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An a priori error analysis for discontinuous Lagrangian finite elements applied to nonconforming dual mixed formulations: Poisson and Stokes problems

Tomás P. Barrios* and Rommel Bustinza†

Abstract

In this paper, we discuss the well posedness of a mixed discontinuous Galerkin (DG) scheme for the Poisson and Stokes problems in 2D, considering only piecewise Lagrangian finite elements. The difficulty here relies on the fact that the classical Babuška-Brezzi theory is not easy to check for low order finite elements, so we proceed in a non-standard way. First, we prove uniqueness, and then we apply a discrete version of Fredholm's alternative theorem to ensure existence. The a priori error analysis is done by introducing suitable projections of exact solution. As a result, we prove that the method is convergent, and under standard additional regularity assumptions on the exact solution, the optimal rate of convergence of the method is guaranteed.

Key words: Discontinuous Galerkin, Lagrange shape functions, a-priori error estimates.

Mathematics subject classifications (1991): 65N30; 65N12; 65N15

1 Introduction

Let Ω be a bounded and simply connected domain in $\mathbb{R}^2$ with polygonal boundary Γ. Then, given $f \in L^2(\Omega)$ and $g \in H^{1/2}(\Gamma)$, we look for $u \in H^1(\Omega)$ such that

$$-\Delta u = f \text{ in } \Omega, \; u = g \text{ on } \Gamma_D, \tag{1.1}$$

where $\boldsymbol{\nu}$ denotes the unit outward normal to $\partial\Omega$.

We follow [2] and introduce the gradient $\boldsymbol{\sigma} := -\nabla u$ in Ω as additional unknown. In this way, (1.1) can be reformulated as the following problem in $\bar{\Omega}$: *Find* $(\boldsymbol{\sigma}, u)$ *in appropriate spaces such that, in the distributional sense*

$$\boldsymbol{\sigma} + \nabla u = 0 \quad \text{in} \quad \Omega, \quad \operatorname{div} \boldsymbol{\sigma} = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g \quad \text{on} \quad \Gamma. \tag{1.2}$$

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We remark that problem (1.2) has been already analysed in [18] using the so-called local discontinuous Galerkin (LDG) method, where the unknowns $\boldsymbol{\sigma}$ and u are approximated in piecewise polynomial spaces that locally belong to H^1 and L^2, respectively. In the present work, we are interested in approximating the vector unknown $\boldsymbol{\sigma}$ in a properly discrete space, such that it locally belongs to $H(\mathrm{div})$. This motivates us to consider the employment of local Raviart-Thomas spaces, to approximate $\boldsymbol{\sigma}$. This kind of approaches has been also applied in the previous works [3] [4] [5] and [7]. All of them consider the standard definition of numerical fluxes for LDG scheme, whose parameters α and $\boldsymbol{\beta}$ behave as $\mathcal{O}(\frac{1}{\mathtt{h}})$ and $\mathcal{O}(1)$, respectively. Recently, in the framework of the nonconforming pseudostress - velocity formulation for the Stokes system, in [8] we study an unusual DG approach which requires a vector numerical flux parameter $\boldsymbol{\beta}$ having the standard behaviour $\mathcal{O}(1)$, and two scalar numerical fluxes parameters (α and γ). Here, one of them (γ) behaves as $\mathcal{O}(\frac{1}{\mathtt{h}})$, while the other one ($\alpha$), as $\mathcal{O}(\mathtt{h})$. The well posedness of the proposed scheme is established using discontinuous Raviart-Thomas finite elements for the pseudostress unknown, and discontinuos polynomials for the velocity one. A particularity of this approach is that when it is tested with continuous functions, we obtain the standard conforming dual mixed formulation. In this sense, the analysis developed in [8] can be seen as an extension of the one described in [12], where it has been proved that the pair of conforming Raviart-Thomas finite elements with discontinuos polynomials is stable for the Stokes system, in dual-mixed form. Of course, the DG scheme presented in [8] could be approximated by other finite elements, thus this will be the aim of this article, i.e., in this paper we extend the analysis developed in [8] to use others pair of finite element spaces.

The paper is organized as follows. In Section 2, by simplicity, we take into account the analysis described in [8] for Poisson problem, and adapt/generalize it to study the a priori error analysis for this boundary value problem, considering only finite element spaces of the Lagrange type for all unknowns.

In Section 3, we extend the results obtained in the previous section to the Stokes problem, approximating the unknowns by polynomials of suitable degrees. Numerical examples that validate our theoretical results are shown and discussed in Section 4. Final remarks and conclusions are described in Section 5.

In the rest of the paper we will use the following notation. Given any Hilbert space H, we denote by H^2 the space of vectors of order 2 with entries in H, and by $H^{2\times 2}$ the space of square tensors of order 2 with entries in H. In particular, given $\boldsymbol{\tau} := (\tau_{ij})$, $\boldsymbol{\zeta} := (\zeta_{ij}) \in \mathbb{R}^{2\times 2}$, we write, as usual, $\boldsymbol{\tau}^{\mathtt{t}} := (\tau_{ji})$, $\mathrm{tr}(\boldsymbol{\tau}) := \tau_{11} + \tau_{22}$ and $\boldsymbol{\tau} : \boldsymbol{\zeta} := \sum_{i,j=1}^{2} \tau_{ij}\,\zeta_{ij}$. For vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ in $\mathbb{R}^2$, we denote by $\boldsymbol{v} \otimes \boldsymbol{w}$ the matrix whose ij-th entry is $v_i\,w_j$. We also use the standard notation for Sobolev spaces and norms. We denote $H(\mathrm{div};\,\Omega) := \{\boldsymbol{v} \in [L^2(\Omega)]^2 : \quad \mathrm{div}(\boldsymbol{v}) \in L^2(\Omega)\}$, $H = H(\mathbf{div};\,\Omega) := \{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} : \quad \mathbf{div}(\boldsymbol{\tau}) \in [L^2(\Omega)]^2\}$, and $H_0 := \{\boldsymbol{\tau} \in H : \quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\}$. Note that $H = H_0 \oplus \mathbb{R}\,\mathbf{I}$, that is, for any $\boldsymbol{\tau} \in H$ there exists unique $(\boldsymbol{\tau}_0, \rho) \in H_0 \times \mathbb{R}$ such that $\boldsymbol{\tau} = \boldsymbol{\tau}_0 + \rho\,\mathbf{I}$. In addition, we define the deviator of the tensor $\boldsymbol{\tau} \in H$ by $\boldsymbol{\tau}^d := \boldsymbol{\tau} - \frac{1}{2}\,\mathrm{tr}(\boldsymbol{\tau})\mathbf{I}$. We remark that $\mathrm{tr}(\boldsymbol{\tau}^d) = 0$ in Ω, and thus $\boldsymbol{\tau}^d \in H_0$ for any $\boldsymbol{\tau} \in H$. Finally, we use C or c, with or without subscripts, to denote generic constants, independent of the discretization parameters, which may take different values at different occurrences.

2 A modified LDG formulation

In this section, we derive a discrete formulation for the linear model (1.1), applying an unusual discontinuous Galerkin method in divergence form. We begin with some definitions and notations.

2.1 Meshes

We let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape-regular triangulations of $\bar{\Omega}$ (with possible hanging nodes) made up of straight-side triangles T with diameter h_T and unit outward normal to ∂T denoted by $\boldsymbol{\nu}_T$. As usual, the index h also denotes $h := \max\limits_{T\in\mathcal{T}_h} h_T$. Then, given $\mathcal{T}_h$, its edges are defined as follows. An *interior edge* of $\mathcal{T}_h$ is the (nonempty) interior of $\partial T \cap \partial T'$, where T and T' are two adjacent elements of $\mathcal{T}_h$, not necessarily matching. We denote by $\mathcal{E}_I$ the list of all interior edges of $\mathcal{T}_h$ (counted only once) in Ω, by $\mathcal{E}_\Gamma$ the list of all boundary edges, respectively, and set $\mathcal{E} := \mathcal{E}_I \cup \mathcal{E}_\Gamma$ the skeleton inherited by the triangulation $\mathcal{T}_h$. Moreover, for each $e \in \mathcal{E}$, h_e represents its length. In addition, in what follows we assume that $\mathcal{T}_h$ is of *bounded variation*, which means that there exists a constant $l > 1$, independent of the meshsize h, such that $l^{-1} \leq \frac{h_T}{h_{T'}} \leq l$ for each pair T, $T' \in \mathcal{T}_h$ sharing an interior edge.

2.2 Averages and jumps

Here, we define average and jump operators. Next, in order to define average and jump operators, we let T and T' be two adjacent elements of $\mathcal{T}_h$ and $\boldsymbol{x}$ be an arbitrary point on the interior edge $e = \partial T \cap \partial T' \in \mathcal{E}_I$. In addition, let q , $\boldsymbol{v}$ and $\boldsymbol{\tau}$ be scalar-, vector-, and matrix-valued functions, respectively, that are smooth inside each element $T \in \mathcal{T}_h$. We denote by $(q_{T,e}, \boldsymbol{v}_{T,e}, \boldsymbol{\tau}_{T,e})$ the restriction of $(q_T, \boldsymbol{v}_T, \boldsymbol{\tau}_T)$ to e. Then, we define the averages at $\boldsymbol{x} \in e$ by:

$$\{q\} := \frac{1}{2}\big(q_{T,e} + q_{T',e}\big)\,, \quad \{\boldsymbol{v}\} := \frac{1}{2}\big(\boldsymbol{v}_{T,e} + \boldsymbol{v}_{T',e}\big)\,, \quad \{\boldsymbol{\tau}\} := \frac{1}{2}\big(\boldsymbol{\tau}_{T,e} + \boldsymbol{\tau}_{T',e}\big)\,.$$

Similarly, the jumps at $\boldsymbol{x} \in e$ are given by

$$[\![q]\!] := q_{T,e}\,\boldsymbol{\nu}_T + q_{T',e}\,\boldsymbol{\nu}_{T'}\,, \quad [\![\boldsymbol{v}]\!] := \boldsymbol{v}_{T,e}\cdot\boldsymbol{\nu}_T + \boldsymbol{v}_{T',e}\cdot\boldsymbol{\nu}_{T'}\,,$$

$$\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}_{T,e}\otimes\boldsymbol{\nu}_T + \boldsymbol{v}_{T',e}\otimes\boldsymbol{\nu}_{T'}\,, \quad [\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}_{T,e}\,\boldsymbol{\nu}_T + \boldsymbol{\tau}_{T',e}\,\boldsymbol{\nu}_{T'}\,.$$

On boundary edges e, we set $\{q\} := q$, $\{\boldsymbol{v}\} := \boldsymbol{v}$, $\{\boldsymbol{\tau}\} := \boldsymbol{\tau}$, as well as $[\![q]\!] := q\,\boldsymbol{\nu}$, $[\![\boldsymbol{v}]\!] := \boldsymbol{v}\cdot\boldsymbol{\nu}$, $\underline{[\![\boldsymbol{v}]\!]} := \boldsymbol{v}\otimes\boldsymbol{\nu}$ and $[\![\boldsymbol{\tau}]\!] := \boldsymbol{\tau}\,\boldsymbol{\nu}$. Hereafter, $\mathbf{div}_h$ and ∇_h denote the piecewise divergence and gradient operators, respectively. Associated to these operators, for $\epsilon > 1/2$, we also introduced the broken Sobolev spaces $H^\epsilon(\mathcal{T}_h)$, $H(\mathrm{div};\mathcal{T}_h)$ and $H(\mathbf{div};\mathcal{T}_h)$, which are defined in the standard way, and in order to short the notation, we set $\boldsymbol{\Sigma} := H(\mathrm{div};\mathcal{T}_h) \cap [H^\epsilon(\Omega)]^2$ and $\underline{\boldsymbol{\Sigma}} := H(\mathbf{div};\mathcal{T}_h) \cap [H^\epsilon(\Omega)]^{2\times 2}$.

2.3 A discontinuous discrete formulation for the Poisson problem

Given a mesh $\mathcal{T}_h$, we proceed as in [18] (or [10]) and multiply each one of the equations (introduced at the introduction) by suitable test functions. We wish to approximate the exact solution $(\boldsymbol{\sigma}, u)$ of (1.2) by discrete functions $(\boldsymbol{\sigma}_h, u_h)$ in appropriate finite element space $\boldsymbol{\Sigma}_h \times V_h$ such that, for each $T \in \mathcal{T}_h$, we have

$$\begin{aligned} \int_T \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} - \int_T u_h \operatorname{div} \boldsymbol{\tau} + \int_{\partial T} \widehat{u}\, \boldsymbol{\tau} \cdot \boldsymbol{\nu}_T &= 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,, \\ -\int_T \boldsymbol{\sigma}_h \cdot \nabla v + \int_{\partial T} v\, \widehat{\boldsymbol{\sigma}} \cdot \boldsymbol{\nu}_T &= \int_T f v \qquad \forall\, v \in V_h\,. \end{aligned} \tag{2.1}$$

Here, the *numerical fluxes* $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$, which usually depend on u_h, $\boldsymbol{\sigma}_h$, and the boundary data, are defined in terms of averages and jumps of the discrete unknowns, so that some compatibility conditions are satisfied (see [2]).

We are now ready to complete the DG formulation (2.1). Indeed, using the approach from [18] and [13], we define the numerical fluxes $\widehat{u}$ and $\widehat{\boldsymbol{\sigma}}$ for each $T \in \mathcal{T}_h$ as follows:

$$\widehat{u}_{T,e} := \begin{cases} \{u_h\} - [\![u_h]\!] \cdot \boldsymbol{\beta} + \gamma [\![\boldsymbol{\sigma}_h]\!] & \text{if } e \in \mathcal{E}_I, \\ g & \text{if } e \in \mathcal{E}_\Gamma \end{cases} \tag{2.2}$$

and

$$\widehat{\boldsymbol{\sigma}}_{T,e} := \begin{cases} \{\boldsymbol{\sigma}_h\} + \boldsymbol{\beta} [\![\boldsymbol{\sigma}_h]\!] + \alpha [\![u_h]\!] & \text{if } e \in \mathcal{E}_I, \\ \boldsymbol{\sigma}_h + \alpha (u_h - g) \boldsymbol{\nu} & \text{if } e \in \mathcal{E}_\Gamma, \end{cases} \tag{2.3}$$

where the scalar parameters α and γ, as well as the vector one $\boldsymbol{\beta}$, to be chosen appropriately, are single valued on each edge $e \in \mathcal{E}$ and such that they allow us to prove the optimal rates of convergence of our approximation. To this aim, we set $\alpha := \widehat{\alpha}\, \mathtt{h}$, $\gamma := \frac{\widehat{\gamma}}{\mathtt{h}}$, and $\boldsymbol{\beta} \in [L^\infty(\mathcal{E}_I)]^2$ as an arbitrary vector in $\mathbb{R}^2$. Hereafter, $\widehat{\alpha} > 0$ and $\widehat{\gamma} > 0$ are arbitrary, while $\mathtt{h}$ is defined by

$$\mathtt{h} := \begin{cases} \max\{h_T, h_{T'}\} & \text{if } e \in \mathcal{E}_I\,, \\ h_T & \text{if } e \in \mathcal{E}_\Gamma\,. \end{cases}$$

Then, integrating by parts in the second equation in (2.1), summing up over all $T \in \mathcal{T}_h$, and applying a well-known algebraic identity, we arrive to the following discrete dual mixed discontinuous Galerkin formulation: *Find* $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ *such that*

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau}, u_h) &= G_{DG}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \\ b_{DG}(\boldsymbol{\sigma}_h, v) + c_{DG}(u_h, v) &= F_{DG}(v) \quad \forall\, v \in V_h, \end{aligned} \tag{2.4}$$

where the bilinear forms $a_{DG} : \boldsymbol{\Sigma} \times \boldsymbol{\Sigma} \to \mathbb{R}$, $c_{DG} : H^\epsilon(\mathcal{T}_h) \times H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ and $b_{DG} : \boldsymbol{\Sigma} \times H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ are defined by

$$a_{DG}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \int_\Omega \boldsymbol{\sigma}_h \cdot \boldsymbol{\tau} + \int_{\mathcal{E}_I} \gamma [\![\boldsymbol{\sigma}_h]\!] [\![\boldsymbol{\tau}]\!] \quad , \quad c_{DG}(w, v) := \int_{\mathcal{E}} \alpha\, [\![v]\!] [\![w]\!]\,,$$

$$\text{and} \quad b_{DG}(\boldsymbol{\tau}, v) := \int_\Omega v \, \mathrm{div}_h \, \boldsymbol{\tau} - \int_{\mathcal{E}_I} \big(\{v\} - \boldsymbol{\beta} \cdot [\![v]\!]\big)[\![\boldsymbol{\tau}]\!] \,,$$

and the linear functionals $G_{DG} : \boldsymbol{\Sigma} \to \mathbb{R}$ and $F_{DG} : H^\epsilon(\mathcal{T}_h) \to \mathbb{R}$ are given by

$$G_{DG}(\boldsymbol{\tau}) := - \int_{\mathcal{E}_\Gamma} g \, \boldsymbol{\tau} \cdot \boldsymbol{\nu} \qquad \text{and} \qquad F_{DG}(v) := \int_\Omega f \, v + \int_{\mathcal{E}_\Gamma} \alpha \, g \, v \,.$$

Now, the space $\boldsymbol{\Sigma}_h$ is provided with the usual norm of $H(\mathrm{div}; \mathcal{T}_h)$, which is denoted by $\| \cdot \|_{\boldsymbol{\Sigma}}$, that is

$$\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \, := \, \big(||\boldsymbol{\tau}||^2_{0,\Omega} + || \, \mathrm{div}_h \, \boldsymbol{\tau}||^2_{0,\Omega} + \|\gamma^{1/2} [\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{E}_I} \big)^{1/2} \quad \forall \, \boldsymbol{\tau} \in \boldsymbol{\Sigma} \,,$$

while for V_h we introduce its standard L^2-norm. In addition, we define $||(\cdot, \cdot)||_{DG} : \boldsymbol{\Sigma} \times L^2(\Omega) \to \mathbb{R}$ by

$$||(\boldsymbol{\tau}, v)||_{DG} \, := \, \Big(||\boldsymbol{\tau}||^2_{\boldsymbol{\Sigma}} \, + \, \|v\|^2_{L^2(\Omega)} \Big)^{1/2} \quad \forall \, (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma} \times L^2(\Omega) \,.$$

Finally, we introduce the application $|| \cdot ||_{\boldsymbol{\Sigma},0} \, : \, \boldsymbol{\Sigma} \to \mathbb{R}$, given by

$$||\boldsymbol{\tau}||_{\boldsymbol{\Sigma},0} \, : \, \big(||\boldsymbol{\tau}||^2_{0,\Omega} + \|\gamma^{1/2} [\![\boldsymbol{\tau}]\!]\|^2_{0,\mathcal{E}_I} \big)^{1/2} \quad \forall \, \boldsymbol{\tau} \in \boldsymbol{\Sigma} \,,$$

which will be helpful for our purposes.

The boundedness of the bilinear forms and functionals are reported in the next lemma

Lemma 2.1 *There exists $C > 0$, independent of the mesh size, such that*

$$a_{DG}(\boldsymbol{\tau}, \boldsymbol{\zeta}) \leq C \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma},0} \|\boldsymbol{\zeta}\|_{\boldsymbol{\Sigma},0} \quad \forall \, (\boldsymbol{\tau}, \boldsymbol{\zeta}) \in \boldsymbol{\Sigma} \times \boldsymbol{\Sigma} \,, \tag{2.5}$$

$$b_{DG}(\boldsymbol{\tau}, v) \leq C \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|v\|_{L^2(\Omega)} \quad \forall \, (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma} \times V_h \,, \tag{2.6}$$

$$c_{DG}(v, w) \leq C \|v\|_{L^2(\Omega)} \|w\|_{L^2(\Omega)} \quad \forall \, (v, w) \in V_h \times V_h \,. \tag{2.7}$$

$$|F_{DG}(v)| \leq C \bigg(\|f\|_{L^2(\Omega)} + \|\alpha^{1/2} g\|_{L^2(\mathcal{E}_\Gamma)} \bigg) \|v\|_{L^2(\Omega)} \quad \forall \, v \in V_h \,, \tag{2.8}$$

$$|G_{DG}(\boldsymbol{\tau})| \leq C \|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}} \|\gamma^{1/2} g\|_{L^2(\Gamma)} \quad \forall \, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \,, \tag{2.9}$$

Proof. It is analogous to the proofs of Lemmas 3.2 and 3.3 in [8]. We omit further details. □

The well posedness of Problem (2.4) is established in the next theorem.

Theorem 2.1 *Under the assumption that $\nabla_h V_h$ is a subspace of $\boldsymbol{\Sigma}_h$, there exists a unique $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ solution of the Problem (2.4).*

Proof. Since the discrete system is square, it is enough to check that the corresponding homogeneous problem: *Find $(\boldsymbol{\sigma}_h, u_h) \in \boldsymbol{\Sigma}_h \times V_h$ such that*

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau}, u_h) &= \; 0 \quad \forall \, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h \,, \\ b_{DG}(\boldsymbol{\sigma}_h, v) + c_{DG}(u_h, v) &= \; 0 \quad \forall \, v \in V_h \,, \end{aligned} \tag{2.10}$$

has only the trivial solution. To this end, taking $\boldsymbol{\tau} = \boldsymbol{\sigma}_h$, $v = u_h$ in (2.10), and after adding these two equations, we deduce

$$\|\boldsymbol{\sigma}_h\|^2_{[L^2(\Omega)]^2} + \|\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]\|^2_{L^2(\mathcal{E}_I)} + \|\alpha^{1/2}[\![u_h]\!]\|^2_{[L^2(\mathcal{E})]^2} = 0\,,$$

which implies that $\boldsymbol{\sigma}_h \in H(\text{div},\Omega)$, $\boldsymbol{\sigma}_h = \mathbf{0}$ in Ω, $u_h \in C(\overline{\Omega})$, and $u_h = 0$ on $\partial\Omega$. Now, if elements of V_h are piecewise constant, the proof is concluded. Otherwise, since $\nabla_h V_h$ is a subspace of $\boldsymbol{\Sigma}_h$, by integrating by parts the first equation of (2.10), we obtain $\nabla_h u_h = \mathbf{0}$ in Ω and thus $u_h = 0$ in Ω. In other words, Problem (2.10) admits only the trivial solution. Therefore, existence is consequence of a finite dimensional Fredholm's alternative theorem. $\square$

Our next concern is the stability of the scheme (2.4). In order to set the approximation spaces, we denote by $\mathbf{P}_\kappa(T)$ the space of polynomials of degree at most κ on T, for a given integer $\kappa \geq 0$ and for each $T \in \mathcal{T}_h$. Now, in what follows we consider $\boldsymbol{\Sigma}_h$ and V_h as

$$\boldsymbol{\Sigma}_h := \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^2 : \quad \boldsymbol{\tau}\big|_T \in [\mathbf{P}_r(T)]^2 \quad \forall T \in \mathcal{T}_h\Big\},$$

$$V_h := \Big\{v_h \in L^2(\Omega) : \quad v_h\big|_T \in \mathbf{P}_k(T) \quad \forall T \in \mathcal{T}_h\Big\}\,,$$

with $k \geq 0$ and $r \geq 1$. We notice that, In order to verify the well known *mild condition*: $\nabla_h V_h$ is a subspace of $\boldsymbol{\Sigma}_h$, we require $k \leq r+1$. This choice of spaces allows us to establish the following discrete inf-sup condition, whose proof follows the ideas given in the proof of Lemma 3.4 in [8], and requires the introduction of the local Raviart-Thomas space of order κ (cf. [19]), $RT_\kappa(T) := [\mathbf{P}_\kappa(T)]^2 \oplus \boldsymbol{x}\mathbf{P}_\kappa(T) \subseteq [\mathbf{P}_{\kappa+1}(T)]^2$.

Lemma 2.2 *Let V_h a finite dimensional subspace of $L^2(\Omega)$ such that V_h is a subspace of $\mathbf{P}_r(\mathcal{T}_h)$. Then for all $v \in V_h$, there exists $\tilde{c} > 0$, independent of the mesh size, such that*

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\setminus\{0\}} \frac{b_{DG}(\boldsymbol{\tau}, v)}{\|\boldsymbol{\tau}\|_{\boldsymbol{\Sigma}}} \geq \tilde{c}\|v\|_{L^2(\Omega)}\,.$$

Proof. We take $v \in V_h$, and define the auxiliary problem: Find $w \in H^1_0(\Omega)$ such that, in distributional sense, $-\Delta w = v$ in Ω, $w = 0$ on Γ. Now, we set $\boldsymbol{\zeta} := -\nabla w$ in Ω. Then, we have that $\text{div}(\boldsymbol{\zeta}) = v$ in Ω, which implies $\boldsymbol{\zeta} \in \boldsymbol{\Sigma}$ and $||\boldsymbol{\zeta}||_{0,\Omega} \leq ||v||_{0,\Omega}$. Now, let $\boldsymbol{\zeta}_h := RT_{r-1}\boldsymbol{\zeta} \in \boldsymbol{\Sigma}_h \cap H(\text{div};\Omega)$, which means that $\text{div}(\boldsymbol{\zeta}_h) = v$ and $||\gamma^{1/2}[\![\boldsymbol{\zeta}_h]\!]||_{0,\mathcal{E}_I} = 0$. These relations imply that

$$\sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\setminus\{\mathbf{0}\}} \frac{b_{DG}(\boldsymbol{\tau}, v)}{||\boldsymbol{\tau}||_{\boldsymbol{\Sigma}}} \geq \frac{b_{DG}(\boldsymbol{\zeta}, v)}{||\boldsymbol{\zeta}||_{\boldsymbol{\Sigma}}} \geq \frac{1}{\sqrt{2}}\frac{||v||^2_{0,\Omega}}{||v||_{0,\Omega}} = \frac{1}{\sqrt{2}}||v||_{0,\Omega}\,,$$

which ends the proof. $\square$

In order to obtain the a priori error estimates for the scheme (2.4) we need the following lemmas, which establish local approximation properties of piecewise polynomials approximations.

Lemma 2.3 *Let $\mathcal{T}_h$ be an element of a shaped-regular triangulation family $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a nonnegative integer m, let $\Pi_T^m : L^2(T) \to \mathbf{P}_m(T)$ be the linear and bounded operator given by the $L^2(T)-$orthogonal projection, which satisfies $\Pi_T^m(p) = p$ for all $p \in \mathbf{P}_m(T)$. Then there exists $C > 0$, independent of the meshsize, such that for each s, t satisfying $0 \le s \le m+1$ and $0 \le s < t$, there holds*

$$|(I - \Pi_T^m)(w)|_{s,T} \le C\, h_T^{\min\{t,m+1\}-s} ||w||_{t,T} \quad \forall\, w \in H^t(T)\,, \tag{2.11}$$

and for each $t > 1/2$ there holds

$$|(I - \Pi_T^m)(w)|_{0,\partial T} \le C\, h_T^{\min\{t,m+1\}-1/2} ||w||_{t,T} \quad \forall\, w \in H^t(T)\,, \tag{2.12}$$

Proof. We refer to [14], [16]. □

Lemma 2.4 *Let $\mathcal{T}_h$ be an element of a shape-regular family of triangulations $\{\mathcal{T}_h\}_{h>0}$, and let $T \in \mathcal{T}_h$. Given a positive integer k, let $\mathcal{E}_T^k : [H^1(T)]^2 \to RT_{k-1}(T)$ be the local interpolation operator, which satisfies $\mathrm{div}(\mathcal{E}_T^k(\boldsymbol{\tau})) = \Pi_T^{k-1}(\mathrm{div}(\boldsymbol{\tau}))$ for all $\boldsymbol{\tau} \in [H^1(T)]^2$. There exists $C > 0$, independent of the meshsize but depending on integers $l > 0$ and $s \ge 0$, such that for all $\boldsymbol{\tau} \in [H^l(T)]^2$ with $\mathrm{div}(\boldsymbol{\tau}) \in H^s(T)$ there hold*

$$||\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau})||_{[L^2(T)]^2} \le C\, h_T^l\, |\boldsymbol{\tau}|_{[H^l(T)]^2} \quad 1 \le l \le k\,, \tag{2.13}$$

and

$$||\,\mathrm{div}(\boldsymbol{\tau} - \mathcal{E}_T^k(\boldsymbol{\tau}))||_{L^2(T)} \le C\, h_T^s\, |\boldsymbol{\tau}|_{[H^s(T)]^2} \quad 0 \le s \le k\,. \tag{2.14}$$

Proof. We refer to [1]. □

Now, we introduce the application

$$|||(\boldsymbol{\tau}, v)|||_{DG} := \left(||\boldsymbol{\tau}||_{0,\Omega}^2 + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|_{0,\mathcal{E}_I}^2 + \|v\|_{0,\Omega}^2\right)^{1/2} \quad \forall\, (\boldsymbol{\tau}, v) \in \boldsymbol{\Sigma} \times L^2(\Omega)\,.$$

Through the rest of this section, we denote by $(\boldsymbol{\sigma}, u)$ and $(\boldsymbol{\sigma}_h, u_h)$ the unique solutions of (1.2) and (2.4), respectively. The strategy we propose will be, then, to obtain error estimates for $|||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|||_{DG}$ and for $\|\,\mathrm{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)\|_{0,\Omega}$.

The optimal rate of convergence of $|||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|||_{DG}$ is reported in the following result, whose proof requires the assumption: $\mathrm{div}_h\, \boldsymbol{\Sigma}_h$ is a subspace of V_h. This is satisfied when $r \le k+1$, which together with the *mild condition*, let us to conclude that $r = k+1$.

Theorem 2.2 *Assuming that $\boldsymbol{\sigma}|_T \in [H^t(T)]^2$ and $u|_T \in H^{1+t}(T)$ with $t > 1/2$, for any $T \in \mathcal{T}_h$, there exists $C_{\mathtt{err}} > 0$ is independent of the mesh size, such that*

$$\begin{aligned} &|||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|||_{DG}^2 \\ &\le\, C_{\mathtt{err}} \sum_{T \in \mathcal{T}_h} h_T^{2\min\{t,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|_{[H^t(T)]^2}^2 + \|u\|_{H^1(T)}^2 \Big\}\,. \end{aligned} \tag{2.15}$$

Proof. First, we notice that our discrete scheme (2.4) is consistent with the exact solution $(\boldsymbol{\sigma}, u)$ of (1.2). This means that

$$\left[\begin{array}{rl} a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) - b_{DG}(\boldsymbol{\tau}, u - u_h) = & 0 \quad \forall\, \boldsymbol{\tau} \in \boldsymbol{\Sigma}_h\,, \\ \\ b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v) + c_{DG}(u - u_h, v) = & 0 \quad \forall\, v \in V_h\,. \end{array}\right. \tag{2.16}$$

Now, let $\Pi\boldsymbol{\sigma} \in \boldsymbol{\Sigma}_h$ and $\Pi u \in V_h$ be suitable projections of $\boldsymbol{\sigma}$ and u, respectively. By the triangle inequality, we have

$$|||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)|||_{DG} \leq |||(\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}, u-\Pi u)|||_{DG} + |||(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \Pi u-u_h)|||_{DG} \quad (2.17)$$

Our aim is to bound $|||(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \Pi u-u_h)|||_{DG}$. To this end, we consider $\Pi\boldsymbol{\sigma}$ as the L^2-orthogonal projection of $\boldsymbol{\sigma}$ onto $\boldsymbol{\Sigma}_h \cap [C(\bar{\Omega})]^2$ and Πu as the L^2- orthogonal projection of u onto V_h. We also introduce $(e_h^{\boldsymbol{\sigma}}, e_h^u) := (\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \Pi u-u_h) \in \boldsymbol{\Sigma}_h \times V_h$.

In what follows, we first use the definition of the bilinear forms $a_{DG}(\cdot,\cdot)$ and $c_{DG}(\cdot,\cdot)$. Hence, by adding and subtracting the exact solution, and after doing some algebraic manipulations, we deduce that

$$\begin{aligned} ||e_h^{\boldsymbol{\sigma}}||_{0,\Omega}^2 + ||\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]||_{0,\mathcal{E}_I}^2 + ||\alpha^{1/2}[\![e_h^u]\!]||_{0,\mathcal{E}}^2 &= a_{DG}(e_h^{\boldsymbol{\sigma}}, e_h^{\boldsymbol{\sigma}}) + c_{DG}(e_h^u, e_h^u) \\ &= a_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}}) + b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi u - u) \\ &\quad + b_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}, e_h^u) + c_{DG}(\Pi u - u, e_h^u)\,. \end{aligned} \quad (2.18)$$

Now, we aim to bound each one of the four terms on the right hand side. To this end, we observe that $[\![\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}]\!] = 0$ on $\mathcal{E}_I$, and then, after applying Cauchy-Schwarz inequality, we deduce

$$\left|a_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}, e_h^{\boldsymbol{\sigma}})\right| \leq ||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega}||e_h^{\boldsymbol{\sigma}}||_{0,\Omega}\,, \quad (2.19)$$

$$|c_{DG}(\Pi u - u, e_h^u)| \leq \|\alpha^{1/2}[\![\Pi u - u]\!]\|_{0,\mathcal{E}}\|\alpha^{1/2}[\![e_h^u]\!]\|_{0,\mathcal{E}}\,. \quad (2.20)$$

In addition, since $\operatorname{div}_h \boldsymbol{\Sigma}_h$ is a subspace of V_h and Πu is the L^2- orthogonal projection of u onto V_h, using the definition of $b_{DG}(\cdot,\cdot)$, we deduce that

$$\begin{aligned} \left|b_{DG}(e_h^{\boldsymbol{\sigma}}, \Pi u - u)\right| &= \left|\int_{\mathcal{E}_I} \gamma^{-1/2}\big(\{\Pi u - u\} - \boldsymbol{\beta}\cdot[\![\Pi u - u]\!]\big)\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\right| \\ &\leq c\left(\|\gamma^{-1/2}\{\Pi u - u\}\|_{0,\mathcal{E}_I} + \|\gamma^{-1/2}[\![\Pi u - u]\!]\|_{0,\mathcal{E}_I}\right)\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I}\,. \end{aligned} \quad (2.21)$$

with $C > 0$ being a constant independent of the mesh size. For the remaining term, we need to introduce the Raviart-Thomas projection of $\boldsymbol{\sigma}$ of order k, $RT_k(\boldsymbol{\sigma}) \in \boldsymbol{\Sigma}_h \cap H(\operatorname{div};\Omega)$. Thus, we have $\operatorname{div}(RT_k(\boldsymbol{\sigma})) = \Pi(\operatorname{div}(\boldsymbol{\sigma})) \in V_h$ and $[\![RT_k(\boldsymbol{\sigma})]\!] = 0$ on $\mathcal{E}_I$ (see Section 3 in [15]). Introducing $\tilde{e}_h^{\boldsymbol{\sigma}} := \Pi\boldsymbol{\sigma} - RT_k(\boldsymbol{\sigma}) \in \boldsymbol{\Sigma}_h \cap H(\operatorname{div};\Omega)$, and using (2.16), we realize that

$$\begin{aligned} b_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}, e_h^u) &= \int_\Omega \operatorname{div}_h(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma})e_h^u = \int_\Omega \operatorname{div}_h(\tilde{e}_h^{\boldsymbol{\sigma}})e_h^u \\ &= \int_\Omega \operatorname{div}_h(\tilde{e}_h^{\boldsymbol{\sigma}})(\Pi u - u_h) = \int_\Omega \operatorname{div}_h(\tilde{e}_h^{\boldsymbol{\sigma}})(u-u_h) = b_{DG}(\tilde{e}_h^{\boldsymbol{\sigma}}, u-u_h) \\ &= -a_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, \tilde{e}_h^{\boldsymbol{\sigma}}) = -a_{DG}(e_h^{\boldsymbol{\sigma}}, \tilde{e}_h^{\boldsymbol{\sigma}}) \\ &\leq C\|e_h^{\boldsymbol{\sigma}}\|_{0,\Omega}\|\tilde{e}_h^{\boldsymbol{\sigma}}\|_{0,\Omega} \leq C\|e_h^{\boldsymbol{\sigma}}\|_{0,\Omega}\left(\|\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})\|_{0,\Omega} + \|\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}\|_{0,\Omega}\right). \end{aligned} \quad (2.22)$$

Thus, applying (2.19)-(2.22) in (2.18), we conclude that there exists $C_* > 0$, independent of the mesh size, such that

$$||e_h^{\boldsymbol{\sigma}}||_{0,\Omega} \,+\, ||\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]||_{0,\mathcal{E}_I} \,+\, ||\alpha^{1/2}[\![e_h^u]\!]||_{0,\mathcal{E}}$$

$$\leq\, C_*\,\left(||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega} \,+\, \|\alpha^{1/2}\{\Pi u-u\}\|_{0,\mathcal{E}_I} \,+\, ||\alpha^{1/2}[\![u-\Pi u]\!]||_{0,\mathcal{E}} \,+\, ||\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})||_{0,\Omega}\right)\,. \tag{2.23}$$

Now, we focus on $||e_h^u||_{0,\Omega}$. To this end, we first notice that for any $\boldsymbol{\tau} \in \boldsymbol{\Sigma}_h$

$$b_{DG}(\boldsymbol{\tau}, e_h^u) = b_{DG}(\boldsymbol{\tau}, \Pi u - u) + b_{DG}(\boldsymbol{\tau}, u - u_h)\,.$$

Thanks to the first equation in (2.16), we obtain that

$$b_{DG}(\boldsymbol{\tau}, e_h^u) = b_{DG}(\boldsymbol{\tau}, \Pi u - u) + a_{DG}(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}, \boldsymbol{\tau}) - a_{DG}(e_h^{\boldsymbol{\sigma}}, \boldsymbol{\tau})\,.$$

Taking into account the inf-sup condition given by Lemma 2.2, and bounding each term in bilinear forms a_{DG} and b_{DG}, we estimate

$$\begin{aligned}\tilde{c}\,||e_h^u||_{0,\Omega} \,\leq\, & \sup_{\boldsymbol{\tau}\in\boldsymbol{\Sigma}_h\setminus\{0\}} \frac{b_{DG}(\boldsymbol{\tau}, e_h^u)}{||\boldsymbol{\tau}||_{\boldsymbol{\Sigma}}} \\ \leq\, & \hat{c}\Bigg(||u-\Pi u||_{0,\Omega} \,+\, ||\alpha^{1/2}\{u-\Pi u\}||_{0,\mathcal{E}_I} \,+\, ||\alpha^{1/2}[\![u-\Pi u]\!]||_{0,\mathcal{E}} \\ & \quad +\, ||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega} + ||\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})||_{0,\Omega}\Bigg)\,,\end{aligned} \tag{2.24}$$

where we have also taken into account the bound for $e_h^{\boldsymbol{\sigma}}$ given in (2.23). Finally, the conclusion follows from (2.17), (2.23), (2.24) and the well-known approximation results of the projection operators $\Pi\boldsymbol{\sigma}$, $RT_k(\boldsymbol{\sigma})$ and Πu that we have introduced in Lemmas 2.3 and 2.4. □

The error estimate for $\|\operatorname{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}$ is presented in the next theorem.

Theorem 2.3 *Assuming that $\boldsymbol{\sigma}|_T \in [H^t(T)]^2$, $\operatorname{div}(\boldsymbol{\sigma}) \in H^t(T)$ and $u|_T \in H^1(T)$ with $t > 1/2$, for each $T \in \mathcal{T}_h$, , there exists $C_2 > 0$ is independent of the mesh size, such that*

$$\begin{aligned}&\|\operatorname{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega}^2 \\ &\leq\, C_2 \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}}\Big\{\|\boldsymbol{\sigma}\|_{[H^t(T)]^2}^2 + \|\operatorname{div}(\boldsymbol{\sigma})\|_{H^t(T)}^2 + \|u\|_{H^1(T)}^2\Big\}\,.\end{aligned} \tag{2.25}$$

Proof. First, we denote again by $RT_k(\boldsymbol{\sigma})$ the Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k, onto $\boldsymbol{\Sigma}_h \cap H(\mathrm{div};\Omega)$. Then, applying the triangle inequality, we deduce

$$\|\operatorname{div}_h(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega} \leq \|\operatorname{div}_h(\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\operatorname{div}_h(RT_k(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h)\|_{0,\Omega}\,.$$

A straightforward application of Lemma 2.4, implies that

$$\|\operatorname{div}_h(\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma}))\|_{0,\Omega} \leq c\sum_{T\in\mathcal{T}_h} h_T^t\|\operatorname{div}(\boldsymbol{\sigma})\|_{t,T}$$

For the second term, we denote by $\hat{e}_h^{\boldsymbol{\sigma}} := RT_k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h$. Let $v \in V_h$. Then we have

$$\int_\Omega \mathrm{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v = \int_\Omega \mathrm{div}_h(RT_k(\boldsymbol{\sigma}))v \;-\; \int_\Omega \mathrm{div}_h(\boldsymbol{\sigma}_h)v\,.$$

Since $\displaystyle\int_\Omega \mathrm{div}_h(RT_k(\boldsymbol{\sigma}))v = \int_{\mathcal{T}_h} \Pi_k(\mathrm{div}_h(\boldsymbol{\sigma}))v = \int_\Omega \mathrm{div}(\boldsymbol{\sigma})v$, we deduce

$$\int_\Omega \mathrm{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v = \int_\Omega \mathrm{div}_h(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)v \;=\; b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v) \;+\; \int_{\mathcal{E}_I} (\{v\} - [\![v]\!] \cdot \boldsymbol{\beta})[\![\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!] \;\;.$$

Furthermore, using (2.16) we note that

$$b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v) = c_{DG}(u - u_h, v) = c_{DG}(\Pi u - u_h, v) + c_{DG}(u - \Pi u, v)$$

hence, replacing this identity in the above equality, we deduce

$$\begin{aligned}\int_\Omega \mathrm{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v \;\;&= b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, v) \;+\; \int_{\mathcal{E}_I} (\{v\} - [\![v]\!] \cdot \boldsymbol{\beta})[\![e_h^{\boldsymbol{\sigma}}]\!] \\ &= c_{DG}(\Pi u - u_h, v) + c_{DG}(u - \Pi u, v) \;+\; \int_{\mathcal{E}_I} \gamma^{-1/2}(\{v\} - [\![v]\!] \cdot \boldsymbol{\beta})\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!] \;\;. \\ &= c_{DG}(e_h^u, v) + c_{DG}(u - \Pi u, v) \;+\; \int_{\mathcal{E}_I} \gamma^{-1/2}(\{v\} - [\![v]\!] \cdot \boldsymbol{\beta})\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\end{aligned}$$

In this way, bounding each term of the bilinear forms and using (2.23), we deduce

$$||\,\mathrm{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})||_{0,\Omega} \le \sup_{v \in V_h \setminus \{0\}} \frac{\displaystyle\int_\Omega \mathrm{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})v}{||v||_{0,\Omega}}$$

$$\le C\, \left(||\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}||_{0,\Omega} \;+\; ||\alpha^{1/2}[\![u - \Pi u]\!]||_{0,\mathcal{E}} \;+\; ||\boldsymbol{\sigma} - RT_k(\boldsymbol{\sigma})||_{0,\Omega}\right)\;.$$

Then, the proof follows from Lemmas 2.3 and 2.4. $\square$

Remark 2.1 *In summary, the analysis developed in this section shows us that* $[\mathbf{P}_{k+1}(\mathcal{T}_h)]^2 \times \mathbf{P}_k(\mathcal{T}_h)$*, with* $k \geq 0$*, define a set of stable pairs for the dual mixed DG approach (2.4).*

Remark 2.2 *Let* $\rho > 0$ *be the density,* $\mathbf{g}$ *the gravity vector,* g_c *a conversion constant,* φ *the volumetric flow rate source or sink, and* ψ *the normal component of the velocity field on the boundary such that the data* φ *and* ψ *satisfy the compatibility constraint* $\int_\Omega \varphi = \int_\Gamma \psi$*. Denoting by* $\boldsymbol{f} := -\frac{\rho}{g_c}\mathbf{g}$*, we have that a version of the Darcy problem reads: Find the Darcy velocity vector* $\boldsymbol{v} : \Omega \to \mathbb{R}^2$ *and the pressure* $p : \Omega \to \mathbb{R}$ *such that*

$$\left\{\begin{array}{c} \boldsymbol{v} + \mathcal{K}\nabla p = \boldsymbol{f} \quad in \quad \Omega\,, \quad \mathrm{div}(\boldsymbol{v}) = \varphi \quad in \quad \Omega, \\ \\ and \quad \boldsymbol{v} \cdot \boldsymbol{\nu} = \psi \quad on \quad \Gamma\,, \end{array}\right. \tag{2.26}$$

where $\mathcal{K} \in [L^\infty(\Omega)]^{2\times 2}$ *in general is a given symmetric and uniformly positive definite matrix-valued function. However, in many applications it is assumed that the medium is* *isotropic. This allows us to set $\mathcal{K} = \frac{\kappa}{\mu}\mathbf{I}$, where $\kappa > 0$ and $\mu > 0$ denote, respectively, the permeability and the viscosity of the porous medium, and $\mathbf{I}$ being the identity matrix. Then, it is not difficult to see that the treatment of the model Problem (1.1) is similar to a Poisson problem with Neumann boundary condition, when considering a dual mixed formulation. Therefore, the results in Sections 2 can be extended to Darcy flow (1.1) in a natural way, once the analysis of Poisson problem with Dirichlet boundary condition is extended to Neumann boundary condition.*

3 The Stokes system

In this section, we concentrate our efforts in the extension of the results developed in the previous Section 2 to the incompressible Stokes problem: given the source terms $\boldsymbol{f} \in [L^2(\Omega)]^2$ and $\boldsymbol{g} \in [H^{1/2}(\Gamma)]^2$ we look for the velocity $\boldsymbol{u}$ and the pressure p that satisfy

$$-\nu\, \Delta \boldsymbol{u} + \nabla p = \boldsymbol{f} \quad \text{in} \quad \Omega\,, \quad \operatorname{div}(\boldsymbol{u}) = 0 \quad \text{in} \quad \Omega\,, \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma\,. \tag{3.1}$$

Hereafter, Ω is a bounded and simply connected domain in the plane with polygonal boundary Γ.

Next, we proceed to write (3.1) as a linear system of first order. To this aim, we proceed as in [8] (cf. Section 2), and introduce the pseudostress $\boldsymbol{\sigma} := \nu\,\nabla\boldsymbol{u} - p\,\boldsymbol{I}$ in Ω. This allows us to eliminate the pressure in (3.1), since it is not difficult to deduce $p = -\frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma})$. In order to ensure the uniqueness of the solution of (3.1), we require that $p \in L^2_0(\Omega)$ which is equivalent to ask that $\boldsymbol{\sigma}$ lives in $\underline{\boldsymbol{\Sigma}}_0 := \{\boldsymbol{\tau} \in H(\mathbf{div};\Omega) \,:\, \int_\Omega \mathrm{tr}(\boldsymbol{\sigma}) = 0\}$.

Then, we arrive to the dual mixed formulation: *Find* $(\boldsymbol{\sigma}, \boldsymbol{u}) \in \underline{\boldsymbol{\Sigma}}_0 \times [H^1(\Omega)]^2$

$$\boldsymbol{\sigma}^d = \nu\nabla\boldsymbol{u} \quad \text{in} \quad \Omega\,, \quad \mathbf{div}(\boldsymbol{\sigma}) = -\boldsymbol{f} \quad \text{in} \quad \Omega, \quad \text{and} \quad \boldsymbol{u} = \boldsymbol{g} \quad \text{on} \quad \Gamma\,. \tag{3.2}$$

In order to approximate the solution of the Problem (3.2), we consider the DG scheme introduced and analized in [8]. To this end, we propose the discrete spaces $\underline{\boldsymbol{\Sigma}}_h$, $\underline{\boldsymbol{\Sigma}}_{h,0}$ and $\boldsymbol{\mathcal{V}}_h$ as follows

$$\underline{\boldsymbol{\Sigma}}_h := \Big\{\boldsymbol{\tau} \in [L^2(\Omega)]^{2\times 2} \,:\quad \boldsymbol{\tau}\big|_T \in [\mathbf{P}_r(T)]^{2\times 2} \quad \forall\, T \in \mathcal{T}_h\Big\},$$

$$\underline{\boldsymbol{\Sigma}}_{h,0} := \Big\{\boldsymbol{\tau} \in \underline{\boldsymbol{\Sigma}}_h \,:\quad \int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = 0\Big\},$$

$$\boldsymbol{\mathcal{V}}_h := \Big\{\boldsymbol{v}_h \in [L^2(\Omega)]^2 \,:\quad \boldsymbol{v}_h\big|_T \in [\mathbf{P}_k(T)]^2 \quad \forall\, T \in \mathcal{T}_h\Big\}\,,$$

with $k \geq 0$ and $r \geq 1$.

Then, problem (3.2) reads: *Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \underline{\boldsymbol{\Sigma}}_{h,0} \times \boldsymbol{\mathcal{V}}_h$ *such that*

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u}_h) &= \quad G_{DG}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau} \in \underline{\boldsymbol{\Sigma}}_{h,0}\,, \\ -b_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h, \boldsymbol{v}) &= \quad F_{DG}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h\,. \end{aligned} \tag{3.3}$$

Here the bilinear forms $a_{DG} : \underline{\boldsymbol{\Sigma}} \times \underline{\boldsymbol{\Sigma}} \to \mathbb{R}$, $c_{DG} : [H^\epsilon(\mathcal{T}_h)]^2 \times [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$ and $b_{DG} : \underline{\boldsymbol{\Sigma}} \times [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$ are defined by

$$a_{DG}(\boldsymbol{\sigma}, \boldsymbol{\tau}) := \frac{1}{\nu}\int_\Omega \boldsymbol{\sigma}^d : \boldsymbol{\tau}^d + \int_{\mathcal{E}_I} \gamma [\![\boldsymbol{\sigma}]\!] \cdot [\![\boldsymbol{\tau}]\!]\,, \qquad c_{DG}(\boldsymbol{w}, \boldsymbol{v}) := \int_{\mathcal{E}} \alpha\, \underline{[\![\boldsymbol{v}]\!]} \,:\, \underline{[\![\boldsymbol{w}]\!]}\,,$$

$$\text{and} \quad b_{DG}(\boldsymbol{\tau},\boldsymbol{v}) := \int_\Omega \boldsymbol{v}\cdot\mathbf{div}_h(\boldsymbol{\tau}) - \int_{\mathcal{E}_I} \big(\{\boldsymbol{v}\} + \underline{[\![\boldsymbol{v}]\!]}\boldsymbol{\beta}\big)\cdot[\![\boldsymbol{\tau}]\!]\,,$$

while the functionals $G_{DG} : \underline{\boldsymbol{\Sigma}} \to \mathbb{R}$ and $F_{DG} : [H^\epsilon(\mathcal{T}_h)]^2 \to \mathbb{R}$ are given by

$$G_{DG}(\boldsymbol{\tau}) := \int_{\mathcal{E}_\Gamma} \boldsymbol{g}\cdot\boldsymbol{\tau}\boldsymbol{\nu} \qquad \text{and} \qquad F_{DG}(\boldsymbol{v}) := \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v} + \int_{\mathcal{E}_\Gamma} \alpha\big(\boldsymbol{g}\otimes\boldsymbol{\nu}\big) : \big(\boldsymbol{v}\otimes\boldsymbol{\nu}\big)\,.$$

We point out that the parameters α, γ and $\boldsymbol{\beta}$, introduced here to define the numerical fluxes, are at our disposal. Indeed, they will be defined as in the previous Section. Now, we introduce the seminorm

$$|(\boldsymbol{\tau},\boldsymbol{v})|_{\underline{DG}} := \Big(||\boldsymbol{\tau}^{\mathrm{d}}||^2_{[L^2(\Omega)]^{2\times 2}} + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\boldsymbol{v}\|^2_{[L^2(\Omega)]^2}\Big)^{1/2} \quad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \underline{\boldsymbol{\Sigma}}_0 \times [L^2(\Omega)]^2,$$

and the norm

$$||(\boldsymbol{\tau},\boldsymbol{v})||_{\underline{DG}} := \big(|(\boldsymbol{\tau},\boldsymbol{v})|^2_{\underline{DG}} + \|\,\mathbf{div}_h\,\boldsymbol{\tau}\|^2_{0,\Omega}\big)^{1/2} \quad \forall\,(\boldsymbol{\tau},\boldsymbol{v}) \in \underline{\boldsymbol{\Sigma}}_0 \times [L^2(\Omega)]^2.$$

Remark 3.1 *At this point, thanks to Lemma 3.1 in [6] (see also Lemma 3.10 in [8]), we note that the norm* $||(\boldsymbol{\tau},\boldsymbol{v})||_{\underline{DG}}$ *is equivalent, on* $\underline{\boldsymbol{\Sigma}}_0 \times [L^2(\Omega)]^2$, *with the standard one, defined by*

$$|||(\boldsymbol{\tau},\boldsymbol{v})|||_{\underline{DG}} := \Big(\|\boldsymbol{\tau}\|^2_{0,\Omega} + \|\,\mathbf{div}_h\,\boldsymbol{\tau}\|^2_{0,\Omega} + \|\gamma^{1/2}[\![\boldsymbol{\tau}]\!]\|^2_{[L^2(\mathcal{E}_I)]^2} + \|\boldsymbol{v}\|^2_{[L^2(\Omega)]^2}\Big)^{1/2}\,.$$

Now with the aim to ensure existence, uniqueness, hereafter we assume that $\nabla_h\boldsymbol{\mathcal{V}}_h$ is a subspace of $\underline{\boldsymbol{\Sigma}}_h$. Then, the proof of the well posedness of (3.3), under this assumption, is very similar to the one developed in Section 3 in [8].

Theorem 3.1 *Under the assumption that* $\nabla_h\boldsymbol{\mathcal{V}}_h$ *is a subspace of* $\underline{\boldsymbol{\Sigma}}_h$, *problem* (3.3) *has one and only one solution.*

Proof. Since the linear system (3.3) is square, it is enough to show that the corresponding homogeneous system: *Find* $(\boldsymbol{\sigma}_h,\boldsymbol{u}_h) \in \underline{\boldsymbol{\Sigma}}_{h,0} \times \boldsymbol{\mathcal{V}}_h$ *such that*

$$\begin{aligned} a_{DG}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau},\boldsymbol{u}_h) &= 0 \quad \forall\,\boldsymbol{\tau} \in \underline{\boldsymbol{\Sigma}}_{h,0}\,, \\ -b_{DG}(\boldsymbol{\sigma}_h,\boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h,\boldsymbol{v}) &= 0 \quad \forall\,\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h\,. \end{aligned} \tag{3.4}$$

has only the trivial solution. To this end, we replace $\boldsymbol{\tau} := \boldsymbol{\sigma}_h$ and $\boldsymbol{v} := \boldsymbol{u}_h$ in (3.4) and, after summing the equations, we deduce

$$\frac{1}{\nu}||\boldsymbol{\sigma}_h^d||^2_{0,\Omega} + ||\gamma^{1/2}[\![\boldsymbol{\sigma}_h]\!]||^2_{0,\mathcal{E}_I} + ||\alpha^{1/2}\underline{[\![\boldsymbol{u}_h]\!]}||^2_{0,\mathcal{E}} = 0\,.$$

This let us to infer that

$$\boldsymbol{\sigma}_h^d \;\Leftrightarrow\; \boldsymbol{\sigma}_h = \frac{1}{2}\mathrm{tr}(\boldsymbol{\sigma}_h)\boldsymbol{I}\,, \tag{3.5}$$

$$[\![\boldsymbol{\sigma}_h]\!] = \mathbf{0} \quad \text{on} \quad \mathcal{E}_I \;\Leftrightarrow\; \boldsymbol{\sigma}_h \in H(\mathbf{div};\Omega)\,, \tag{3.6}$$

$$\underline{[\![\boldsymbol{u}_h]\!]} = \mathbf{0} \quad \text{on} \quad \mathcal{E} \;\Leftrightarrow\; \Big(\boldsymbol{u}_h \in C(\overline{\Omega}) \wedge \boldsymbol{u}_h = \mathbf{0} \quad \text{on} \quad \mathcal{E}_\Gamma\Big)\,. \tag{3.7}$$

Then, system (3.3) reduced to

$$\begin{aligned} \int_\Omega \boldsymbol{\nabla} \boldsymbol{u}_h : \boldsymbol{\tau} &= 0 \quad \forall \boldsymbol{\tau} \in \boldsymbol{\Sigma}_{h,0} && (3.8)\\ \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) \cdot \operatorname{div}(\boldsymbol{v}) &= 0 \quad \forall \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h \,. && (3.9) \end{aligned}$$

Let now $\boldsymbol{\tau}_h := \boldsymbol{\nabla} \boldsymbol{u}_h \in \boldsymbol{\Sigma}_h$, by hypothesis. Since

$$\int_\Omega \operatorname{tr}(\boldsymbol{\tau}_h) \,=\, \int_\Omega \operatorname{div}_h(\boldsymbol{u}_h) \,=\, \sum_{T \in \mathcal{T}_h} \int_T \operatorname{div}(\boldsymbol{u}_h) \,=\, \sum_{T \in \mathcal{T}_h} \int_{\partial T} \boldsymbol{u}_h \cdot \boldsymbol{\nu} \,=\, \int_{\mathcal{E}_\Gamma} \boldsymbol{u}_h \cdot \boldsymbol{\nu} \,=\, 0 \,,$$

we conclude that $\boldsymbol{\tau}_h \in \boldsymbol{\Sigma}_{h,0}$. Then, replacing $\boldsymbol{\tau}_h$ in (3.9) we obtain $\boldsymbol{\nabla} \boldsymbol{u}_h = \mathbf{0}$ in Ω, which together with (3.7), implies that $\boldsymbol{u}_h \in \mathbf{P}_0(\overline{\Omega})$. Since $\boldsymbol{u}_h = \mathbf{0}$ on Γ, it is concluded that $\boldsymbol{u}_h = \mathbf{0}_{\boldsymbol{\mathcal{V}}_h}$. Now, since $\boldsymbol{\sigma}_h \in H(\mathbf{div};\Omega)$, there exists a unique $\boldsymbol{w} \in [H_0^1(\Omega)]^2$ such that $\operatorname{div}(\boldsymbol{w}) = \operatorname{tr}(\boldsymbol{\sigma}_h)$. Therefore, setting $\boldsymbol{w}_h$ as the local Raviart-Thomas projection of $\boldsymbol{w}$ of order k (i.e. $\boldsymbol{w}_h|_T = \Pi_{RT}^k(\boldsymbol{w})$ for each $T \in \mathcal{T}_h$), and replacing it in (3.9), we derive that $\operatorname{tr}(\boldsymbol{\sigma}_h) = 0$ in Ω. Thus $\boldsymbol{\sigma}_h = \mathbf{0}_{\boldsymbol{\Sigma}_{h,0}}$, and we conclude the proof.

□

To ensure the stability of the discrete scheme, we require that $\mathbf{div}_h \, \underline{\boldsymbol{\Sigma}}_h$ is a subspace of $\boldsymbol{\mathcal{V}}_h$. This condition, together with the *mild condition* required in Theorem 3.1, allow us to conclude that they are valid when $r = k+1$. This yield us to deal with the pair of approximation spaces: $[\mathbf{P}_{k+1}(\mathcal{T}_h)]^{2\times 2} \times [\mathbf{P}_k(\mathcal{T}_h)]^2$, as for Poisson problem in Section 2.

Then, from now on $(\boldsymbol{\sigma}, \boldsymbol{u})$ and $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h)$ will be the unique solutions of (3.2) and (3.3), respectively.

Theorem 3.2 *Assuming, in addition, that* $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$ *and* $\boldsymbol{u}|_T \in [H^1(T)]^2$ *with* $t > 1/2$, *for all* $T \in \mathcal{T}_h$, *then we have*

$$\begin{aligned} &|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)|^2_{\underline{DG}} \\ &\leq C_{\mathtt{err}} \sum_{T \in \mathcal{T}_h} h_T^{2\min\{t,k+1\}} \Big\{ \|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times 2}} + \|\boldsymbol{u}\|^2_{[H^1(T)]^2} \Big\} \,, \end{aligned} \tag{3.10}$$

where $C_{\mathtt{err}} > 0$ *is independent of* h.

Proof. First, we notice that our discrete scheme (3.3) is consistent, i.e, if $(\boldsymbol{\sigma}, \boldsymbol{u})$ is the exact solution of (3.2), then

$$\left[\begin{aligned} a_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u} - \boldsymbol{u}_h) &= 0 \quad \forall\, \boldsymbol{\tau} \in \underline{\boldsymbol{\Sigma}}_{h,0} \,, \\ -b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) &= 0 \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h \,. \end{aligned} \right. \tag{3.11}$$

Let $\Pi\boldsymbol{\sigma} \in \underline{\boldsymbol{\Sigma}}_{h,0}$ and $\Pi\boldsymbol{u} \in \boldsymbol{\mathcal{V}}_h$ be suitable projections of $\boldsymbol{\sigma}$ and $\boldsymbol{u}$, respectively. By the triangle inequality, we have

$$|(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)|_{\underline{DG}} \,\leq\, |(\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}, \boldsymbol{u} - \Pi\boldsymbol{u})|_{\underline{DG}} \,+\, |(\Pi\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \Pi\boldsymbol{u} - \boldsymbol{u}_h)|_{\underline{DG}} \,. \tag{3.12}$$

Our aim is to bound $|(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\Pi\boldsymbol{u}-\boldsymbol{u}_h)|_{\underline{DG}}$. To this end, we let $\Pi\boldsymbol{\sigma}$ be the L^2-orthogonal projection of $\boldsymbol{\sigma}$ onto $\underline{\boldsymbol{\Sigma}}_h\cap[C(\overline{\Omega})]^{2\times 2}$, while $\Pi\boldsymbol{u}$ denotes the L^2-projection of $\boldsymbol{u}$ onto $\boldsymbol{\mathcal{V}}_h$. We also introduce $(e_h^{\boldsymbol{\sigma}},e_h^{\boldsymbol{u}}):=(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\Pi\boldsymbol{u}-\boldsymbol{u}_h)$. We notice that

$$\int_\Omega \operatorname{tr}(e_h^{\boldsymbol{\sigma}})=\int_\Omega e_h^{\boldsymbol{\sigma}}:\mathbf{I}=\int_\Omega(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):\mathbf{I}=\int_\Omega(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):\mathbf{I}=\int_\Omega \operatorname{tr}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)=0\,,$$

and then, $e_h^{\boldsymbol{\sigma}}\in\underline{\boldsymbol{\Sigma}}_{h,0}$.

Next, we test (3.11) with $(\boldsymbol{\tau},\boldsymbol{v}):=(e_h^{\boldsymbol{\sigma}},e_h^{\boldsymbol{u}})$. After adding all the equations, we deduce

$$\begin{aligned}\nu^{-1}||\big(e_h^{\boldsymbol{\sigma}}\big)^d||_{0,\Omega}^2+||\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]||_{0,\mathcal{E}_I}^2+||\alpha^{1/2}\underline{[\![e_h^{\boldsymbol{u}}]\!]}||_{0,\mathcal{E}}^2 &= a_{DG}(e_h^{\boldsymbol{\sigma}},e_h^{\boldsymbol{\sigma}})+c_{DG}(e_h^{\boldsymbol{u}},e_h^{\boldsymbol{u}})\\ &= a_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma},e_h^{\boldsymbol{\sigma}})+b_{DG}(e_h^{\boldsymbol{\sigma}},\Pi\boldsymbol{u}-\boldsymbol{u})\\ &- b_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma},e_h^{\boldsymbol{u}})+c_{DG}(\Pi\boldsymbol{u}-\boldsymbol{u},e_h^{\boldsymbol{u}})\,,\end{aligned}$$

Now, we bound each term on the right hand side deducing that

$$\big|a_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma},e_h^{\boldsymbol{\sigma}})\big|\le\frac{2}{\nu}||(e_h^{\boldsymbol{\sigma}})^{\mathtt{d}}||_{0,\Omega}||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega},$$

$$|c_{DG}(\Pi\boldsymbol{u}-\boldsymbol{u},e_h^{\boldsymbol{u}})|\le\|\alpha^{1/2}[\![\Pi\boldsymbol{u}-\boldsymbol{u}]\!]\|_{0,\mathcal{E}}\|\alpha^{1/2}[\![e_h^{\boldsymbol{u}}]\!]\|_{0,\mathcal{E}}$$

In addition, using that $\mathbf{div}(\underline{\boldsymbol{\Sigma}}_h)$ is a subspace of $\boldsymbol{\mathcal{V}}_h$ and $\Pi\boldsymbol{u}$ is the L^2- orthogonal projection of $\boldsymbol{u}$, from the definition of $b_{DG}(\cdot,\cdot)$, we deduce that

$$\begin{aligned}\big|b_{DG}(e_h^{\boldsymbol{\sigma}},\Pi\boldsymbol{u}-\boldsymbol{u})\big| &= \left|\int_{\mathcal{E}_I}\gamma^{-1/2}\big(\{\Pi\boldsymbol{u}-\boldsymbol{u}\}-\boldsymbol{\beta}\cdot[\![\Pi\boldsymbol{u}-\boldsymbol{u}]\!]\big)\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\right|\\ &\le c\left(\|\gamma^{-1/2}\{\Pi\boldsymbol{u}-\boldsymbol{u}\}\|_{0,\mathcal{E}_I}+\|\gamma^{-1/2}[\![\Pi\boldsymbol{u}-\boldsymbol{u}]\!]\|_{0,\mathcal{E}_I}\right)\|\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]\|_{0,\mathcal{E}_I}\end{aligned}$$

For the last term, we introduce the normal component continuous Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k, $RT_k(\boldsymbol{\sigma})$, then we have $\mathbf{div}(RT_k(\boldsymbol{\sigma}))\in\boldsymbol{\mathcal{V}}_h$ and $[\![RT_k(\boldsymbol{\sigma})]\!]=0$ on $\mathcal{E}_I$ (see Section 3 in [15]). Denoting by $\tilde{e}_h^{\boldsymbol{\sigma}}:=\Pi\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})$ and using (3.11) we note that

$$\begin{aligned}b_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma},e_h^{\boldsymbol{u}}) &= \int_\Omega\mathbf{div}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma})\cdot e_h^{\boldsymbol{u}}=\int_\Omega\mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot e_h^{\boldsymbol{u}}\\ &= \int_\Omega\mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot(\Pi\boldsymbol{u}-\boldsymbol{u}_h)=\int_\Omega\mathbf{div}(\tilde{e}_h^{\boldsymbol{\sigma}})\cdot(\boldsymbol{u}-\boldsymbol{u}_h)\\ &= -a_{DG}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h,\tilde{e}_h^{\boldsymbol{\sigma}})=-\frac{1}{\nu}\int_\Omega(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)^{\mathtt{d}}:(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathtt{d}}=-\frac{1}{\nu}\int_\Omega(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathtt{d}}\\ &= -\frac{1}{\nu}\int_\Omega(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h):(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathtt{d}}=-\frac{1}{\nu}\int_\Omega(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)^{\mathtt{d}}:(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathtt{d}}=-a_{DG}(e_h^{\boldsymbol{\sigma}},\tilde{e}_h^{\boldsymbol{\sigma}})\\ &\le\frac{1}{\nu}||(e_h^{\boldsymbol{\sigma}})^{\mathtt{d}}||_{0,\Omega}\|(\tilde{e}_h^{\boldsymbol{\sigma}})^{\mathtt{d}}\|_{0,\Omega}\le\frac{1}{\nu}||(e_h^{\boldsymbol{\sigma}})^{\mathtt{d}}||_{0,\Omega}\,||e_h^{\boldsymbol{\sigma}}||_{0,\Omega}\\ &\le\frac{2}{\nu}||(e_h^{\boldsymbol{\sigma}})^{\mathtt{d}}||_{0,\Omega}\bigg(\|\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})\|_{0,\Omega}+\|\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}\|_{0,\Omega}\bigg).\end{aligned}$$

In this way, we deduce that there exists $C_* > 0$, independent of the mesh size, such that

$$\frac{1}{\sqrt{\nu}}||\left(e_h^{\boldsymbol{\sigma}}\right)^d||_{0,\Omega} + ||\gamma^{1/2}[\![e_h^{\boldsymbol{\sigma}}]\!]||_{0,\mathcal{E}_I} + ||\alpha^{1/2}\underline{[\![e_h^{\boldsymbol{u}}]\!]}||_{0,\mathcal{E}}$$

$$\leq C_*\Bigg(||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega} + \|\gamma^{-1/2}\{\Pi\boldsymbol{u}-\boldsymbol{u}\}\|_{0,\mathcal{E}_I} + ||\alpha^{1/2}\underline{[\![\boldsymbol{u}-\Pi\boldsymbol{u}]\!]}||_{0,\mathcal{E}} \tag{3.13}$$

$$+||\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})||_{0,\Omega}\Bigg).$$

On the other hand, concerning to $||e_h^{\boldsymbol{u}}||_{0,\Omega}$, we first notice that for any $\boldsymbol{\tau}\in\underline{\boldsymbol{\Sigma}}_{h,0}$

$$b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}}) = b_{DG}(\boldsymbol{\tau}, \Pi\boldsymbol{u}-\boldsymbol{u}) + b_{DG}(\boldsymbol{\tau}, \boldsymbol{u}-\boldsymbol{u}_h),$$

and using the first equation in (3.11), we deduce that

$$b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}}) = b_{DG}(\boldsymbol{\tau}, \Pi\boldsymbol{u}-\boldsymbol{u}) + a_{DG}(\Pi\boldsymbol{\sigma}-\boldsymbol{\sigma}, \boldsymbol{\tau}) - a_{DG}(e_h^{\boldsymbol{\sigma}}, \boldsymbol{\tau}).$$

Thanks to an analogous inf-sup condition to the Lemma 3.4 in [8] (whose proof should be quite similar to the one of Lemma 2.2), and bounding each term in bilinear forms a_{DG} and b_{DG}, we estimate

$$\tilde{c}\,||e_h^{\boldsymbol{u}}||_{0,\Omega} \leq \sup_{\boldsymbol{\tau}\in\underline{\boldsymbol{\Sigma}}_{h,0}\backslash\{\boldsymbol{\theta}\}} \frac{b_{DG}(\boldsymbol{\tau}, e_h^{\boldsymbol{u}})}{||\boldsymbol{\tau}||_{\underline{\boldsymbol{\Sigma}}}} \leq \hat{c}\Bigg(\|\gamma^{-1/2}\{\Pi\boldsymbol{u}-\boldsymbol{u}\}\|_{0,\mathcal{E}_I} + ||\alpha^{1/2}\underline{[\![\boldsymbol{u}-\Pi\boldsymbol{u}]\!]}||_{0,\mathcal{E}}$$

$$+ ||\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma})||_{0,\Omega} + ||\boldsymbol{\sigma}-\Pi\boldsymbol{\sigma}||_{0,\Omega}\Bigg),$$

where we have also taken into account the bound for $||(e_h^{\boldsymbol{\sigma}})^{\mathtt{d}}||_{0,\Omega}$ given in (3.13).

Finally, the conclusion follows from (3.12) and the well-known approximation results of the projection operators $\Pi\boldsymbol{\sigma}$, $RT_k(\boldsymbol{\sigma})$ and $\Pi\boldsymbol{u}$ we have introduced. $\square$

The error in the divergence of $\boldsymbol{\sigma}$ is presented in the next theorem.

Theorem 3.3 *Assuming that $\boldsymbol{\sigma}|_T \in [H^t(T)]^{2\times 2}$, $\mathbf{div}(\boldsymbol{\sigma}) \in [H^t(T)]^2$ and $\boldsymbol{u}|_T \in [H^1(T)]^2$ with $t > 0$, for each $T \in \mathcal{T}_h$, , there exists $C_2 > 0$ is independent of the mesh size, such that*

$$\|\,\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|^2_{0,\Omega}$$

$$\leq C_2 \sum_{T\in\mathcal{T}_h} h_T^{2\min\{t,k+1\}}\Big\{\|\boldsymbol{\sigma}\|^2_{[H^t(T)]^{2\times2}} + \|\,\mathbf{div}(\boldsymbol{\sigma})\|^2_{[H^t(T)]^2} + \|\boldsymbol{u}\|^2_{[H^1(T)]^2}\Big\}. \tag{3.14}$$

Proof. First, we denote again by $RT_k(\boldsymbol{\sigma})$ continuous Raviart-Thomas interpolation of $\boldsymbol{\sigma}$ of order k. Then, applying the triangle inequality, we deduce

$$\|\,\mathbf{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)\|_{0,\Omega} \leq \|\,\mathbf{div}(\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma}))\|_{0,\Omega} + \|\,\mathbf{div}(RT_k(\boldsymbol{\sigma})-\boldsymbol{\sigma}_h)\|_{0,\Omega}.$$

A straightforward application of the local interpolation property, implies that

$$\|\,\mathbf{div}_h(\boldsymbol{\sigma}-RT_k(\boldsymbol{\sigma}))\|_{0,\Omega} \leq c\sum_{T\in\mathcal{T}_h} h_T^t\|\,\mathbf{div}(\boldsymbol{\sigma})\|_{t,T}$$

For the second term, we denote by $\hat{e}_h^{\boldsymbol{\sigma}} := RT_k(\boldsymbol{\sigma}) - \boldsymbol{\sigma}_h$. Let $\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h$. Then we have

$$\int_\Omega \mathbf{div}(\hat{e}_h^{\boldsymbol{\sigma}}) \cdot \boldsymbol{v} = \int_\Omega \mathbf{div}(RT_k(\boldsymbol{\sigma})) \cdot \boldsymbol{v} \, - \, \int_\Omega \mathbf{div}(\boldsymbol{\sigma}_h) \cdot \boldsymbol{v} \, .$$

Since $\displaystyle\int_\Omega \mathbf{div}(RT_k(\boldsymbol{\sigma})) \cdot \boldsymbol{v} = \int_{\mathcal{T}_h} \Pi_k(\mathbf{div}(\boldsymbol{\sigma})) \cdot \boldsymbol{v} = \int_\Omega \mathbf{div}(\boldsymbol{\sigma}) \cdot \boldsymbol{v}$, we deduce

$$\int_\Omega \mathbf{div}(\hat{e}_h^{\boldsymbol{\sigma}}) \cdot \boldsymbol{v} = \int_\Omega \mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h) \cdot \boldsymbol{v} \; = \; b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{v}) \quad .$$

Furthermore, using (3.11) we note that

$$b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{v}) = c_{DG}(\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) = c_{DG}(\Pi\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u} - \Pi\boldsymbol{u}, \boldsymbol{v})$$

hence, replacing this identity in the above equality, we deduce

$$\begin{aligned} \int_\Omega \mathbf{div}(\hat{e}_h^{\boldsymbol{\sigma}}) \cdot \boldsymbol{v} \quad &= b_{DG}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{v}) \\ &= c_{DG}(\Pi\boldsymbol{u} - \boldsymbol{u}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u} - \Pi\boldsymbol{u}, \boldsymbol{v}) \\ &= c_{DG}(e_h^{\boldsymbol{u}}, \boldsymbol{v}) + c_{DG}(\boldsymbol{u} - \Pi\boldsymbol{u}, \boldsymbol{v}) \end{aligned}$$

In this way, bounding each term of the bilinear forms and using (3.13), we deduce

$$|| \operatorname{div}_h(\hat{e}_h^{\boldsymbol{\sigma}})||_{0,\Omega} \leq \sup_{\boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h \backslash \{0\}} \frac{\displaystyle\int_\Omega \operatorname{div}(\hat{e}_h^{\boldsymbol{\sigma}}) \cdot \boldsymbol{v}}{||\boldsymbol{v}||_{0,\Omega}}$$

$$\leq C \, \left(\|\gamma^{-1/2}\{\Pi\boldsymbol{u} - \boldsymbol{u}\}\|_{0,\mathcal{E}_I} + ||\boldsymbol{\sigma} - \Pi\boldsymbol{\sigma}||_{0,\Omega} \, + \, ||\alpha^{1/2}[\![\boldsymbol{u} - \Pi\boldsymbol{u}]\!]||_{0,\mathcal{E}} \, + \, ||\boldsymbol{\sigma} - RT_k(\boldsymbol{\sigma})||_{0,\Omega} \right) \, .$$

Then, the proof follows from approximation properties. $\square$

Remark 3.2 *In summary, the analysis developed in this section allows us to consider the set of pairs* $[\mathbf{P}_{k+1}(\mathcal{T}_h)]^{2\times 2} \times [\mathbf{P}_k(\mathcal{T}_h)]^2$*, with* $k \geq 0$*, since each one of them is stable for the dual mixed DG approach (3.3) of Stokes system.*

4 Numerical examples

In this section we present several examples illustrating our results for the Poisson problem (cf. (2.10)) and Stokes one (cf. (3.3)). All the numerical results given below have been obtained using a Matlab code. In addition, the errors on each triangle are computed applying a 7-points quadrature rule. We consider, for both problems, the lowest polynomial approximation spaces: $[\mathbf{P}_1(\mathcal{T}_h)]^2 - \mathbf{P}_0(\mathcal{T}_h)$ and $[\mathbf{P}_1(\mathcal{T}_h)]^{2\times 2} - [\mathbf{P}_0(\mathcal{T}_h)]^2$ for Poisson and Stokes problems, respectively (which means that in this case $k = 0$). Concerning the parameters that defines both two discrete schemes, we set $\boldsymbol{\beta} := (1\ 1)^t$, $\gamma := \frac{1}{\mathsf{h}}$ and $\alpha := \mathsf{h}$.

Example	Ω	$u(x_1,x_2)$
1	$]0,1[^2$	$\frac{1}{3}(x_1^3x_2 - x_2^3x_1)$
2	$]-1,1[^2 \setminus [0,1]\times[-1,0]$	$r^{2/3}\sin\left(\frac{2}{3}\theta\right)$
3	$\{(x_1,x_2) \,:\, x_1^2+x_2^2=1\} \setminus [0,1]\times[-1,0]$	$r^{2/3}\sin\left(\frac{2}{3}\theta\right)$

Table 1: Examples considered for Poisson problem

4.1 Numerical examples for Poisson equation

Here we first introduce some useful notations for errors and experimental rates of convergence. Let N the number of degrees of freedom, $\boldsymbol{e}_0(u) := ||u-u_h||_{0,\Omega}$, $\boldsymbol{e}_0(\boldsymbol{\sigma}) := \left(||\boldsymbol{\sigma}-\boldsymbol{\sigma}_h||^2_{0,\Omega} + ||\gamma^{1/2}[\![\boldsymbol{\sigma}-\boldsymbol{\sigma}_h]\!]||^2_{0,\mathcal{E}_I}\right)^{1/2}$, $\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma}) := ||\operatorname{div}(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h)||_{0,\Omega}$, and $\boldsymbol{e} := ||(\boldsymbol{\sigma}-\boldsymbol{\sigma}_h, u-u_h)||_{DG}$. We point out that in this case we have $N = 8\times\mathrm{card}(\mathcal{T}_h)$. Considering that in 2D h behaves as $N^{-1/2}$, we set the so called experimental rate of convergence of the global error $\boldsymbol{e}$ as

$$r := -\frac{1}{2}\frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')},$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ denote the corresponding errors at two consecutive triangulations with number of degrees of freedom N and N', respectively. The experimental rates of convergence for the other errors are defined in analogous way.

We present three examples. Their domain Ω as well as their corresponding exact solution u are given in Table 1. We notice that the first example has a smooth solution. Then, it is expected that rate of convergence for global error $\boldsymbol{e}$ be close to 1, as well as for $\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma})$, since we are using the lowest order of discrete approximation space for each unknown. The results shown in Table 3 are in agreement with this. The exact solution for examples 2 and 3 is the same, is given in polar coordinates, and lives in $H^{1+2/3}(\Omega)$, since its gradient has a singularity at the origin. We point out that the results for Example 3 are not covered by the current work, since the corresponding domain does not have a polygonal boundary. The rates of convergence for each one of the introduced errors behave as Theorems 3.1 and 3.2 predicts: $\mathcal{O}(h)$, since in these cases $\operatorname{div}(\boldsymbol{\sigma}) = 0$. These are shown in Tables 4 and 5.

4.2 Numerical examples for Stokes system

We first note that since the search of a suitable basis of $\boldsymbol{\Sigma}_{h,0}$ is very difficult, we introduce the zero mean value condition of trace of elements of $\boldsymbol{\Sigma}_h$ with the help of a Lagrange multiplier. This allows us to establish the following result.

Theorem 4.1 *Consider the problem: Find* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \lambda) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h \times \mathbb{R}$ *such that*

$$a_{DG}(\boldsymbol{\sigma}_h,\boldsymbol{\tau}) + b_{DG}(\boldsymbol{\tau},\boldsymbol{u}_h) + \lambda\int_\Omega \mathrm{tr}(\boldsymbol{\tau}) = \quad G_{DG}(\boldsymbol{\tau}) \quad \forall\, \boldsymbol{\tau}\in\underline{\boldsymbol{\Sigma}}_h\,,$$

$$
\begin{aligned}
-b_{DG}(\boldsymbol{\sigma}_h, \boldsymbol{v}) + c_{DG}(\boldsymbol{u}_h, \boldsymbol{v}) &= F_{DG}(\boldsymbol{v}) \quad \forall\, \boldsymbol{v} \in \boldsymbol{\mathcal{V}}_h \\
\mu \int_\Omega \operatorname{tr}(\boldsymbol{\sigma}_h) &= 0 \qquad \forall\, \mu \in \mathbb{R}\,.
\end{aligned} \tag{4.1}
$$

Then, we have

1. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, \lambda) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h \times \mathbb{R}$ *is a solution of* (4.1)*, then* $\lambda = 0$ *and* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{\mathcal{V}}_h$ *is a solution of* (3.3)*.*
2. *If* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h) \in \boldsymbol{\Sigma}_{h,0} \times \boldsymbol{\mathcal{V}}_h$ *is a solution of* (3.3)*, then* $(\boldsymbol{\sigma}_h, \boldsymbol{u}_h, 0) \in \boldsymbol{\Sigma}_h \times \boldsymbol{\mathcal{V}}_h \times \mathbb{R}$ *is a solution of* (4.1)*.*

Then, we proceed to implement (4.1). As for Poisson problem, we need to introduce some useful notations for the errors and experimental rates of convergence. We let N be the number of degrees of freedom, that in our case corresponds to $N = 14 \times \operatorname{card}(\mathcal{T}_h) + 1$. We also introduce $\boldsymbol{e}_0(\boldsymbol{u}) := ||\boldsymbol{u} - \boldsymbol{u}_h||_{0,\Omega}$, $\boldsymbol{e}_0(\boldsymbol{\sigma}) := \left(||\boldsymbol{\sigma}^d - \boldsymbol{\sigma}_h^d||^2_{0,\Omega} + ||\gamma^{1/2} [\![\boldsymbol{\sigma} - \boldsymbol{\sigma}_h]\!]||^2_{0,\mathcal{E}_I} \right)^{1/2}$, $\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma}) := ||\,\mathbf{div}(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)||_{0,\Omega}$, and $\boldsymbol{e} := |(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, u - u_h)|_{\underline{DG}}$. The so called experimental rate of convergence of the seminorm of the total error, $\boldsymbol{e}$, is computed by

$$
r := -\frac{1}{2} \frac{\log(\boldsymbol{e}/\boldsymbol{e}')}{\log(N/N')}\,,
$$

where $\boldsymbol{e}$ and $\boldsymbol{e}'$ denote the corresponding errors at two consecutive triangulations with number of degrees of freedom N and N', respectively. The experimental rates of convergence for the other errors are defined in analogous way.

We consider two smooth examples. Their domain Ω as well as their corresponding exact solution $(\boldsymbol{u}, p)$ are given in Table 2. Concerning Example 1, we resume our results in Table 1, where the total error and their components goes to zero as $\mathcal{O}(h)$. This is in agreement with our expectations. In addition, we observe that the L^2 norms of the stress error $(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h)$ and of the pressure $(p - p_h)$, have larger rates of convergence: $\mathcal{O}(h^2)$.

Example 2 is taken from [17], where the parameter λ is given by

$$
\lambda \,:=\, -\frac{8\pi^2}{\nu^{-1} + \sqrt{\nu^{-2} + 16\pi^2}}\,.
$$

It will help us to test the robustness of our method, for different values of viscosity: $\nu \in \{1, 0.1, 0.059\}$. Numerical results for each one of these values for ν, are shown in Tables 7, 8 and 9. From these tables, we realize that the individual error $\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$ dominates the seminorm $\boldsymbol{e}$, and then it will determine the behavior $\mathcal{O}(h)$ of the total error $||(\boldsymbol{\sigma} - \boldsymbol{\sigma}_h, \boldsymbol{u} - \boldsymbol{u}_h)||_{\underline{\boldsymbol{\Sigma}}}$, as expected. We also notice that the errors $\boldsymbol{e}_0(\boldsymbol{\sigma})$ and $\boldsymbol{e}_0(p)$ have a better behavior than expected: $\mathcal{O}(h^2)$, for each one of the considered values for ν here.

5 Final comments and conclusions

In this paper, we have first extended the techniques shown in [8] to the case of Lagrangian finite elements, to approximate each unknown, for a mixed discontinuous Galerkin formulation of the Poisson equation. We have proved that the method is stable and converges

Example	Ω	$u(x_1,x_2)$	$p(x_1,x_2)$
1	$]-1,1[^2$	$\begin{pmatrix} -e^{x_1}(x_2\,\cos(x_2)+\sin(x_2)) \\ e^{x_1}x_2\,\sin(x_2) \end{pmatrix}$	$2e^{x_1}\,\sin(x_2)$
2	$]-1/2,3/2[\times]0,2[$	$\begin{pmatrix} 1-e^{\lambda x_1}\cos(2\pi x_2) \\ \frac{\lambda}{2\pi}e^{\lambda x_1}\sin(2\pi x_2) \end{pmatrix}$	$-\frac{1}{2}e^{2\lambda x_1}-\bar{p}$

Table 2: Examples considered for Stokes system

with the optimal rate of convergence, when reasonable additional regularity of exact solution is assumed. The main relevance of the analysis relies on the fact that we have been able to obtain a discrete approximation of local $H(div)$ functions using the standard discontinuous polynomial space, instead of traditional Raviart-Thomas space. The results shown in Tables 3, 4 and 5, for $[\mathbf{P}_1(\mathcal{T}_h)]^2\times\mathbf{P}_0(\mathcal{T}_h)$ approximation spaces, are in agreement with the conclusions of the a priori error analysis we derived.

Next, we extend the approach to solve a Stokes system. We recall that in [8] we have analized a pseudostress-velocity mixed discontinuous formulation, considering the pair $[RT_k(\mathcal{T}_h)]^2\times[\mathbf{P}_k(\mathcal{T}_h)]^2$, with $k\geq 0$. We have proved that this family of approximation spaces is stable. This can be seen as a generalization of the scheme studied earlier in [12]. We point out that here we have developed an a priori error analysis for an unusual nonconforming dual-mixed variational formulation for the Poisson and Stokes problem, considering piecewise polynomial approximation spaces for each unknown. In this sense, we have circumvented the well-known Rham commutative diagram when a local subspace of $H(div)$ is used, proving the optimal convergence of the method in a non usual way. We would like to emphasize that in this paper we have proved that the pair $[\mathbf{P}_{k+1}(\mathcal{T}_h)]^{2\times 2}\times[\mathbf{P}_k(\mathcal{T}_h)]^2$, with $k\geq 0$, is stable (in pseudostress velocity formulation). In particular, for this nonconforming scheme, surprisingly we have proved that $[\mathbf{P}_1(\mathcal{T}_h)]^{2\times 2}\times[\mathbf{P}_0(\mathcal{T}_h)]^2$ is a stable pair for the Stokes problem, whereas for the corresponding conforming scheme it is well known that the pair $[\mathbf{P}_1(\Omega)]^{2\times 2}\times[\mathbf{P}_0(\mathcal{T}_h)]^2$ is not stable, and it needs some stabilization procedure in order to use it. Tables 6, 7, 8 and 9 show us that the method converges for each case, with order $\mathcal{O}(h)$, as predicted by the theory we developed here. On the other hand, the results in these tables give us numerical evidence that the L^2 error of pseudostress and pressure behave as $\mathcal{O}(h^2)$. This could be the subject of a future work.

Finally, it is important to remark that the analysis has been obtained with the optimal regularity assumptions, which give us good omens in order to think to develop the corresponding a posteriori error estimate, which will be reported in a separate work. Furthermore, the fact that the same formulation works for Stokes and Darcy should simplify its coupling, therefore this topic will be explored in another work.

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N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\text{div}}(\boldsymbol{\sigma})$	$r_{\text{div}}(\boldsymbol{\sigma})$
28	0.0225	——	0.0958	——	0.0984	——	0.2951	——
112	0.0183	0.2964	0.0541	0.8251	0.0571	0.7855	0.1771	0.7364
448	0.0099	0.8916	0.0361	0.5838	0.0374	0.6100	0.1103	0.6836
1792	0.0050	0.9758	0.0204	0.8239	0.0210	0.8335	0.0617	0.8384
7168	0.0025	0.9941	0.0107	0.9228	0.0110	0.9267	0.0327	0.9173
28672	0.0013	0.9985	0.0055	0.9640	0.0057	0.9658	0.0168	0.9574
114688	0.0006	0.9996	0.0028	0.9826	0.0029	0.9835	0.0085	0.9783

Table 3: Errors and experimental rates of convergence for Example 1 (Poisson)

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\text{div}}(\boldsymbol{\sigma})$	$r_{\text{div}}(\boldsymbol{\sigma})$
42	0.3146	——	0.7500	——	0.8133	——	0.9572	——
168	0.1635	0.9445	0.5635	0.4125	0.5867	0.4711	0.9501	0.0107
672	0.0809	1.0155	0.3253	0.7927	0.3352	0.8077	0.5062	0.9085
2688	0.0401	1.0125	0.1734	0.9073	0.1780	0.9130	0.2304	1.1353
10752	0.0200	1.0041	0.0903	0.9411	0.0925	0.9442	0.1042	1.1447
43008	0.0100	1.0011	0.0469	0.9458	0.0479	0.9483	0.0485	1.1038
172032	0.0050	1.0003	0.0245	0.9354	0.0250	0.9381	0.0232	1.0636

Table 4: Errors and experimental rates of convergence for Example 2 (Poisson)

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N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathrm{div}}(\boldsymbol{\sigma})$	$r_{\mathrm{div}}(\boldsymbol{\sigma})$
42	0.2027	——	0.4690	——	0.5109	——	0.7088	——
168	0.1090	0.8958	0.3506	0.4195	0.3672	0.4766	0.7321	——
672	0.0550	0.9859	0.2017	0.7978	0.2091	0.8125	0.4108	0.8335
2688	0.0275	0.9990	0.1084	0.8953	0.1119	0.9021	0.2060	0.9956
10752	0.0138	1.0000	0.0574	0.9176	0.0590	0.9223	0.1018	1.0177
43008	0.0069	1.0001	0.0305	0.9104	0.0313	0.9150	0.0504	1.0142
172032	0.0034	1.0001	0.0165	0.8876	0.0169	0.8926	0.0250	1.0085

Table 5: Errors and experimental rates of convergence for Example 3 (Poisson)

N	$\boldsymbol{e}_0(\boldsymbol{u})$	$r_0(\boldsymbol{u})$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
29	2.3999	——	4.6090	——	4.5215	——
113	1.2536	0.9550	3.5506	0.3836	7.6949	——
449	0.6395	0.9757	2.3368	0.6064	4.8937	0.6561
1793	0.3215	0.9935	1.3147	0.8309	2.4012	1.0284
7169	0.1610	0.9979	0.6921	0.9260	1.1352	1.0811
28673	0.0805	0.9994	0.3544	0.9655	0.5449	1.0589
114689	0.0403	0.9999	0.1793	0.9834	0.2661	1.0340
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
29	4.1739	——	2.4102	——		
113	4.0573	0.0417	2.7151	——		
449	2.0405	0.9963	1.4032	0.9569		
1793	0.6270	1.7044	0.4343	1.6941		
7169	0.1672	1.9077	0.1161	1.9032		
28673	0.0428	1.9657	0.0298	1.9633		
114689	0.0108	1.9852	0.0075	1.9839		

Table 6: Errors and experimental rates of convergence for Ex. 1 (Stokes), for $\nu = 1$

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	19.6523	——	81.5980	——	469.6606	——
225	10.1306	0.9652	65.5374	0.3193	380.7820	0.3056
897	4.9803	1.0269	28.3148	1.2137	301.9265	0.3356
3585	2.5719	0.9540	13.1790	1.1040	187.0733	0.6910
14337	1.3105	0.9728	6.4181	1.0382	99.9140	0.9050
57345	0.6585	0.9930	3.2116	0.9989	50.8263	0.9751
229377	0.3296	0.9982	1.6119	0.9946	25.5173	0.9941
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	307.4529	——	211.3465	——		
225	81.5107	1.9338	37.0654	2.5357		
897	36.2827	1.1705	19.3759	0.9381		
3585	12.1414	1.5803	7.1110	1.4470		
14337	3.1585	1.9429	1.8808	1.9190		
57345	0.7931	1.9936	0.4745	1.9869		
229377	0.1979	2.0025	0.1186	2.0004		

Table 7: Errors and experimental rates of convergence for Ex. 2 (Stokes), for $\nu = 1$

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	4.9442	——	7.2237	——	27.6457	——
225	2.6894	0.8869	3.8397	0.9205	17.7333	0.6468
897	1.4207	0.9229	3.8946	——	18.3476	——
3585	0.6790	1.0657	2.7499	0.5024	10.6941	0.7792
14337	0.3372	1.0101	1.5354	0.8409	5.4225	0.9799
57345	0.1665	1.0184	0.7939	0.9516	2.7213	0.9947
229377	0.0828	1.0077	0.4009	0.9856	1.3583	1.0025
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	14.9810	——	10.5309	——		
225	3.5498	2.0974	2.3011	2.2154		
897	2.3914	0.5713	1.6136	0.5133		
3585	1.2373	0.9512	0.8460	0.9320		
14337	0.3613	1.7761	0.2458	1.7838		
57345	0.0936	1.9493	0.0633	1.9575		
229377	0.0235	1.9916	0.0159	1.9954		

Table 8: Errors and experimental rates of convergence for Ex. 2 (Stokes), for $\nu = 0.1$

N	$\boldsymbol{e}_0(u)$	$r_0(u)$	$\boldsymbol{e}$	r	$\boldsymbol{e}_{\mathbf{div}}(\boldsymbol{\sigma})$	$r_{\mathbf{div}}(\boldsymbol{\sigma})$
57	3.2095	——	4.3081	——	13.2613	——
225	1.8459	0.8057	2.2457	0.9489	7.0122	0.9281
897	1.0290	0.8452	2.4596	—-	10.2963	——
3585	0.5340	0.9469	1.7508	0.4906	5.8493	0.8163
14337	0.2585	1.0467	1.0116	0.7916	3.0870	0.9222
57345	0.1189	1.1200	0.5325	0.9259	1.6129	0.9366
229377	0.0569	1.0640	0.2704	0.9774	0.8179	0.9796
N	$\boldsymbol{e}_0(\boldsymbol{\sigma})$	$r_0(\boldsymbol{\sigma})$	$\boldsymbol{e}_0(p)$	$r_0(p)$		
57	6.7667	——	4.7644	——		
225	2.5420	1.4261	1.7484	1.4602		
897	1.5787	0.6889	1.0874	0.6868		
3585	0.7923	0.9953	0.5423	1.0042		
14337	0.2446	1.6959	0.1647	1.7200		
57345	0.0653	1.9063	0.0433	1.9266		
229377	0.0166	1.9749	0.0110	1.9834		

Table 9: Errors and experimental rates of convergence for Ex. 2 (Stokes), for $\nu = 0.059$

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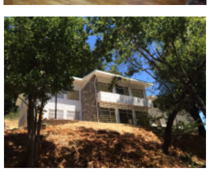

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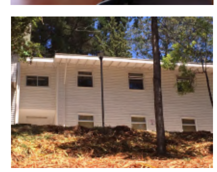

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