}+\left\|\overline{\partial}_{tt}\boldsymbol{u}^n-\left(\partial_{tt}\boldsymbol{u}\right)^{n,\theta}\right\|_{\boldsymbol{H}}.$$

Hence, by proceeding as in the proof of (Karaa, 2011, Lemma 2) and using Lemma 3.3 to estimate $(\mathbf{P}_h - \mathbf{I})$, we arrive at

$$\|\boldsymbol{r}^n\|_{\boldsymbol{H}} \leqslant C\left[\Delta t^{-1} h^{\min\{\alpha,\beta\}} \int_{t_{n-1}}^{t_{n+1}} \|\partial_{tt}\boldsymbol{u}(s)\|_{\boldsymbol{X}}\, ds + \Delta t \int_{t_{n-1}}^{t_{n+1}} \|\partial_{tttt}\boldsymbol{u}(s)\|_{\boldsymbol{H}}\, ds\right].$$

Then, the result follows from the previous inequality and the definition of $\boldsymbol{R}^m$. □

The last two terms on the right-hand side of (5.5) can be bounded by using the following results:

LEMMA 5.3 Let $\boldsymbol{w}^n = \left(\boldsymbol{w}_S^n, \boldsymbol{w}_F^n\right) \in \boldsymbol{V}_h$, $n = 1, \ldots, N$. For $1 \leqslant n \leqslant N$, it follows that

$$\Delta t \sum_{m=1}^{n-1} \Theta^m \left(\boldsymbol{w}^{m+1} - \boldsymbol{w}^m\right) \leqslant C\left[\Delta t \max_{1\leqslant n\leqslant N} |\boldsymbol{w}_{\mathrm{S}}^n|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right]\left[h\,\|\partial_{tt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})}\right].$$

Proof. First notice that

$$\Delta t \sum_{m=1}^{n-1} \Theta^m \left(\boldsymbol{w}^{m+1} - \boldsymbol{w}^m\right) = \Delta t^2 \sum_{m=1}^{n-1} \left(A_m, b_{m+1} - b_m\right)_{\Gamma_{\mathrm{I}}},$$

where

$$A_m := \sum_{l=1}^{m} a_l, \qquad a_l := \rho_{\mathrm{F}} c^2 \operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{l,\theta} \qquad \text{and} \qquad b_m := (\boldsymbol{w}_F^m - \boldsymbol{w}_S^m)\cdot\boldsymbol{n}.$$

By summation by parts we arrive at

$$\begin{aligned}
\Delta t \sum_{m=1}^{n-1} \Theta^m \left(\Phi^{m+1} - \Phi^m\right) &= \Delta t^2 \left(A_n, b_n\right)_{\Gamma_{\mathrm{I}}} - \Delta t^2 \sum_{m=1}^{n} \left(a_m, b_m\right)_{\Gamma_{\mathrm{I}}} \\
&= \Delta t^2 \left(\rho_{\mathrm{F}} c^2 \sum_{m=1}^{n} \operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}, (\boldsymbol{w}_F^n - \boldsymbol{w}_S^n)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} - \Delta t^2 \sum_{m=1}^{n} \left(\rho_{\mathrm{F}} c^2 \operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}, (\boldsymbol{w}_F^m - \boldsymbol{w}_S^m)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}.
\end{aligned}$$

We apply Lemma 3.1 to both terms on the right-hand side of the previous equation and (2.2) to write

$$\begin{aligned}
\Delta t^2 \left(\rho_{\mathrm{F}} c^2 \sum_{m=1}^{n} \operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}, (\boldsymbol{w}_F^n - \boldsymbol{w}_S^n)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} &\leqslant Ch\Delta t^2 |\boldsymbol{w}_{\mathrm{S}}^n|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d} \sum_{m=1}^{n} \left|\operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}\right|_{\mathrm{H}^1(\Omega_{\mathrm{F}})} \\
&\leqslant Ch\Delta t \left(\max_{1\leqslant n\leqslant N} |\boldsymbol{w}_{\mathrm{S}}^n|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right) \|\partial_{tt}\boldsymbol{u}\|^2_{C(0,T;\boldsymbol{H})}.
\end{aligned}$$

$$\begin{aligned}
\Delta t^2 \sum_{m=1}^{n} \left(\rho_{\mathrm{F}} c^2 \operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}, (\boldsymbol{w}_F^m - \boldsymbol{w}_S^m)\cdot\boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} &\leqslant Ch\Delta t^2 \sum_{m=1}^{n} |\boldsymbol{w}_{\mathrm{S}}^m|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d} \left|\operatorname{div}\left(\boldsymbol{u}^{\mathrm{F}}\right)^{m,\theta}\right|_{\mathrm{H}^1(\Omega_{\mathrm{F}})} \\
&\leqslant Ch\Delta t \left(\max_{1\leqslant n\leqslant N} |\boldsymbol{w}_{\mathrm{S}}^n|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d}\right) \|\partial_{tt}\boldsymbol{u}\|^2_{C(0,T;\boldsymbol{H})}.
\end{aligned}$$

The result follows from the last three equations. □

LEMMA 5.4 For all $\boldsymbol{v}_h \in \boldsymbol{V}_h$ it follows that

$$\Delta t^2 \mathscr{L}(\boldsymbol{v}_h) \leqslant C\left[h^{\min\{\alpha,\beta\}} \|\partial_t \boldsymbol{u}\|_{C(0,T;\boldsymbol{X})} + \Delta t^2 \|\partial_{ttt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})} + \Delta t^2 \|\partial_t \boldsymbol{f}\|_{C\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)}\right] \Delta t \|\boldsymbol{v}_h\|_{\boldsymbol{H}}$$
$$+ C\Delta t^3 \left[\|\partial_t \boldsymbol{g}\|_{C\left(0,T;\mathrm{L}^2(\Gamma_\mathrm{N})^d\right)} + h \|\partial_{ttt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})}\right] \|\boldsymbol{v}_h\|_{\boldsymbol{X}},$$

with α and β as in Remark 3.1.

Proof. We recall that

$$\mathscr{L}(\boldsymbol{v}_h) = \Delta t^{-2}\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right)_\rho + \theta a\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right) \qquad \forall \boldsymbol{v}_h \in \boldsymbol{V}_h.$$

For the first term on the right-hand side, we have that

$$\begin{aligned}
\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right)_\rho &= \left(\mathbf{P}_h \boldsymbol{u}^1 - \boldsymbol{u}_h^1, \boldsymbol{v}_h\right)_\rho - \left(\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right)_\rho \\
&= \left(\mathbf{P}_h \boldsymbol{u}^1 - \boldsymbol{u}^1, \boldsymbol{v}_h\right)_\rho + \left(\boldsymbol{u}^1 - \boldsymbol{u}_h^1, \boldsymbol{v}_h\right)_\rho - \left(\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}^0, \boldsymbol{v}_h\right)_\rho - \left(\boldsymbol{u}^0 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right)_\rho \\
&= \left((\mathbf{P}_h - I)\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho + \left(\boldsymbol{u}^1 - \boldsymbol{u}^0, \boldsymbol{v}_h\right)_\rho - \left(\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right)_\rho,
\end{aligned}$$

whereas, for the second one, by using the definition of $\mathbf{P}_h$ (cf. (3.2)) we have

$$\begin{aligned}
a\left(\boldsymbol{\phi}^1 - \boldsymbol{\phi}^0, \boldsymbol{v}_h\right) &= a\left(\mathbf{P}_h \boldsymbol{u}^1 - \boldsymbol{u}_h^1, \boldsymbol{v}_h\right) - a\left(\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right) \\
&= a\left(\mathbf{P}_h \boldsymbol{u}^1 - \boldsymbol{u}^1, \boldsymbol{v}_h\right) + a\left(\boldsymbol{u}^1 - \boldsymbol{u}_h^1, \boldsymbol{v}_h\right) - a\left(\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}^0, \boldsymbol{v}_h\right) - a\left(\boldsymbol{u}^0 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right) \\
&= \left((\mathbf{P}_h - I)\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho + a\left(\boldsymbol{u}^1 - \boldsymbol{u}^0, \boldsymbol{v}_h\right) - a\left(\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right).
\end{aligned}$$

On the other hand, from Taylor's theorem we obtain

$$\boldsymbol{u}^1 - \boldsymbol{u}^0 = \Delta t\left(\partial_t \boldsymbol{u}\right)^0 + \frac{\Delta t^2}{2}\partial_{tt}\boldsymbol{u}^0 + \frac{1}{2}\int_0^{t_1} (\Delta t - s)^2 \partial_{ttt}\boldsymbol{u}(s)\, ds,$$

whereas evaluating (4.8) at $t = t_0$ and $t = t_1$, we obtain

$$\begin{aligned}
a\left(\boldsymbol{u}^1 - \boldsymbol{u}^0, \boldsymbol{v}_h\right) = \left(\boldsymbol{f}^1 - \boldsymbol{f}^0, \boldsymbol{v}_h^\mathrm{S}\right)_\mathrm{S} + \left(\boldsymbol{g}^1 - \boldsymbol{g}^0, \boldsymbol{v}_h^\mathrm{S}\right)_{\Gamma_\mathrm{N}} - \left(\left(\partial_{tt}\boldsymbol{u}\right)^1 - \left(\partial_{tt}\boldsymbol{u}\right)^0, \boldsymbol{v}_h\right)_\rho \\
+ \left(\rho_\mathrm{F} c^2 \operatorname{div}\left(\left(\boldsymbol{u}^\mathrm{F}\right)^1 - \left(\boldsymbol{u}^\mathrm{F}\right)^0\right), \left(\boldsymbol{v}_h^\mathrm{F} - \boldsymbol{v}_h^\mathrm{S}\right)\cdot \boldsymbol{n}\right)_{\Gamma_\mathrm{I}}.
\end{aligned}$$

Thus, from the previous two equations we arrive at

$$\begin{aligned}
&\left(\boldsymbol{u}^1 - \boldsymbol{u}^0, \boldsymbol{v}_h\right)_\rho + \theta\Delta t^2 a\left(\boldsymbol{u}^1 - \boldsymbol{u}^0, \boldsymbol{v}_h\right) \\
&\quad = \Delta t\left(\boldsymbol{u}_1, \boldsymbol{v}_h\right)_\rho + \frac{\Delta t^2}{2}\left(\partial_{tt}\boldsymbol{u}^0, \boldsymbol{v}_h\right)_\rho + \frac{1}{2}\int_0^{t_1} (\Delta t - s)^2 \left(\partial_{ttt}\boldsymbol{u}(s), \boldsymbol{v}_h\right)_\rho ds \\
&\quad - \theta\Delta t^2\left(\left(\partial_{tt}\boldsymbol{u}\right)^1 - \left(\partial_{tt}\boldsymbol{u}\right)^0, \boldsymbol{v}_h\right)_\rho + \theta\Delta t^2\left(\boldsymbol{f}^1 - \boldsymbol{f}^0, \boldsymbol{v}_h^\mathrm{S}\right)_\mathrm{S} + \theta\Delta t^2\left(\boldsymbol{g}^1 - \boldsymbol{g}^0, \boldsymbol{v}_h^\mathrm{S}\right)_{\Gamma_\mathrm{N}} \\
&\quad + \theta\Delta t^2\left(\rho_\mathrm{F} c^2 \operatorname{div}\left(\left(\boldsymbol{u}^\mathrm{F}\right)^1 - \left(\boldsymbol{u}^\mathrm{F}\right)^0\right), \left(\boldsymbol{v}_h^\mathrm{F} - \boldsymbol{v}_h^\mathrm{S}\right)\cdot \boldsymbol{n}\right)_{\Gamma_\mathrm{I}}.
\end{aligned}$$

On the other hand, we recall that $\boldsymbol{u}_h^1$ is the solution to (5.1), namely, $\forall \boldsymbol{v}_h \in \boldsymbol{V}_h$

$$\left(\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right)_\rho + \theta\Delta t^2 a\left(\boldsymbol{u}_h^1 - \boldsymbol{u}_h^0, \boldsymbol{v}_h\right) = \Delta t\left(\boldsymbol{u}_1, \boldsymbol{v}_h\right)_\rho + \frac{\Delta t^2}{2}\left(\left(\boldsymbol{f}^0, \boldsymbol{v}_h^\mathrm{S}\right)_\mathrm{S} + \left(\boldsymbol{g}^0, \boldsymbol{v}_h^\mathrm{S}\right)_{\Gamma_\mathrm{N}} - a\left(\boldsymbol{u}^0, \boldsymbol{v}_h\right)\right).$$

Thus, from the previous two equations and (3.2) we arrive at

$$\begin{aligned}
\Delta t^2 \mathscr{L}(\boldsymbol{v}_h) &= \left((\mathbf{P}_h - \mathbf{I})\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho + \theta \Delta t^2 \left((\mathbf{P}_h - \mathbf{I})\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho \\
&\quad + \frac{1}{2}\int_0^{t_1} (\Delta t - s)^2 \left(\partial_{ttt}\boldsymbol{u}(s), \boldsymbol{v}_h\right)_\rho ds + \theta \Delta t^2 \left(\boldsymbol{f}^1 - \boldsymbol{f}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\mathrm{S}} + \theta \Delta t^2 \left(\boldsymbol{g}^1 - \boldsymbol{g}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}} \\
&\quad - \theta \Delta t^2 \left((\partial_{tt}\boldsymbol{u})^1 - (\partial_{tt}\boldsymbol{u})^0, \boldsymbol{v}_h\right)_\rho + \theta \Delta t^2 \left(\rho_{\mathrm{F}} c^2 \operatorname{div}\left(\left(\boldsymbol{u}^{\mathrm{F}}\right)^1 - \left(\boldsymbol{u}^{\mathrm{F}}\right)^0\right), \left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot \boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} \\
&= \left(1 + \theta \Delta t^2\right)\left((\mathbf{P}_h - \mathbf{I})\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho + \frac{1}{2}\int_0^{t_1} (\Delta t - s)^2 \left(\partial_{ttt}\boldsymbol{u}(s), \boldsymbol{v}_h\right)_\rho ds \\
&\quad + \theta \Delta t^2 \left(\boldsymbol{f}^1 - \boldsymbol{f}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\mathrm{S}} + \theta \Delta t^2 \left(\boldsymbol{g}^1 - \boldsymbol{g}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}} - \theta \Delta t^2 \left((\partial_{tt}\boldsymbol{u})^1 - (\partial_{tt}\boldsymbol{u})^0, \boldsymbol{v}_h\right)_\rho \\
&\quad + \theta \Delta t^2 \left(\rho_{\mathrm{F}} c^2 \operatorname{div}\left(\left(\boldsymbol{u}^{\mathrm{F}}\right)^1 - \left(\boldsymbol{u}^{\mathrm{F}}\right)^0\right), \left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot \boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}}.
\end{aligned}$$

The next step is to estimate each term on the right-hand side above. For the former we use Lemma 3.3 and Taylor's formula to obtain

$$\left((\mathbf{P}_h - \mathbf{I})\left(\boldsymbol{u}^1 - \boldsymbol{u}^0\right), \boldsymbol{v}_h\right)_\rho \leqslant C h^{\min\{\alpha,\beta\}} \Delta t \left\|\partial_t \boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{H}}.$$

For the latter, we resort to Lemma 3.1 and differentiate in time (2.2). Thus, we have

$$\begin{aligned}
\left(\operatorname{div}\left(\left(\boldsymbol{u}^{\mathrm{F}}\right)^1 - \left(\boldsymbol{u}^{\mathrm{F}}\right)^0\right), \left(\boldsymbol{v}_h^{\mathrm{F}} - \boldsymbol{v}_h^{\mathrm{S}}\right)\cdot \boldsymbol{n}\right)_{\Gamma_{\mathrm{I}}} &\leqslant C h \Delta t \left|\partial_t \left(\operatorname{div} \boldsymbol{u}^{\mathrm{F}}\right)\right|_{C\left(0,T;\mathrm{H}^1(\Omega_{\mathrm{F}})\right)} \left\|\boldsymbol{v}_h^{\mathrm{S}}\right\|_{\mathrm{H}^1(\Omega_{\mathrm{S}})^d} \\
&\leqslant C h \Delta t \left\|\partial_{ttt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{H})} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{X}}.
\end{aligned}$$

For the rest of the terms we have the following bounds which are easy to check:

$$\begin{aligned}
\int_0^{t_1} (\Delta t - s)^2 \left(\partial_{ttt}\boldsymbol{u}(s), \boldsymbol{v}_h\right)_\rho ds &\leqslant C \Delta t^3 \left\|\partial_{ttt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{H})} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{H}}, \\
\left(\boldsymbol{f}^1 - \boldsymbol{f}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\mathrm{S}} &\leqslant C \Delta t \left\|\partial_t \boldsymbol{f}\right\|_{C\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)} \left\|\boldsymbol{v}_h^{\mathrm{S}}\right\|_{\mathrm{L}^2(\Omega_{\mathrm{S}})^d}, \\
\left(\boldsymbol{g}^1 - \boldsymbol{g}^0, \boldsymbol{v}_h^{\mathrm{S}}\right)_{\Gamma_{\mathrm{N}}} &\leqslant C \Delta t \left\|\partial_t \boldsymbol{g}\right\|_{C\left(0,T;\mathrm{L}^2(\Gamma_{\mathrm{N}})^d\right)} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{X}}, \\
\left(\partial_{tt}\boldsymbol{u}^1 - \partial_{tt}\boldsymbol{u}^0, \boldsymbol{v}_h\right)_\rho &\leqslant C \Delta t \left\|\partial_{ttt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{H})} \left\|\boldsymbol{v}_h\right\|_{\boldsymbol{H}}.
\end{aligned}$$

The lemma follows from the above equations. □

Putting together the above lemmas we obtain the following result.

LEMMA 5.5 There exist $C > 0$, independent of h and Δt, such that

$$\begin{aligned}
\max_{0 \leqslant n \leqslant N} \left\|\boldsymbol{\phi}^n\right\|_{\boldsymbol{H}} &\leqslant C\left\{\left\|\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}_{0h}\right\|_{\boldsymbol{X}} + h^{\min\{\alpha,\beta\}}\left[\left\|\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})} + \left\|\partial_t \boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})} + \left\|\partial_{tt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{X})}\right]\right\} \\
&\quad + C \Delta t^2 \left[\left\|\partial_{ttt}\boldsymbol{u}\right\|_{C(0,T;\boldsymbol{H})} + \left\|\partial_{tttt}\boldsymbol{u}\right\|_{\mathrm{L}^1(0,T;\boldsymbol{H})} + \left\|\partial_t \boldsymbol{f}\right\|_{C\left(0,T;\mathrm{L}^2(\Omega_{\mathrm{S}})^d\right)} + \left\|\partial_t \boldsymbol{g}\right\|_{C\left(0,T;\mathrm{L}^2(\Gamma_{\mathrm{N}})^d\right)}\right]
\end{aligned}$$

with α and β as in Remark 3.1.

Proof. We begin with equation (5.5) and use the previous lemmas to bound each term on its right-hand side. With this end, first notice that, from the definition of Φ^n, $n = 1, \ldots, N$, we have

$$\Delta t \left\|\Phi^n\right\|_{\boldsymbol{H}} \leqslant C \max_{0 \leqslant m \leqslant n} \left\|\boldsymbol{\phi}^m\right\|_{\boldsymbol{H}} \qquad \text{and} \qquad \Delta t^2 \left\|\Phi^n\right\|_{\boldsymbol{X}}^2 \leqslant C\left[\left(\max_{0 \leqslant m \leqslant n} \left\|\boldsymbol{\phi}^m\right\|_{\boldsymbol{H}}\right)^2 + \Delta t^2 a\left(\Phi^n, \Phi^n\right)\right].$$

Then, from (5.9), Lemmas 5.2, 5.3 and 5.4 and Young's inequality, it follows that for all $\gamma > 0$

$$\Delta t \sum_{m=1}^{n-1} \left(\boldsymbol{R}^m + \boldsymbol{P}^m, \boldsymbol{\phi}^{m+1} + \boldsymbol{\phi}^m\right)_\rho \leqslant \gamma \max_{1\leqslant n\leqslant N} \|\boldsymbol{\phi}^n\|_{\boldsymbol{H}}^2 + \frac{C}{\gamma}\left\{h^{\min\{\alpha,\beta\}}\left[\|\partial_{tt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{X})} + \|\boldsymbol{u}\|_{C(0,T;\boldsymbol{X})}\right] + \Delta t^2 \|\partial_{tttt}\boldsymbol{u}\|_{\mathrm{L}^1(0,T;\boldsymbol{H})}\right\}^2,$$

$$2\Delta t \sum_{m=1}^{n-1} \Theta^m \left(\Phi^{m+1} - \Phi^m\right) \leqslant \frac{1}{8}\Delta t^2 \max_{1\leqslant n\leqslant N} a\left(\Phi^n, \Phi^n\right) + Ch^2 \|\partial_{tt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})}^2$$

and

$$2\Delta t^2 \mathscr{L}\left(\Phi^n - \Phi^0\right) \leqslant \gamma \max_{0\leqslant m\leqslant N} \|\boldsymbol{\phi}^m\|_{\boldsymbol{H}}^2 + \frac{1}{8}\Delta t^2 a\left(\Phi^n, \Phi^n\right) + C\left[\Delta t \left\|\Phi^0\right\|_{\boldsymbol{X}} + \frac{h^{\min\{\alpha,\beta\}}}{\gamma}\|\partial_t \boldsymbol{u}\|_{C(0,T;\boldsymbol{X})} + \Delta t^2\left(\frac{1}{\gamma} + h\right)\|\partial_{ttt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})} + \frac{\Delta t^2}{\gamma}\|\partial_t \boldsymbol{f}\|_{C\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)} + \Delta t^2 \|\partial_t \boldsymbol{g}\|_{C\left(0,T;\mathrm{L}^2(\Gamma_\mathrm{N})^d\right)}\right]^2.$$

Adding all these inequalities we arrive at

$$\Delta t \sum_{m=1}^{n-1} \left(\boldsymbol{R}^m + \boldsymbol{P}^m, \boldsymbol{\phi}^{m+1} + \boldsymbol{\phi}^m\right)_\rho + 2\Delta t \sum_{m=1}^{n-1} \Theta^m\left(\Phi^{m+1} - \Phi^m\right) + 2\Delta t^2 \mathscr{L}\left(\Phi^n - \Phi^0\right)$$
$$\leqslant 2\gamma \max_{0\leqslant n\leqslant N} \|\boldsymbol{\phi}^n\|_{\boldsymbol{H}}^2 + \frac{1}{4}\Delta t^2 \max_{0\leqslant n\leqslant N} a\left(\Phi^n, \Phi^n\right) + C_\gamma\left[\Delta t^4 \left\|\Phi^0\right\|_{\boldsymbol{X}}^2 + h^{2\min\{\alpha,\beta\}}\|\partial_{tt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{X})}^2 + \Delta t^4 \|\partial_{tttt}\boldsymbol{u}\|_{\mathrm{L}^1(0,T;\boldsymbol{H})}^2 + \Delta t^4 \|\partial_{ttt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})}^2 + \Delta t^4 \|\partial_t \boldsymbol{f}\|_{C\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)}^2 + \Delta t^4 \|\partial_t \boldsymbol{g}\|_{C\left(0,T;\mathrm{L}^2(\Gamma_\mathrm{N})^d\right)}^2\right],$$

where $C_\gamma > 0$ depends on γ. Using that $\boldsymbol{\phi}^0 = \mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}_{0h}$ and recalling that $\Phi^0 = -\frac{1}{2}\phi^0$ and $\theta \geqslant 1/4$, straightforward computations allow us to conclude the proof. □

We are now in a position to write the main result of this section, which establishes error estimate for the fully-discrete scheme in the L^2-norm.

THEOREM 5.2 Let $\boldsymbol{u}$ and $\left\{\boldsymbol{u}_h^n\right\}_{n=1}^N$ be the solutions to Problems 2.1 and 5.1, respectively. Then,

$$\max_{0\leqslant n\leqslant N} \|\boldsymbol{u}^n - \boldsymbol{u}_h^n\|_{\boldsymbol{H}} \leqslant C\left\{\left\|\mathbf{P}_h \boldsymbol{u}^0 - \boldsymbol{u}_{0h}\right\|_{\boldsymbol{X}} + h^{\min\{\alpha,\beta\}}\|\boldsymbol{u}\|_{C^2(0,T;\boldsymbol{X})} + \Delta t^2\left[\|\partial_{tttt}\boldsymbol{u}\|_{\mathrm{L}^1(0,T;\boldsymbol{H})} + \|\partial_{ttt}\boldsymbol{u}\|_{C(0,T;\boldsymbol{H})} + \|\partial_t \boldsymbol{g}\|_{C\left(0,T;\mathrm{L}^2(\Gamma_\mathrm{N})^d\right)} + \|\partial_t \boldsymbol{f}\|_{C\left(0,T;\mathrm{L}^2(\Omega_\mathrm{S})^d\right)}\right]\right\},$$

where $\alpha \in \left(\frac{1}{2}, 1\right]$ and $\beta \in (0,1]$ are as in Remark 3.1.

Proof. The result is a consequence of the decomposition (5.4) and Lemmas 3.3 and 5.5. □

6. Numerical examples

In this section we will report a couple of numerical tests performed with the method analyzed in this paper. First, we will check that the resulting numerical solution converges as the discretization parameters h and Δt go to zero. Secondly, we will compare our results with those arising from an alternative approach usual in the engineering practice.

We have considered a 2D geometry as shown in Fig. 2 (left). We have taken unit physical parameters, namely, $\rho_{\mathrm{F}} = \rho_{\mathrm{S}} = \lambda_{\mathrm{S}} = \mu_{\mathrm{S}} = c = 1$. We have solved the problem in the time interval $[0, T]$ with $T := \frac{3\pi}{8}$ and an initial condition $(0, u_0)$, where $u_0(\boldsymbol{x}) := (\max\{0.125 - \|\boldsymbol{x} - \boldsymbol{x}_0\|_1, 0\})^2$ with $\boldsymbol{x}_0$ as shown in Fig. 2. This initial condition is shown in Fig. 3 where it can be seen that is localized in the surroundings of the point $\boldsymbol{x}_0$. We have also taken $\boldsymbol{f} := \boldsymbol{g} := \boldsymbol{0}$ and $\Gamma_{\mathrm{N}} := \partial\Omega$. For the Newmark scheme we have chosen the parameter $\theta := \frac{3}{8} > \frac{1}{4}$, in order to improve stability.

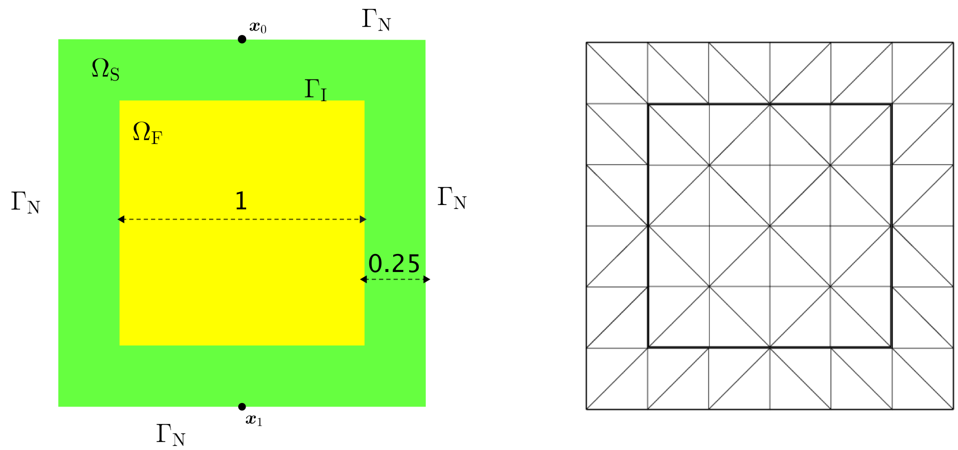


FIG. 2: Physical domain (left) and initial mesh (right).

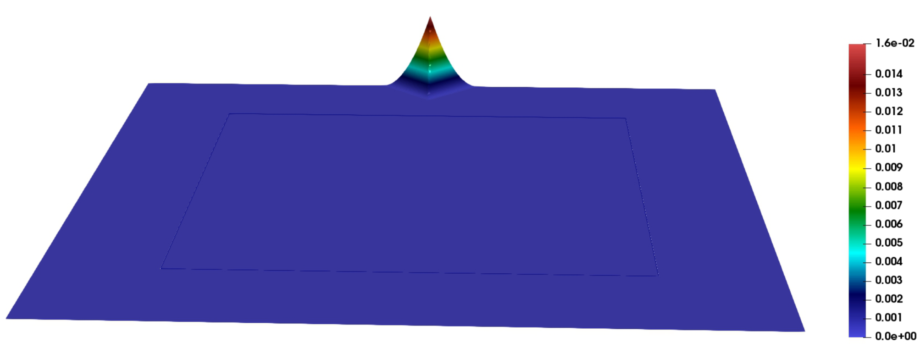


FIG. 3: Initial condition u_0.

To observe the convergence of the proposed method, we have solved the problem with different time-steps and mesh-sizes. We have taken as initial mesh that shown in Fig. 2 (right) (whose mesh-size we denote by h_0) and an initial partition of the time interval with time-step $\Delta t_0 := T/100$. We have computed the numerical solution on the meshes and partitions of the time interval obtained by uniformly subdividing the initial ones, so that the mesh-size and the time steps are h_0/M and $\Delta t_0/M$, respectively, for several values of $M \in \mathbb{N}$.

Figure 4 shows the second component of the displacement at the point $\boldsymbol{x}_1$ (which is shown in Fig. 2). Notice that, because of the symmetry of the problem and of the used meshes, the first component of the displacement vanishes identically.

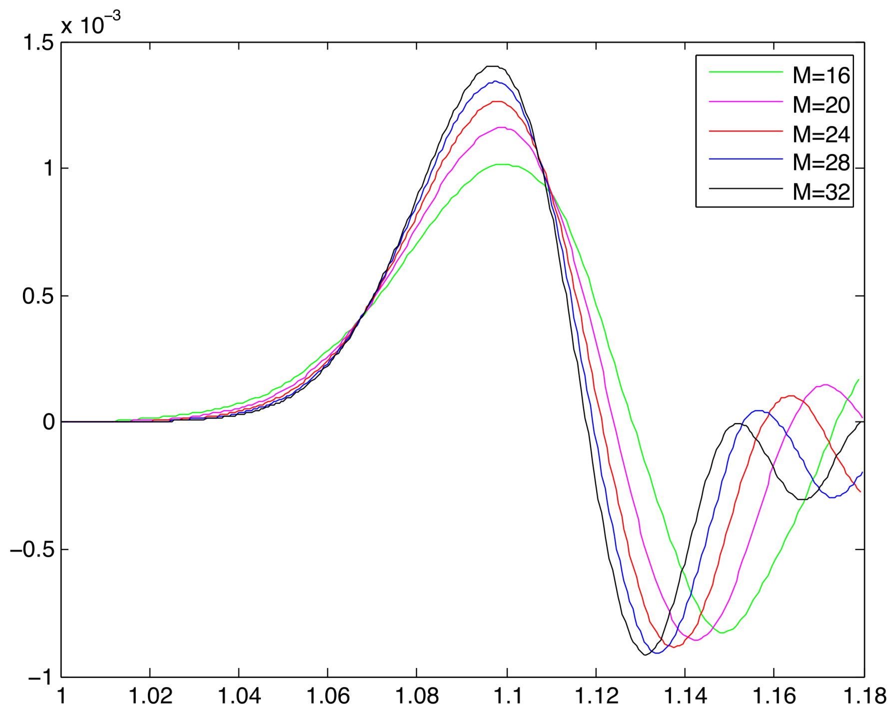
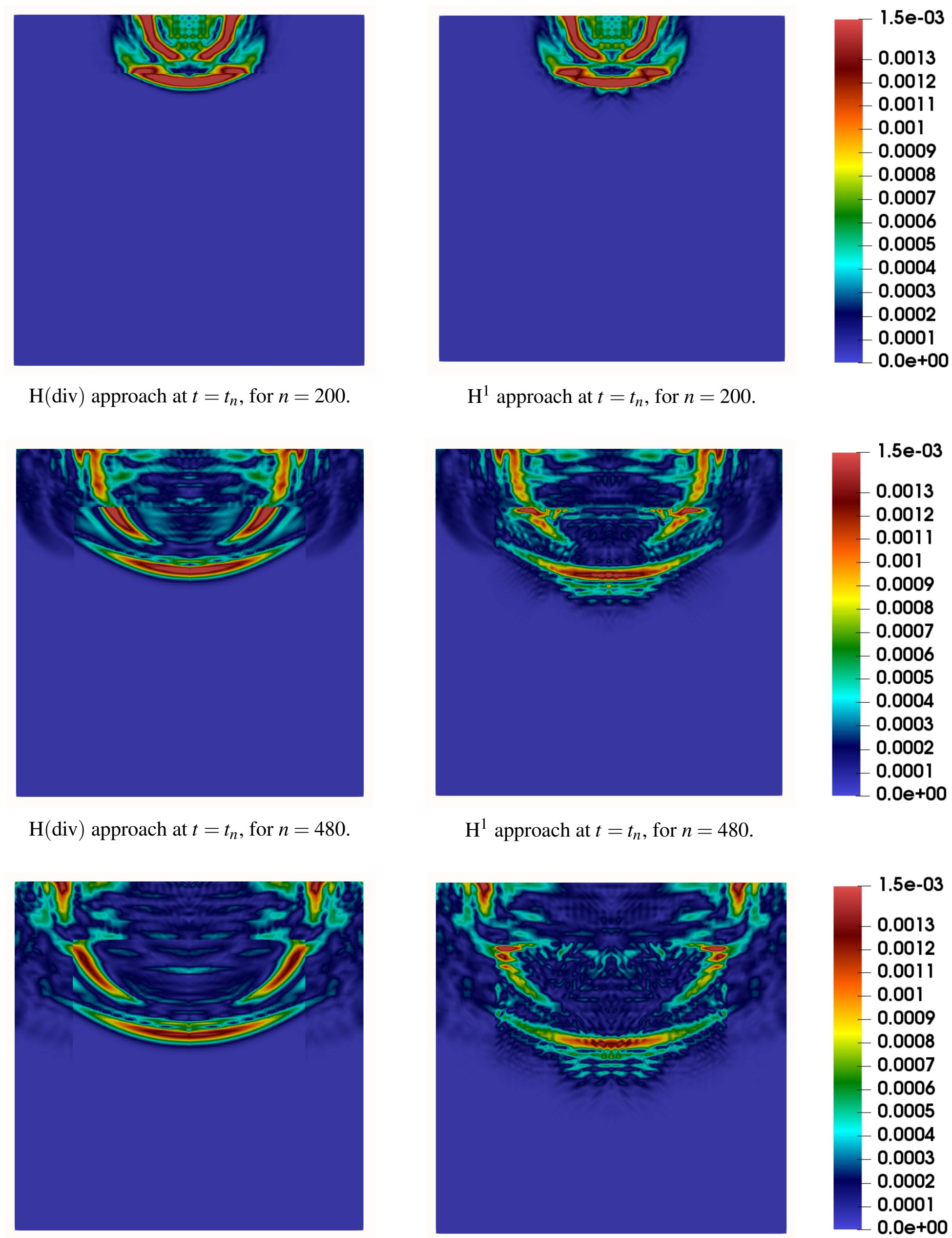


FIG. 4: Second component of the displacement at the point $\boldsymbol{x}_1$ versus time for different levels of refinement of the initial mesh. Here $\Delta t := \dfrac{\Delta t_0}{M} = \dfrac{3\pi}{800M}$ and $h := h_0/M$, where h_0 is the mesh-size of the mesh shown in Fig. 2.

It can be seen from Fig. 4 that the computed solutions converge as the refinement parameter M goes to infinity (and, hence, the discretization parameters h and Δt go to zero).

A common procedure in fluid-solid computations is to consider the fluid as a solid without resistance to shear strain; namely, with Lamé coefficient $\mu_{\mathrm{F}} := 0$, the other Lamé coefficient being taken as $\lambda_{\mathrm{F}} := \rho_{\mathrm{F}} c^2$ (see Kiefling & Feng (1976), for instance). This leads to search a displacement field $\boldsymbol{u}: [0,T] \to \mathrm{H}^1(\Omega)^d$ (with $\Omega := \overline{\Omega}_{\mathrm{F}} \cup \Omega_{\mathrm{S}}$) by means of a standard code for linear elasticity based on using standard Lagrangian elements in the whole domain Ω. However, when this approach is used for the free-vibration problem, it is well known that it leads to spurious modes (Hamdi *et al.* (1978); Bermúdez & Rodríguez (1994)). To the best of the authors knowledge, the behavior of this approach applied to the time domain problem has not been reported. The aim of the second test is to compare the results obtained with this common engineering practice (that we will call H^1) and with the method analyzed in this paper (that we will call H(div)). Figures 5 and 6 show the Euclidean norm of the displacement field at several times computed with both approaches.

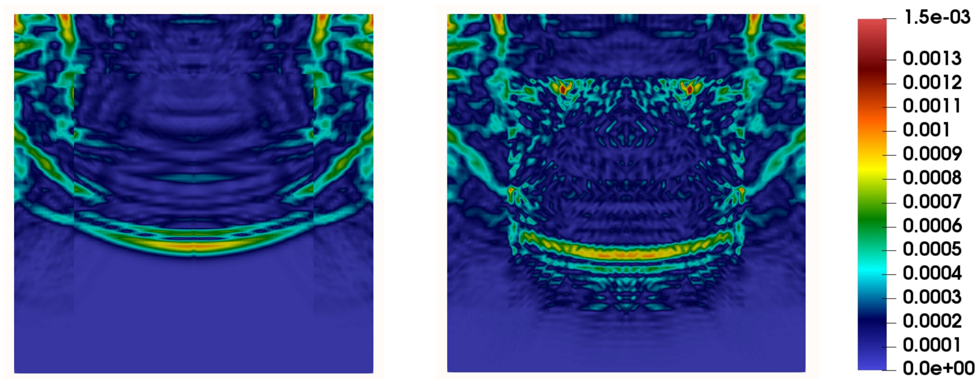


$\mathrm{H}(\mathrm{div})$ approach at $t = t_n$, for $n = 200$. H^1 approach at $t = t_n$, for $n = 200$.

$\mathrm{H}(\mathrm{div})$ approach at $t = t_n$, for $n = 480$. H^1 approach at $t = t_n$, for $n = 480$.

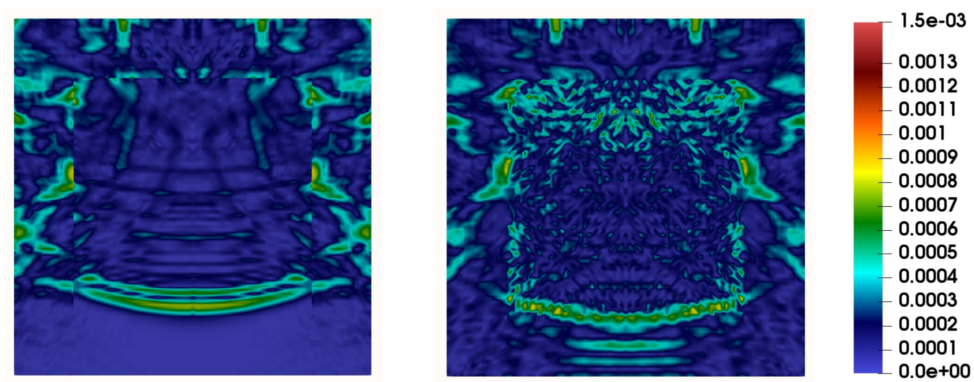
$\mathrm{H}(\mathrm{div})$ approach at $t = t_n$, for $n = 640$. H^1 approach at $t = t_n$, for $n = 640$.

FIG. 5: Comparison between our method and the H^1-approach for different times $t_n := n\Delta t$ with $\Delta t := \dfrac{\Delta t_0}{16} = \dfrac{3\pi}{12800}$ and $h := h_0/16$, where h_0 is the mesh-size of the mesh shown in Fig. 2. Here we plot the magnitude of the displacement field.



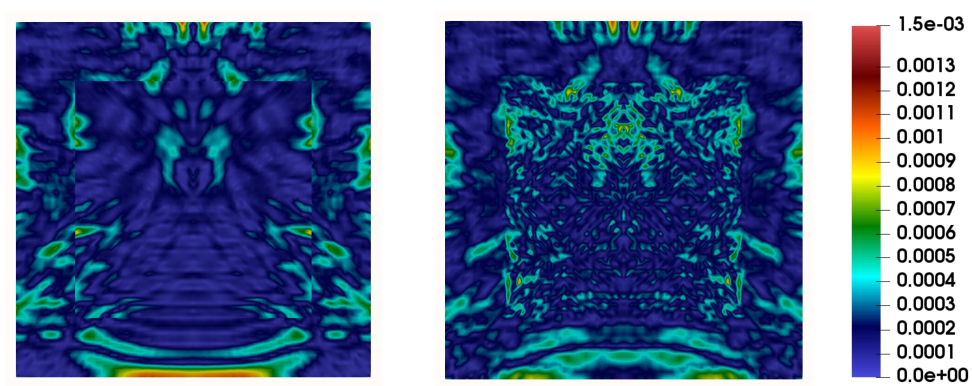
H(div) approach at $t = t_n$, for $n = 1000$.

H^1 approach at $t = t_n$, for $n = 1000$.



H(div) approach at $t = t_n$, for $n = 1280$.

H^1 approach at $t = t_n$, for $n = 1280$.



H(div) approach at $t = t_n$, for $n = 1500$.

H^1 approach at $t = t_n$, for $n = 1500$.

FIG. 6: Comparison between our method and the H^1-approach for different times $t_n := n\Delta t$ with $\Delta t := \dfrac{\Delta t_0}{16} = \dfrac{3\pi}{12800}$ and $h := h_0/16$, where h_0 is the mesh-size of the mesh shown in Fig. 2. Here we plot the magnitude of the displacement field.

It can be seen that the wave propagation is much more noisy for the H^1-approach than for the method analyzed in this paper. Moreover, the wave arrives at the point $\boldsymbol{x}_1$ earlier with the H^1-approach. This can also be seen in Fig. 7, which shows the vertical displacement computed with both methods on the finest mesh ($M = 32$) on a longer time interval $[0, \frac{\pi}{2}]$.

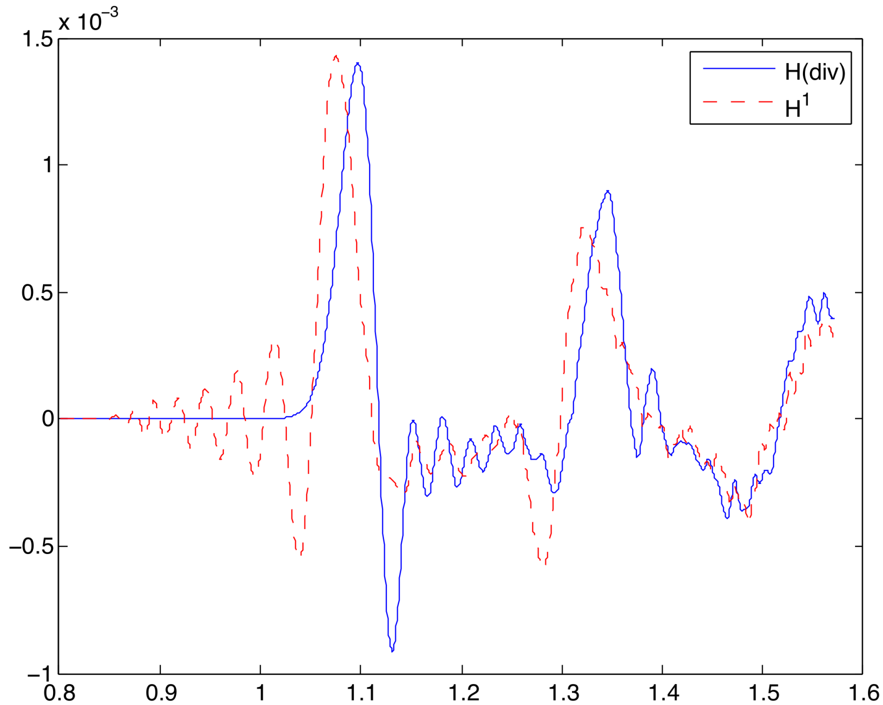


FIG. 7: Second component of the displacement at the point $\boldsymbol{x}_1$ computed with our scheme (solid blue) and the H^1-approach (dashed red) in a refined mesh ($M = 32$) versus time.

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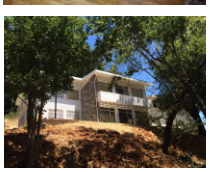

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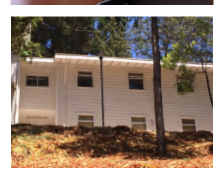
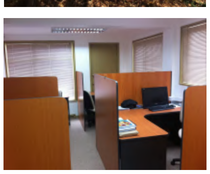

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