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A new mixed finite element method for the n-dimensional Boussinesq problem with temperature-dependent viscosity*

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Abstract

In this paper, we propose a new mixed-primal finite element method for heat-driven flows with temperature-dependent viscosity modeled by the stationary Boussinesq equations. The motivation for this work is to overcome a drawback found by the authors in a recent work where, in order to derive the mixed formulation for the momentum equation, the reciprocal of the viscosity appears multiplied to a tensor product of velocities, making the analysis more restrictive, as it is necessary to use a continuous injection that is guaranteed only in 2D. Therefore, we show in this work that by adding the strain rate tensor as a new unknown in the problem, we get more flexibility in our reasoning and are able to consider the n-dimensional case, as the viscosity now appears multiplied by this new term only. The rest of the analysis is again based on the introduction of the pseudostress and vorticity tensors, the elimination of the pressure (which can be recovered later on via postprocessing), the incorporation of augmented Galerkin-type terms in the mixed formulation for the momentum equations, and the definition of the normal heat flux as a suitable Lagrange multiplier in the primal formulation employed for the energy equation. The resulting problem is analysed by means of the Banach and Brouwer fixed-point theorems, and several numerical examples illustrating the performance of the new scheme and confirming the theoretical rates of convergence are presented.

Key words: Boussinesq equations, augmented mixed-primal formulation, fixed-point theory, finite element methods, a priori error analysis

Mathematics subject classifications (2000): 65N30, 65N12, 65N15, 35Q79, 80A20, 76R05, 76D07

1 Introduction

The description of a variety of natural phenomena and engineering problems deal with incompressible quasi-Newtonian flows with viscous heating and buoyancy terms (natural convection of fluids). Mantle convection with very large viscosities, waves and currents near shorelines, heat transfer in nanoparticle

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fluids, creeping thermal plumes, stratified oceanic flows, chemical reactors, and many other examples can be invoked. Here we advocate the study of mixed finite element schemes to approximate the solution of the Boussinesq equations with thermally-dependent viscosity. In the recent contribution [3] the authors construct an augmented mixed-primal finite element method for such a problem restricting the analysis to two-dimensional bounded domains with polygonal boundary. More precisely, in the problem at hand one seeks a velocity field $\mathbf{u}$, a pressure field p and a temperature field φ such that

$$\begin{aligned} -\mathbf{div}\,(\mu(\varphi)\mathbf{e}(\mathbf{u})) + (\nabla\mathbf{u})\mathbf{u} + \nabla p - \varphi\mathbf{g} = \mathbf{0} &\quad \text{in } \Omega, &(1.1a)\\ \operatorname{div}\mathbf{u} = 0 &\quad \text{in } \Omega, &(1.1b)\\ -\operatorname{div}(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi = 0 &\quad \text{in } \Omega, &(1.1c)\\ \mathbf{u} = \mathbf{u}_D &\quad \text{on } \Gamma, &(1.1d)\\ \varphi = \varphi_D &\quad \text{on } \Gamma, &(1.1e) \end{aligned}$$

where $\Omega \subset \mathbb{R}^2$, $\Gamma := \partial\Omega$, the symbol $\mathbf{e}(\mathbf{u})$ denotes the strain rate tensor (symmetric part of the velocity gradient tensor $\nabla\mathbf{u}$), $-\mathbf{g} \in \mathbf{L}^\infty(\Omega)$ is a body force per unit mass (e.g., gravity), $\mathbb{K} \in \mathbb{L}^\infty(\Omega)$ is a uniformly positive definite tensor describing thermal conductivity and $\mu : R \to R^+$ is a temperature-dependent viscosity function, which is assumed to be bounded and Lipschitz continuous, that is, there exist constants $\mu_2 \geq \mu_1 > 0$ and $L_\mu > 0$ such that

$$\mu_1 \leq \mu(s) \leq \mu_2 \quad \forall\ s \in R,$$

and

$$|\mu(s) - \mu(t)| \leq L_\mu |s - t| \quad \forall\ s, t \in R.$$

With respect to the boundary conditions for (1.1), we assume that $\mathbf{u}_D \in \mathbf{H}^{1/2}(\Gamma)$, $\varphi_D \in H^{1/2}(\Gamma)$ and that $\mathbf{u}_D$ verifies the compatibility condition

$$\int_\Gamma \mathbf{u}_D \cdot \boldsymbol{\nu} = 0\,, \tag{1.2}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal on Γ.

The construction of the mixed-primal formulation considered in [3] begins with the introduction of the pseudostress and vorticity tensors, respectively defined as

$$\boldsymbol{\sigma} := \mu(\varphi)\mathbf{e}(\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - p\mathbb{I} \qquad \text{and} \qquad \boldsymbol{\gamma} := \boldsymbol{\omega}(\mathbf{u}) \in \mathbb{L}^2_{\mathtt{skew}}(\Omega), \tag{1.3}$$

where $\boldsymbol{\omega}(\mathbf{u})$ is the skew-symmetric part of the velocity gradient tensor $\nabla\mathbf{u}$ and

$$\mathbb{L}^2_{\mathtt{skew}}(\Omega) := \Big\{\boldsymbol{\eta} \in \mathbb{L}^2(\Omega) : \boldsymbol{\eta} + \boldsymbol{\eta}^{\mathtt{t}} = \mathbf{0}\Big\}. \tag{1.4}$$

Therefore, after eliminating the pressure p, problem (1.1) is rewritten as: Find $(\boldsymbol{\sigma}, \mathbf{u}, \boldsymbol{\gamma}, \varphi)$ such that

$$\begin{aligned} \nabla\mathbf{u} - \boldsymbol{\gamma} - \frac{1}{\mu(\varphi)}\,(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} = \frac{1}{\mu(\varphi)}\,\boldsymbol{\sigma}^{\mathtt{d}} &\quad \text{in } \Omega, &(1.5a)\\ -\mathbf{div}\,\boldsymbol{\sigma} - \varphi\mathbf{g} = \mathbf{0} &\quad \text{in } \Omega, &(1.5b)\\ -\operatorname{div}(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi = 0 &\quad \text{in } \Omega, &(1.5c)\\ \mathbf{u} = \mathbf{u}_D &\quad \text{on } \Gamma, &(1.5d)\\ \varphi = \varphi_D &\quad \text{on } \Gamma, &(1.5e)\\ \int_\Omega \operatorname{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) = 0\,, && (1.5f) \end{aligned}$$

where (1.5f) constitutes a uniqueness condition for the pressure. At this point we remark that, with the term $\nabla\mathbf{u}$ free in (1.5a) thanks to the division by $\mu(\varphi)$ in this equation, integration by parts upon multiplication by a test function is now possible. However, it can be seen that this leads to the usage of a continuous injection from $H^1(\Omega)$ into $L^8(\Omega)$, as required by the following estimate:

$$\begin{aligned}\int_\Omega \left|\varphi(\mathbf{u}\otimes\mathbf{w})^{\mathrm{d}} : \boldsymbol{\tau}\right| &\le \|\varphi\|_{L^4(\Omega)}\|\mathbf{u}\|_{\mathbf{L}^8(\Omega)}\|\mathbf{w}\|_{\mathbf{L}^8(\Omega)}\|\boldsymbol{\tau}\|_{0,\Omega}\\ &\le C(\Omega)\,\|\varphi\|_{1,\Omega}\|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}\|\boldsymbol{\tau}\|_{1,\Omega}\,,\end{aligned}$$

with $C(\Omega) > 0$ and valid for any $\varphi \in H^1(\Omega)$; $\mathbf{u},\mathbf{w} \in \mathbf{H}^1(\Omega)$; $\boldsymbol{\tau} \in \mathbb{L}^2(\Omega)$, and more important, for $\Omega \subset R^d$, $d \in \{1,2\}$, according to the Sobolev embedding theorem (cf., e.g., [16, Theorem 1.3.5]). This estimate is used in several ways throughout [3], at both continuous and discrete levels (see, [3, Lemmas 3.8, 4.5 and 5.3]), and its main purpose is to help in the proof of Lipschitz continuity of the fixed-point operator $\mathbf{T}$ (respectively $\mathbf{T}_h$) that consequently provides well-posedness of the continuous formulation (respectively the Galerkin scheme).

The purpose of this work is to derive a new augmented mixed-primal finite element method for the Boussinesq problem (1.1) considering an n-dimensional domain, $n \in \{2,3\}$. To this end, we have a look at works such as [7, 8, 14] where the strain rate tensor $\mathbf{e}(\mathbf{u})$ is considered as a new variable in the system, in addition to the vorticity and pseudostress tensors. This fact provides more flexibility in the scheme, as it is no longer necessary (nor advisable) to divide in (1.5a) by the viscosity to set the gradient free. Instead, the decomposition of the velocity gradient tensor into its symmetric and skew-symmetric parts provides equation to be integrated by parts. We will see that this problem can be analysed by suitably modifying the approach from [3], thus yielding an augmented mixed-primal finite element method that uses discontinuous piecewise polynomial functions of degree $\le k$ to approximate the strain rate, vorticity and normal heat flux, Raviart-Thomas elements of order k for the pseudostress, and Lagrange elements of order $k+1$ for the velocity and temperature of the fluid.

The rest of this work is organized as follows. In Section 2 we rewrite the Boussinesq problem (1.1) considering the strain rate, pseudostress and vorticity tensors as new variables, to then derive an augmented mixed-primal formulation, whose well-posedness will be proved by means of a fixed-point approach. Similarly, in Section 3 we provide the corresponding Galerkin scheme and its associated well-posedness result, to then, in Section 4, proceed to derive a priori error estimates and state the rates of convergence of the scheme when a particular choice of finite element subspaces is made. Finally, to complement our theoretical results, we present in Section 5 a set of numerical examples that serve to confirm the properties of the proposed schemes.

2 The continuous formulation

In order to avoid the division by $\mu(\varphi)$ in (1.5a), we have a look at recent work [7], where the authors develop a augmented mixed finite element method for the Navier-Stokes equations with nonlinear viscosity. This approach relies on the definition of the strain rate tensor as a new unknown

$$\mathbf{t} := \mathbf{e}(\mathbf{u}) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega),$$

where

$$\mathbb{L}^2_{\mathtt{tr}}(\Omega) = \Big\{\mathbf{s} \in \mathbb{L}^2(\Omega) : \quad \mathbf{s} = \mathbf{s}^{\mathrm{t}} \quad \text{and} \quad \mathrm{tr}(\mathbf{s}) = 0\Big\},$$

which together with the pseudostress and vorticity tensors already defined in (1.3), allows us to rewrite (1.5) as: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$ such that

$$
\begin{aligned}
\mathbf{t} + \boldsymbol{\gamma} &= \nabla \mathbf{u} \quad \text{in } \Omega, && (2.1a)\\
\mu(\varphi)\mathbf{t} - (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} &= \boldsymbol{\sigma}^{\mathtt{d}} \quad \text{in } \Omega, && (2.1b)\\
-\mathbf{div}\,\boldsymbol{\sigma} - \varphi\mathbf{g} &= \mathbf{0} \quad \text{in } \Omega, && (2.1c)\\
-\mathrm{div}\,(\mathbb{K}\nabla\varphi) + \mathbf{u}\cdot\nabla\varphi &= 0 \quad \text{in } \Omega, && (2.1d)\\
\mathbf{u} &= \mathbf{u}_D \quad \text{on } \Gamma, && (2.1e)\\
\varphi &= \varphi_D \quad \text{on } \Gamma, && (2.1f)\\
\int_\Omega \mathrm{tr}(\boldsymbol{\sigma} + \mathbf{u}\otimes\mathbf{u}) &= 0. && (2.1g)
\end{aligned}
$$

2.1 An augmented mixed-primal formulation

Multiplying (2.1a) by a test function $\boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega)$, integrating by parts and using the Dirichlet condition (2.1e), we obtain

$$
\int_\Omega \mathbf{t} : \boldsymbol{\tau}^{\mathtt{d}} + \int_\Omega \boldsymbol{\gamma} : \boldsymbol{\tau} + \int_\Omega \mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau} = \langle \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \rangle_\Gamma \quad \forall\, \boldsymbol{\tau} \in \mathbb{H}(\mathbf{div};\Omega).
$$

Then, we multiply (2.1b) and (2.1c) by appropriate test functions, imposing at the same time the symmetry of the pseudostress tensor $\boldsymbol{\sigma}$, thus obtaining

$$
\int_\Omega \mu(\varphi)\mathbf{t} : \mathbf{s} - \int_\Omega (\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}} : \mathbf{s} - \int_\Omega \boldsymbol{\sigma}^{\mathtt{d}} : \mathbf{s} = 0 \quad \forall\, \mathbf{s} \in \mathbb{L}^2_{\mathtt{tr}}(\Omega), \tag{2.2}
$$

and

$$
-\int_\Omega \mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma} - \int_\Omega \boldsymbol{\sigma} : \boldsymbol{\eta} = \int_\Omega \varphi\mathbf{g}\cdot\mathbf{v} \quad \forall\, (\mathbf{v},\boldsymbol{\eta}) \in \mathbf{L}^2(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega).
$$

The equations associated with the primal formulation of the energy equation are recalled next from [3]:

$$
\int_\Omega \mathbb{K}\nabla\varphi\cdot\nabla\psi + \langle \lambda, \psi \rangle_\Gamma = -\int_\Omega \psi\mathbf{u}\cdot\nabla\varphi \qquad \forall\, \psi \in H^1(\Omega), \tag{2.3}
$$

$$
\langle \xi, \varphi \rangle_\Gamma = \langle \xi, \varphi_D \rangle_\Gamma \qquad \forall\, \xi \in H^{-1/2}(\Gamma)\,, \tag{2.4}
$$

where $\lambda := \mathbb{K}\nabla\varphi\cdot\boldsymbol{\nu} \in H^{-1/2}(\Gamma)$ is introduced as the Lagrange multiplier taking care of the Dirichlet boundary condition on Γ. Notice that, due to the second term in (2.2) and the right hand side of (2.3), the velocity $\mathbf{u}$ must live in $\mathbf{H}^1(\Omega)$, since appealing to the continuous injection of $H^1(\Omega)$ into $L^4(\Omega)$, there exist positive constants $c_1(\Omega)$ and $c_2(\Omega)$ such that

$$
\left|\int_\Omega (\mathbf{u}\otimes\mathbf{w})^{\mathtt{d}} : \mathbf{s}\right| \le c_1(\Omega)\|\mathbf{u}\|_{1,\Omega}\|\mathbf{w}\|_{1,\Omega}\|\mathbf{s}\|_{0,\Omega} \quad \forall\, \mathbf{u},\mathbf{w} \in \mathbf{H}^1(\Omega),\ \forall\, \mathbf{s} \in \mathbb{L}^2(\Omega), \tag{2.5}
$$

and

$$
\left|\int_\Omega \psi\mathbf{u}\cdot\nabla\varphi\right| \le c_2(\Omega)\|\mathbf{u}\|_{1,\Omega}\|\psi\|_{1,\Omega}|\varphi|_{1,\Omega} \quad \forall\, \mathbf{u} \in \mathbf{H}^1(\Omega) \quad \forall\, \varphi,\psi \in H^1(\Omega). \tag{2.6}
$$

Also, obeying to the orthogonal decomposition

$$
\mathbb{H}(\mathbf{div};\Omega) = \mathbb{H}_0(\mathbf{div};\Omega) \oplus R\mathbb{I},
$$

where

$$
\mathbb{H}_0(\mathbf{div};\Omega) := \left\{ \boldsymbol{\zeta} \in \mathbb{H}(\mathbf{div};\Omega) : \int_\Omega \mathrm{tr}(\boldsymbol{\zeta}) = 0 \right\},
$$

we can consider $\boldsymbol{\sigma}$ and $\boldsymbol{\tau}$ in $\mathbb{H}_0(\mathbf{div};\Omega)$ (see [3, Lemma 3.1] for a detailed justification of this change). Having in mind these considerations, at a first glance, the weak formulation reads: Find $(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma},\varphi,\lambda)\in\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)\times H^1(\Omega)\times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned}
\int_\Omega \mu(\varphi)\mathbf{t}:\mathbf{s}-\int_\Omega(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}}:\mathbf{s}-\int_\Omega\boldsymbol{\sigma}^{\mathtt{d}}:\mathbf{s}&=0 &&\forall\,\mathbf{s}\in\mathbb{L}^2_{\mathtt{tr}}(\Omega),\\
\int_\Omega\mathbf{t}:\boldsymbol{\tau}^{\mathtt{d}}+\int_\Omega\boldsymbol{\gamma}:\boldsymbol{\tau}+\int_\Omega\mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}&=\langle\boldsymbol{\tau\nu},\mathbf{u}_D\rangle_\Gamma &&\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
-\int_\Omega\mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}-\int_\Omega\boldsymbol{\sigma}:\boldsymbol{\eta}&=\int_\Omega\varphi\mathbf{g}\cdot\mathbf{v} &&\forall\,(\mathbf{v},\boldsymbol{\eta})\in\mathbf{L}^2(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega),\\
\int_\Omega\mathbb{K}\nabla\varphi\cdot\nabla\psi+\langle\lambda,\psi\rangle_\Gamma&=-\int_\Omega\psi\mathbf{u}\cdot\nabla\varphi &&\forall\,\psi\in H^1(\Omega),\\
\langle\xi,\varphi\rangle_\Gamma&=\langle\xi,\varphi_D\rangle_\Gamma &&\forall\,\xi\in H^{-1/2}(\Gamma).
\end{aligned}\tag{2.7}$$

To achieve a conforming scheme, and to properly analyse (2.7), we augment this variational formulation using redundant Galerkin terms arising from equations of the strong problem (2.1), but tested differently from (2.7), namely:

$$\begin{aligned}
\kappa_1\int_\Omega\Big\{\boldsymbol{\sigma}^{\mathtt{d}}+(\mathbf{u}\otimes\mathbf{u})^{\mathtt{d}}-\mu(\varphi)\mathbf{t}\Big\}:\boldsymbol{\tau}^{\mathtt{d}}&=0 &&\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_2\int_\Omega\Big\{\mathbf{div}\,\boldsymbol{\sigma}+\varphi\mathbf{g}\Big\}\cdot\mathbf{div}\,\boldsymbol{\tau}&=0 &&\forall\,\boldsymbol{\tau}\in\mathbb{H}_0(\mathbf{div};\Omega),\\
\kappa_3\int_\Omega\Big\{\mathbf{e}(\mathbf{u})-\mathbf{t}\Big\}:\mathbf{e}(\mathbf{v})&=0 &&\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),\\
\kappa_4\int_\Omega\Big\{\boldsymbol{\gamma}-\boldsymbol{\omega}(\mathbf{u})\Big\}:\boldsymbol{\eta}&=0 &&\forall\,\boldsymbol{\eta}\in\mathbb{L}^2_{\mathtt{skew}}(\Omega),\\
\kappa_5\int_\Gamma\mathbf{u}\cdot\mathbf{v}&=\kappa_5\int_\Gamma\mathbf{u}_D\cdot\mathbf{v} &&\forall\,\mathbf{v}\in\mathbf{H}^1(\Omega),
\end{aligned}$$

where κ_j, $j\in\{1,\dots,5\}$ are stabilization (or augmentation) positive constants to be specified later on. In this way, and denoting by $\mathcal{H}:=\mathbb{L}^2_{\mathtt{tr}}(\Omega)\times\mathbb{H}_0(\mathbf{div};\Omega)\times\mathbf{H}^1(\Omega)\times\mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $\vec{\mathbf{t}}:=(\mathbf{t},\boldsymbol{\sigma},\mathbf{u},\boldsymbol{\gamma})$ and $\vec{\mathbf{s}}:=(\mathbf{s},\boldsymbol{\tau},\mathbf{v},\boldsymbol{\eta})$, we arrive to the following augmented mixed-primal formulation: Find $(\vec{\mathbf{t}},(\varphi,\lambda))\in\mathcal{H}\times H^1(\Omega)\times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned}
\mathbf{A}_\varphi(\vec{\mathbf{t}},\vec{\mathbf{s}})+\mathbf{B}_{\mathbf{u}}(\vec{\mathbf{t}},\vec{\mathbf{s}})&=F_\varphi(\vec{\mathbf{s}})+F_D(\vec{\mathbf{s}}) &&\forall\,\vec{\mathbf{s}}\in\mathcal{H},\\
\mathbf{a}(\varphi,\psi)+\mathbf{b}(\psi,\lambda)&=F_{\mathbf{u},\varphi}(\psi) &&\forall\,\psi\in H^1(\Omega),\\
\mathbf{b}(\varphi,\xi)&=G(\xi) &&\forall\,\xi\in H^{-1/2}(\Gamma),
\end{aligned}\tag{2.8}$$

where, given an arbitrary $(\mathbf{w},\phi)\in\mathbf{H}^1(\Omega)\times H^1(\Omega)$, the forms $\mathbf{A}_\phi$, $\mathbf{B}_{\mathbf{w}}$, $\mathbf{a}$, $\mathbf{b}$, and the functionals F_D, F_ϕ, $F_{\mathbf{w},\phi}$ and G are defined as

$$\begin{aligned}
\mathbf{A}_\phi(\vec{\mathbf{t}},\vec{\mathbf{s}}):=&\int_\Omega\mu(\phi)\mathbf{t}:\Big\{\mathbf{s}-\kappa_1\boldsymbol{\tau}^{\mathtt{d}}\Big\}+\int_\Omega\mathbf{t}:\Big\{\boldsymbol{\tau}^{\mathtt{d}}-\kappa_3\mathbf{e}(\mathbf{v})\Big\}-\int_\Omega\boldsymbol{\sigma}^{\mathtt{d}}:\Big\{\mathbf{s}-\kappa_1\boldsymbol{\tau}^{\mathtt{d}}\Big\}\\
&+\int_\Omega\mathbf{u}\cdot\mathbf{div}\,\boldsymbol{\tau}-\int_\Omega\mathbf{v}\cdot\mathbf{div}\,\boldsymbol{\sigma}+\int_\Omega\boldsymbol{\gamma}:\boldsymbol{\tau}-\int_\Omega\boldsymbol{\sigma}:\boldsymbol{\eta}-\kappa_4\int_\Omega\boldsymbol{\omega}(\mathbf{u}):\boldsymbol{\eta}\\
&+\kappa_2\int_\Omega\mathbf{div}\,\boldsymbol{\sigma}\cdot\mathbf{div}\,\boldsymbol{\tau}+\kappa_3\int_\Omega\mathbf{e}(\mathbf{u}):\mathbf{e}(\mathbf{v})+\kappa_4\int_\Omega\boldsymbol{\gamma}:\boldsymbol{\eta}+\kappa_5\int_\Gamma\mathbf{u}\cdot\mathbf{v},
\end{aligned}\tag{2.9}$$

$$\mathbf{B}_{\mathbf{w}}(\vec{\mathbf{t}},\vec{\mathbf{s}}):=\int_\Omega(\mathbf{u}\otimes\mathbf{w})^{\mathtt{d}}:\Big\{\kappa_1\boldsymbol{\tau}^{\mathtt{d}}-\mathbf{s}\Big\},\tag{2.10}$$

for all $\vec{\mathbf{t}}, \vec{\mathbf{s}} \in \mathcal{H}$;

$$\mathbf{a}(\varphi, \psi) := \int_\Omega \mathbb{K}\nabla\varphi \cdot \nabla\psi, \tag{2.11}$$

for all $\varphi, \psi \in H^1(\Omega)$;

$$\mathbf{b}(\psi, \xi) := \langle\, \xi, \psi \,\rangle_\Gamma, \tag{2.12}$$

for all $(\psi, \xi) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$;

$$F_D(\vec{\mathbf{s}}) := \langle\, \boldsymbol{\tau}\boldsymbol{\nu}, \mathbf{u}_D \,\rangle_\Gamma + \kappa_5 \int_\Gamma \mathbf{u}_D \cdot \mathbf{v}, \tag{2.13}$$

$$F_\phi(\vec{\mathbf{s}}) := \int_\Omega \phi\mathbf{g} \cdot \Big\{\mathbf{v} - \kappa_2 \mathbf{div}\, \boldsymbol{\tau}\Big\}, \tag{2.14}$$

for all $\vec{\mathbf{s}} \in \mathcal{H}$;

$$F_{\mathbf{w},\phi}(\psi) = -\int_\Omega \psi\mathbf{w} \cdot \nabla\phi, \tag{2.15}$$

for all $\psi \in H^1(\Omega)$; and

$$G(\xi) = \langle\, \xi, \varphi_D \,\rangle_\Gamma, \tag{2.16}$$

for all $\xi \in H^{-1/2}(\Gamma)$.

2.2 The fixed-point argument

A crucial tool in [3] to prove the well-posedness of the continuous and discrete formulations is a technique that decouples the problem into the mixed formulation of the momentum equation and the primal formulation of the energy equation, which further enables us to rewrite the formulation as a fixed-point problem. Hence, we denote $\mathbf{H} := \mathbf{H}^1(\Omega) \times H^1(\Omega)$ and consider in what follows the operator $\mathbf{S} : \mathbf{H} \to \mathcal{H}$ defined by

$$\mathbf{S}(\mathbf{w}, \phi) = (\mathbf{S}_1(\mathbf{w}, \phi), \mathbf{S}_2(\mathbf{w}, \phi), \mathbf{S}_3(\mathbf{w}, \phi), \mathbf{S}_4(\mathbf{w}, \phi)) := \vec{\mathbf{t}},$$

where $\vec{\mathbf{t}}$ is the solution of the problem: Find $\vec{\mathbf{t}} \in \mathcal{H}$ such that

$$\mathbf{A}_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}}) + \mathbf{B}_\mathbf{w}(\vec{\mathbf{t}}, \vec{\mathbf{s}}) = F_\phi(\vec{\mathbf{s}}) + F_D(\vec{\mathbf{s}}) \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}. \tag{2.17}$$

In addition, let $\widetilde{\mathbf{S}} : \mathbf{H} \to H^1(\Omega)$ be the operator defined by

$$\widetilde{\mathbf{S}}(\mathbf{w}, \phi) := \varphi,$$

where φ is the first component of the solution of the problem: Find $(\varphi, \lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ such that

$$\begin{aligned} \mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) &= F_{\mathbf{w},\phi}(\psi) && \forall\, \psi \in H^1(\Omega), \\ \mathbf{b}(\varphi, \xi) &= G(\xi) && \forall\, \xi \in H^{-1/2}(\Gamma)\,. \end{aligned} \tag{2.18}$$

In this way, by introducing the operator $\mathbf{T} : \mathbf{H} \to \mathbf{H}$ as

$$\mathbf{T}(\mathbf{w}, \phi) = \Big(\mathbf{S}_3(\mathbf{w}, \phi), \widetilde{\mathbf{S}}(\mathbf{S}_3(\mathbf{w}, \phi), \phi)\Big) \quad \forall\, (\mathbf{w}, \phi) \in \mathbf{H}, \tag{2.19}$$

we realize that (2.8) can be rewritten as the fixed-point problem: Find $(\mathbf{u}, \varphi) \in \mathbf{H}$ such that

$$\mathbf{T}(\mathbf{u}, \varphi) = (\mathbf{u}, \varphi). \tag{2.20}$$

As in [3], the objective is to use the Banach fixed-point theorem to prove existence and uniqueness of (2.20). We recall that the key difference in the present work with respect to [3] is in the problem that defines the operator $\mathbf{S}$, and therefore, those results associated to the operator $\widetilde{\mathbf{S}}$ and the primal formulation of the energy equation will be considered here as well, but we only cite them.

2.3 Well-posedness of the uncoupled problems

In what follows, we consider

$$\| \vec{\mathbf{s}} \| := \Big\{ \| \mathbf{s} \|_{0,\Omega}^2 + \| \boldsymbol{\tau} \|_{\mathbf{div};\Omega}^2 + \| \mathbf{v} \|_{1,\Omega}^2 + \| \boldsymbol{\eta} \|_{0,\Omega}^2 \Big\}^{1/2} \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H},$$

and

$$\| (\psi, \xi) \| := \Big\{ \| \psi \|_{1,\Omega}^2 + \| \xi \|_{-1/2,\Gamma}^2 \Big\}^{1/2} \quad \forall\, (\psi, \xi) \in H^1(\Omega) \times H^{-1/2}(\Gamma).$$

We first recall some results that will be used for ellipticity purposes.

Lemma 2.1. *There exists $c_3(\Omega) > 0$ such that*

$$c_3(\Omega)\, \| \boldsymbol{\tau}_0 \|_{0,\Omega}^2 \le \| \boldsymbol{\tau}^{\mathrm{d}} \|_{0,\Omega}^2 + \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2 \quad \forall\, \boldsymbol{\tau} = \boldsymbol{\tau}_0 + c\mathbb{I} \in \mathbb{H}(\mathbf{div};\Omega).$$

Proof. See [6, Proposition 3.1], [13, Lemma 2.3]. □

Lemma 2.2. *There exists $\kappa_0(\Omega) > 0$ such that*

$$\kappa_0(\Omega)\, \| \mathbf{v} \|_{1,\Omega}^2 \le \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 + \| \mathbf{v} \|_{0,\Gamma}^2 \quad \forall\, \mathbf{v} \in \mathbf{H}^1(\Omega).$$

Proof. See [12, Lemma 3.1]. □

The following results establish sufficient conditions for the operators $\mathbf{S}$ and $\widetilde{\mathbf{S}}$ being well-defined, equivalently, (2.17) and (2.18) being well-posed.

Lemma 2.3. *Assume that for $\delta_1 \in \Big(0, \frac{2}{\mu_2}\Big)$, $\delta_2, \delta_3 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_5 \in (0,\infty),$$

$$\kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right) \quad \text{and} \quad \kappa_4 \in \left(0, 2\delta_3\kappa_0(\Omega)\, \min\left\{\kappa_3\left(1 - \frac{\delta_2}{2}\right), \kappa_5\right\}\right).$$

Then, there exists $r_0 > 0$ such that for each $r \in (0, r_0)$, the problem (2.17) has a unique solution $\vec{\mathbf{t}} := \mathbf{S}(\mathbf{w},\phi) \in \mathcal{H}$ for each $(\mathbf{w},\phi) \in \mathbf{H}$ such that $\| \mathbf{w} \|_{1,\Omega} \le r$. Moreover, there exists a constant $C_{\mathbf{S}} > 0$, independent of $(\mathbf{w},\phi)$ such that there holds

$$\| \mathbf{S}(\mathbf{w},\phi) \| = \| \vec{\mathbf{t}} \| \le C_{\mathbf{S}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\}. \tag{2.21}$$

Proof. Given $(\mathbf{w},\phi) \in \mathbf{H}$, we notice from (2.9) that $\mathbf{A}_\phi$ is bilinear. Then, by using the upper bound of the viscosity function, the Cauchy-Schwarz inequality and the trace theorem with constant $c_0(\Omega)$, we see that for any $\vec{\mathbf{t}}, \vec{\mathbf{s}} \in \mathcal{H}$,

$$\begin{aligned}
|\mathbf{A}_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}})| &\le \mu_2(1+\kappa_1^2)^{1/2}\, \| \mathbf{t} \|_{0,\Omega} \| \vec{\mathbf{s}} \| + (1+\kappa_3^2)^{1/2}\, \| \mathbf{t} \|_{0,\Omega} \| \vec{\mathbf{s}} \| + (1+\kappa_1^2)^{1/2}\, \| \boldsymbol{\sigma}^{\mathrm{d}} \|_{0,\Omega} \| \vec{\mathbf{s}} \| \\
&\quad + \| \mathbf{u} \|_{0,\Omega} \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega} + \| \boldsymbol{\gamma} \|_{0,\Omega} \| \boldsymbol{\tau} \|_{0,\Omega} + \| \mathbf{div}\, \boldsymbol{\sigma} \|_{0,\Omega} \| \mathbf{v} \|_{0,\Omega} + \| \boldsymbol{\sigma} \|_{0,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} \\
&\quad + \kappa_4 \| \mathbf{u} \|_{1,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} + \kappa_2 \| \mathbf{div}\, \boldsymbol{\sigma} \|_{0,\Omega} \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega} + \kappa_3 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{v} \|_{1,\Omega} \\
&\quad + \kappa_4 \| \boldsymbol{\gamma} \|_{0,\Omega} \| \boldsymbol{\eta} \|_{0,\Omega} + \kappa_5 c_0(\Omega)^2 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{v} \|_{1,\Omega}\,,
\end{aligned}$$

and therefore, there exists a constant $C_{\mathbf{A}} > 0$ depending only on μ_2, $c_0(\Omega)$ and the stabilization parameters κ_j, such that

$$|\mathbf{A}_\phi(\vec{\mathbf{t}}, \vec{\mathbf{s}})| \le C_{\mathbf{A}} \| \vec{\mathbf{t}} \| \| \vec{\mathbf{s}} \| \quad \forall\, \vec{\mathbf{t}}, \vec{\mathbf{s}} \in \mathcal{H}. \tag{2.22}$$

On the other hand, from (2.10), $\mathbf{B_w}$ is a bilinear form, and by using the estimate (2.5), we obtain for any $\vec{\mathbf{t}}, \vec{\mathbf{s}} \in \mathcal{H}$ that

$$|\mathbf{B_w}(\vec{\mathbf{t}}, \vec{\mathbf{s}})| \leq c_1(\Omega)(1+\kappa_1)^2 \| \mathbf{u} \|_{1,\Omega} \| \mathbf{w} \|_{1,\Omega} \| \vec{\mathbf{s}} \|, \tag{2.23}$$

which together with (2.22) implies the existence of a positive constant denoted by $\| \mathbf{A}_\phi + \mathbf{B_w} \|$, independent of $(\mathbf{w}, \phi)$ such that

$$|(\mathbf{A}_\phi + \mathbf{B_w})(\vec{\mathbf{t}}, \vec{\mathbf{s}})| \leq \| \mathbf{A}_\phi + \mathbf{B_w} \| \| \vec{\mathbf{t}} \| \| \vec{\mathbf{s}} \|. \tag{2.24}$$

To prove that $\mathbf{A}_\phi + \mathbf{B_w}$ is elliptic, we first prove that $\mathbf{A}_\phi$ is elliptic. Indeed, for any $\vec{\mathbf{s}} \in \mathcal{H}$ we have

$$\begin{aligned} \mathbf{A}_\phi(\vec{\mathbf{s}}, \vec{\mathbf{s}}) = & \int_\Omega \mu(\phi) \mathbf{s} : \mathbf{s} - \kappa_1 \int_\Omega \mu(\phi) \mathbf{s} : \boldsymbol{\tau}^{\mathrm{d}} - \kappa_3 \int_\Omega \mathbf{s} : \mathbf{e}(\mathbf{v}) - \kappa_4 \int_\Omega \boldsymbol{\omega}(\mathbf{v}) : \boldsymbol{\eta} \\ & + \kappa_1 \| \boldsymbol{\tau}^{\mathrm{d}} \|_{0,\Omega}^2 + \kappa_2 \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2 + \kappa_3 \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 + \kappa_4 \| \boldsymbol{\eta} \|_{0,\Omega}^2 + \kappa_5 \| \mathbf{v} \|_{0,\Gamma}^2 , \end{aligned}$$

and then, using the bounds for the viscosity and the Cauchy-Schwarz and Young inequalities, we obtain for any $\delta_1, \delta_2, \delta_3 > 0$ and any $\vec{\mathbf{s}} \in \mathcal{H}$ that

$$\begin{aligned} \mathbf{A}_\phi(\vec{\mathbf{s}}, \vec{\mathbf{s}}) \geq & \mu_1 \| \mathbf{s} \|_{0,\Omega}^2 - \frac{\kappa_1 \mu_2}{2\delta_1} \| \mathbf{s} \|_{0,\Omega}^2 - \frac{\kappa_1 \mu_2 \delta_1}{2} \| \boldsymbol{\tau}^{\mathrm{d}} \|_{0,\Omega}^2 - \frac{\kappa_3}{2\delta_2} \| \mathbf{s} \|_{0,\Omega}^2 - \frac{\kappa_3 \delta_2}{2} \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 \\ & - \frac{\kappa_4}{2\delta_3} \| \boldsymbol{\omega}(\mathbf{v}) \|_{0,\Omega}^2 - \frac{\kappa_4 \delta_3}{2} \| \boldsymbol{\eta} \|_{0,\Omega}^2 + \kappa_1 \| \boldsymbol{\tau}^{\mathrm{d}} \|_{0,\Omega}^2 + \kappa_2 \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2 + \kappa_3 \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 \\ & + \kappa_4 \| \boldsymbol{\eta} \|_{0,\Omega}^2 + \kappa_5 \| \mathbf{v} \|_{0,\Gamma}^2 \\ \geq & \left(\mu_1 - \frac{\kappa_1 \mu_2}{2\delta_1} - \frac{\kappa_3}{2\delta_2} \right) \| \mathbf{s} \|_{0,\Omega}^2 + \kappa_1 \left(1 - \frac{\mu_2 \delta_1}{2} \right) \| \boldsymbol{\tau}^{\mathrm{d}} \|_{0,\Omega}^2 + \kappa_2 \| \mathbf{div}\, \boldsymbol{\tau} \|_{0,\Omega}^2 \\ & + \kappa_3 \left(1 - \frac{\delta_2}{2} \right) \| \mathbf{e}(\mathbf{v}) \|_{0,\Omega}^2 + \kappa_5 \| \mathbf{v} \|_{0,\Gamma}^2 - \frac{\kappa_4}{2\delta_3} |\mathbf{v}|_{1,\Omega}^2 + \kappa_4 \left(1 - \frac{\delta_3}{2} \right) \| \boldsymbol{\eta} \|_{0,\Omega}^2 . \end{aligned}$$

Then, defining the following constants:

$$\begin{aligned} \alpha_1 := \mu_1 - \frac{\kappa_1 \mu_2}{2\delta_1} - \frac{\kappa_3}{2\delta_2}, \quad \alpha_2 := \min \left\{ \kappa_1 \left(1 - \frac{\mu_2 \delta_1}{2} \right), \frac{\kappa_2}{2} \right\}, \quad \alpha_3 := \min \left\{ \kappa_3 \left(1 - \frac{\delta_2}{2} \right), \kappa_5 \right\}, \\ \alpha_4 := \kappa_4 \left(1 - \frac{\delta_3}{2} \right), \quad \alpha_5 := \min \left\{ \alpha_2 c_3(\Omega), \frac{\kappa_2}{2} \right\}, \quad \alpha_6 := \alpha_3 \kappa_0(\Omega) - \frac{\kappa_4}{2\delta_3}, \end{aligned}$$

and using Lemmas 2.1 and 2.2, it is possible to find a constant $\alpha(\Omega) := \min\{\alpha_1, \alpha_4, \alpha_5, \alpha_6\}$, independent of $(\mathbf{w}, \phi)$, such that

$$\mathbf{A}_\phi(\vec{\mathbf{s}}, \vec{\mathbf{s}}) \geq \alpha(\Omega) \| \vec{\mathbf{s}} \|^2 \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}.$$

Then, from the foregoing inequality, together with the definition of $\mathbf{B_w}$ (cf. (2.10)) and the estimation (2.5), we get that, for any $\vec{\mathbf{s}} \in \mathcal{H}$, there holds

$$(\mathbf{A}_\phi + \mathbf{B_w})(\vec{\mathbf{s}}, \vec{\mathbf{s}}) \geq \left(\alpha(\Omega) - c_1(\Omega)(1+\kappa_1)^{1/2} \| \mathbf{w} \|_{1,\Omega} \right) \| \vec{\mathbf{s}} \|^2 .$$

Therefore, we easily see that

$$(\mathbf{A}_\phi + \mathbf{B_w})(\vec{\mathbf{s}}, \vec{\mathbf{s}}) \geq \frac{\alpha(\Omega)}{2} \| \vec{\mathbf{s}} \|^2 \quad \forall\, \vec{\mathbf{s}} \in \mathcal{H}, \tag{2.25}$$

provided that

$$\frac{\alpha(\Omega)}{2} \geq c_1(\Omega)(1+\kappa_1^2)^{1/2} \| \mathbf{w} \|_{1,\Omega},$$

that is,

$$\| \mathbf{w} \|_{1,\Omega} \leq \frac{\alpha(\Omega)}{2c_1(\Omega)(1+\kappa_1^2)^{1/2}} =: r_0, \tag{2.26}$$

thus proving ellipticity for $\mathbf{A}_\phi + \mathbf{B}_\mathbf{w}$ under the requirement (2.26). Finally, the linearity of the functionals F_D and F_ϕ is clear, and from (2.13), (2.14), using the Cauchy-Schwarz inequality, the trace estimates in $\mathbb{H}(\mathbf{div};\Omega)$ and $\mathbf{H}^1(\Omega)$, with constants 1 and $c_0(\Omega)$, and the continuous injection from $H^{1/2}(\Gamma)$ into $L^2(\Gamma)$ with constant $C_{1/2}$ we have

$$|F_D(\vec{\mathbf{s}})| \leq \Big(1+\kappa_5 c_0(\Omega) C_{1/2}\Big) \| \mathbf{u}_D \|_{1/2,\Gamma} \| \vec{\mathbf{s}} \|,$$

and

$$|F_\phi(\vec{\mathbf{s}})| \leq (1+\kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega} \| \vec{\mathbf{s}} \|, \tag{2.27}$$

for all $\vec{\mathbf{s}} \in \mathcal{H}$. Thus, there exists a constant $M_\mathbf{S} := \max\Big\{(1+\kappa_2^2)^{1/2}, 1+\kappa_5 c_0(\Omega) C_{1/2}\Big\}$ such that

$$\| F_\phi + F_D \| \leq M_\mathbf{S} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} \| \phi \|_{0,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\},$$

and by the Lax-Milgram theorem (see, e.g. [13, Theorem 1.1]), there exists a unique $\vec{\mathbf{t}} \in \mathcal{H}$ solution of (2.17), and the corresponding dependence result (2.21) is satisfied with $C_\mathbf{S} := \frac{2M_\mathbf{S}}{\alpha(\Omega)}$, a constant clearly independent of $(\mathbf{w},\phi)$. □

Lemma 2.4. *For each $(\mathbf{w},\phi) \in \mathbf{H}$ there exists a unique pair $(\varphi,\lambda) \in H^1(\Omega) \times H^{-1/2}(\Gamma)$ solution of the problem (2.18). Moreover, there exists $C_{\widetilde{\mathbf{S}}} > 0$ such that*

$$\Big\| \widetilde{\mathbf{S}}(\mathbf{w},\phi) \Big\| \leq \| (\varphi,\lambda) \| \leq C_{\widetilde{\mathbf{S}}} \Big\{ \| \mathbf{w} \|_{1,\Omega} | \phi |_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \Big\}. \tag{2.28}$$

Proof. The results comes from a direct application of the Babuška-Brezzi theory (see [3, Lemma 3.6]). In particular, the right hand side of (2.28) is obtained after bounding the functionals $F_{\mathbf{w},\phi}$ (cf. (2.15)) and G (cf. (2.16)), respectively. □

From the previous two lemmas, it is now clear that $\mathbf{T}$ is well-defined for any element $(\mathbf{w},\phi) \in \mathbf{W}$, where

$$\mathbf{W} := \Big\{ (\mathbf{w},\phi) \in \mathbf{H} : \| (\mathbf{w},\phi) \| \leq r \Big\}, \tag{2.29}$$

which is nothing but the closed ball in $\mathbf{H}$ with center $(\mathbf{0},0)$ and radius r, with $r \in (0, r_0)$ and r_0 defined as in (2.26). We also notice that, for computational purposes, a particular choice of stabilization parameters is necessary. Therefore, we choose them so that $\alpha(\Omega)$ is as large as possible. We begin by selecting the middle points of the ranges for δ_1, δ_2, δ_3, κ_1 and κ_3, that is,

$$\delta_1 = \frac{1}{\mu_2}, \quad \delta_2 = \delta_3 = 1, \quad \kappa_1 = \frac{\mu_1 \delta_1}{\mu_2} = \frac{\mu_1}{\mu_2^2}, \quad \kappa_3 = \delta_2 \left(\mu_1 - \frac{\kappa_1 \mu_2}{2\delta_1} \right) = \frac{\mu_1}{2},$$

to then pick κ_2 and κ_5 so that α_2 and α_3 can attain the largest value possible, that is

$$\kappa_2 = \frac{\mu_1}{\mu_2^2}, \quad \kappa_5 = \frac{\mu_1}{4},$$

and hence, we can pick κ_4 as the middle point of its range,

$$\kappa_4 = \kappa_0(\Omega) \frac{\mu_1}{4}.$$

Notice that $\kappa_0(\Omega)$, the constant in the Korn-type Lemma 2.2, is still unknown. Nevertheless, works such as [3, 7] suggest that a heuristic choice of this parameter is enough.

2.4 Further-regularity assumption

Although the problem that defines the operator $\mathbf{S}$, that is (2.17), is well-posed, a small further-regularity assumption has to be made in order to continue with the analysis. More precisely, and inspired by [4, 5], we assume that $\mathbf{u}_D \in \mathbf{H}^{1/2+\varepsilon}(\Gamma)$ for some $\varepsilon \in (0,1)$ when $n=2$, or $\varepsilon \in \left[\frac{1}{2},1\right)$ when $n=3$, and that for each $(\mathbf{z},\psi) \in \mathbf{H}$ with $\|\mathbf{z}\|_{1,\Omega} \le r$, $r>0$ given, there hold $(\mathbf{q},\boldsymbol{\zeta},\mathbf{v},\boldsymbol{\chi}) := \mathbf{S}(\mathbf{z},\psi) \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) \cap \mathbb{H}^\varepsilon(\Omega) \times \mathbb{H}_0(\mathbf{div};\Omega) \cap \mathbb{H}^\varepsilon(\Omega) \times \mathbf{H}^{1+\varepsilon}(\Omega) \times \mathbb{L}^2_{\mathtt{skew}}(\Omega) \cap \mathbb{H}^\varepsilon(\Omega)$ and

$$\|\mathbf{q}\|_{\varepsilon,\Omega} + \|\boldsymbol{\zeta}\|_{\varepsilon,\Omega} + \|\mathbf{v}\|_{1+\varepsilon,\Omega} + \|\boldsymbol{\chi}\|_{\varepsilon,\Omega} \le \widetilde{C}_{\mathbf{S}}(r)\Big\{\|\mathbf{g}\|_{\infty,\Omega}\|\psi\|_{0,\Omega} + \|\mathbf{u}_D\|_{1/2+\varepsilon,\Gamma}\Big\}, \tag{2.30}$$

with $\widetilde{C}_{\mathbf{S}}(r)>0$ independent of $\mathbf{z}$ but depending on the upper bound r of its $\mathbf{H}^1$-norm.

2.5 Solvability analysis of the fixed-point equation

We now proceed to directly fulfill the hypotheses of the Banach fixed-point theorem. The following result shows that $\mathbf{T}$ can map the ball $\mathbf{W}$ into itself.

Lemma 2.5. *Consider the closed ball $\mathbf{W}$ defined in (2.29) with $r \in (0,r_0)$ and r_0 as given in (2.26). Suppose the data satisfy*

$$c(r)\Big\{\|\mathbf{g}\|_{\infty,\Omega} + \|\mathbf{u}_D\|_{1/2,\Gamma}\Big\} + C_{\widetilde{\mathbf{S}}}\|\varphi_D\|_{1/2,\Gamma} \le r,$$

where

$$c(r) := \big(1 + C_{\widetilde{\mathbf{S}}}r\big)\,C_{\mathbf{S}}\max\{1,r\},$$

and $C_{\mathbf{S}}$, $C_{\widetilde{\mathbf{S}}}$ are given in Lemmas 2.3 and 2.4, respectively. Then, there holds $\mathbf{T}(\mathbf{W}) \subseteq \mathbf{W}$.

Proof. It follows as a consequence of the continuous dependence results (2.21) and (2.28), much in an identical way as in [3, Lemma 3.7]. □

Next, we prove some results that will help us to arrive to the Lipschitz continuity of $\mathbf{T}$.

Lemma 2.6. *Let $r \in (0,r_0)$ with r_0 as given in (2.26). Then, there exists a positive constant $\widehat{C}_{\mathbf{S}}$, independent of r, such that*

$$\begin{aligned}\|\mathbf{S}(\mathbf{w}_1,\phi_1) - \mathbf{S}(\mathbf{w}_2,\phi_2)\| \le \widehat{C}_{\mathbf{S}}\Big\{&\|\mathbf{S}_1(\mathbf{w}_1,\phi_1)\|_{\varepsilon,\Omega}\|\phi_1-\phi_2\|_{L^{n/\varepsilon}(\Omega)}\\ &+ \|\mathbf{S}_3(\mathbf{w}_1,\phi_1)\|_{1,\Omega}\|\mathbf{w}_1-\mathbf{w}_2\|_{1,\Omega} + \|\mathbf{g}\|_{\infty,\Omega}\|\phi_1-\phi_2\|_{0,\Omega}\Big\},\end{aligned} \tag{2.31}$$

for all $(\mathbf{w}_1,\phi_1),(\mathbf{w}_2,\phi_2) \in \mathbf{H}$ such that $\|\mathbf{w}_1\|_{1,\Omega}, \|\mathbf{w}_2\|_{1,\Omega} \le r$.

Proof. Let $(\mathbf{w}_1,\phi_1),(\mathbf{w}_2,\phi_2) \in \mathbf{H}$ as indicated and let $\vec{\mathbf{t}}_j := (\mathbf{t}_j,\boldsymbol{\sigma}_j,\mathbf{u}_j,\boldsymbol{\gamma}_j) = \mathbf{S}(\mathbf{w}_j,\phi_j) \in \mathcal{H}$, $j \in \{1,2\}$ be the corresponding solutions of (2.17). Then, adding and subtracting the equality

$$\mathbf{A}_{\phi_1}(\vec{\mathbf{t}}_1,\vec{\mathbf{s}}) + \mathbf{B}_{\mathbf{w}_1}(\vec{\mathbf{t}}_1,\vec{\mathbf{s}}) = F_{\phi_1}(\vec{\mathbf{s}}) + F_D(\vec{\mathbf{s}}) \qquad \forall\,\vec{\mathbf{s}} \in \mathcal{H}\,,$$

we find that

$$(\mathbf{A}_{\phi_2} + \mathbf{B}_{\mathbf{w}_2})(\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2,\vec{\mathbf{s}}) = \mathbf{A}_{\phi_2}(\vec{\mathbf{t}}_1,\vec{\mathbf{s}}) - \mathbf{A}_{\phi_1}(\vec{\mathbf{t}}_1,\vec{\mathbf{s}}) + \mathbf{B}_{\mathbf{w}_2-\mathbf{w}_1}(\vec{\mathbf{t}}_1,\vec{\mathbf{s}}) + F_{\phi_1-\phi_2}(\vec{\mathbf{s}}) \qquad \forall\,\vec{\mathbf{s}} \in \mathcal{H}\,.$$

Thus, using the ellipticity of $\mathbf{A}_{\phi_2} + \mathbf{B}_{\mathbf{w}_2}$ (cf. (2.25)) and the foregoing expression, we obtain

$$\begin{aligned}\frac{\alpha(\Omega)}{2}\|\,\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2\,\| &\leq (\mathbf{A}_{\phi_2} + \mathbf{B}_{\mathbf{w}_2})(\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2, \vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2)\\ &= \int_\Omega \Big[\mu(\phi_2) - \mu(\phi_1)\Big]\mathbf{t}_1 : \Big\{(\mathbf{t}_1 - \mathbf{t}_2) - \kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}}\Big\}\\ &+ \int_\Omega \Big[\mathbf{u}_1 \otimes (\mathbf{w}_2 - \mathbf{w}_1)\Big]^{\mathtt{d}} : \Big\{\kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}} - (\mathbf{t}_1 - \mathbf{t}_2)\Big\}\\ &+ \int_\Omega (\phi_1 - \phi_2)\mathbf{g}\cdot\Big\{(\mathbf{u}_1 - \mathbf{u}_2) - \kappa_2\mathbf{div}\,(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\Big\}.\end{aligned} \tag{2.32}$$

First, we bound the last two terms in the same way as Lemma 2.3 (that is, using the estimates (2.23) and (2.27)):

$$\begin{aligned}&\left|\int_\Omega \Big[\mathbf{u}_1 \otimes (\mathbf{w}_2 - \mathbf{w}_1)\Big]^{\mathtt{d}} : \Big\{\kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}} - (\mathbf{t}_1 - \mathbf{t}_2)\Big\}\right|\\ &\quad\leq\; c_1(\Omega)(1+\kappa_1^2)^{1/2}\,\|\,\mathbf{u}_1\,\|_{1,\Omega}\|\,\mathbf{w}_1 - \mathbf{w}_2\,\|_{1,\Omega}\|\,\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2\,\|,\end{aligned} \tag{2.33}$$

and

$$\begin{aligned}&\left|\int_\Omega (\phi_1 - \phi_2)\mathbf{g}\cdot\Big\{(\mathbf{u}_1 - \mathbf{u}_2) - \kappa_2\mathbf{div}\,(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)\Big\}\right|\\ &\quad\leq\; (1+\kappa_2^2)^{1/2}\,\|\,\mathbf{g}\,\|_{\infty,\Omega}\,\|\,\phi_1 - \phi_2\,\|_{0,\Omega}\,\|\,\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2\,\|,\end{aligned} \tag{2.34}$$

Next, for the first term, we use the Lipschitz continuity of μ and the Cauchy-Schwarz and Hölder inequalities to show in a similar way to [3, Eq. (3.56)] that

$$\begin{aligned}&\left|\int_\Omega \Big[\mu(\phi_2) - \mu(\phi_1)\Big]\mathbf{t}_1 : \Big\{(\mathbf{t}_1 - \mathbf{t}_2) - \kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}}\Big\}\right|\\ &\quad\leq\; L_\mu\|\,(\phi_2 - \phi_1)\mathbf{t}_1\,\|_{0,\Omega}\|\,(\mathbf{t}_1 - \mathbf{t}_2) - \kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}}\,\|_{0,\Omega}\\ &\quad\leq\; L_\mu(1+\kappa_1^2)^{1/2}\|\,\phi_2 - \phi_1\,\|_{L^{2q}(\Omega)}\|\,\mathbf{t}_1\,\|_{\mathbb{L}^{2p}(\Omega)}\|\,\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2\,\|,\end{aligned} \tag{2.35}$$

with $p, q \in [1, +\infty)$ such that $\frac{1}{p} + \frac{1}{q} = 1$. At this point, we take into consideration the further-regularity assumption in Section 2.4 and we recall from the Sobolev embedding theorem that $H^\varepsilon(\Omega)$ is continuously embedded into $L^{2p}(\Omega)$, with

$$2p = \begin{cases}\dfrac{2}{1-\varepsilon} & \text{if } n = 2,\\[2mm] \dfrac{6}{3-2\varepsilon} & \text{if } n = 3,\end{cases}$$

(cf. [1, Theorem 4.12], [16, Theorem 1.3.4]) meaning that there exists $C_\varepsilon > 0$ such that

$$\|\,\mathbf{t}\,\|_{\mathbb{L}^{2p}(\Omega)} \leq C_\varepsilon\|\,\mathbf{t}\,\|_{\varepsilon,\Omega} \quad \forall\, \mathbf{t} \in \mathbb{H}^\varepsilon(\Omega).$$

In this way,

$$2q = \frac{2p}{p-1} = \begin{cases}\dfrac{2}{\varepsilon} & \text{if } n = 2,\\[2mm] \dfrac{3}{\varepsilon} & \text{if } n = 3\end{cases} = \frac{n}{\varepsilon},$$

and (2.35) now yields

$$\begin{aligned}&\left|\int_\Omega \Big[\mu(\phi_2) - \mu(\phi_1)\Big]\mathbf{t}_1 : \Big\{(\mathbf{t}_1 - \mathbf{t}_2) - \kappa_1(\boldsymbol{\sigma}_1 - \boldsymbol{\sigma}_2)^{\mathtt{d}}\Big\}\right|\\ &\quad\leq\; L_\mu(1+\kappa_1^2)^{1/2}C_\varepsilon\|\,\mathbf{t}_1\,\|_{\varepsilon,\Omega}\|\,\phi_1 - \phi_2\,\|_{L^{n/\varepsilon}(\Omega)}\|\,\vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2\,\|.\end{aligned} \tag{2.36}$$

Therefore, putting (2.33), (2.34) and (2.36) together into (2.32), we obtain

$$\| \vec{\mathbf{t}}_1 - \vec{\mathbf{t}}_2 \| \leq \widehat{C}_{\mathbf{S}} \Big\{ \| \mathbf{t}_1 \|_{\varepsilon,\Omega} \| \phi_1 - \phi_2 \|_{L^{n/\varepsilon}(\Omega)} + \| \mathbf{u}_1 \|_{1,\Omega} \| \mathbf{w}_1 - \mathbf{w}_2 \|_{1,\Omega} + \| \mathbf{g} \|_{\infty,\Omega} \| \phi_1 - \phi_2 \|_{0,\Omega} \Big\}, \quad (2.37)$$

with

$$\widehat{C}_{\mathbf{S}} := \frac{2}{\alpha(\Omega)} \max \Big\{ L_\mu (1+\kappa_1^2)^{1/2} C_\varepsilon, c_1(\Omega)(1+\kappa_1^2)^{1/2}, (1+\kappa_2^2)^{1/2} \Big\},$$

and since $\mathbf{t}_1 = \mathbf{S}_1(\mathbf{w}_1, \phi_1)$ and $\mathbf{u}_1 = \mathbf{S}_3(\mathbf{w}_1, \phi_1)$, the last inequality is exactly the estimate (2.31). □

Notice how the foregoing Lemma shows the main difference with respect to the previous work [3]: the tensor-product term in (2.33) no longer appears multiplied by an H^1-term, thus avoiding the use of the injection $H^1(\Omega) \hookrightarrow L^8(\Omega)$ (not ensured for $\Omega \subset R^3$) when splitting them as in [3, Eq. (3.54)], yielding a more robust formulation for both two and three-dimensional cases. On the other hand, the analogous result for $\widetilde{\mathbf{S}}$ remains intact.

Lemma 2.7. *There exists a positive constant* $\widehat{C}_{\widetilde{\mathbf{S}}}$ *such that*

$$\Big\| \widetilde{\mathbf{S}}(\mathbf{w}_1, \phi_1) - \widetilde{\mathbf{S}}(\mathbf{w}_2, \phi_2) \Big\| \leq \widehat{C}_{\widetilde{\mathbf{S}}} \Big\{ \| \mathbf{w}_1 \| | \phi_1 - \phi_2 |_{1,\Omega} + \| \mathbf{w}_1 - \mathbf{w}_2 \|_{1,\Omega} | \phi_2 |_{1,\Omega} \Big\}. \quad (2.38)$$

Proof. See [3, Lemma 3.9]. □

As a consequence of the previous Lemmas, $\mathbf{T}$ is a Lipschitz continuous operator, as shown next.

Lemma 2.8. *Let $r \in (0, r_0)$, with r_0 given as in (2.26). Then, there exists a constant $C_{\mathbf{T}} > 0$ such that*

$$\| \mathbf{T}(\mathbf{w}_1, \phi_1) - \mathbf{T}(\mathbf{w}_2, \phi_2) \| \leq C_{\mathbf{T}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} \Big\} \| (\mathbf{w}_1, \phi_1) - (\mathbf{w}_2, \phi_2) \|$$

Proof. The result comes from the definition of $\mathbf{T}$ (cf. (2.19)) and the estimates obtained in the previous two lemmas (cf. (2.31) and (2.38)) in an identical way to [3, Lemma 3.10]. We omit further details. □

In summary, from Lemmas 2.3 and 2.4, $\mathbf{T} : \mathbf{W} \to \mathbf{W}$ is well-defined and does map the ball into itself (thanks to Lemma 2.5). Then, Lemma 2.8 shows that $\mathbf{T}$ is Lipschitz continuous, so that when its Lipschitz constant is < 1, it becomes a contraction, yielding the existence and uniqueness of a fixed point, thanks to the Banach fixed-point theorem. By what has been explained in Section 2.2, this fact is equivalent to the well-posedness of the augmented mixed-primal formulation (2.8), thus providing us the main result for this section.

Theorem 2.9. *Assume that for $\delta_1 \in \left(0, \frac{2}{\mu_2}\right)$, $\delta_2, \delta_3 \in (0,2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_5 \in (0,\infty),$$

$$\kappa_3 \in \left(0, 2\delta_2\left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right) \quad \textit{and} \quad \kappa_4 \in \left(0, 2\delta_3\kappa_0(\Omega) \min\left\{\kappa_3\left(1 - \frac{\delta_2}{2}\right), \kappa_5\right\}\right).$$

and consider the ball $\mathbf{W}$ (cf. (2.29)) with radius $r \in (0, r_0)$ and r_0 as in (2.26). In addition, assume that the data satisfy

$$c(r)\Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\} + C_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma} \leq r,$$

with $c(r)$ as in Lemma 2.5, and

$$C_{\mathbf{T}} \Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} \Big\} < 1.$$

Then, the problem (2.8) has a unique solution $(\vec{\mathbf{t}}, (\varphi, \lambda)) \in \mathcal{H} \times H^1(\Omega) \times H^{-1/2}(\Gamma)$, with $(\mathbf{u}, \varphi) \in \mathbf{W}$. Moreover, there hold

$$\| \vec{\mathbf{t}} \| \leq C_{\mathbf{S}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\} \tag{2.39}$$

and

$$\| (\varphi, \lambda) \| \leq C_{\widetilde{\mathbf{S}}} \Big\{ r \| \mathbf{u} \|_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \Big\}, \tag{2.40}$$

with $C_{\mathbf{S}}$ and $C_{\widetilde{\mathbf{S}}}$ as in Lemmas 2.3 and 2.4, respectively.

3 The Galerkin scheme

In this section, we derive the corresponding Galerkin scheme for the augmented mixed-primal formulation (2.8). To begin with, let us consider $\mathcal{T}_h$ a regular triangulation of $\bar{\Omega}$ by triangles T (when $n = 2$) or tetrahedra T (when $n = 3$) of diameter h_T and define the meshsize $h := \max\{h_T : T \in \mathcal{T}_h\}$. Then, consider arbitrary finite-dimensional subspaces $\mathbb{H}_h^{\mathbf{t}} \subset \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\sigma}} \subset \mathbb{H}_0(\mathbf{div};\Omega)$, $\mathbf{H}_h^{\mathbf{u}} \subset \mathbf{H}^1(\Omega)$, $\mathbb{H}_h^{\boldsymbol{\gamma}} \subset \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, $H_h^{\varphi} \subset H^1(\Omega)$, $H_h^{\lambda} \subset H^{-1/2}(\Gamma)$ and denote by $\mathcal{H}_h := \mathbb{H}_h^{\mathbf{t}} \times \mathbb{H}_h^{\boldsymbol{\sigma}} \times \mathbf{H}_h^{\mathbf{u}} \times \mathbb{H}_h^{\boldsymbol{\gamma}}$, $\vec{\mathbf{t}}_h := (\mathbf{t}_h, \boldsymbol{\sigma}_h, \mathbf{u}_h, \boldsymbol{\gamma}_h)$ and $\vec{\mathbf{s}}_h := (\mathbf{s}_h, \boldsymbol{\tau}_h, \mathbf{v}_h, \boldsymbol{\eta}_h)$. Hence, according to (2.8), the Galerkin scheme reads: Find $(\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \in \mathcal{H}_h \times H_h^{\varphi} \times H_h^{\lambda}$ such that

$$\begin{aligned} \mathbf{A}_{\varphi_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) + \mathbf{B}_{\mathbf{u}_h}(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) &= F_{\varphi_h}(\vec{\mathbf{s}}_h) + F_D(\vec{\mathbf{s}}_h) && \forall\, \vec{\mathbf{s}}_h \in \mathcal{H}_h, \\ \mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) &= F_{\mathbf{u}_h,\varphi_h}(\psi_h) && \forall\, \psi_h \in H_h^{\varphi}, \\ \mathbf{b}(\varphi_h, \xi_h) &= G(\xi_h) && \forall\, \xi_h \in H_h^{\lambda}, \end{aligned} \tag{3.1}$$

where the forms $\mathbf{A}_{\varphi_h}$, $\mathbf{B}_{\mathbf{u}_h}$, $\mathbf{a}$ and $\mathbf{b}$; and the functionals F_{φ_h}, F_D, $F_{\mathbf{u}_h,\varphi_h}$ are defined by (2.9)-(2.16). Since the proof of well-posedness follows the steps of the previous section, and moreover, analogously to [3, Section 4], we only state the requirements to be imposed over the finite-dimensional subspaces and the main result, which is analogous to [3, Theorem 4.8].

3.1 Well-posedness of the Galerkin scheme

It can be seen that no restrictions have to be added to $\mathbb{H}_h^{\mathbf{t}}$, $\mathbb{H}_h^{\boldsymbol{\sigma}}$, $\mathbf{H}_h^{\mathbf{u}}$ and $\mathbb{H}_h^{\boldsymbol{\gamma}}$ other than being finite-dimensional subspaces of the described spaces, however, for ellipticity purposes of $\mathbf{a}$ in the discrete kernel of the operator induced by $\mathbf{b}$ (according the Babuška-Brezzi theory), the following inf-sup conditions must be met:

(H.1) There exists a constant $\widehat{\alpha} > 0$, independent of h such that

$$\sup_{\substack{\psi_h \in V_h \\ \psi_h \neq 0}} \frac{\mathbf{a}(\psi_h, \phi_h)}{\| \psi_h \|_{1,\Omega}} \geq \widehat{\alpha} \| \phi_h \|_{1,\Omega} \quad \forall\, \phi_h \in V_h, \tag{3.2}$$

where

$$V_h := \Big\{ \psi_h \in H_h^{\varphi} : \mathbf{b}(\psi_h, \xi_h) = 0 \quad \forall\, \xi_h \in H_h^{\lambda} \Big\},$$

and,

(H.2) There exists a constant $\widehat{\beta} > 0$, independent of h such that

$$\sup_{\substack{\psi_h \in H_h^{\varphi} \\ \psi_h \neq 0}} \frac{\mathbf{b}(\psi_h, \xi_h)}{\| \psi_h \|_{1,\Omega}} \geq \widehat{\beta} \| \xi_h \|_{-1/2,\Gamma} \quad \forall\, \xi_h \in H_h^{\lambda}. \tag{3.3}$$

Then, denoting by $\mathbf{W}_h$ the closed ball in $\mathbf{H}_h := \mathbf{H}_h^{\mathbf{u}} \times H_h^{\varphi}$ of radius r and center $(\mathbf{0}, 0)$, that is

$$\mathbf{W}_h := \Big\{ (\mathbf{w}_h, \phi_h) \in \mathbf{H}_h : \| (\mathbf{w}_h, \phi_h) \| \le r \Big\},$$

the main result of this section reads as follows.

Theorem 3.1. *Assume that for $\delta_1 \in \Big(0, \frac{2}{\mu_2}\Big)$, $\delta_2, \delta_3 \in (0, 2)$ we choose*

$$\kappa_1 \in \left(0, \frac{2\mu_1\delta_1}{\mu_2}\right), \quad \kappa_2, \kappa_5 \in (0, \infty),$$

$$\kappa_3 \in \left(0, 2\delta_2 \left(\mu_1 - \frac{\kappa_1\mu_2}{2\delta_1}\right)\right) \quad \text{and} \quad \kappa_4 \in \left(0, 2\delta_3\kappa_0(\Omega) \min\left\{\kappa_3\left(1 - \frac{\delta_2}{2}\right), \kappa_5\right\}\right),$$

and consider the ball $\mathbf{W}_h$ with radius $r \in (0, r_0)$, r_0 as in (2.26). Then, there exist positive constants $C_{\mathbf{S}}$, $\widetilde{C}_{\widetilde{\mathbf{S}}}$ and $\widetilde{c}(r)$ such that, if the data satisfy

$$\widetilde{c}(r)\Big\{ \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\} + \widetilde{C}_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma} \le r,$$

then the problem (3.1) has at least one solution $(\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \in \mathcal{H}_h \times H_h^{\varphi} \times H_{\widetilde{h}}^{\lambda}$ with $(\mathbf{u}_h, \varphi_h) \in \mathbf{W}_h$. Moreover, there hold

$$\| \vec{\mathbf{t}}_h \| \le C_{\mathbf{S}} \Big\{ r \| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma} \Big\} \tag{3.4}$$

and

$$\| (\varphi_h, \lambda_h) \| \le \widetilde{C}_{\widetilde{\mathbf{S}}} \Big\{ r \| \mathbf{u}_h \|_{1,\Omega} + \| \varphi_D \|_{1/2,\Gamma} \Big\}.$$

Proof. We only mention that (3.1) is transformed into a fixed-point problem that is analysed by means of the Brouwer fixed-point theorem in the convex and compact set $\mathbf{W}_h$ (see [3, Theorem 4.8]). $\square$

3.2 Specific choice of finite element subspaces

Given a set $S \subset \mathbf{R} := R^n$ and an integer $k \ge 0$, we define $P_k(S)$ as the space of polynomial functions on S of degree $\le k$, and for each $T \in \mathcal{T}_h$, we define the local Raviart-Thomas spaces of order k as

$$\mathbf{RT}_k(T) := \mathbf{P}_k(T) + P_k(T)\mathbf{x},$$

where $\mathbf{x}$ is a generic vector in $\mathbf{R}$. Hence, the strain rate, pseudostress, velocity, vorticity and temperature variables can be approximated using the following finite element subspaces:

$$\mathbb{H}_h^{\mathbf{t}} := \Big\{ \mathbf{s}_h \in \mathbb{L}^2_{\mathtt{tr}}(\Omega) : \quad \mathbf{s}_h\Big|_T \in \mathbb{P}_k(T) \quad \forall\, T \in \mathcal{T}_h \Big\}, \tag{3.5}$$

$$\mathbb{H}_h^{\boldsymbol{\sigma}} := \big\{ \boldsymbol{\tau}_h \in \mathbb{H}_0(\mathbf{div}; \Omega) : \quad \mathbf{c}^{\mathtt{t}}\boldsymbol{\tau}_h\big|_T \in \mathbf{RT}_k(T), \quad \forall\, \mathbf{c} \in \mathbf{R},\ \forall\, T \in \mathcal{T}_h \big\}, \tag{3.6}$$

$$\mathbf{H}_h^{\mathbf{u}} := \big\{ \mathbf{v}_h \in \mathbf{C}(\bar{\Omega}) : \quad \mathbf{v}_h\big|_T \in \mathbf{P}_{k+1}(T), \quad \forall\, T \in \mathcal{T}_h \big\}, \tag{3.7}$$

$$\mathbb{H}_h^{\boldsymbol{\gamma}} := \big\{ \boldsymbol{\eta}_h \in \mathbb{L}^2_{\mathtt{skew}}(\Omega) : \quad \boldsymbol{\eta}_h\big|_T \in \mathbb{P}_k(T), \quad \forall\, T \in \mathcal{T}_h \big\}, \tag{3.8}$$

$$H_h^{\varphi} := \big\{ \psi_h \in C(\bar{\Omega}) : \quad \psi_h\big|_T \in P_{k+1}(T), \quad \forall\, T \in \mathcal{T}_h \big\}, \tag{3.9}$$

whereas for the normal component of the heat flux, we let $\{\widetilde{\Gamma}_1, \widetilde{\Gamma}_2, \ldots, \widetilde{\Gamma}_m\}$ be an independent triangulation of Γ (made of straight segments in R^2 or triangles in R^3), and define $\widetilde{h} := \max_{j \in \{1,\ldots,m\}} |\widetilde{\Gamma}_j|$. Then, with the same integer $k \ge 0$ used in definitions (3.5)-(3.9), we approximate λ by piecewise polynomials of degree $\le k$ over this new mesh, that is

$$H_{\widetilde{h}}^{\lambda} := \Big\{ \xi_{\widetilde{h}} \in L^2(\Gamma) : \quad \xi_{\widetilde{h}}\big|_{\widetilde{\Gamma}_j} \in P_k(\widetilde{\Gamma}_j) \quad \forall\, j \in \{1, \ldots, m\} \Big\}. \tag{3.10}$$

It can be seen that H_h^{φ} and $H_{\widetilde{h}}^{\lambda}$ do satisfy the inf-sup conditions (3.2) and (3.3) as long as $h \le C_0 \widetilde{h}$ for some $C_0 > 0$ (cf. [3, Section 4.3]), and from [6, 13], the approximation properties for these subspaces are

$(\mathbf{AP}_h^{\mathbf{t}})$ There exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{t} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathtt{tr}}(\Omega)$, there holds

$$\operatorname{dist}\left(\mathbf{t}, \mathbb{H}_h^{\mathbf{t}}\right) \le C h^s \| \mathbf{t} \|_{s,\Omega},$$

$(\mathbf{AP}_h^{\boldsymbol{\sigma}})$ There exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega) \cap \mathbb{H}_0(\mathbf{div}; \Omega)$ with $\mathbf{div}\, \boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, there holds

$$\operatorname{dist}(\boldsymbol{\sigma}, \mathbb{H}_h^{\boldsymbol{\sigma}}) \le C h^s \left\{ \| \boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{div}\, \boldsymbol{\sigma} \|_{s,\Omega} \right\},$$

$(\mathbf{AP}_h^{\mathbf{u}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, there holds

$$\operatorname{dist}(\mathbf{u}, \mathbf{H}_h^{\mathbf{u}}) \le C h^s \| \mathbf{u} \|_{s+1,\Omega},$$

$(\mathbf{AP}_h^{\boldsymbol{\gamma}})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega) \cap \mathbb{L}^2_{\mathtt{skew}}(\Omega)$, there holds

$$\operatorname{dist}\left(\boldsymbol{\gamma}, \mathbb{H}_h^{\boldsymbol{\gamma}}\right) \le C h^s \| \boldsymbol{\gamma} \|_{s,\Omega},$$

$(\mathbf{AP}_h^{\varphi})$ there exists $C > 0$, independent of h, such that for each $s \in (0, k+1]$, and for each $\varphi \in H^{s+1}(\Omega)$, there holds

$$\operatorname{dist}\left(\varphi, H_h^{\varphi}\right) \le C h^s \| \varphi \|_{s+1,\Omega},$$

$(\mathbf{AP}_{\widetilde{h}}^{\lambda})$ there exists $C > 0$, independent of $\widetilde{h}$, such that for each $s \in (0, k+1]$, and for each $\lambda \in H^{-1/2+s}(\Gamma)$, there holds

$$\operatorname{dist}\left(\lambda, H_{\widetilde{h}}^{\lambda}\right) \le C \widetilde{h}^s \| \lambda \|_{-1/2+s,\Gamma}.$$

4 A priori error analysis

Let $(\vec{\mathbf{t}}, (\varphi, \lambda)) \in \mathcal{H} \times H^1(\Omega) \times H^{-1/2}(\Gamma)$ with $(\mathbf{u}, \varphi) \in \mathbf{W}$ be the solution of the continuous problem (2.8), and $(\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \in \mathcal{H}_h \times H_h^{\varphi} \times H_h^{\lambda}$ with $(\mathbf{u}_h, \varphi_h) \in \mathbf{W}_h$ be a solution of the discrete problem (3.1), that is,

$$(\mathbf{A}_{\varphi} + \mathbf{B}_{\mathbf{u}})(\vec{\mathbf{t}}, \vec{\mathbf{s}}) = (F_{\varphi} + F_D)(\vec{\mathbf{s}}) \qquad \forall\, \vec{\mathbf{s}} \in \mathcal{H}, \tag{4.1a}$$

$$(\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{t}}_h, \vec{\mathbf{s}}_h) = (F_{\varphi_h} + F_D)(\vec{\mathbf{s}}_h) \quad \forall\, \vec{\mathbf{s}}_h \in \mathcal{H}_h, \tag{4.1b}$$

and

$$\begin{aligned} \mathbf{a}(\varphi, \psi) + \mathbf{b}(\psi, \lambda) &= F_{\mathbf{u},\varphi}(\psi) && \forall\, \psi \in H^1(\Omega), \\ \mathbf{b}(\varphi, \xi) &= G(\xi) && \forall\, \xi \in H^{-1/2}(\Gamma); \\ \mathbf{a}(\varphi_h, \psi_h) + \mathbf{b}(\psi_h, \lambda_h) &= F_{\mathbf{u}_h,\varphi_h}(\psi_h) && \forall\, \psi_h \in H_h^{\varphi}, \\ \mathbf{b}(\varphi_h, \xi_h) &= G(\xi_h) && \forall\, \xi_h \in H_h^{\lambda}. \end{aligned}$$

In what follows, we denote as usual

$$\operatorname{dist}\left(\vec{\mathbf{t}}, \mathcal{H}_h\right) := \inf_{\vec{\mathbf{s}}_h \in \mathcal{H}_h} \left\| \vec{\mathbf{t}} - \vec{\mathbf{s}}_h \right\|$$

and

$$\operatorname{dist}\Big((\varphi,\lambda), H_h^{\varphi}\times H_h^{\lambda}\Big) := \inf_{(\psi_h,\xi_h)\in H_h^{\varphi}\times H_h^{\lambda}} \|\,(\varphi,\lambda)-(\psi_h,\xi_h)\,\|.$$

First, the error estimate related to the variables of the momentum equation is obtained by means of the Strang Lemma, applied to the pair (4.1). We recall the Lemma, and its consequent result next.

Lemma 4.1 (Strang)**.** *Let V be a Hilbert space, $F\in V'$, and $A: V\times V\to R$ be a bounded and V-elliptic bilinear form. In addition, let $\{V_h\}_{h>0}$ be a sequence of finite-dimensional subspaces of V, and for each $h>0$, consider a bounded bilinear form $A_h: V_h\times V_h\to R$ and a functional $F_h\in V_h'$. Assume that the family $\{A_h\}_{h>0}$ is uniformly elliptic in V_h, that is, there exists a constant $\widetilde{\alpha}>0$, independent of h, such that*

$$A_h(v_h,v_h)\geq \widetilde{\alpha}\|\,v_h\,\|_V^2 \quad \forall\ v_h\in V_h,\ \forall\ h>0.$$

In turn, let $u\in V$ and $u_h\in V_h$ such that

$$A(u,v)=F(v)\quad \forall\ v\in V \qquad \text{and} \qquad A_h(u_h,v_h)=F(v_h)\quad \forall\ v_h\in V_h.$$

Then, for each $h>0$, there holds

$$\|\,u-u_h\,\|_V\leq C_{ST}\left\{\sup_{\substack{w_h\in V_h\\ w_h\neq 0}}\frac{|F(w_h)-F_h(w_h)|}{\|\,w_h\,\|_V} + \inf_{\substack{v_h\in V_h\\ v_h\neq 0}}\left(\|\,u-v_h\,\|_V+\sup_{\substack{w_h\in V_h\\ w_h\neq 0}}\frac{|A(v_h,w_h)-A_h(v_h,w_h)|}{\|\,w_h\,\|_V}\right)\right\},$$

where $C_{ST}:=\widetilde{\alpha}^{-1}\max\{1,\|\,A\,\|\}$.

Proof. See [17, Theorem 11.1]. □

Lemma 4.2. *Let $C_{ST}:=\frac{2}{\alpha(\Omega)}\max\{1,\|\,\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u}\,\|\}$, where $\frac{\alpha(\Omega)}{2}$ is the ellipticity constant of $\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u}$ (cf. (2.25)). Then, there holds*

$$\begin{aligned}\|\,\vec{\mathbf{t}}-\vec{\mathbf{t}}_h\,\|\leq C_{ST}\bigg\{&\Big(1+2\|\,\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u}\,\|\Big)\operatorname{dist}\big(\vec{\mathbf{t}},\mathcal{H}_h\big)+c_1(\Omega)(1+\kappa_1^2)^{1/2}\|\,\mathbf{u}\,\|_{1,\Omega}\|\,\mathbf{u}-\mathbf{u}_h\,\|\\ &+\Big\{L_\mu C_\varepsilon\widetilde{C}_\varepsilon(1+\kappa_1^2)^{1/2}\|\,\mathbf{t}\,\|_{\varepsilon,\Omega}+(1+\kappa_2^2)^{1/2}\|\,\mathbf{g}\,\|_{\infty,\Omega}\Big\}\|\,\varphi-\varphi_h\,\|_{1,\Omega}\bigg\}.\end{aligned} \tag{4.2}$$

Proof. From Lemma 2.3, we see that $\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u}$ and $\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h}$ are bilinear, bounded (both with constant $\|\,\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u}\,\|$, w.l.o.g. since it is independent of $(\mathbf{u},\varphi)$) and uniformly elliptic (both with constant $\frac{\alpha(\Omega)}{2}$). Also, $F_\varphi+F_D$ and $F_{\varphi_h}+F_D$ are linear bounded functionals in $\mathcal{H}$ and $\mathcal{H}_h$, respectively. Hence, a straightforward application of Lemma 4.1 to the pair (4.1) yields

$$\|\,\vec{\mathbf{t}}-\vec{\mathbf{t}}_h\,\|\leq C_{ST}\left\{\sup_{\substack{\vec{\mathbf{s}}_h\in\mathcal{H}_h\\ \vec{\mathbf{s}}_h\neq\vec{\mathbf{0}}}}\frac{|F_\varphi(\vec{\mathbf{s}}_h)-F_{\varphi_h}(\vec{\mathbf{s}}_h)|}{\|\,\vec{\mathbf{s}}_h\,\|}+\inf_{\substack{\vec{\mathbf{q}}_h\in\mathcal{H}_h\\ \vec{\mathbf{q}}_h\neq\vec{\mathbf{0}}}}\left(\|\,\vec{\mathbf{t}}-\vec{\mathbf{q}}_h\,\| + \sup_{\substack{\vec{\mathbf{s}}_h\in\mathcal{H}_h\\ \vec{\mathbf{s}}_h\neq\vec{\mathbf{0}}}}\frac{|(\mathbf{A}_\varphi+\mathbf{B}_\mathbf{u})(\vec{\mathbf{q}}_h,\vec{\mathbf{s}}_h)-(\mathbf{A}_{\varphi_h}+\mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{q}}_h,\vec{\mathbf{s}}_h)|}{\|\,\vec{\mathbf{s}}_h\,\|}\right)\right\}, \tag{4.3}$$

where $C_{ST} := \frac{2}{\alpha(\Omega)} \max\big\{1, \| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|\big\}$. First, we notice that

$$|F_\varphi(\vec{\mathbf{s}}_h) - F_{\varphi_h}(\vec{\mathbf{s}}_h)| = |F_{\varphi-\varphi_h}(\vec{\mathbf{s}}_h)| \leq (1+\kappa_2^2)^{1/2} \| \mathbf{g} \|_{\infty,\Omega} \| \varphi - \varphi_h \|_{1,\Omega} \| \vec{\mathbf{s}}_h \| \quad \forall\, \vec{\mathbf{s}}_h \in \mathcal{H}_h. \tag{4.4}$$

Then, in order to estimate the last supremum in (4.3), we add and subtract suitable terms to write

$$\begin{aligned}
&(\mathbf{A}_\varphi + \mathbf{B}_\mathbf{u})(\vec{\mathbf{q}}_h, \vec{\mathbf{s}}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{q}}_h, \vec{\mathbf{s}}_h) \\
&\quad = (\mathbf{A}_\varphi + \mathbf{B}_\mathbf{u})(\vec{\mathbf{q}}_h - \vec{\mathbf{t}}, \vec{\mathbf{s}}_h) + (\mathbf{A}_\varphi - \mathbf{A}_{\varphi_h})(\vec{\mathbf{t}}, \vec{\mathbf{s}}_h) \\
&\quad + (\mathbf{B}_\mathbf{u} - \mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{t}}, \vec{\mathbf{s}}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{q}}_h - \vec{\mathbf{t}}, \vec{\mathbf{s}}_h)\,,
\end{aligned}$$

and so, using the boundedness of the bilinear forms $\mathbf{A}_\varphi + \mathbf{B}_\mathbf{u}$ and $\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h}$ (cf. (2.24)), the estimate (2.5), the continuous embedding $H^1(\Omega) \hookrightarrow L^{n/\varepsilon}(\Omega)$ with constant $\widetilde{C}_\varepsilon$ and the further-regularity assumption in a similar way to (2.36), we obtain

$$\begin{aligned}
|(\mathbf{A}_\varphi + \mathbf{B}_\mathbf{u})(\vec{\mathbf{q}}_h, \vec{\mathbf{s}}_h) - (\mathbf{A}_{\varphi_h} + \mathbf{B}_{\mathbf{u}_h})(\vec{\mathbf{q}}_h, \vec{\mathbf{s}}_h)| &\leq 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \| \| \vec{\mathbf{q}}_h - \vec{\mathbf{t}} \| \| \vec{\mathbf{s}}_h \| \\
&+ \left| \int_\Omega \Big[\mu(\varphi) - \mu(\varphi_h)\Big] \mathbf{t} : \Big\{ \mathbf{s}_h - \kappa_1 \boldsymbol{\tau}_h^{\mathtt{d}} \Big\} \right| + \left| \int_\Omega \Big[\mathbf{u} \otimes (\mathbf{u} - \mathbf{u}_h)\Big]^{\mathtt{d}} : \Big\{ \kappa_1 \boldsymbol{\tau}_h^{\mathtt{d}} - \mathbf{s}_h \Big\} \right| \\
&\leq \Big\{ 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \| \| \vec{\mathbf{t}} - \vec{\mathbf{q}}_h \| + L_\mu C_\varepsilon \widetilde{C}_\varepsilon (1+\kappa_1^2)^{1/2} \| \mathbf{t} \|_{\varepsilon,\Omega} \| \varphi - \varphi_h \|_{1,\Omega} \\
&+ c_1(\Omega)(1+\kappa_1^2)^{1/2} \| \mathbf{u} \|_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} \Big\} \| \vec{\mathbf{s}}_h \|.
\end{aligned}$$

The previous inequality, together with (4.4), back into (4.3), results in (4.2), concluding this way the proof. □

Next, we recall from [3] the error estimate of the variables in the energy equation.

Lemma 4.3. *There exists a positive constant $\widehat{C}_{ST}$, depending only on $\| \mathbf{a} \|$, $\| \mathbf{b} \|$, $\widehat{\alpha}$ and $\widehat{\beta}$ (cf. (3.2), (3.3)), such that*

$$\begin{aligned}
&\| (\varphi, \lambda) - (\varphi_h, \lambda_h) \| \\
&\quad \leq \widehat{C}_{ST} \Big\{ c_2(\Omega) \Big(| \varphi |_{1,\Omega} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} + \| \mathbf{u}_h \|_{1,\Omega} | \varphi - \varphi_h |_{1,\Omega} \Big) + \operatorname{dist}\Big((\varphi, \lambda), H_h^\varphi \times H_h^\lambda \Big) \Big\}.
\end{aligned}$$

Proof. See [3, Lemma 5.4]. □

Hence, adding the estimates obtained in the previous two lemmas, we have a preliminary estimate for the total error:

$$\begin{aligned}
\big\| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \big\| &\leq C_{ST}(1 + 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|) \operatorname{dist}(\vec{\mathbf{t}}, \mathcal{H}_h) + \widehat{C}_{ST} \operatorname{dist}\Big((\varphi, \lambda), (H_h^\varphi \times H_h^\lambda) \Big) \\
&+ \Big\{ C_1 \| \mathbf{u} \|_{1,\Omega} + C_2 | \varphi |_{1,\Omega} \Big\} \| \mathbf{u} - \mathbf{u}_h \|_{1,\Omega} + \Big\{ C_3 \| \mathbf{t} \|_{\varepsilon,\Omega} + C_4 \| \mathbf{g} \|_{\infty,\Omega} + C_2 \| \mathbf{u}_h \|_{1,\Omega} \Big\} \| \varphi - \varphi_h \|_{1,\Omega},
\end{aligned} \tag{4.5}$$

where

$$C_1 := C_{ST} c_1(\Omega)(1+\kappa_1^2)^{1/2}, \quad C_2 := \widehat{C}_{ST} c_2(\Omega),$$
$$C_3 := C_{ST} L_\mu C_\varepsilon \widetilde{C}_\varepsilon (1+\kappa_1^2)^{1/2}, \quad C_4 = C_{ST}(1+\kappa_2^2)^{1/2}.$$

Then, bounding the terms $\| \mathbf{u} \|_{1,\Omega}$, $| \varphi |_{1,\Omega}$ and $\| \mathbf{u}_h \|_{1,\Omega}$ using the continuous dependence results (2.39), (2.40) and (3.4), and the further-regularity assumption (2.30) to bound $\| \mathbf{t} \|_{\varepsilon,\Omega}$, (4.5) becomes

$$\begin{aligned} \| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \| &\leq C_{ST}(1 + 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|) \text{ dist}\left(\vec{\mathbf{t}}, \mathcal{H}_h\right) + \widehat{C}_{ST} \text{ dist}\Big((\varphi, \lambda), (H_h^\varphi \times H_h^\lambda)\Big) \\ &+ \Big\{(C_1 + C_2 C_{\widetilde{\mathbf{S}}} r + C_2) C_\mathbf{S} \Big(r\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2,\Gamma}\Big) + C_3 \widetilde{C}_\mathbf{S}(r) \Big(r\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma}\Big) \\ &+ C_4 \| \mathbf{g} \|_{\infty,\Omega} + C_2 C_{\widetilde{\mathbf{S}}} \| \varphi_D \|_{1/2,\Gamma}\Big\} \| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \|. \end{aligned} \tag{4.6}$$

Therefore, using the continuous injection $H^{1/2+\varepsilon}(\Gamma) \hookrightarrow H^{1/2}(\Gamma)$ with constant $\widehat{C}_i$ and denoting by

$$C_5 := C_2 C_{\widetilde{\mathbf{S}}}, \quad C_6 := (C_1 + C_5 r + C_2) C_\mathbf{S}, \quad C_7 := C_6 r + C_3 \widetilde{C}_\mathbf{S}(r) r + C_4, \quad C_8 := C_6 \widehat{C}_i + C_3 \widetilde{C}_\mathbf{S}(r),$$

and

$$\mathbf{C}_0 := \max\{C_5, C_7, C_8\},$$

we see from (4.6) that

$$\begin{aligned} \| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \| &\leq C_{ST}(1 + 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|) \text{ dist}\left(\vec{\mathbf{t}}, \mathcal{H}_h\right) + \widehat{C}_{ST} \text{ dist}\Big((\varphi, \lambda), (H_h^\varphi \times H_h^\lambda)\Big) \\ &+ \mathbf{C}_0 \Big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} + \| \varphi_D \|_{1/2,\Gamma}\Big) \| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \|, \end{aligned} \tag{4.7}$$

which leads us the main result of this section.

Theorem 4.4. *Assume that*

$$\mathbf{C}_0 \Big(\| \mathbf{g} \|_{\infty,\Omega} + \| \mathbf{u}_D \|_{1/2+\varepsilon,\Gamma} + \| \varphi_D \|_{1/2,\Gamma}\Big) < \frac{1}{2}. \tag{4.8}$$

Then, there exists $C > 0$ depending only on parameters, data and other constants, all of them independent of h, such that

$$\| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \| \leq C \text{ dist}\Big((\vec{\mathbf{t}}, (\varphi, \lambda)), \mathcal{H}_h \times H_h^\varphi \times H_h^\lambda\Big). \tag{4.9}$$

Proof. The assumption (4.8) allows us to subtract the total error term in the right-hand side of (4.7), thus verifying the Céa's estimate with $C = 2 \max\Big\{C_{ST}(1 + 2\| \mathbf{A}_\varphi + \mathbf{B}_\mathbf{u} \|), \widehat{C}_{ST}\Big\}$. □

Finally, we state the rates of convergence of the Galerkin scheme (3.1) when the finite element subspaces (3.5)-(3.10) are used.

Lemma 4.5. *In addition to the hypotheses of Theorems 2.9, 3.1 and 4.4, assume that there exists $s > 0$ such that $\mathbf{t} \in \mathbb{H}^s(\Omega)$, $\boldsymbol{\sigma} \in \mathbb{H}^s(\Omega)$, $\mathbf{div}\, \boldsymbol{\sigma} \in \mathbf{H}^s(\Omega)$, $\mathbf{u} \in \mathbf{H}^{s+1}(\Omega)$, $\boldsymbol{\gamma} \in \mathbb{H}^s(\Omega)$, $\varphi \in H^{s+1}(\Omega)$ and $\lambda \in H^{-1/2+s}(\Gamma)$. Then, there exists $\widehat{C} > 0$, independent of h and $\widetilde{h}$ such that for all $h \leq C_0 \widetilde{h}$ there holds*

$$\begin{aligned} \| (\vec{\mathbf{t}}, (\varphi, \lambda)) - (\vec{\mathbf{t}}_h, (\varphi_h, \lambda_h)) \| &\leq \widehat{C} \widetilde{h}^{\min\{s,k+1\}} \| \lambda \|_{-1/2+s,\Gamma} \\ &+ \widehat{C} h^{\min\{s,k+1\}} \Big\{ \| \mathbf{t} \|_{s,\Omega} + \| \boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{div}\, \boldsymbol{\sigma} \|_{s,\Omega} + \| \mathbf{u} \|_{s+1,\Omega} + \| \boldsymbol{\gamma} \|_{s,\Omega} + \| \varphi \|_{s+1,\Omega} \Big\}. \end{aligned}$$

Proof. It follows from Céa's estimate (4.9) and the approximation properties $(\mathbf{AP}_h^{\mathbf{t}})$, $(\mathbf{AP}_h^{\boldsymbol{\sigma}})$, $(\mathbf{AP}_h^{\mathbf{u}})$, $(\mathbf{AP}_h^{\boldsymbol{\gamma}})$, $(\mathbf{AP}_h^{\varphi})$, and $(\mathbf{AP}_{\widetilde{h}}^{\lambda})$ described in Section 3.2. □

5 Numerical Results

We now present three numerical examples that will show the performance of the augmented mixed-primal finite element method (3.1) with the subspaces specified in Section 3.2. The computational implementation uses the finite element library `FreeFem++` (cf. [15]), and the inversion of the linear systems arising at each Picard step is performed employing the unsymmetric multi-frontal direct solvers MUMPS (cf. [2]) and UMFPACK (cf. [10]).

Here, the iterative method mimics the fixed-point strategy presented in Section 2.2: it begins with a given initial point (in all the subsequent examples, this point is $(\mathbf{u},\varphi)=(\mathbf{0},0)$) and it stops whenever the relative error between two consecutive iterations of the complete coefficient vector measured in the discrete ℓ^2 norm is sufficiently small, i.e.,

$$\frac{\|\,\mathbf{coeff}^{\,m+1}-\mathbf{coeff}^{\,m}\,\|_{\ell^2}}{\|\,\mathbf{coeff}^{\,m+1}\,\|_{\ell^2}}<\mathtt{tol},$$

where $\mathtt{tol}$ is a specified tolerance (in this case, $\mathtt{tol}=10^{-8}$). We also recall that the pressure is post-processed as

$$p_h:=-\frac{1}{n}\mathrm{tr}(\boldsymbol{\sigma}_h+\mathbf{u}_h\otimes\mathbf{u}_h),$$

and, as explained in [3, Section 5.2], the computed pressure converges to the exact one at the same rate as the other variables (cf. Theorem 4.5). In this way, we define the error per variable

$$e(\mathbf{t}):=\|\,\mathbf{t}-\mathbf{t}_h\,\|_{0,\Omega},\quad e(\boldsymbol{\sigma}):=\|\,\boldsymbol{\sigma}-\boldsymbol{\sigma}_h\,\|_{\mathbf{div};\Omega},\quad e(\mathbf{u}):=\|\,\mathbf{u}-\mathbf{u}_h\,\|_{1,\Omega},\quad e(p):=\|\,p-p_h\,\|_{0,\Omega},$$
$$e(\boldsymbol{\gamma}):=\|\,\boldsymbol{\gamma}-\boldsymbol{\gamma}_h\,\|_{0,\Omega},\quad e(\varphi):=\|\,\varphi-\varphi_h\,\|_{1,\Omega},\quad e(\lambda):=\|\,\lambda-\lambda_{\widetilde{h}}\,\|_{0,\Gamma},$$

as well as their corresponding rates of convergence

$$r(\mathbf{t}):=\frac{\log(e(\mathbf{t})/e'(\mathbf{t}))}{\log(h/h')},\quad r(\boldsymbol{\sigma}):=\frac{\log(e(\boldsymbol{\sigma})/e'(\boldsymbol{\sigma}))}{\log(h/h')},\quad r(\mathbf{u}):=\frac{\log(e(\mathbf{u})/e'(\mathbf{u}))}{\log(h/h')},$$
$$r(p):=\frac{\log(e(p)/e'(p))}{\log(h/h')},\quad r(\boldsymbol{\gamma}):=\frac{\log(e(\boldsymbol{\gamma})/e'(\boldsymbol{\gamma}))}{\log(h/h')},$$
$$r(\varphi):=\frac{\log(e(\varphi)/e'(\varphi))}{\log(h/h')},\quad r(\lambda):=\frac{\log(e(\lambda)/e'(\lambda)),}{\log(\widetilde{h}/\widetilde{h}')},$$

where h and h' (respectively $\widetilde{h}$ and $\widetilde{h}'$) denote two consecutive mesh sizes with errors e and e'.

5.1 Example 1

We first consider $\Omega:=(-1,1)^2$, viscosity, thermal conductivity and body force given by $\mu(\varphi)=\exp(-0.25\,\varphi)$, $\mathbb{K}=\mathbb{I}$, $\mathbf{g}=(0,1)^{\mathtt{t}}$, and boundary conditions such that the exact solution to (1.1) is given by

$$\mathbf{u}=\begin{pmatrix}\sin(\pi x)\cos(\pi y)\\ -\cos(\pi x)\sin(\pi y)\end{pmatrix},\quad \mathbf{t}=\mathbf{e}(\mathbf{u}),\quad \boldsymbol{\gamma}=\nabla\mathbf{u}-\mathbf{e}(\mathbf{u}),\quad p=x^2-y^2,$$
$$\boldsymbol{\sigma}=\mu(\varphi)\mathbf{e}(\mathbf{u})-\mathbf{u}\otimes\mathbf{u}-p\mathbb{I},\quad \varphi=-0.6944\,y^4+1.6944\,y^2,\quad \lambda=-\mathbb{K}\nabla\varphi\cdot\boldsymbol{\nu}.$$

Notice that nonzero source terms appear in the right-hand sides of the momentum and energy equations. Nevertheless, well-posedness is still ensured, since the smoothness of the exact solution makes these terms immediately belong to $L^2(\Omega)$, thus requiring only a minor modification in the functionals

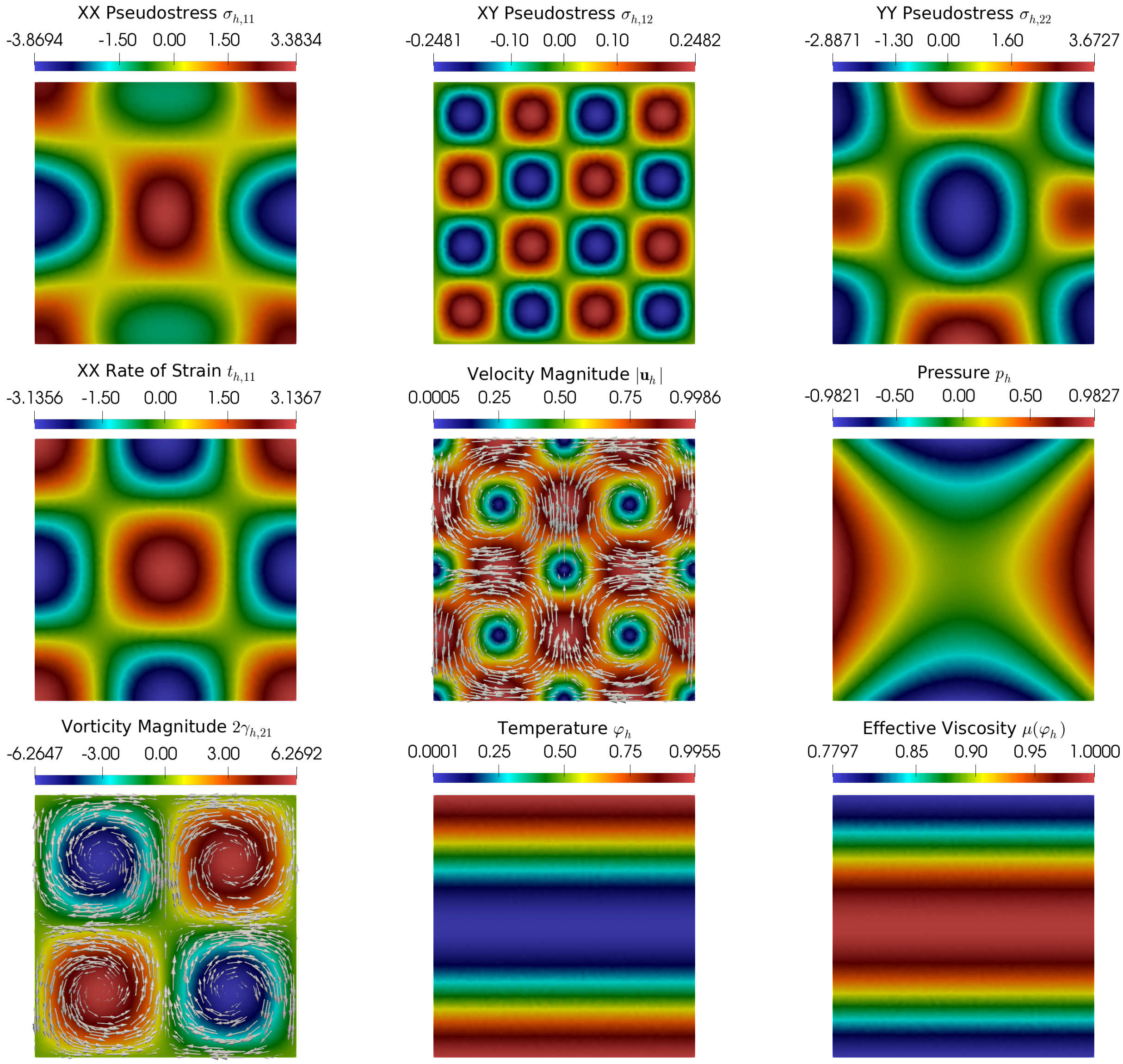

Figure 5.1: Numerical results for Example 1. From left to right and from up to down: XX, XY and YY pseudostress, XX strain rate, velocity magnitude (with the corresponding vector field overlapped), postprocessed pressure, true vorticity magnitude (computed as twice the YX component of the full vorticity tensor; velocity vector field overlapped), temperature and effective viscosity fields. Snapshots obtained from a simulation with 243,540 DOF and a second order approximation.

of the variational formulation. Concerning the stabilization parameters, these are taken as pointed out in Section 2.3, where the viscosity bounds are estimated in $\mu_1 = 0.5$, $\mu_2 = 1.25$ and $\kappa_0(\Omega)$ is simply taken as $\kappa_0(\Omega) = 1$, thus resulting in $\kappa_1 = \kappa_2 = 0.32$, $\kappa_3 = 0.25$ and $\kappa_4 = 0.125$. We also remark that the trace condition on the stress is enforced through a penalization strategy, not only in the present case but also in the upcoming examples.

In Figure 5.1 we show part of the obtained numerical solution with 171,402 DOF and a second order approximation, whereas in Table 5.1 we show the convergence history given the specified data and the finite element spaces from Section 3.2 with successive quasi-uniform mesh refinements. In both cases, it can be seen that the rates of convergence are the expected ones according to Theorem 4.5, that is, $\mathcal{O}(h)$ for the first case, and $\mathcal{O}(h^2)$ for the second one.

Finite Element: $\mathbb{P}_0$ - $\mathbb{RT}_0$ - $\mathbf{P}_1$ - P_0 - P_1 - P_0								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
1,283	0.4129	1.3213	3.5473	2.0880	0.5528	3.1251	0.3535	0.1602
4,845	0.1901	0.6133	1.7652	0.9576	0.2597	1.6453	0.1733	0.0724
18,629	0.0968	0.2987	0.8684	0.4758	0.1219	0.9224	0.0827	0.0286
73,422	0.0527	0.1488	0.4413	0.2390	0.0637	0.4286	0.0426	0.0112
294,878	0.0307	0.0740	0.2181	0.1187	0.0301	0.2301	0.0211	0.0035
1,165,980	0.0150	0.0365	0.1088	0.0584	0.0146	0.1155	0.0105	0.0014
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
10	0.5000	-	-	-	-	-	-	-
10	0.2500	0.9892	0.8996	1.0048	0.9735	0.8270	0.9187	1.1466
10	0.1250	1.0652	1.0506	1.0358	1.1207	0.8570	1.0962	1.3385
10	0.0625	1.1475	1.1150	1.1340	1.0677	1.2625	1.0938	1.3471
10	0.0312	1.2886	1.3002	1.2919	1.3846	1.1474	1.2979	1.6812
10	0.0156	0.9902	0.9742	0.9930	1.0104	0.9657	0.9690	1.2759
Finite Element: $\mathbb{P}_1$ - $\mathbb{RT}_1$ - $\mathbf{P}_2$ - P_1 - P_2 - P_1								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
4,122	0.4129	0.1732	0.5091	0.2737	0.1058	0.2728	0.0409	0.0076
15,846	0.1940	0.0409	0.1199	0.0642	0.0287	0.0773	0.0097	0.0031
61,494	0.0995	0.0102	0.0305	0.0159	0.0071	0.0236	0.0023	0.0009
243,540	0.0527	0.0026	0.0077	0.0041	0.0018	0.0054	0.0006	0.0002
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
10	0.5000	-	-	-	-	-	-	-
10	0.2500	1.9095	1.9145	1.9203	1.7272	1.6691	1.9046	1.3089
10	0.1250	2.0857	2.0509	2.0902	2.0988	1.7794	2.1711	1.7274
10	0.0625	2.1427	2.1662	2.1350	2.1851	2.3063	2.0953	1.8947

Table 5.1: Convergence history for Example 1, with a quasi-uniform mesh refinement and approximations of first and second order.

5.2 Example 2

We next consider $\Omega := (0,1)^2$; viscosity, thermal conductivity and body force as in Example 1, and boundary conditions and source terms such that the exact solution is the one considered in [11] for viscoelastic flow, that is,

$$u_1(x,y) = \left[1 - \cos\left(\frac{2\pi(e^{r_1 x} - 1)}{e^{r_1} - 1}\right)\right] \sin\left(\frac{2\pi(e^{r_2 y} - 1)}{e^{r_2} - 1}\right) \frac{r_2}{2\pi} \frac{e^{r_2 y}}{e^{r_2} - 1},$$

$$u_2(x,y) = -\left[1 - \cos\left(\frac{2\pi(e^{r_2 y} - 1)}{e^{r_2} - 1}\right)\right] \sin\left(\frac{2\pi(e^{r_1 x} - 1)}{e^{r_1} - 1}\right) \frac{r_1}{2\pi} \frac{e^{r_1 x}}{e^{r_1} - 1},$$

$$p(x,y) = r_1 r_2 \sin\left(\frac{2\pi(e^{r_1 x} - 1)}{e^{r_1} - 1}\right) \sin\left(\frac{2\pi(e^{r_2 y} - 1)}{e^{r_2} - 1}\right) \frac{e^{r_1 x + r_2 y}}{(e^{r_1} - 1)(e^{r_2} - 1)},$$

where r_1 and r_2 are positive parameters, and

$$\mathbf{u} = \begin{pmatrix} u_1(x,y) \\ u_2(x,y) \end{pmatrix}, \quad \mathbf{t} = \mathbf{e}(\mathbf{u}), \quad \boldsymbol{\gamma} = \nabla\mathbf{u} - \mathbf{e}(\mathbf{u}), \quad \boldsymbol{\sigma} = \mu(\varphi)\mathbf{e}(\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - p\mathbb{I}$$

$$\varphi = u_1(x,y) + u_2(x,y), \quad \lambda = -\mathbb{K}\nabla\varphi\cdot\boldsymbol{\nu}.$$

It is expected to find a counter-clockwise rotating vortex with center $(\hat{x},\hat{y})$, where

$$\hat{x} = \frac{1}{r_1}\log\left(\frac{e^{r_1}+1}{2}\right), \quad \hat{y} = \frac{1}{r_2}\log\left(\frac{e^{r_2}+1}{2}\right).$$

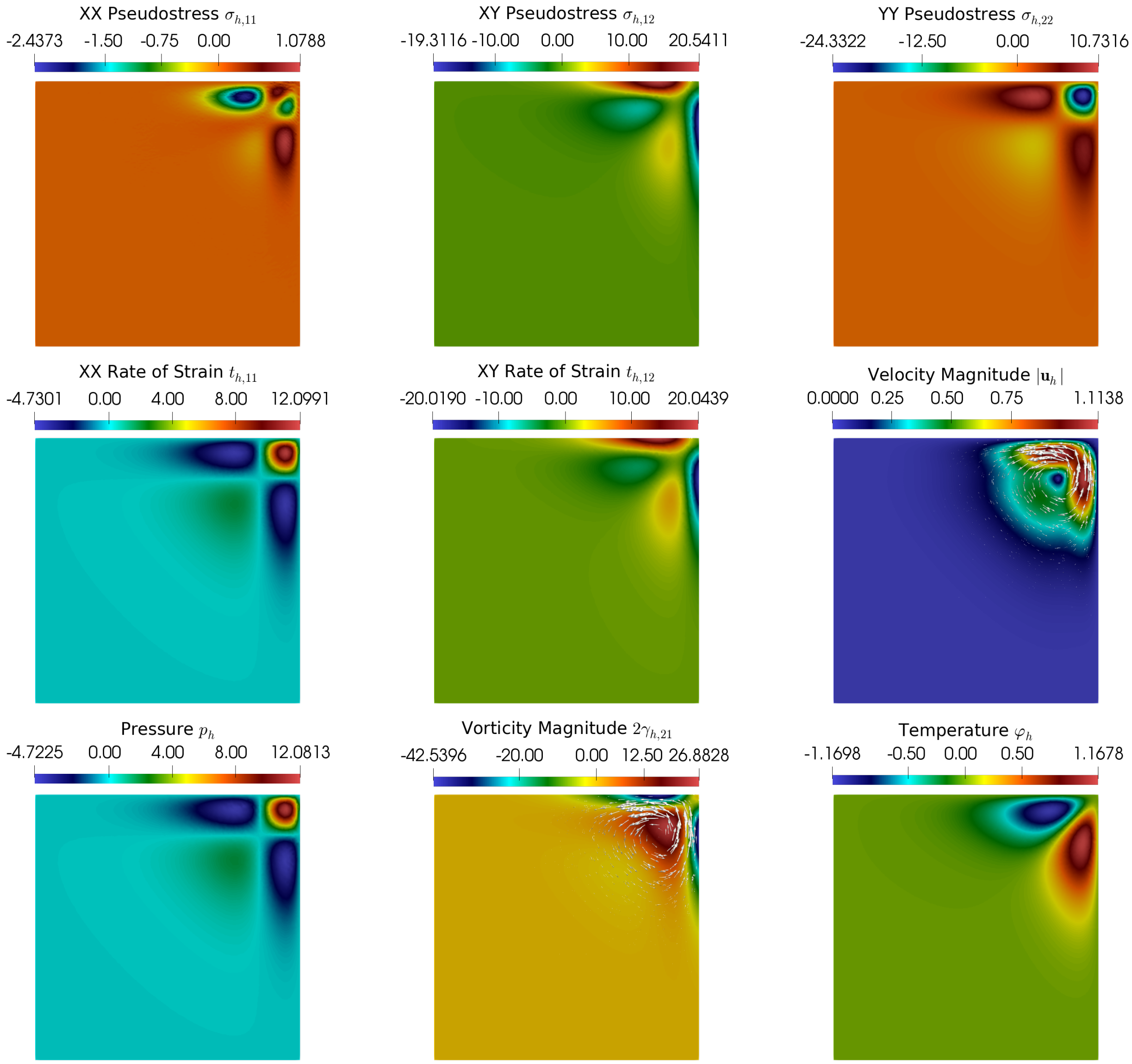

Figure 5.2: Numerical results for Example 2. From left to right and from up to down: XX, XY and YY pseudostress, XX and XY strain rate, velocity magnitude (with the corresponding vector field overlapped), postprocessed pressure, true vorticity magnitude (computed as twice the YX component of the full vorticity tensor; velocity vector field overlapped) and temperature fields. Snapshots obtained from a simulation with 1,165,005 DOF and a first order approximation.

In particular, taking $r_1 = r_2 = 4.5$, the center of the vortex is expected to appear at the top-right corner of the cavity (that is, $(\hat{x}, \hat{y}) = (0.829, 0.829)$) in a similar way to the examples shown in [9]. Then, considering the stabilization parameters as in Section 2.3, estimating the viscosity bounds in $\mu_1 = 0.74$, $\mu_2 = 1.35$ and taking $\kappa_0(\Omega) = 1$, we obtain the following values: $\kappa_1 = \kappa_2 = 0.406$, $\kappa_3 = 0.37$ and $\kappa_4 = 0.185$.

Part of the solution is shown in Figure 5.2, where a first order of approximation has been used with 1,165,005 DOF, whereas in Table 5.2 we show the corresponding convergence history. As expected, when using the finite-element subspaces of Section 3.2 with $k = 0$ and $k = 1$, the rates of convergence are near 1 and 2, respectively. Notice that a high degree of refinement was needed to obtain a good solution; a drawback that can be easily overcome by implementing an adaptive algorithm, instead of

Finite Element: $\mathbb{P}_0$ - $\mathbb{RT}_0$ - $\mathbf{P}_1$ - P_0 - P_1 - P_0								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
1,268	0.1901	2.5548	49.1289	9.5647	1.1838	3.7866	3.8763	85.8576
4,740	0.0950	1.3353	28.3875	2.9504	0.7207	2.3406	2.2414	56.0199
18,494	0.0490	0.6594	12.3390	1.1265	0.3370	1.2941	1.0635	24.0197
73,167	0.0244	0.3416	6.4760	0.5563	0.1809	0.6812	0.5513	11.9313
294,323	0.0140	0.1650	3.1796	0.2638	0.0866	0.3404	0.2687	5.9119
1,165,005	0.0078	0.0793	1.6011	0.1278	0.0395	0.1796	0.1304	2.9356
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
7	0.2500	-	-	-	-	-	-	-
6	0.1250	0.9360	0.7913	1.6968	0.7159	0.6940	0.7903	0.6160
6	0.0625	1.0657	1.2584	1.4542	1.1481	0.8950	1.1260	1.2217
6	0.0312	0.9445	0.9258	1.0134	0.8938	0.9215	0.9435	1.0095
6	0.0156	1.3016	1.2721	1.3342	1.3179	1.2406	1.2857	1.0131
6	0.0078	1.2501	1.1701	1.2362	1.3399	1.0903	1.2334	1.0100
Finite Element: $\mathbb{P}_1$ - $\mathbb{RT}_1$ - $\mathbf{P}_2$ - P_1 - P_2 - P_1								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
4,072	0.1901	0.9820	16.9574	1.9820	0.7614	1.2657	2.2098	47.0700
15,496	0.1025	0.2914	6.5567	0.4721	0.2827	0.3505	0.5027	9.0859
61,044	0.0490	0.0638	1.5747	0.0939	0.0517	0.0922	0.1020	4.1253
242,690	0.0256	0.0174	0.4668	0.0252	0.0137	0.0266	0.0266	1.2049
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
6	0.2500	-	-	-	-	-	-	-
6	0.1250	1.9686	1.5396	2.3244	1.6055	2.0805	2.3991	2.3731
6	0.0625	2.0583	1.9327	2.1882	2.3013	1.8088	2.1610	1.1391
6	0.0312	2.0002	1.8724	2.0245	2.0429	1.9167	2.0705	1.7756

Table 5.2: Convergence history for Example 2, with a quasi-uniform mesh refinement and approximations of first and second order.

the quasi-uniform mesh refinement that was considered.

5.3 Example 3

The implementation of the numerical scheme and the accuracy for the three-dimensional case are assessed with this last computational test. The domain is the unit cube $\Omega = (0,1)^3$ and we consider the following closed-form solutions to the governing equations (1.1)

$$\mathbf{u} = \begin{pmatrix} \sin(\pi x)\cos(\pi y)\cos(\pi z) \\ -2\cos(\pi x)\sin(\pi y)\cos(\pi z) \\ \cos(\pi x)\cos(\pi y)\sin(\pi z) \end{pmatrix}, \quad \mathbf{t} = \mathbf{e}(\mathbf{u}), \quad \boldsymbol{\gamma} = \nabla\mathbf{u} - \mathbf{e}(\mathbf{u}), \quad p = \sin(\pi x)\sin(\pi y)\sin(\pi z),$$

$$\boldsymbol{\sigma} = \mu(\varphi)\mathbf{e}(\mathbf{u}) - \mathbf{u}\otimes\mathbf{u} - p\mathbb{I}, \quad \varphi = 1 - \sin(\pi x)\cos(\pi y)\sin(\pi z), \quad \lambda = -\mathbb{K}\nabla\varphi\cdot\boldsymbol{\nu},$$

with $\mathbb{K} = \mathbb{I}$, $\mu(\varphi) = \exp(-0.25\,\varphi)$, and $\mathbf{g} = (0,0,1)^{\mathbf{t}}$. The manufactured velocity is divergence free, it satisfies the compatibility condition (1.2) and it is used as Dirichlet datum on Γ. The exact temperature is uniformly bounded and it is also exploited as Dirichlet datum. In this configuration the viscosity bounds can be set as $\mu_1 = 0.5$, $\mu_2 = 1$ and the augmentation constants are again chosen according to Section (2.3), leading to the values $\kappa_1 = \kappa_2 = 0.5$, $\kappa_3 = 0.25$, $\kappa_5 = 0.125$, and as κ_4 depends on the (unknown) Korn constant, we simply take $\kappa_4 = \kappa_5$. The error history, associated with the schemes of order one and two, are performed using six steps of uniform mesh refinement applied to an initial structured tetrahedral mesh. On each level we compute approximate solutions, as well as errors and convergence rates defined as above. The boundary partition is considered conforming with the interior mesh, for sake of convenience and simplicity of the 3D mesh generation. Our findings are collected in Table 5.3, where errors and Picard iteration count are tabulated by number of degrees of freedom and meshsize. As in the 2D case, optimal error decay is observed for all individual errors, and we also note that the stress and velocity errors $e(\boldsymbol{\sigma}), e(\mathbf{u})$ are dominant. One can also see that (perhaps assisted by the conformity between the interior and boundary meshes, and only noticed for the lowest-order method) the error associated with the boundary heat flux exhibits a convergence slightly better than the optimal predicted by Lemma 4.5. Finally, we portray in Figure 5.3 a sample of the approximate solutions generated by the lowest-order mixed method on a relatively coarse mesh.

Finite Element: $\mathbb{P}_0$ - $\mathbb{RT}_0$ - $\mathbf{P}_1$ - P_0 - P_1 - P_0								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
900	0.7071	1.9873	5.6367	4.2163	0.6140	1.3454	1.9131	0.0137
2,848	0.4714	1.1368	4.0751	2.9781	0.3299	1.0208	1.3774	0.0065
12,564	0.2828	0.7463	2.5402	1.8903	0.2071	0.7146	0.7827	0.0027
71,068	0.1571	0.3888	1.4463	1.1242	0.1267	0.4591	0.4332	0.0011
451,690	0.0882	0.1925	0.7612	0.6791	0.0698	0.2300	0.2179	0.0006
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
7	0.7071	-	-	-	-	-	-	-
7	0.4714	0.8933	0.9000	0.9574	1.0935	0.8809	0.8378	1.4345
8	0.2828	0.8937	0.9525	0.9898	0.9877	0.8982	1.1064	1.6061
8	0.1571	0.9199	0.9852	0.9840	0.9828	0.9525	1.0064	1.5084
8	0.0882	0.9382	0.9874	1.0522	0.9703	0.9554	0.9849	1.4690
Finite Element: $\mathbb{P}_1$ - $\mathbb{RT}_1$ - $\mathbf{P}_2$ - P_1 - P_2 - P_1								
DOF	h	$e(\mathbf{t})$	$e(\boldsymbol{\sigma})$	$e(\mathbf{u})$	$e(p)$	$e(\boldsymbol{\gamma})$	$e(\varphi)$	$e(\lambda)$
3,693	0.7071	0.7084	2.5493	2.8720	0.2803	0.7668	1.0241	0.0069
11,741	0.4714	0.2268	0.8202	0.9132	0.0846	0.1949	0.3093	0.0023
51,825	0.2828	0.0603	0.2192	0.2609	0.0217	0.0625	0.0794	0.0006
286,905	0.1571	0.0169	0.0516	0.0689	0.0575	0.0164	0.0197	0.0002
1,879,712	0.0882	0.0052	0.0135	0.0186	0.0167	0.0043	0.0051	0.0001
IT	$\widetilde{h}$	$r(\mathbf{t})$	$r(\boldsymbol{\sigma})$	$r(\mathbf{u})$	$r(p)$	$r(\boldsymbol{\gamma})$	$r(\varphi)$	$r(\lambda)$
6	0.7071	-	-	-	-	-	-	-
7	0.4714	1.8533	1.8196	1.8459	1.8286	1.7980	1.8655	2.4821
7	0.2828	1.9406	1.8871	1.8970	1.9550	1.9123	1.8493	2.6157
8	0.1571	1.9807	1.9501	1.8931	1.9715	1.9837	1.9916	2.4768
8	0.0882	1.9463	1.9192	1.9054	1.9469	1.9508	1.9377	2.0842

Table 5.3: Convergence history for Example 3, with a quasi-uniform mesh refinement and approximations of first and second order.

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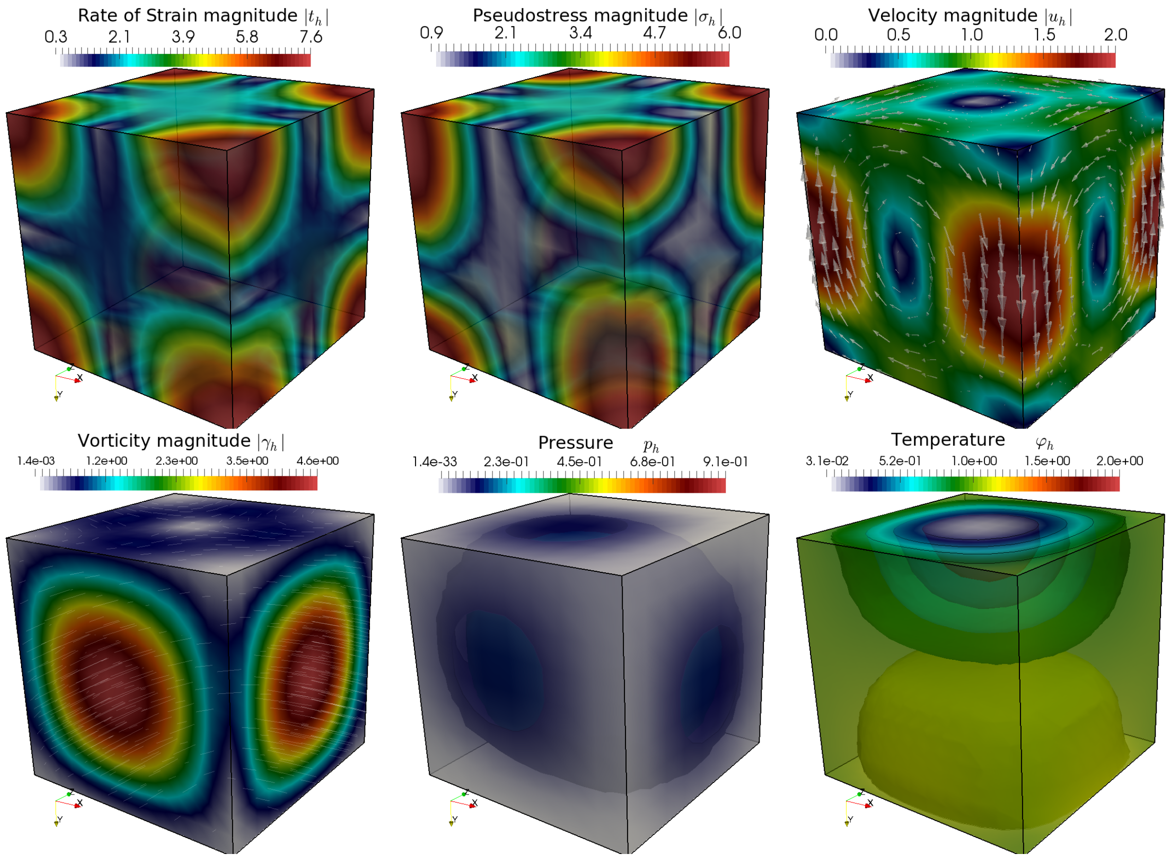

Figure 5.3: Example 3. Approximate solutions (from left to right and from up to down): magnitude of strain rate, pseudostress, velocity magnitude and arrows, vorticity magnitude, postprocessed pressure, and temperature. Snapshots obtained from a simulation with a lowest-order approximation.

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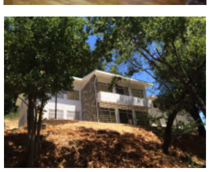

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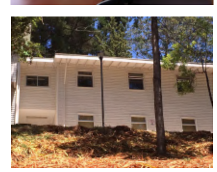

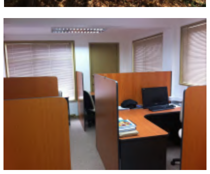

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